ON APPROXIMATIONS OF FIRST INTEGRALS FOR STRONGLY NONLINEAR OSCILLATORS

S. B. Waluya † and W.T. van Horssen

Department of Applied Mathematical Analysis,
Faculty of Information Technology and Systems, Delft University of Technology,
Mekelweg 4, 2628 CD Delft,
The Netherlands

(S.B.Waluya@its.tudelft.nl, W.T.vanHorssen@its.tudelft.nl)

Abstract. In this paper strongly nonlinear oscillator equations will be studied. It will be shown that the recently developed perturbation method based on integrating factors can be used to approximate first integrals. Not only approximations of first integrals will be given, but it will also be shown how in a rather efficient way the existence and stability of time-periodic solutions can be obtained from these approximations. In particular the generalized Rayleigh oscillator equation $\ddot{X} + 9X + \mu X^2 + \lambda X^3 = \epsilon\left(\dot{X} - \dot{X}^3\right)$ will be studied in detail, and it will be shown that at least five limit cycles can occur.

Key words. Integrating factor, integrating vector, first integral, perturbation method, asymptotic approximation of first integral, periodic solution, bifurcations, elliptic function, elliptic integrals, generalized Rayleigh oscillator equation.

1. Introduction. In [13, 14, 22, 23, 24, 25] a perturbation method based on integrating factors and vectors has been presented for regularly or singularly perturbed systems of ordinary differential equations (ODEs). When approximations of integrating vectors have been obtained an approximation of a first integral can be given. Also an error-estimate for this approximation of a first integral can be given on a time-scale. It has also been shown in [13, 23, 24, 25] how in a rather efficient way the existence and stability of time-periodic solutions can be obtained from these approximations for the first integrals. In this paper it will be shown explicitly how the perturbation method can be applied to the following strongly nonlinear oscillator equation

$$\ddot{X} + c_1 X + c_2 X^2 + c_3 X^3 = \epsilon f(X, \dot{X}), \tag{1.1}$$

where c_1, c_2, c_3 are parameters, where $0 < \epsilon \ll 1$, and where the dot represents differentiation with respect to t. Recently equation (1.1) obtained a lot of attention. For example in [1] Doelman and Verhulst studied (1.1) with $f(X, \dot{X}) = \left(1 - X^2\right)\dot{X}$ (a Van der Pol type of perturbation) by using a Melnikov/Poincaré return map technique. For $c_2 = 0$ and $f(X, \dot{X}) = b\dot{X}$ Yuste and Bejarano [15] applied a Krylov-Bogoliubov method. Coppola and Rand [19], and Roy [11] used an averaging method which is based on elliptic functions to study (1.1) with $c_2 = 0$. Also for $c_2 = 0$ Chen and Cheung [16, 17] used a Lindstedt-Poincaré method to study (1.1) with $f(X, \dot{X}) = \left(a - bX^2\right)\dot{X}$, where a and b are constants. By using Melnikov functions and a Picard-Fuchs analysis Iliev and Perko [6] studied (1.1) with $c_2 = 0$, $c_1 = \pm 1$, $c_3 = \pm 1$, and $f(X, \dot{X}) = a\dot{X} + bX^2 + cX\dot{X} + dX^2\dot{X}$, where a, b, c, and d are parameters. For equations like (1.1) Blows and Perko [12] wrote an interesting survey paper on Melnikov/Poincaré techniques. Margallo and Bejarano [8, 9], and Lynch [18] studied

† On leave from: Mathematics Department, Semarang State University, Indonesia

(1.1) with $c_2 = 0$ and $f(X, \dot{X}) = \dot{X} - \dot{X}^3$, and showed that at least one limit cycle can occur. Waluya and Van Horssen [13] studied (1.1) with $c_3 = 0$ and $f(X, \dot{X}) = \dot{X} - \dot{X}^3$ (a Rayleigh type of perturbation) by using the perturbation method based on integrating factors. It has been shown in [1, 13] that for (1.1) with $c_3 = 0$ and $f(X, \dot{X}) = \dot{X} - \dot{X}^3$ two limit cycles can occur. The case $c_3 \neq 0$, $c_2 \neq 0$ and $f(X, \dot{X}) = \dot{X} - \dot{X}^3$ has not yet been studied, and will be studied in this paper using the perturbation method based on integrating factors. It will turn out in this paper that five limit cycles can occur. Equation (1.1) with $f(X, \dot{X}) = \dot{X} - \dot{X}^3$ plays an important role in applications, for instance in flow-induced vibrations of cables in a windfield. For details of this application we refer the readers to the papers of Van der Beek [2, 3]. Using the perturbation method based on integrating vectors and some numerical calculations we will study (1.1) with $c_3 \neq 0$ and $c_2 \neq 0$ and $f(X, \dot{X}) = \dot{X} - \dot{X}^3$ in detail, that is, the existence and stability, and the bifurcation of time-periodic solutions will be investigated in detail. In this paper we restrict ourselves to autonomous differential equations. The presented perturbation method, however, can also be extended to nonautonomous equations. In relation to these nonautonomous equations we refer the readers for instance to the work of Roy [11], Brothers and Haberman [7], and Bosley [5], who used averaging and matching techniques to obtain insight in the solution structure for a class of non-autonomous equations. This paper is organized as follows. In section 2 of this paper the perturbation method based on integrating vectors and an asymptotic theory will be given briefly. It will be shown in section 3 of this paper how approximations of first integrals can be constructed for the strongly nonlinear oscillator equation

$$\ddot{X} + \frac{dU(X)}{dX} = \epsilon f(X, \dot{X}), \tag{1.2}$$

where $U(X)$ is the potential energy of the unperturbed (that is, $\epsilon = 0$), nonlinear oscillator, and where $X = X(t)$, $\dot{X} = \frac{dX}{dt}$, ϵ is a small parameter satisfying $0 < \epsilon \ll 1$, and where f is a sufficiently smooth function. Approximations of first integrals for the oscillator equation (1.1) will be presented in section 4 of this paper. Using these approximations it will be shown in section 5 how the existence and stability of time-periodic solutions for the oscillator equation (1.1) can be obtained. The bifurcation(s) of limit cycles will be studied in detail, and a complete set of topological different phase portraits will be presented. Finally in section 6 of this paper some conclusions will be drawn and some remarks will be made.

2. Integrating vectors and an asymptotic theory. In this section we briefly outline the perturbation method based on integrating vectors as given in [13, 22, 23, 24]. We consider the following system of n first order ODEs

$$\frac{d\underline{y}}{dt} = \underline{f}(\underline{y}, t; \epsilon), \tag{2.1}$$

where ϵ is a small parameter, and where the function $\underline{f}$ has the form $\underline{f}(\underline{y}, t; \epsilon) = \underline{f}_0(\underline{y}, t) + \epsilon \underline{f}_1(\underline{y}, t)$. An integrating vector $\underline{\mu} = \underline{\mu}(\underline{y}, t; \epsilon)$ for system (2.1) has to satisfy

$$\begin{cases} \frac{\partial \mu_i}{\partial y_j} = \frac{\partial \mu_j}{\partial y_i}, & 1 \leq i < j \leq n, \\ \\ \frac{\partial \underline{\mu}}{\partial t} = -\nabla(\underline{\mu} \cdot \underline{f}). & \end{cases} \tag{2.2}$$

Assume that $\underline{\mu}$ can be expanded in a power series in ϵ, that is, $\underline{\mu}(\underline{y},t;\epsilon) = \underline{\mu}_0(\underline{y},t) + \epsilon\underline{\mu}_1(\underline{y},t) + \ldots + \epsilon^m\underline{\mu}_m(\underline{y},t) + \ldots$. We determine an integrating vector up to $\mathcal{O}(\epsilon^m)$. An approximation F_{app} of F in the first integral $F = constant$ can be obtained from:

$$\left\{\begin{array}{lcl} \nabla F_{app} & = & \underline{\mu}_0 + \epsilon\underline{\mu}_1 + \ldots + \epsilon^m\underline{\mu}_m, \\ \frac{\partial F_{app}}{\partial t} & = & -\left[\left(\underline{\mu}_0 + \epsilon\underline{\mu}_1 + \ldots + \epsilon^m\underline{\mu}_m\right)\cdot\underline{f}\right]_*, \end{array}\right. \tag{2.3}$$

where the * indicates that terms of order ϵ^{m+1} and higher have been neglected. Then we obtain $F_{app}(\underline{y},t;\epsilon) = F_0(\underline{y},t) + \epsilon F_1(\underline{y},t) + \ldots + \epsilon^m F_m(\underline{y},t)$. It should be observed that an approximation up to $\mathcal{O}(\epsilon^m)$ of an integrating vector $\underline{\mu}$ has been used to obtain an exact ODE up to $\mathcal{O}(\epsilon^{m+1})$, that is,

$$\begin{array}{lcl} \frac{dF_{app}}{dt} & = & \left[\left(\underline{\mu}_0 + \epsilon\cdot\underline{\mu}_1 + \ldots + \epsilon^m\underline{\mu}_m\right)\cdot\underline{f}\right]_{**} \\ & = & \epsilon^{m+1}R_{m+1}(\underline{y},t,\underline{\mu}_0,\ldots,\underline{\mu}_m;\epsilon), \end{array} \tag{2.4}$$

where the ** indicates that only terms of order ϵ^{m+1} and higher are included. How well F_{app} approximates $F(\underline{y},t;\epsilon) = constant$ can be determined from (2.4), that is, error estimates can be given on time-scales depending on the boundedness properties of R_{m+1}.

3. Approximations of First Integrals. In this section we will show how the perturbation method based on integrating vectors can be applied to approximate first integrals for a strongly nonlinear oscillator equation. We consider the class of non-linear oscillators described by the equation

$$\ddot{X} + \frac{dU(X)}{dX} = \epsilon f(X,\dot{X}), \tag{3.1}$$

where $U(X)$ is a potential, $X = X(t)$, $\dot{X} = \frac{dX}{dt}$, ϵ is a small parameter satisfying $0 < \epsilon \ll 1$, and where f is assumed to be sufficiently smooth. We assume that the unperturbed (that is, $\epsilon = 0$) solutions of (3.1) form a family of periodic orbits. This family may cover the entire "phase plane" $(X,\dot{X})$, or a bounded region $\mathcal{D}$ of the phase plane. Each periodic orbit corresponds to a constant energy level $E = \frac{1}{2}\dot{X}^2 + U(X)$. With each constant energy level E corresponds a phase angle ψ, which is defined to be

$$\psi = \int_0^X \frac{dr}{\sqrt{2E - 2U(r)}}. \tag{3.2}$$

From (3.1)-(3.2) a transformation $(X,\dot{X}) \longmapsto (E,\psi)$ can then be defined with

$$\left\{\begin{array}{lcl} \dot{E} = \epsilon\dot{X}f & = & g_1(E,\psi), \\ \dot{\psi} = 1 + \epsilon\left[-\int_0^X \frac{dr}{(2E-2U(r))^{\frac{3}{2}}}\dot{X}f\right] & = & g_2(E,\psi). \end{array}\right. \tag{3.3}$$

Multiplying the first and the second equation in (3.3) with $\mu_1(E,\psi,t)$ and $\mu_2(E,\psi,t)$ respectively, it follows from (2.2) that the integrating factors $\mu_1(E,\psi,t)$ and $\mu_2(E,\psi,t)$

have to satisfy

$$\left\{\begin{aligned} \frac{\partial \mu_1}{\partial \psi} &= \frac{\partial \mu_2}{\partial E}, \\ \frac{\partial \mu_1}{\partial t} &= -\frac{\partial}{\partial E}\left(\mu_1 g_1 + \mu_2 g_2\right), \\ \frac{\partial \mu_2}{\partial t} &= -\frac{\partial}{\partial \psi}\left(\mu_1 g_1 + \mu_2 g_2\right). \end{aligned}\right. \tag{3.4}$$

Expanding μ_1 and μ_2 in formal power series in ϵ, that is,

$$\mu_i(E,\psi,t;\epsilon) = \mu_{i,0}(E,\psi,t) + \epsilon\mu_{i,1}(E,\psi,t) + \dots$$

for $i = 1$ and 2, substituting g_1, g_2 and the expansions for μ_1 and μ_2 into (3.4) and by taking together terms of equal powers in ϵ, we finally obtain the following $\mathcal{O}(\epsilon^n)$-problems: for $n = 0$

$$\left\{\begin{aligned} \frac{\partial \mu_{1,0}}{\partial \psi} &= \frac{\partial \mu_{2,0}}{\partial E}, \\ \frac{\partial \mu_{1,0}}{\partial t} &= -\frac{\partial \mu_{2,0}}{\partial E}, \\ \frac{\partial \mu_{2,0}}{\partial t} &= -\frac{\partial \mu_{2,0}}{\partial \psi}, \end{aligned}\right. \tag{3.5}$$

and for $n \geq 1$

$$\left\{\begin{aligned} \frac{\partial \mu_{1,n}}{\partial \psi} &= \frac{\partial \mu_{2,n}}{\partial E}, \\ \frac{\partial \mu_{1,n}}{\partial t} &= -\frac{\partial}{\partial E}\left(\mu_{1,n-1} g_{1,1} + \mu_{2,n-1} g_{2,1} + \mu_{2,n}\right), \\ \frac{\partial \mu_{2,n}}{\partial t} &= -\frac{\partial}{\partial \psi}\left(\mu_{1,n-1} g_{1,1} + \mu_{2,n-1} g_{2,1} + \mu_{2,n}\right), \end{aligned}\right. \tag{3.6}$$

where $\epsilon g_{1,1} = g_1$, $\epsilon g_{2,1} = g_2 - 1$. The $\mathcal{O}(\epsilon^0)$-problem (3.5) can readily be solved, yielding $\mu_{1,0} = h_{1,0}(E,\psi - t)$ and $\mu_{2,0} = h_{2,0}(E,\psi - t)$ with $\frac{\partial h_{1,0}}{\partial \psi} = \frac{\partial h_{2,0}}{\partial E}$. The functions $h_{1,0}$ and h_{20} are still arbitrary and will now be chosen as simple as possible. We choose $h_{1,0} \equiv 1$ and $h_{2,0} \equiv 0$, and so (see also [13, 23])

$$\mu_{1,0} = 1,\ \mu_{2,0} = 0. \tag{3.7}$$

Then it follows from the order ϵ-problem (3.6) that $\mu_{1,1}$ and $\mu_{2,1}$ have to satisfy

$$\left\{\begin{aligned} \frac{\partial \mu_{1,1}}{\partial t} + \frac{\partial \mu_{1,1}}{\partial \psi} &= -\frac{\partial}{\partial E}\left(g_{1,1}\right), \\ \frac{\partial \mu_{2,1}}{\partial t} + \frac{\partial \mu_{2,1}}{\partial \psi} &= -\frac{\partial}{\partial \psi}\left(g_{1,1}\right). \end{aligned}\right. \tag{3.8}$$

By using the method of characteristics for first order PDEs we then obtain

$$\left\{\begin{aligned} \mu_{1,1} &= h_{1,1}(E,\psi - t) - \int_0^t \left(\frac{\partial}{\partial E}(g_{1,1})\right) d\bar{t}, \\ \mu_{2,1} &= h_{2,1}(E,\psi - t) - \int_0^t \left(\frac{\partial}{\partial \psi}(g_{1,1})\right) d\bar{t}, \end{aligned}\right. \tag{3.9}$$

where $h_{1,1}$, $h_{2,1}$ are arbitrary functions which have to satisfy

$$\frac{\partial h_{1,1}}{\partial \psi} - \frac{\partial}{\partial \psi}\int_0^t \left(\frac{\partial}{\partial E}(g_{1,1})\right) d\bar{t} = \frac{\partial h_{2,1}}{\partial E} - \frac{\partial}{\partial E}\int_0^t \left(\frac{\partial}{\partial \psi}(g_{1,1})\right) d\bar{t}. \tag{3.10}$$

We choose $h_{1,1}$ and $h_{2,1}$ as simple as possible, that is, we take $h_{1,1} = 0$, $h_{2,1} = 0$. We then obtain for $\mu_{1,1}$ and $\mu_{2,1}$

$$\left\{ \begin{array}{l} \mu_{1,1} = -\frac{\partial}{\partial E}\left(\int_0^t g_{1,1} d\bar{t}\right), \\ \\ \mu_{2,1} = -\frac{\partial}{\partial \psi}\left(\int_0^t g_{1,1} d\bar{t}\right). \end{array} \right. \tag{3.11}$$

An approximation F_1 of a first integral $F = \textit{constant}$ of system (3.3) can now be obtained from (3.7), (3.11), and (2.3), yielding

$$F_1(E, \psi, t) = E - \epsilon \left[\int_0^t g_{1,1} d\bar{t}\right]. \tag{3.12}$$

How well F_1 approximates a first integral $F = \textit{constant}$ follows from (2.4). In this case we have

$$\begin{aligned} \frac{dF_1}{dt} &= [(1 + \epsilon\mu_{1,1})g_1 + \epsilon\mu_{2,1}g_2]_{**} \\ &= \epsilon\mu_{1,1}g_1 + \epsilon\mu_{2,1}(g_2 - 1) = \epsilon^2 \mathcal{R}_1(E, \psi, t), \end{aligned} \tag{3.13}$$

where g_1, g_2, and $\mu_{1,1}$, $\mu_{2,1}$ are given by (3.3) and (3.11) respectively. From the existence and uniqueness theorems for ODEs we know that initial value problems for (3.1) (with sufficiently smooth potential $U(X)$ and nonlinearity $f(X, \dot{X})$) are well-posed on a time-scale of order $\frac{1}{\epsilon}$. This implies that also an initial-value problem for system (3.3) is well-posed on this time-scale. From (3.3) it then follows on this time-scale that if $E(0)$ is bounded then $E(t)$ is bounded and $\psi(t)$ is bounded by a constant plus t. Since $|\mathcal{R}_1| \leq c_0 + c_1 t$ on a time scale of order $\frac{1}{\epsilon}$, where c_0, c_1 are constants, it follows from (3.13) that

$$F_1(E(t), \psi(t), t; \epsilon) = \textit{constant} + \epsilon^2 \int_0^t \mathcal{R}_1(E(s), \psi(s), s; \epsilon) ds,$$

and so,

$$\begin{aligned} F_1(E(t), \psi(t), t; \epsilon) &= \textit{constant} + \mathcal{O}(\epsilon^2),\ 0 \leq t \leq T_0 < \infty, \\ F_1(E(t), \psi(t), t; \epsilon) &= \textit{constant} + \mathcal{O}(\epsilon),\ 0 \leq t \leq \frac{L}{\sqrt{\epsilon}}, \end{aligned} \tag{3.14}$$

where T_0 and L are ϵ-independent constants. Another (functionally independent) approximation of a first integral can be obtained by putting in (3.6)

$$\mu_{2,0} = 1,\ \mu_{1,0} = 0, \tag{3.15}$$

instead of (3.7). The $\mathcal{O}(\epsilon)$-problem (3.6) can now be solved, yielding

$$\left\{ \begin{array}{l} \mu_{1,1} = k_{1,1}(E, \psi - t) - \int_0^t \left(\frac{\partial}{\partial E}(g_{2,1})\right) d\bar{t}, \\ \\ \mu_{2,1} = k_{2,1}(E, \psi - t) - \int_0^t \left(\frac{\partial}{\partial \psi}(g_{2,1})\right) d\bar{t}, \end{array} \right. \tag{3.16}$$

where the functions $k_{1,1}$ and $k_{2,1}$ are arbitrary functions which have to satisfy

$$\frac{\partial k_{1,1}}{\partial \psi} - \frac{\partial}{\partial \psi}\int_0^t \left(\frac{\partial}{\partial E}(g_{2,1})\right) d\bar{t} = \frac{\partial k_{2,1}}{\partial E} - \frac{\partial}{\partial E}\int_0^t \left(\frac{\partial}{\partial \psi}(g_{2,1})\right) d\bar{t}. \tag{3.17}$$

We choose these functions as simple as possible, that is $k_{1,1} = 0$, and $k_{2,1} = 0$. Then we obtain

$$\begin{cases} \mu_{1,1} = -\frac{\partial}{\partial E}\left(\int_0^t g_{2,1} d\bar{t}\right), \\ \\ \mu_{2,1} = -\frac{\partial}{\partial \psi}\left(\int_0^t g_{2,1} d\bar{t}\right). \end{cases} \tag{3.18}$$

An approximation F_2 of a first integral $F = constant$ of system (3.3) can now be obtained from (3.15), (3.18), and (2.3), yielding

$$F_2(E, \psi, t) = (\psi - t) - \epsilon \left[\int_0^t g_{2,1} d\bar{t}\right]. \tag{3.19}$$

How well F_2 approximates a first integral $F = constant$ follows from (2.4). In this case we have

$$\begin{aligned} \frac{dF_2}{dt} &= [\epsilon\mu_{1,1}g_1 + (1 + \epsilon\mu_{2,1})g_2]_{**} \\ &= \epsilon\mu_{1,1}g_1 + \epsilon\mu_{2,1}(g_2 - 1) = \epsilon^2\mathcal{R}_2(E, \psi, t), \end{aligned}$$

where g_1, g_2, and $\mu_{1,1}$, $\mu_{2,1}$ are given by (3.3) and (3.18) respectively. In the following section we will treat some examples to show how this perturbation method can be applied.

4. Example of a Strongly Nonlinear Oscillator. In this section we will consider the following strongly nonlinear oscillator equation

$$\ddot{X} + \frac{dU(X)}{dX} = \epsilon f(X, \dot{X}), \tag{4.1}$$

where $\frac{dU(X)}{dX} = 9X + \mu X^2 + \lambda X^3$ with μ and λ parameters, where the function $f(X, \dot{X})$ is a so-called Rayleigh perturbation, that is, $f(X, \dot{X}) = \dot{X} - \dot{X}^3$, and where ϵ is a small parameter with $0 < \epsilon \ll 1$. In [2] Van der Beek introduced (4.1) with $\mu = \mathcal{O}(\sqrt{\epsilon})$ and $\lambda = 0$ as a model equation to describe the vibrations of an oscillator in a uniform windfield. This model equation is related to the phenomenon of galloping of overhead power transmission lines on which ice has accreted. Using first order normal form techniques it has been shown in [2] that (4.1) with $\mu = \mathcal{O}(\sqrt{\epsilon})$ and $\lambda = 0$ has a unique (stable) periodic solution. Doelman and Verhulst [1], and Waluya and Van Horssen [13] showed that a stable and an unstable periodic solution can occur simultaneously for (4.1) with $\mu = \mathcal{O}(1)$ and $\lambda = 0$. For (4.1) with $\mu = 0$ and $\lambda > 0$ Garcia-Margallo and Bejarano [9] showed that at least one limit cycle can occur. In this section we will construct approximations of first integrals for (4.1) with μ and λ arbitrary. To give a complete analysis of (4.1) we have to consider two main cases: (i) $\mu = 0$ and λ arbitrary, and (ii) $\mu > 0$ and λ arbitrary. It should be observed that the case $\mu < 0$ and λ arbitrary is included in case (ii) (just replace X by $-X$ in (4.1)). The constructed approximations of the first integrals will be used in section 5 to determine the number of periodic solutions for (4.1).

4.1. The case $\mu = 0$ and λ arbitrary. To study (4.1) with $\mu = 0$ in detail we have to consider three subcases: $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. These three cases will be studied in the following three subsections.

4.1.1. The case $\mu = 0$ and $\lambda = 0$. By putting $X(t) = \tilde{X}(\tau)$ with $t = \frac{\tau}{3}$ in (4.1) the following weakly nonlinear Rayleigh oscillator equation is obtained

$$\tilde{X}'' + \tilde{X} = \tilde{\epsilon} g(\tilde{X}'), \tag{4.2}$$

where $\tilde{\epsilon} = \frac{1}{3}\epsilon$, $\tilde{X}' = \frac{d\tilde{X}}{d\tau}$, and where $g(\tilde{X}') = \tilde{X}' - 9(\tilde{X}')^3$. By introducing the transformation $(\tilde{X}, \tilde{X}') \longmapsto (E, \psi)$ as defined by

$$\begin{cases} E &= \frac{1}{2}(\tilde{X}')^2 + \frac{1}{2}(\tilde{X})^2, \\ \\ \psi &= \int_0^{\tilde{X}} \frac{dr}{\sqrt{2E - r^2}} = \sin^{-1}\left(\frac{\tilde{X}}{\sqrt{2E}}\right), \end{cases} \tag{4.3}$$

(where E and ψ are the energy and the phase angle of the unperturbed oscillator ($\epsilon = 0$) respectively) we obtain the following system of ODEs

$$\begin{cases} E' = \tilde{\epsilon}\tilde{X}' g &= \xi_1(E, \psi) = \tilde{\epsilon}\xi_{1,1}(E, \psi), \\ \\ \psi' = 1 + \tilde{\epsilon}\left[-\int_0^{\tilde{X}} \frac{dr}{(2E - r^2)^{\frac{3}{2}}} \tilde{X}' g\right] &= \xi_2(E, \psi) = 1 + \tilde{\epsilon}\xi_{2,1}(E, \psi). \end{cases} \tag{4.4}$$

From the calculations as presented in section 3 of this paper it follows that two functionally independent approximations of first integrals for system (4.4) are given by

$$\begin{aligned} F_1(E, \psi, \tau) &= E - \epsilon \int_0^{\tau} \xi_{1,1} d\bar{t} = E - \tilde{\epsilon} \int_0^{\tau} \left((\tilde{X}')^2 - 9(\tilde{X}')^4\right) d\bar{t} \\ &= E - \tilde{\epsilon} \int_0^{\tau} \left(2E\cos(\psi)^2 - 36E^2\cos(\psi)^4\right) d\bar{t} \\ &= E - \tilde{\epsilon}\left(\left(E - \frac{27}{2}E^2\right)\psi - 9E^2\sin(2\psi) - \frac{9}{8}E^2\sin(4\psi)\right), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} F_2(E, \psi, \tau) &= (\psi - \tau) - \tilde{\epsilon} \int_0^{\tau} \xi_{2,1} d\bar{t} \\ &= (\psi - \tau) + \frac{\tilde{\epsilon}}{2E} \int_0^{\tau} \left(2E\sin(\psi)\cos(\psi) - 36E^2\sin(\psi)\cos(\psi)^3\right) d\bar{t} \\ &= (\psi - \tau) + \tilde{\epsilon}\left(-\frac{1}{4}\cos(2\psi) + \frac{1}{4}E\cos(2\psi) + \frac{1}{16}E\cos(4\psi)\right). \end{aligned} \tag{4.6}$$

How well F_1 and F_2 approximate a first integral $F = \textit{constant}$ follows from (2.4). In this case for $j = 1, 2$ we have

$$\frac{dF_j}{d\tau} = \tilde{\epsilon}\mu_{1,1}\xi_1 + \tilde{\epsilon}\mu_{2,1}(\xi_2 - 1) = \tilde{\epsilon}^2\mathcal{R}_j(E, \psi), \tag{4.7}$$

where ξ_1 and ξ_2 are given by (4.4). It follows from (4.7) that for $j = 1, 2$ (see also (3.13)-(3.14))

$$F_j(E(\tau), \psi(\tau), \tau; \tilde{\epsilon}) = \textit{constant} + \tilde{\epsilon}^2 \int_0^{\tau} \mathcal{R}_j(E(s), \psi(s), s; \tilde{\epsilon}) ds, \tag{4.8}$$

and so,

$$
\begin{aligned}
F_j(E(\tau),\psi(\tau),\tau;\tilde{\epsilon}) &= constant + \mathcal{O}(\tilde{\epsilon}^2),\ 0 \le \tau \le T_0 < \infty, \\
F_j(E(\tau),\psi(\tau),\tau;\tilde{\epsilon}) &= constant + \mathcal{O}(\tilde{\epsilon}),\ 0 \le \tau \le \frac{L}{\sqrt{\tilde{\epsilon}}},
\end{aligned} \tag{4.9}
$$

where T_0 and L are $\tilde{\epsilon}$-independent constants.

4.1.2. The case $\mu = 0$ and $\lambda > 0$. By putting $X(t) = \tilde{X}(\tau)$ with $t = \frac{\tau}{3}$ in (4.1) the following nonlinear Rayleigh oscillator equation is obtained

$$
\tilde{X}'' + \tilde{X} + \beta\tilde{X}^3 = \tilde{\epsilon} g(\tilde{X}'), \tag{4.10}
$$

where $\tilde{\epsilon} = \frac{1}{3}\epsilon$, $\beta = \frac{\lambda}{9}$, $\tilde{X}' = \frac{d\tilde{X}}{d\tau}$, and where $g(\tilde{X}') = \tilde{X}' - 9(\tilde{X}')^3$. By introducing the transformation $(\tilde{X},\tilde{X}') \longmapsto (E,\psi)$ as defined by

$$
\left\{
\begin{aligned}
E &= \tfrac{1}{2}(\tilde{X}')^2 + \tfrac{1}{2}(\tilde{X})^2 + \tfrac{1}{4}\beta(\tilde{X})^4, \\
\psi &= \textstyle\int_0^{\tilde{X}} \frac{dr}{\sqrt{2E - r^2 - \frac{1}{2}\beta r^4}},
\end{aligned}
\right. \tag{4.11}
$$

(where E and ψ are the energy and the phase angle of the unperturbed oscillator (that is, (4.1) with $\epsilon = 0$)) we obtain the following system of ODEs

$$
\left\{
\begin{aligned}
E' = \tilde{\epsilon}\tilde{X}'g \qquad &= \xi_3(E,\psi) = \tilde{\epsilon}\xi_{3,1}(E,\psi), \\
\psi' = 1 + \tilde{\epsilon}\left[-\textstyle\int_0^{\tilde{X}} \frac{dr}{(2E - r^2 - \frac{1}{2}\beta r^4)^{\frac{3}{2}}}\tilde{X}'g\right] &= \xi_4(E,\psi) = 1 + \tilde{\epsilon}\xi_{4,1}(E,\psi).
\end{aligned}
\right. \tag{4.12}
$$

The solution of the unperturbed equation (4.10) is $\tilde{X} = A_0 cn(\vartheta, k)$ with $\vartheta = \omega_0\psi$, where $\psi = \tau + constant$, k is a modulus given by $k^2 = \frac{\beta A_0^2}{2\omega_0^2}$, and $\omega_0^2 = 1 + \beta A_0^2$ (see also [4, 10, 11, 16, 17, 19]). The relationship between the energy E and the "amplitude" A_0 is given by $E = \frac{1}{2}A_0^2 + \frac{1}{4}\beta A_0^4$. The function $cn(\vartheta,k)$ is a Jacobian elliptic function with argument ϑ and modulus k. From the calculations as presented in section 3 of this paper it follows that two functionally independent approximations of first integrals for system (4.12) are given by

$$
\begin{aligned}
F_3(E,\psi,\tau) &= E - \tilde{\epsilon}\int_0^{\tau} \xi_{3,1} d\bar{t} = E - \tilde{\epsilon}\int_0^{\tau} \left((\tilde{X}')^2 - 9(\tilde{X}')^4\right) d\bar{t} \\
&= E - \tilde{\epsilon}\left[\int_0^{\tau} (\omega_0^2 A_0^2 sn(\vartheta,k)^2 dn(\vartheta,k)^2 - \eta\omega_0^4 A_0^4 sn(\vartheta,k)^4 dn(\vartheta,k)^4)\frac{d\vartheta}{\omega_0}\right],
\end{aligned} \tag{4.13}
$$

and

$$
\begin{aligned}
F_4(E,\psi,\tau) &= (\psi - \tau) - \tilde{\epsilon}\int_0^{\tau} \xi_{4,1} d\bar{t} \\
&= (\psi - \tau) + \tilde{\epsilon}\left[\int_0^{\tau} P_1(\vartheta,k)\,(\omega_0 A_0 sn(\vartheta,k) dn(\vartheta,k)\right. \\
&\qquad \left. - \eta\omega_0^3 A_0^3 sn(\vartheta,k)^3 dn(\vartheta,k)^3)\,\frac{d\vartheta}{\omega_0}\right],
\end{aligned} \tag{4.14}
$$

where $P_1(\vartheta, k) = \frac{\partial A_0}{\partial E} cn(\vartheta, k) - A_0 \psi sn(\vartheta, k) dn(\vartheta, k) \frac{\partial \omega_0}{\partial E} + A_0 \frac{\partial}{\partial k} cn(\vartheta, k) \frac{\partial k}{\partial E}$, in which $sn(\vartheta, k)$, and $dn(\vartheta, k)$ are elliptic functions , and where $\frac{\partial A_0}{\partial E}$, $\frac{\partial \omega_0}{\partial E}$, and $\frac{\partial k}{\partial E}$ are given by

$$\frac{\partial A_0}{\partial E} = \frac{1}{A_0 + \beta A_0^3}, \quad \frac{\partial \omega_0}{\partial E} = \frac{\beta A_0}{\omega_0 \left(A_0 + \beta A_0^3\right)}, \quad \frac{\partial k}{\partial E} = \frac{\beta A_0 \left(1 - 2k^2\right)}{2k\omega_0^2 \left(A_0 + \beta A_0^3\right)}.$$

How well F_3 and F_4 approximate a first integral $F = constant$ follows from (2.4). In this case for $j = 3, 4$ we have

$$\frac{dF_j}{d\tau} = \tilde{\epsilon}\mu_{1,1}\xi_3 + \tilde{\epsilon}\mu_{2,1}(\xi_4 - 1) = \tilde{\epsilon}^2 \mathcal{R}_j(E, \psi), \tag{4.15}$$

where ξ_3 and ξ_4 are given by (4.12). It follows from (4.15) that for $j = 3, 4$ (see also (3.13)-(3.14))

$$F_j(E(\tau), \psi(\tau), \tau; \tilde{\epsilon}) = constant + \tilde{\epsilon}^2 \int_0^\tau \mathcal{R}_j(E(s), \psi(s), s; \tilde{\epsilon}) ds, \tag{4.16}$$

and so,

$$\begin{aligned} F_j(E(\tau), \psi(\tau), \tau; \tilde{\epsilon}) &= constant + \mathcal{O}(\tilde{\epsilon}^2), \ 0 \le \tau \le T_0 < \infty, \\ F_j(E(\tau), \psi(\tau), \tau; \tilde{\epsilon}) &= constant + \mathcal{O}(\tilde{\epsilon}), \ 0 \le \tau \le \frac{L}{\sqrt{\tilde{\epsilon}}}, \end{aligned} \tag{4.17}$$

where T_0 and L are $\tilde{\epsilon}$-independent constants.

4.1.3. The case $\mu = 0$ and $\lambda < 0$. By putting $X(t) = \tilde{X}(\tau)$ with $t = \frac{\tau}{3}$ in (4.1) the following nonlinear oscillator equation is obtained

$$\tilde{X}'' + \tilde{X} - \gamma \tilde{X}^3 = \tilde{\epsilon} g(\tilde{X}'), \tag{4.18}$$

where $\gamma = \frac{-\lambda}{9} > 0$, $\tilde{\epsilon} = \frac{1}{3}\epsilon$, $\tilde{X}' = \frac{d\tilde{X}}{d\tau}$, and where $g(\tilde{X}') = \tilde{X}' - 9(\tilde{X}')^3$. By introducing the transformation $(X, X') \longmapsto (E, \psi)$ as defined by

$$\left\{ \begin{array}{rcl} E &=& \frac{1}{2}(\tilde{X}')^2 + \frac{1}{2}(\tilde{X})^2 - \frac{1}{4}\gamma(\tilde{X})^4, \\ \psi &=& \int_0^{\tilde{X}} \frac{dr}{\sqrt{2E - r^2 + \frac{1}{2}\gamma r^4}}, \end{array} \right. \tag{4.19}$$

(where E and ψ are the energy and the phase angle of the unperturbed oscillator (that is, (4.1) with $\epsilon = 0$)) we obtain the following system of ODEs

$$\left\{ \begin{array}{lcl} E' = \tilde{\epsilon}\tilde{X}' g &=& \xi_5(E, \psi) = \tilde{\epsilon}\xi_{5,1}(E, \psi), \\ \psi' = 1 + \tilde{\epsilon}\left[- \int_0^{\tilde{X}} \frac{dr}{(2E - r^2 + \frac{1}{2}\gamma r^4)^{\frac{3}{2}}} \tilde{X}' g \right] &=& \xi_6(E, \psi) = 1 + \tilde{\epsilon}\xi_{6,1}(E, \psi). \end{array} \right. \tag{4.20}$$

The solution of the unperturbed equation (4.18) is given by $\tilde{X} = A_0 sn(\vartheta, k)$ with $\vartheta = \omega_0 \psi$, where $\psi = t + constant$, $k^2 = \frac{\gamma A_0^2}{2\omega_0^2}$, $\omega_0^2 = 1 - \frac{1}{2}\gamma A_0^2$, and $E = \frac{1}{2}A_0^2 - \frac{1}{4}\gamma A_0^4$. From the calculations as presented in section 3 of this paper it follows that two functionally independent approximations of first integrals for system (4.20) are given by

$$
\begin{aligned}
F_5(E,\psi,\tau) &= E-\tilde{\epsilon}\int_0^{\tau}\xi_{5,1}d\bar{t} = E-\tilde{\epsilon}\int_0^{\tau}\Big((\tilde{X}')^2-9(\tilde{X}')^4\Big)\,d\bar{t}\\
&= E-\tilde{\epsilon}\left[\int_0^{\tau}(\omega_0^2A_0^2cn(\vartheta,k)^2dn(\vartheta,k)^2-\omega_0^4A_0^4cn(\vartheta,k)^4dn(\vartheta,k)^4)\frac{d\vartheta}{\omega_0}\right],
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
F_6(E,\psi,\tau) &= (\psi-\tau)-\tilde{\epsilon}\int_0^{\tau}\xi_{6,1}d\bar{t}\\
&= (\psi-\tau)+\tilde{\epsilon}\left[\int_0^{\tau}P_2(\vartheta,k)\,(\omega_0A_0cn(\vartheta,k)dn(\vartheta,k)\right.\\
&\qquad\left.-\eta\omega_0^3A_0^3cn(\vartheta,k)^3dn(\vartheta,k)^3)\,\frac{d\vartheta}{\omega_0}\right],
\end{aligned}
\tag{4.22}
$$

where $P_2(\vartheta,k)=\frac{\partial A_0}{\partial E}sn(\vartheta,k)+A_0\psi cn(\vartheta,k)dn(\vartheta,k)\frac{\partial\omega_0}{\partial E}+A_0\frac{\partial}{\partial k}sn(\vartheta,k)\frac{\partial k}{\partial E}$ in which

$$
\frac{\partial A_0}{\partial E}=\frac{1}{A_0-\gamma A_0^3},\quad \frac{\partial\omega_0}{\partial E}=\frac{-\gamma A_0}{2\omega_0\left(A_0-\gamma A_0^3\right)},\quad \frac{\partial k}{\partial E}=\frac{\gamma A_0\left(1+k^2\right)}{2k\omega_0^2\left(A_0-\gamma A_0^3\right)}.
$$

How well F_5 and F_6 approximate a first integral $F=constant$ follows from (2.4). In this case for $j=5,6$ we have

$$
\frac{dF_j}{d\tau} = \tilde{\epsilon}\mu_{1,1}\xi_5+\tilde{\epsilon}\mu_{2,1}(\xi_6-1)=\tilde{\epsilon}^2\mathcal{R}_j(E,\psi),
\tag{4.23}
$$

where ξ_5 and ξ_6 are given by (4.20). It follows from (4.23) that for $j=5,6$ (see also (3.13)-(3.14))

$$
F_j(E(\tau),\psi(\tau),\tau;\tilde{\epsilon})=constant+\tilde{\epsilon}^2\int_0^{\tau}\mathcal{R}_j(E(s),\psi(s),s;\tilde{\epsilon})ds,
\tag{4.24}
$$

and so,

$$
\begin{aligned}
F_j(E(\tau),\psi(\tau),\tau;\tilde{\epsilon}) &= constant+\mathcal{O}(\tilde{\epsilon}^2),\ 0\le\tau\le T_0<\infty,\\
F_j(E(\tau),\psi(\tau),\tau;\tilde{\epsilon}) &= constant+\mathcal{O}(\tilde{\epsilon}),\ 0\le\tau\le\frac{L}{\sqrt{\tilde{\epsilon}}},
\end{aligned}
\tag{4.25}
$$

where T_0 and L are $\tilde{\epsilon}$-independent constants.

4.2. The case $\mu>0$ and λ arbitrary. By putting

$$
X(t) = \frac{9}{\mu}Z(\tau),\quad t=\frac{\tau}{3}.
\tag{4.26}
$$

in (4.1) the following nonlinear oscillator equation is obtained

$$
Z''+Z+Z^2+\xi Z^3=\tilde{\epsilon}g(Z'),
\tag{4.27}
$$

where $\xi = \frac{9\lambda}{\mu^2}$, $\tilde{\epsilon} = \frac{1}{3}\epsilon$, $Z' = \frac{dZ}{d\tau}$, and $g(Z') = Z' - \eta(Z')^3$ with $\eta = \frac{9^3}{\mu^2}$. By introducing the transformation $(Z, Z') \longmapsto (E, \psi)$ as defined by

$$\left\{ \begin{array}{lcl} E & = & \frac{1}{2}Z'^2 + \frac{1}{2}Z^2 + \frac{1}{3}Z^3 + \frac{1}{4}\xi Z^4, \\ \psi & = & \int_0^Z \frac{dr}{\sqrt{2E - r^2 - \frac{2}{3}r^3 - \frac{1}{2}\xi r^4}}, \end{array} \right. \tag{4.28}$$

(where E and ψ are the energy and the phase angle of the unperturbed oscillator (that is, (4.1) with $\epsilon = 0$)) we obtain the following system of ODEs

$$\left\{ \begin{array}{lcl} E' = \tilde{\epsilon} Z' g & = & \zeta_1(E, \psi) = \tilde{\epsilon}\zeta_{1,1}, \\ \psi' = 1 + \tilde{\epsilon}\left[-\int_0^Z \frac{dr}{(2E - r^2 - \frac{2}{3}r^3 - \frac{1}{2}\xi r^4)^{\frac{3}{2}}} Z' g \right] & = & \zeta_2(E, \psi) = 1 + \tilde{\epsilon}\zeta_{2,1}. \end{array} \right. \tag{4.29}$$

From the calculations as presented in section 3 of this paper it follows (see also section 4.1) that two functionally independent approximations of first integrals for system (4.29) are given by

$$\begin{aligned} F_7(E, \psi) &= E - \tilde{\epsilon} \int_0^\tau \zeta_{1,1} d\bar{t} \\ &= E - \tilde{\epsilon} \int_0^\tau (G_{[2]}^2 - \eta G_{[2]}^4) d\bar{t}, \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} F_8(E, \psi) &= (\psi - \tau) - \tilde{\epsilon} \int_0^\tau \zeta_{2,1} d\bar{t} \\ &= (\psi - \tau) - \tilde{\epsilon} \int_0^\tau F_{[2]}(G_{[2]} - \eta G_{[2]}^3) d\bar{t}, \end{aligned} \tag{4.31}$$

where $F_{[2]} = \frac{\partial Z}{\partial E}$ and $G_{[2]} = Z'$ are elliptic functions, which are defined by (4.28). How well F_7 and F_8 approximate a first integral $F = constant$ follows from (2.4). For $j = 7, 8$ we have

$$\frac{dF_j}{d\tau} = \tilde{\epsilon}\mu_{1,1}\zeta_1 + \tilde{\epsilon}\mu_{2,1}(\zeta_2 - 1) = \tilde{\epsilon}^2 \mathcal{R}_j(E, \psi), \tag{4.32}$$

where ζ_1 and ζ_2 are given by (4.29). It follows from (4.32) that for $j = 7, 8$ (see also (3.13)-(3.14))

$$F_j(E(\tau), \psi(\tau), \tau; \tilde{\epsilon}) = constant + \tilde{\epsilon}^2 \int_0^\tau \mathcal{R}_j(E(s), \psi(s), s; \tilde{\epsilon}) ds, \tag{4.33}$$

and so,

$$\begin{aligned} F_j(E(\tau), \psi(\tau), \tau; \tilde{\epsilon}) &= constant + \mathcal{O}(\tilde{\epsilon}^2),\ 0 \le \tau \le T_0 < \infty, \\ F_j(E(\tau), \psi(\tau), \tau; \tilde{\epsilon}) &= constant + \mathcal{O}(\tilde{\epsilon}),\ 0 \le \tau \le \frac{L}{\sqrt{\tilde{\epsilon}}}, \end{aligned} \tag{4.34}$$

where T_0 and L are $\tilde{\epsilon}$-independent constants.

5. Time-periodic solutions and a bifurcation analysis. In the previous section it has been shown how asymptotic approximations of first integrals can be obtained. In this section we will show how the existence of non-trivial, time-periodic solutions can be determined from the asymptotic approximations of the first integrals. We will also present phase portraits and a bifurcation analysis. In section 5.1 we will show that (4.1) with $\mu = 0$ can have (at least) two limit cycles, and in section 5.2 we will give strong numerical evidence that (4.1) with $\mu \neq 0$ can have (at least) five limit cycles.

5.1. The case $\mu = 0$ and λ arbitrary. To determine the non-trivial, time-periodic solutions from the asymptotic approximations of the first integrals we have to consider (as in section 4.1) three subcases: $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. These three cases will be studied in the following three subsections.

5.1.1. The case $\mu = 0$ and $\lambda = 0$. Let $T < \infty$ be the period of a periodic solution and let c_1 be a constant in the first integral $F(E, \psi, \tau\,; \epsilon) = \mathit{constant}$ for which a periodic solution exists. Consider $F = c_1$ for $\tau = 0$ and $\tau = T$. Approximating F by F_1 (as given by (4.5)), eliminating c_1 by subtraction, we then obtain (using the fact that $E(0) = E(T)$ for a periodic solution)

$$\tilde{\epsilon}\left(\int_0^T \left((\tilde{X}')^2 - 9(\tilde{X}')^4\right) d\bar{t}\right) = \mathcal{O}(\tilde{\epsilon}^2) \Leftrightarrow \tilde{\epsilon}\left(\int_{\tilde{X}(0)}^{\tilde{X}(T)} \left(\tilde{X}' - 9(\tilde{X}')^3\right) d\tilde{X}\right) = \mathcal{O}(\tilde{\epsilon}^2). \tag{5.1}$$

Because of the symmetry of the unperturbed orbits in the phase plane (5.1) can be rewritten as

$$\tilde{\epsilon} I(E) = \mathcal{O}(\tilde{\epsilon}^2), \tag{5.2}$$

where

$$I(E) = 4\int_0^A \left(\tilde{X}' - 9(\tilde{X}')^3\right) d\tilde{X}, \tag{5.3}$$

with $A = \tilde{X}(\frac{1}{2}T) = \sqrt{2E}$. To have a periodic solution for (4.2) we have to find an energy E such that $I(E)$ is equal to zero (see also [13, 20, 21]). To find this energy E we rewrite $I(E)$ in

$$I(E) = 4I_1(E)\left(1 - 9\frac{I_2(E)}{I_1(E)}\right), \tag{5.4}$$

where

$$\begin{cases} I_1(E) &= \int_0^A \left(2E - \tilde{X}^2\right)^{\frac{1}{2}} d\tilde{X} = \frac{E\pi}{2}, \\ I_2(E) &= \int_0^A \left(2E - \tilde{X}^2\right)^{\frac{3}{2}} d\tilde{X} = \frac{3E^2\pi}{4}. \end{cases} \tag{5.5}$$

It now easily follows from (5.4) and (5.5) that $I(E) = 0$ for $E = \frac{2}{27}$ or $E = 0$. Putting $Q = \frac{I_2}{I_1}$ (where I_1 and I_2 are as defined in (5.5)) it easily follows from (5.5) that $Q = \frac{3}{2}E$. Since Q is strictly monotonically increasing in E we can conclude that there exists a unique, nontrivial, stable periodic solution for (4.2). The standard arguments leading to this conclusion can for instance be found in [[13], section 4.2] or in [6, 12].

5.1.2. The case $\mu = 0$ and $\lambda > 0$. Let $T < \infty$ be the period of a periodic solution and let c_2 be a constant in the first integral $F(E, \psi, \tau; \epsilon) = \textit{constant}$ for which a periodic solution exists. Consider $F = c_2$ for $\tau = 0$ and $\tau = T$. Approximating F by F_3 (as given by (4.13)), eliminating c_2 by subtraction, we then obtain (using the fact that $E(0) = E(T)$ for a periodic solution)

$$\tilde{\epsilon}\left(\int_0^T \left((\tilde{X}')^2 - 9(\tilde{X}')^4\right) d\bar{t}\right) = \mathcal{O}(\tilde{\epsilon}^2) \Leftrightarrow \tilde{\epsilon}\left(\int_{\tilde{X}(0)}^{\tilde{X}(T)} \left(\tilde{X}' - 9(\tilde{X}')^3\right) d\tilde{X}\right) = \mathcal{O}(\tilde{\epsilon}^2). \tag{5.6}$$

Because of the symmetry of the unperturbed orbits in the phase plane (5.6) can be rewritten as

$$\tilde{\epsilon} I(E, \beta) = \mathcal{O}(\tilde{\epsilon}^2), \tag{5.7}$$

where

$$I(E, \beta) = 4\int_0^A \left(\tilde{X}' - 9(\tilde{X}')^3\right) d\tilde{X}, \tag{5.8}$$

with $A = \tilde{X}(\frac{1}{2}T)$. To have a periodic solution for (4.10) we have to find an energy E such that $I(E, \beta)$ is equal to zero (see also [13, 20, 21]). To find this constant energy E we rewrite $I(E, \beta)$ in

$$I(E, \beta) = 4I_1(E, \beta)\left(1 - 9\frac{I_2(E, \beta)}{I_1(E, \beta)}\right), \tag{5.9}$$

where

$$\begin{cases} I_1(E, \beta) & = & \int_0^A \left(2E - \tilde{X}^2 - \frac{1}{2}\beta\tilde{X}^4\right)^{\frac{1}{2}} d\tilde{X}, \\ & & \\ I_2(E, \beta) & = & \int_0^A \left(2E - \tilde{X}^2 - \frac{1}{2}\beta\tilde{X}^4\right)^{\frac{3}{2}} d\tilde{X}. \end{cases} \tag{5.10}$$

It should be observed that the unperturbed equation (4.10) with $\tilde{\epsilon} = 0$ has one equilibrium point $(0, 0)$ which is center point. The phase portrait of the unperturbed equation (4.10) is given in Figure 5.1. It should be observed from (4.11) that $E \geq 0$. Putting $Q(E, \beta) = \frac{I_2(E,\beta)}{I_1(E,\beta)}$ (where I_1 and I_2 are given by (5.10)) it can be shown elementarily that $\lim_{E\downarrow 0} Q(E, \beta) = 0$, and that $\frac{\partial Q}{\partial E}(0, \beta) = \frac{3}{2} > 0$. There is strong numerical evidence (see figure 5.3 (g) and (h)) that Q is monotonically increasing. So we can conclude that there exists a unique, nontrivial value for E such that $I(E, \beta) = 0$ or equivalently $Q(E, \beta) = \frac{1}{9}$. From these results it can be concluded (see also for instance [[13], section 4.2]) that there exists a unique, nontrivial, stable time-periodic solution for (4.10). The period T of this periodic solution is given by

$$T = \int_A^B \frac{2d\tilde{X}}{\sqrt{\left(2E^* - \tilde{X}^2 - \frac{1}{2}\beta\tilde{X}^4\right)}}, \tag{5.11}$$

where $A = -\sqrt{-\frac{1}{\beta} + \frac{1}{\beta}\sqrt{1 + 4\beta E^*}}$ and $B = \sqrt{-\frac{1}{\beta} + \frac{1}{\beta}\sqrt{1 + 4\beta E^*}}$ in which E^* is the energy for which the periodic solution occurs.

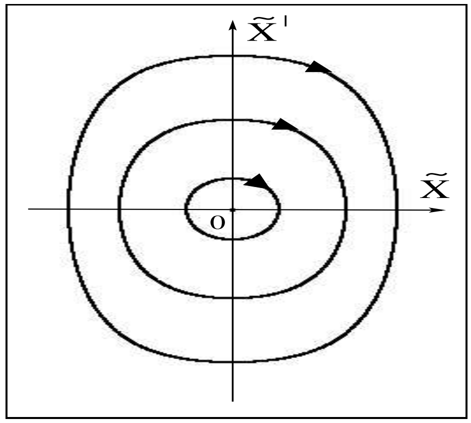


FIG. 5.1. *Phase portrait of the unperturbed equation (4.10) with* $\tilde{\epsilon} = 0$*,* $\mu = 0$*, and* $\lambda > 0$*.*

5.1.3. The case $\mu = 0$ and $\lambda < 0$. Let $T < \infty$ be the period of a periodic solution and let c_3 be a constant in the first integral $F(E, \psi, t\,;\,\epsilon) = constant$ for which a periodic solution exists. Consider $F = c_3$ for $\tau = 0$ and $\tau = T$. Approximating F by F_5 (as given by (4.21)), eliminating c_3 by subtraction, we then obtain (using the fact that $E(0) = E(T)$ for a periodic solution)

$$\tilde{\epsilon}\left(\int_0^T \left((\tilde{X}')^2 - 9(\tilde{X}')^4\right) d\bar{t}\right) = \mathcal{O}(\tilde{\epsilon}^2) \Leftrightarrow \tilde{\epsilon}\left(\int_{\tilde{X}(0)}^{\tilde{X}(T)} \left(\tilde{X}' - 9(\tilde{X}')^3\right) d\tilde{X}\right) = \mathcal{O}(\tilde{\epsilon}^2). \tag{5.12}$$

Because of the symmetry of the unperturbed orbits in phase plane (5.12) can be rewritten as

$$\tilde{\epsilon} I(E,\gamma) = \mathcal{O}(\tilde{\epsilon}^2), \tag{5.13}$$

where

$$I(E,\gamma) = 4\int_0^A \left(\tilde{X}' - 9(\tilde{X}')^3\right) d\tilde{X}, \tag{5.14}$$

with $A = \tilde{X}(\frac{1}{2}T)$. To have a periodic solution for (4.18) we have to find an energy E such that $I(E,\gamma)$ is equal to zero (see also [13, 20, 21]). To find this energy E we rewrite $I(E,\gamma)$ in

$$I(E,\gamma) = 4I_1(E)\left(1 - 9\frac{I_2(E,\gamma)}{I_1(E,\gamma)}\right), \tag{5.15}$$

where

$$\begin{cases} I_1(E,\gamma) &= \int_0^A \left(2E - \tilde{X}^2 + \frac{1}{2}\gamma\tilde{X}^4\right)^{\frac{1}{2}} d\tilde{X}, \\ I_2(E,\gamma) &= \int_0^A \left(2E - \tilde{X}^2 + \frac{1}{2}\gamma\tilde{X}^4\right)^{\frac{3}{2}} d\tilde{X}. \end{cases} \tag{5.16}$$

It should be observed that the unperturbed equation (4.18) with $\tilde{\epsilon} = 0$ has three equilibrium points: a center point in $(0,0)$ and two saddles in $(\pm\frac{1}{\sqrt{\gamma}}, 0)$. The phase

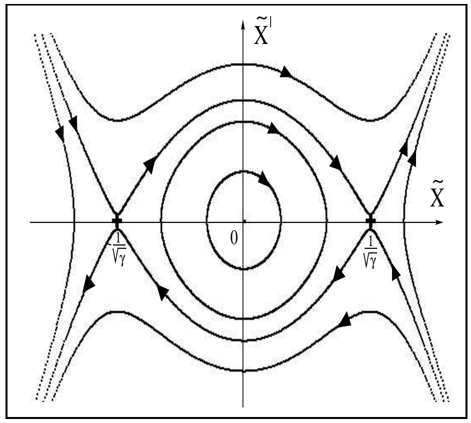


FIG. 5.2. *Phase portrait of the unperturbed equation (4.18) with $\tilde{\epsilon} = 0$, $\mu = 0$, and $\lambda < 0$.*

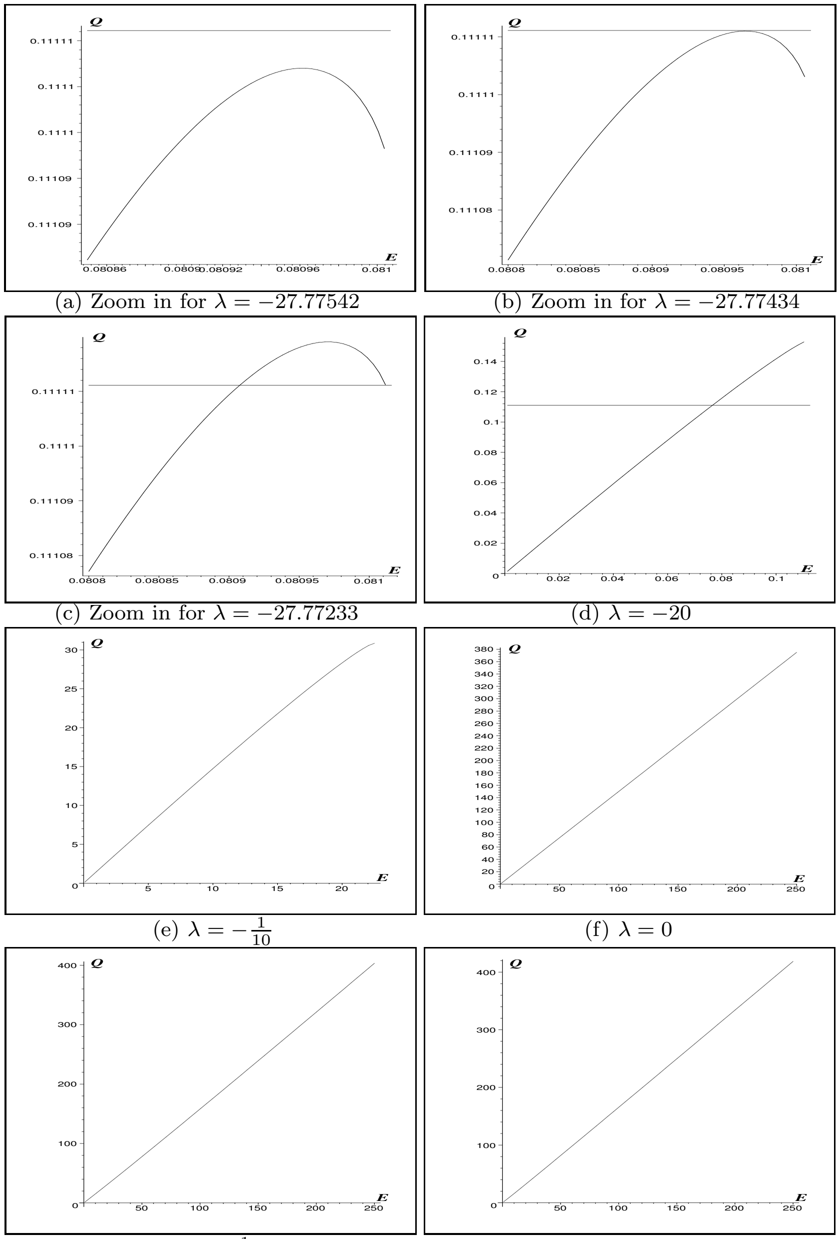
portrait of the unperturbed equation (4.18) is given in Figure 5.2. For the periodic (that is, closed) orbits of the unperturbed equation (4.18) it can be deduced from (4.19) that $0 \leq E \leq E_{max} = \frac{1}{4\gamma}$. Putting $Q(E,\gamma) = \frac{I_2(E,\gamma)}{I_1(E,\gamma)}$ (where I_1 and I_2 are given by (5.16)) it can be shown analytically that $\lim_{E\downarrow 0} Q(E,\gamma) = 0$, $\lim_{E\uparrow \frac{1}{4\gamma}} Q(E,\gamma) = \frac{12}{35\gamma}$, $\frac{\partial Q}{\partial E}(0,\gamma) = \frac{3}{2} > 0$, and $\frac{\partial Q}{\partial E}(\frac{1}{4\gamma},\gamma) = -\infty$. By using an adaptive Clenshaw-Curtis quadrature scheme $Q(E,\gamma)$ has been calculated numerically for different values of the parameter γ (or λ). These numerical results can be found in figure 5.3 (a)-(e). Using these numerical results we can try to find nontrivial values of E such that $I(E,\gamma) = 0$ or equivalently $Q(E,\gamma) = \frac{1}{9}$ for a given value of $\gamma = \frac{-\lambda}{9} > 0$. The numerical results can be summarized as follows: for $\lambda < -27.77434...$ there are no nontrivial values of E such that $Q = \frac{1}{9}$, and so, there are no limit cycles; for $\lambda = -27.77434...$ there are two coinciding nontrivial values of E such that $Q = \frac{1}{9}$, and so, there is a semi-stable limit cycle; for $-27.77434... < \lambda < -27.77233...$ there are two different, nontrivial values of E such that $Q = \frac{1}{9}$, and so, there are two limit cycles (one stable and one unstable); for $-27.77233... < \lambda < 0$ there is exactly one nontrivial value of E such that $Q = \frac{1}{9}$, and so, there is exactly one (stable) limit cycle. In Figure 5.4 a sketch of the appearance and disappearance of limit cycles is given for decreasing values of the parameter λ (and $\mu = 0$).

5.2. The case $\mu > 0$ and λ arbitrary. The case $\mu < 0$ and λ arbitrary can be treated similarly by simply replacing X by $-X$. Let $T < \infty$ be the period of a periodic solution and let c_4 be a constant in a first integral $F = constant$ for which a periodic solution exists. Consider $F = c_4$ for $\tau = 0$ and $\tau = T$. Approximating F by F_7 (as given by (4.30)) and eliminating c_4 by subtraction, we then obtain

$$\tilde{\epsilon}\left(\int_0^T G_{[2]}^2\left(1 - \eta G_{[2]}^2\right) d\bar{t}\right) = \mathcal{O}(\tilde{\epsilon}^2), \tag{5.17}$$

or equivalently

$$\tilde{\epsilon}\left(\int_{Z(0)}^{Z(T)} G_{[2]}\left(1 - \eta G_{[2]}^2\right) dZ\right) = \mathcal{O}(\tilde{\epsilon}^2) \Leftrightarrow \tilde{\epsilon} I(E,\xi,\eta) = \mathcal{O}(\tilde{\epsilon}^2), \tag{5.18}$$



(a) Zoom in for $\lambda = -27.77542$

(b) Zoom in for $\lambda = -27.77434$

(c) Zoom in for $\lambda = -27.77233$

(d) $\lambda = -20$

(e) $\lambda = -\frac{1}{10}$

(f) $\lambda = 0$

(g) $\lambda = \frac{1}{10}$

(h) $\lambda = 1$

FIG. 5.3. *Plot of Q as function of E for $\mu = 0$ and for different values of λ.*

where

$$I(E, \xi, \eta) = 2 \int_A^B G_{[2]} \left(1 - \eta G_{[2]}^2\right) dZ, \tag{5.19}$$

with $A = Z(0)$ and $B = Z(\frac{1}{2}T)$. To have a periodic solution for equation (4.27) we have to find an energy E such that $I(E, \xi, \eta)$ is equal to zero (see also [13, 20, 21]).

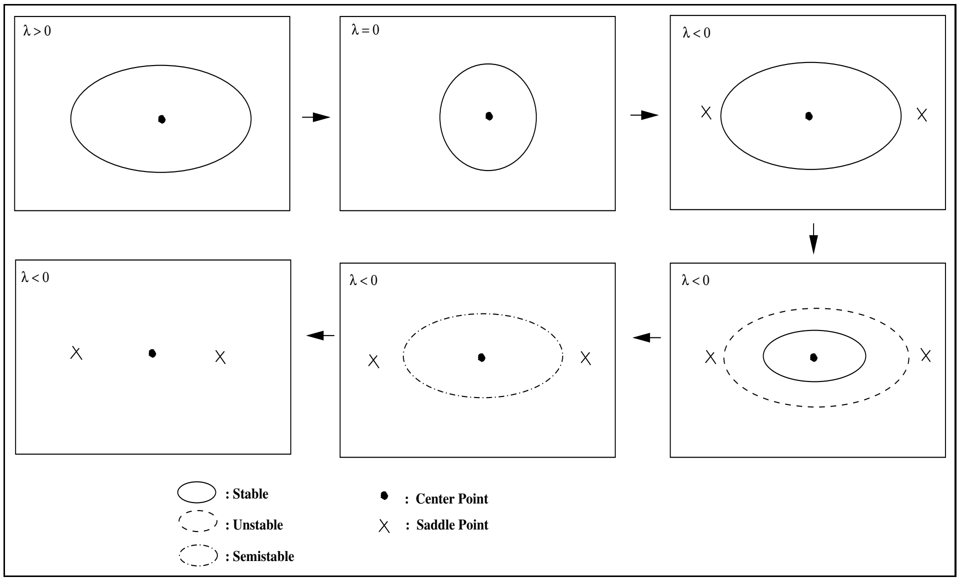


FIG. 5.4. *Sketch of the appearance and disappearance of limit cycles for $\mu = 0$ and for decreasing values of the parameter λ.*

To find this energy E we rewrite $I(E, \xi, \eta)$ in

$$I(E,\xi,\eta) = 2I_1(E,\xi)\left(1 - \eta\frac{I_2(E,\xi)}{I_1(E,\xi)}\right), \tag{5.20}$$

where

$$\begin{cases} I_1(E,\xi) &=& \int_A^B \left(2E - Z^2 - \frac{2}{3}Z^3 - \frac{1}{2}\xi Z^4\right)^{\frac{1}{2}} dZ, \\ I_2(E,\xi) &=& \int_A^B \left(2E - Z^2 - \frac{2}{3}Z^3 - \frac{1}{2}\xi Z^4\right)^{\frac{3}{2}} dZ. \end{cases} \tag{5.21}$$

Throughout this section $Q(E,\xi)$ is equal to $\frac{I_2(E,\xi)}{I_1(E,\xi)}$. The equilibrium points in the (Z, Z') phase plane of the unperturbed equation (4.27) with $\tilde{\epsilon} = 0$ are listed in Table 5.1, and the corresponding phase portraits of the unperturbed equation are given in Figure 5.5. To determine whether or not $I(E,\xi,\eta)$ can be (nontrivially) equal to zero (or equivalently $Q(E,\xi) = \frac{1}{\eta}$) we have to distinguish five cases: $\xi < 0$, $\xi = 0$, $0 < \xi < \frac{1}{4}$, $\xi = \frac{1}{4}$, and $\xi > \frac{1}{4}$. These five cases will be studied in the following five subsections.

5.2.1. The case $\xi < 0$. The unperturbed equation (4.27) with $\tilde{\epsilon} = 0$ and $\xi < 0$ has only periodic orbits surrounding the center point $(0,0)$. For these periodic orbits the energy E satisfies: $0 = E_{min} \le E \le E_{max} = \frac{1}{2}z_1^2 + \frac{1}{3}z_1^3 + \frac{1}{4}\xi z_1^4$, where z_1 is $-\frac{1}{2\xi} + \frac{1}{2\xi}\sqrt{1-4\xi}$. Putting $Q(E,\xi) = \frac{I_2(E,\xi)}{I_1(E,\xi)}$ (where I_1 and I_2 are given by (5.21)) it can be shown elementarily that $\lim_{E \downarrow E_{min}} Q = 0$, and that $\lim_{E \uparrow E_{max}} Q = P(\xi)$. P as function of ξ can be calculated numerically and is given in Figure 5.6. It can also be shown analytically that $\frac{\partial Q}{\partial E}(E_{min},\xi) = \frac{3}{2} > 0$, and that $\frac{\partial Q}{\partial E}(E_{max},\xi) = -\infty$ (see also Figure 5.7). For several values of ξ we have calculated $Q(E,\xi)$ numerically by using an adaptive recursive Simpson rule. The numerical results are presented in

ξ	Type and position of Equilibrium point(s)
$\xi < 0$	a center in (0,0), a saddle in $\left(-\frac{1}{2\xi} - \frac{1}{2\xi}\sqrt{1-4\xi}, 0\right)$, and a saddle in $\left(-\frac{1}{2\xi} + \frac{1}{2\xi}\sqrt{1-4\xi}, 0\right)$
$\xi = 0$	a center in (0,0), and a saddle in (-1,0)
$0 < \xi < \frac{1}{4}$	a center in (0,0), a center in $\left(-\frac{1}{2\xi} - \frac{1}{2\xi}\sqrt{1-4\xi}, 0\right)$, and a saddle in $\left(-\frac{1}{2\xi} + \frac{1}{2\xi}\sqrt{1-4\xi}, 0\right)$
$\xi = \frac{1}{4}$	a center in $(0,0)$, and a higher order singularity in $(-2,0)$
$\xi > \frac{1}{4}$	a center in $(0,0)$

TABLE 5.1

Type of equilibrium points of the unperturbed equation (4.27) with $\tilde{\epsilon} = 0$ in the (Z, Z') phase plane.

Figure 5.7. From Figure 5.7 it is clear that there are cases for which we can find two nontrivial values of E such that $I(E,\xi,\eta) = 0$ or equivalently $Q(E,\xi) = \frac{1}{\eta}$. So, we can conclude that for (4.27) with $\xi < 0$ at least two limit cycles can occur. It should be remarked that when two limit cycles are "bifurcated" out of the periodic orbits around the center point then these two limit cycles are very close each other. In fact the energy level of the stable periodic solution is only a little bit less then the energy level of the unstable periodic solution. The period T of the periodic solution(s) can be determined from

$$\frac{dZ}{d\tau} = Z' = \sqrt{\left(2E^* - Z^2 - \frac{2}{3}Z^3 - \frac{1}{2}\xi Z^4\right)}, \tag{5.22}$$

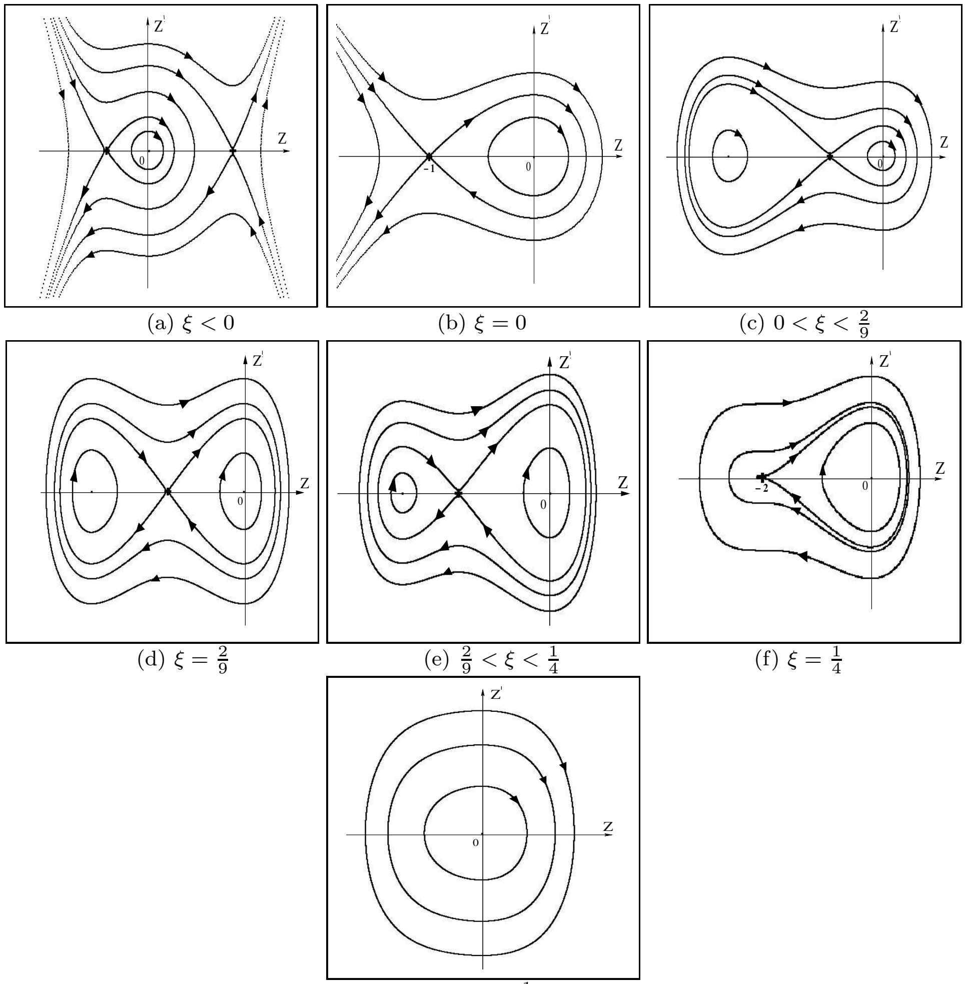
or equivalently from

$$T = \int_A^B \frac{2dZ}{\sqrt{\left(2E^* - Z^2 - \frac{2}{3}Z^3 - \frac{1}{2}\xi Z^4\right)}}, \tag{5.23}$$

where $\left(-\frac{1}{2\xi} + \frac{1}{2\xi}\sqrt{1-4\xi}\right) \leq A < B \leq \left(-\frac{1}{6\xi} - \frac{1}{2\xi}\sqrt{1-4\xi} + \frac{1}{3\xi}\sqrt{1+3\sqrt{1-4\xi}}\right)$, and where A and B satisfy $\frac{1}{2}Z^2 + \frac{1}{3}Z^3 + \frac{1}{4}\xi Z^4 = E^*$ in which E^* is the energy for which a periodic solution occurs.

5.2.2. The case $\xi = 0$. This case has already been studied in [13], and we refer the reader to this paper for detailed calculations. It has been shown in [13] that at most two limit cycles can bifurcated out of the periodic orbits surrounding the center point. A sketch of the appearance and disappearance of limit cycles for $\xi = 0$ and decreasing values of η is given in Figure 5.8.

5.2.3. The case $0 < \xi < \frac{1}{4}$. The unperturbed equation (4.27) with $\tilde{\epsilon} = 0$ has in this case three equilibrium points in the phase plane: a center point in $\underline{\sigma}_1 = (0,0)$, a center point in $\underline{\sigma}_2 = \left(-\frac{1}{2\xi} - \frac{1}{2\xi}\sqrt{1-4\xi}, 0\right)$, and a saddle point in $\underline{\sigma}_3 = \left(-\frac{1}{2\xi} + \frac{1}{2\xi}\sqrt{1-4\xi}, 0\right)$ (see also Table 5.1 and Figure 5.5). The maximum energy level for the periodic orbit inside the saddle loop connection (around the center point in $\underline{\sigma}_1$, and around the center point in $\underline{\sigma}_2$), and the minimum energy level for the periodic orbits outside the saddle loop connection are equal to $E_{loop} = \frac{1}{2}z_1^2 + \frac{1}{3}z_1^3 + \frac{1}{4}\xi z_1^4$,



(a) $\xi < 0$ (b) $\xi = 0$ (c) $0 < \xi < \frac{2}{9}$

(d) $\xi = \frac{2}{9}$ (e) $\frac{2}{9} < \xi < \frac{1}{4}$ (f) $\xi = \frac{1}{4}$

(g) $\xi > \frac{1}{4}$

FIG. 5.5. *Phase portrait of the unperturbed equation (4.27) with $\tilde{\epsilon} = 0$ for several values of ξ.*

where z_1 is $-\frac{1}{2\xi} + \frac{1}{2\xi}\sqrt{1-4\xi}$. The minimum energy level $E_{min[\underline{\sigma}_2]}$ for periodic orbits surrounding the center point in $\underline{\sigma}_2$ is $E_{min[\underline{\sigma}_2]} = \frac{1}{2}z_2^2 + \frac{1}{3}z_2^3 + \frac{1}{4}\xi z_2^4$, where z_2 is $-\frac{1}{2\xi} - \frac{1}{2\xi}\sqrt{1-4\xi}$. Putting $Q(E,\xi) = \frac{I_2(E,\xi)}{I_1(E,\xi)}$ (where I_1 and I_2 are given by (5.21)) it can be shown analytically that:

(i) for the periodic orbits surrounding the center point in $\underline{\sigma}_1$:
$\lim_{E \downarrow E_{min[\underline{\sigma}_1]}} Q(E,\xi) = 0$, $\lim_{E \uparrow E_{loop}} Q(E,\xi) = R_{\underline{\sigma}_1}(\xi)$, $\frac{\partial Q}{\partial E}(E_{min[\underline{\sigma}_1]},\xi) = \frac{3}{2}$, and $\frac{\partial Q}{\partial E}(E_{loop},\xi) = -\infty$,

(ii) for the periodic orbits surrounding the center point in $\underline{\sigma}_2$:
$\lim_{E \downarrow E_{min[\underline{\sigma}_2]}} Q(E,\xi) = 0$, $\lim_{E \uparrow E_{loop}} Q(E,\xi) = R_{\underline{\sigma}_2}(\xi)$, $\frac{\partial Q}{\partial E}(E_{min[\underline{\sigma}_2]},\xi) = \frac{3}{2}$, and $\frac{\partial Q}{\partial E}(E_{loop},\xi) = -\infty$, and

(iii) for the periodic orbits outside the saddle loop connection:
$\lim_{E \downarrow E_{loop}} Q(E,\xi) = R_{\underline{\sigma}_3}(\xi)$, $\lim_{E \uparrow \infty} Q(E,\xi) = \infty$, $\frac{\partial Q}{\partial E}(E_{loop},\xi) = -\infty$,

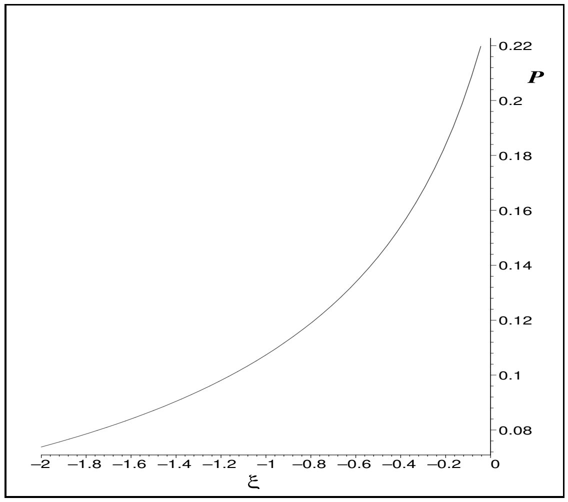


FIG. 5.6. *Plot of P as function of ξ for $\xi < 0$.*

where $R_{\underline{\sigma}_i}(\xi)$ for $i = 1, 2$, and 3 can be determined numerically and are given in Figure 5.9. Using an adaptive recursive Simpson rule $Q(E, \xi)$ has been calculated numerically for several values of ξ. Plots of Q are given in Figure 5.10 for $\xi = \frac{1}{9}$, in Figure 5.11 for $\xi = \frac{2}{9}$, and in Figure 5.12 for $\xi = \frac{17}{72}$. From these figures it is obvious that there are always non-trivial E-values such that $I(E, \xi, \eta) = 0$ or equivalently $Q(E, \xi) = \frac{1}{\eta}$. Each non-trivial E-value corresponds to a limit cycle in the phase plane. It also follows from these numerical calculations that at most two limit cycles can be bifurcated out of the periodic orbits surrounding $\underline{\sigma}_1$. The same result also holds for the periodic orbits surrounding $\underline{\sigma}_2$, and for the periodic orbits outside the saddle loop. However, out of all the periodic orbits at most five limit cycles can be bifurcated simultaneously. Numerical calculations give the following results:

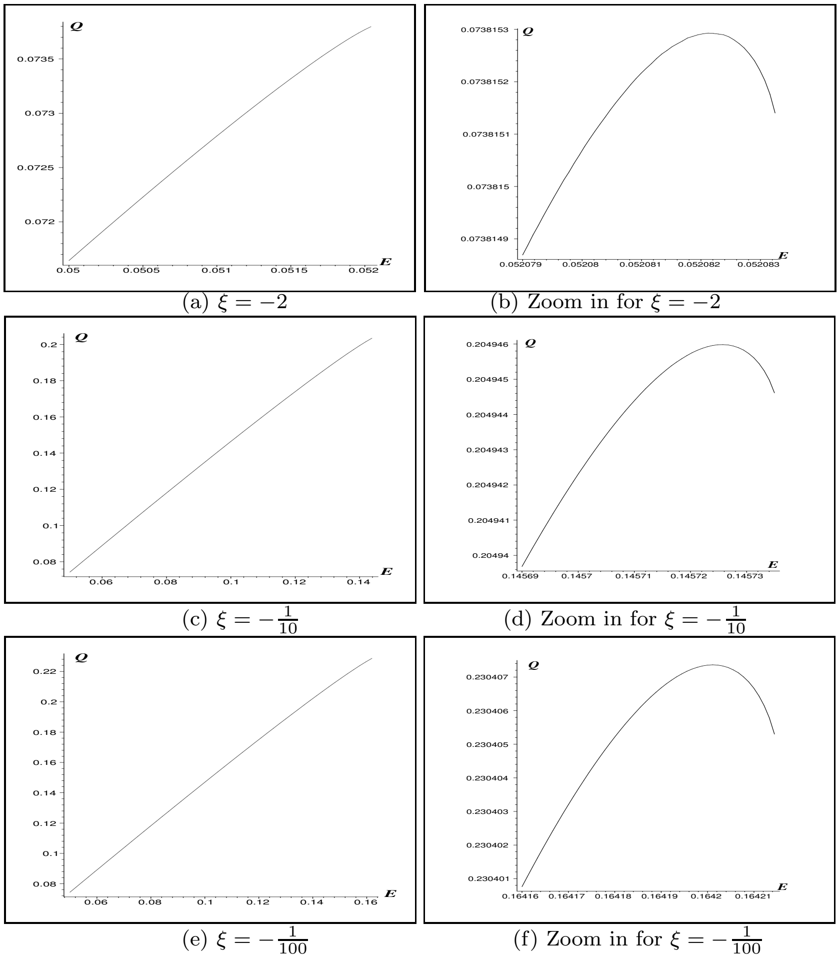
(i) for $0 < \xi < \frac{2}{9} - 569... \times 10^{-6}$at most three limit cycles can be bifurcated out of the periodic orbits,

(ii) for $\frac{2}{9} - 5.69... \times 10^{-6} < \xi < \frac{2}{9} + 5.69... \times 10^{-6}$ at most five limit cycles can be bifurcated out of the periodic orbits, and

(iii) for $\frac{2}{9} + 5.69... \times 10^{-6} < \xi < \frac{1}{4}$ at most three limit cycles can be bifurcated out of the periodic orbits.

For $\xi = \frac{1}{9}$ and decreasing values of η and for $\xi = \frac{2}{9}$ and decreasing values of η sketches of the appearance and disappearance of limit cycles are presented in Figure 5.13 and in Figure 5.14 respectively. The period T of a periodic solution can again be determined as is indicated in section 5.2.1.

5.2.4. The case $\xi = \frac{1}{4}$. The unperturbed equation (4.27) with $\tilde{\epsilon} = 0$ has in this case two equilibrium points in the phase plane: a center point in $\underline{\sigma}_1 = (0, 0)$, and a higher order singularity in $\underline{\sigma}_2 = (-2, 0)$(see also Table 5.1 and Figure 5.5). The maximum energy level for the periodic orbits inside the loop connection (that is, the orbit which starts in $(-2, 0)$ and ends in $(-2, 0)$), and the minimum energy level for the periodic orbits outside the loop connection are equal to $E_{loop} = \frac{1}{3}$. The minimum energy level for the periodic orbits surrounding the center point in $\underline{\sigma}_1 = (0, 0)$ is 0. Putting $Q(E, \xi) = \frac{I_2(E,\xi)}{I_1(E,\xi)}$ (where I_1 and I_2 are given by (5.21)) it can be shown analytically that:



(a) $\xi = -2$ (b) Zoom in for $\xi = -2$

(c) $\xi = -\frac{1}{10}$ (d) Zoom in for $\xi = -\frac{1}{10}$

(e) $\xi = -\frac{1}{100}$ (f) Zoom in for $\xi = -\frac{1}{100}$

FIG. 5.7. *Plot of $Q(E;\xi)$ as function of E for several values of $\xi < 0$.*

(i) for the periodic orbits inside the loop connection: $\lim_{E\downarrow 0} Q(E,\frac{1}{4}) = 0$, $\lim_{E\uparrow E_{loop}} Q(E,\frac{1}{4}) = \frac{4}{9}$, $\frac{\partial Q}{\partial E}(0,\frac{1}{4}) = \frac{3}{2}$, and $\frac{\partial Q}{\partial E}(E_{loop},\frac{1}{4}) = -\infty$, and

(ii) for the periodic orbits outside the loop connection: $\lim_{E\downarrow E_{loop}} Q(E,\frac{1}{4}) = \frac{4}{9}$, $\lim_{E\uparrow\infty} Q(E,\frac{1}{4}) = \infty$, $\frac{\partial Q}{\partial E}(E_{loop},\frac{1}{4}) = -\infty$.

Again using an adaptive recursive Simpson rule $Q(E,\frac{1}{4})$ has been calculated numerically. Plots of $Q(E,\frac{1}{4})$ are given in Figure 5.15. From this figure it is clear that there are always non-trivial E-values such that $I(E,\xi,\eta) = 0$ or equivalently $Q(E,\xi) = \frac{1}{\eta}$. Each non-trivial E-value corresponds to a limit cycle in the phase plane. It also follows from these numerical calculations that for $\xi = \frac{1}{4}$ at most three limit cycles can be bifurcated out of the periodic orbits. More explicitly in this case: for $0.4318... < \frac{1}{\eta} < 0.44548...$ three limit cycles will occur, and for $\frac{1}{\eta} < 0.4318...$ or for $\frac{1}{\eta} > 0.44548...$ exactly one limit cycle will occurs. A sketch of the appearance and disappearance of limit cycles for $\xi = \frac{1}{4}$ and for decreasing values of η is given in Figure 5.16.

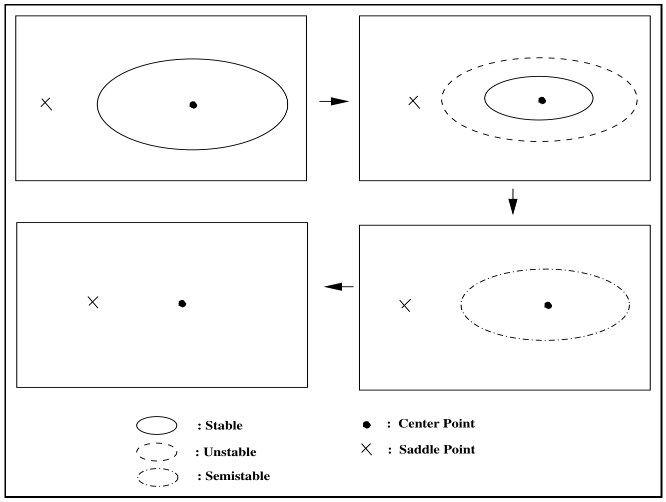
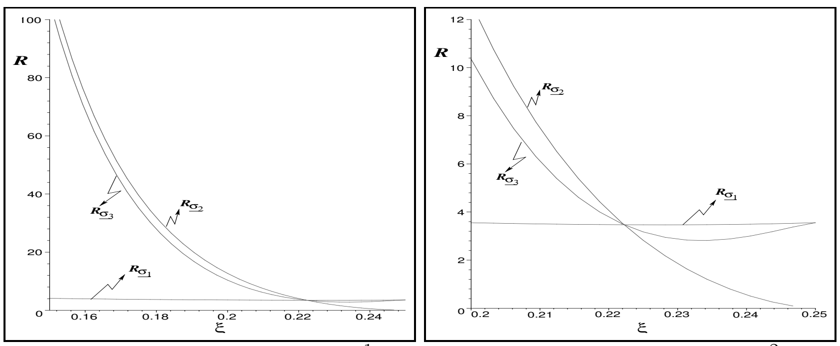


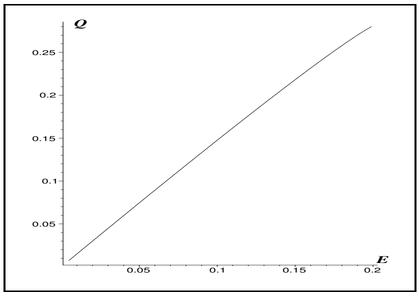
FIG. 5.8. *Sketch of the appearance and disappearance of limit cycles for $\xi = 0$ and for decreasing value of η.*



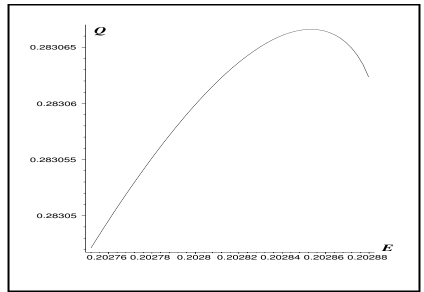
(a) $R_{\underline{\sigma}_i}(\xi)$ for $0 \leq \xi \leq \frac{1}{4}$. (b) Zoom in near $\xi = \frac{2}{9}$.

FIG. 5.9. *Plot of R as function of ξ.*

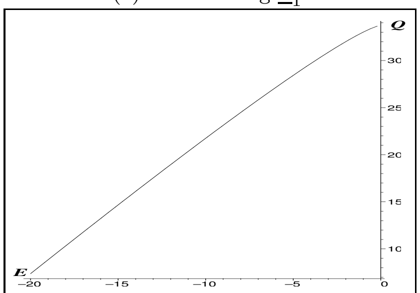
5.2.5. The case $\xi > \frac{1}{4}$. The unperturbed equation (4.27) with $\tilde{\epsilon} = 0$ has in this case only one equilibrium point in the phase plane: a center in (0,0) (see also Table 5.1 and Figure 5.5). The minimum and the maximum energy level for the periodic orbits are in this case 0 and ∞ respectively. Putting $Q(E, \xi) = \frac{I_2(E,\xi)}{I_1(E,\xi)}$ (where I_1 and I_2 are given by (5.21)) it can be shown analytically that $\lim_{E \downarrow E_{min}} Q(E, \xi) = 0$, $\lim_{E \uparrow \infty} Q(E, \xi) = \infty$, and $\frac{\partial Q}{\partial E}(0, \xi) = \frac{3}{2}$. Again $Q(E, \xi)$ has been calculated numerically and plots of $Q(E, \xi)$ are given in Figure 5.17 for different values of ξ. From this figure it is clear that there are always non-trivial E-values such that $I(E, \eta, \xi) = 0$ or equivalently $Q(E, \xi) = \frac{1}{\eta}$. It also follows from these numerical calculations that for $\frac{1}{4} < \xi < \frac{1}{4} + 0.004488...$ at most three limit cycles, and for $\xi > \frac{1}{4} + 0.004488...$ at most one limit cycle can be bifurcated out of the periodic orbits. A sketch of the appearance and disappearance of limit cycles for $\frac{1}{4} < \xi < \frac{1}{4} + 0.004488...$ and for decreasing values of η is given in Figure 5.16.



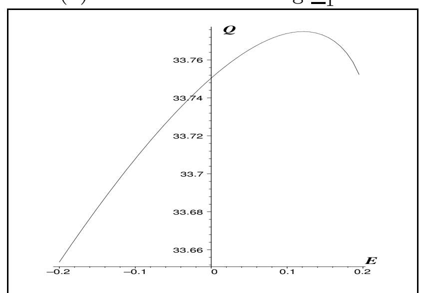
(a) Surrounding $\underline{\sigma}_1$



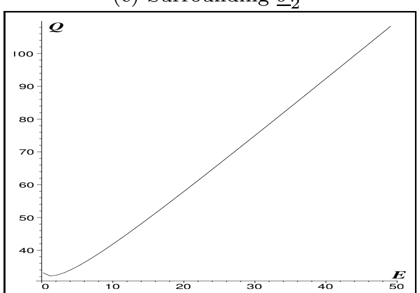
(b) Zoom in surrounding $\underline{\sigma}_1$



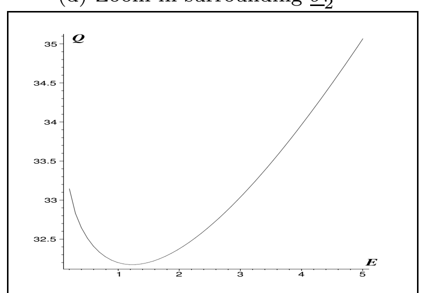
(c) Surrounding $\underline{\sigma}_2$



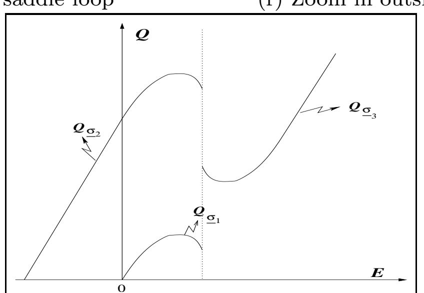
(d) Zoom in surrounding $\underline{\sigma}_2$



(e) Outside saddle loop



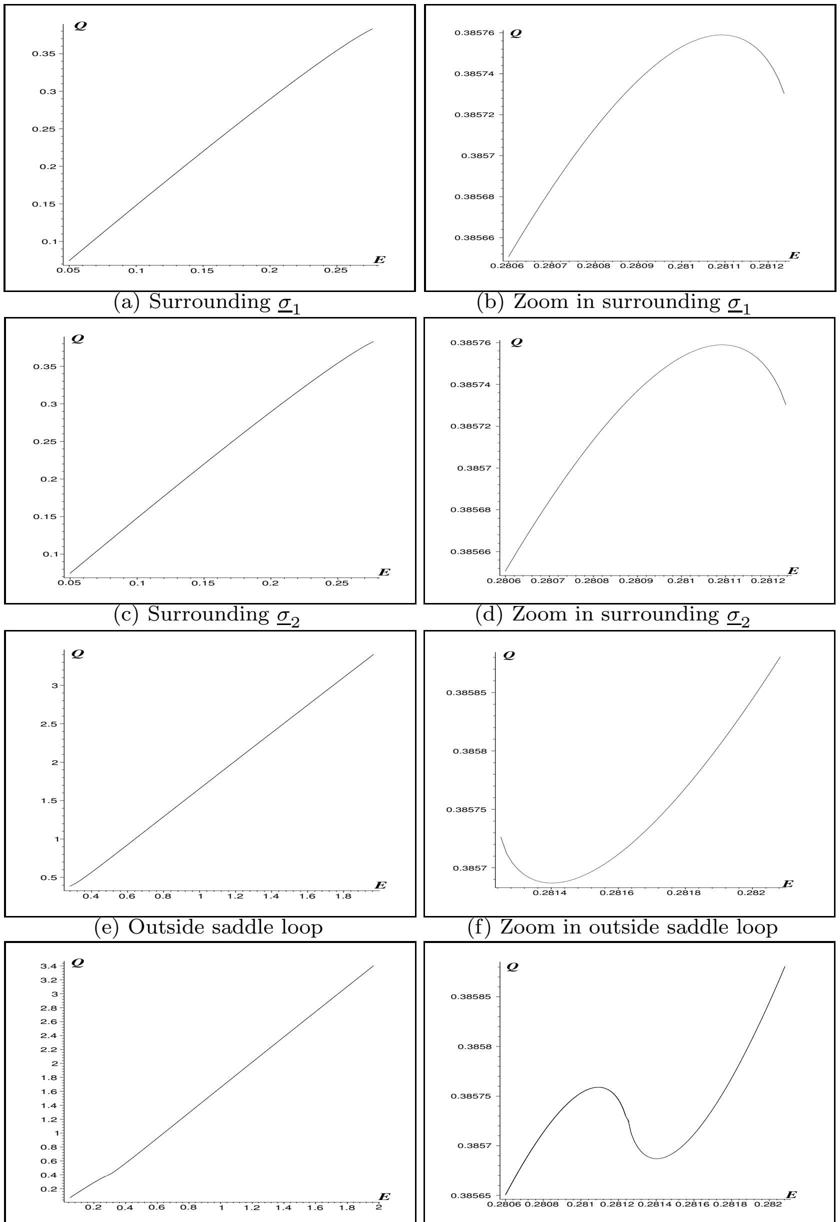
(f) Zoom in outside saddle loop



(g) Sketch for all

FIG. 5.10. *Plot of* $Q(E;\xi)$ *as function of* E *for* $\xi = \frac{1}{9}$.

6. Conclusions and remarks. In this paper it has been shown that the perturbation method based on integrating factors can be used efficiently to approximate first integrals for strongly nonlinear oscillator equations. In section 2 (and 3) of this paper an asymptotic justification of the presented perturbation method has been given. It has been shown how the existence and stability of time-periodic solutions can be deduced from the approximations of the first integrals. In section 4 it has been shown explicitly how approximations of first integrals can be constructed for the generalized



(a) Surrounding $\underline{\sigma}_1$ (b) Zoom in surrounding $\underline{\sigma}_1$

(c) Surrounding $\underline{\sigma}_2$ (d) Zoom in surrounding $\underline{\sigma}_2$

(e) Outside saddle loop (f) Zoom in outside saddle loop

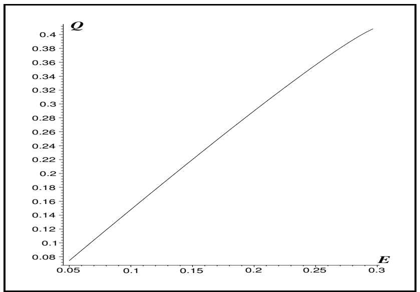
(g) For all (h) Zoom in for all

FIG. 5.11. *Plot of $Q(E;\xi)$ as function of E for $\xi = \frac{2}{9}$.*

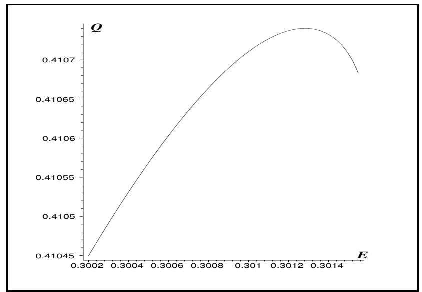
Rayleigh oscillator equation

$$\ddot{X} + 9X + \mu X^2 + \lambda X^3 = \epsilon(\dot{X} - \dot{X}^3), \tag{6.1}$$

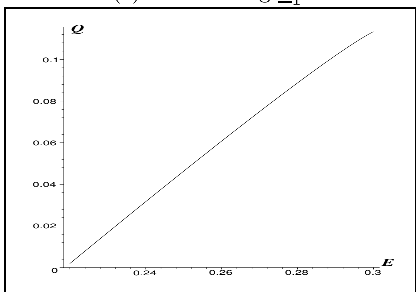
where μ and λ are parameters, and where ϵ is a small parameter satisfying $0 < \epsilon \ll 1$. In section 5 it has been shown how the existence of time-periodic solutions of (6.1) can be determined from the approximations of first integrals. The following results for the generalized Rayleigh oscillator equation (6.1) have been obtained in this paper:



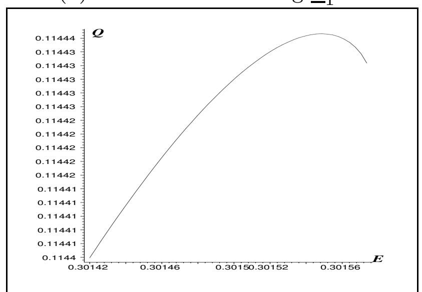
(a) Surrounding $\underline{\sigma}_1$



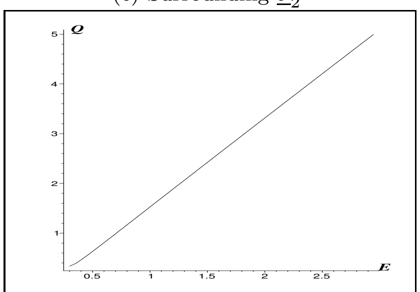
(b) Zoom in surrounding $\underline{\sigma}_1$



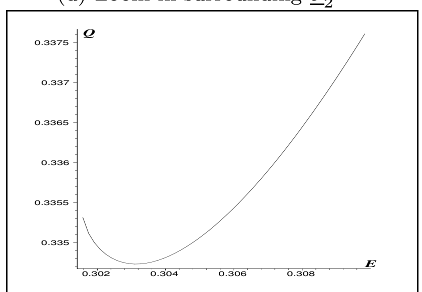
(c) Surrounding $\underline{\sigma}_2$



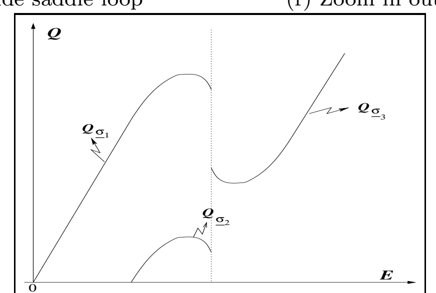
(d) Zoom in surrounding $\underline{\sigma}_2$



(e) Outside saddle loop



(f) Zoom in outside saddle loop



(g) Sketch for all

FIG. 5.12. *Plot of $Q(E;\xi)$ as function of E for $\xi = \frac{17}{72}$.*

(partially based on strong numerical evidence)

(i) for $\mu = 0$ and $\lambda \geq 0$, and for $\mu \neq 0$ and $\frac{9\lambda}{\mu^2} > \frac{1}{4} + 0.004488...$: exactly one limit cycle is bifurcated out of the periodic orbits (of the unperturbed equation (6.1) with $\epsilon = 0$).

(ii) for $\lambda < 0$, and for $\mu \neq 0$ and $\lambda = 0$: at most two limit cycles can be bifurcated out of the periodic orbits.

(iii) for $\mu \neq 0$ and $0 < \frac{9\lambda}{\mu^2} < \frac{2}{9} - 5.69... \times 10^{-6}$, and for $\mu \neq 0$ and $\frac{2}{9} + 5.69... \times 10^{-6} <$

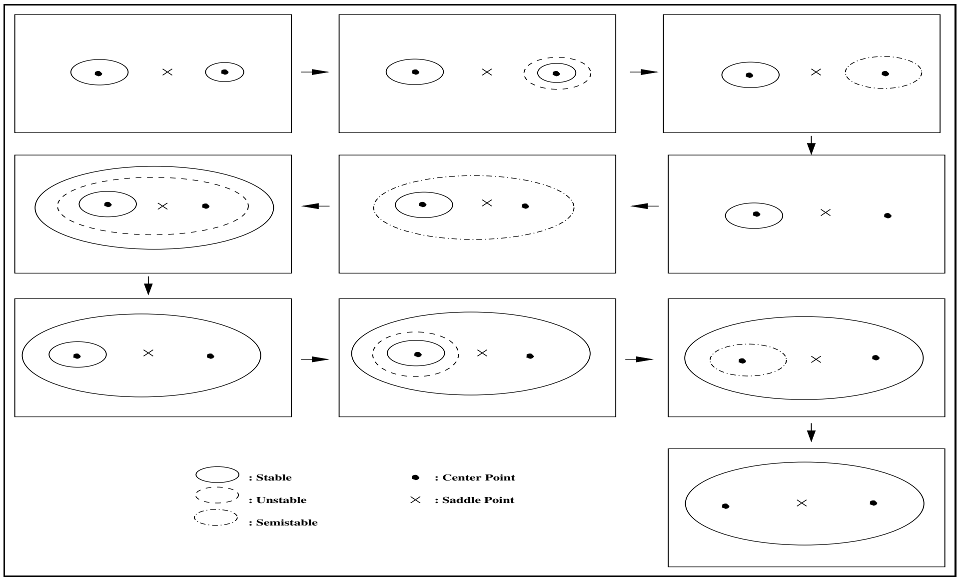


FIG. 5.13. *Sketch of the appearance and disappearance of limit cycles for* $\xi = \frac{1}{9}$ *and for decreasing values of* η.

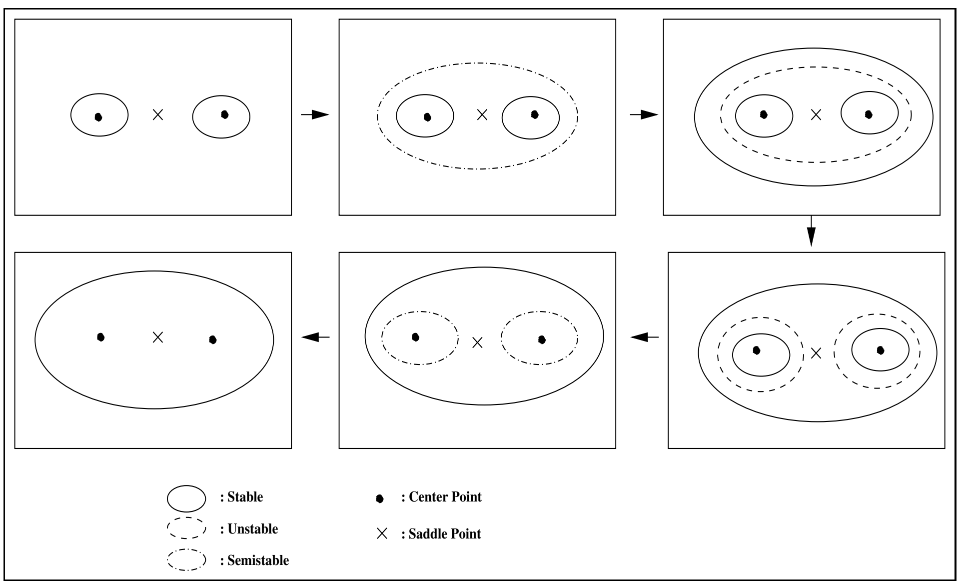
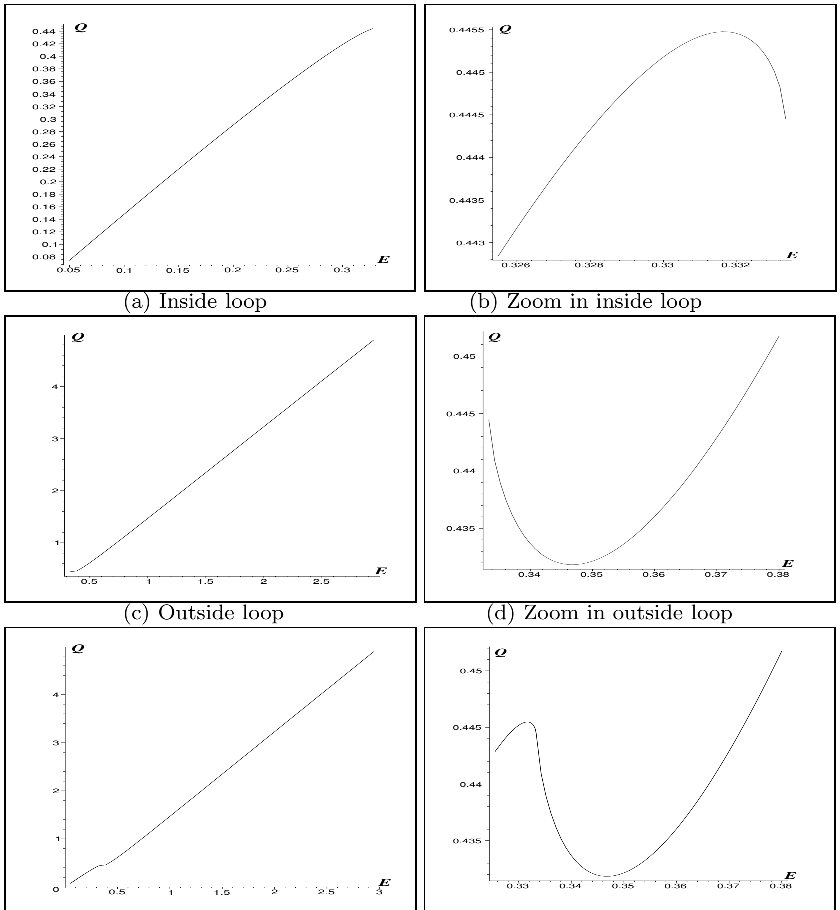


FIG. 5.14. *Sketch of the appearance and disappearance of limit cycles for* $\xi = \frac{2}{9}$ *and for decreasing values of* η.

$\frac{9\lambda}{\mu^2} \leq \frac{1}{4} + 0.004488...$: at most three limit cycles can be bifurcated out of the periodic orbits.

(iv) for $\mu \neq 0$ and $\frac{2}{9} - 5.69... \times 10^{-6} < \frac{9\lambda}{\mu^2} < \frac{2}{9} + 5.69... \times 10^{-6}$: at most five limit cycles can be bifurcated out of the periodic orbits.



(a) Inside loop (b) Zoom in inside loop

(c) Outside loop (d) Zoom in outside loop

(e) For all combined (f) Zoom in

FIG. 5.15. *Plot of $Q(E;\xi)$ as function of E for $\xi = \frac{1}{4}$.*

Acknowledgment–This research project was sponsored by Secondary Teacher Development Programme (PGSM)(Indonesia) and The University of Technology in Delft (The Netherlands).

REFERENCES

[1] A. Doelman and F. Verhulst, *Bifurcations of Strongly Non-linear Self-excited Oscillations*, Mathematical Methods in the Applied Sciences, Vol. 17, 1994, pp. 189-207.

[2] C.G.A. van der Beek, *Normal form and periodic solutions in the theory of non-linear oscillations. Existence and asymptotic theory*, International journal of non-linear mechanics,Vol. 24, No. 4, 1989, pp. 263-279.

[3] C.G.A. van der Beek, *Analysis of a system of two weakly nonlinear coupled harmonic oscillators arising from the field of wind-induced vibrations*, International journal of non-linear mechanics,Vol. 27, No. 4, 1992, pp. 679-704.

[4] D.F. Lawden, *Elliptic Functions and Applications*, Springer-Verlag, New York, 1989.

[5] D.L. Bosley, *An improved matching procedure for transient resonance layers in weakly nonlinear oscillatory systems*, SIAM J. Appl. Math., vol. 56, No. 2, 1996, pp. 420-445.

[6] I.D. Iliev and L.M. Perko, *Higher Order Bifurcations of Limit Cycles*, Journal of Differential

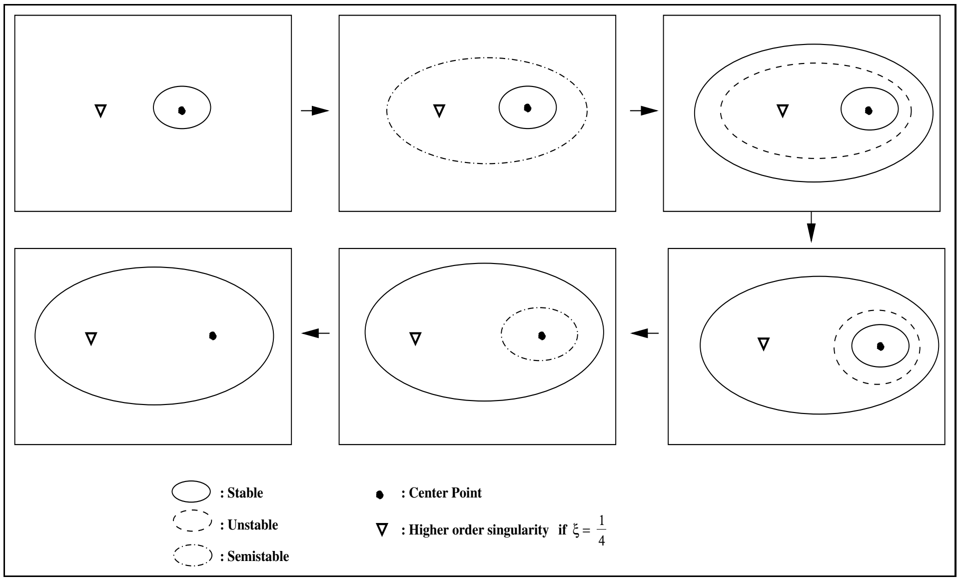


FIG. 5.16. *Sketch of the appearance and disappearance of limit cycles for* $\frac{1}{4} \leq \xi < \frac{1}{4}+0.004488...$ *and for decreasing values of* η.

Equations, 154, 1999, pp. 339-363.

[7] J.D. Brothers and R. Haberman, *Accurate Phase after Passage through Subharmonic Resonance*, SIAM J.Appl. Math., Vol. 59, No. 1, 1998, pp. 347-364.

[8] J.G. Margallo and J.D. Bejarano, *Stability of limit cycles and bifurcations of generalized van der Pol oscillator:*$\ddot{X} + AX - 2BX^3 + \epsilon(z_3 + z_2X^2 + z_1X^4)\dot{X} = 0$, Int. J. Non-Linear Mechanics, Vol. 25, No. 6, 1990, pp. 663-675.

[9] J.G. Margallo and J.D. Bejarano, *The limit cycles of the generalized Rayleigh-Liénard oscillator*, Journal of Sound and Vibration 156(2), 1992, pp. 283-301.

[10] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd, Berlin Springer, Berlin,1971.

[11] R.V. Roy, *Averaging Methods for strongly Non-linear Oscillators with Periodic Excitations*, Int.J. Non-linear Mechanics, Vol.29 No 5, 1994, pp. 737-753.

[12] T.R. Blows and L.M. Perko, *Bifurcation of limit cycles from center and sparatrix cycles of planar analytic systems*, SIAM Review, Vol.36 No 3, 1994, pp. 341-376.

[13] S.B. Waluya and W.T. van Horssen, *Asymptotic Approximations of First Integrals for a Nonlinear Oscillator*, to be published in Nonlinear Analysis TMA.

[14] S.B. Waluya and W.T. van Horssen, *On Approximations of First Integrals for a System of Weakly Nonlinear, Coupled Harmonic Oscillators*, to be published in Nonlinear Dynamics.

[15] S.B. Yuste and J.D. Bejarano, *Extension and improvement to the Krylov-Bogoliubov methods using elliptic functions*, Int.J. Control, Vol.49 No 4, 1989, pp. 1127-1141.

[16] S.H. Chen and Y.K. Cheung, *An Elliptic Lindstedt-Poincaré Method for certain strongly Nonlinear Oscillators*, Nonlinear Dynamics 12, 1997, pp. 199-213.

[17] S.H. Chen and Y.K. Cheung, *An Elliptic Perturbation Method for certain strongly Non-linear Oscillators*, Journal of Sound and Vibration 192(2), 1996, pp. 453-464.

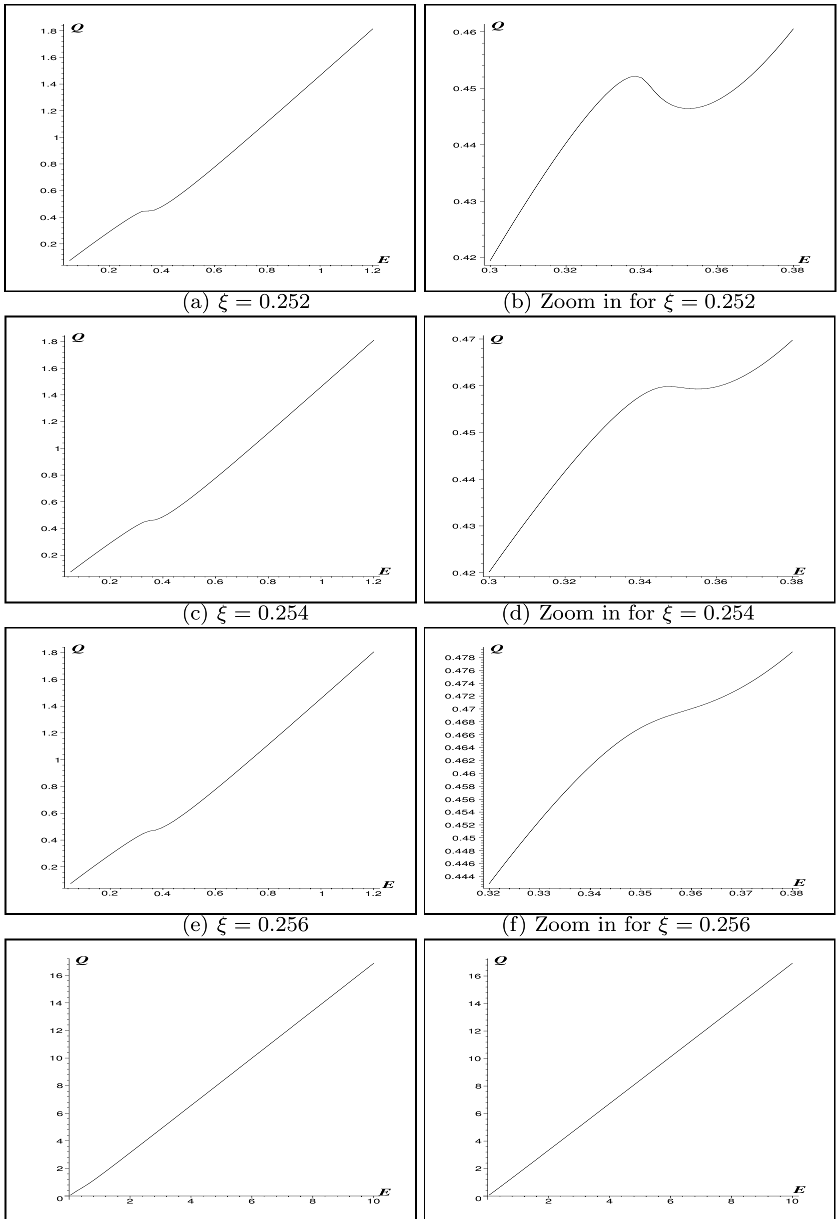
[18] S. Lynch, *Small amplitude limit cycles of the generalized mixed Rayleigh-Liénard oscillator*, Journal of Sound and Vibration 178(5), 1994, pp. 615-620.

[19] V.T. Coppola and R.H. Rand, *Averaging using Elliptic Functions:approximation of limit cycles*, Acta Mechanica 81, 1990, pp. 125-142.

[20] W.T. van Horssen and R. E. Kooij, *Bifurcation of limit cycles in a Particular class of Quadratic systems with Two Centers*, Journal of Differential Equations, Vol. 114, No. 2, 1994, pp.538-569.

[21] W.T. van Horssen and J.W. Reyn, *Bifurcation of limit cycles in a particular class of quadratic systems*, Differential and Integral Equations, Vol. 8, No. 4, 1995, pp. 907-920.

[22] W.T. van Horssen, *On Integrating Factors for Ordinary Differential Equations*, Nieuw Archief



(a) $\xi = 0.252$ (b) Zoom in for $\xi = 0.252$

(c) $\xi = 0.254$ (d) Zoom in for $\xi = 0.254$

(e) $\xi = 0.256$ (f) Zoom in for $\xi = 0.256$

(g) $\xi = 0.3$ (h) $\xi = 10$

FIG. 5.17. *Plot of $Q(E;\xi)$ as function of E for $\xi > \frac{1}{4}$.*

voor Wiskunde, Vol. 15, 1997, pp. 15-26.

[23] W.T. van Horssen, *A Perturbation Method Based on Integrating Factors*, SIAM journal on Applied Mathematics, Vol. 59, No. 4, 1999, pp. 1427-1443.

[24] W.T. van Horssen, *A Perturbation Method Based on Integrating Vectors and Multiple Scales*, SIAM journal on Applied Mathematics, Vol. 59, No. 4, 1999, pp. 1444-1467.

[25] W.T. van Horssen, *On integrating vectors and multiple scales for singularly perturbed ordinary differential equations*, Nonlinear Analysis TMA, Vol. 46, 2001, pp. 19-43.