Department of Precision and Microsystems Engineering

Techniques to synthesize Inherently Force Balanced Mechanisms

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TECHNIQUES TO SYNTHESIZE INHERENTLY FORCE BALANCED MECHANISMS

by

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A Giovi e Marchino

Preface

This thesis represents the end of a long journey which started in Milan with my bachelor. There, I have developed a strong interest in the mechanical design for high-precision tasks. When I found the "High-Tech Engineering" track at TU Delft, I immediately realized that it was the right path for me. In particular, the expectations that I had towards the "Precision Mechanisms Design" course of Prof. Just Herder were exceeded, and I found in dynamic balancing a potential topic for my thesis. Indeed, after the first talks with my supervisor Volkert van der Wijk, I realized that working on the Synthesis of Inherently Dynamically Balanced mechanisms represented an opportunity for me to combine my interests with a new dynamic balancing approach based on classical mechanics theories.

I would like to express my sincere gratitude to my supervisor Volkert van der Wijk for everything he has done for me, all the helpful comments and encouragement during our meetings and for having introduced me to the *world* of Inherent Dynamic Balance and all its potential applications.

Furthermore, I would like to thank all the people who helped me to reach this goal. I cannot thank enough my friends Giuliano and Yanni: you guys have always been there for me and honestly I don't know how I could have made it without you. Special thanks go to Antonio and Mylo for all the moments that we shared (not only) in Delft, thank you for making me feel at home.

Also, I would like to thank the Department of Precision and Microsystem Engineering for creating a friendly environment and my colleagues for the connection we created and all the time that we spent together.

Last but not least, I would like to express my deepest gratitude towards my family for the constant encouragement and for having supported me in all my decisions during these years at university.

Lorenzo Girgenti
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1

Introduction

In a variety of applications, when machine elements are actuated inertia forces and moments are generated. They produce reaction forces and moments on the base which are called *shaking forces* and *shaking moments* and induce undesired base vibrations. These result in noise, wear and fatigue problems [1] and can compromise the accuracy of the machines [2]. In addition, issues related to the stability of the systems can occur when machines are placed on bases which are not fixed, an example consists in manipulators which are mounted on satellites [3]. When dynamic balance is achieved, *shaking forces* and *shaking moments* are canceled as well as vibrations on the base: dynamically balanced mechanisms are designed. Force balanced and moment balanced mechanisms are created, respectively, when only *shaking forces* and *shaking moments* are canceled.
Dynamic balance is generally achieved by introducing additional elements which do not affect mechanisms' motions. Counterweights are introduced to achieve force balance: the center of mass (CoM) of a mechanism is kept on a stationary point and the linear momentum is constant, therefore no external forces are applied on the mechanism. On the other hand, moment balance is achieved by introducing counter-rotating elements in order to keep the angular momentum constant, therefore no external moments are applied. Both counterweights and counter-rotating elements can be introduced in mechanisms and connected to the links in different ways. Counterweights can be mounted on links or connected to them through pantograph linkages [4]. These, in particular have been widely used thanks to their copying properties [5, 6]. On the other hand, counter-rotating elements can be introduced on the base and connected to their respective links through either gears [7], belt transmissions [8], or idler loops [9]. In addition, counter-rotating elements can be introduced on their respective links and used as counterweights as well. In this case, they are called counter-rotary counterweights and are driven by either gears [10], belts or chains [11] in order to make their rotations opposite to the ones of the respective links. Also, dynamic balance can be achieved by introducing axial and mirror symmetric duplicate mechanisms which perform opposite motions with respect to the initial one [11].
The introduction of additional elements, whose function consists in only dynamic balancing, implies the introduction of additional mass and inertia leading to an increase of the power required to drive the resulting mechanisms. Its design results to be more complex as well. This because dynamic balance is generally considered after the design process. Nevertheless, *inherent dynamic balancing* [12] considers dynamic balance as a design principle. It aims at designing mechanisms in which all the links contribute to both the kinematics and the dynamic balance, therefore additional elements are not included. These mechanisms are called *inherently dynamically balanced* (IDB) mechanisms. By considering dynamic balance as a design principle, IDB mechanisms result to be synthesized from inherently force balanced linkages. These are called *principal vector linkages* as their design is based on the method of *principal vectors* [1]. Force balance is achieved by keeping the common CoM on a stationary point and relating all the links' positions and motions to this point. The overall kinematics results to be a combination of those of the parallelograms forming *principal vector linkages*. As a result, when only a link rotates, its parallel links rotate while the others only translate. The force balance conditions, which are derived from the linear momentum equations as described in [12], are based on this essential kinematics of *principal vector linkages*.
Although several examples of synthesized linkages were provided [12–14], issues can be identified and can constitute a limitation in real applications. Even though the kinematic properties of parallelograms are es-

sential to maintain force balance, these linkages can constitute a problem. In particular, they are change point mechanisms [15]. When all the links become collinear, the change point state is reached and the mechanism's motions are not uniquely determined: parallelograms can switch to their anti-parallelogram configuration and force balance is not maintained. In addition, links can potentially overlap each other: this can represent a problem for those applications in which making overlapping links is complex. Furthermore, the single design of *principal vector linkages* can represent a limitation in their functionality and applicability. As they consist of only links, which need to form parallelograms to maintain the force balance, their design can not be adapted to specific motion and space requirements.
This thesis presents techniques to modify *principal vector linkages* in a variety of ways and can be used to synthesize inherently force balanced mechanisms. They can consist of additional machine elements like sliders, gears, belt and chain drives to reduce the number of DoFs and to replace the original links. Potential advantages can consist in creating new inherently force balanced designs. These are still based on the kinematics of the original *principal vector linkages*, but they can be adapted to specific motions and space requirements while preventing issues like singularities of parallelograms.
All the techniques maintain the force balance of *principal vector linkages* as they do not modify their essential kinematics, on which the force balance conditions are based. This implies that, given any *principal vector linkage* together with the force balance conditions, it can be modified by applying the techniques. The force balance conditions do not need to be derived for the resulting linkage because the kinematics of the initial *principal vector linkage* is maintained. Therefore, the same equations are considered. Only possible adjustments can be required when links' position are modified and additional machine elements are introduced.

Approach and outline

In order to realize how additional machine elements could be introduced in *principal vector linkages*, the literature study, presented in chapter 2, focused on these linkages' design and on how machine elements are generally introduced in mechanisms. In addition to describing the synthesized IDB mechanisms including additional elements, two categorizations are presented and describe how rotating elements of gear, belt and chain drives and sliders are generally introduced in mechanisms.
The techniques to modify *principal vector linkages* are presented in chapter 3. They were derived by observing how the machine elements presented in chapter 2 could be introduced and how similar outcomes can be achieved without their introduction. As a result, the number of DoFs can be reduced by constraining links' motions. Furthermore the position of the links can be modified and the parallelograms forming *principal vector linkages* result to be resized. Links can be also replaced by machine elements like sliders, gear, belt and chain drives. These substitutions do not affect the kinematics of the parallelograms and therefore the essential kinematics of the overall *principal vector linkages* is maintained. Potential advantages are presented as well as a inherently force balanced mechanism which was synthesized by applying these techniques. Its design results to be different from the original *principal vector linkage* and, by performing a simulation, it is proven that the force balance is maintained.
Chapter 4 describes how the force balance conditions are adjusted when techniques are applied in *principal vector linkages*, especially when mass-asymmetric links and machine elements are considered. The equations which are presented in this chapter apply for *principal vector linkages* having mass-symmetric links and machine elements: mass symmetry is indeed a particular case of mass asymmetry. In addition, when mass-asymmetric additional elements are considered, force balance conditions can be complex to adjust, depending on how the elements move with respect to the other links.
After the discussion and the conclusion which are presented in chapters 5 and 6, respectively, Appendix A shows the force balance conditions of the synthesized inherently force balanced mechanism which was presented in chapter 3 and describes how the techniques' application prevents issues regarding parallelogram linkages. Appendix B, on the other hand, presents calculations in order to introduce mass-asymmetric gear trains in *principal vector linkages*.

Since *principal vector linkages* are only force balanced, the techniques only aim at maintaining force balance and not at defining moment balanced solutions. Moreover, all the links and machine elements are assumed rigid. In addition, no backlash is considered in gear and chain drives as well as no slippage of the belt is considered in belt drives.

2

Methods to introduce gears and sliders in Inherently Dynamically Balanced mechanisms

Dynamic balance is normally considered as the next step after mechanical design: mechanisms are designed first and subsequently balance is considered. Additional elements acting as counterweights and counter-inertias are generally introduced. On the other hand, *Inherent Dynamic Balancing* considers dynamic balance as a design principle: balanced mechanisms are synthesized from linkage architectures which are force balanced. As a result, no additional element is introduced, leading to practical and low mass and low inertia solutions. This is achieved by designing *principal vector linkages*, which are force balanced during all their motions. Inherently Dynamically Balanced mechanisms results to be synthesized from these linkages. However, not much is known about all the possibilities for mechanisms' synthesis, especially those regarding the application of additional machine elements like gears and sliders in *principal vector linkages*. In order to investigate how mechanisms can be synthesized, while possibly including these additional elements, a study was conducted. At the beginning, it focused on the design principles and the different types of *principal vector linkages*. Issues regarding the parallelogram structures within these linkages were identified. Then, an investigation in the literature, focused on how elements such gears and sliders are normally used in mechanisms, was performed. Categories were defined basing on how these elements are connected to the other links of the mechanisms and the base. It was supposed that the introduction of these elements can be advantageous in order to prevent issues related to parallelogram linkages.

1 Introduction

Dynamic balance has always been a serious issue in designing mechanical systems. Due to the motion of machine components, reaction forces and moments are generated on the base. This leads to base vibrations which can cause noise, wear, fatigue problems, together with a reduction of the performances. In manufacturing and metrology systems, for example, base vibrations due to unbalanced machines compromise the accuracy of all the systems which are placed on the same base. This problem is more serious in high-tech industry, where high speed and high precision are required. Many balancing methods have been formulated [1]: they include the addition of counterweights and counter-inertias. However, the introduction of these elements increases the total mass and total inertia of the resulting mechanisms. As a result, the power required to drive these mechanisms can significantly increase as well as their design complexity. This is due to the fact that the traditional approach consists in balancing mechanisms after their design. However, the approach proposed by Van der Wijk [2], known as *Inherent Dynamic Balancing*, consists in considering dynamic balancing as a design principle, in order to design mechanisms that are already balanced. These, which are called *Inherently Dynamically Balanced* (IDB) mechanisms, are synthesized from force balanced linkages architectures which will be shown section 2.
A problem which was identified consists in the parallelogram linkages constituting these architectures. Parallelogram linkages are called change point mechanisms [3]: when all the revolute joints become collinear, the change point state is reached and the mechanism' motions can not be uniquely determined. Singularities can occur and the force balance of the overall architectures is not maintained. In addition, overlapping links may not be possible in many real applications and therefore the range of motion of the synthesized mechanisms results to be limited. In order to prevent this problem, different machine elements, such gears and sliders, could be introduced. By constraining the mechanisms, overlapping links could be removed and singularities could be prevented. However, only two examples of synthesized IDB mechanisms were provided in which gears and sliders were used to add constraints and replace links. This led to the question *"Are there any other ways in which gears and sliders can be introduced in IDB linkages?"*. An investigation in the literature was made in order to find mechanisms in which gears and sliders were introduced in different ways. The results of this investigation, shown in section 3, consist in two categorizations describing, respectively, how gears and sliders are introduced in mechanisms. As several ways to introduce these elements were observed, the research proposal which is carried out consists in *"Synthesizing IDB mechanisms having gears and sliders"* by introducing these elements as they are presented in the categories described in section 3.
After the categorizations, which present potential advantages and disadvantages of the different categories, the intended approach to the research is discussed in the fourth section, together with potential risks and potential benefits of introducing additional machine elements.

2 Inherently dynamically balanced (IDB) linkage architectures

In order to reduce the additional mass and inertia, Van der Wijk [2] proposed an approach which considered dynamic balancing as a principle in mechanisms design. This led to the design of inherently force balanced linkage architectures from which mechanisms can be synthesized. They are called *principal vector linkages* and are force balanced as their center of mass (CoM) is stationary during all the motions. If the CoM of a system is stationary or moves with constant velocity with respect to the base, the system is force balanced and therefore the reaction forces to the base are zero. This means that the summation of external forces acting on the system is zero and linear momentum is constant during all the motions of the system, according to Newton's second law. On the other hand, these linkage architectures are not moment balanced. Therefore, the summation of external moments is not zero and there are moments acting on the base when the architectures move. This means that the angular momentum of the system is not constant. However, it was shown that moment balance conditions can be defined and solutions can be obtained by adding constraints.

2.1 Principal vector linkages

Principal vector linkages are force balanced linkages consisting in *principal elements*, which are connected together through revolute joints called *principal joints*, and *principal vector links*, which allow the relative motion of the the *principal elements* with respect to the common CoM. This is possible thanks to the *principal vectors method.* Vectors having constant magnitude describe the position of *principal points* with respect to the common CoM: for each *principal element*, a *principal point* represents the CoM of the reduced-mass model of the linkage when only the rotation of that *principal element* is allowed and the masses of the moving elements are projected on it. The force balance conditions, consisting in the derivatives of linear momentum equations which are set equal to zero, describe the position of the *principal points* within the respective *principal elements. Principal vectors* describe the position of the *principal points* with respect to the common CoM. Their magnitudes, which are called *principal dimensions*, are derived from the force balance conditions as well. Since their lengths are constant, *principal vectors* are transformed in rigid links, the *principal vector links*, whose masses are taken into account when the position of the common CoM and the reduced-mass models are defined.
It can be stated that in *principal vector linkages* the CoMs of all the links are related to the common CoM. Force balance is achieved by keeping this point invariant. This property is found in pantograph linkages. The pantograph in Figure 1 can be considered as a 2 DoF *principal vector linkage* where S is the common CoM and the *principal vectors* correspond to links SP_1 and SP_2 having *principal dimensions*, respectively, a_2 and a_1. *Principal elements* AP_1 and AP_2, having *principal points* P_1 and P_2 and CoMs in Q and R, are connected through *principal joint* A. Considerations about balance conditions and similarity properties were made in [2].

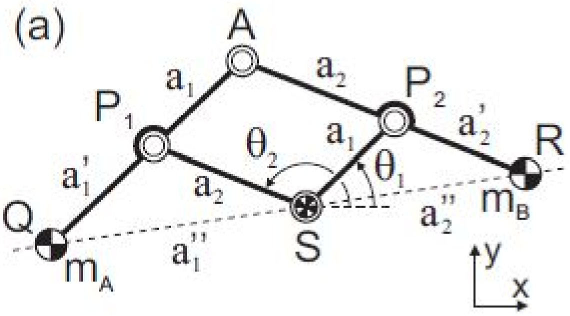


Figure 1: 2 DoF force balanced pantograph. Common CoM fixed in joint S. Source [2]

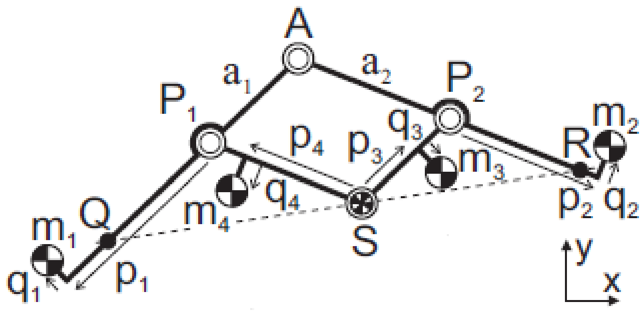


Figure 2: 2 DoF force balanced pantograph. Links having mass. Source [2]

When *principal vectors* are replaced by *principal vector links* and the links' CoMs are placed in generic positions (Figure 2), reduced-mass models are made when each DoF is singularly actuated (Figures 3 and 4). Thanks to parallelogram SP_1AP_2, links whose angle is not actuated solely translate: their masses can be easily projected on the rotating *principal element*. The force balance conditions are derived from the linear momentum equations in each model. Equations (1) are defined when θ_1 is actuated and equations (2) when θ_2 is actuated.

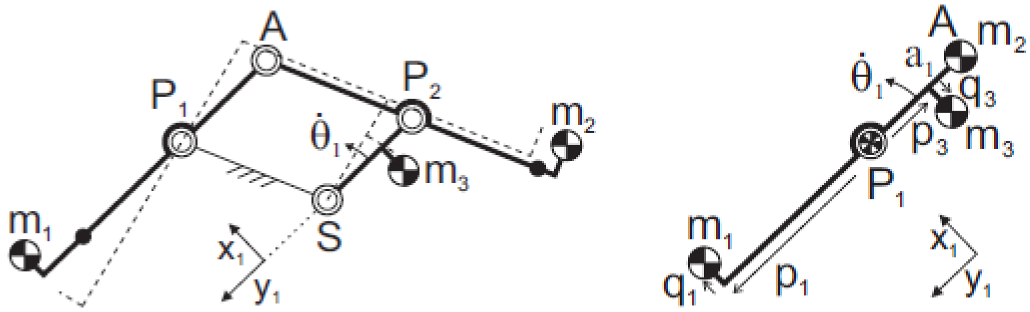


Figure 3: Pantograph of Figure 2. θ_1 actuated, reduced-mass model. Source [2]

$$
\begin{aligned}
m_1p_1 &= m_2a_1 + m_3p_3 \\
m_1q_1 &= m_3q_3
\end{aligned} \tag{1}
$$

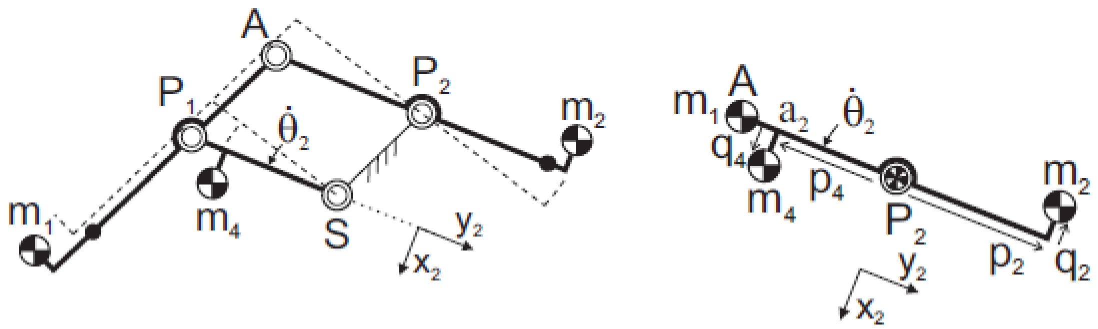


Figure 4: Pantograph of Figure 2. θ_2 actuated, reduced-mass model. Source [2]

$$\begin{aligned} m_2 p_2 &= m_1 a_2 + m_4 p_4 \\ m_2 q_2 &= m_4 q_4 \end{aligned} \tag{2}$$

2.1.1 Open chains principal vector linkages

The presented pantograph linkage is a 2 DoF open chain *principal vector linkage* having two *principal elements*. Other open chains *principal vector linkages* were designed having more *principal elements* as well. The design of these linkages is based on the principle that the CoMs of two *principal elements* can be related their common CoM with one pantograph, whose links can be incorporated in the *principal elements* themselves. These pantographs are connected with others until one single pantograph is centered on the common CoM of the overall *principal vector linkage*.

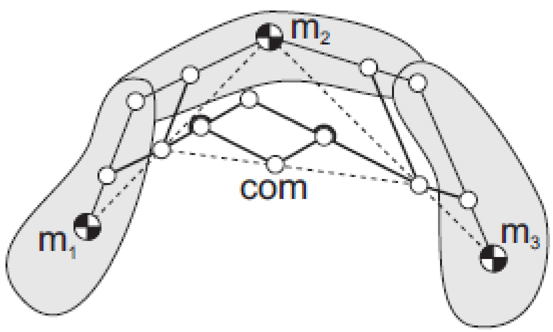


Figure 5: 3 DoF principal vector linkage, three pantographs. Source [2]

Figure 5 shows a 3 DoF *principal vector linkage* having three *principal elements*. Two pantographs have links incorporated in the *principal elements*: they trace, respectively, the common CoM related to m_1 and part of m_2, and the common CoM related to m_3 and part of m_2. Another pantogrpah connects the previous ones with the common CoM, which is always in an invariant point to achieve force balance. Thanks to the kinematic property of the parallelograms which are included in the pantographs, they can be combined and the result consists in the linkage in Figure 6 which has always 3 DoF and less links than the one in Figure 5.

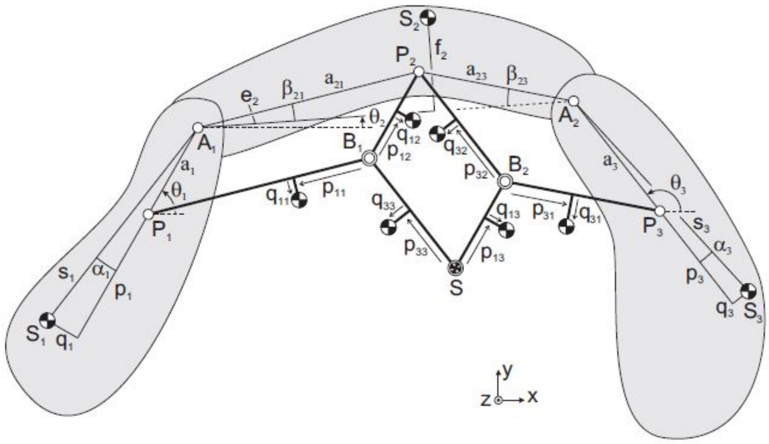


Figure 6: 3 DoF principal vector linkage, combined parallelograms. Links CoMs in generic positions. Source [2]

With respect to Figure 6, the *principal vectors* describing the *principal points* P_1, P_2 and P_3 are shown in Figure 7. The force balanced conditions are derived from the linear momentum equations written for each DoF. The same procedure seen for the pantograph linkage was carried out [2].

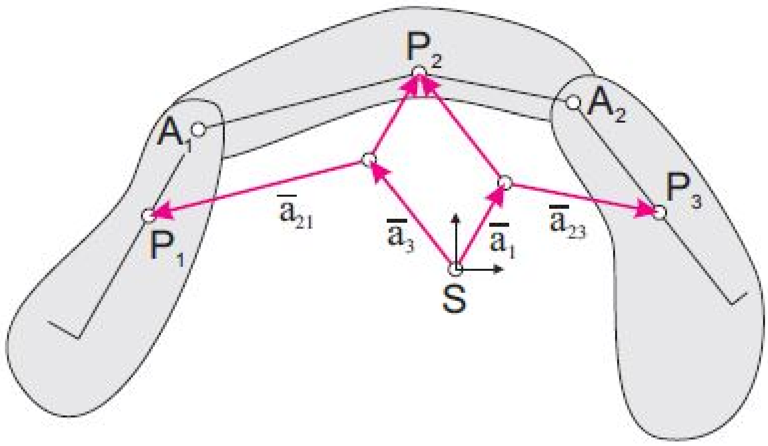


Figure 7: 3 DoF principal vector linkage of Figure 6. Principal vectors. Source [2]

Principal vector linkages having more *principal elements* were designed by following the same procedure. As an example, Figures 8 and 9 show, respectively, a 4 DoF *principal vector linkage* having four *principal elements* and its relative *principal vectors* configuration.

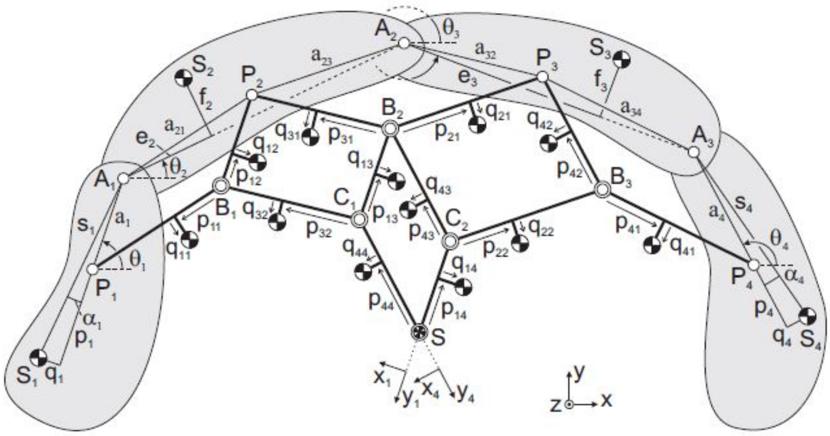


Figure 8: 4 DoF principal vector linkage. Links CoMs in generic positions. Source [2]

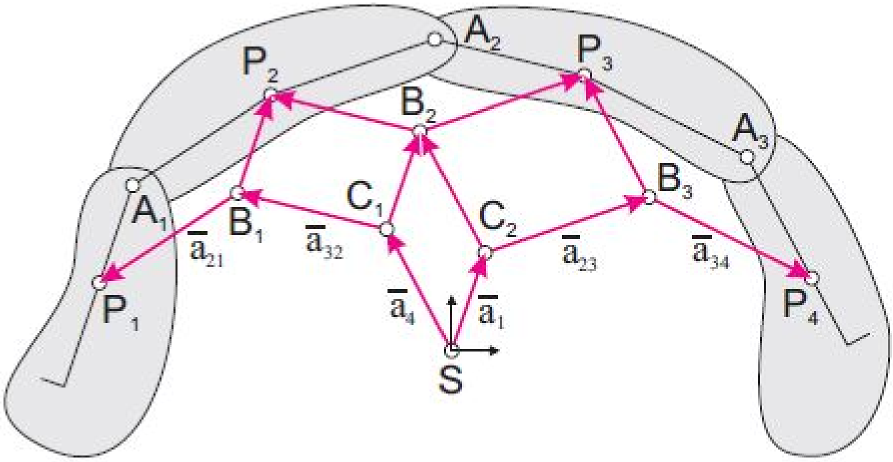


Figure 9: 4 DoF principal vector linkage of Figure 8. Principal vectors. Source [2]

2.1.2 Closed chains principal vector linkages

In addition to open chains, closed chains *principal vector linkages* were designed. One method to derive these linkages consists in closing an open chain, that is, connecting the outer *principal elements* through a revolute joint. By connecting the outer elements of Figure 8, the 2 DoF closed chain shown in Figure10 is obtained.

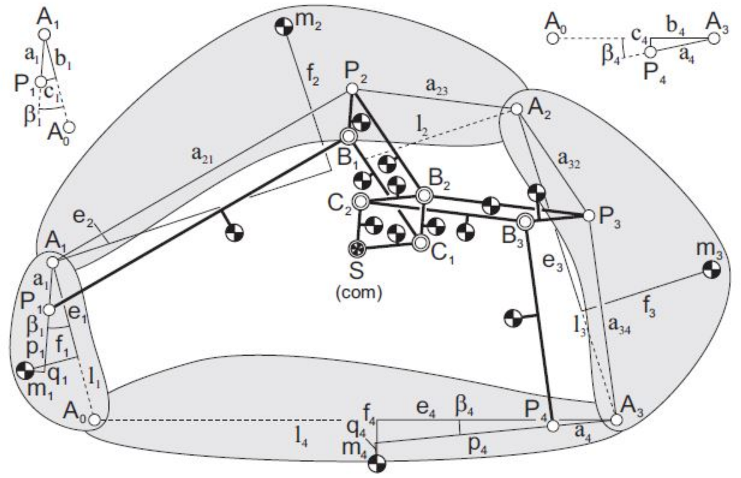


Figure 10: Closed chain principal vector linkage derived by closing open chain. Source [2]

It is possible to locate the common CoM in principal element A_0A_3 as shown in Figure 11. In doing so, the linkage becomes overconstrained but yet movable because of the similarity of quadrilaterals $A_0A_1A_2A_3$, $A_0P_1B_1C_1$ and $C_1B_2P_3A_3$. This is because parallelograms $P_1A_1P_2B_1$, $B_1P_2B_2C_1$ and $C_2B_3P_3B_2$ are movable as well.

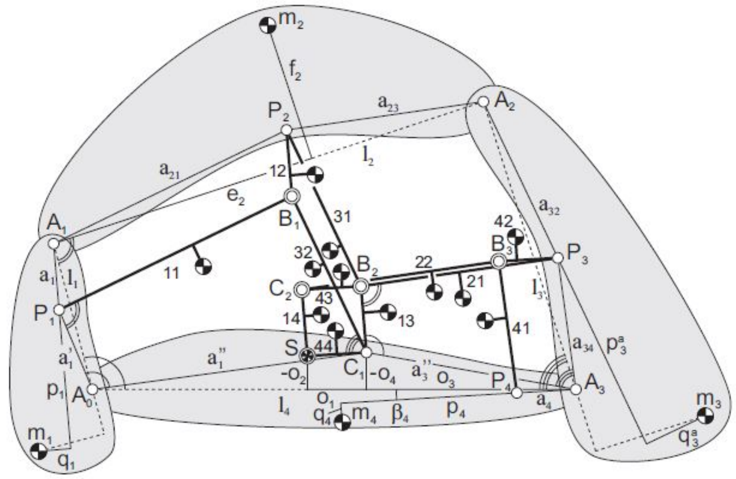


Figure 11: Closed chain principal vector linkage, common CoM in principal element A_0A_3. Source [2]

Another method to derive closed chains *principal vector linkages* consists in making a mass equivalent model of a *principal element* and, therefore, studying the remaining open chain *principal vector linkage* while considering the equivalent masses. For example, by considering the closed chain of *principal elements* in Figure 10, a mass model related to A_0A_3 can be defined and can replace the *principal element* itself. As shown in Figure 12, two real equivalent masses m_4^b and m_4^a are located, respectively, on the *principal joints* A_0 and A_3, and two virtual equivalent masses m_4^c are located on points J_{41} and J_{42}. All the equivalent masses are considered in the remaining open chain *principal vector linkage*. In particular, the real equivalent masses are located on A_0 and A_3, while the virtual equivalent masses are considered on each *principal element*. The resulting linkage is shown in Figure

13. It can be observed that the configuration of the *principal vector links* is similar to the one of the 3 DoF *principal vector linkage* shown in Figure 6.

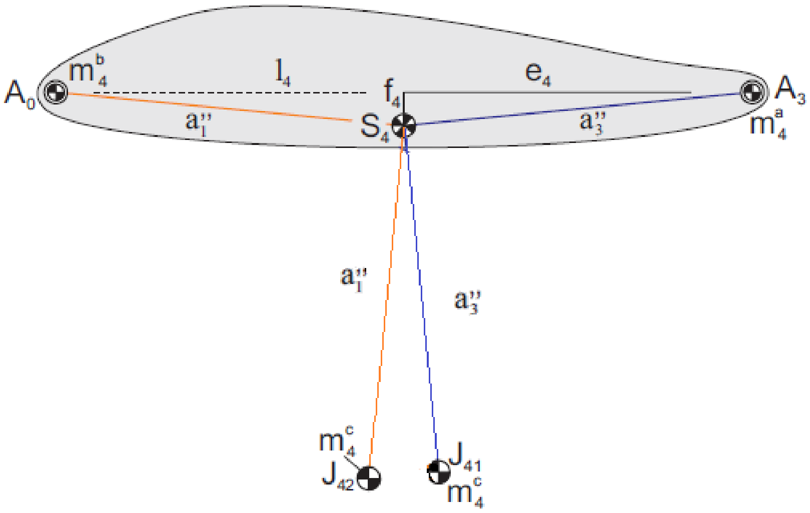


Figure 12: Principal element A_0A_3 of the linkage in Figure 10. Mass equivalent model. Source [2]

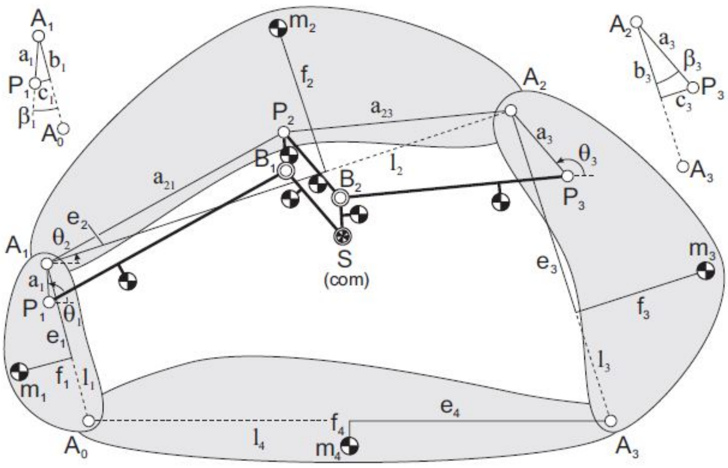


Figure 13: Closed chain principal vector linkage of Figure 10 with A_0A_3 modeled with equivalent masses. Source [2]

This procedure is used to design closed chain *principal vector linkages* having more than four *principal elements* as well.

By looking at Figure 13, the parallelograms $A_1P_2B_1P_1$, $B_1P_2B_2S$ and $P_2A_2P_3B_2$ do not involve *principal element* A_0A_3, as a mass equivalent model was made. Therefore, it is possible to make different design alternatives by making a mass equivalent model for each *principal element*. All the four alternatives can be included together in a single linkage, as Figure 14 shows. As it can be observed, mass-symmetric links can be considered as well.

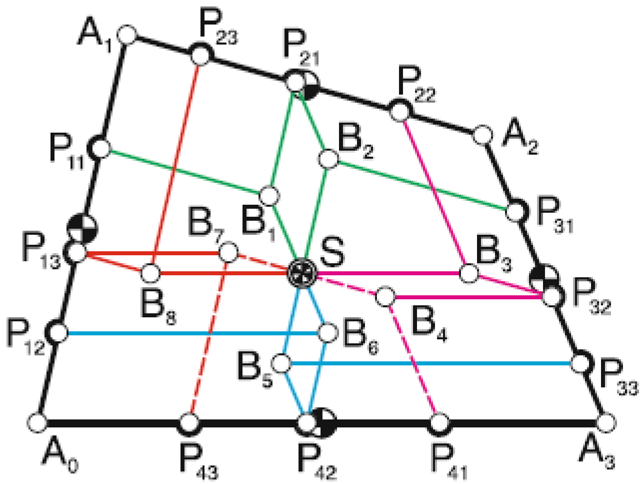


Figure 14: Combination of principal vector configurations for closed chain principal vector linkage. Source [4]

Then, it was discovered that some *principal vector links* could be extended and connected to others. The resulting linkage, shown in Figure 15, was called *Grand 4R Four-bar Based Inherently Balanced Linkage Architecture*, [4]. As it is overconstrained but yet movable, IDB mechanisms can be synthesized from this architecture by removing links, as shown in [4] and [5].

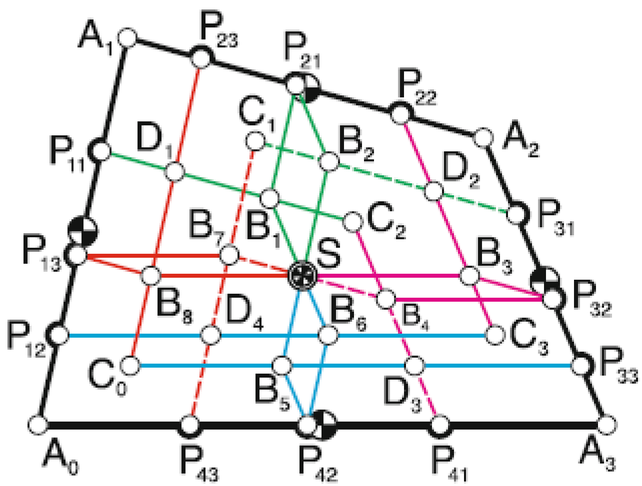


Figure 15: Grand 4R Four-bar Based Inherently Balanced Linkage Architecture. Source [4]

Another method to design closed chain *principal vector linkages* consists in using similar linkages: the common CoM of a three moving links 4-bar linkage describes a curve which is similar to the coupler's curve of the linkage [6]. Figure 16 shows two similar 4-bar linkages: the coupler's curve described by point T is similar to the curve described by point S, which is the common CoM.

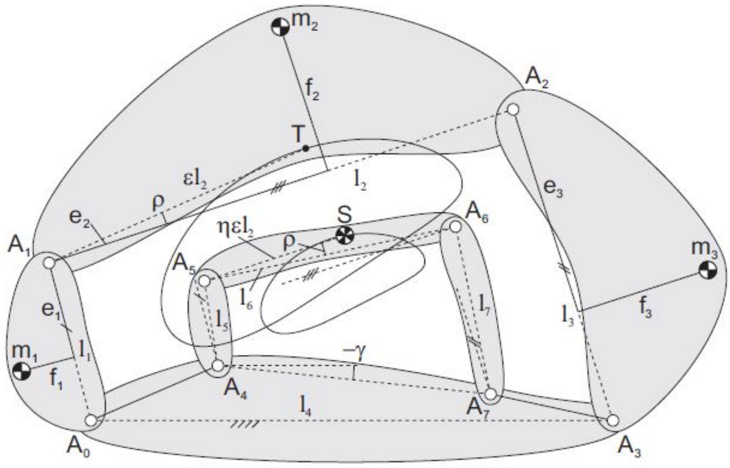


Figure 16: Similar three moving links four-bar linkages. Source [2]

Principal vector linkage architectures were designed from this linkage. In order to constrain the motions of similar linkages to be similar, solutions based on the method of *principal vectors* were derived (Figures 17 and 18).

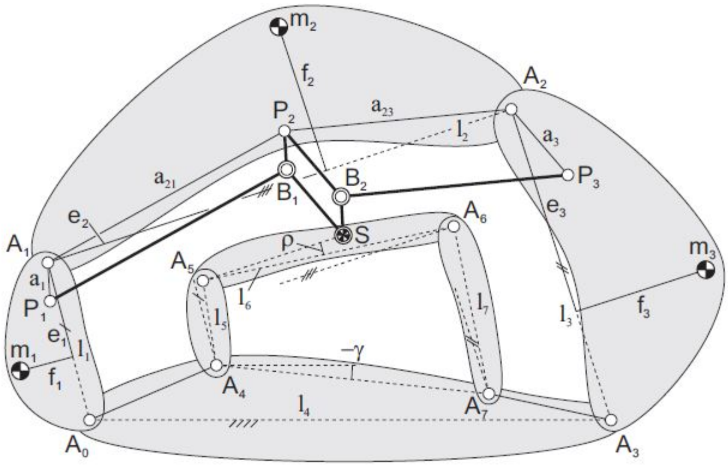


Figure 17: Closed chain principal vector linkage including similar linkages. Source [2]

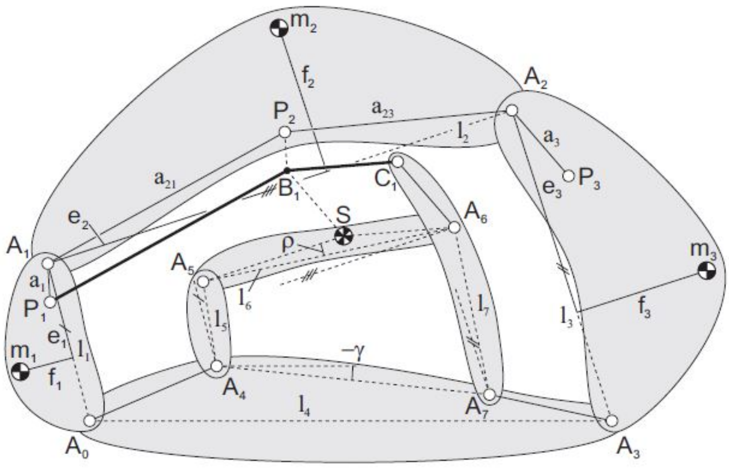


Figure 18: closed chain principal vector linkage of Figure 17. Design alternative. Source [2]

Making parallelograms which include similar links is sufficient to constrain the motions. One example is shown in Figure 19 where, by adding links D_8E_8 and D_9E_9, parallelograms $A_0D_8E_8A_4$ and $A_7E_9D_9A_3$ were made: only one of them is sufficient to obtain a 2 DoF closed chain *principal vector linkage* with similar elements. Although the linkage in Figure 19 is overconstrained, it is still movable.

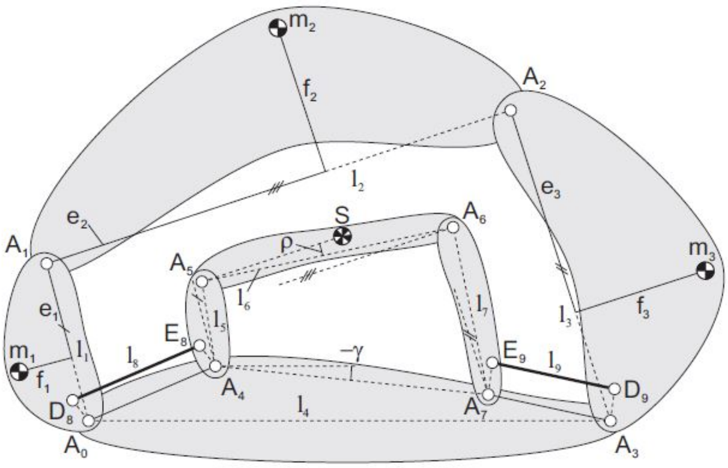


Figure 19: Closed chain principal vector linkage with similar linkages connected by parallelograms. Overconstrained but yet movable. Source [2]

Mechanisms in Figures 17, 18 and 19 can be derived from the *Grand 4R Four-bar Based Inherently Balanced Linkage Architecture* in Figure 15 as well.

2.2 Examples of IDB mechanisms with different elements

All the *principal vector linkages* which have been presented include parallelogram linkages. As previously stated, this can represent a problem in all the applications in which links can not overlap and singularities can occur. Nevertheless, it is possible to prevent overlapping links and singularities by introducing gears and sliders to constrain *principal vector linkages* and replace links. In [2] two examples were shown and are now presented.

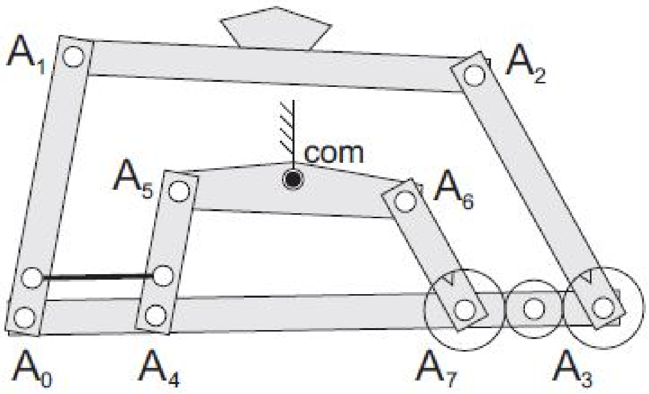


Figure 20: Principal vector linkage with gear train. Source [2]

The mechanism in Figure 20 represents the *principal vector linkage* in Figure 19, where link E_9D_9 has been replaced by a gear train located between *principal joints* A_7 and A_3. In particular, the gears whose axes coincide with these joints are fixed to *principal elements* A_6A_7 and A_2A_3. Therefore, the rotations of these elements are kept the same and the kinematic properties of parallelogram $A_7E_9D_9A_3$ are maintained. In addition, the gears masses can be included in those of the elements to which they are pivoted. As two gears are fixed with principal elements, their inertia can be considered together with the one of elements A_6A_7 and A_2A_3 in the angular momentum. On the other hand, tha mass of the gear in the middle can be included in the one of A_0A_3. In addition, this gear has an opposite rotation with respect to the others: its inertia can not be included in any other element.

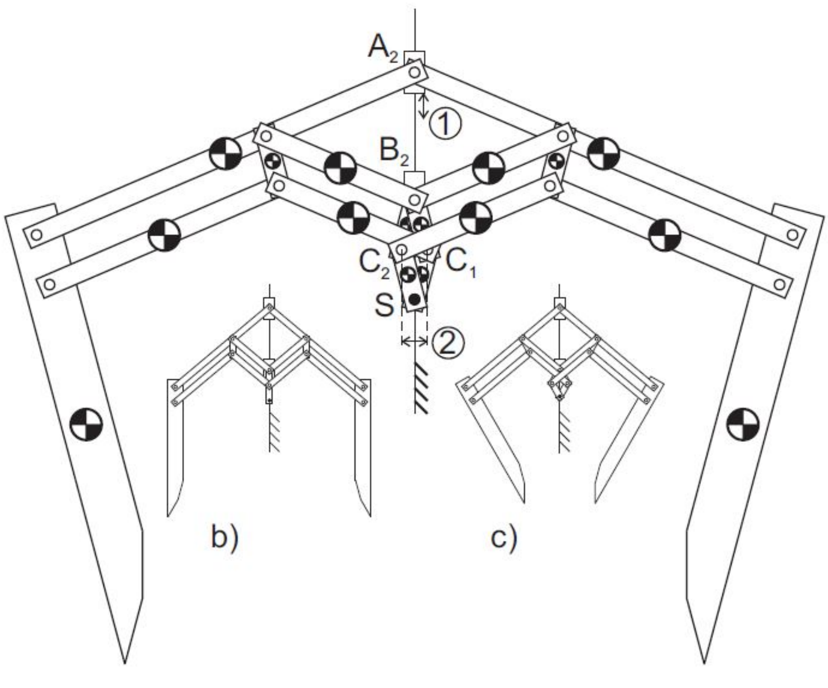


Figure 21: IDB 2 DoF grasper. Source [2]

On the other hand, Figure 21 shows a IDB 2 DoF grasper which was synthesized from the 4 DoF *principal vector linkage* in Figure 8. Two DoFs were constrained by introducing sliders in joints A_2 and B_2. Their paths consist in a straight line through S, which is the common CoM of the mechanism. Sliders are non-rotating elements, so they have no inertia, while their masses can be included in one of the links connected to them, [7]. Dynamic balance is achieved because, together with the symmetric design, the sliders make both the sides of the mechanism move similarly and oppositely. In particular, slider in A_2, whose motion is made by an actuator between A_2 and S, allows the links pivoted in A_2 rotate oppositely. The motion of the slider in B_2 is made by an actuator between joints C_1 and C_2: this allows the links connected between S and B_2 to rotate similarly and oppositely together with all their parallel links.

These two mechanisms are examples of how gears and sliders can be introduced in *principal vector linkages*. In addition to removing links, as seen in Figure 20, the introduction of these elements can reduce the number of DoFs to make the resulting mechanisms perform specific motions. However, by looking at the different *principal vector linkages*, it is clear that gears and sliders can be introduced in different ways.

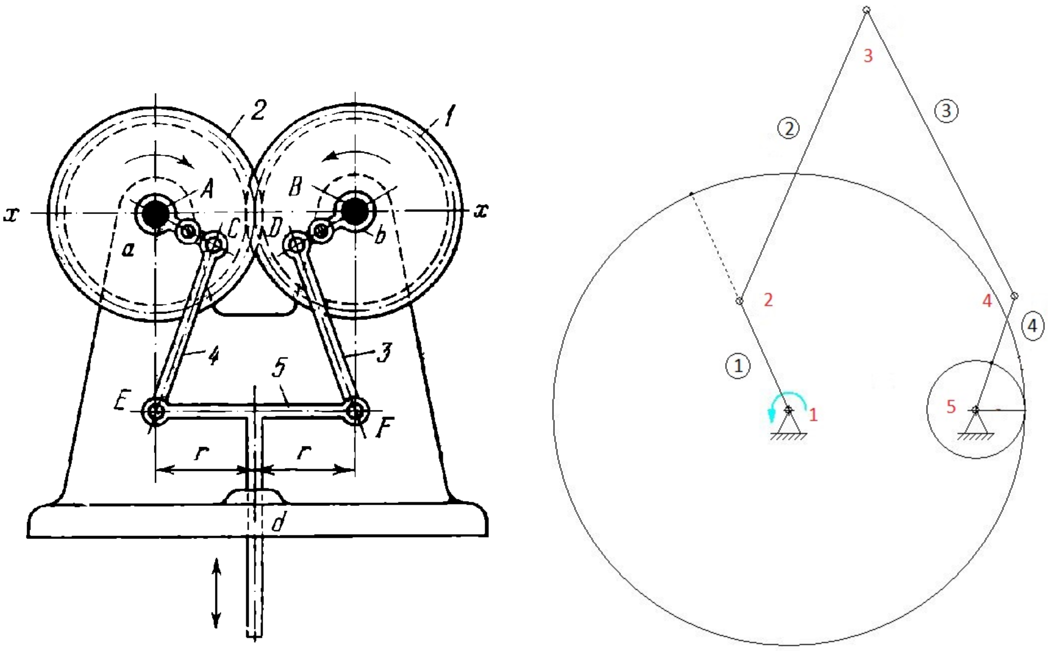
3 Categorizations. How elements are introduced in mechanisms

After having described how gears and sliders had been already introduced in *principal vector linkages*, an investigation was performed in the literature in order to find different ways of introducing gears and sliders in mechanisms. This led to two categorizations describing where these elements can be introduced and how they are connected to the mechanisms. In all the presented categories, gears and sliders were assumed having their CoM placed on their geometric center. Considerations were made about how the additional elements' masses and inertias affect the common CoM and the angular momentum of each mechanism.

3.1 Gears categorization

It was observed that gears are introduced in mechanisms for several functions. For example, they are introduced to constrain and make path generation mechanisms, to generate dwell motions and convert uniform rotary motions in nonuniform or reciprocating motions. All the categorized mechanisms include a couple of gears, while the categorization was based on the number of gears which are pivoted to the base. In each category, possible situations in which one of the gears' rotation needs to be defined, even equal to zero, were investigated separately. Each category was investigated with both external and internal gears. If any mechanism had not been found in the literature, a SAM model was designed. In Appendix A, a table is presented showing an overview of the different categories.

1. **Both gears pivoted to the base**
   It was observed that both external and internal gears are introduced to constrain mechanisms and design particular paths which can be modified by changing the gear ratios.
   Figure 22(a) shows a 1 DoF 6-bar linkage in which two external gears are fixed to the cranks AC and BD, pivoted to the base. This creates one constraint as the two cranks rotate according to the gear ratio. In particular, since the gears have the same radius, the rotations are equal and opposite. These are converted in a reciprocating



(a) *External gears.*

(b) *Internal gears.*

Figure 22: Mechanisms with both gears pivoted to the base. (a) Source [8]. (b) SAM model

vertical translation of link 5. Since the gears are pivoted to the base, their masses are not considered in the common CoM of the mechanism. On the other hand, their inertias have to be taken into account in the angular momentum. In this particular situation, as the gears have the same radius and counter-rotate, their inertias balance each other.

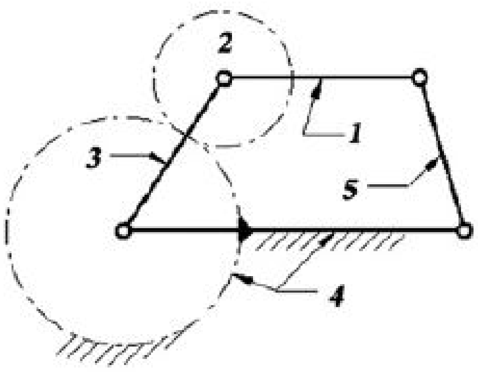
Figure 22(b) shows a 1 DoF 5-bar linkage where two internal gears are fixed to links 1 and 4, which are pivoted to the base, so their rotations are related. Introducing internal gears requires a large space and makes the two links rotate in the same direction. This effect can be made by introducing either two pulleys driven by a belt, two sprockets driven by a chain, or a train of three gears where the outer ones are fixed to the links. These alternatives require less space. As seen for the external gears, the masses of the internal gears are not considered in the common CoM because they are pivoted to the base. On the other hand, their inertias have to be taken into account and, since internal gears rotate in the same direction, they could never balance each other.

2. **One gear pivoted to the base**

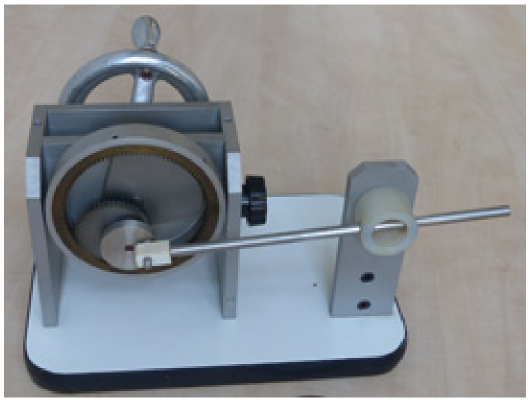
   For both internal and external gears, the situations in which either only one gear rotates or both rotate were investigated separately.

   (a) *Rotation equal to zero. Gear fixed to the base*

   In both the mechanisms in Figure 23 (a) and (b), the rotations of the gears pivoted to the base are fixed. Figure 23(a) shows a 5-bar linkage consisting in four links pivoted to each other. The gear pivoted to the base is fixed to the link representing the base itself: both of them constitute link 4. Gear 2, on the other hand, is considered as a link. The set of gears is a planetary one: since the sun gear's rotation is fixed, the rotation of the planetary pinion (gear 2) depends on the one of the planetary carrier (link 3). This gear configuration, in which gear 2 is not fixed with other links, does not make any additional constraint: the rotations of all the links are related. As the sun gear is fixed to the base, it does not affect both the common CoM and the angular momentum. On the other hand, the mass of gear 2 can be considered on its joint and as part of the connecting links, while its inertia is taken into account in the angular momentum.



(a) *External gears.*

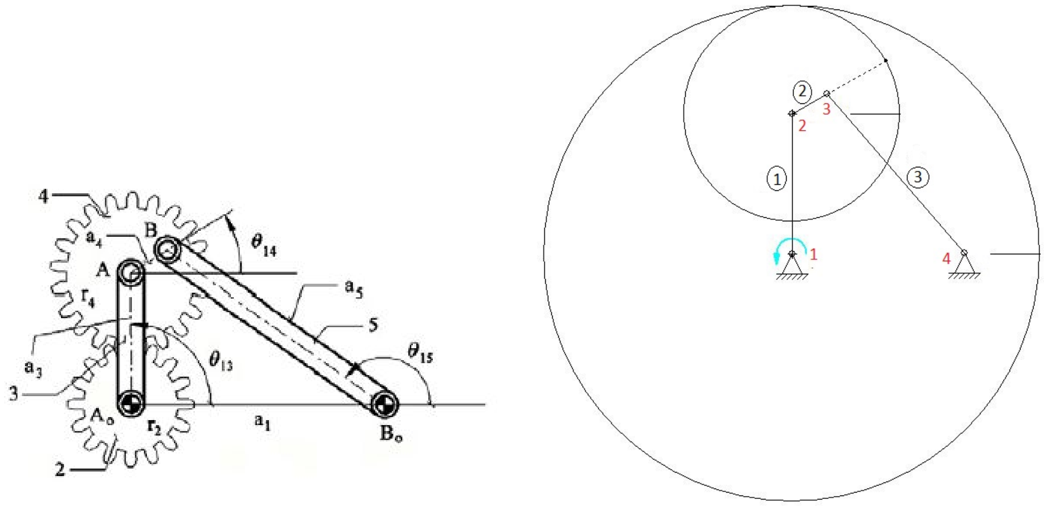


(b) *Internal gears.*

Figure 23: Mechanisms having one gear fixed the base. (a) Source [9]. (b) Source [10]

Figure 23 (b) shows a different planetary gear set. The planetary pinion is connected trhough a revolute joint to a crank pivoted to the base, while the ring gear is fixed. On the pinion, an inverted-slider-crank mechanism is built with the inverted-slider pivoted to the base and the crank fixed to the pinion. Since the pinion's diameter is half the one of the ring gear, the joint connecting the crank on the pinion and the coupler describes a straight line. The mass and the inertia of the ring gear are not considered in the common CoM and the angular momentum of the mechanism, while the mass and the inertia of the pinion are taken into account. The inertia of the crank driving the pinion can be balanced with the inertia of the pinion, as the elements rotates in opposite directions. In addition, by modifying the position of the inverted slider, the coupler would only translate along a straight line and its inertia would not be considered. An experimental study was performed by making a Hypocycloidal Mechanism with this property for an air compressor [11]: after having added counterweight for the pinion, the results of the test showed lower vibrations reaction forces on the base and therefore lower vibrations.

(b) *Rotation defined. Both gear rotates*



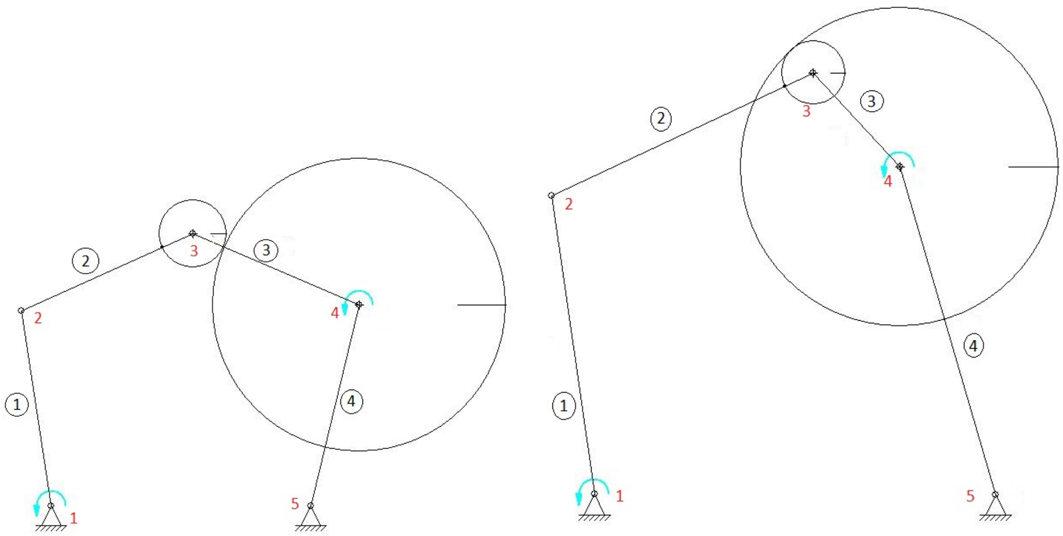
(a) *External gears.* (b) *Internal gears.*

Figure 24: Mechanisms with one gear pivoted to the base. Both gear rotate. (a) Source [9]. (b) SAM model

Figure 24 (a) and (b) show the same 4-bar linkage in which different planetary gear sets are added. In both the cases the pinion is fixed to the coupler which can perform a complete rotation. Therefore, the mechanisms are 5-bar linkages having 1 DoF and the rotations of the gears pivoted to the base are considered the inputs. Thanks to the gear sets, which transmits the input rotation, the complete rotation of a link which is not pivoted to the base is allowed. In Figure 24 (a) the gear set consists in a sun gear and the pinion: as they counter-rotate, their inertias can balance each other. As the sun is pivoted to the base, its mass is not considered in the common CoM, while the mass of gear 4 can be considered in joint A and therefore as part of link 3. For the mechanism in Figure 24 (b) a SAM model was made. The ring gear and the pinion rotate in the same direction, so their inertias can not balance each other. An equivalent kinematic solution can be made by replacing the gears with pulleys driven by a belt or sprockets driven by a chain.

3. **Both gears not pivoted to the base**
For both internal and external gears, the situations in which either only one gear's rotation is defined or no rotation is defined were investigated separately. Several SAM models have been made because only one example was found in the literature.

   (a) *One gear's rotation defined*

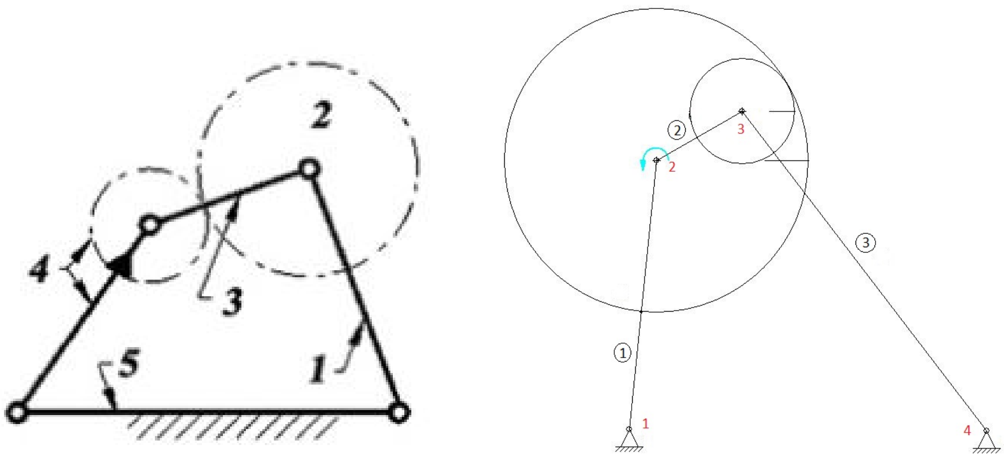


(a) *External gears.* (b) *Internal gears.*

Figure 25: Mechanisms with no gear pivoted to the base. One gear's rotation defined. (a) SAM model. (b) SAM model

Figures 25 (a) and (b) show a 5-bar linkage having 2 DoF where a couple of gears is added. In both the mechanisms, the gear pivoted in joint 3 is fixed to link 2, while the rotation of the gear pivoted in joint 4 is defined. In both the mechanisms, different paths are made either by fixing different links to the gears or by changing the gear ratios. Furthermore, the masses of both the gears are considered in the common CoM. They can be considered as point masses placed on the revolute joints connecting the gears to the links. Only the inertias of the rotating gears has to be taken into account in the angular momentum.

(b) *No gear's rotation defined*



(a) *External gears.* (b) *Internal gears.*

Figure 26: Mechanisms with no gear pivoted to the base. No gear's rotation defined. (a) Source [9]. (b) SAM model

> Figures 26 (a) and (b) present 4-bar linkages to which a couple of gears is added. In (a) one of the gears is fixed to link 4, while in (b) the ring gear is fixed to link 1. The linkages have 1 DoF: if any link is actuated the others rotate. In both the linkages, as one of the gears is not fixed to any link and can rotate, changing the gear ratios would only modify the relative rotation between the gears. Therefore, for different ratios, different values in the angular momentum would be made. As seen previously, in both the mechanisms the masses of all the gears are considered in the common CoM and all the inertias are considered in the angular momentum, as all the gears rotate.

It was observed that, in order to replace internal gears requiring more space, either gear trains, pulleys driven by belts, or sprockets driven by chains can be introduced.
In order to introduce rotating elements in IDB structures, it was observed that potential purposes, in addition to replacing links, could be introducing constraints or transmitting motions along different links, as seen in Figure 24(a). It could be seen how introducing counter-rotating gears in IDB linkages could be useful for moment balance by looking at several balancing techniques [12, 13, 14]. For the same purpose, links' rotations can be transmitted by using rotating elements driven by belts or chain [1, 15]: the rotation is therefore transmitted to the base where counter-rotating elements are placed. Also, Arakelian and Smith [16] proposed several balancing solutions based on pantographs including gears, since it can be hard to have parallel links in a mechanism where some links perform a complete rotation.
On the other hand, about constraining mechanisms, the number of constraints introduced by gears can vary. In fact, it was shown how a couple of gears introduces one additional DoF if they are free to rotate with respect to the other links. The number of DoFs of the resulting mechanism, on the other hand, does not change if one gear is fixed to either a link or the base. One constraint is made when gears are fixed to different links. In the cases in which gears' rotations were defined, the number of DoFs of the resulting mechanisms was the same of the initial one. Also, the gear ratio can affect the kinematics of the mechanism. However, a potential risk of introducing gears in IDB structures consists in increasing the mass and the inertia of the system. Therefore, more power would be required to drive the mechanisms.

3.2 Sliders categorization

The following categorization was based on where the sliders had been located, either on the base or on the links, and which kinds of kinematic pairs had been used, provided that a slider has always a sliding pair. In Appendix B, a table is presented showing an overview of the different cases.

1. **Sliders connected to the base**

    (a) *Through a sliding pair*

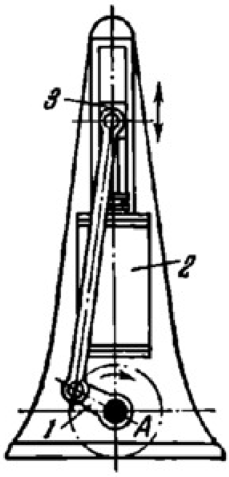


Figure 27: Crank-slider mechanism. Slider translating on the base. Source [17]

The most common mechanism including a slider translating on the base is the crank-slider mechanism. Figure 27 shows an aligned crank-slider engine mechanism where the cylinder 2 actuates slider 3 which allows the crank to rotate. The mechanism could be considered as a 4-bar linkage where the slider is a link connected to the coupler by a revolute joint, and to the base by a sliding pair. Without the slider, the mechanism could be seen as a double pendulum having 2 DoF. The introduction of the slider constrains one DoF. The slider is a non-rotating link and therefore it has no inertia to be considered in the angular momentum. On the other hand, as its CoM can be located on the joint, its mass can be considered as part of the coupler.
Sliders can perform circular trajectories on the base as well [17]. The only difference with the slider presented in Figure 27 consists in the element's motion: the slider rotates and therefore its inertia needs to be included in the angular momentum.

(b) *Through a revolute joint*

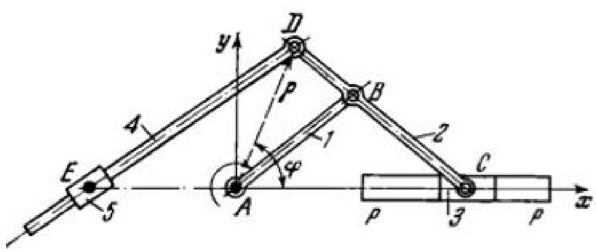


Figure 28: 6-bar mechanism. Slider pivoted to the base. Source [17]

Figure 28 shows a 6-bar mechanism consisting in 3 links and two sliders, all of them connected through revolute joints. Point D describes an elliptic curve when links $AB = BC$. It is not a pantograph as links AB and DE are not always parallel. In point E, a slider is pivoted to the base and adds one constraint to the system: link 4 can translate only along the slider's guide and rotate about its axis on point E. As the slider is connected to the base, its mass is not considered in the common CoM. On the other hand, because it can rotate, its inertia has to be taken into account in the angular momentum. In particular, it can be considered together with the inertia of link 4 as their rotation is the same.

2. **Sliders connected between links**

   (a) *Trough revolute joint (two mechanisms in parallel)*

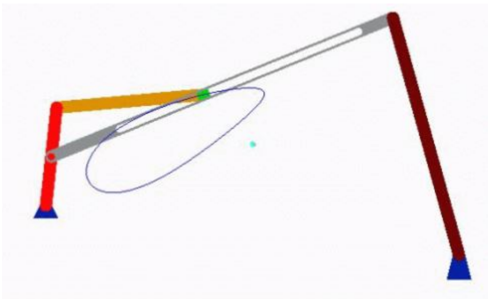


Figure 29: 6-bar mechanism. Slider translating along a link. Two mechanisms in parallel. Source [18]

The 6-bar linkage shown in Figure 29 consists in a 4-bar linkage in parallel with a crank-slider mechanism. The red link acts as the crank of both the mechanisms, while the slider, in green, translates along a guide made in the grey coupler of the 4-bar linkage. The slider would describe different paths if the lengths of the links were changed. Both the inertia and the mass of the slider have to be considered: this element is not connected to the base and its angle is the same of the coupler of the 4-bar linkage. Therefore, the inertia of the slider can be considered together with the one of the grey link, while the mass can be considered as part of the yellow one.

   (b) *Through revolute joint*

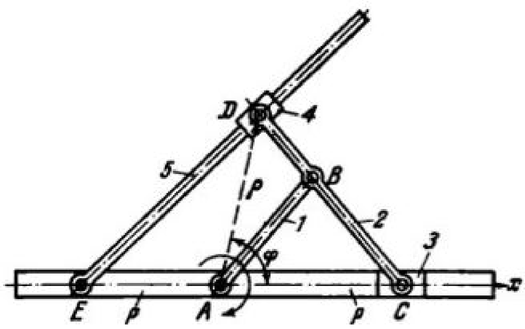


Figure 30: 6-bar mechanism. Slider translating along a link. Source [17]

The mechanism shown in Figure 30 is a 6-bar linkage similar to the one in Figure 28. In point D, which describes an elliptic curve when $AB = BC$, a slider is connected between links 5 and 2. Because of the revolute joint connecting the slider to link 2, the relative rotation between link 5 and link 2 is allowed, together with link 2 translation along link 5. The mass of the slider can be considered as part of link 2, while its inertia can be considered together with the one of link 5, as the two elements have the same rotation. It was observed that point D moves along link 5. This could represent a problem if a slider was introduced in IDB linkage architectures between two links: the *principal vector links* would have a varying length and the condition about the constant magnitude of the *principal vectors* would not be respected.

(c) *Slider fixed to a link*

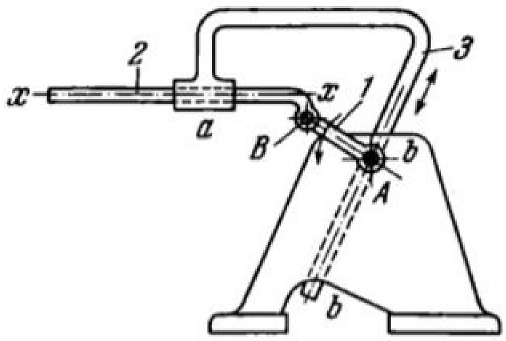


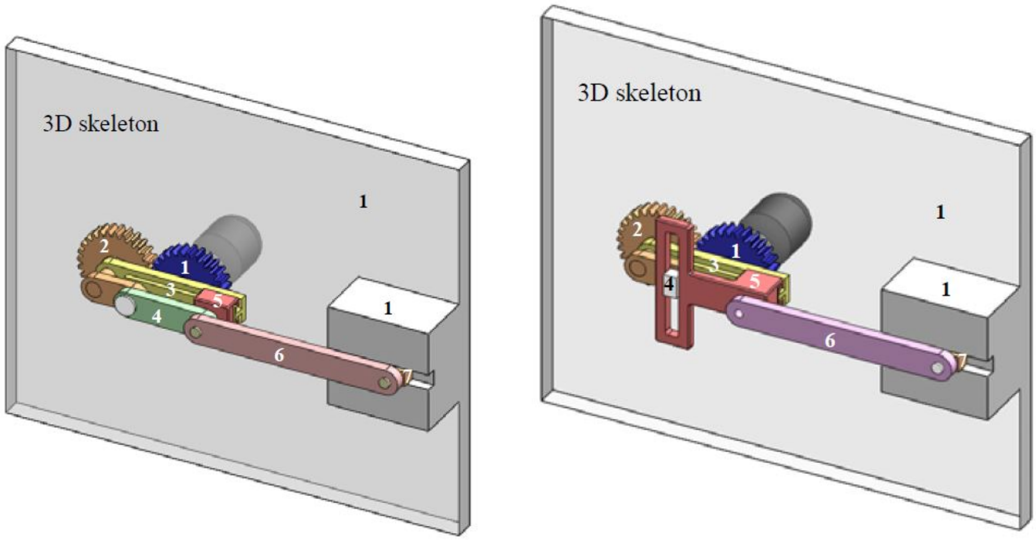
Figure 31: 4-bar linkage. Slider fixed to a link. Source [17]

Figure 31 shows a 4-bar linkage where a sliding pair is introduced between links 2 and 3. This can be considered as a slider fixed to link 3, so only the translation along link 2 is allowed. Therefore, a constant angle is fixed between links 2 and 3. As crank AB rotates, link 2 performs a circular translational motion while link 3 moves along guide b-b made on the base. As the slider is fixed to link 3, they can be considered as a single element with proper mass and inertia. Although link 3 does not rotate in the mechanism, its inertia and the one of the slider could be considered together with the inertia of link 2 in the angular momentum, as the angle between the links is always constant. On the other hand, it was observed that, as the sliding pair moves along link 2, the distance between the revolute joint in point B and the sliding pair is not constant. This could represent the same problem mentioned in the previous case about introducing sliders within IDB linkage architectures.

It was observed that cases 2(a) and 2(b) are similar. Although the function of both mechanism is path generation, it was decided to consider them separately because the linkage discussed in case 2(a) is a combination of two mechanisms in parallel, and there could be room for investigation of such mechanisms in IDB architectures.

It was observed that sliders are introduced to create path generation mechanisms and to add constraints. These functions can be potentially used in IDB linkage architectures. About the number of constraints that are introduced, this depends on how the slider is connected either to the other links or to the base. Together with the sliding pair, if a slider is connected through a revolute joint, it adds one constraint, while it adds two constraints if it is fixed to a link. On the other hand, it was observed that adding sliders between links could represent a problem for IDB linkage architectures. *Principal vector links* having varying length would be created, in contrast with the *principal vector method* that requires vectors having constant magnitude.

It is possible to combine different cases of the two categorizations in the same mechanism. Soong, in [19], proposed two variations of a "geared linkage mechanism", shown in Figure 32.



(a) *Geared linkage mechanism, crank-slider mechanism in parallel.* (b) *Geared linkage mechanism, scotch-yoke mechanism in parallel.*

Figure 32: Geared linkage mechanism, two variations. Source [19]

Both Figures 32 (a) and (b) show a 1 DoF 7-bar linkage which consists in a planetary gear set where the rotation of the sun (gear 1) is fixed (like in case 2(a) in gears categorization) while link 3, the carrier, is the input link and drives gear 2. On link 3, a crank-slider and a scotch-yoke mechanisms are built, respectively, in Figure 32 (a) and (b). In both the mechanisms, the crank is fixed to gear 2 and it is connected to slider 5, which slides along link 3 (like in case 2(a) in sliders categorization), through either coupler 4 (Figure 32 (a)) or slider 4 (Figure 32 (b)). Link 6 is then connected through a revolute joint to slider 5 and to the output slider 7. Both the mechanisms are considered as crank-slider mechanisms with a variable-length crank. This is constituted by the planetary gear set together with either the crank-slider, in Figure 32 (a), or the scotch-yoke, in Figure 32 (b). Thanks to the variable crank, when the gear ratio is equal to one, the output slider 7 can generate two required motion cycles while the input link 3 completes a single motion cycle. Although synthesizing an IDB mechanism having this property is supposed to be complex, it was decided to report these two variations of the same mechanism in order to show that combining gears and sliders in the same mechanism can lead to different mechanisms' functions and properties.

4 Discussion and conclusion

At the end of each categorization, potential purposes of introducing gears and sliders were presented. Potential disadvantages consist in the additional mass and inertia of the additional elements: in addition to modifying the position of the common CoM, the total mass and inertia of the mechanisms results to be increased such that higher power is required to drive the resulting mechanisms. About the sliders, it was observed that a potential issue consists in not maintaining the force balance conditions if links with varying length are created.

A first approach to the research project consists in introducing the elements, as described in the different categories, in given *principal vector linkage* architectures. Possible differences are supposed to be observed between open chain and closed chain *principal vector linkages*. At the beginning, the elements of each category will be introduced separately, and its outcomes about the kinematics, the position of the common CoM and the linear momentum equation will be observed. Then, combinations of these categories will be introduced and their outcomes will be studied. By having an overview of the different ways in which additional elements can be introduced in IDB architectures, it is assumed to be possible to design several IDB linkages including sliders and gears, together with other rotating elements like pulleys driven by belts and sprockets driven by chains. Since some links could be removed, it will be also studied how the force balance conditions will change. Potential benefits consist in improving the range of motion of synthesized mechanisms by preventing overlapping links and singularities in parallelogram linkages, as previously stated. Further potential benefits consist in new mechanisms functions and properties that can be generated by introducing gears and sliders, as seen in [19] and Figure 32.

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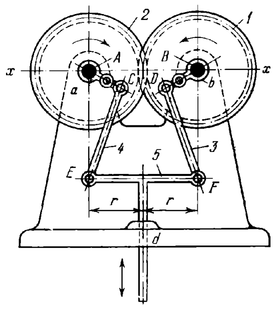
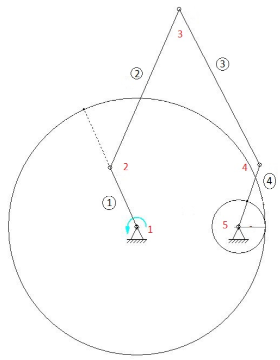
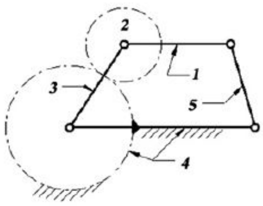
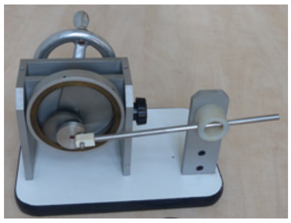
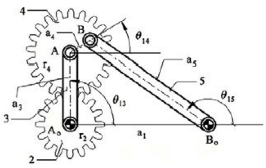
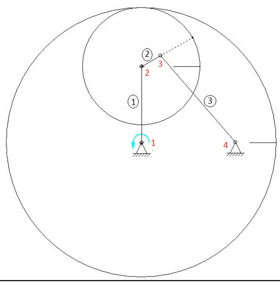
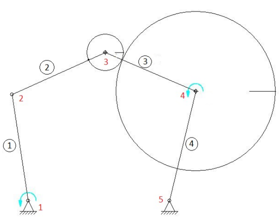
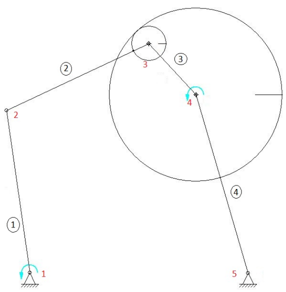
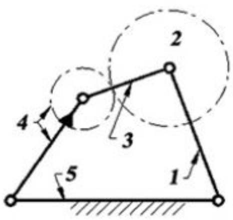
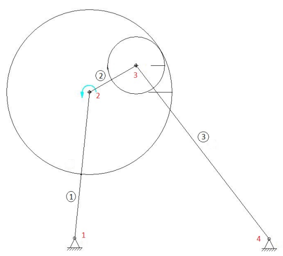
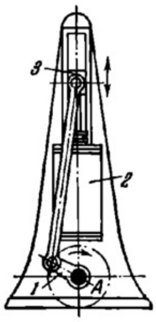
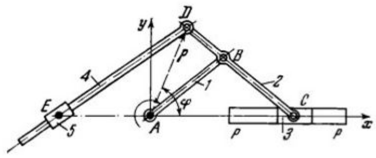
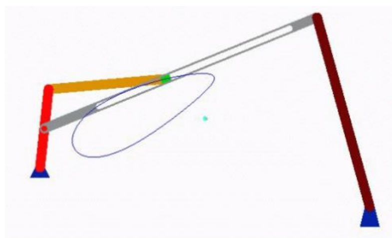
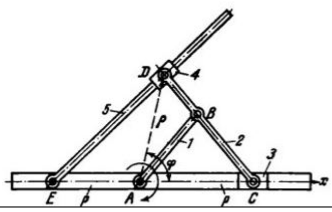
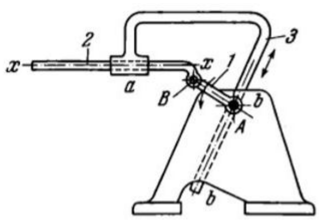
A Gears categorization overview

		External gears	Internal gears
Both gears pivoted to the base			
One gear pivoted to the base	One gear rotates		
	Both gears rotate		
No gear pivoted to the base	One gear's rotation defined		
	No gear's rotation defined		

B Sliders categorization overview

Slider connected to the base	Through Sliding pair	
	Through revolute joint	
Slider connected between links	Through revolute joint (parallel mech.)	
	Through revolute joint	
	Through clamping	

3

Techniques to modify Principal Vector Linkages by keeping force balance

Techniques to modify Principal Vector Linkages by keeping force balance

Lorenzo Girgenti

Volkert van der Wijk

ABSTRACT

Inherent dynamic balancing is a principle consisting in designing mechanisms whose links contribute to both the kinematics and the dynamic balance, in order to avoid counterweights and counter-rotating elements which are typical of the other principles. Inherently dynamically balanced mechanisms can be synthesized from force balanced linkages, namely principal vector linkages. This article presents different synthesis techniques aiming to modify the principal vector linkages while keeping the kinematics equal. Constraints can be introduced to modify the number of degrees of freedom and to create specific motions. Links can be changed in position to meet specific space requirements. They can be also replaced by other machine elements to prevent possible issues related to the original principal vector linkages' design. Techniques can be combined or applied on different points of the same linkage. A synthesized inherently force balanced mechanism in which various techniques have been combined is finally presented. With a simulation it is proven that the techniques indeed lead to a new inherently balanced design while maintaining the force balance of the original principal vector linkage.

1 Introduction

In machine design, dynamic balancing plays an important role in order to provide reliable and accurate mechanical systems. If machine components were dynamically unbalanced, their motions would generate reaction forces and moments on their bases, leading to base vibrations [1]. These vibrations cause noise, wear and fatigue issues which compromise the accuracy and reliability of all the systems which are placed on the same base [2].

Most of the common balancing principles include the introduction of counterweights and counter-rotating elements (counter-rotations) to achieve, respectively, force balance and moment balance, according to the conservation of linear momentum and angular momentum [3]. In particular, the principles differ in the ways the elements are introduced. There can be separate counterweights and counter-rotations: either gear trains [4, 5] or pantograph linkages [6] can be introduced, depending on where the counter-rotations are placed within the mechanisms. In addition, thanks to their copying properties, pantograph linkages can be used to achieve force balance [7, 8]. It is also possible that the same elements act as counterweights and counter-rotations at the same time: they are called Counter-rotary Counterweights [9, 10]. Also, dynamic balancing can be achieved by adding both axial and mirror symmetric duplicate mechanisms [11]. Nevertheless, a new principle, which is called *inherent dynamic balancing* [12], is based on designing mechanisms which are already balanced: these are called *inherently dynamically balanced* (IDB) mechanisms. Compared to the other principles, which involve additional elements to already designed mechanisms, *inherent dynamic balancing* considers dynamic balance as a design principle: every link contributes to both the kinematics and the balance of the mechanisms. As a result, no additional element, therefore no additional mass and inertia, is introduced and less power is required to drive IDB mechanisms.

In particular, these mechanisms are synthesized from force balanced linkages known as *Principal Vector Linkages* (PVL). They consist in multi-degree-of-freeedom (multi-DoF) kinematic chains whose elements are connected to their common center of mass (common CoM) through several parallelogram linkages, representing pantographs. According to the method of *principal vectors* [2], these linkages are designed such that the common CoM is always placed on the same point. Therefore, PVL result to be force balanced by simply fixing this point. As a result, IDB mechanisms are synthesized from force balanced linkages which are constituted by parallelograms. This can represent a limitation for designers, as singularities can occur within parallelogram linkages. These are called change point mechanisms [13] since their motions can not be determined when their links become collinear. Furthermore, in some applications, it is hard to make links overlap, since they theoretically could within PVL. Moreover, the single design of the different PVL represents a limitation in their functionality and therefore in their application. Mechanisms consisting in only links can be adapted to a limited number of design requirements, especially when links need to form specific structures like parallelograms within PVL. However, it was investigated how the PVL's overall design can be modified while maintaining the essential kinematics, which is based on those of the parallelograms. This represents a considerable advantage because the force balance conditions, which are derived as described in [12], are based on the PVL's essential kinematics: if this is maintained, the force balance conditions do not have to be derived again when the PVL's overall design is modified.

The goal of this paper is to present different techniques which modify *principal vector linkages* while maintaining the same kinematics and can be used to synthesize inherently force balanced mechanisms. After a brief introduction about PVL in Section 2, next sections will describe the different techniques. Section 3 presents techniques based on reducing the number of DoFs of PVL, which can possibly include machine elements like sliders, gear, belt and chain drives, in order for the resulting linkages to perform specific motions. Section 4 presents a technique which modifies links' positions and section 5 presents a technique in which links are replaced by machine elements. Finally, Section 6 presents how the different techniques can be combined, together with a synthesized inherently force balanced mechanism.

As the techniques modify PVL's overall design while maintaining the essential kinematics, the force balance conditions of the initial PVL are still considered: they do not need to be derived for the new linkage. Only slight adjustments can be required when links' positions are modified and additional elements are introduced.

Moreover, since PVL are only force balanced, the techniques only aim at maintaining the force balance and do not aim at achieving moment balanced solutions. Nevertheless, as moment balance conditions for each PVL can be defined from the angular momentum equations, as described in [12], it can be possible to consider the inertia of the additional elements. These result to be included in the equations constituting the moment balance conditions of the original *principal vector linkage*.

2 Principal Vector Linkages

Principal vector linkages are force balanced linkages since their common CoM is always kept fixed on the same point. They are constituted by a kinematic chain whose elements, which are called *principal elements*, are connected through revolute joints. When the kinematic chain is open, the resulting number of DoFs of PVL corresponds to the number of *principal elements*, due to the parallelogram linkages connecting them to the common CoM. An example is shown in Fig. 1. A 4 DoF PVL is constituted by the open kinematic chain of *principal elements* P_1A_1, A_1A_2, A_2A_3 and A_3P_4 which are connected to the common CoM placed in S. As each link results to be parallel to one of the *principal elements*, their angles are considered the linkage's DoFs. In particular, the links constituting the parallelograms are called *principal vector links*, since they are designed according to the method of *principal vectors*. This method relates the position, therefore the motion, of each *principal element* to the position of the common CoM. This is made by using vectors having constant magnitude, the *principal vectors*, which are replaced by actual links, i.e. the *principal vector links*. As a consequence, the parallelograms network is derived by connecting all the *principal elements* to the common CoM. In [12] it is described how these parallelograms represent pantograph linkages and how linear momentum equations are involved into the force balance conditions to determine the lengths of *principal vector links*, namely *principal dimensions*. In [12] it is also explained how the parallelograms' kinematic property, consisting in letting a link translate while rotating two parallel links, is essential in making the *principal elements* move relatively to each other, while keeping the common CoM in the same position.

Once *principal elements* are connected to the common CoM through the *principal vector links*, force balanced is achieved by simply fixing the common CoM to the base. In Fig. 1, therefore, a pivot fixed to the base is placed on S. In addition, Fig. 1 shows that the *principal elements* and the *principal vector links* can be mass-asymmetric, which means that their own CoMs are not on the lines connecting their revolute joints. Nevertheless, the force balance conditions described in [12] consider this mass asymmetry as well.

PVL can be constituted by closed chains of *principal elements* as well. In particular, *principal vector links* can connect the *principal elements* to the common CoM in different ways, leading to different designs: these can be all included in an extensive force balanced linkage architecture, namely a *grand* inherently balanced linkage architecture.

In [14] and [15], the *Grand 4R four-bar based inherently balanced linkage architecture*, shown in Fig. 2, was presented together with many derived inherently force balanced solutions. The linkage is overconstrained but yet movable, as all the links are parallel to each other and the overall motion is based on the parallelograms kinematics. Different design solutions are derived by removing different links while maintaining the same kinematics. In addition, in the linkage architecture shown in Fig. 2, all the links are mass-symmetric. Nevertheless, [15] presented the same linkage architecture having mass-asymmetric links as well. However, more attention is generally paid to linkages having mass-symmetric links because this kind of links is more common in real applications.

By looking at Figs. 1 and 2, it can be observed that the essential kinematics of the overall linkages depends on those of the parallelograms forming PVL. As a result, when a link rotates, its parallel links rotate by the same amount while the others only translate. The force balance conditions, which are derived as described in [12], are based on this kinematics. As all the synthesized mechanisms, which were presented in [12, 14, 15], maintained the essential kinematics of the initial PVL, it

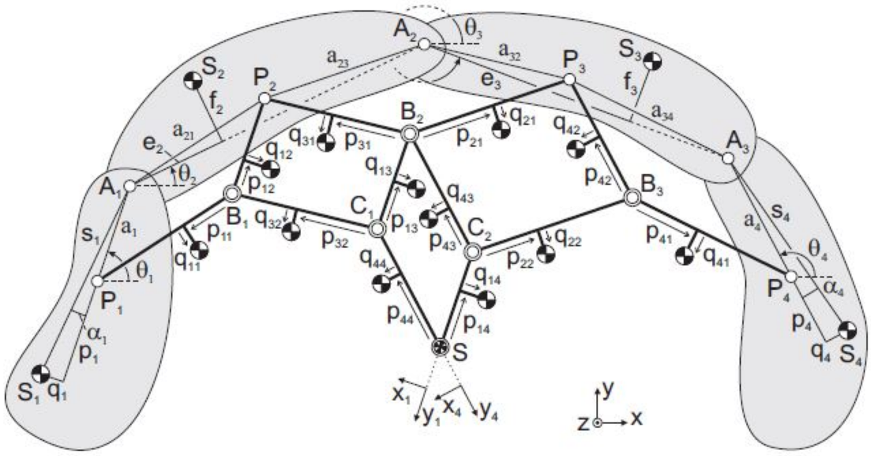


Fig. 1. 4 DoF Principal Vector Linkage having mass-asymmetric links in which the common CoM of all links is in S for any pose. Source [12].

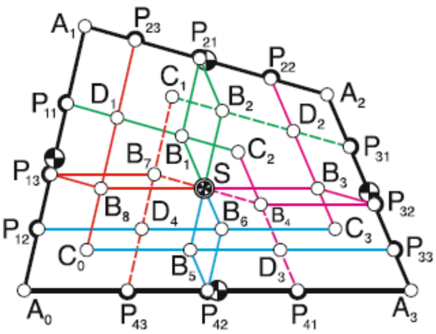


Fig. 2. Grand 4R four-bar based inherently balanced linkage architecture. The common CoM is in S for any pose. The numerous internal links make the overall linkage overconstrained yet movable. Source [14]

can be observed that the overall design of any *principal vector linkage* can be modified without changing this kinematics. This implies that the force balance conditions, which were derived for the initial PVL, do not need to be derived again: the same equations are considered. Therefore, given any *principal vector linkage* and its force balance conditions, the techniques which are presented in the next sections can be applied to modify its overall design while maintaining its essential kinematics. The force balance conditions of the original linkage are still considered: only slight modifications can be required when links' positions are modified and additional machine elements are introduced. As the force balance conditions do not need to be derived when techniques are applied, they are not presented in the next sections, which focus on how PVL can be modified. Chapter 4 will describe how the equations of the force balance conditions are adjusted when links' positions are modified and additional elements are introduced. An initial PVL will be presented, together with its force balance conditions, then the techniques and the adjustments in the equations will be applied. A general overview will be presented by considering mass-asymmetric links and machine elements.

In order to make simple figures describing each technique in the next sections, mass-symmetric links will be considered in the PVL. It is assumed that each technique can be applied to mass-asymmetric links as well. In addition, both *principal elements* and *principal vector links* will be assumed having mass, even if their CoM will not be shown in the figures. PVL constituted by open chains of *principal elements* will be considered. This because the effects of the techniques constraining PVL are more clear if the initial linkages have a high number of DoFs. In fact, for the same number of *principal elements*, open chains PVL have more DoFs than closed chains. Nevertheless, it is assumed that the same techniques can be applied to closed chains PVL as well.

3 Constraining Principal Vector Linkages

A straightforward approach to synthesize IDB mechanisms consists in constraining PVL, then reducing the number of DoFs, to create the desired motions. This basically consists in constraining the links, which can be *principal vector links* and *principal elements*. The constraints can be applied to either links' absolute motions, with respect to the base, or the relative

ones. It is important to notice that this kind of techniques does not affect the force balance conditions unless the mass of additional elements has to be considered. The common CoM is always kept on the same point and the kinematic properties of the parallelograms are maintained: only the motions of the overall linkages change. On the other hand, when the mass of additional elements needs to be considered, the force balance conditions shall be adjusted: in doing so, the linkage has to be considered having the original number of DoFs.
Examples will show the different types of constraints that can be introduced. A 4 DoF PVL is considered. The angles of its *principal elements*, θ_1 for A_0A_1, θ_2 for A_1A_2, θ_3 for A_2A_3 and θ_4 for A_3A_4, will be considered as the initial DoFs. The common CoM S will be kept pivoted to the base and considered as the origin point of the reference frame. In each example, kinematic relations will be defined: they describe how the original DoFs are constrained. Furthermore, specific cases regarding the introduction of each constraint will be described as well.

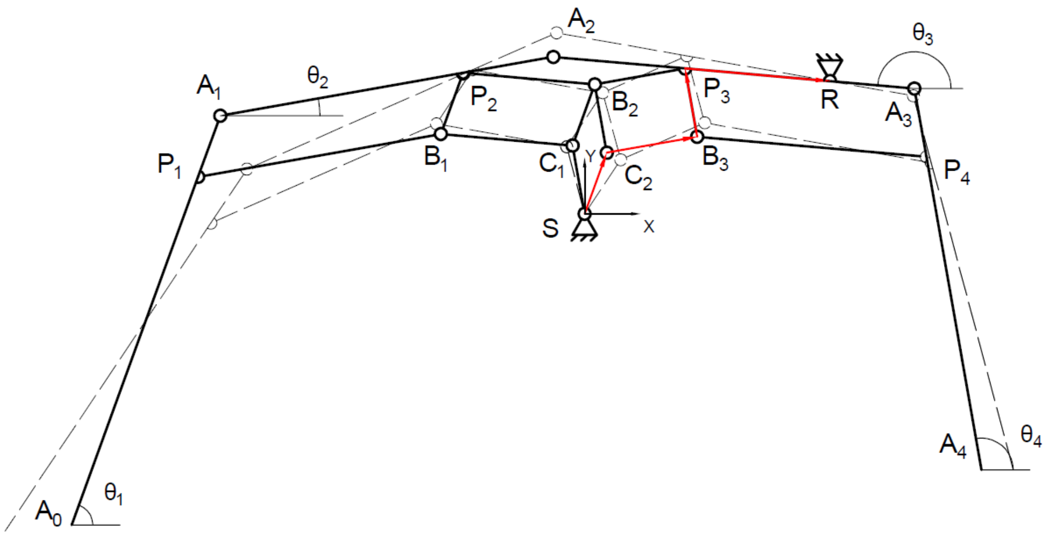


Fig. 3. Principal Vector Linkage having 2 DoF. A_2A_3 is pivoted to the base in R, therefore it is allowed to only rotate.

3.1 Pivoting links to the base

Figure 3 shows a PVL in which *principal element* A_2A_3 has been pivoted to the base on point R. As a result, A_2A_3 is only allowed to rotate around R and the resulting PVL has 2 DoF. It can be proved by using Grübler's equation. As the number of DoFs has been reduced by two, it is possible to derive kinematic relations describing two of the original DoFs as functions of the remaining two. They are derived by describing the position of R with respect to S using a vector loop, as a summation of vectors $\vec{SC_2}$, $\vec{C_2B_3}$,$\vec{B_3P_3}$, $\vec{P_3R}$. These have constant magnitudes and varying angles, which are the original DoFs. In particular, the relations are derived from the system constituted by Eqns.(1) and (2), representing the scalar equations of the vector summation along X and Y components.

$$SR_x = SC_2\cos\theta_1 + C_2B_3\cos\theta_2 + B_3P_3\cos\theta_3 - P_3R\cos\theta_4 \tag{1}$$

$$SR_y = SC_2\sin\theta_1 + C_2B_3\sin\theta_2 + B_3P_3\sin\theta_3 - P_3R\sin\theta_4 \tag{2}$$

By considering, for example, θ_3 and θ_4 as the remaining DoFs, the solution of the system will give the relations for θ_1 and θ_2 as $\theta_1 = \theta_1(\theta_3,\theta_4)$ and $\theta_2 = \theta_2(\theta_3,\theta_4)$. As Eqns. (1) and (2) show, the relations depend on the position of R.
Fixed pivots can be introduced in almost all the links of PVL. It is important that the same link is not pivoted twice, in order to prevent redundant constraints: pivots can not be introduced on links SC_1 and SC_2, as they are already pivoted in S. Moreover, if two contiguous links are pivoted to the base, they will result to be fixed and can be removed. For example, this occurs when one of links C_1B_1, C_1B_2, C_2B_2 or C_2B_3 is pivoted. In particular, links of parallelogram $SC_1B_2C_2$ can be removed if either C_1B_2 or C_2B_2 is pivoted to the base. The revolute joints belonging to the removed links need to be pivoted to the base, if they are still connected to other links.

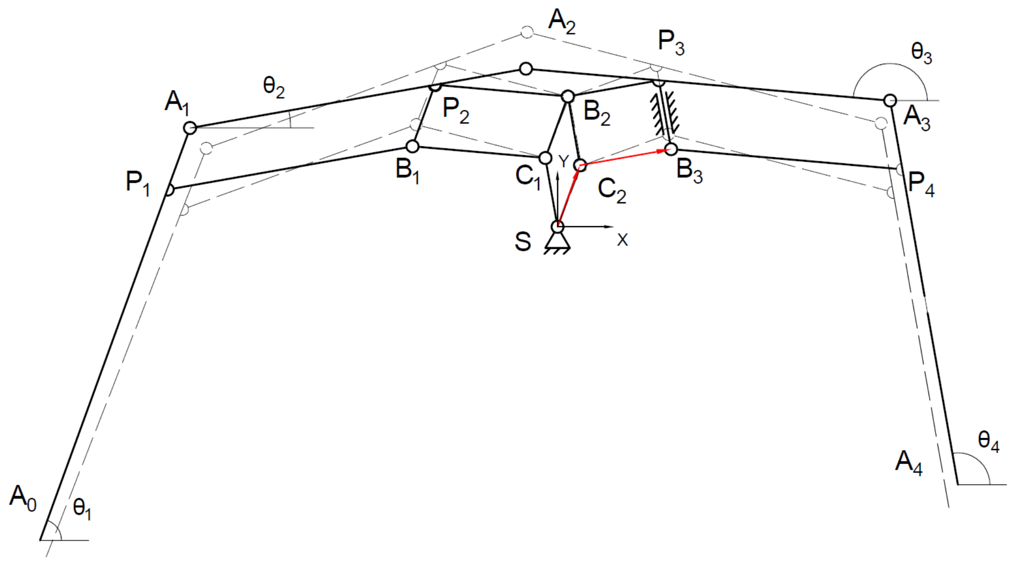


Fig. 4. Principal Vector Linkage having 2 DoF. B_3P_3 is only allowed to translate along a straight sliding guide on the base.

3.2 Constraining links' rotation

In Fig. 4, the rotation of B_3P_3 has been constrained by making this link translate along its orientation. The resulting PVL has 2 DoF, as can be proved by using Grübler's equation. It is possible to derive kinematic relations between the original DoFs by using a vector loop from S to any point of the constrained link, for example B_3. As θ_4 is fixed and B_3 can move, the remaining DoFs can be chosen among θ_1, θ_2, θ_3 and B_3 displacement $\Delta\vec{B}_3$; the latter is defined with respect to a known position $\vec{SB}_{3,0}$. The vector loop shall describe a random position of B_3, given by the sum of $\vec{SB}_{3,0}$ and $\Delta\vec{B}_3$, as the summation of $\vec{SC}_2$ and $\vec{C_2B}_3$. Equations (3) and (4) constitute a system and represent the scalar equations of the vector loop along X and Y components.

$$SB_{3,0x} + \Delta B_3 \cos\theta_4 = SC_2 \cos\theta_1 + C_2B_3 \cos\theta_2 \tag{3}$$

$$SB_{3,0y} + \Delta B_3 \sin\theta_4 = SC_2 \sin\theta_1 + C_2B_3 \sin\theta_2 \tag{4}$$

As Eqns. (3) and (4) show, θ_3 is not included. This because the constraint lies on a *principal vector link* which is connected to *principal element* A_2A_3. When only θ_3 is actuated, A_2A_3 rotates around its *principal point* P_3 while the other *principal elements* only translate: this can be seen in Fig. 4, as the translation of A_3A_4 is partly due to A_2A_3 rotation. Therefore, it is recommended to consider θ_3 as one of the two remaining DoFs. The system of Eqns. (3) and (4) will give the relations between the others: by considering ΔB_3 the second remaining DoF, for example, the results will be $\theta_1 = \theta_1(\Delta B_3)$ and $\theta_2 = \theta_2(\Delta B_3)$. These relations depend on the fixed value of θ_4, as it is included in Eqns. (3) and (4).
B_3P_3 has been connected to the base through a straight sliding guide. This can be applied to almost all the links, including *principal elements*. Exceptions about introducing this constraint regard links which are already pivoted to the base. If a fixed sliding guide is connected to either SC_1 or SC_2, redundant constraints will be created. Similarly to the previous section, if the rotations of two contiguous links are constrained, the links will result to be fixed and can be removed. The same result is achieved if one of these links is pivoted to the base. As a consequence, the same observations made in the previous section for links C_1B_1, C_1B_2, C_2B_2 and C_2B_3 apply.
When a link's rotation is constrained, its parallel links will only translate. If the translation of a link is not allowed, like SC_1 in Fig. 4, this link will result to be fixed and can be removed. On the other hand, it is possible to constrain rotations while making the links translate along given trajectories. However, this can be achieved by introducing sliders.

3.3 Introducing sliders

Sliders can be introduced to constrain linkages as they allow links to rotate and translate along either a given trajectory or a specific direction. These different motions depend on how these elements are connected to the links and the base. Examples will show how sliders can be introduced in PVL. They are assumed to be mass-symmetric, then their CoM is placed on their geometric center. The revolute joint connecting them to either the base or the links, depending on the different example, is assumed to be placed on the geometric center as well.

3.3.1 Slider pivoted to the base

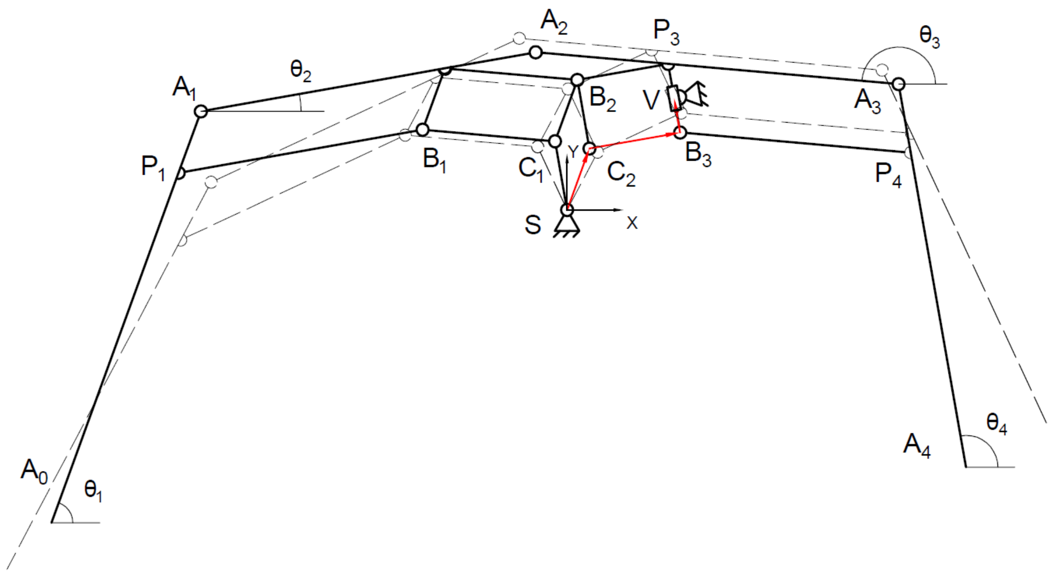


Fig. 5. Principal Vector Linkage having 3 DoF. Slider pivoted to the base in V allows B_3P_3 to rotate around this point and to translate along its sliding pair.

In Fig. 5 a slider has been pivoted to the base in point V and connected to link B_3P_3 through its sliding pair. The resulting PVL has 3 DoF, as can be proved by using Grübler's equation. In order to derive the kinematic relations, the vector loop from S to V can be defined as the summation of vectors $\vec{SC_2}$, $\vec{C_2B_3}$ and $\vec{B_3V}$. The magnitude of the latter, in particular, can vary and can be considered as one of the remaining DoFs. Equations (5) and (6) represent the scalar equations along X and Y components and constitute a system. Its results will consist in the kinematic relations among θ_1, θ_2, θ_4 and B_3V.

$$SV_x = SC_2 \cos\theta_1 + C_2B_3 \cos\theta_2 + B_3V \cos\theta_4 \tag{5}$$

$$SV_y = SC_2 \sin\theta_1 + C_2B_3 \sin\theta_2 + B_3V \sin\theta_4 \tag{6}$$

By considering θ_4 and B_3V as the remaining DoFs, the system will give $\theta_1 = \theta_1(\theta_4, B_3V)$ and $\theta_2 = \theta_2(\theta_4, B_3V)$. For the same reason seen in the previous section, θ_3 does not appear in the equations and it is recommended to be considered as the third remaining DoF.

A slider pivoted to the base can be placed on every point within a PVL. Particular situations occur when it is connected to links which have been already constrained. If a link is pivoted to the base, the introduction of this kind of slider will completely constrain the link. It is the case of links SC_1 and SC_2: they will result to be fixed, together with their angles, therefore they can be removed. On the other hand, if a link's rotation is already constrained as seen in the previous section, the introduction of a slider pivoted to the base will create a redundant constraint: the link is already allowed to move only along its orientation and the slider pivoted to the base does not introduce any further constraint. The same motion can be made by introducing two pivoted sliders to the same link; in this case no redundant constraints are introduced.

As the slider is assumed to be pivoted to the base on its own CoM, its mass does not affect the force balance of the linkage. On the other hand, this element rotates, therefore its inertia needs to be considered in the angular momentum of the linkage if moment balance conditions have to be defined. Since the slider rotates together with the link to which it is connected, its inertia can be added to the link's one.

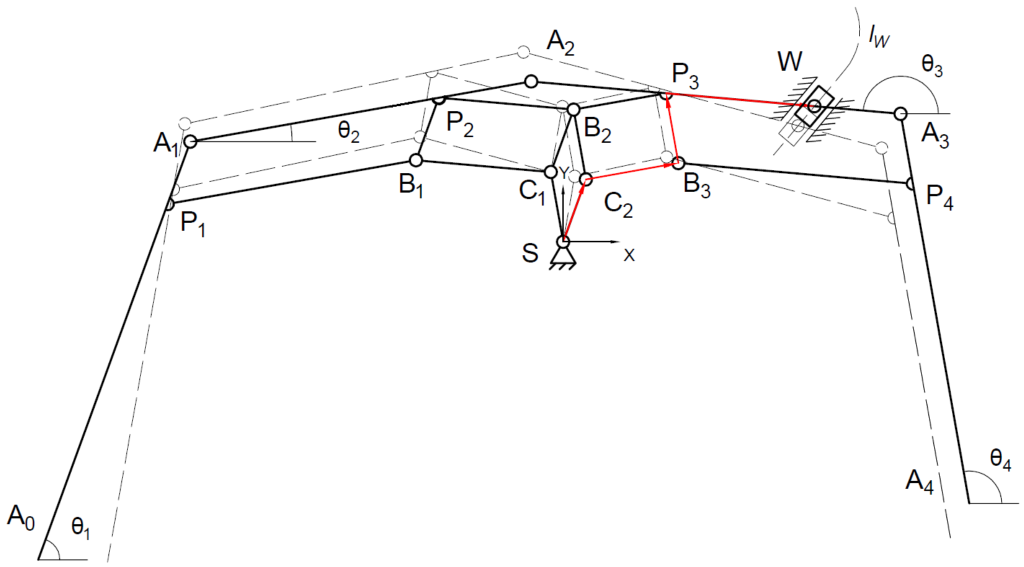


Fig. 6. Principal Vector Linkage having 3 DoF. Slider pivoted on W to A_2A_3 allows this link to rotate around this point and to translate along its trajectory on the base.

3.3.2 Slider moving on the base

Sliders can be also connected to PVL through a revolute joint and to the base through their sliding pair. Their masses have to be considered in the total mass, and therefore in the force balance conditions. They are assumed to move along paths that can be curved: in this case, their inertia shall be considered in the angular momentum of the linkage.
Figure 6 shows a resulting 3 DoF PVL where a slider has been connected to A_2A_3 through a revolute joint in point W. This moves along a given path. The vector loop describing W position with respect to S can be defined: three variables can be chosen as remaining DoFs among θ_1, θ_2, θ_3, θ_4 and the position of the slider along its path. It is assumed that each point of this path, therefore every position of W, is known and determined by a variable called l_W. In doing so, the vector describing W position shall be in the form $\vec{SW}(l_W)$; this will be equal to the summation of vectors $\vec{SC_2}$, $\vec{C_2B_3}$, $\vec{B_3P_3}$, $\vec{P_3W}$.

$$SW(l_W)_x = SC_2\cos\theta_1 + C_2B_3\cos\theta_2 + B_3P_3\cos\theta_4 - P_3W\cos\theta_3 \tag{7}$$

$$SW(l_W)_y = SC_2\sin\theta_1 + C_2B_3\sin\theta_2 + B_3P_3\sin\theta_4 - P_3W\sin\theta_3 \tag{8}$$

If θ_3, θ_4 and l_W, for example, are considered the remaining DoFs, the system constituted by the scalar equations of the vector loop, Eqns. (7) and (8), will give, as a result, $\theta_1 = \theta_1(\theta_3, \theta_4, l_W)$ and $\theta_2 = \theta_2(\theta_3, \theta_4, l_W)$.
Similarly to pivoted sliders, this kind of sliders introduces one constraint and can be placed in every point of PVL. Depending on the paths that they are allowed to move along, different situations can occur. If the paths are straight, these sliders can perfectly constrain a link which is already pivoted to the base and that can be therefore removed. The same result can be reached if a link's rotation is already constrained as seen in section 3.2: in this case, the slider's path must not be parallel to the link's orientation, otherwise it represents a redundant constraint. On the other hand, sliders having a path which is curved can generate infinite solutions depending on the specific path. This kind of sliders can be used to constrain links' rotations as well. By introducing two sliders performing identical trajectories and pivoted on the same link, this will translate while keeping the same angle.

3.4 Fixing links

Both rotation and translations can be constrained by fixing links, which indeed can be considered as the base and can be removed. This implies the introduction of three constraints in a *principal vector linkage*. In the PVL shown in Fig. 7, *principal element* A_2A_3 has been fixed to the base. Its angle θ_3 is fixed and its parallel *principal vector links* are allowed to only translate. The resulting PVL has 1 DoF. Furthermore, A_2A_3 can be removed while introducing fixed pivots on A_2, P_3 and A_3. In order to derive the kinematic relations between θ_1, θ_2 and θ_4, a vector loop describing the position of one of these points can be defined. As an example, A_3 position is described by Eqns. (9) and (10), which represent the summation of

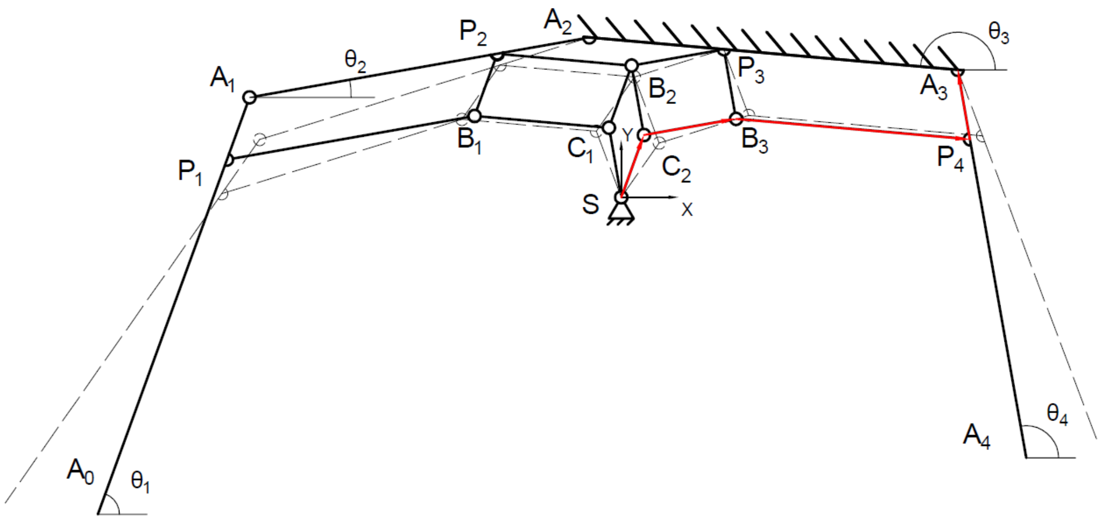


Fig. 7. Principal Vector Linkage having 1 DoF. A_2A_3 is fixed to the base. All its motions are constrained and its parallel links can only translate.

vectors $\vec{SC_2}$, $\vec{C_2B_3}$, $\vec{B_3P_4}$ and $\vec{P_4A_3}$, equal to $\vec{SA_3}$, along X and Y components.

$$SA_{3,x} = SC_2 \cos\theta_1 + C_2B_3 \cos\theta_2 - B_3P_4 \cos\theta_3 + P_4A_3 \cos\theta_4 \tag{9}$$

$$SA_{3,y} = SC_2 \sin\theta_1 + C_2B_3 \sin\theta_2 - B_3P_4 \sin\theta_3 + P_4A_3 \sin\theta_4 \tag{10}$$

Both the equations constitute a system. If θ_4 is considered the remaining DoF, for example, the results will be $\theta_1 = \theta_1(\theta_4)$ and $\theta_2 = \theta_2(\theta_4)$. It is clear that the relations depend on where *principal element* A_2A_3 has been fixed.
Particular situations occur when this kind of constraint is introduced in *principal vector links*. In particular, if these are connected to any *principal point*, the only remaining DoF is the rotation of the *principal element* to which the point belongs. On the other hand, links which are already constrained can not be fixed, as redundant constraints would be introduced. For the same reason, constraining links which are contiguous to a fixed one could result in adding redundant constraints: they only rotate about the revolute joints connected to the fixed link. Only the sliders of sections 3.3.1 and 3.3.2 can be introduced: they will fix these contiguous links. However, redundant constraints are introduced if sliders are made rotate around the revolute joints connected to the fixed link. Section 5 will show how this redundancy can be prevented by removing the links to which these sliders are introduced.

3.5 Constraining angles between links

Sections 3.1 to 3.4 showed examples about how PVL can be constrained by limiting links' absolute motions. On the other hand, it is possible to constrain relative motions between links. Due to the parallelograms network constituting PVL, constraining links' relative motions means constraining the angles between them. This consists in either fixing these angles or creating certain relations between them by introducing machine elements like gear, belt and chain drives.

3.5.1 Fixing angles

Figure 8 shows a PVL in which the angle between *principal elements* A_2A_3 and A_3A_4 has been fixed. This implies that both of them will have the same rotation together with *principal vector links* B_3P_3 and B_3P_4, therefore parallelogram $P_3A_3P_4B_3$ will not change its shape. It is possible to prove that the resulting linkage has 3 DoF by using Grübler's equation and considering B_3P_3, B_3P_4, A_2A_3 and A_3A_4 as a rigid body connected to the rest of the PVL through revolute joints in A_2, P_3 and B_3. The relation describing how the angles θ_3 and θ_4 are constrained can be defined by using angle properties. As the fixed angle in A_3 is assumed to be known, Eqn. (11) shows the relation between the angles.

$$\angle A_3 = \pi - \theta 3 + \theta 4 \tag{11}$$

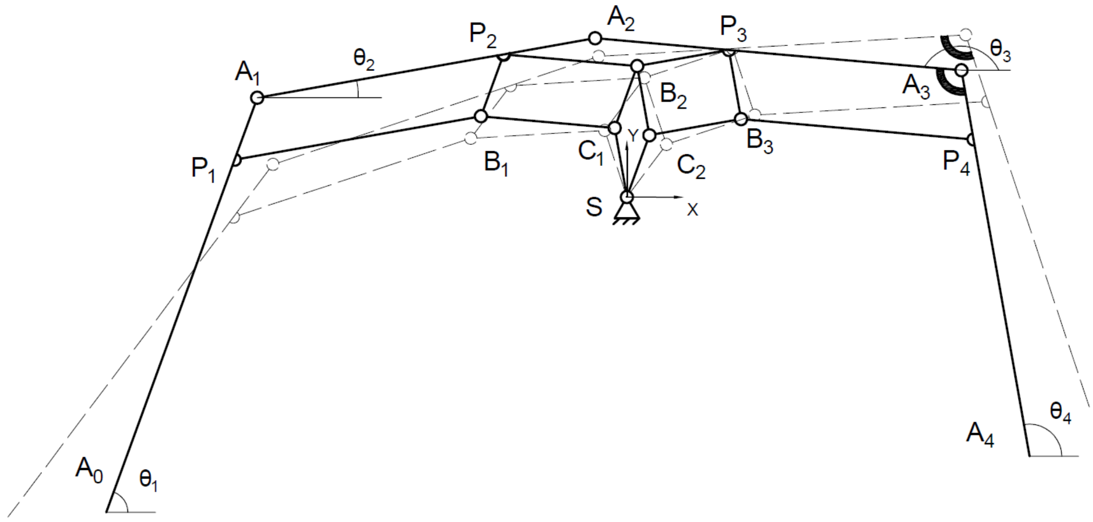


Fig. 8. Principal Vector Linkage having 3 DoF. Angle between A_2A_3 and A_3A_4 is fixed. Parallelogram $P_3A_3P_4B_3$ is rigid.

Due to the parallelograms network of PVL, fixing an angle between two links means fixing the angles between contiguous links which are parallel to them. Therefore, in Fig. 8 the angle between SC_1 and C_1B_1 and the one between C_2B_2 and B_2P_2 are fixed.
It is also possible to fix an angle between two *principal vector links* and therefore constraining two angles which do not belong to contiguous *principal elements*. For example, by fixing the angle between C_1B_2 and C_2B_2, θ1 and θ4 will be constrained making A_0A_1 and A_3S_4 have the same rotation; as a result, parallelogram $SC_1B_2C_2$ will be considered as a rigid body pivoted to the base on S.

3.5.2 Introducing gear, belt and chain drives

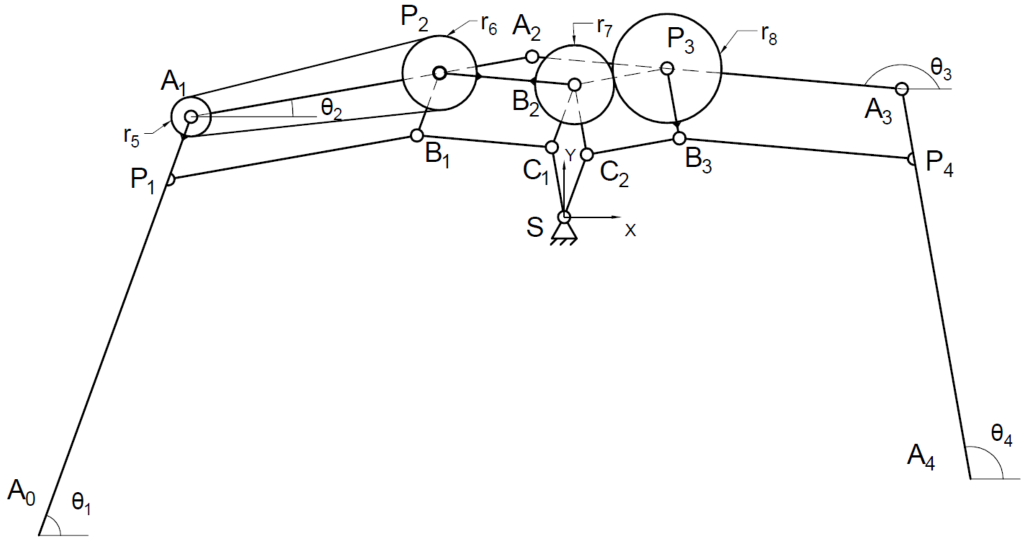


Fig. 9. Principal Vector Linkage having 2 DoF. Belt (or chain) drive on A_1P_2, pulleys (or sprockets) are fixed to A_1S_1 and B_2P_2. Meshed gears on B_2P_3 are fixed to B_2P_2 and P_3B_3. The rotations of the links fixed to the elements of the same drive are constrained.

On the other hand, it is possible to define relations describing relative rotations between links. Compared to the previous section in which the angle between links, therefore the relative rotation, has been fixed, here the angle varies as the links can rotate relatively to each other. This can be achieved by introducing gears, belt or chain drives. After pivoting the rotating elements, i.e. gears or pulleys or sprockets, to the ends of a link, they have to be fixed to one of their contiguous links.

In particular, the CoM of each rotating element is assumed to be placed on the geometric center, on which the element is pivoted to the links.
In Fig. 9, pulleys 5 and 6 have been pivoted on A_1 and P_2 and fixed to A_0A_1 and B_1P_2, while gears 7 and 8 have been pivoted on B_2 and P_3 and fixed to B_2P_2 and B_3P_3. The pulleys could be even sprockets of a chain drive, as both the elements rotate similarly. Since each drive adds one constraint, the resulting PVL has 2 DoF. The relations describing how the original DoFs are constrained are based on the drives' properties. Compared to the previous sections where the angles are involved in the kinematic relations, here the angular velocities $\dot{\theta}$ are involved: the relations will therefore include angular velocities. Nevertheless, this is not a problem as the derived equations are still kinematic relations.
The drives that can be introduced are planetary, and therefore three angular velocities are involved in each relation: those of the rotating elements and the one of the link on which they are pivoted. For example, in deriving the relation about the belt drive and, in particular, the velocity of pulley 6 $\dot{\theta}_3$, the superposition principle is used: pulley 5 and link A_1P_2 are made rotate separately: their effects are the first and the second terms, respectively, on the right side of Eqn. (12).

$$\dot{\theta}_3 = \frac{r_5}{r_6}\dot{\theta}_1 + \left(1 - \frac{r_5}{r_6}\right)\dot{\theta}_2 \tag{12}$$

A similar procedure is made for the gear drive. For example, the angular velocity of gear 8 $\dot{\theta}_4$ can be derived by using superposition principle: the first and the second term on the right side of Eqn. (13) are, respectively, the effects of gear 7 ($\dot{\theta}_3$) and B_2P_3 ($\dot{\theta}_2$) velocities.

$$\dot{\theta}_4 = -\frac{r_7}{r_8}\dot{\theta}_3 + \left(1 + \frac{r_7}{r_8}\right)\dot{\theta}_2 \tag{13}$$

Equations (12) and (13) constitute a system. Any couple of angular velocity can be considered as the DoFs of the resulting *principal vector linkage*.
Particular attention needs to be paid in introducing the drives. The rotating elements need to be fixed to links that are contiguous to the one on which they are pivoted. This because fixing one element to this link will result in fixing the rotation of the two elements, therefore fixing angles between links. Furthermore, drives can be introduced on links pivoted to the common CoM as well. In doing so, it is possible that the rotating element pivoted to the common CoM will be fixed to the base. In this case, the resulting kinematic relation will involve two angular velocities: the one of the link pivoted to the base and the one of the link to which the other rotating element is fixed.
Furthermore, if the rotating elements of any drive are fixed to parallel links, the constraint will be redundant and different scenarios can occur. If the transmission ratio is not 1, a conflict would be generated between the drive and the parallelogram linkage to which the parallel links belong: these would not be allowed to rotate both by the same amount. On the other hand, if the transmission ratio is 1, the parallel links will rotate by the same amount although the constraint will be still redundant. Section 5 will show that this redundancy can be prevented by removing the link opposite to the one on which the drive is introduced.
Gear drives can include internal gears as well. It is clear that, in this case, more space is required and certain transmission ratios can not be reached. For example, it is impossible for internal gears to have a transmission ratio equal to 1.
As additional elements are introduced and fixed to the links, their masses can be considered as part of the links' ones, together with their inertias. As a consequence, the force balance conditions need to be adjusted and the *principal dimensions* are expected to change.

4 Modifying links position

Another technique that can be used to synthesize IDB mechanisms from PVL consists in modifying the position of the links without any change in the number of DoFs.
The 3 DoF PVL shown in Fig. 10 is considered. The *principal elements* A_0A_1, A_1A_2 and A_2A_3 are connected to the common CoM S through the *principal vector links* constituting parallelograms. The overall linkage's kinematics depends on the combination of the different parallelograms' ones and on the *principal dimensions*. As the parallelograms' kinematic property, which was mentioned in section 2, does not depend on the links' length, it is possible to change the parallelograms' sizes while keeping the essential kinematics and the force balance of the linkage. As a result, Fig. 11 shows the linkage in which links B_1P_1, SB_2 and B_2P_3 have been collocated at certain distances from, respectively, A_1P_2, B_1P_2 and P_2A_2. The new links F_1F_2, K_1C_1, C_1C_2 and P_2C_1 in Fig.11 can no longer be considered *principal vector links*, since their positions have been changed. In addition, other links can be modified as well, like link P_2C_1. The *principal dimensions* have been modified as the force balance conditions had to be adjusted. Figure 11 shows that, when A_2A_3 rotates, links K_1C_1 and C_1C_2 translate, while

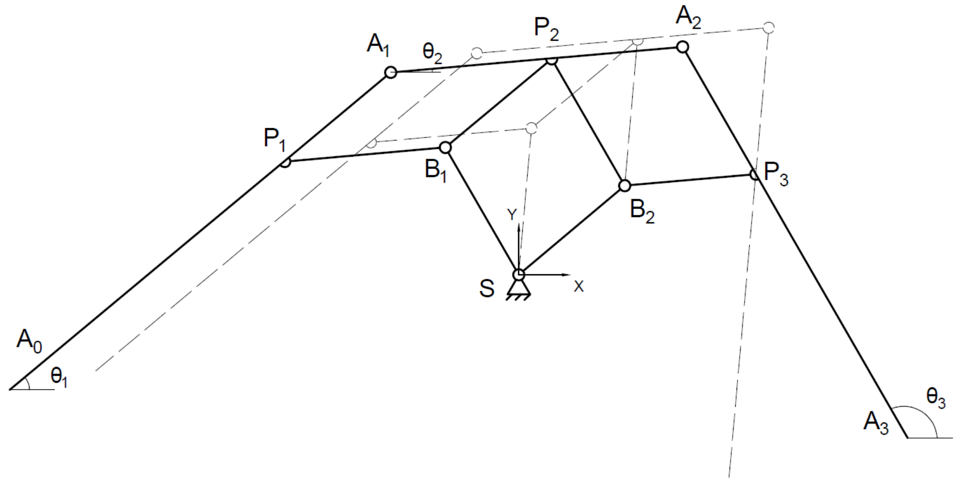


Fig. 10. 3 DoF Principal Vector Linkage.

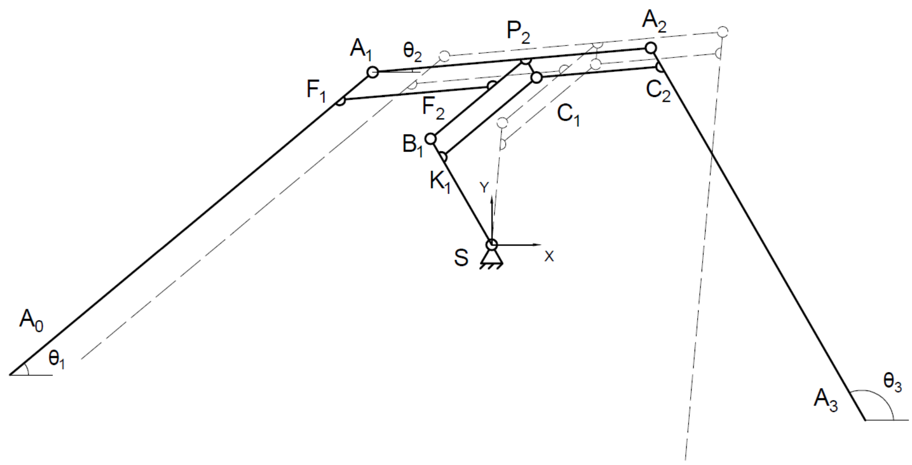


Fig. 11. 3 DoF Principal Vector Linkage of Fig. 10 with moved links. Kinematics is maintained.

their respective links in Fig. 10, SB_1 and B_1P_3, do not move. Therefore, the masses of these links have to be considered in the linear momentum equation regarding θ_3. As a result, the *principal dimension* which is derived from this equation has been changed. Thus, since SB_1 is still a *principal vector link* having this particular *principal dimension*, it has a different length in Fig.11. A similar procedure has been made for linear momentum regarding θ_1, leading to a change of the relative *principal dimension* and B_1P_2 length: this is still a *principal vector link* in the new linkage in Fig.11.
In conclusion, the resulting linkage is force balanced like the original PVL, although several links have been moved. The linkage's essential kinematics is maintained as well as force balance. Adjustments have been made to the force balance conditions and *principal dimensions* have been changed. The overall linkage's motions results to be similar to the initial one's, as shown by the dashed lines in Figs. 10 and 11. Moreover, the new *principal dimensions* do not correspond to the new sizes of the parallelograms. In fact, the parallelograms are essential because of their kinematic properties, not their specific sizes. On the other hand, the *principal dimensions* still determine the overall motions. This is the main reason why certain links, like SB_1 and B_1B_2, still correspond to *principal vectors* in order to keep the correct references from the common CoM together with the essential kinematics of PVL.
This technique can be applied to *principal elements* as well, but further adjustments are required. When a link which is part of a *principal element* is moved, it is divided into two links which have to be kept parallel. This is possible by introducing either gear, belt or chain drives whose transmission ratio is 1, and fixing the rotating elements to the divided links, as Fig. 12 shows. As the rotating elements are fixed to the links, their mass and inertia will be considered together with the ones of the links. In this case, force balance conditions need to be adjusted as they have to include both the new position of the link and the additional mass of the rotating elements.

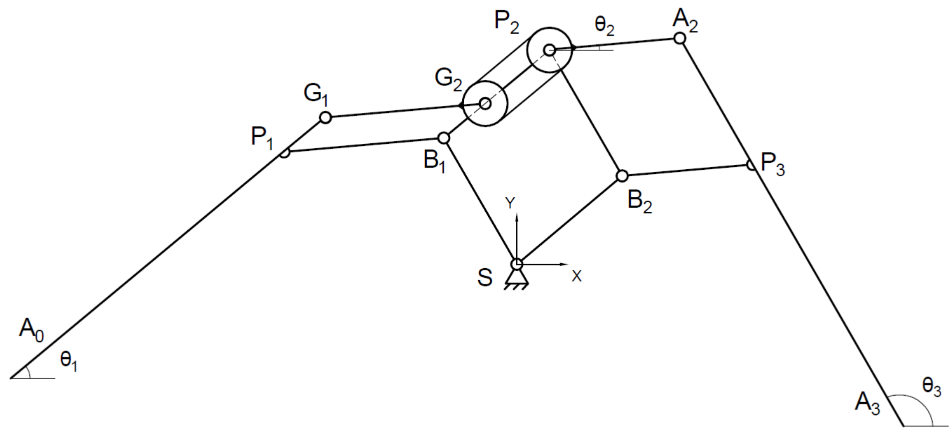


Fig. 12. 3 DoF Principal Vector Linkage of Fig. 10: *principal element* A_1A_2 is divided in G_1G_2 and P_1A_2. Belt (or chain) drive keeps them parallel. Kinematics is maintained.

5 Substituting links

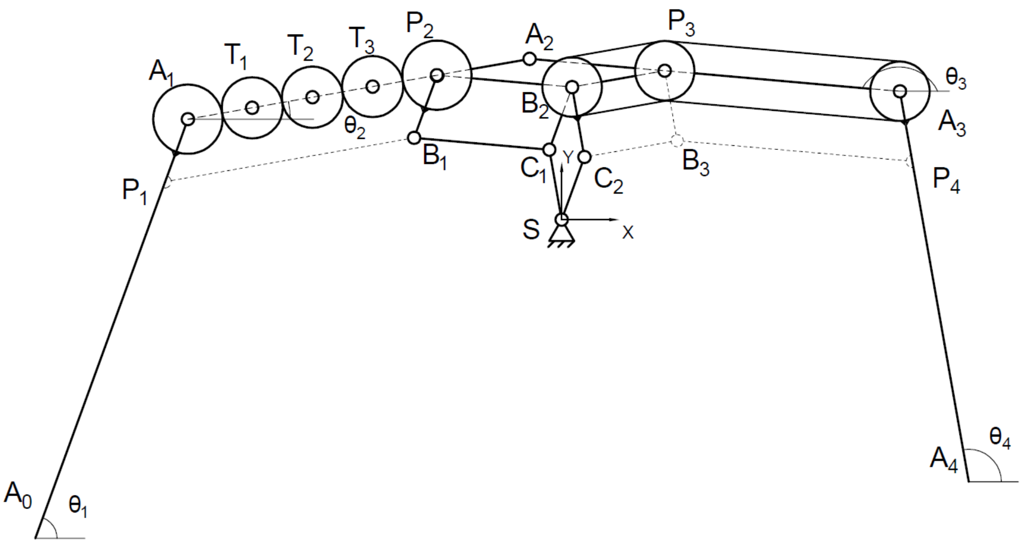


Fig. 13. 4 DoF Principal Vector Linkage. B_3P_4, P_3B_3 and C_2B_3 are substituted by belt (or chain) drive: outer elements are fixed to C_2B_2. and A_3A_4. B_1P_1 is substituted by gear train: outer elements are fixed to A_0A_1 and B_1P_2.

In addition to modifying links' position, within PVL links can be even substituted by different machine elements such sliders and gear, belt and chain drives. This section will describe how these elements need to be introduced in order to maintain the kinematic properties which are essential for force balance. Like in the previous sections, the additional elements are assumed be mass-symmetric. The CoM is placed on the geometric center of each element, on which the revolute joint is placed.

5.1 Introducing gear, belt or chain drives

Section 3.5.2 stated that a redundant constraint is created when the rotating elements of a drive having transmission ratio equal to 1 are fixed to parallel links. This redundancy can be deleted by removing the link opposite to the one on which the drive is introduced. This already describes how gear, belt and chain drives can be can be used to replace links within *principal vector linkages*. The parallelogram's kinematic property results to be maintained because the rotating elements which are fixed to the links keep them parallel and make them rotate always by the same amount.
An example is shown in Fig. 13. In the 4 DoF PVL, B_1P_1 has been replaced by a gear train placed on A_1P_2. The external gears are fixed to A_0A_1 and B_1P_2: as they have the same radius, therefore the transmission ratio is 1, their rotations are equal

and the links are kept parallel. Furthermore, it is possible to replace more links since contiguous parallelograms have parallel links, like C_2B_2, B_3P_3 and A_3A_4 in Fig. 13: by fixing rotating elements having equal radius to these links, C_2B_3 and B_3P_4 can be removed. As a result, B_3P_3 can be removed as well since no other link is connected to B_3.
When links constituting *principal elements* are replaced, on the other hand, an additional drive is required for the same reason seen in section 4. The PVL in Fig.14 presents a gear train on B_1P_1 whose external gears have been fixed to A_0P_1 and B_1P_2, in order to remove A_1P_2. As links B_1P_1 and P_2A_2 need to be kept parallel, a belt (or chain) drive is added on B_1P_2 and the pulleys (or sprockets) are fixed to B_1P_1 and P_2A_2. In addition, P_1A_1 can be removed, since no other link is connected to A_1.
When links are substituted, their contributions in the force balance conditions are removed, while those of the additional

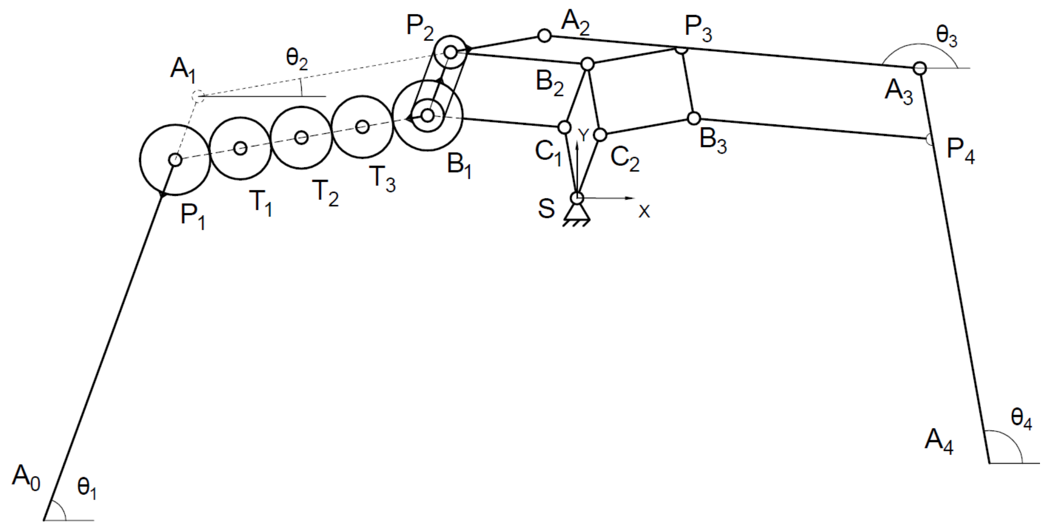


Fig. 14. 4 DoF Principal Vector Linkage. A_1P_2 is substituted by gear train: outer elements are fixed to A_0P_1 and B_1P_2. Belt (or chain) drive on B_1P_2: elements are fixed to P_2A_2 and B_1P_1 to keep them parallel.

elements are included. In addition, the inertia of the rotating elements need to be considered in the angular momentum equations. Both the mass and the inertia of the elements which are fixed to the links can be considered together with the links' ones. On the other hand, when additional elements are only pivoted on links, like the gear trains in Figs. 13 and 14, only their masses can be considered as part of the links'. As a consequence of the adjustments of the force balance conditions, the *principal dimensions* will change.

5.2 Introducing sliders

Due to the parallelograms network constituting PVL, introducing sliders to replace links seems unfeasible at first sight. Sliders can not be connected between links through their sliding pairs: these would make contiguous links able to translate relatively to each other. Therefore, the overall linkage's kinematics would be lost as well as force balance. However, section 3.4 stated that introducing a slider creates a redundant constraint if it performs a circular trajectory around a fixed pivot. This redundancy can be prevented by removing the link rotating around the same pivot. As a result, a slider replacing a link is created. Potential issues regarding the introduction between links are prevented since sliders move along fixed trajectories: their sliding pairs are connected to the base. Although the fixed pivots which have been discussed in section 3.4 were those created by fixing a link, every PVL includes this kind of pivots on their common CoM. This implies that all the links pivoted to the common CoM can be replaced by sliders.
Figure 15 shows this substitution in a 3 DoF PVL. Links SB_1 and SB_2 have been replaced by sliders pivoted on B_1 and B_2 and performing circular trajectories around S having radii equal to, respectively, SB_1 and SB_2. The kinematics of parallelogram $SB_1P_2B_2$ has not been chaged, as the radii are equal to the length of the remaining links. As a result, sliders rotate by the same amount of the substituted links , therefore their contributions in the force balance conditions replace those of the original links. In addition, their inertia shall be considered in the angular momentum equations.
By considering the PVL presented in section 3.4 and shown in Fig. 7, B_2P_3 and B_3P_3 can be replaced by sliders pivoted on B_2 and B_3 and moving along circular trajectories around P_3: the radii of the trajectories are equal to the lengths of the replaced links. In addition, sliders can replace P_2A_2 and A_3A_4 since they rotate around A_2 and A_3, respectively. However, they need to be fixed to the remaining parts of the *principal elements*, i.e. A_1P_2 and P_4A_4, in order to maintain the overall linkage's kinematics. As sliders substitute only parts of principal elements, they have to rotate in accordance with the remaining parts.

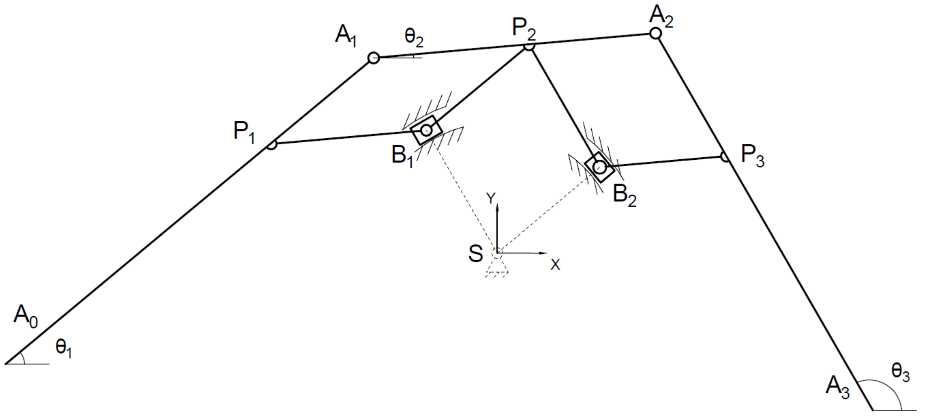


Fig. 15. 3 DoF Principal Vector Linkage. Sliders on B_1 and B_2 replace SB_1 and SB_2 by moving along circular trajectories centered on S.

The resulting linkage is presented in Fig. 16. The force balance conditions need to be adjusted. In particular, sliders on P_2 and P_4 can be considered together with links A_1P_2 and P_4A_4, as they are fixed to them.
In conclusion, it is possible to state that, within a *principal vector linkage*, sliders can replace links if these are pivoted to the base. They shall perform circular trajectories having radii equal to the distance between the sliders and the fixed pivots. If only part of a link is substituted, the slider needs to be fixed to the remaining part in order to maintain the overall linkage's kinematics. It is important that links are pivoted to the base on one of their ends. If a slider replaces a link which is pivoted on a different point, it will replace only part of the link: the slider's motion will not be in accordance with the remaining part of the original link. Fixing the introduced element with the remaining part, as previously described, is not possible since they are not connected through any joint.
It is supposed that more links can be replaced by sliders when constraints are introduced in *principal vector linkages*. Section 6 will show an example.

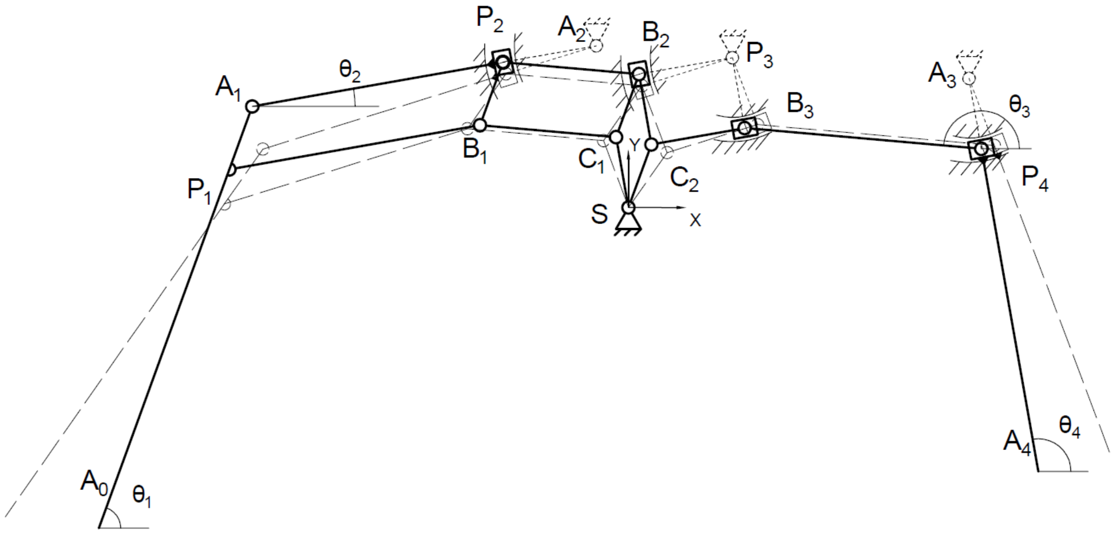


Fig. 16. Principal Vector Linkage of Fig. 7. A_2A_3 is fixed to the base and removed. P_2A_2, B_2P_3, B_3P_3 and A_3P_4 are replaced by sliders. Sliders on P_2 and P_4 are fixed to A_1P_2 and P_4A_4 respectively, to keep the same rotations.

6 Techniques combined

All the techniques do not modify PVL in the same way. Section 3 presented techniques which modify the number of DoFs, while sections 4 and 5 showed techniques which modify the linkage's design. As a consequence, many solutions can be derived, as the techniques can be applied on different points of PVL, therefore many mechanisms can be synthesized. Moreover, in addition to applying different techniques on different points, some of them can be applied together on the same

points in order to create new additional solutions. This procedure is considered as a combination of techniques.
An example consists in combining techniques described in sections 3.1 to 3.4, which constrain the links' absolute motions, with the technique described in section 4. By constraining a link whose position has been modified, the kinematic relations depend on this new position. The derived motions of the resulting linkage are therefore unique as they can be achieved only by the combination of those specific techniques. Figure 17 shows the 3 DoF PVL of Fig.11 in which link C_1C_2 has

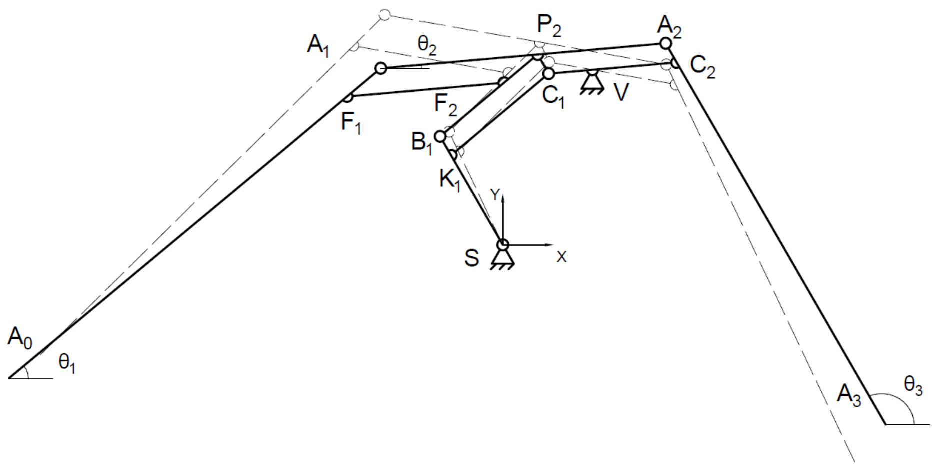


Fig. 17. Principal Vector Linkage of Fig.11. C_1C_2 is pivoted to the base in V.

been pivoted to the base in point V. The resulting linkage has 1 DoF. The dashed lines show the resulting motion which is characteristic of this particular linkage. As described in section 4, C_1C_2 corresponds to *principal vector link* B_2P_3 of Fig.10 which has been moved. If the original link B_2P_3 was pivoted to the base, the motion of the resulting 1 DoF PVL would consist in the one shown by dashed lines in Fig. 10. The clear difference between the two motions is due to having pivoted a link which has been previously moved. By looking at the scalar equations of the vector loop from S to V, it is possible to notice that the position of the moved link and the one of the fixed pivot affect the relations between the original DoFs.

$$SV_x = SK_1 \cos\theta_3 + K_1C_1 \cos\theta_1 + C_1V \cos\theta_2 \tag{14}$$

$$SV_y = SK_1 \sin\theta_3 + K_1C_1 \sin\theta_1 + C_1V \sin\theta_2 \tag{15}$$

Equations (14) and (15) represent the scalar equations of the loop. The position of V is described as the summation of vectors $\vec{SK_1}$, $\vec{K_1C_1}$ and $\vec{C_1V}$, which have constant magnitudes and varying angles. SK_1 indicates how much the links have been moved from their original positions, while SV and C_1V show how the position of the fixed pivot affects the kinematic relations. The system constituted by these equations can be solved by considering any angle the remaining DoF: if θ_3 is considered, for example, the kinematic relations will be in the form $\theta_1 = \theta_1(\theta_3)$ and $\theta_2 = \theta_2(\theta_3)$.

Although the previous example has shown how new kinematic solutions can be derived, a different type of solutions resulting from the techniques combinations regards the elements constituting PVL.
The following example shows how links can be removed after having constrained a PVL and how this leads to the substitution of links with sliders. It involves, in particular, the resulting 2 DoF PVL described in section 3.2 and shown in Fig. 4. The rotation of B_3P_3 was constrained: the link was allowed to translate only along its orientation and its parallel links could not rotate. Thus, *principal vector link* SC_1 resulted to be fixed. By removing this link, revolute joint C_1 needs to be pivoted to the base, as it is still connected to links C_1B_1 and C_1B_2, while the pivot in S is still connected to SC_2. Since these three links rotate around fixed pivots, according to section 5.2, it is possible to replace them with sliders placed on B_1, B_2 and C_2 and moving along circular trajectories having radii equal to, respectively, C_1B_1, C_1B_2 and SC_2.
The resulting PVL, which is shown in Fig. 18, is a clear example of how constraining links, B_3P_3 in this particular case, can lead to the removal of links, SC_1, and the substitution of others with machine elements, C_1B_1, C_1B_2 and SC_2 replaced by

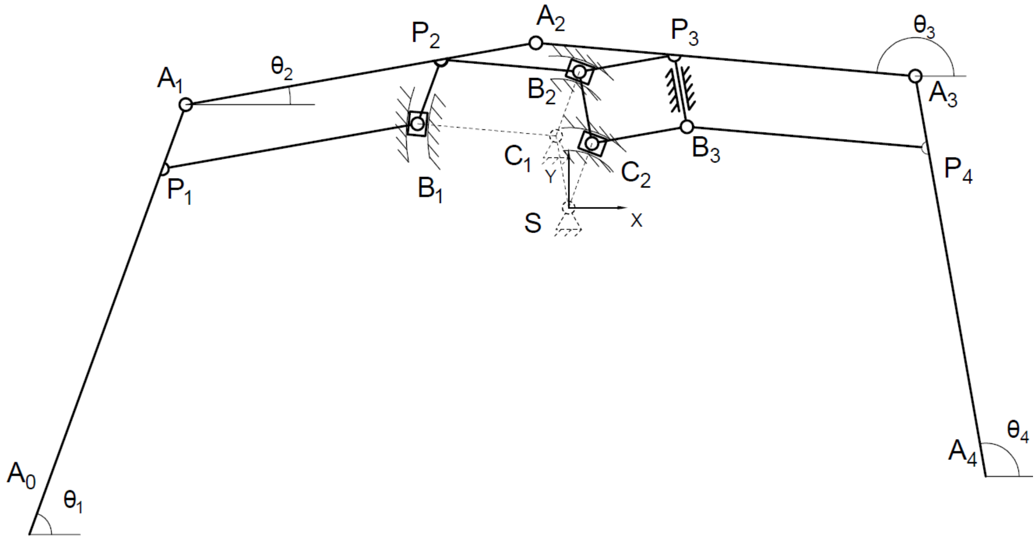


Fig. 18. Principal Vector Linkage of Fig. 16. SC_1 is removed. Sliders in B_1, B_2 and C_2 replace C_1B_1, C_1B_2 and SC_2, respectively.

sliders. The resulting linkage is supposed to have 2 DoF like the one in Fig.4. However, it has a redundant constraint. After having removed SC_1 and having pivoted C_1 to the base, link C_2B_2 was supposed to be removed as well. Nevertheless, it has not been removed as singularities can occur whenever S, C_2 and B_3 become collinear, as well as C_1, B_2 and P_3: parallelism between B_2P_3 and C_2B_3 could be lost, together with the one between SC_2 and C_1B_2. Link C_2B_2 prevents this issue by always keeping B_2 and C_2 at the right distance between each other. It is clear, on the other hand, that this link is supposed to be necessary only if the singularity can be reached: if points S, C_2 and B_3 can never be made collinear, e.g. because of specific design requirements, there is no reason to keep link C_2B_2 within the linkage. As links have been replaced by sliders, the force balance conditions need to be revised. In deriving the new conditions, the *principal vector linkage* shall be considered having the original number of DoFs without any additional constraint.

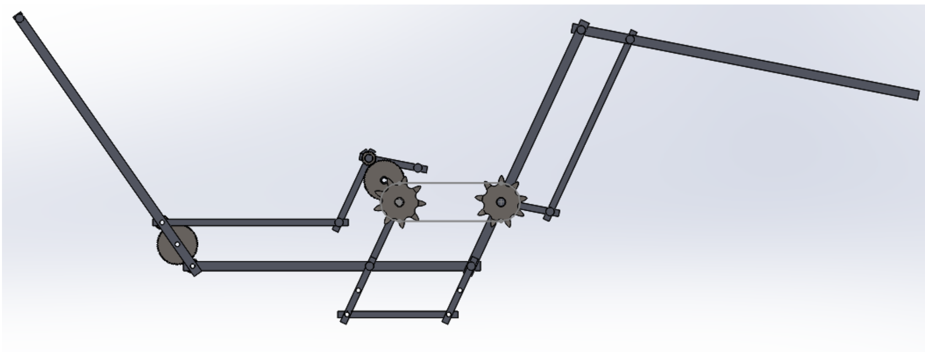


Fig. 19. 2 DoF inherently force balanced mechanism synthesized from 4 DoF Principal Vector Linkage. CAD model.

As previously stated, many techniques can be simply applied to the same PVL on different points and therefore many solutions can be derived. As an example, Fig. 19 shows a 2 DoF inherently force balance mechanism which has been synthesized from a 4 DoF PVL. Figure 20 presents the scheme of the mechanism. The common CoM is pivoted to the base on S. In addition to the fixed pivot introduced on point A_0, it can be noticed that, in order to replace links, different drives have been introduced. Link B_1P_2 has been replaced by the gear drive on P_1A_1, whose external gears are fixed to B_1P_1 and A_1A_2. In addition, a drive having the same gears has been introduced on C_1B_2 while a chain drive has been introduced on B_2P_3. The sprocket on P_3 is fixed to B_3P_3 while the other is fixed to the external gear on B_2; the gear on C_1 is fixed to SC_1. Therefore, SC_1 and B_3P_3 are kept parallel. The rotation of A_3A_4 results to be transmitted to SC_1 through several transmission elements: parallelogram $A_3P_4B_3P_3$, the chain drive and the gear train. Furthermore, points P_3 and B_2 are not connected by any link: the distance between them is maintained as links A_2A_3 and K_1B_2 are kept parallel by parallelogram $P_2A_2K_2K_1$, which has been created by having moved link B_2P_3 and created K_1K_2.

A simulation was performed in order to prove that, after having applied the different techniques, together with the relative

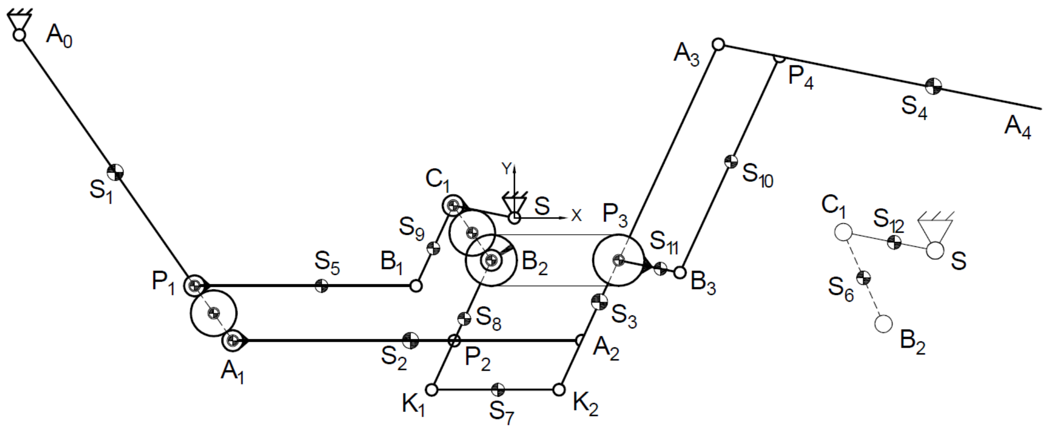


Fig. 20. Scheme of the 2 DoF inherently force balanced mechanism of Fig. 19. A_0A_1 pivoted to the base on A_0. Gear and chain drives replace links. Original link B_2P_3 moved to K_1K_2.

adjustments of the balance conditions, the resulting mechanism is force balanced. The mechanism shown in Fig. 20 was considered. All the elements were assumed having mass. All the links' parameters such mass, length and CoM position are presented in Table 1, together with the drives' parameters. The values of P_1S_1, P_2S_2, P_3S_3 and P_4S_4, in particular, were calculated with Matlab from the force balance conditions, which are presented in Appendix A. Then, the mechanism was modelled with the multy-body dynamic simulation software package SPACAR. The CoM of each link, gear and sprocket was introduced as a point mass on each element. Constant torques of 2 Nm and 0.3 Nm were applied, respectively, to links SC_1 and A_0A_1, in particular on the fixed pivots S and A_0, for a simulation time of 0.5s which was divided in 1000 time steps. Figure 23 shows the modeled mechanism at the final time step. The reaction forces on the fixed pivots were calculated by the simulation package. Then, they were summed up and plotted with respect to the simulation time in Fig. 21. The position of the common CoM was derived from those of each link's CoM, which were provided by the simulation. The X and Y coordinates were calculated for each time step and plotted in Fig. 22. Both the plots were expected to show data equal to zero. The difference between the expected values and the ones shown in the plots is due to the computation accuracy of both the simulation package and Matlab. Therefore, both the reaction forces and the position of the common CoM can be considered equal to zero.

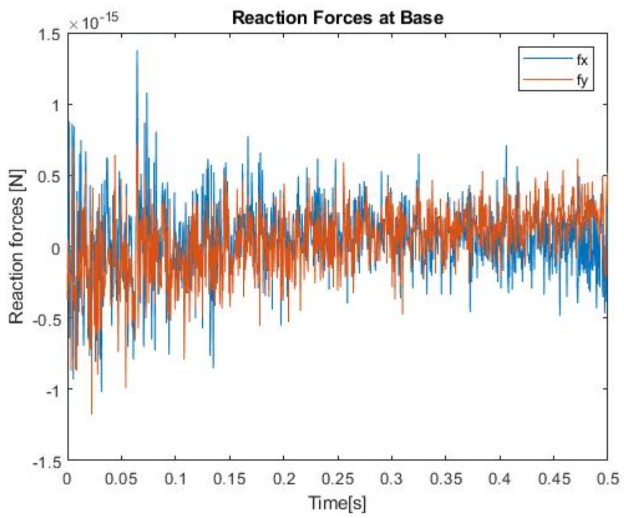


Fig. 21. Horizontal and vertical reaction forces calculated from the simulation performed on the mechanism of Fig. 20

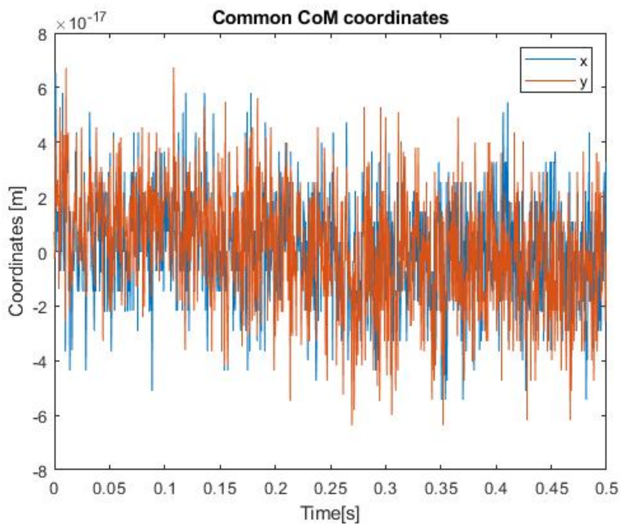


Fig. 22. X and Y coordinates of the common CoM derived from the results of the simulation performed on the mechanism of Fig. 20.

Masses [kg]	Principal dimensions [mm]	Links parameters [mm]	Elements parameters [mm]
$m_1 = 0.360$	$P_1A_1 = 28$	$A_0A_1 = A_3F = 200$	$R_{CentrGear} = 11$
$m_2 = 0.270$	$A_1P_2 = 90$	$P_1S_1 = 117.5222$	$R_{ExtGear} = 3$
$m_3 = 0.300$	$P_2A_2 = 60$	$A_1A_2 = 150$	$R_{Sprocket} = 14$
$m_4 = 0.450$	$A_2P_3 = 40$	$P_2S_2 = 77.1667$	
$m_5 = 0.165$	$P_3A_3 = 100$	$K_2A_3 = 160$	
$m_6 = 0.060$	$A_3P_4 = 20$	$P_3S_3 = 36.0333$	
$m_7 = 0.105$		$A_3S_4 = 200$	
$m_8 = 0.105$		$P_4S_4 = 67.6444$	
$m_9 = 0.075$		$B_1S_5 = 45$	
$m_{10} = 0.180$		$C_1S_6 = 14$	
$m_{11} = 0.030$		$K_1S_7 = 30$	
$m_{12} = 0.030$		$B_2S_8 = 30$	
$m_{CentrGear} = 0.009$		$C_1S_9 = 20$	
$m_{ExtGear} = 0.001$		$B_3S_{10} = 50$	
$m_{Sprocket} = 0.015$		$B_3S_{11} = SS_{12} = 10$	

Table 1. Links and drive's parameters of the 2 DoF inherently force balanced mechanism shown in Fig. 20

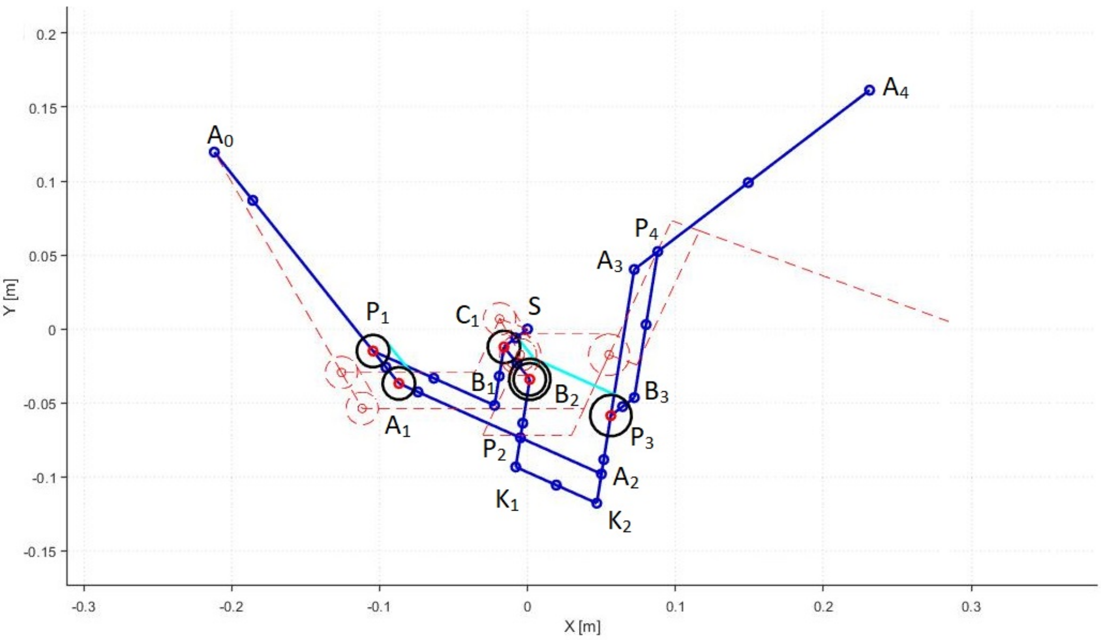


Fig. 23. 2 DoF inherently force balanced mechanism of Fig. 20 at the end of the simulation time. Dashed lines: initial position.

7 Discussion

The mechanism in Figs. 19 and 20 clearly showed how the techniques presented in the previous sections can be useful to synthesized inherently force balanced mechanisms from *principal vector linkages*. These can be modified to meet specific design requirements concerning, for example, the motions and the available space. In addition, the possibility to modify links' positions and to replace with other machine elements can be helpful as overlapping links can be prevented when these can constitute a limitation for the motions of the real mechanisms. Furthermore, it was shown how combining different techniques can create more specific solutions in terms of motions and space requirements. This is due to the fact that techniques modify PVL in different ways: the ones presented in section 3 modify the motions of the overall linkages while those presented in sections 4 and 5 modify their design, in terms of links' positions and substitutions with machine elements.

Basically, the techniques presented in section 3 reduce the number of DoFs, then the overall linkage's mobility. This kind of techniques could seem useless at first sight because different PVL have been designed, each one having a different number of DoFs: rather than adding constraints, linkages having less DoFs could be used. However, the main potential advantage of constraining linkages consists in performing specific motions which can be difficult to be performed by those having no constraints. Nevertheless, linkages having different number of DoFs can be constrained differently and can represent design alternatives. For example, if a 2 DoF mechanism is required, this can be synthesized by either a 4 DoF PVL or a 3 DoF PVL in which different constraints have been introduced. In addition, different motions can be performed by a constrained PVL depending on the constraints' types and where these have been introduced within the linkage.
This vast range of solutions that can be derived by constraining *principal vector linkages* depending on the initial number of DoFs, the type of constraints and where these are introduced, shows the significant versatility of this kind of techniques.

On the other hand, the main potential advantage of the techniques presented in sections 4 and 5 consists in adapting PVL to specific space requirements while maintaining the overall linkage's kinematics. While the techniques presented in section 3 considerably modify the motions by reducing the number of DoFs, the ones presented in sections 4 and 5 only modify the *principal dimensions*, then the links' lengths.
Moreover, about the technique presented in section 5, it can be stated that different potential advantages are derived depending on the type of introduced machine elements.
Replacing links with gear, belt or chain drives can be derived from a specific case of constraining PVL, as described in section 3.5.2, but it can be considered as an improvement of moving links as well: the resulting linkages in section 4 present parallelogram linkages and therefore singularities can still occur. On the other hand, the drives introduced to replace links prevent singularities as the links which are fixed to the rotating elements are made rotate always by the same amount. In addition, overlapping links are prevented as well, particularly when drives replace more links: in Fig.13, *principal elements* A_2A_3 and A_3S_4 can be collinear without creating any singularity or overlapping links. This can be helpful in real mechanisms in which overlapping links can not be made and, more generally, it can increase their range of motions.
However, the introduction of sliders to substitute links does not prevent singularities but shows a considerable change in terms of space which can potentially represent a significant advantage. In particular, if fixed pivots, including the common CoM, can not be physically connected to the linkage, introducing sliders to replace links constitutes a valid alternative.

All the techniques have been presented for *principal vector linkages* having mass-symmetric links. In fact, in section 2 it was assumed that the techniques can be applied to mass-asymmetric links as well. In principle it is possible since literature shows that the theory for *principal vector linkages* also applies for mass asymmetry [12] and the presented techniques do not modify links' mass distribution. This applies in particular when additional machine elements are not introduced. As stated in section 3, constraining PVL without modifying the total mass does not affect the force balance conditions. On the other hand, links' position can be modified within PVL even if these include mass-asymmetric links. It was stated that when links are moved, then the parallelograms' sizes are changed, the force balance conditions need to be adjusted and result in different values of the *principal dimensions*. However, the equations related to links' mass asymmetry are not modified. As Chapter 4 will show, the distance between links' CoM and the line connecting the joints needs to be balanced separately from the links' lengths and the *principal dimensions*.
On the other hand, when additional elements are introduced, the eventual adjustments of the force balance conditions depend on their mass distribution as well. In the previous sections, all the additional elements were assumed having their CoM on the geometric center, therefore to be mass-symmetric. This assumption was made because this kind of elements is more common in reality and force balance conditions are simple to adjust: only the *principal dimensions* result to be modified, regardless of whether the initial linkage includes mass-symmetric or mass-asymmetric links. However, when additional elements are introduced to replace mass-asymmetric links, further adjustments in the force balance conditions are required, especially in the equation regarding the mass asymmetry of the links parallel to the replaced one. As the contribution of the latter has been removed, the CoM of each remaining link will result to be placed at a different distance from the line connecting the joints. Chapter 4 will explain in detail how the force balance conditions change in this case.
Moreover, it is possible to introduce mass-asymmetric machine elements, i.e. elements whose CoM is placed on any point but the geometric center. This introduction will generally affect the force balance conditions concerning the mass asymmetry of the links, regardless of whether these are mass-symmetric or mass-asymmetric. In particular, depending on the type of the introduced elements and on how these are connected to the links, different scenarios can occur. Chapter 4 will present the different scenarios and how the additional elements can be included in the force balance conditions. These can result to be significantly complex to adjust, depending on the specific scenarios.

All the presented techniques can be applied to closed chains *principal vector linkages* as their design is similar to the one of the open chains. Since they include parallelogram linkages, the techniques presented in sections 4 and 5.1 can be applied. In addition, links rotating around the common CoM can be replaced by sliders as shown in section 5.2. Closed chain PVL can be constrained as well, even though the smaller number of DoFs constitutes a limitation in the types of constraints that can be introduced.

8 Conclusions

After a brief introduction of the *principal vector linkages*, this article presented different techniques which can be used to synthesize inherently force balanced mechanisms. They modify *principal vector linkages* while maintaining their overall kinematics based on parallelogram linkages.

The overall motions, therefore the number of DoFs, can be modified by constraining links' motions. Two techniques constrain, respectively, the links' absolute translations and rotations. The former are constrained by pivoting links to the base, while the latter are constrained by making links translate along straight sliding guides on the base. Both these techniques reduce the number of DoFs by two. Two other techniques reduce 1 DoF and consist in introducing sliders between the base and the links. One consists in pivoting the slider to the base and connecting it to a link through its sliding pair; the other consists in connecting the slider to a link through a revolute joint and to the base through its sliding pair. In both the techniques, the constrained links are allowed to rotate around the revolute joints of the sliders and to translate along their sliding pairs. Furthermore, a technique reducing 3 DoF consists in fixing links to the base in order to constrain both their absolute translations and rotations. In addition, a technique based on fixing angles between links reduces 1 DoF and makes them rotate by the same amount. Another technique defines relations between links' rotations by introducing gear, belt and chain drives. When a drive is introduced, the rotating elements are fixed to the links and 1 DoF is reduced.

Other techniques modify the links of *principal vector linkages*. While a technique modifies their positions and changes the sizes of the parallelograms, two techniques replace links with machine elements. One considers gear, belt and chain drives whose elements are fixed to parallel links: parallelism is maintained by setting the transmission ratio equal to 1. The other technique consists in introducing sliders to replace links which are pivoted to the base.

It was also explained how different techniques can be combined in the same *principal vector linkage* in order to create more specific solutions in terms of overall motions and design. A 2 DoF inherently force balanced mechanism was presented. It was synthesized by applying several techniques in order to reduce the number of DoFs and modify the design of the initial *principal vector linkage*: specific motions were created by pivoting a link while other links were moved and replaced by gear and chain drives. A simulation was performed in order to prove that the mechanism was still force balanced.

The different modifications that can be made to *principal vector linkages* suggest that they could be adapted to several applications. Potential limitations related to the original design of *principal vector linkages*, like the use of only links and the parallelograms' singularities that can occur, can be overcome by applying the presented techniques. The new designs that can be synthesized significantly differ from those of the original linkages and can be adapted to more specific motions and space requirements.

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4

The influence of mass asymmetry

Principal vector linkages can consist of mass-asymmetric links. These are bodies whose CoM is not placed on the line connecting the revolute joints because of the mass distribution. As a consequence, the force balance conditions consider links' mass asymmetry as well. These conditions are presented in this chapter together with their eventual modifications due to the relocation and replacement of links within *principal vector linkages.* Similarly, mass-asymmetric elements such gears, pulleys, sprockets and sliders can be introduced in *principal vector linkages.* This leads to modifications of the force balance conditions. Depending on how they are connected and move with respect to the other links, the force balance conditions can be potentially complex to modify. When the elements' motions can be related with respect to the other links', new force balance conditions can be defined.

1. Introduction

The techniques to synthesize mechanisms were presented in chapter 3 on *principal vector linkages* having mass-symmetric links. It was stated that techniques not including additional machine elements can be applied to *principal vector linkages* having mass-asymmetric links as well. In particular, when constraints are introduced without modifying the total mass, the force balance conditions do not change. It was also stated that, when parallelograms' sizes are modified, the force balance conditions change but no modification is made in those equations regarding links' mass asymmetry. These equations are modified when links are replaced by other machine elements, which can be mass-asymmetric as well. In this case, the force balance conditions are therefore modified, including those equations regarding links' mass asymmetry.
This chapter presents an overview of how force balance conditions change when techniques are applied to *principal vector linkages.* Mass-asymmetric links and machine elements are considered in order to show in detail how the mass distribution of the different components can affect the force balance conditions. After having presented in section 2 a linkage on which the different techniques will be applied, section 3 shows how force balance conditions change in both the cases in which parallelograms are resized and links are replaced by machine elements. These, in particular, are considered mass-symmetric in this section in order to focus only on the effects of removing mass-asymmetric links. Section 4 presents how mass-asymmetric machine elements can be included in the force balance conditions. The results which are observed in this section clearly show how force balance conditions are modified when mass-asymmetric links are substituted by mass-asymmetric machine elements.

2. Principal Vector Linkage

Figure 4.1 shows the 3 DoF *principal vector linkage* on which the different techniques will be applied. Both the *principal elements* A_0A_1, A_1A_2 and A_2A_3 and the *principal vector links* are mass-asymmetric. With respect to points S, B_1, B_2, P_1, P_2 and P_3, the position of each CoM is described by a p coordinate along the line connecting the joints and a q coordinate, which is perpendicular. The latter, in particular, describes in each link the distance between the line connecting the revolute joints and the CoM itself and it is characteristic of mass-asymmetric links. It is clear that, the q coordinate of a mass-symmetric link is zero. The lengths of the *principal vector links* are the *principal dimensions* a_x which are calculated from the force balance conditions

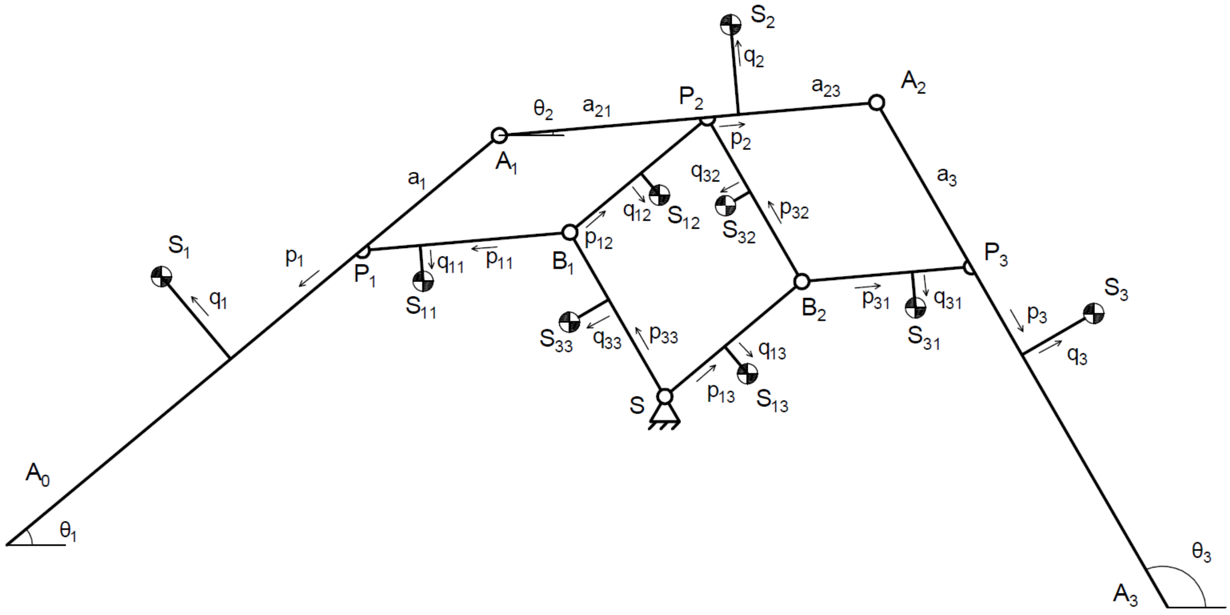

Figure 4.1: 3 DoF Principal Vector Linkage. Mass-asymmetric links.

shown in Equations 4.1 to 4.6. These are derived from the time derivative of the linear momentum equations by actuating each DoF, θ_1, θ_2 and θ_3, separately. The common CoM is placed in point S which is pivoted to the base.

$$m_1 p_1 = (m_2 + m_3 + m_{31} + m_{32}) a_1 + m_{12} p_{12} + m_{13} p_{13} \tag{4.1}$$
$$m_1 q_1 = m_{12} q_{12} + m_{13} q_{13} \tag{4.2}$$
$$m_2 p_2 + m_3 a_{23} + m_{31} p_{31} = m_1 a_{21} + m_{11} p_{11} \tag{4.3}$$
$$m_2 q_2 = m_{11} q_{11} + m_{31} q_{31} \tag{4.4}$$
$$m_3 p_3 = (m_1 + m_2 + m_{11} + m_{12}) a_3 + m_{32} p_{32} + m_{33} p_{33} \tag{4.5}$$
$$m_3 q_3 = m_{32} q_{32} + m_{33} q_{33} \tag{4.6}$$

By looking at the equations, it is possible to notice that those presenting the *principal dimensions* include only the p coordinates, while the q ones are included in different equations. These consider only the contributions of parallel links which rotate together when their angle is actuated. This implies that, when the links' masses are fixed parameters, the mass asymmetry, referred as the distance between the CoM and the line connecting the joints in each link, is balanced separately: the values of the *principal dimensions* and the p coordinates do not depend on links' mass asymmetry.

3. Techniques application

As previously stated, techniques modifying the number of DoFs do not affect the force balance conditions, unless additional elements are introduced. Therefore, this section analyzes techniques modifying the linkages' design by moving links, then changing the parallelograms' sizes, and substituting them with machine elements, which are assumed mass-symmetric in this section.

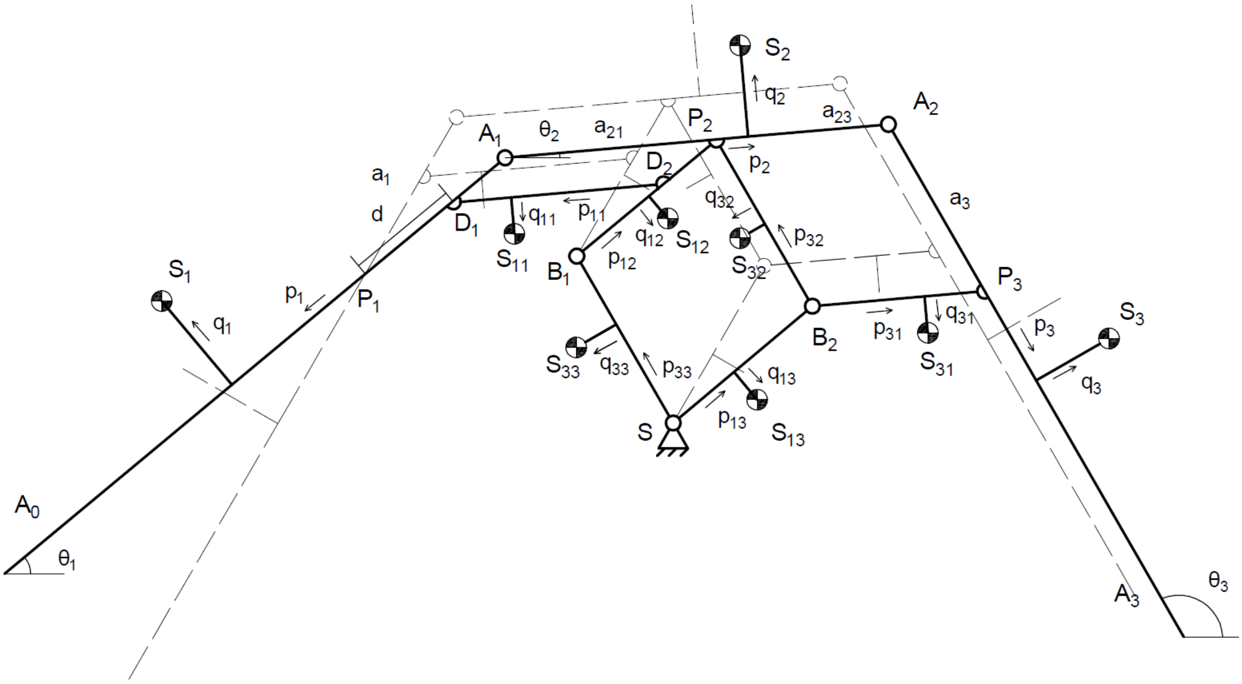

Figure 4.2: 3 DoF Principal Vector Linkage. D_1D_2 created by moving B_1P_1 by d. Dashed lines: θ_1 actuated.

3.1. Moving links

Figure 4.2 shows the 3 DoF *principal vector linkage* of Figure 4.1 in which B_1P_1 has been moved by a distance d, generating link D_1D_2. The force balance conditions are presented in Equations 4.7 to 4.12.

$$m_1p_1 = (m_2 + m_3 + m_{31} + m_{32})a_1 + m_{12}p_{12} + m_{13}p_{13} + m_{11}d \tag{4.7}$$

$$m_1q_1 = m_{12}q_{12} + m_{13}q_{13} \tag{4.8}$$

$$m_2p_2 + m_3a_{23} + m_{31}p_{31} = m_1a_{21} + m_{11}p_{11} \tag{4.9}$$

$$m_2q_2 = m_{11}q_{11} + m_{31}q_{31} \tag{4.10}$$

$$m_3p_3 = (m_1 + m_2 + m_{11} + m_{12})a_3 + m_{32}p_{32} + m_{33}p_{33} \tag{4.11}$$

$$m_3q_3 = m_{32}q_{32} + m_{33}q_{33} \tag{4.12}$$

By comparing these equations with those regarding the original linkage, it can be noticed that the only difference consists in Equation 4.7: this involves mass m_{11}, multiplied by the distance d of link D_1D_2 from its original position. This because, when θ_1 is actuated, this link translates, and therefore its mass needs to be considered in the linear momentum. Since a *principal vector link* has been simply moved without modifying its mass properties, i.e. its CoM coordinates and the mass value, the force balance conditions regarding the mass asymmetry are not modified. For the same reason, Equation 4.9, in which m_{11} and p_{11} are involved, has not been changed as well. As described in chapter 3, the parallelogram's kinematic properties are still maintained as well as force balance.

As the force balance conditions have been modified, the results derived from Equations 4.7 to 4.12 can be different from those derived from Equations 4.1 to 4.6. In particular, when the equations are decoupled, therefore each unknown parameter is involved in only one equation, the only difference consists in the different values calculated from Equations 4.1 and 4.7.

3.2. Substituting links

In the linkage shown in Figure 4.3, a belt drive has been introduced to replace link B_1P_1: the pulleys, each one having mass m_p, have been pivoted on A_1 and P_2 and fixed to A_0A_1 and B_1P_2, respectively. In addition, a slider, having mass m_s, has been pivoted on B_2 to replace link SB_2: it performs a circular trajectory having radius equal to the length of the replaced link.

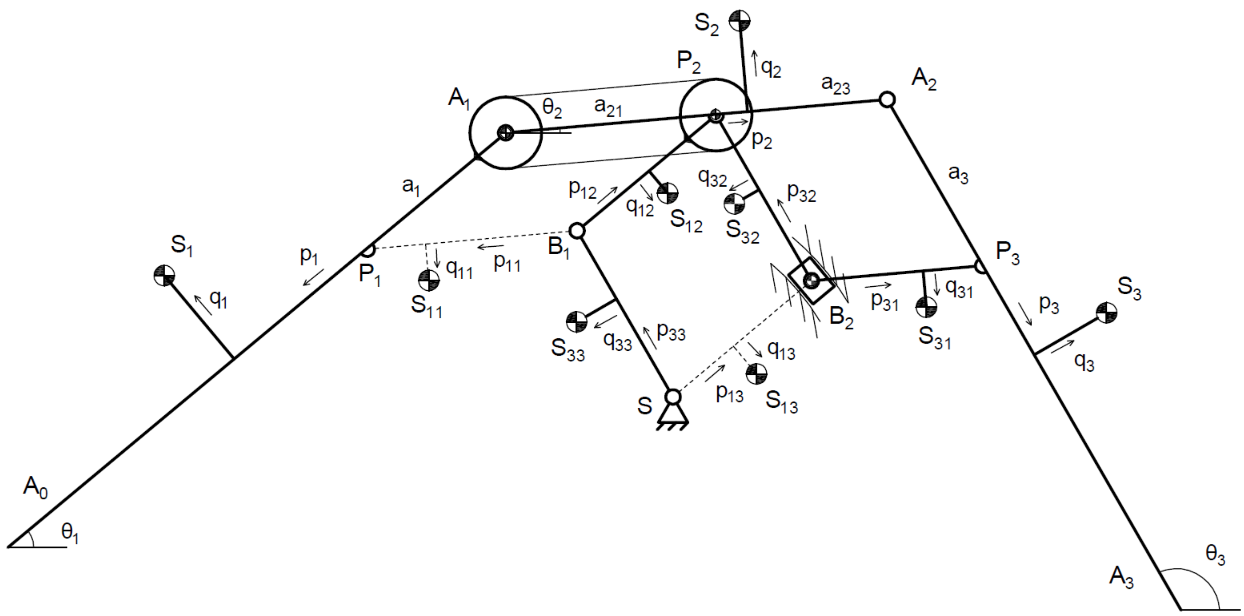

Figure 4.3: 3 DoF Principal Vector Linkage. B_1P_1 replaced by belt drive on A_1P_2. SB_2 replaced by slider pivoted on B_2 rotating around S.

$$m_1p_1 = (m_2 + m_3 + m_{31} + m_{32} + 2m_p + m_s)a_1 + m_{12}p_{12} \tag{4.13}$$

$$m_1q_1 = m_{12}q_{12} \tag{4.14}$$

$$m_2p_2 + m_3a_{23} + m_{31}p_{31} = (m_1 + m_p)a_{21} \tag{4.15}$$

$$m_2q_2 = m_{31}q_{31} \tag{4.16}$$

$$m_3p_3 = (m_1 + m_2 + m_{12} + 2m_p)a_3 + m_{32}p_{32} + m_{33}p_{33} \tag{4.17}$$

$$m_3q_3 = m_{32}q_{32} + m_{33}q_{33} \tag{4.18}$$

Equations 4.13 to 4.18 represent the force balance conditions. By comparing them with those of the original linkage, it can be noticed that different equations have been modified because the elements constituting the linkage have been changed. The terms related to the removed links have been replaced by those related to the additional elements: $m_{11}p_{11}$ in Equation 4.3 has been replaced by m_pa_{21} in Equation 4.15, while $m_{13}p_{13}$ in Equation 4.1 has been replaced by m_sa_1 in Equation 4.13. In addition, the terms related to the mass asymmetry of the replaced links have been removed as well. However, they have not been replaced by any term of the machine elements because the pulleys do not rotate together with the links parallel to B_1P_1 and the slider, although it rotates like the replaced link, is mass-symmetric.
The results derived from the force balance conditions can be significantly different from those of the initial linkage. Several equations have been modified, including those regarding the mass asymmetry of the links parallel to the replaced ones. In particular, these equations will modify either the links' mass values (m_x) or the distances between the CoMs and the lines connecting the joints, i.e. the q coordinates. Adjustments regarding links' mass properties, including mass distribution, could be necessary.

4. Introducing mass-asymmetric machine elements

This section describes how introducing mass-asymmetric machine elements affects the force balance conditions in *principal vector linkages*, especially the equations regarding links' mass asymmetry. Different scenarios will be presented, each one showing a specific type of elements, either drives or sliders, which are connected to the links in different ways. The drives' rotating elements can be either fixed or pivoted to links, while sliders can be either pivoted or connected through their sliding pairs to the links.

4.1. Mass-asymmetric rotating elements

When additional mass-asymmetric elements are introduced, it is important to describe their CoM with respect to a point of the linkage in order to include them in the force balance conditions. These, in particular, need to be maintained during any motion of the linkage. Figure 4.4 shows a mass-asymmetric rotating ele-

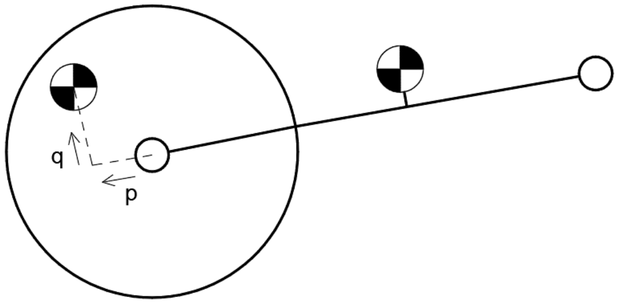

Figure 4.4: Mass-asymmetric rotating elements pivoted to a link. p and q coordinates describe the CoM position with respect to the revolute joint.

ments, representing either a gear, pulley or sprocket, which is pivoted to a link. Its CoM is placed at a certain distance from the revolute joint. Its position can be described by the same kind of coordinates describing the CoM of each link. The p coordinate describes the distance between the CoM and the pivot along the line connecting the link's joints, while the q coordinate describes the distance between the CoM and this line.
It is clear that, whether the rotating elements are fixed or simply pivoted to links, the coordinates can be either constant or varying. Therefore, the force balance conditions can be complex to define. The cases in which the elements are either fixed or simply pivoted are presented separately.

Rotating elements fixed to links
As drives' rotating elements rotate together with the links to which they are fixed, both p and q coordinates describing the CoM are constant. In particular, their values can be defined when the elements are added to the linkages. It is possible to set one of the coordinates equal to zero as well. Figure 4.5 shows the cases in

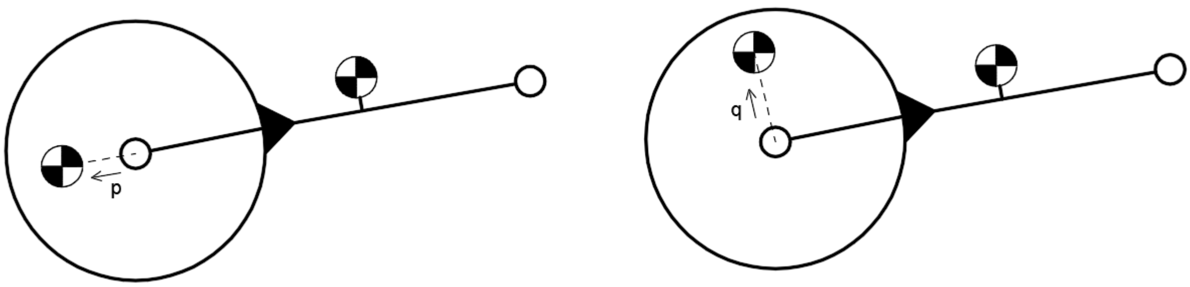

Figure 4.5: Mass-asymmetric rotating elements fixed to a link. On the left, q coordinate equal to zero. On the right, p coordinate equal to zero.

which one of the coordinates is set equal to zero. Particular attention needs to be paid when the q one is zero: in this case, the introduction of a mass-asymmetric rotating element does not affect the force balance conditions regarding the mass asymmetry: its CoM is placed along the line connecting the link's joints. On the other hand, when the p coordinate is set equal to zero, the element's introduction still affects the equations regarding the *principal dimension*, as an additional element is introduced. The following example will show how the force balance conditions change when rotating elements having these different mass distributions are introduced.

Figure 4.6 shows a resulting 2 DoF *principal vector linkage* in which two drives have been introduced. In order to substitute B_1P_1, the pulleys of a belt drive are pivoted on A_1 and P_2 and fixed to A_0A_1 and B_1P_2, respectively. On the other hand, two meshed gears are pivoted on B_2 and P_3 and fixed to SB_2 and A_2A_3, respectively, to constrain their rotations. A chain drive can be introduced instead of the belt one, as the

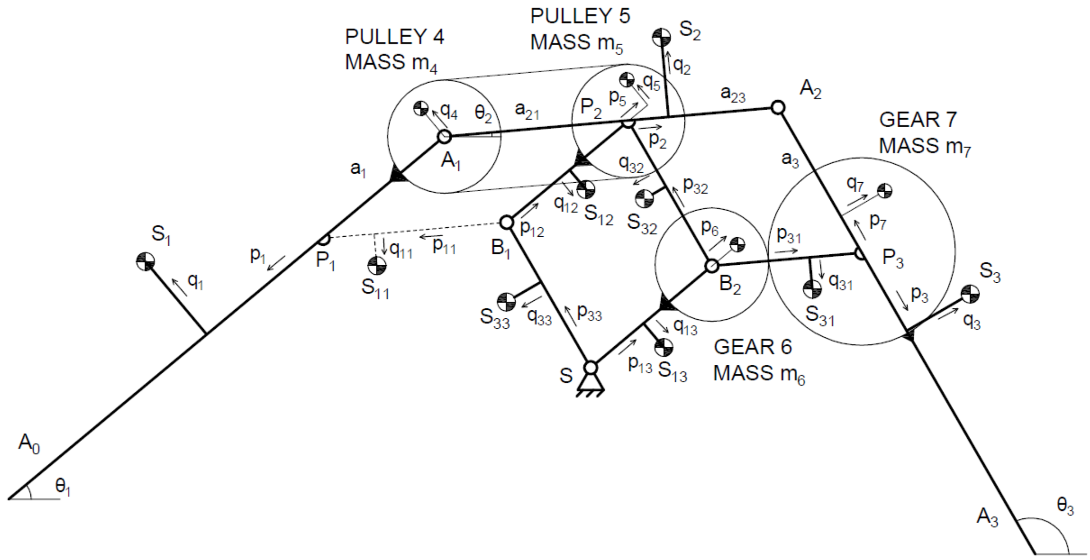

Figure 4.6: Resulting 2 DoF *principal vector linkage*. Belt drive on A_1P_2 replacing B_1P_1. Gear drive on B_2P_3 relating SB_2 and A_2A_3 rotations. Mass-asymmetric rotating elements.

motion of their rotating elements is similar.

$$m_1p_1 = (m_2 + m_3 + m_{31} + m_{32} + m_4 + m_7)a1 + m_{12}p_{12} + m_{13}p_{13} + m_5(a_1 + p_5) + m_6(a_1 + p_6) \tag{4.19}$$

$$m_1q_1 + m_4q_4 + m_5q_5 = m_{12}q_{12} + m_{13}q_{13} \tag{4.20}$$

$$m_2p_2 + (m_3 + m_7)a_{23} + m_{31}p_{31} = (m_1 + m_4)a_{21} \tag{4.21}$$

$$m_2q_2 = m_{31}q_{31} \tag{4.22}$$

$$m_3p_3 = (m_1 + m_2 + m_{12} + m_4 + m_5)a_3 + m_{32}p_{32} + m_{33}p_{33} + m_7p_7 \tag{4.23}$$

$$m_3q_3 + m_7q_7 = m_{32}q_{32} + m_{33}q_{33} \tag{4.24}$$

The force balance conditions, corresponding to Equations 4.19 to 4.24, are valid for all the motion of the resulting linkage.
By looking at both the equations and the linkage, different observations can be made. Although the CoM of pulley 4 has its p coordinate equal to zero, its mass m_4 is still included in Equation 4.19 and multiplied by a_1. Both the pulleys are included in Equation 4.20 regarding the mass asymmetry of links parallel to A_0A_1. However, gear 6, which rotates together with these links, is not included in this equation as its CoM is placed along the line connecting the joints: q_6 is equal to zero. On the other hand, gear 7 rotates together with A_2A_3 and, since its CoM has both p and q coordinates different than zero, it is included in Equations 4.23 and 4.24. The contributions of the replaced link B_1P_1 have been removed as previously described.

Rotating elements pivoted to links

Compared to the previous case, the definition of the force balance conditions results more complex when drives' rotating elements are only pivoted to links. This situation occurs when gear trains are introduced in *principal vector linkages*. The CoM of each gear moves with respect to the link on which they are pivoted, and therefore the p and q coordinates vary. Figure 4.7 shows an example of gear train that can be introduced. The link on which the gears are pivoted has an angle θ_0 and rotates with an angular velocity ω_0. All the gears have a diameter D_x and an angular velocity ω_x. For simplicity, the angular velocities are assumed to be constant. These are all described with respect to the absolute reference system XY placed on point O, on the center of gear 1. The outer gears, having diameters D_1 and D_6, are fixed to other links and can be included in the force balance conditions as described in the previous example. Therefore, only the masses of the pivoted gears are considered in this study. The angular velocities of these gears can be defined in the same form presented in

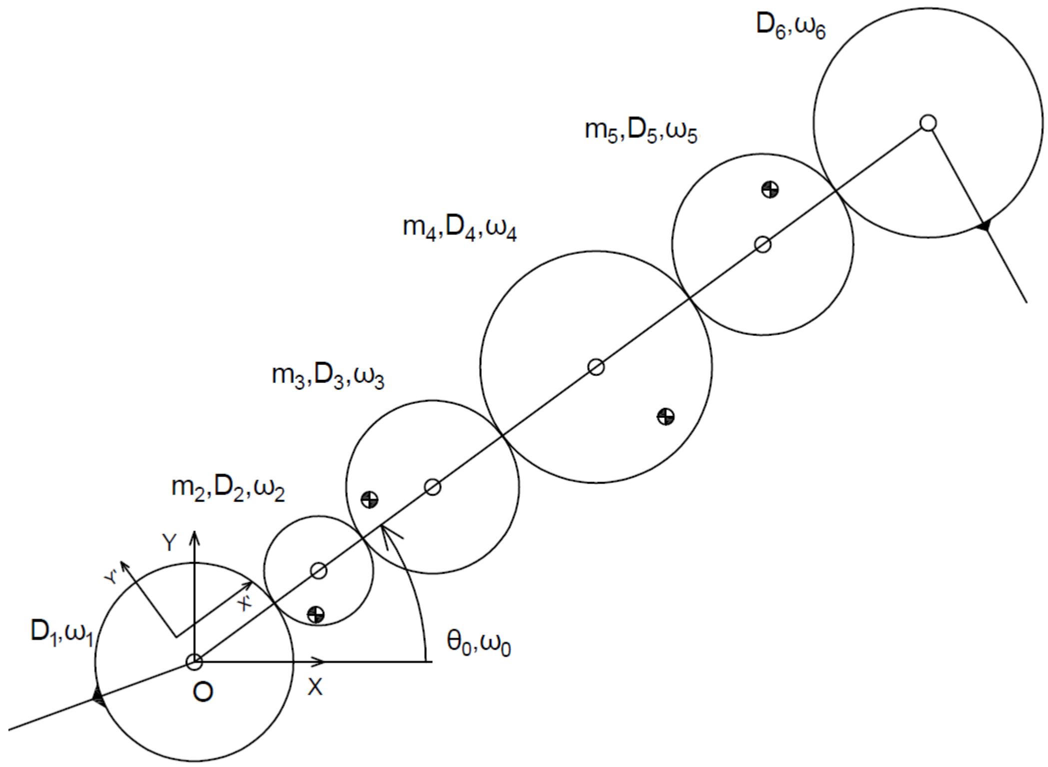

Figure 4.7: Mass-asymmetric gear train introduced on a link. Absolute reference system.

chapter 3. They are presented in Equations 4.26 to 4.29.

$$\omega_2 = -\frac{D_1}{D_2}\omega_1 + \left(1 + \frac{D_1}{D_2}\right)\omega_0 \tag{4.25}$$

$$\omega_3 = \frac{D_1}{D_3}\omega_1 + \left(1 - \frac{D_1}{D_3}\right)\omega_0 \tag{4.26}$$

$$\omega_4 = -\frac{D_1}{D_4}\omega_1 + \left(1 + \frac{D_1}{D_4}\right)\omega_0 \tag{4.27}$$

$$\omega_5 = \frac{D_1}{D_5}\omega_1 + \left(1 - \frac{D_1}{D_5}\right)\omega_0 \tag{4.28}$$

Each gear's CoM rotates around the respective revolute joint. Therefore, the p and q coordinates change during the motion and can not be included in the force balance conditions.
However, it can be possible to define a common CoM which considers only the pivoted gears. By setting its position invariant with respect to the link, it can be included in the force balance conditions. For this purpose, a relative reference system X'Y' has been defined: it is placed on O and rotates together with the link. Each parameter that will be presented is defined with respect to this reference system. Figure 4.8 shows the gear train with respect to the relative reference system X'Y', in which X' direction corresponds to the link's axis. Each element rotates with an angular velocity ω_{xrel} which is different from the respective one which was previously defined. With respect to the link, which rotate with an angular velocity ω_0, the relative angular velocities of the gears are obtained by subtracting the one of the reference system, ω_0, to the absolute ones

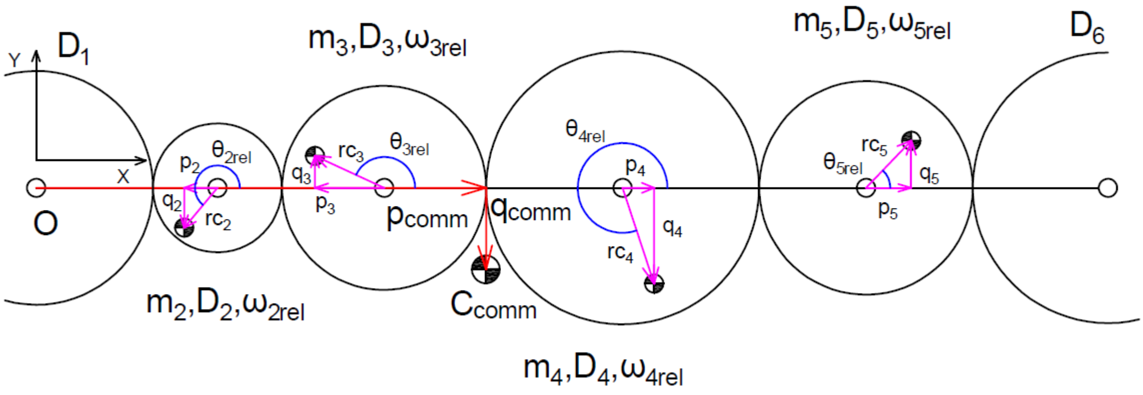

Figure 4.8: Mass-asymmetric gear train of Figure 4.7. Reference system relative to the link.

defined in Equations 4.26 to 4.29. The results are shown in Equations 4.30 to 4.33.

$$\omega_{2rel} = \frac{D_1}{D_2}(\omega_0 - \omega_1) \tag{4.29}$$

$$\omega_{3rel} = \frac{D_1}{D_3}(\omega_1 - \omega_0) \tag{4.30}$$

$$\omega_{4rel} = \frac{D_1}{D_4}(\omega_0 - \omega_1) \tag{4.31}$$

$$\omega_{5rel} = \frac{D_1}{D_5}(\omega_1 - \omega_0) \tag{4.32}$$

Moreover, in each gear, the position of the CoM can be defined by a vector rc_x with respect to the revolute joint. The angle between this vector and the link is defined as θ_{xrel}. It is important to notice that this angle changes during the motion. For each vector rc_x, a p_x and a q_x coordinate can be defined, respectively, along and perpendicularly to the link.

$$p_x = rc_x \cos\theta_{xrel} \qquad q_x = rc_x \sin\theta_{xrel} \qquad x = 2,3,4,5 \tag{4.33}$$

With respect to the origin of the reference system O, the position of each gear's CoM is defined with q_x along Y', and with a vector pos_x along X'. The latter consists in the summation of the vector describing the position of each revolute joint plus p_x, as shown in Equations 4.35 to 4.38.

$$pos_2 = \frac{D_1 + D_2}{2} + p_2 \tag{4.34}$$

$$pos_3 = \frac{D_1 + 2D_2 + D_3}{2} + p_3 \tag{4.35}$$

$$pos_4 = \frac{D_1 + 2D_2 + 2D_3 + D_4}{2} + p_4 \tag{4.36}$$

$$pos_5 = \frac{D_1 + 2D_2 + 2D_3 + 2D_4 + D_5}{2} + p_5 \tag{4.37}$$

The position of the common CoM C_{comm} can be defined with a p_{comm} and a q_{comm} coordinates along X' and Y', respectively. In order to be included in the force balance conditions, they have to be constant. Equations 4.39 and 4.40 show how these coordinates depend on the position of each gear's CoM.

$$p_{comm} = \frac{m_2 pos_2 + m_3 pos_3 + m_4 pos_4 + m_5 pos_5}{m_2 + m_3 + m_4 + m_5} = constant \tag{4.38}$$

$$q_{comm} = \frac{m_2 q_2 + m_3 q_3 + m_4 q_4 + m_5 q_5}{m_2 + m_3 + m_4 + m_5} = constant \tag{4.39}$$

In order to keep these coordinates constant, their time derivatives are set equal to zero. As the only time dependent parameters are the angles θ_{xrel} within the p_x and q_x coordinates, the time derivatives of p_{comm}

and q_{comm} are defined and set equal to zero.

$$\frac{dp_{comm}}{dt} = \frac{1}{m_2+m_3+m_4+m_5}\left(m_2\frac{dp_2}{dt}+m_3\frac{dp_3}{dt}+m_4\frac{dp_4}{dt}+m_5\frac{dp_5}{dt}\right)=0 \tag{4.40}$$

$$\frac{dq_{comm}}{dt} = \frac{1}{m_2+m_3+m_4+m_5}\left(m_2\frac{dq_2}{dt}+m_3\frac{dq_3}{dt}+m_4\frac{dq_4}{dt}+m_5\frac{dq_5}{dt}\right)=0 \tag{4.41}$$

A solution for Equations 4.41 and 4.42 is found when the parentheses are set equal to zero. In addition, time derivatives of p_x and q_x can be defined as Equations 4.43 and 4.44 show.

$$\frac{dp_x}{dt} = -rc_x\sin\theta_{xrel}\frac{d\theta_{xrel}}{dt} \qquad x=2,3,4,5 \tag{4.42}$$

$$\frac{dq_x}{dt} = rc_x\cos\theta_{xrel}\frac{d\theta_{xrel}}{dt} \qquad x=2,3,4,5 \tag{4.43}$$

As the time derivative of θ_{xrel} is ω_{xrel}, it can be replaced by Equations 4.30 to 4.33 for any value of x. Therefore, Equations 4.43 and 4.44 can be included in 4.41 and 4.42. Equations 4.45 and 4.46 correspond to the parentheses set equal to zero.

$$\begin{aligned}-m_2rc_2\sin\theta_{2rel}\frac{D_1}{D_2}(\omega_0-\omega_1)-m_3rc_3\sin\theta_{3rel}\frac{D_1}{D_3}(\omega_1-\omega_0)-m_4rc_4\sin\theta_{4rel}\frac{D_1}{D_4}(\omega_0-\omega_1)+\\-m_5rc_5\sin\theta_{5rel}\frac{D_1}{D_5}(\omega_1-\omega_0)=0\end{aligned} \tag{4.44}$$

$$\begin{aligned}m_2rc_2\cos\theta_{2rel}\frac{D_1}{D_2}(\omega_0-\omega_1)+m_3rc_3\cos\theta_{3rel}\frac{D_1}{D_3}(\omega_1-\omega_0)+m_4rc_4\cos\theta_{4rel}\frac{D_1}{D_4}(\omega_0-\omega_1)+\\+m_5rc_5\cos\theta_{5rel}\frac{D_1}{D_5}(\omega_1-\omega_0)=0\end{aligned} \tag{4.45}$$

These equations can be written as shown in 4.47 and 4.48.

$$D_1(\omega_0-\omega_1)\left[-m_2\frac{rc_2}{D_2}\sin\theta_{2rel}+m_3\frac{rc_3}{D_3}\sin\theta_{3rel}-m_4\frac{rc_4}{D_4}\sin\theta_{rel4}+m_5\frac{rc_5}{D_5}\sin\theta_{5rel}\right]=0 \tag{4.46}$$

$$D_1(\omega_0-\omega_1)\left[m_2\frac{rc_2}{D_2}\cos\theta_{2rel}-m_3\frac{rc_3}{D_3}\cos\theta_{3rel}+m_4\frac{rc_4}{D_4}\cos\theta_{rel4}-m_5\frac{rc_5}{D_5}\cos\theta_{5rel}\right]=0 \tag{4.47}$$

By setting the square parentheses equal to zero, conditions to keep the common CoM in an invariant position are defined. When ω_0 is equal to ω_1, therefore gear 1 and the link rotate with the same velocity, the pivoted gears do not rotate relatively to the link itself. In addition, a solution consisting in D_1 equal to zero does not make sense in practice.

It is clear that the number of terms in the parentheses depends on the number of pivoted gears. Moreover, it can be noticed that terms in sine and cosine change during time in different ways. In each gear, θ_{xrel} describes the angle between the rc_x vector and the link. As each gear rotates, θ_{xrel} values are different in each time instant. However, it is possible to define this angle with respect to the time.

$$\theta_{xrel}(t) = \omega_{xrel}t+\theta_{xrel,in} \qquad x=2,3,4,5 \tag{4.48}$$

By having assumed ω_{xrel} constant, Equation 4.49 shows that each angle is equal to its angular velocity multiplied by the time, plus an initial value $\theta_{xrel,in}$. This, in particular, can be the angle between the rc_x vector and the link when each gear was assembled.

Therefore, Equations 4.47 and 4.48 include constant parameters which are multiplied by sines and cosines of angles changing in time with different velocities. These velocities depend on the diameter of each gear and their directions depend on gears' position within the train, as these are counter-rotating elements. Solutions can theoretically be achieved if each term is balanced by another in each instant of time in both the equations. For this purpose, a considerable number of gears is supposed to be required and, on the other hand, solutions can be hard to achieved when, for example, only four gears are pivoted to a link.

Nevertheless, if the angular velocities of gears rotating in the same direction are assumed equal, as well as their diameter, a solution can be obtained. In gears rotating in the same direction, their CoMs would, therefore, rotate with the same velocity. As a result, solutions would depend on the constant parameters m_x, rc_x and $\theta_{xrel,in}$.

A formulation can be obtained by substituting Equation 4.49 to each respective term in Equations 4.47 and 4.48. By using trigonometric addition formulas, each term in parentheses can be written as shown in Equations 4.50 and 4.51.

$$m_x \frac{rc_x}{D_x} \sin\theta_{xrel} = m_x \frac{rc_x}{D_x} \left[\sin(\omega_{xrel} t)\cos\theta_{xrel,in} + \cos(\omega_{xrel} t)\sin\theta_{xrel,in}\right] \qquad x = 2,3,4 \tag{4.49}$$

$$m_x \frac{rc_x}{D_x} \cos\theta_{xrel} = m_x \frac{rc_x}{D_x} \left[\cos(\omega_{xrel} t)\cos\theta_{xrel,in} - \sin(\omega_{xrel} t)\sin\theta_{xrel,in}\right] \qquad x = 2,3,4 \tag{4.50}$$

With respect to Figure 4.8, by considering gears 2 and 4 having the same angular velocity ω_{2rel} and diameter D_2, and gears 3 and 5 having velocity ω_{3rel} and diameter D_3, the square parentheses in Equations 4.47 and 4.48 are written and set equal to zero in 4.52 and 4.53.

$$\begin{aligned}
&-m_2 \frac{rc_2}{D_2}\left[\sin(\omega_{2rel} t)\cos\theta_{2rel,in} + \cos(\omega_{2rel} t)\sin\theta_{2rel,in}\right] + \\
&+m_3 \frac{rc_3}{D_3}\left[\sin(\omega_{3rel} t)\cos\theta_{3rel,in} + \cos(\omega_{3rel} t)\sin\theta_{3rel,in}\right] + \\
&-m_4 \frac{rc_4}{D_2}\left[\sin(\omega_{2rel} t)\cos\theta_{4rel,in} + \cos(\omega_{2rel} t)\sin\theta_{4rel,in}\right] + \\
&+m_5 \frac{rc_5}{D_3}\left[\sin(\omega_{3rel} t)\cos\theta_{5rel,in} + \cos(\omega_{3rel} t)\sin\theta_{5rel,in}\right] = 0
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
&m_2 \frac{rc_2}{D_2}\left[\cos(\omega_{2rel} t)\cos\theta_{2rel,in} - \sin(\omega_{2rel} t)\sin\theta_{2rel,in}\right] + \\
&+m_3 \frac{rc_3}{D_3}\left[\cos(\omega_{3rel} t)\cos\theta_{3rel,in} - \sin(\omega_{3rel} t)\sin\theta_{3rel,in}\right] + \\
&+m_4 \frac{rc_4}{D_2}\left[\cos(\omega_{2rel} t)\cos\theta_{4rel,in} - \sin(\omega_{2rel} t)\sin\theta_{4rel,in}\right] + \\
&+m_5 \frac{rc_5}{D_3}\left[\cos(\omega_{3rel} t)\cos\theta_{5rel,in} - \sin(\omega_{3rel} t)\sin\theta_{5rel,in}\right] = 0
\end{aligned} \tag{4.52}$$

These equations can be written by considering together those terms having sine and cosine of the same angular velocities, as shown in Equations 4.54 and 4.55.

$$\begin{aligned}
&-\frac{\sin(\omega_{2rel} t)}{D_2}\left[m_2 rc_2 \cos\theta_{2rel,in} + m_4 rc_4 \cos\theta_{4rel,in}\right] - \frac{\cos(\omega_{2rel} t)}{D_2}\left[m_2 rc_2 \sin\theta_{2rel,in} + m_4 rc_4 \sin\theta_{4rel,in}\right] + \\
&+\frac{\sin(\omega_{3rel} t)}{D_3}\left[m_3 rc_3 \cos\theta_{3rel,in} + m_5 rc_5 \cos\theta_{5rel,in}\right] + \frac{\cos(\omega_{3rel} t)}{D_3}\left[m_3 rc_3 \sin\theta_{3rel,in} + m_5 rc_5 \sin\theta_{5rel,in}\right] = 0
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
&\frac{\cos(\omega_{2rel} t)}{D_2}\left[m_2 rc_2 \cos\theta_{2rel,in} + m_4 rc_4 \cos\theta_{4rel,in}\right] - \frac{\sin(\omega_{2rel} t)}{D_2}\left[m_2 rc_2 \sin\theta_{2rel,in} + m_4 rc_4 \sin\theta_{4rel,in}\right] + \\
&+\frac{\cos(\omega_{3rel} t)}{D_3}\left[m_3 rc_3 \cos\theta_{3rel,in} + m_5 rc_5 \cos\theta_{5rel,in}\right] - \frac{\sin(\omega_{3rel} t)}{D_3}\left[m_3 rc_3 \sin\theta_{3rel,in} + m_5 rc_5 \sin\theta_{5rel,in}\right] = 0
\end{aligned} \tag{4.54}$$

In both the equations, each square parenthesis contains constant terms. By setting each parenthesis equal to zero, the conditions for a common CoM having invariant position are defined. These, in particular, are valid during all the motions. It can be noticed that the conditions derived from Equation 4.54 are the same derived from Equation 4.55.

It is clear that a solution can be made by assuming $\theta_{2rel,in}$ different from $\theta_{4rel,in}$ by an angle of 180°, as well as $\theta_{3rel,in}$ from $\theta_{5rel,in}$. The resulting conditions are presented in Equations 4.56 and 4.57.

$$m_2 rc_2 - m_4 rc_4 = 0 \tag{4.55}$$

$$m_3 rc_3 - m_5 rc_5 = 0 \tag{4.56}$$

In addition, Equations 4.54 and 4.55 clearly shows how defining a solution including the all terms is complex. The t parameter within the sine and cosine terms represents the time dependence: for any value of t, then for any instant of time, both the equations have to be zero. Moreover, although gears rotating in the same direction have been set having the same diameters and angular velocities, the sine and cosine terms include

different velocities which have opposite directions, as gears are counter-rotating elements.
The presented calculations have been made with respect to the gear train of Figures 4.7 and 4.8. It is clear that more gears can be included: their diameters shall be set equal to those of the gears rotating in the same direction in order to define equations similar to 4.54 and 4.55. On the other hand, it can be observed that a minimum number of mass-asymmetric pivoted gears is required in order to define conditions: at least two gears rotating in the same direction at the same velocity are required. Therefore, conditions can be defined with at least three pivoted gears, in which the one rotating oppositely to the others is mass-symmetric, or four, in which each gear rotating in the same direction has the same diameter.

4.2. Mass-asymmetric sliders

As stated in section 4.1, in order to include additional elements in the force balance conditions, it is important to describe their CoM with respect to a point of the linkage. The position of a mass-asymmetric slider's CoM can be described with a p and a q coordinate, as described in the previous sections. However, as Figure 4.9 shows, these coordinates can change during the motion. Nevertheless, depending on the trajectory

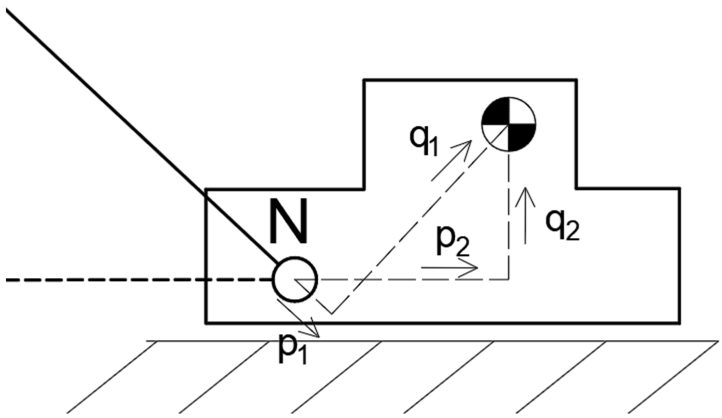

Figure 4.9: Mass-asymmetric slider. p and q coordinates describing CoM change during motion.

performed by the slider, the position of its CoM can be defined such that the element can be included in the force balance conditions. It can be noticed in Figure 4.9 that the slider could be considered as a mass-symmetric element if it was pivoted on its CoM. However, this case will not be discussed as it is not relevant for the purpose of this chapter, which consists in presenting how the force balance conditions change when mass-asymmetric links and elements are considered. The sliders described in chapter 3 are presented.

Sliders pivoted to the links
It was shown that these sliders can be introduced to either constrain or substituting links in *principal vector linkages*. Although the CoM's p and q coordinates can change, as shown in Figure 4.9, the elements can be easily included in the force balance conditions when these perform specific motions. In particular, the trajectories have to be either straight or circular. The latter are characteristic when sliders replace links.

Sliders having circular and straight trajectories. Figure 4.10 shows a resulting 2 DoF *principal vector linkage*, whose force balance conditions are presented in Equations 4.58 to 4.63. A slider, having its CoM on S_4 and mass m_4, has been introduced on B_2 to replace SB_2. It performs a circular trajectory around S having radius equal to SB_2. In addition, a slider, having its CoM on S_5 and mass m_5, has been pivoted on A_3 and performs a straight trajectory. Therefore, it is a non-rotating element and its CoM can be assumed to be placed on any point. Assumed its CoM is placed on A_3, the slider can be considered as a point mass on *principal element* A_2A_3 and its mass m_5 can be easily included in the force balanced conditions, especially in Equations

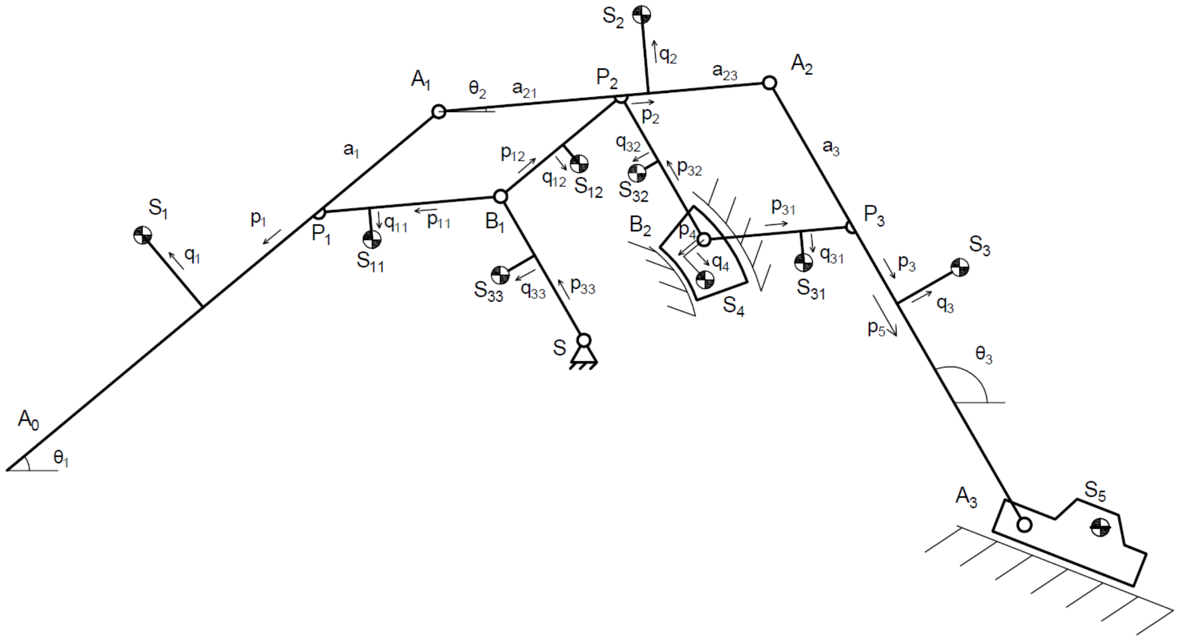

Figure 4.10: Resulting 2 DoF *principal vector linkage*. Slider pivoted on B_2 replacing SB_2. Slider pivoted on A_3 moves along straight trajectory.

4.58, 4.60 and 4.62.

$$m_1 p_1 = (m_2 + m_3 + m_5 + m_{31} + m_{32}) a_1 + m_{12} p_{12} + m_4 (a_1 - p_4) \tag{4.57}$$

$$m_1 q_1 = m_{12} q_{12} + m_4 q_4 \tag{4.58}$$

$$m_2 p_2 + (m_3 + m_5) a_{23} + m_{31} p_{31} = m_1 a_{21} + m_{11} p_{11} \tag{4.59}$$

$$m_2 q_2 = m_{11} q_{11} + m_{31} q_{31} \tag{4.60}$$

$$m_3 p_3 + m_5 p_5 = (m_1 + m_2 + m_{11} + m_{12}) a_3 + m_{32} p_{32} + m_{33} p_{33} \tag{4.61}$$

$$m_3 q_3 = m_{32} q_{32} + m_{33} q_{33} \tag{4.62}$$

On the other hand, slider pivoted on B_2 rotates together with the links parallel to the replaced one. Therefore, the position of its CoM S_4 can be described with respect to B_2 with p and q coordinates which are always parallel to the respective ones of links B_1P_2 and A_0A_1. As a consequence, slider's contributions can be easily included in the force balance conditions, as seen in Equations 4.58 and 4.59.

Sliders having random trajectories. On the other hand, sliders can perform random trajectories, which can be partially curved. Figure 4.11 shows a resulting 2 DoF *principal vector linkage* in which a slider has been pivoted on A_3 and performs a trajectory which is partially curved. In this case, the force balance conditions are complex to define, as the additional element can both translate and rotate. A single translation could be considered like in the previous example, but the resulting equations do not consider the slider's rotation. This suggests that it is impossible to modify the balance conditions of the initial *principal vector linkage* in order to include the slider's contribution during all the motions. In addition, slider's rotation can be different from the one of any other link: when this element rotates, the motion of its CoM is not considered by the common CoM S.

Since force balance conditions including the additional element can not be defined, it is possible to consider the slider as a *principal element*. Therefore, it can be connected to the other links through *principal vector links*, represented as dashed lines in Figure 4.11. The new common CoM S' considers the slider motions. Furthermore, new force balance conditions need to be defined. The length of all the links, even those of the initial linkage, needs to be calculated.

The resulting linkage can be considered as a 4 DoF *principal vector linkage* in which a *principal element* is constrained to move along a given trajectory, leading to a decrease of 2 DoF.

The proposed solution significantly changes the design of the initial *principal vector linkage*. This results to

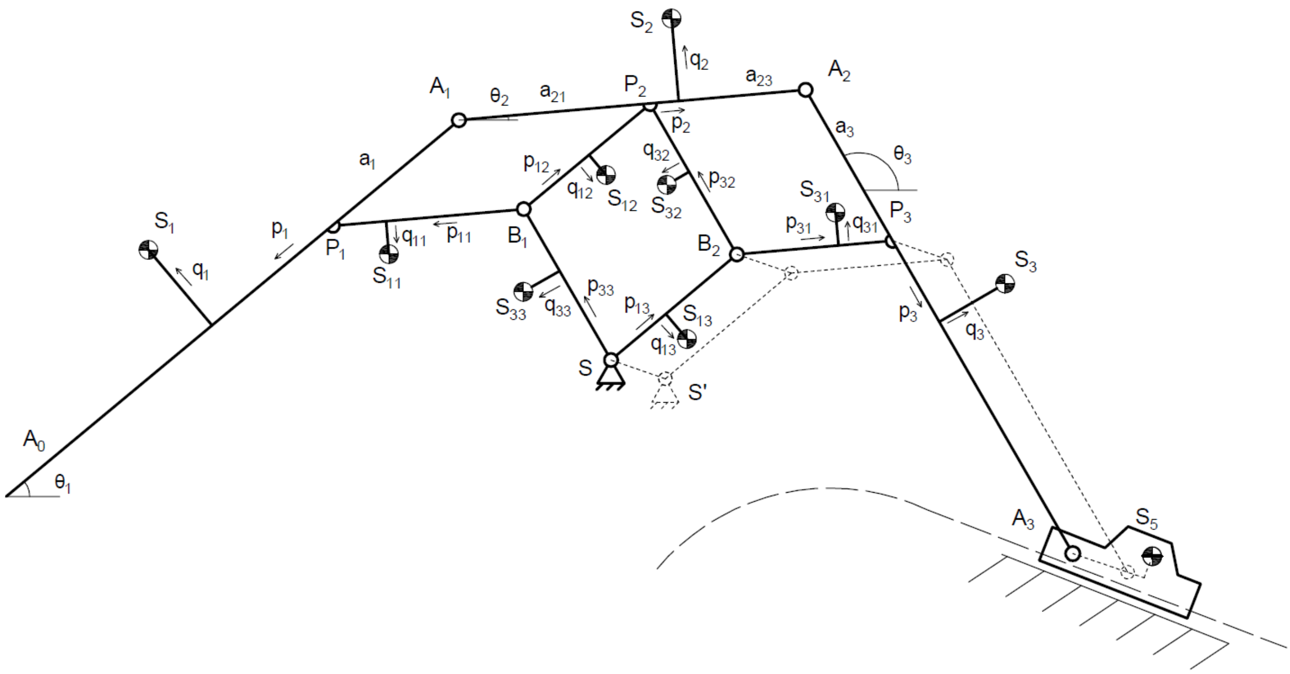

Figure 4.11: Resulting 2 DoF *principal vector linkage*. Slider pivoted on A_3 moves along random trajectory. Dashed lines: possible *principal vector links* connecting the slider to the new common CoM S'

be necessary as the motion of a mass-asymmetric element's CoM can not be related to those of the other links.

Sliders pivoted to the base

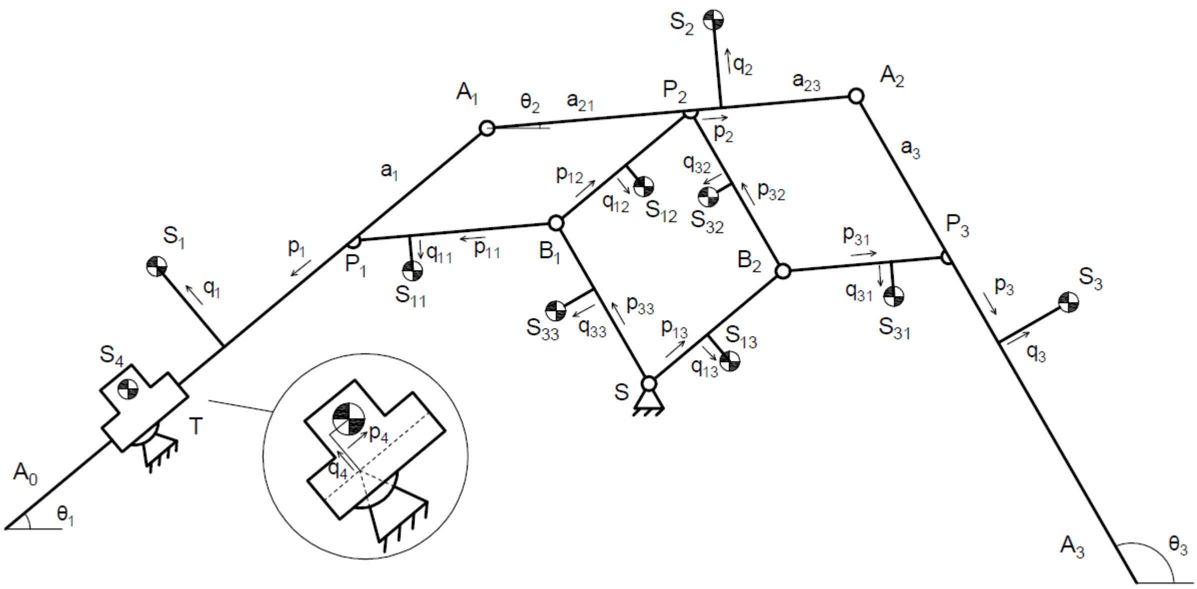

Figure 4.12: Resulting 2 DoF *principal vector linkage*. Slider pivoted to the base on T. A_0A_1 rotates around T and translates only along its orientation

Figure 4.12 shows a resulting 2 DoF *principal vector linkage* in which a slider, having mass m_4, has been pivoted to the base on point T. Its sliding pair is connected to *principal element* A_0A_1: this can therefore rotate around T and translate along its orientation. The slider's CoM S_4 is placed at a certain distance from the center of rotation: p_4 and q_4 coordinates describe this position. In particular, p_4 is defined as the coordinate parallel to A_0A_1, while q_4 is perpendicular. Since these coordinates are constant during all the linkage's

motions and the slider rotates together with a link, it is possible to include this element in the force balance conditions. However, particular attention has to be paid since the slider can move relatively to *principal element* A_0A_1: the distance between its CoM S_4 and *principal point* P_1 changes during the motion. Nevertheless, the slider can still be included in the force balance conditions.
In particular, it is an element which rotates with respect to the base by the same amount of A_0A_1 and its parallel links. This implies that, regardless of whether this element is placed, it can be considered in the linear momentum equation when its angle θ_1 is actuated. When this is the case, A_0A_1, B_1P_2 and SB_2 rotate around P_1, B_1 and S, respectively, which are fixed; by considering the slider, it rotates around the fixed pivot T. Although B_1 and P_1 are not fixed when the slider is introduced, the linear momentum equation has to consider only the contributions related to θ_1. As mentioned in chapter 3, when additional elements are introduced to constrain the linkage, the force balance conditions have to be defined by considering the linkage having the original number of DoFs. Therefore, regardless of where the slider is pivoted, it can be included in the force balance conditions by being considered as a link parallel to A_0A_1: its CoM's p_4 and q_4 coordinates correspond to the respective ones of the other links. Therefore, slider's mass multiplied by these coordinates is included in the force balance conditions, represented by Equations 4.64 to 4.69, especially in 4.64 and 4.65, regarding the actuation of θ_1. As the slider does not translate, it is not included in the other equations.

$$m_1p_1 = (m_2+m_3+m_{31}+m_{32})a_1 + m_{12}p_{12} + m_{13}p_{13} + m_4p_4 \tag{4.63}$$

$$m_1q_1 + m_4q_4 = m_{12}q_{12} + m_{13}q_{13} \tag{4.64}$$

$$m_2p_2 + m_3a_{23} + m_{31}p_{31} = m_1a_{21} + m_{11}p_{11} \tag{4.65}$$

$$m_2q_2 = m_{11}q_{11} + m_{31}q_{31} \tag{4.66}$$

$$m_3p_3 = (m_1+m_2+m_{11}+m_{12})a_3 + m_{32}p_{32} + m_{33}p_{33} \tag{4.67}$$

$$m_3q_3 = m_{32}q_{32} + m_{33}q_{33} \tag{4.68}$$

The presented example is important in order to understand that, regardless of where a machine element is placed with respect to a *principal vector linkage*, it can be included in the force balance conditions. It is sufficient that this element rotates together with one of the *principal elements*. This clearly modifies the position of the common CoM. In Figure 4.12, it is not placed on S, as an additional element has been introduced. However, its position does not change during the overall linkage's motions as the force balance conditions assure that the linear momentum is always constant.

5. Discussion and Conclusion

This chapter presented how the force balance conditions change when techniques modifying *principal vector linkages*' are applied. Mass-asymmetric links and additional elements were considered. The resulting force balance conditions in each example constitute a general form: they can be considered for mass-symmetric links and elements by setting links' q coordinates and additional elements' p and q coordinates equal to zero. Furthermore, it is important to notice that specific equations are defined to balance links' mass asymmetry. They do not change when links are simply moved, like in section 3.1. They change when links are replaced or mass-asymmetric elements are introduced. When mass-symmetric elements were introduced in Figure 4.3 to replace links, equations regarding mass asymmetry changed since the replaced links' contributions had been removed: as the elements were mass-symmetric, they were not included in these equations.
On the other hand, when additional elements are mass-asymmetric, they can be included in the equations regarding links' mass asymmetry, depending on how they are connected to the linkage and which motions they perform. For example, mass-asymmetric rotating elements are not included in these equations if they are fixed to links and their CoM has only a p coordinate. This is not the only case, as it was shown that a slider moving along a straight path does not affect the equations regarding links' mass asymmetry. In addition, it can be observed that in other cases additional elements could not be included in these equations. For example, in Figure 4.10, if the CoM of the slider replacing SB_2 had only its p coordinate, the slider would have been mass-asymmetric but it would not have been included in Equation 4.59. A similar reasoning can be made about the slider in Figure 4.12: if its CoM had its q coordinate equal to zero, the element would not have been included in Equation 4.65.
Moreover, introducing mass-asymmetric elements can lead to conditions which are complex to adjust and can require specific parameters in order to achieve a solution. This is the case for gears which are only pivoted to links: their CoMs rotating at different velocities and in different directions constitutes a problem in which time dependency plays a role. Nevertheless, by defining certain parameters like the diameters, it was shown

that conditions can be defined. Appendix B will show how conditions can be defined while considering different number of gears. In particular, it will be shown that the q_{comm} coordinate of the common CoM C_{comm} result to be zero. Therefore, although pivoted mass-asymmetric gears can be included in the *principal vector linkages*' force balance conditions, they do not affect the equations regarding links' mass asymmetry.
However, due to the motions performed by the additional elements, it is possible that conditions can not be adjusted. This was presented in Figure 4.11, in which the motions of a slider performing a random trajectory can not be considered by the common CoM of the linkage. The proposed solution consisted in introducing *principal vector links* making the new common CoM consider these motions. As previously stated, this implies a change in the actual design of the initial *principal vector linkage* as well as the definition of new force balance conditions. An alternative solution can be identified by placing the revolute joint on the slider's CoM. However, as stated at the beginning of section 4.2, this case was not considered since it is irrelevant for the study conducted in this chapter.
In conclusion, the techniques presented in chapter 3 can involve mass-asymmetric links and machine elements. If the latter are introduced in linkages having mass-symmetric links, equations regarding the mass asymmetry need to be considered in the resulting force balance conditions. Since machine elements can potentially perform different types of motion, the force balance conditions can be complex to modify. Nevertheless, it can be stated that, if these motions can be related to those of the other links, and therefore considered by the common CoM, force balance conditions can be adjusted.

5

Discussion

Techniques which modify *principal vector linkages* and can be used to synthesize inherently force balanced mechanisms were presented in chapter 3. Potential advantages that were observed consist in making the resulting mechanisms perform specific motions and adapting them to certain space requirements. In addition, it was observed that constraining *principal vector linkages* can lead to possible links' removal. This can result in potential advantages like the reduction of the mechanisms' total mass.
The techniques were derived by following the approach described at the end of chapter 2. This consisted in introducing machine elements in *principal vector linkages* as described in the categorizations of chapter 2. Observations were made while introducing the elements. These led to the definition of the different techniques and are now presented. Subsequently, the considerations made in chapter 4 are discussed. Finally, potential guidelines for applying techniques to *principal vector linkages* are presented.

Introducing machine elements in principal vector linkages

In the mechanisms presented in chapter 2, machine elements were introduced principally to constrain linkages, therefore to reduce the number of DoFs. It was observed that the same results can be achieved by introducing these elements in *principal vector linkages.* Gears, together with belt and chain drives, need to be fixed to different links in order to relate their rotations. On the other hand, sliders need to be connected to the base to constrain links' absolute motions: they can perform curved paths as well. Moreover, it was observed that constraints can be introduced without the introduction of additional elements. DoFs can be reduced by pivoting links, constraining their rotations, and fixing them to the base: links' motions with respect to the base result to be either partially or completely constrained. While gear, belt and chain drives relate different links' rotations, these can be kept equal by fixing angles between links. All of these alternatives constitute the techniques which modify *principal vector linkages*' motions.
In addition, machine elements can be introduced to replace links. An example of gears replacing a link within a *principal vector linkage* was already found in the literature: this can be made with belt or chain drives as well, provided that their transmission ratio is equal to 1. By fixing the outer rotating elements to parallel links, the parallelogram kinematic properties, which are essential for force balance, are maintained. On the other hand, as sliders can perform circular paths on the base, it was observed that they can replace links which are pivoted to the base. The parallelogram properties are maintained since the radii of the circular paths are equal to the lengths of the replaced links.
Although chapter 2 stated that possible differences could be observed between open and closed chains *principal vector linkages,* the only one which was observed consists in the different number of DoFs for a given number of *principal elements.* The design principle of both the kinds of kinematic chains is the same: force balance conditions are derived in the same way and parallelogram linkages are included. Therefore, all the presented techniques and the considerations which were made in chapters 3 and 4 for open chains *principal vector linkages* can be applied to closed chains as well.
However, not all the elements described in the categories of chapter 2 can be applied to *principal vector linkages.* It was observed that sliders can not be introduced between links as these would not be kept parallel: the parallelograms' kinematic properties would not be maintained as well as force balance. On the other hand,

it was observed that introducing drives whose elements are all placed on the base seems unfeasible, since the common CoM is the only point pivoted to the base of *principal vector linkages*. Nevertheless, when constraints are introduced, it is possible that more links result to be pivoted to the base: their rotations can be related by introducing drives whose elements are all pivoted to the base. On the other hand, it is clear that drives having one element pivoted to the base can be always introduced. This element result to be pivoted to the common CoM and, in order to reduce the number of DoFs, it needs to be fixed either to the base or to the link to which the other drive's elements are not pivoted.

Mass-asymmetric links and elements and force balance conditions

Chapter 4 showed how the force balance conditions are adjusted when techniques are applied to *principal vector linkages*. By considering mass-asymmetric links and machine elements, it was observed that solutions are achieved when machine elements' motions, especially their rotations, can be related to those of either the other links or the base.
The force balance conditions presented for each example represent an exhaustive overview of how they are generally adjusted when the different techniques are applied to *principal vector linkages*. In particular, they do not change when the applied techniques modify the number of DoFs and do not include additional elements. They are modified when machine elements are introduced, regardless of whether they replace links or constrain linkages, and when links' positions change. In addition, the equations regarding mass-asymmetry are modified when mass-asymmetric machine elements and links are, respectively, introduced and removed. All the presented force balance conditions, which were related to *principal vector linkages* including mass-asymmetric links and machine elements, represent a general form. As stated in chapter 4, these conditions still apply when mass-symmetric links and machine elements are included. A mass-symmetric distribution is a particular case of mass asymmetry: the distance between the geometric center of a body and its CoM is equal to zero.
The main difference between including mass-symmetric or mass-asymmetric links and elements consists in the different inertias: those of mass-asymmetric bodies are generally higher. Therefore, more power would be required to drive linkages having mass-asymmetric links and machine elements. Nevertheless, a potential advantage of using this kind of elements can consist in setting different inertia values in order to potentially achieve moment balance solutions.

Guidelines for applying techniques to principal vector linkages

As stated in chapter 3, the presented techniques modify the overall design of *principal vector linkages* but maintain their essential kinematics, which is based on those of the parallelograms forming the linkages. This is important because, when techniques are applied, the force balance conditions, which are based on this kinematics, do not need to be derived for the resulting linkages. As stated in chapter 3 and shown in chapter 4, the equations need to be simply adjusted when links' positions are modified and additional elements are introduced. Possible guidelines can be defined in order to synthesize mechanisms. They consist in, first, considering a *principal vector linkage* together with its force balance conditions. Then, the different techniques can be applied in order to adapt the linkage to specific design requirements, which can include the motions that the resulting linkage needs to perform. If the applied techniques modify links' positions or introduce machine elements, regardless of whether they constrain motions or replace links, the equations constituting the force balance conditions of the initial *principal vector linkage* need to be adjusted. The parameters concerning the mass and the length of all the links are defined in order to meet the force balance conditions and the resulting mechanism can be manufactured.

6
Conclusion

This thesis presented ten techniques which modify *principal vector linkages* and can be used to synthesize inherently force balanced mechanisms.
Linkages' overall motions can be modified. One technique consists in pivoting links in order to constrain their absolute translations, with respect to the base: 2 DoF of the initial *principal vector linkage* are reduced when a link is pivoted to the base. Another technique consists in constraining links' absolute rotations by making them translate along a straight guide on the base: 2 DoF are reduced when a link is constrained. While a technique consists in introducing a slider pivoted to the base and connected to a link through its sliding pair, another one introduces a slider moving on the base and pivoted to a link. Both these techniques reduce 1 DoF and the constrained links are allowed to translate along the sliding pair and rotating around the sliders' pivots. A different technique constraining both the absolute translations and rotation of a link is based on fixing the link to the base: the number of DoFs results to be reduced by 3. Links' relative rotations can be constrained with two techniques. One consists in fixing the angle between the links, allowing them to rotate always by the same amount. The other technique requires the introduction of gear, belt or chain drives in order to define a relation between the rotations of the two links: these are fixed to the rotating elements of the drives. 1 DoF is reduced both when one angle between links is fixed and when a drive is introduced.
A different technique modifies the position of the links within *principal vector linkages*. The sizes of the parallelograms within the linkages result to be modified while the overall kinematics and motions do not change. Two other techniques consist in replacing links with machine elements like sliders, gear, belt and chain drives. While the drives need to have a transmission ratio equal to 1 in order to keep links parallel, the sliders can only replace links pivoted to the base. These elements shall perform circular trajectories around the fixed pivot of the original links.
Furthermore, it was described how the techniques can be applied to *principal vector linkages* having mass-asymmetric links and how mass-asymmetric machine elements can be introduced and considered in their force balance conditions.
It was shown how techniques can be combined on the same *principal vector linkage*. A simulation was performed on a mechanism which was synthesized by combining different techniques. The results proved that the force balance of the initial *principal vector linkage* was maintained.
In addition, the synthesized mechanism clearly showed how the application of techniques can lead to considerable modifications of the overall design of *principal vector linkages*. The applicability of these linkages was initially limited, as they consist of only links which need to form parallelograms to achieve force balance. With the new different design solutions that can be synthesized by applying the presented techniques, the balancing principle based on *principal vector linkages* can be adapted to a variety of additional motion and space requirements and applications.

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A

Synthesized inherently force balanced mechanism

The force balance conditions of the synthesized 2 DoF inherently force balanced mechanism, which was presented in chapter 3, are presented. Subsequently it will be presented how the singularities due to the parallelograms' change point are prevented by substituting links with gear, belt or chain drives.

Force balance conditions

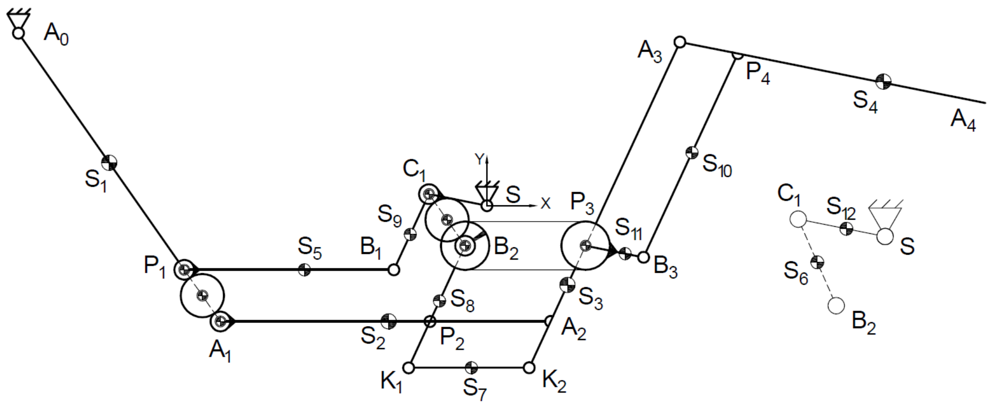

Figure A.1: 2 DoF inherently force balanced mechanism synthesized from 4DoF Principal Vector Linkage

Figure A.1 shows the synthesized mechanism. As described in chapter 3, the force balance conditions of the initial *principal vector linkage* are adjusted in order to consider links' new positions and additional machine elements. Therefore, they consist in four equations corresponding to the rotations of each *principal element*, without considering the fixed pivot in A_0 which reduces the number of DoFs by two.
The rotations of each *principal element* are now presented. They are around *principal points* P_1, P_2, P_3 and P_4, respectively. For each rotation, only one equation, regarding the distance between links' CoM and the *principal points*, is presented as all the links and machine elements are mass-symmetric. The parameters shown in Table A.1 are included in these equations.

A_0A_1 rotated

Figure A.2 shows the mechanism when only *principal element* A_0A_1 rotates. Since C_1B_2 is the only parallel link, it rotates around C_1 while the other links translate except for SC_1, C_1B_1 and B_1P_1. The mass of all the additional rotating elements, i.e. gears and sprockets, are considered except for those of the gears pivoted on

Masses [kg]	Principal dimensions [mm]	Links parameters [mm]	Elements parameters [mm]
$m_1 = 0.360$	$P_1A_1 = 28$	$A_0A_1 = A_3F = 200$	$R_{CentrGear} = 11$
$m_2 = 0.270$	$A_1P_2 = 90$	$P_1S_1 = 117.5222$	$R_{ExtGear} = 3$
$m_3 = 0.300$	$P_2A_2 = 60$	$A_1A_2 = 150$	$R_{Sprocket} = 14$
$m_4 = 0.450$	$A_2P_3 = 40$	$P_2S_2 = 77.1667$	
$m_5 = 0.165$	$P_3A_3 = 100$	$K_2A_3 = 160$	
$m_6 = 0.060$	$A_3P_4 = 20$	$P_3S_3 = 36.0333$	
$m_7 = 0.105$		$A_3S_4 = 200$	
$m_8 = 0.105$		$P_4S_4 = 67.6444$	
$m_9 = 0.075$		$B_1S_5 = 45$	
$m_{10} = 0.180$		$C_1S_6 = 14$	
$m_{11} = 0.030$		$K_1S_7 = 30$	
$m_{12} = 0.030$		$B_2S_8 = 30$	
$m_{CentrGear} = 0.009$		$C_1S_9 = 20$	
$m_{ExtGear} = 0.001$		$B_3S_{10} = 50$	
$m_{Sprocket} = 0.015$		$B_3S_{11} = SS_{12} = 10$	

Table A.1: Links and drive's parameters of the 2 DoF inherently force balanced mechanism shown in Fig. A.1

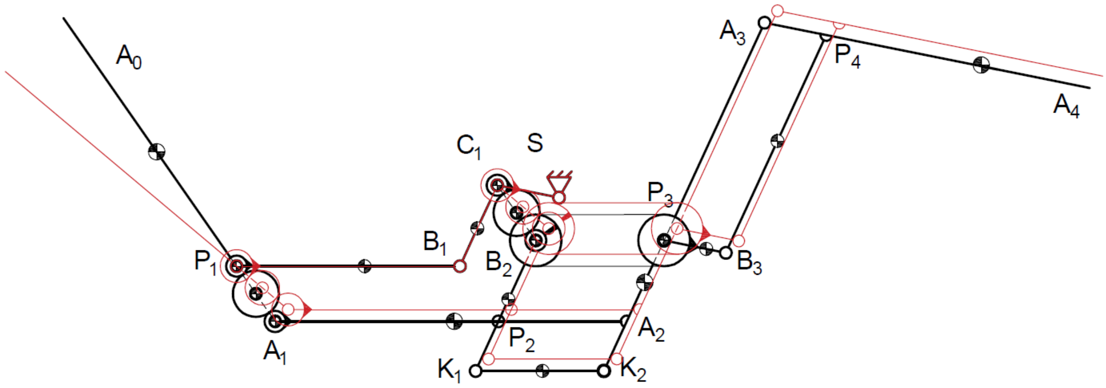

Figure A.2: Synthesized inherently force balanced mechanism of Fig. A.1. Principal element A_0A_1 rotated

P_1 and C_1.

$$m_1P_1S_1 = (m_2 + m_3 + m_4 + m_7 + m_8 + m_{10} + m_{11} + 2m_{Sprocket} + 2m_{ExtGear})P_1A_1 + \\ +(m_6 + m_{CentrGear})C_1S_6 + m_{CentrGear}\frac{P_1A_1}{2} \tag{A.1}$$

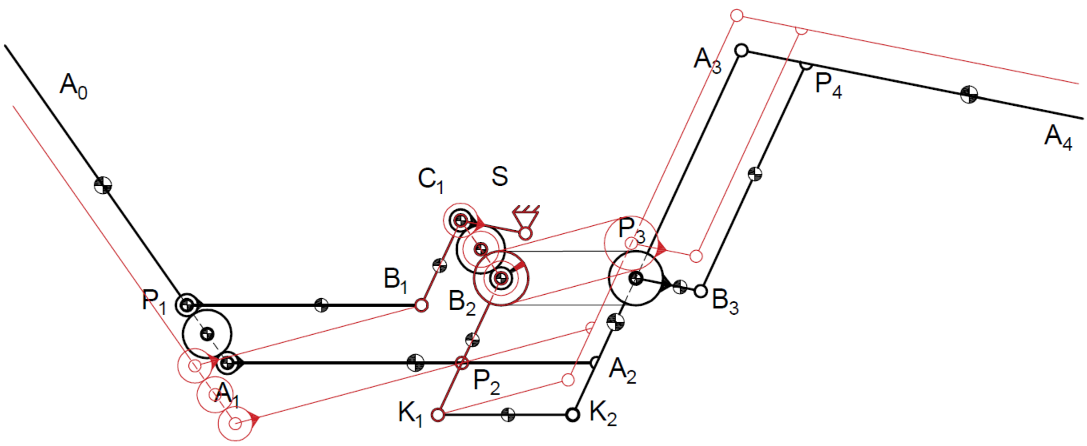

Figure A.3: Synthesized inherently force balanced mechanism of Fig. A.1. Principal element A_1A_2 rotated

$A_1 A_2$ rotated

Figure A.3 shows the mechanism when only *principal element* $A_1 A_2$ rotates together with its parallel links $B_1 P_1$ and $K_1 K_2$. All the other links translate except for SC_1, $C_1 B_1$, $C_1 B_2$ and $B_2 K_1$. The mass of the gears on $P_1 A_1$ and the sprocket on P_3 are considered. The gear drive on $C_1 B_2$ and the sprocket on B_2 are not considered since $C_1 B_2$ does not move.

$$m_2 P_2 S_2 + (m_1 + 2m_{ExtGear} + m_{CentrGear}) P_2 A_1 + m_5 B_1 S_5 = \\ = (m_3 + m_4 + m_{10} + m_{11} + m_{Sprocket}) P_2 A_2 + m_7 K_1 S_7 \quad \text{(A.2)}$$

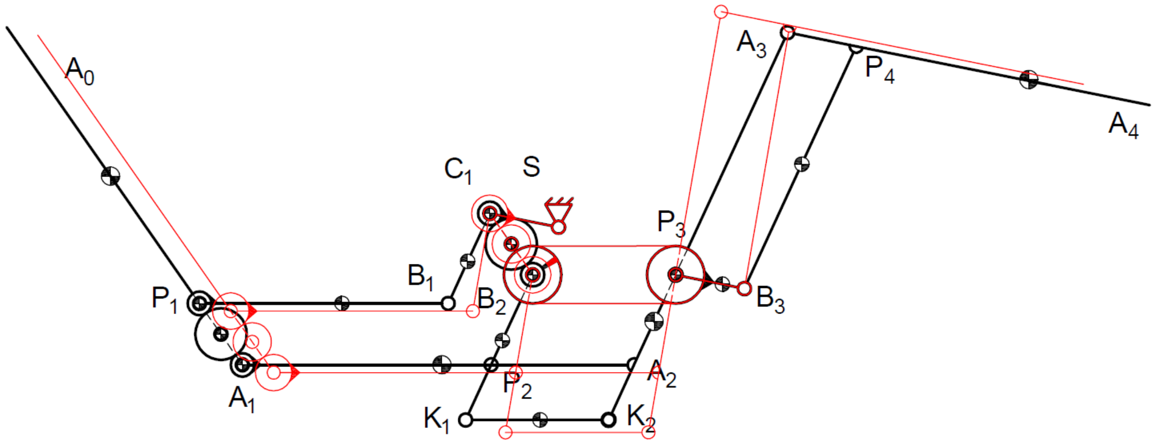

Figure A.4: Synthesized inherently force balanced mechanism of Fig. A.1. Principal element $A_2 A_3$ rotated

$A_2 A_3$ rotated

Figure A.4 shows the mechanism when only *principal element* $A_2 A_3$ rotates around its *principal point* P_3. Its parallel links $B_2 K_1$, $C_1 B_1$ and $B_3 P_4$ rotate around B_2, C_1 and B_3, respectively. All the other links translate except for SC_1 and $C_1 B_2$. Only the mass of the gears on $P_1 A_1$ are considered as well as the position of link $K_1 K_2$ which was moved from $B_2 P_3$.

$$m_3 P_3 S_3 + (m_1 + m_2 + 2m_{ExtGear} + m_{CentrGear} + m_5) P_3 A_2 + m_7 P_3 K_2 + m_8 B_2 S_8 + m_9 C_1 S_9 = \\ = m_4 P_3 A_3 + m_1 0 B_3 S_1 0 \quad \text{(A.3)}$$

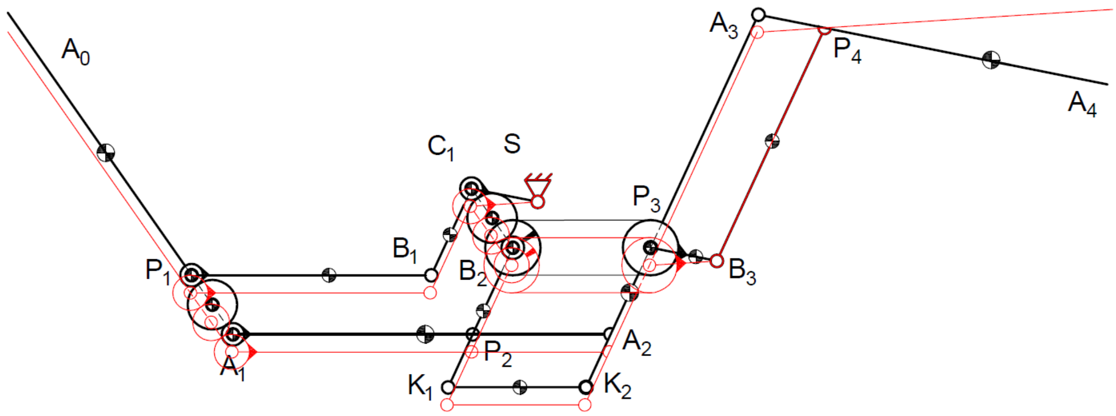

Figure A.5: Synthesized inherently force balanced mechanism of Fig. A.1. Principal element $A_3 A_4$ rotated

$A_3 A_4$ rotated

Figure A.5 shows the mechanism when only *principal element* $A_3 A_4$ rotates. Only its parallel links $B_3 P_3$ and SC_1 rotate around $B_3$and S, respectively. All the other links translate except for $B_3 P_4$. The mass of the rotating elements of all the drives are considered.

$$m_4P_4S_4 = (m_1 + m_2 + m_3 + m_5 + m_6 + m_7 + m_8 + m_9 + 2m_{Sprocket} + 4m_{ExtGear} + 2m_{CentrGear})P_4A_3 + \\ + m_{11}B_3S_{11} + m_{12}SS_{12} \tag{A.4}$$

Singularities prevention

Equations A.1, A.2 A.3 and A.4 represent the force balance conditions of the mechanism. It can be observed that the additional elements were included in the equations according to how they move when each *principal element* rotates. By comparing Figure A.1 with Figure A.6, which represent the mechanism without links'

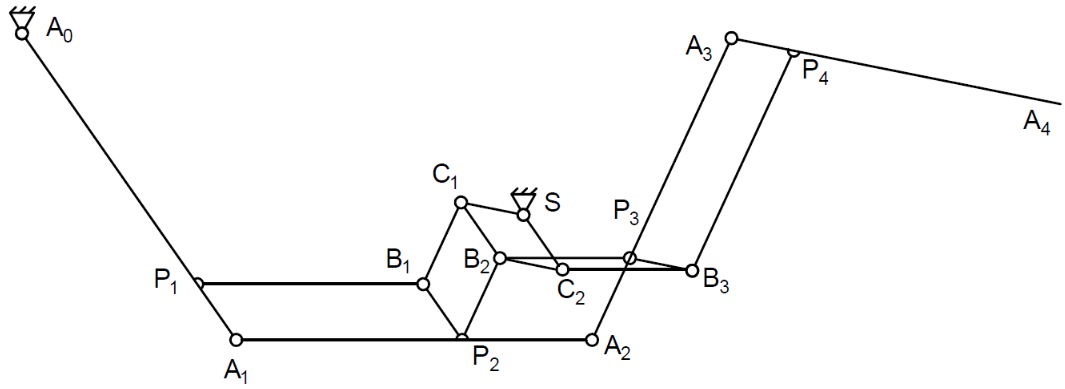

Figure A.6: Synthesized inherently force balanced mechanism of Figure A.1. No link moved nor replaced by machine elements

substitutions, it can be observed that different links overlap others. In particular, link C_2B_3 overlaps A_2A_3. The chain drive was introduced to avoid the overlapping between these links and to prevent singularities due to parallelogram $B_2P_3B_3C_2$. During the simulation performed in chapter 3, the change point state of this parallelogram was reached and singularity could have occurred: the parallelogram could have changed to the anti-parallelogram configuration. This was prevented thanks to the chain drive and the gear one on C_1B_2 which kept SC_1 and B_3P_3 parallel to each other. As a consequence, no values that could be related to singularities were observed in the plots showing the reaction forces on the base and the position of the common CoM in chapter 3.

Nevertheless, it was decided to investigate how the values of the reactions forces and the position of the common CoM change when singularities occur. By considering the synthesized mechanism in Figure A.1 and the parameters shown in Table A.1, a simulation on SPACAR was performed while applying a torque equal to 0 Nm on A_0 and one equal to 3 Nm on S for a simulation time of 0.5s divided in 1000 time steps. By looking at Figure A.7, parallelogram $P_3A_3P_4B_3$ reached the change point when its revolute joints were aligned. Although the linkage maintained the parallelogram configuration, values which can be related to singularities were observed. By looking at Figures A.8 and A.9, when the change point is reached, the plots of both the reaction forces and the common CoM position present values which are significantly different than zero. The parallelism between the parallelogram's links could have been lost as well as force balance.

Conclusion

The force balance conditions related to the 2 DoF inherently force balanced mechanism mentioned in chapter 3 were presented. Like those of the initial *principal vector linkage*, they are related to the rotation of each single *principal element*. On the other hand, it was shown how introducing machine elements is advantageous to replace links which overlap each other and to prevent singularities due to parallelogram linkages, which can compromise mechanisms' force balance.

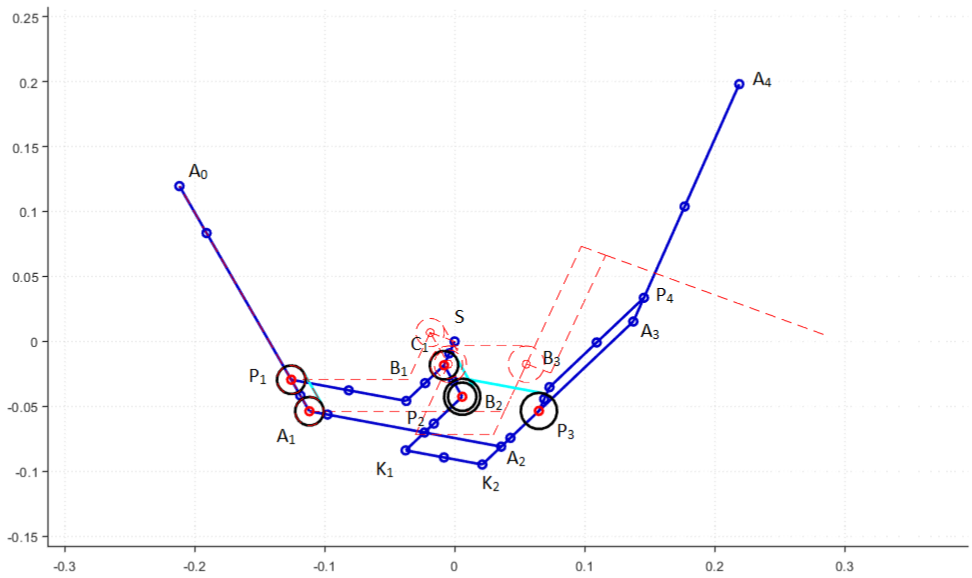

Figure A.7: 2 DoF IDB mechanism at the end of the simulation time. Dashed lines: initial position.

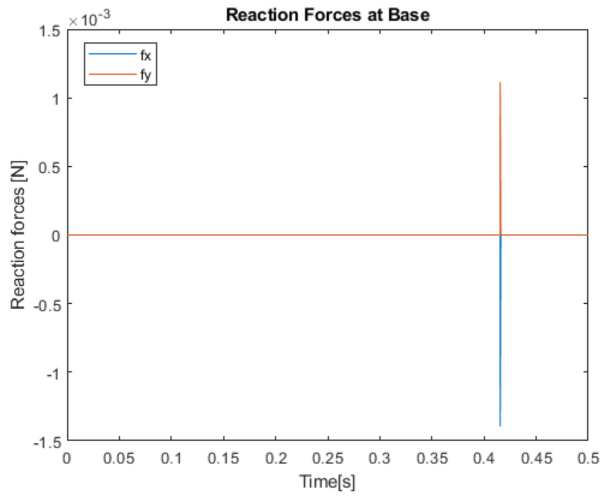

Figure A.8: Horizontal and vertical reaction forces calculated from the simulation.

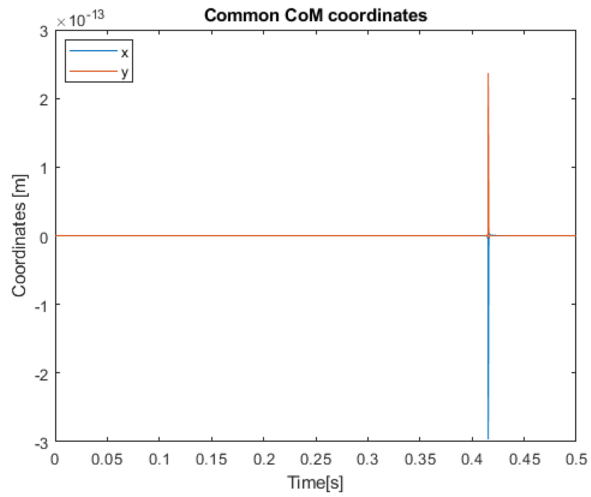

Figure A.9: X and Y coordinates of the common CoM derived from the simulation results.

B

Mass-asymmetric gear trains

Chapter 4 showed how mass-asymmetric gears which are only pivoted to links can be included in the force balance conditions. This is possible by defining a common CoM, considering only the pivoted gears, whose position does not change with respect to the link on which the gears are pivoted. Specific conditions were defined by setting the same angular velocity for gears rotating in the same direction. It was stated that, depending on the number of gears involved in a train, different equations can be defined as well as different solutions. In particular, at least two mass-asymmetric gears rotating in the same direction are required in order to define a solution. Conditions can be defined with at least three pivoted gears, in which the one rotating oppositely to the others is mass-symmetric. An example considering this configuration of pivoted gears will be provided, together with two cases in which four and five mass-asymmetric gears are pivoted.
Each gear train will be presented with respect to the link on which the elements are pivoted. Their angular velocities will be defined relatively to the link as well as the angles between the link and the vectors describing each gear's CoM. It will be shown that the values of the angular velocities does not affect the resulting conditions.

Three pivoted gears

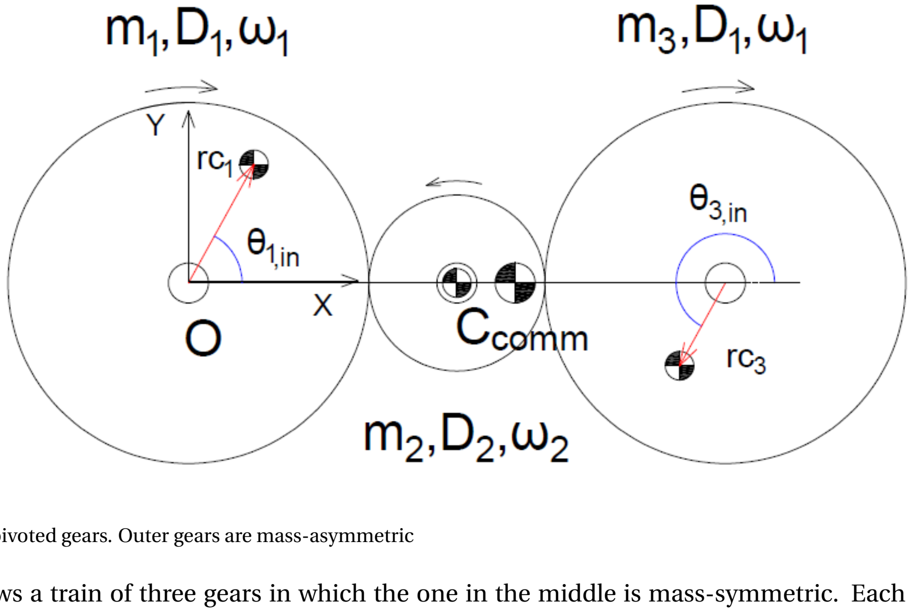

Figure B.1: Three pivoted gears. Outer gears are mass-asymmetric

Figure B.1 shows a train of three gears in which the one in the middle is mass-symmetric. Each gear has a mass m_x a diameter D_x and an angular velocity ω_x. Since the outer ones rotate in the same direction, they are assumed to have the same diameter D_1 and angular velocity ω_1. Their CoMs are at distances rc_1 and rc_3 from their respective revolute joints. The parameters are presented in Table B.1. The angular velocities are not presented as their values do not affect the solution. For simplicity in the calculations, they were assumed

to be constant.
When the train was introduced on the link, the gears were assembled such that vectors rc_1 and rc_3 created

Mass [g]	Diameter [mm]	Distance CoM [mm]
$m_1 = 10$	$D_1 = 70$	$rc_1 = 27$
$m_2 = 8$	$D_2 = 35$	
$m_3 = 13.5$	$D_3 = D_1 = 70$	$rc_3 = 20$

Table B.1: Gear train of Figure B.1. Gears parameters

with the link an angle of, respectively, $\theta_{1,in} = 60°$ and $\theta_{1,in} = 240°$. It was stated in chapter 4 that a solution is defined for two gears rotating in the same direction when the angles between the rc_x vectors and the link differ by 180°.
The common CoM C_{comm} can be defined with p_{comm} and q_{comm} coordinates, with respect to the origin O of the reference system XY, as described in chapter 4.

$$p_{comm} = \frac{m_1 pos_1 + m_2 pos_2 + m_3 pos_3}{m_1 + m_2 + m_3} = constant \tag{B.1}$$

$$q_{comm} = \frac{m_1 q_1 + m_2 q_2 + m_3 q_3}{m_1 + m_2 + m_3} = constant \tag{B.2}$$

Where pos_x and q_x are defined with respect to the origin as follows.

$$pos_1 = rc_1 \cos(\omega_1 t + \theta_{1,in}) \qquad q_1 = rc_1 \sin(\omega_1 t + \theta_{1,in}) \tag{B.3}$$

$$pos_2 = \frac{D_1 + D_2}{2} \qquad q_2 = 0 \tag{B.4}$$

$$pos_3 = \frac{D_1 + 2D_2 + D_3}{2} + rc_3 \cos(\omega_1 t + \theta_{3,in}) \qquad q_3 = rc_3 \sin(\omega_1 t + \theta_{3,in}) \tag{B.5}$$

In order to keep p_{comm} and q_{comm} constant, their time derivatives are set equal to zero.

$$\frac{dp_{comm}}{dt} = \frac{1}{m_1 + m_2 + m_3}\left(-m_1 rc_1 \omega_1 \sin(\omega_1 t + \theta_{1,in}) - m_3 rc_3 \omega_1 \sin(\omega_1 t + \theta_{3,in})\right) = 0 \tag{B.6}$$

$$\frac{dq_{comm}}{dt} = \frac{1}{m_1 + m_2 + m_3}\left(m_1 rc_1 \omega_1 \cos(\omega_1 t + \theta_{1,in}) + m_3 rc_3 \omega_1 \cos(\omega_1 t + \theta_{3,in})\right) = 0 \tag{B.7}$$

A solution is obtained when the parentheses are equal to zero. It can be observed that the ω_1 outside the sine and cosine terms can be removed. As previously stated, its value does not affect the solution. By using the trigonometric addition formulas, Equations B.6 and B.7 can be written as follows.

$$-\sin(\omega_1 t)\left[m_1 rc_1 \cos\theta_{1,in} + m_3 rc_3 \cos\theta_{3,in}\right] - \cos(\omega_1 t)\left[m_1 rc_1 \sin\theta_{1,in} + m_3 rc_3 \sin\theta_{3,in}\right] = 0 \tag{B.8}$$

$$\cos(\omega_1 t)\left[m_1 rc_1 \cos\theta_{1,in} + m_3 rc_3 \cos\theta_{3,in}\right] - \sin(\omega_1 t)\left[m_1 rc_1 \sin\theta_{1,in} + m_3 rc_3 \sin\theta_{3,in}\right] = 0 \tag{B.9}$$

By setting each square parenthesis equal to zero in both the equations, the derived results are the following.

$$m_1 rc_1 \cos\theta_{1,in} + m_3 rc_3 \cos\theta_{3,in} = 0 \tag{B.10}$$

$$m_1 rc_1 \sin\theta_{1,in} + m_3 rc_3 \sin\theta_{3,in} = 0 \tag{B.11}$$

By substituting the values of the gears' parameters and the angles, it can be observed that the equations are verified. In particular, it is important that an angle of 180° is created between $\theta_{1,in}$ and $\theta_{3,in}$. Therefore, $\cos\theta_{3,in}$ results to be equal to $-\cos\theta_{1,in}$ and $\sin\theta_{3,in}$ results equal to $-\sin\theta_{3,in}$. As a result, Equations B.10 and B.11 can be written with a single equation.

$$m_1 rc_1 - m_3 rc_3 = 0 \tag{B.12}$$

The common CoM C_{comm} has its p_{comm} coordinate equal to 58.3333mm, while its q_{comm} coordinate is equal to zero. These values are constant while the gears rotate.

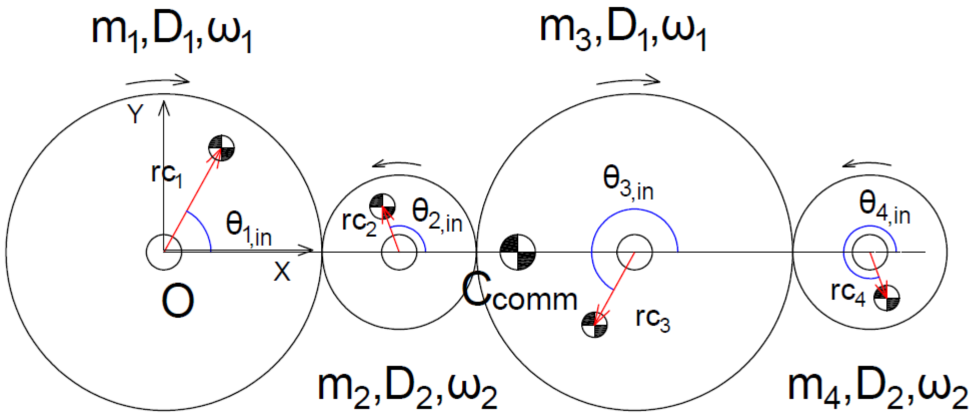

Figure B.2: Four pivoted mass-asymmetric gears.

Mass [g]	Diameter [mm]	Distance CoM [mm]
$m_1 = 10$	$D_1 = 70$	$rc_1 = 27$
$m_2 = 8$	$D_2 = 35$	$rc_2 = 10$
$m_3 = 13.5$	$D_3 = D_1 = 70$	$rc_3 = 20$
$m_4 = 8$	$D_4 = D_2 = 35$	$rc_4 = 10$

Table B.2: Gear train of Figure B.2. Gears parameters

Four pivoted gears

Figure B.2 shows a gear train having four mass-asymmetric gears. The first and the third ones are the same of Figure B.1, while the second is now mass-asymmetric and its mass m_2 and diameter D_2 are equal to those of the fourth gear m_4 and D_4. In addition, the distances between their CoMs and their respective revolute joints rc_2 and rc_4 are equal. Like in Figure B.1, $\theta_{1,in} = 60°$ and $\theta_{3,in} = 240°$. Therefore, they differ from each other by 180° as well as $\theta_{2,in}$ and $\theta_{4,in}$ which are equal to, respectively, 110° and 290°.
The common CoM C_{comm} can be defined with p_{comm} and q_{comm} coordinates.

$$p_{comm} = \frac{m_1 pos_1 + m_2 pos_2 + m_3 pos_3 + m_4 pos_4}{m_1 + m_2 + m_3 + m_4} = constant \tag{B.13}$$

$$q_{comm} = \frac{m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4}{m_1 + m_2 + m_3 + m_4} = constant \tag{B.14}$$

Where pos_x and q_x are defined as follows.

$$pos_1 = rc_1 \cos(\omega_1 t + \theta_{1,in}) \qquad q_1 = rc_1 \sin(\omega_1 t + \theta_{1,in}) \tag{B.15}$$

$$pos_2 = \frac{D_1 + D_2}{2} + rc_2 \cos(\omega_2 t + \theta_{2,in}) \qquad q_2 = rc_2 \sin(\omega_2 t + \theta_{2,in}) \tag{B.16}$$

$$pos_3 = \frac{D_1 + 2D_2 + D_3}{2} + rc_3 \cos(\omega_1 t + \theta_{3,in}) \qquad q_3 = rc_3 \sin(\omega_1 t + \theta_{3,in}) \tag{B.17}$$

$$pos_4 = \frac{D_1 + 2D_2 + 2D_3 + D_4}{2} + rc_4 \cos(\omega_2 t + \theta_{4,in}) \qquad q_4 = rc_4 \sin(\omega_2 t + \theta_{4,in}) \tag{B.18}$$

Time derivatives of p_{comm} and q_{comm} are set equal to zero.

$$\frac{dp_{comm}}{dt} = \frac{1}{m_1 + m_2 + m_3 + m_4}[-m_1 rc_1 \omega_1 \sin(\omega_1 t + \theta_{1,in}) - m_2 rc_2 \omega_2 \sin(\omega_2 t + \theta_{2,in}) + \\ -m_3 rc_3 \omega_1 \sin(\omega_1 t + \theta_{3,in}) - m_4 rc_4 \omega_2 \sin(\omega_2 t + \theta_{4,in})] = 0 \tag{B.19}$$

$$\frac{dq_{comm}}{dt} = \frac{1}{m_1 + m_2 + m_3 + m_4}[m_1 rc_1 \omega_1 \cos(\omega_1 t + \theta_{1,in}) + m_2 rc_2 \omega_2 \cos(\omega_2 t + \theta_{2,in}) + \\ +m_3 rc_3 \omega_1 \cos(\omega_1 t + \theta_{3,in}) + m_4 rc_4 \omega_2 \cos(\omega_2 t + \theta_{4,in})] = 0 \tag{B.20}$$

In both Equations B.19 and B.20, each term within square parentheses has either ω_1 or ω_2. Compared to the previous case, in Equations B.6 and B.7 only ω_1 was included and, since it was in each term, it was removed. In

B.19 and B.20 neither ω_1 nor ω_2 can be removed, but the terms having the same parameters can be considered together in order to derive the solutions. Equations B.21 and B.22 are derived, respectively, from B.19 and B.20. Solutions are derived by setting the square parentheses equal to zero.

$$-\omega_1\left[m_1 rc_1 \sin(\omega_1 t+\theta_{1,in}) + m_3 rc_3 \sin(\omega_1 t+\theta_{3,in})\right]+ \\ -\omega_2\left[m_2 rc_2 \sin(\omega_2 t+\theta_{2,in}) + m_4 rc_4 \sin(\omega_2 t+\theta_{4,in})\right]=0 \tag{B.21}$$

$$\omega_1\left[m_1 rc_1 \cos(\omega_1 t+\theta_{1,in}) + m_3 rc_3 \cos(\omega_1 t+\theta_{3,in})\right]+ \\ +\omega_2\left[m_2 rc_2 \cos(\omega_2 t+\theta_{2,in}) + m_4 rc_4 \cos(\omega_2 t+\theta_{4,in})\right]=0 \tag{B.22}$$

It can be observed that the parentheses multiplied by ω_1 lead to the same equations defined in the previous case, as they are related to the first and third gears in the train. The parentheses multiplied by ω_2 are related to the second and fourth gears. By following the same steps shown in the previous case, solutions are defined for these gears.

$$\sin(\omega_2 t)\left[m_2 rc_2 \cos\theta_{2,in} + m_4 rc_4 \cos\theta_{4,in}\right] + \cos(\omega_2 t)\left[m_2 rc_2 \sin\theta_{2,in} + m_4 rc_4 \sin\theta_{4,in}\right]=0 \tag{B.23}$$

$$\cos(\omega_2 t)\left[m_2 rc_2 \cos\theta_{2,in} + m_4 rc_4 \cos\theta_{4,in}\right] - \sin(\omega_2 t)\left[m_2 rc_2 \sin\theta_{2,in} + m_4 rc_4 \sin\theta_{4,in}\right]=0 \tag{B.24}$$

Equations B.23 and B.24 are derived by using the trigonometric addition formulas. The square parentheses in both the equations give the same results, which are shown in Equations B.25 and B.26.

$$m_2 rc_2 \cos\theta_{2,in} + m_4 rc_4 \cos\theta_{4,in} = 0 \tag{B.25}$$

$$m_2 rc_2 \sin\theta_{2,in} + m_4 rc_4 \sin\theta_{4,in} = 0 \tag{B.26}$$

Like in the previous example, one equation can be derived as $\theta_{2,in}$ and $\theta_{4,in}$ differ by 180°.

$$m_2 rc_2 - m_4 rc_4 = 0 \tag{B.27}$$

Equations B.12 and B.27 represent the conditions for the gear train of Figure B.2. By substituting their parameters with the values shown in Table B.2, it can be observed that the equations are verified. By including the angles, Equations B.10, B.11, B.25 and B.26, which are verified as well, have to be considered. The coordinates of the common CoM p_{comm} and q_{comm} are, respectively, 78.4177mm and zero.

Five pivoted gears

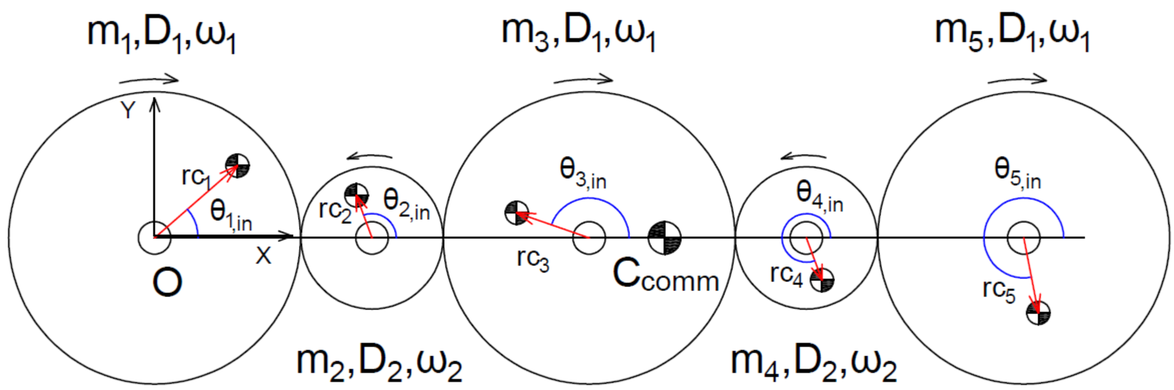

Figure B.3: Five pivoted mass-asymmetric gears.

A train of five mass-asymmetric gears is shown in Figure B.3. The first four gears are the same of Figure B.2 while the fifth is equal to the third one. All the gears parameters are shown in Table B.3.

While $\theta_{2,in}$ and $\theta_{4,in}$ are the same of the previous case, i.e. 110° and 290° respectively, $\theta_{1,in}$, $\theta_{3,in}$ and $\theta_{5,in}$ are equal to, respectively, 40°, 160° and 280°. This value were set to prove that, when more than two mass-asymmetric gears rotate with the same velocity, the vectors describing their CoMs do not have to differ always by 180°.

Mass [g]	Diameter [mm]	Distance CoM [mm]
$m_1 = 10$	$D_1 = 70$	$rc_1 = 27$
$m_2 = 8$	$D_2 = 35$	$rc_2 = 10$
$m_3 = 13.5$	$D_3 = D_1 = 70$	$rc_3 = 20$
$m_4 = 8$	$D_4 = D_2 = 35$	$rc_4 = 10$
$m_5 = 13.5$	$D_5 = D_1 = 70$	$rc_5 = rc_3 = 20$

Table B.3: Gear train of Figure B.3. Gears parameters

The common CoM of the gears can be defined with p_{comm} and q_{comm} coordinates.

$$p_{comm} = \frac{m_1 pos_1 + m_2 pos_2 + m_3 pos_3 + m_4 pos_4 + m_5 pos_5}{m_1 + m_2 + m_3 + m_4 + m_5} = constant \tag{B.28}$$

$$q_{comm} = \frac{m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4 + m_5 q_5}{m_1 + m_2 + m_3 + m_4 + m_5} = constant \tag{B.29}$$

Where pos_x and q_x, for $x = 1,2,3,4$, are defined in Equations B.15 to B.18, while pos_5 and q_5 are defined as follows.

$$pos_5 = \frac{D_1 + 2D_2 + 2D_3 + 2D_4 + D_5}{2} + rc_5 \cos(\omega_1 t + \theta_{5,in}) \qquad q_5 = rc_5 \sin(\omega_1 t + \theta_{5,in}) \tag{B.30}$$

The time derivatives of p_{comm} and q_{comm} are set equal to zero.

$$\begin{aligned}\frac{dp_{comm}}{dt} = \frac{1}{m_1 + m_2 + m_3 + m_4 + m_5}&[-m_1 rc_1 \omega_1 \sin(\omega_1 t + \theta_{1,in}) - m_2 rc_2 \omega_2 \sin(\omega_2 t + \theta_{2,in}) + \\ &-m_3 rc_3 \omega_1 \sin(\omega_1 t + \theta_{3,in}) - m_4 rc_4 \omega_2 \sin(\omega_2 t + \theta_{4,in}) - m_5 rc_5 \omega_1 \sin(\omega_1 t + \theta_{5,in})] = 0\end{aligned} \tag{B.31}$$

$$\begin{aligned}\frac{dq_{comm}}{dt} = \frac{1}{m_1 + m_2 + m_3 + m_4 + m_5}&[m_1 rc_1 \omega_1 \cos(\omega_1 t + \theta_{1,in}) + m_2 rc_2 \omega_2 \cos(\omega_2 t + \theta_{2,in}) + \\ &+m_3 rc_3 \omega_1 \cos(\omega_1 t + \theta_{3,in}) + m_4 rc_4 \omega_2 \cos(\omega_2 t + \theta_{4,in}) + m_5 rc_5 \omega_1 \cos(\omega_1 t + \theta_{5,in})] = 0\end{aligned} \tag{B.32}$$

Like in the previous case, the terms having the same angular velocity can be considered together.

$$\begin{aligned}-\omega_1 \left[m_1 rc_1 \sin(\omega_1 t + \theta_{1,in}) + m_3 rc_3 \sin(\omega_1 t + \theta_{3,in}) + m_5 rc_5 \sin(\omega_1 t + \theta_{5,in})\right] + \\ -\omega_2 \left[m_2 rc_2 \sin(\omega_2 t + \theta_{2,in}) + m_4 rc_4 \sin(\omega_2 t + \theta_{4,in})\right] = 0\end{aligned} \tag{B.33}$$

$$\begin{aligned}\omega_1 \left[m_1 rc_1 \cos(\omega_1 t + \theta_{1,in}) + m_3 rc_3 \cos(\omega_1 t + \theta_{3,in}) + m_5 rc_5 \cos(\omega_1 t + \theta_{5,in})\right] + \\ +\omega_2 \left[m_2 rc_2 \cos(\omega_2 t + \theta_{2,in}) + m_4 rc_4 \cos(\omega_2 t + \theta_{4,in})\right] = 0\end{aligned} \tag{B.34}$$

In Equations B.33 and B.34, the parentheses multiplied by ω_2 lead to the same results presented in the previous case. Therefore, only the parentheses multiplied by ω_1 will be considered. Their terms can be written by using trigonometric addition formulas and set equal to zero, as Equations B.35 and B.36 show.

$$\begin{aligned}\sin(\omega_1 t)\left[m_1 rc_1 \cos\theta_{1,in} + m_3 rc_3 \cos\theta_{3,in} + m_5 rc_5 \cos\theta_{5,in}\right] + \\ +\cos(\omega_1 t)\left[m_1 rc_1 \sin\theta_{1,in} + m_3 rc_3 \sin\theta_{3,in} + m_5 rc_5 \sin\theta_{5,in}\right] = 0\end{aligned} \tag{B.35}$$

$$\begin{aligned}\cos(\omega_1 t)\left[m_1 rc_1 \cos\theta_{1,in} + m_3 rc_3 \cos\theta_{3,in} + m_5 rc_5 \cos\theta_{5,in}\right] + \\ -\sin(\omega_1 t)\left[m_1 rc_1 \sin\theta_{1,in} + m_3 rc_3 \sin\theta_{3,in}\right] = 0\end{aligned} \tag{B.36}$$

The square parentheses are the same in both the equations. Solutions are defined by setting them equal to zero.

$$m_1 rc_1 \cos\theta_{1,in} + m_3 rc_3 \cos\theta_{3,in} + m_5 rc_5 \cos\theta_{5,in} = 0 \tag{B.37}$$

$$m_1 rc_1 \sin\theta_{1,in} + m_3 rc_3 \sin\theta_{3,in} + m_5 rc_5 \sin\theta_{5,in} = 0 \tag{B.38}$$

Since $\theta_{1,in}$, $\theta_{3,in}$ and $\theta_{5,in}$ do not differ by either 0 or 180°, a single equation can not be derived like in the previous examples. When the gears are more than two, it is possible to set specific values for the angle. A rule of thumb can be defined: given a certain number of gears rotating at the same angular velocity, by setting the product $m_x rc_x$ equal for each gear, the angles between each vector rc_x can be derived by diving 360° by the

number of gears. Indeed, in the presented example, $m_1 rc_1 = m_3 rc_3 = m_5 rc_5$ and the angles between each vector were equal to $120° = 360°/3$.
Equations B.37 and B.38, together with B.27, represent the conditions. These are verified by substituting for each parameter the values presented in Table B.3. The coordinates of the common CoM p_{comm} and q_{comm} are, respectively, 111.9340mm and zero.

Conclusion

The examples which were presented showed that it is possible to define conditions to keep the common CoM constant when mass-asymmetric gears rotate. In particular, it was observed that the q_{comm} coordinate resulted to be zero in all the examples. This is due to the fact that each gear's CoM rotates around a point which is placed at a distance equal to zero from the link. In addition, it was shown that the angles between the rc_x vectors and the link don't have to differ by 180° when more than two gears rotate in the same direction.