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DOI
10.1137/21M144709X

Publication date
2022

Document Version
Final published version

Published in
SIAM Journal on Control and Optimization

Citation (APA)
Bartl, D., Kupper, M., Lux, T., & Papapantoleon, A. (2022). Marginal and Dependence Uncertainty: Bounds, Optimal Transport, And Sharpness. *SIAM Journal on Control and Optimization*, *60*(1), 410-434. https://doi.org/10.1137/21M144709X

SIAM J. CONTROL OPTIM.
Vol. 60, No. 1, pp. 410–434

MARGINAL AND DEPENDENCE UNCERTAINTY: BOUNDS, OPTIMAL TRANSPORT, AND SHARPNESS*

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Abstract. Motivated by applications in model-free finance and quantitative risk management, we consider Fréchet classes of multivariate distribution functions where additional information on the joint distribution is assumed, while uncertainty in the marginals is also possible. We derive optimal transport duality results for these Fréchet classes that extend previous results in the related literature. These proofs are based on representation results for convex increasing functionals and the explicit computation of the conjugates. We show that the dual transport problem admits an explicit solution for the function $f = 1_B$, where B is a rectangular subset of $\mathbb{R}^d$, and provide an intuitive geometric interpretation of this result. The improved Fréchet–Hoeffding bounds provide ad hoc bounds for these Fréchet classes. We show that the improved Fréchet–Hoeffding bounds are pointwise sharp for these classes in the presence of uncertainty in the marginals, while a counterexample yields that they are not pointwise sharp in the absence of uncertainty in the marginals, even in dimension 2. The latter result sheds new light on the improved Fréchet–Hoeffding bounds, since Tankov [*J. Appl. Probab.*, 48 (2011), pp. 389–403] has showed that, under certain conditions, these bounds are sharp in dimension 2.

Key words. dependence uncertainty, marginal uncertainty, Fréchet classes, improved Fréchet–Hoeffding bounds, optimal transport duality, relaxed duality, sharpness of bounds

AMS subject classifications. 60E15, 49N15, 28A35

DOI. 10.1137/21M144709X

1. Introduction.

A celebrated result in probability theory is the Fréchet–Hoeffding bounds, which provide a bound on the joint distribution function (or the copula) of a random vector in case only the marginal distributions are known. Let $\mathcal{F}(F_1^*, \dots, F_d^*)$ denote the Fréchet class of cumulative distribution functions (cdfs) on $\mathbb{R}^d$ with (known) univariate marginal distributions $F_1^*, \dots, F_d^*$. Then, the Fréchet–Hoeffding bounds state that the following inequalities hold for all joint distribution functions with the given marginals, i.e., for all $F \in \mathcal{F}(F_1^*, \dots, F_d^*)$ it holds that

$$\left(\sum_{i=1}^d F_i^*(x_i) - (d-1)\right)^+ \le F(x_1, \dots, x_d) \le \min_{i=1,\dots,d} F_i^*(x_i) \tag{1.1}$$

*Received by the editors August 17, 2018; accepted for publication September 17, 2021; published electronically February 14, 2022. The fifth author is the author of the appendix only.
https://doi.org/10.1137/21M144709X
Funding: The work of the first author was supported by the Vienna Science and Technology Fund (WWTF) project MA16-021 and the Austrian Science Fund (FWF), project P28661. The work of the fourth author was supported by the Hellenic Foundation for Research and Innovation, grant HFRI-FM17-2152.

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for all $(x_1, \dots, x_d) \in \mathbb{R}^d$. These bounds can be combined with results on stochastic dominance (see, e.g., [23]), which state that for certain classes of (cost or payoff) functions φ the inequalities on the distribution function are preserved when considering integrals of the form $\int \varphi \, \mathrm{d}F$. In other words, the Fréchet–Hoeffding bounds combined with results from stochastic dominance yield upper and lower bounds on integrals of the form $\int \varphi \, \mathrm{d}F$, over all possible joint distribution functions with given marginals $F_1^*, \dots, F_d^*$ (or, equivalently, over all copulas).

These results have found several applications in financial and insurance mathematics, since they allow the derivation of bounds on the prices of multiasset options and on risk measures, such as the value-at-risk, in the framework of *dependence uncertainty*, i.e., when the marginal distributions are known and the joint distribution is not known; see, e.g., [7, 9, 13, 12, 14, 15, 25]. Analogous results have been also derived using methods from linear programming or optimal transport theory; see, e.g., [10, 4, 6, 16, 17, 18]. The optimal transport duality, also known as pricing-hedging duality in the mathematical finance literature, states, for example, that

$$\sup_{F \in \mathcal{F}(F_1^*, \dots, F_d^*)} \int \varphi \, \mathrm{d}F \; = \inf_{\varphi_1 + \cdots + \varphi_d \geq \varphi} \Big\{ \int \varphi_1 \mathrm{d}F_1^* + \cdots + \int \varphi_d \mathrm{d}F_d^* \Big\}. \tag{1.2}$$

In other words, using the language of mathematical finance, the model-free upper bound on the price of an option with payoff function φ over all joint distributions with given marginals equals the infimum over all hedging strategies which consist of investing according to φ_i in the asset with marginal F_i^*, subject to the condition that $\varphi_1 + \cdots + \varphi_d$ dominates the payoff function φ.

The main pitfall in the framework of dependence uncertainty is that the resulting bounds are too wide to be informative for applications; e.g., the bounds for multiasset option prices may coincide with the trivial no-arbitrage bounds. On the other hand, we can infer from financial and insurance markets partial information on the dependence structure of a random vector which is not utilized in the Fréchet–Hoeffding bounds and more generally the framework of dependence uncertainty, where only information on the marginals is taken into account.

These considerations have led to increased attention on frameworks that could be termed *partial dependence uncertainty*, i.e., when additional information is available on the dependence structure. The additional information available can take several forms; for example, some authors assume that the joint distribution function is known on some subset of its domain, others assume that the correlation (or more generally a measure of association) is known, while others assume that the variance of the sum is known or bounded, and so forth. We refer the reader to [29, 30] for an overview of this literature, with emphasis on applications to value-at-risk bounds.

Analogously to the Fréchet–Hoeffding bounds in the framework of dependence uncertainty, several authors have developed *improved* Fréchet–Hoeffding bounds that correspond to the framework of partial dependence uncertainty; see, e.g., [24, 32, 20, 21, 26]. The improved Fréchet–Hoeffding bounds can accommodate different types of additional information, such as the knowledge of the distribution function on a subset of the domain and the knowledge of a measure of association. In this article, we consider the following Fréchet class under additional information:

$$\mathcal{F}^{S,\pi}(F_1^*, \dots, F_d^*) := \big\{ F \in \mathcal{F}(F_1^*, \dots, F_d^*) \colon F(s) = \pi_s \text{ for all } s \in S \big\}, \tag{1.3}$$

where $S \subset \mathbb{R}^d$ is an arbitrary set and $(\pi_s)_{s \in S}$ a family with values in $[0, 1]$. In other words, we consider all joint distribution functions with known marginals $F_1^*, \dots, F_d^*$

and known value π_s for each $s \in S$, where S is a subset of the domain. The additional information on the joint distribution assumed in this class may not be directly observable in the markets but can be implied from multiasset option prices or other derivatives by arguments analogous to [5]. Improved Fréchet–Hoeffding bounds for this class have been derived in [32, 20] and read as follows:

$$\begin{aligned}(1.4)\qquad &\Big(\sum_{i=1}^d F_i^*(x_i) - (d-1)\Big)^+ \vee \max\Big\{\pi_s - \sum_{i=1}^d \big(F_i^*(s_i) - F_i^*(x_i)\big)^+ : s \in S\Big\} \\ &\quad \le F(x_1,\dots,x_d) \le \min_{i=1,\dots,d} F_i^*(x_i) \wedge \min\Big\{\pi_s + \sum_{i=1}^d \big(F_i^*(x_i) - F_i^*(s_i)\big)^+ : s \in S\Big\}.\end{aligned}$$

The authors in [32, 20] have also used the improved Fréchet–Hoeffding bounds in order to derive bounds for the prices of multiasset derivatives in a framework of partial dependence uncertainty, and showed that the additional information incorporated in the bounds leads to a notable tightening of the option price bounds relative to the case without additional information.

One could ask though whether this framework is realistic for applications, in particular whether the assumption of perfect knowledge of the marginal distributions is supported by empirical evidence or stems from mathematical convenience. We take the view that perfect knowledge of the marginals is *not* a realistic assumption, and are thus interested in frameworks that combine uncertainty in the marginals with partial uncertainty in the dependence structure. Towards this end, we introduce Fréchet classes that correspond to this framework, and we are interested in studying their properties.

The classes we introduce allow us to consider simultaneously uncertainty in the marginal distributions, measured either by 0th or by first order stochastic dominance, and additional information on the dependence structure, provided by values π_s for $s \in S$. Let us thus consider the following relaxed version of the class $\mathcal{F}^{S,\pi}$ in (1.3), provided by

$$(1.5)\quad \mathcal{F}^{S,\pi}_{\preceq_0}(F_1^*,\dots,F_d^*) := \Big\{cF : \begin{array}{l} c \in [0,1] \text{ and } F \text{ cdf on } \mathbb{R}^d \text{ s.t. } cF_i \preceq_0 F_i^* \\ \text{for all } 1 \le i \le d \text{ and } cF(s) \le \pi_s \text{ for all } s \in S \end{array}\Big\},$$

where F_i is the ith marginal distribution of F, and $cF_i \preceq_0 F_i^*$ means that F_i^* dominates cF_i in the 0th stochastic order, i.e., $cF_i \preceq_0 F_i^*$ if and only if $cF_i(t) - cF_i(s) \le F_i^*(t) - F_i^*(s)$ for all $s \le t$.[1] (Note that for $c = 1$, it follows from $F_i \preceq_0 F_i^*$ that $F_i = F_i^*$.) In other words, we consider the class of joint distribution functions (associated with subprobability measures) whose marginals are dominated by $F_1^*,\dots,F_d^*$ in the 0th stochastic order and whose value is smaller than π_s for each point s in a subset S of the domain.

Moreover, we consider the following relaxed versions of the class $\mathcal{F}^{S,\pi}$ given by

$$(1.6)\qquad \overline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*,\dots,F_d^*) := \Big\{F : \begin{array}{l} \text{cdf on } \mathbb{R}^d \text{ s.t. } F_i \le F_i^* \text{ for all } 1 \le i \le d \\ \text{and } F(s) \le \pi_s \text{ for all } s \in S \end{array}\Big\},$$

$$(1.7)\qquad \underline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*,\dots,F_d^*) := \Big\{F : \begin{array}{l} \text{cdf on } \mathbb{R}^d \text{ s.t. } F_i^* \le F_i \text{ for all } 1 \le i \le d \\ \text{and } \pi_s \le F(s) \text{ for all } s \in S \end{array}\Big\}.$$

[1] Denote by μ and μ^* the (sub-)probabilities on the real line associated to cF_j and F_j^*. Then a Dynkin argument shows that $cF_j \preceq_0 F_j^*$ if and only if $\mu(B) \le \mu^*(B)$ for every Borel subset of $\mathbb{R}$.

Here, $F_i \leq F_i^*$ is understood pointwise, which means that F_i first order stochastically dominates F_i^*, i.e., $\mathrm{d}F_i^* \preceq_1 \mathrm{d}F_i$. Further, for $y, z \in \mathbb{R}^d$ we write $y \leq z$ and $y < z$ if $y_i \leq z_i$, respectively, $y_i < z_i$, for all $i = 1, \dots, d$. This class is very similar to the previous one; however, now we consider probability measures on $\mathbb{R}^d$. Let us mention that there exist in the literature tests of first order stochastic dominance; see, e.g., [31].

Let us point out that, since the marginals are fixed, the class $\mathcal{F}^{S,\pi}(F_1^*, \dots, F_d^*)$ can be directly related to a class of copulas. This is the class studied in [32, 20], where these authors assume that the copula is known in some subset S of its domain. On the contrary, the classes $\mathcal{F}^{S,\pi}_{\preceq_0}(F_1^*, \dots, F_d^*)$, $\overline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*, \dots, F_d^*)$, and $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*, \dots, F_d^*)$ cannot be related to a class of copulas, since there exists uncertainty in their marginals. This is the reason for working with distribution functions instead of copulas in this article.

These three classes belong to the framework described above, i.e., they allow us to take into account and combine uncertainty in the marginal distributions with additional partial information on the dependence structure. An easy computation using arguments from copula theory, which is deferred to Appendix B, shows that the improved Fréchet–Hoeffding bounds of [20, 32] hold true also for these classes. An analogous result appears already in [26], which first considered inequality constraints on the copula. The contributions of this article are then threefold:

- We provide optimal transport, or pricing-hedging, duality results for each of the Fréchet classes $\mathcal{F}^{S,\pi}$, $\mathcal{F}^{S,\pi}_{\preceq_0}$, $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$, and $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$. In other words, we show that the optimal transport duality holds in the presence of additional information; this generalizes previous results in the related literature; see, e.g., [27]. Moreover, we provide optimality results for the class $\mathcal{F}^{S,\pi}$ that allow us to characterize the optimal measure. In the context of mathematical finance, we show that the pricing-hedging duality results hold also in the presence of additional information, in which case the hedging portfolio should also consist of positions in multiasset options for which the additional information is available. Moreover, the uncertainty in the marginals translates into trading constraints on the "dual" side, such as shortselling constraints.
- We show that the optimization problem for the function $f = 1_B$, for rectangular sets $B \subset \mathbb{R}^d$, admits an explicit solution in the class $\mathcal{F}^{S,\pi}_{\preceq_0}$. In the language of mathematical finance, we show that the superhedging problem for a multiasset digital option under shortselling constraints admits an explicit solution, and we explain the intuition behind this result.
- Finally, we discuss the pointwise sharpness of the improved Fréchet–Hoeffding bounds for each of the classes $\mathcal{F}^{S,\pi}$, $\mathcal{F}^{S,\pi}_{\preceq_0}$, $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$, and $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$. An (upper) bound $\mathcal{B}$ is called *sharp* for a certain class $\mathcal{C}$ if $\mathcal{B} \in \mathcal{C}$ and is called *pointwise sharp* for $\mathcal{C}$ if $\sup_{F \in \mathcal{C}} F(x) = \mathcal{B}(x)$ for all $x \in \mathrm{dom}(F)$. More specifically, we show that the improved upper Fréchet–Hoeffding bound is pointwise sharp for the classes $\mathcal{F}^{S,\pi}_{\preceq_0}$ and $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$, while the improved lower Fréchet–Hoeffding bound is pointwise sharp for the class $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$. In addition, by means of a counterexample, we show that these bounds are *not* pointwise sharp for the class $\mathcal{F}^{S,\pi}$, even in dimension $d = 2$. The latter result is surprising since [32] has showed that, under certain conditions on the set S, the upper bound is not only pointwise sharp but even sharp, i.e., Tankov actually showed that the upper improved Fréchet–Hoeffding bound belongs to the Fréchet class $\mathcal{F}$.

This article is structured as follows: in section 2, we derive optimal transport duality and optimality results for the Fréchet classes introduced above. In section 3, we show that the improved Fréchet–Hoeffding bounds are pointwise sharp for the classes $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$ and $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$. In section 4, we provide an explicit solution of the dual transport problem for the function $f = 1_B$ in the class $\mathcal{F}^{S,\pi}_{\preceq_0}$, and we deduce the pointwise sharpness of the improved upper Fréchet–Hoeffding bound in this class. Finally, Appendix A contains the aforementioned counterexample, Appendix B contains the derivation of the improved Fréchet–Hoeffding bounds for the classes $\mathcal{F}^{S,\pi}_{\preceq_0}$, $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$, and $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$, and Appendix C and Appendix D contain proofs of results.

2. Transport and relaxed transport duality and optimality under additional information. In this section, we establish our main duality and optimality results for optimal transport problems that feature additional information on the joint measure and, potentially, uncertainty about the marginal measures. More specifically, we derive a dual representation for the transport problem of maximizing the expectation of a d-dimensional cost or payoff function over probability measures whose univariate marginals are given and whose mass is prescribed on certain rectangles in $\mathbb{R}^d$. Using similar arguments, we obtain dual representations for two relaxations of this problem, involving uncertainty in the marginal measures. The uncertainty takes the form of estimates on the marginal measures in the 0th or first stochastic order. As a corollary, we establish duality for a transport problem with constraints on the maximum of random variables. Moreover, we establish optimality results for these transport problems that characterize the optimal measures.

We follow the notation of optimal transport theory in this section and thus work with measures instead of distribution functions. The precise relation between the two notations will be clarified in section 3, when we will make the connection between the optimal transport dualities and the sharpness of the improved Fréchet–Hoeffding bounds. The proofs of the various results are postponed to the appendices in order to improve the presentation of the material and the readability of the article.

2.1. Duality. We start by introducing useful notions and notation. Let us denote by $ca^+(\mathbb{R}^d)$ the set of all finite measures on the Borel σ-field of $\mathbb{R}^d$, $d \geq 1$, and by $ca_1^+(\mathbb{R}^d)$ (resp., $ca^+_{\leq 1}(\mathbb{R}^d)$) the subset of those measures μ satisfying $\mu(\mathbb{R}^d) = 1$ (resp., $\mu(\mathbb{R}^d) \leq 1$). The space $\mathbb{R}^d$ might also be omitted from the notation in case there is no ambiguity.

Let $A_1^i, \dots, A_d^i \in \mathbb{R}$, and define the sets $A^i := (-\infty, A_1^i] \times \cdots \times (-\infty, A_d^i] \subset \mathbb{R}^d$ for $i \in I$, where I is an arbitrary index set. Define for any *cost* or *payoff* function $f\colon \mathbb{R}^d \to \mathbb{R}$ the set

$$\Theta(f) := \Big\{(f_1, \dots, f_d, a) : f_1(x_1) + \cdots + f_d(x_d) + \sum_{i\in I} a^i 1_{A^i}(x) \geq f(x) \text{ for all } x \in \mathbb{R}^d\Big\},$$

where $f_j\colon \mathbb{R} \to \mathbb{R}$, $j \in \{1, \dots, d\}$ are bounded and measurable functions and $a = (a^i) \in \mathbb{R}^I$ such that $a^i = 0$ for all but finitely many $i \in I$. Moreover, consider the measures $\nu_1, \dots, \nu_d \in ca_1^+(\mathbb{R})$, let $0 \leq \underline{\pi}^i \leq \overline{\pi}^i \leq 1$, and define

$$\pi(f_1, \dots, f_d, a) := \int_{\mathbb{R}} f_1 \,\mathrm{d}\nu_1 + \cdots + \int_{\mathbb{R}} f_d \,\mathrm{d}\nu_d + \sum_{i\in I} \left(a^{i+}\overline{\pi}^i - a^{i-}\underline{\pi}^i\right)$$

for every $(f_1, \dots, f_d, a) \in \Theta(f)$, where a^{i+} and a^{i-} denote the positive and negative part of a^i, respectively, i.e., $a^{i+} = \max\{a^i, 0\}$ and $a^{i-} = \max\{-a^i, 0\}$. Denote by

$\Theta_0(f)$ the set of all $(f_1, \dots, f_d, a) \in \Theta(f)$ such that $f_1, \dots, f_d \geq 0$ and $a^i \geq 0$ for all $i \in I$. Now define the functionals

$$\phi(f) := \inf \{\pi(f_1, \dots, f_d, a) : (f_1, \dots, f_d, a) \in \Theta(f)\}$$

and

$$\phi_0(f) := \inf \{\pi(f_1, \dots, f_d, a) : (f_1, \dots, f_d, a) \in \Theta_0(f)\}.$$

Moreover, consider the sets of measures

$$\mathcal{Q} := \{\mu \in ca_1^+(\mathbb{R}^d) : \mu_1 = \nu_1, \dots, \mu_d = \nu_d \text{ and } \underline{\pi}^i \leq \mu(A^i) \leq \overline{\pi}^i \text{ for all } i \in I\} \tag{2.1}$$

and

$$\mathcal{Q}_0 := \{\mu \in ca_{\leq 1}^+(\mathbb{R}^d) : \mu_1 \preceq_0 \nu_1, \dots, \mu_d \preceq_0 \nu_d \text{ and } \mu(A^i) \leq \overline{\pi}^i \text{ for all } i \in I\},$$

where μ_j denotes the jth marginal of the measure μ, while $\mu_j \preceq_0 \nu_j$ should be understood as $\mu_j(B) \leq \nu_j(B)$ for every Borel set $B \subset \mathbb{R}$; the latter condition is also known as stochastic dominance in the 0th order.

The following theorem establishes a Monge–Kantorovich duality under additional constraints in the context of optimal transportation or a pricing-hedging duality under additional information in the context of mathematical finance. Indeed, in the context of optimal transportation we seek to maximize the total cost $\int f \mathrm{d}\mu$ relative to transport plans μ with marginals $\nu_1, \dots, \nu_d$ which in addition satisfy the constraint $\underline{\pi}^i \leq \mu(A^i) \leq \overline{\pi}^i$ for $i \in I$. We also consider a relaxed version of this problem, where we seek to maximize the same total cost relative to transport plans that are dominated by $\nu_1, \dots, \nu_d$ and satisfy the additional constraint $\mu(A^i) \leq \overline{\pi}^i$ for $i \in I$.

In the context of mathematical finance, let f denote the payoff function of an option depending on multiple assets, whose joint distribution is μ. Then, the right-hand side in (2.2) is the model-free superhedging price for this option assuming that the marginal distributions are known (i.e., $\mu_1 = \nu_1, \dots, \mu_d = \nu_d$), while there is also additional information present in the form of the bounds $\underline{\pi}^i, \overline{\pi}^i$ on the price of the multiasset digital options 1_{A^i}, $i \in I$. The left-hand side in (2.2) describes hedging or trading strategies which consist of investing in options with payoff f_i in the ith marginal ν_i, $i \in \{1, \dots, d\}$ and also buying a^{i+} and selling a^{i-} digital options with payoff 1_{A^i} at the prices $\overline{\pi}^i$ and $\underline{\pi}^i$, respectively, $i \in I$, subject to the requirement that the sum of these payoffs dominate f. The duality in (2.3) represents a relaxation of the problem described above, where on the one side uncertainty in the marginals is taken into account, while on the other side the trading strategies are subject to shortselling constraints (i.e., they are positive).

Definition 2.1. *A trading strategy $(f_1, \dots, f_d, a)$ which satisfies*

$$(f_1, \dots, f_d, a) \in \Theta(\varepsilon) \quad \text{and} \quad \pi(f_1, \dots, f_d, a) \leq 0$$

for some $\varepsilon > 0$ is called a uniform strong arbitrage.

Remark 2.2. The strategy described above is called a uniform strong arbitrage because its price at inception is less than or equal to zero, while its outcome is bounded from below by $\varepsilon > 0$. The next theorem relates the absence of uniform strong arbitrage with the existence of an element in $\mathcal{Q}$, the set of probability measures with given marginals that satisfy the condition $\underline{\pi}^i \leq \mu(A^i) \leq \overline{\pi}^i$ for all $i \in \mathcal{I}$. In other words, the absence of arbitrage allows us to conclude something about probability measures with given marginals, and vice versa. Notice that the absence of uniform strong arbitrage is a very weak condition that is implied by the classical no-arbitrage conditions.

THEOREM 2.3. *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be an upper semicontinuous and bounded function. Then, there does not exist a uniform strong arbitrage if and only if $\mathcal{Q}$ is not empty. In this case,*

$$\phi(f) = \max_{\mu\in\mathcal{Q}} \int_{\mathbb{R}^d} f \,\mathrm{d}\mu. \tag{2.2}$$

Moreover,

$$\phi_0(f) = \max_{\mu\in\mathcal{Q}_0} \int_{\mathbb{R}^d} f \,\mathrm{d}\mu. \tag{2.3}$$

Proof. See Appendix C. ∎

Remark 2.4. The optimal transport duality under additional information (2.2) appears in a similar form in [22, Theorem 3.2]. These two results were developed in parallel; however, their proofs are completely different. Moreover, in [22] the authors consider the Fréchet class of d-dimensional probability distributions with given marginals, whose copulas are bounded from below and above by arbitrary quasi-copulas; a quasi-copula generalizes the notion of a copula. Specifically, in the notation of Theorem 2.3, these authors assume that the measures μ are bounded on the entire domain from above and below, i.e.,

$$\underline{\pi}\big(\mu_1((-\infty,A_1]),\dots,\mu_d((-\infty,A_d])\big) \le \mu(A) \le \overline{\pi}\big(\mu_1((-\infty,A_1]),\dots,\mu_d((-\infty,A_d])\big)$$

for all subsets $A=(-\infty,A_1]\times\cdots\times(-\infty,A_d]\subset\mathbb{R}^d$, where $\underline{\pi},\overline{\pi}:[0,1]^d\to[0,1]$ are quasi-copulas, i.e., they are Lipschitz continuous and fulfill certain boundary conditions. These constraints give rise to particular instances of the set $\mathcal{Q}$ considered in (2.1), so that our formulation in Theorem 2.3 is slightly more general than the setting in [22].

Remark 2.5. The appearance of subprobability measures in $\mathcal{Q}_{\preceq 0}$ stems from the duality theorems and can be traced in the constructive proof of these theorems. In terms of mathematical finance, we can view them both on the primal and on the dual side of (2.3). On the primal side, subprobability measures arise once we consider marginals that are not fully known, and their uncertainty is measured with 0th order stochastic dominance. On the dual side, they arise once we consider shortselling constraints on the trading strategies. Both scenarios, uncertainty in the marginals and shortselling constraints, are very realistic.

As a corollary of Theorem 2.3 we derive in the following a duality result for a maximum transport problem. This problem corresponds to the situation where, besides the marginal distributions, the value of the measures is prescribed on an increasing track in $\mathbb{R}^d$. In terms of random variables, this is equivalent to knowing the distribution of the maximum of d random variables.

COROLLARY 2.6 (maximum transport problem). *Let $I=\mathbb{R}$, $A^i=(-\infty,i]^d$ and $\underline{\pi}^i=\overline{\pi}^i=\nu_{\max}((-\infty,i])$ for some measure $\nu_{\max}\in ca_1^+(\mathbb{R})$. Then*

$$\mathcal{Q}=\big\{\mu\in ca_1^+(\mathbb{R}^d):\mu_1=\nu_1,\,\dots,\,\mu_d=\nu_d \text{ and } \mu\circ\max{}^{-1}=\nu_{\max}\big\}, \tag{2.4}$$

and for every upper semicontinuous bounded function $f\colon\mathbb{R}^d\to\mathbb{R}$ one has

$$\phi(f)=\inf\Big\{\sum_{j=1}^d\int_{\mathbb{R}} f_j\,\mathrm{d}\nu_j+\int_{\mathbb{R}} g\,\mathrm{d}\nu_{\max}:f_1,\dots,f_d,g\Big\},$$

where $f_1, \dots, f_d, g \colon \mathbb{R} \to \mathbb{R}$ *are bounded and measurable functions such that*

$$f_1(x_1) + \cdots + f_d(x_d) + g(\max x) \geq f(x) \quad \text{for all } x \in \mathbb{R}^d, \tag{2.5}$$

where $\max x := \max_{j=1,\dots,d} x_j$ *for* $x \in \mathbb{R}^d$.

Proof. See Appendix C. ∎

Next, we provide another relaxation of the duality in (2.2) which follows along the same lines of reasoning as (2.3). In particular, this can be interpreted again as a pricing-hedging duality, where the superhedging problem involves uncertainty both in the marginals and in the joint distribution, while the hedging strategy takes into account bid and ask prices on single-asset options and the trading of multiasset digital options. The uncertainty in the marginals is now measured in terms of first order stochastic dominance.

Let us fix $\underline{\nu}_j, \overline{\nu}_j \in ca_1^+(\mathbb{R})$ for each $j = 1, \dots, d$ such that $\overline{\nu}_j$ first order stochastically dominates $\underline{\nu}_j$. Recall that $\underline{\nu}_j \preceq_1 \overline{\nu}_j$ in the first stochastic order if $\underline{\nu}_j([t,\infty)) \leq \overline{\nu}_j([t,\infty))$ for all $t \in \mathbb{R}$ or, equivalently, if $\int_{\mathbb{R}} f \,\mathrm{d}\underline{\nu}_j \leq \int_{\mathbb{R}} f \,\mathrm{d}\overline{\nu}_j$ for every nondecreasing bounded function $f \colon \mathbb{R} \to \mathbb{R}$.

Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be a cost or payoff function, and define the set

$$\Theta_1(f) := \Big\{ (f_1, g_1, \dots, f_d, g_d, a) \colon \sum_{j=1}^d \big(f_j(x_j) - g_j(x_j)\big) + \sum_{i \in I} a^i 1_{A^i}(x) \geq f(x) \forall x \in \mathbb{R}^d \Big\},$$

where $f_j, g_j \colon \mathbb{R} \to \mathbb{R}$ are nondecreasing and bounded functions, and $a = (a^i)_{i \in I}$ is such that $a^i = 0$ for all but finitely many $i \in I$. Define

$$\pi(f_1, g_1, \dots, f_d, g_d, a) := \sum_{j=1}^d \Big(\int_{\mathbb{R}} f_j \,\mathrm{d}\overline{\nu}_j - \int_{\mathbb{R}} g_j \,\mathrm{d}\underline{\nu}_j \Big) + \sum_{i \in I} \big(a^{i+} \overline{\pi}^i - a^{i-} \underline{\pi}^i \big)$$

for all $(f_1, g_1, \dots, f_d, g_d, a) \in \Theta_1(f)$, and further define the functional

$$\phi_1(f) := \inf \big\{ \pi(f_1, g_1, \dots, f_d, g_d, a) : (f_1, g_1, \dots, f_d, g_d, a) \in \Theta_1(f) \big\}.$$

Moreover, consider the set of measures

$$\mathcal{Q}_1 := \big\{ \mu \in ca_1^+(\mathbb{R}^d) : \underline{\nu}_j \preceq_1 \mu_j \preceq_1 \overline{\nu}_j \text{ and } \underline{\pi}_i \leq \mu(A^i) \leq \overline{\pi}_i \text{ for all } i, j \big\}.$$

Then the following holds.

THEOREM 2.7. *Let* $f \colon \mathbb{R}^d \to \mathbb{R}$ *be an upper semicontinuous and bounded function. Then, if* $\phi_1(\varepsilon) > 0$ *for every* $\varepsilon > 0$*, it holds that*

$$\phi_1(f) = \sup_{\mu \in \mathcal{Q}_1} \int_{\mathbb{R}^d} f \,\mathrm{d}\mu. \tag{2.6}$$

Proof. See Appendix C. ∎

2.2. Optimality. We will provide now results that outline under which conditions the infimum on the left-hand side of the transport duality in (2.2) is attained. This result will also allow us to characterize the optimal measure $\mu^\star$ on the right-hand side of this duality.

Proposition 2.8. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded function, and assume that the set I is finite. Then, there exists $(f_1, \ldots, f_d, a) \in \Theta_0(f)$ such that $\phi_0(f) = \pi(f_1, \ldots, f_d, a)$, that is, the infimum in the definition of ϕ_0 is attained. If, in addition, we assume that $\mathcal{Q}$ is nonempty and that for every $a \in \mathbb{R}^I \setminus \{0\}$ there exists a coupling μ between $\mu_1, \ldots, \mu_d$ such that $\sum_{i\in I} a^i \mu(A^i) > \pi(0, \ldots, 0, a)$, then there exists $(f_1, \ldots, f_d, a) \in \Theta(f)$ such that $\phi(f) = \pi(f_1, \ldots, f_d, a)$.*

Proof. See Appendix D. ∎

Remark 2.9. The assumption that $\sum_{i\in I} a^i \mu(A^i) > \pi(0, \ldots, 0, a)$ can be thought of as (slightly stronger than) requiring that the additional constraints are actual constraints and not automatically satisfied. In case of only one constraint A^1, for example, this simply means that $\inf_\mu \mu(A^1) < \underline{\pi}^1 \le \overline{\pi}^1 < \sup_\mu \mu(A^1)$, where in both cases μ runs through all couplings. In two dimensions, if $\mu_1 = \mu_2$ is the restriction of the Lebesgue measure to $[0, 1]$ and $A = [0, \frac{1}{2}]^2$, then this assumption is satisfied if and only if $0 < \underline{\pi}^1 \le \overline{\pi}^1 < \frac{1}{2}$.

The result we are really after is the following consequence of Proposition 2.8.

Corollary 2.10. *If, in addition to the assumptions of Proposition* 2.8*, we have that $\underline{\pi}^i = \overline{\pi}^i$ for all $i \in I$, then for a probability measure $\mu^\star \in \mathcal{Q}$ the following statements are equivalent:*

(i) *There exists some $(f_1, \ldots, f_d, a) \in \Theta(f)$ such that*

$$f_1(x_1) + \cdots + f_d(x_d) + \sum_{i\in I} a^i 1_{A^i}(x) = f(x) \text{ for } \mu^\star\text{-a.e. } x \in \mathbb{R}^d. \tag{2.7}$$

(ii) *The probability measure $\mu^\star$ is optimal, that is, $\phi(f) = \int f \, \mathrm{d}\mu^\star$.*

Proof. See Appendix D. ∎

Note that in the absence of constraints and in the two-dimensional case ($d = 2$), this readily implies that $\mu^\star \in \mathcal{Q}$ is optimal if and only if it is f-monotone in the sense of [33]. The latter concept was extended to the case of continuous constraints in [34, Theorem 3.6].

Remark 2.11. An optimality result analogous to Corollary 2.10 for the set $\mathcal{Q}_1$ is possible and requires some further technical arguments. We have refrained from providing this here for the sake of brevity.

Remark 2.12. Besides Corollary 2.10, (the proof of) Proposition 2.8 has another consequence, namely, that of duality for *measurable* cost functions. Indeed, the proof shows that for every sequence of uniformly bounded Borel cost functions f^n which increase pointwise to some f, one has that $\phi(f) = \sup_n \phi(f^n)$; similarly $\phi_0(f) = \sup_n \phi_0(f^n)$. An application of Choquet's capacitability theorem in the form of [1, Theorem 2.2] then implies that Theorem 2.3 extends to all bounded Borel f; that is, under the assumptions made in Proposition 2.8, we have

$$\phi(f) = \sup_{\mu\in\mathcal{Q}} \int_{\mathbb{R}^d} f \, \mathrm{d}\mu \quad \text{and} \quad \phi_0(f) = \sup_{\mu\in\mathcal{Q}_0} \int_{\mathbb{R}^d} f \, \mathrm{d}\mu$$

for all $f : \mathbb{R}^d \to \mathbb{R}$ bounded and Borel.

3. Sharpness of the improved Fréchet–Hoeffding bounds for the classes $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$ and $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$. In this and in the following sections, we will present sharpness and nonsharpness results for the improved Fréchet–Hoeffding bounds in the Fréchet classes presented in the introduction of this work. More specifically, in this section we will show that the improved upper Fréchet–Hoeffding bound is pointwise sharp for the Fréchet class $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$ introduced in (1.6), while the improved lower Fréchet–Hoeffding bound is pointwise sharp for the Fréchet class $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$ introduced in (1.7). In section 4, we will show that the improved upper Fréchet–Hoeffding bound is pointwise sharp for the Fréchet class $\mathcal{F}^{S,\pi}_{\preceq_0}$ introduced in (1.5). Finally, the counterexamples in the subsequent Appendix A show that the improved Fréchet–Hoeffding bounds are *not* pointwise sharp for the class $\mathcal{F}^{S,\pi}$.

Before we proceed with the statements and their proofs, let us clarify the relation between the notation used in section 2 and the notation used in the Introduction and also in section 3 and section 4, as well as in Appendix A. The following figure, Figure 1, shall help us clarify the connection between the two notations.

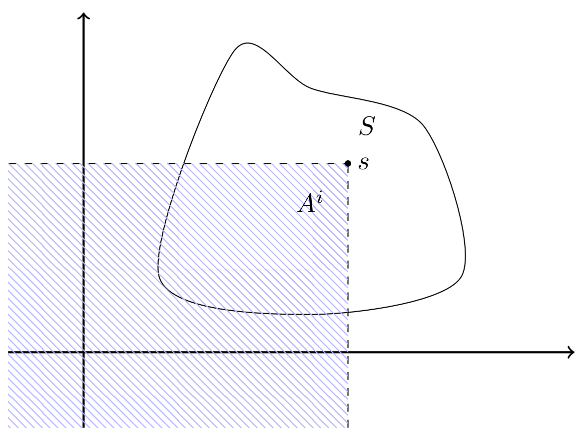

FIG. 1. *Illustration of the relation between the sets* S *and* $(A^i)_{i\in I}$.

Recall the definition of the set $\mathcal{F}^{S,\pi}(F_1^*,\dots,F_d^*)$ in (1.3): for every point $s \in S$ there exists a value $\pi_s \in [0,1]$ such that $\pi_s = F(s)$. Recall also the definition of the set $\mathcal{Q}$ in (2.1) and assume, for simplicity, that $\underline{\pi}^i = \overline{\pi}^i = \pi^i$ for all $i \in I$. Then, for every point $s \in S$ we can define a set A^i via $A^i = \bigtimes_{j=1}^d(-\infty, s_j]$ (see Figure 1); hence $\mathrm{card}(S) = \mathrm{card}(I)$. Then, by construction, we have the following equalities: $\mu(A^i) = \pi^i = \pi_s = F(s)$ for some $i \in I$ and some $s \in S$. Therefore, the set $\mathcal{F}^{S,\pi}$ in (1.3) corresponds exactly to the set $\mathcal{Q}$ in (2.1) in the setting described above. The relation between the sets $\mathcal{F}^{S,\pi}_{\preceq_0}$ and $\mathcal{Q}_{\preceq 0}$ is completely analogous. The relation between the sets $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$ and $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$ and the set $\mathcal{Q}_{\preceq 1}$ is also analogous once we observe that the former have one-sided constraints, while the latter has two-sided constraints.

The question of sharpness or pointwise sharpness of the Fréchet–Hoeffding bounds has a long history in the probability theory literature. The upper Fréchet–Hoeffding bound is a distribution function itself; hence the bound is actually sharp. On the other hand, the lower Fréchet–Hoeffding bound is a distribution function, and thus sharp, only in dimension 2, while in the general case [28] showed that the lower Fréchet–Hoeffding bound is pointwise sharp. Regarding the improved Fréchet–Hoeffding bounds, [32] showed in dimension 2 that the upper bound is a distribution function, and thus also sharp, in case the set S is decreasing (i.e., for $(u_1,u_2),(v_1,v_2) \in S$ holds $(u_1 - v_1)(u_2 - v_2) \le 0$). This result was later strengthened by [3], using weaker assumptions. On the other hand, [20] showed that in the higher-dimensional case

($d > 3$) the upper improved Fréchet–Hoeffding bound is sharp only in trivial cases. The counterexample in Appendix A is therefore surprising because it shows that once the condition of [32] is violated the bound is not even pointwise sharp.

Let us first show that the improved upper Fréchet–Hoeffding bound is pointwise sharp for $\overline{\mathcal{F}}^{S,\pi}_{\preceq 1}$.

PROPOSITION 3.1. *Let S be a bounded subset of $\mathbb{R}^d$. The following holds for every $x \in \mathbb{R}^d$:*

$$\max_{F\in\overline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)} F(x) = \min_{j=1,\dots,d} F_j^*(x_j) \wedge \min\{\pi^s \colon s \in S \text{ such that } x \le s\}. \tag{3.1}$$

Proof. The definition of $\overline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)$ immediately implies that the left-hand side (LHS) of (3.1) is smaller than or equal to its right-hand side (RHS).

In order to show the reverse inequality, fix $x \in \mathbb{R}^d$. Since S is bounded, there exists $r \in \mathbb{R}$ large enough such that $r \ge x_j + 1$ and $r \ge s_j + 1$ for all $j = 1,\dots,d$ and $s \in S$.

Let us distinguish between the following two cases:

Case 1*:* Assume that the RHS in (3.1) is attained at $\min_j F_j^*(x_j)$. Define

$$G_j(t) := F_j^*(x_j)1_{[x_j,r)}(t) + F_j^*(t)1_{[r,\infty)}(t)$$

for $t \in \mathbb{R}$, $j = 1,\dots,d$, and $F(y) = \min_{j=1,\dots,d} G_j(y_j)$ for $y \in \mathbb{R}^d$. One can check that F is a cdf, and it holds that $F_j(t) = G_j(t) \le F_j^*(t)$ for all $t \in \mathbb{R}$ and $j = 1,\dots,d$. To show that $F(s) \le \pi_s$ for all $s \in S$, fix $s \in S$. If $x \le s$, then $x_j \le s_j \le r$ for $j = 1,\dots,d$ and therefore $F(s) = \min_j G_j(s) = \min_j F_j^*(x_j) = \text{RHS} \le \pi_s$. Otherwise, i.e., if there exists some j^* such that $s_{j^*} < x_{j^*}$, one has $F(s) \le G_{j^*}(s_{j^*}) = 0 \le \pi_s$. This shows $F \in \overline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)$. Further, since $F(x) = \min_j F_j^*(x_j) = \text{RHS}$, one obtains that LHS $\ge$ RHS.

Case 2*:* Assume that the RHS is attained at π_{s^*} for some $s^* \in S$. Define

$$G_j(t) := \pi_{s^*}1_{[x_j,r)}(t) + F_j^*(t)1_{[r,\infty)}(t)$$

for $t \in \mathbb{R}$, $j = 1,\dots,d$, and $F(y) = \min_j G_j(y_j)$ for $y \in \mathbb{R}^d$. One can again check that F is a cdf. Since $\pi_{s^*} \le F_j^*(x_j)$, one has $F_j(t) = G_j(t) \le F_j^*(t)$ for all $t \in \mathbb{R}$ and $j = 1,\dots,d$. For $s \in S$ with $x \le s$, it holds that $F(s) = \min_j G_j(s) = \min_j F_j^*(x_j) = \pi_{s^*} \le \pi_s$ since the RHS is attained at π_{s^*}. Otherwise it holds that $F(s) = 0$, so that $F \in \overline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)$. Since $F(x) = \min_j F_j^*(x_j) = \text{RHS}$, one therefore obtains that LHS $\ge$ RHS. ∎

As usual, for a nondecreasing function $g\colon \mathbb{R}^d \to \mathbb{R}$ or $g\colon \mathbb{R} \to \mathbb{R}$, we define its left-continuous version by $g(x-) := \sup_{y<x} g(y)$ for all x. Let us next show that the improved lower Fréchet–Hoeffding bound is pointwise sharp for $\underline{\mathcal{F}}^{S,\pi}_{\preceq 1}$.

THEOREM 3.2. *Let S be a bounded subset of $\mathbb{R}^d$. The following holds for every $x \in \mathbb{R}^d$:*

$$\inf_{F\in\underline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)} F(x-)$$

$$= \max\Big\{1 - d + \sum_{j=1}^d F^*(x_j-), 0\Big\} \vee \sup_{s\in S}\Big\{\pi^s - \sum_{j=1}^d 1_{\{s_j \ge x_j\}}\big(1 - F_i^*(x_j-)\big)\Big\}. \tag{3.2}$$

Moreover, if S is finite, then for every $x \in \mathbb{R}^d$,

$$\inf_{F \in \underline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)} F(x)$$

$$= \max\Big\{1 - d + \sum_{j=1}^d F^*(x_j), 0\Big\} \vee \max_{s\in S}\Big\{\pi^s - \sum_{j=1}^d 1_{\{s_j > x_j\}}\big(1 - F_j^*(x_j)\big)\Big\}. \tag{3.3}$$

Proof. For $y \in \mathbb{R}^d$, we define $A(y) := \{z \in \mathbb{R}^d : z \le y\}$ and $B(y) := \{z \in \mathbb{R}^d : z < y\}$. Fix $x \in \mathbb{R}^d$. Throughout, we assume that $f_1, \dots, f_d : \mathbb{R} \to \mathbb{R}$ are nonincreasing bounded functions and $a^s \ge 0$ for all $s \in S$ and $a^s = 0$ for all but finitely many $s \in S$. The proof is divided into the following steps.

Step 1. We claim that

$$\inf_{F \in \underline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)} F(x-) = \inf_{F \in \underline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)} \int_{\mathbb{R}^d} 1_{B(x)}\,\mathrm{d}F$$

$$= \sup\Big\{\sum_{j=1}^d \int_{\mathbb{R}} f_j\,\mathrm{d}F_j^* + \sum_{s\in S} a^s\pi^s : \sum_{j=1}^d f_j + \sum_{s\in S} a^s 1_{A(s)} \le 1_{B(x)}\Big\}. \tag{3.4}$$

The argumentation is similar to Theorem 2.7; thus we only provide a sketch. For every lower semicontinuous bounded function $f : \mathbb{R}^d \to \mathbb{R}$, we define

$$\underline{\phi}_1(f) := \sup\Big\{\sum_{j=1}^d \int_{\mathbb{R}} f_j\,\mathrm{d}F_j^* + \sum_{i\in I} a^i\pi^i : \sum_{j=1}^d f_j(z_j) + \sum_{s\in S} a^s 1_{A(s)}(z) \le f(z)\,\forall z \in \mathbb{R}^d\Big\}.$$

Since $\underline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)$ is not tight, we restrict to those payoff functions f for which the limit $\lim_{z\to-\infty} f(z)$ exists. For each $N > 0$, we modify the marginals $\tilde{F}_j^* := F_j^* \wedge 1_{[-N,\infty)}$ and consider the approximate version

$$\tilde{\underline{\phi}}_1(f) := \sup\Big\{\sum_{j=1}^d \int_{[-N,\infty)} f_j\,\mathrm{d}\tilde{F}_j^* + \sum_{i\in I} a^i\pi^i :$$
$$\sum_{j=1}^d f_j(z_j) + \sum_{s\in S} a^s 1_{A(s)}(z) \le f(z) \text{ for all } z \in [-N,\infty)^d\Big\}.$$

Similar to Theorem 2.7, it holds that $\tilde{\underline{\phi}}_1(f) = \min_{F \in \underline{\mathcal{F}}^{S,\pi}_{\preceq 1}(\tilde{F}_1^*,\dots,\tilde{F}_d^*)} \int_{[-N,\infty)^d} f\,\mathrm{d}F$. Then, by letting $N \to \infty$, we end up with $\underline{\phi}_1(f) = \inf_{F \in \underline{\mathcal{F}}^{S,\pi}_{\preceq 1}(F_1^*,\dots,F_d^*)} \int_{\mathbb{R}^d} f\,\mathrm{d}F$.

Step 2. We next show that the value of the optimization problem (3.4) is given by the expression in (3.2). This follows from the following partial Steps 2a–2c.

Step 2a. First, we consider the simplified optimization problem

$$\sup \sum_{j=1}^d \int_{\mathbb{R}} f_j\,\mathrm{d}F_j^* \quad \text{such that} \quad \sum_{j=1}^d f_j \le 1_{B(x)}. \tag{3.5}$$

Similar to the classical Fréchet–Hoeffding lower bound, the supremum of (3.5) is attained at

$$\sum_{j=1}^d \bar{f}_j \equiv 0 \quad \text{or} \quad \sum_{j=1}^d \bar{f}_j = 1 - d + \sum_{j=1}^d 1_{(-\infty,x_j)},$$

and the respective optimal value is given by $\max\{1-d+\sum_{j=1}^d F_j^*(x_j-),0\}$.

Step 2b. Next, we consider the optimization problem

$$\sup \sum_{j=1}^d \int_{\mathbb{R}} f_j \,\mathrm{d}F_j^* + \sum_{s\in S} a^s \pi^s : \text{such that } \sum_{j=1}^d f_j + \sum_{s\in S} a^s 1_{A(s)} \leq 1_{B(x)}. \tag{3.6}$$

It is enough to maximize (3.6) over all $(a^s)_{s\in S}$ such that $\sum_s a^s \leq 1$. Indeed, let $f_1,\dots,f_d$ and $(a^s)_{s\in S}$ such that $\sum_s a^s > 1$ and $\sum_j f_j + \sum_s a^s 1_{A(s)} \leq 1_{B(x)}$. Then, for $\tilde f_1,\dots,\tilde f_d$ and $(\tilde a^s)_{s\in S}$ such that $0\leq \tilde a^s \leq a^s$, $\sum_s \tilde a^s = 1$, $\tilde f_1 = f_1 + \sum_s (a^s - \tilde a^s)$, and $\tilde f_j = f_j$ for $j=2,\dots,d$, we have $\sum_j \tilde f_j + \sum_s \tilde a^s 1_{A(s)} \leq 1_{B(x)}$, but $\sum_j \int \tilde f_j \,\mathrm{d}F_j^* + \sum_s \tilde a^s \pi^s \geq \sum_j \int f_j \,\mathrm{d}F_j^* + \sum_s a^s \pi^s$. This shows that it is enough to maximize over those $(a^s)_{s\in S}$ which satisfy $0\leq \sum_s a^s \leq 1$.

Further, in order for the admissibility constraint in (3.6) to hold, one has to compensate $a^s 1_{A(s)}$ with the nonincreasing functions $f_1,\dots,f_d$ whenever $A(s)\not\subseteq B(x)$. Therefore, by replacing $a^s 1_{A(s)}$ with $a^s(1_{A(s)} - \sum_j 1_{\{s_i\geq x_j\}} 1_{[x_j,\infty)})$, the optimal value in (3.6) is given by

$$\begin{aligned} &\sup \sum_{j=1}^d \int_{\mathbb{R}} f_j \,\mathrm{d}F_j^* + \sum_{s\in S} a^s \big(\pi^s - \sum_{j=1}^d 1_{\{s_j\geq x_j\}} F_j^*(x_j-)\big) \\ &\text{such that } \left\{ \begin{array}{l} 0\leq \sum_{s\in S} a^s \leq 1, \\ \sum_{j=1}^d f_j + \sum_{s\in S} a^s \big(1_{A(s)} - \sum_{j=1}^d 1_{\{s_j\geq x_j\}} 1_{[x_j,\infty)}\big) \leq 1_{B(x)}. \end{array}\right. \end{aligned} \tag{3.7}$$

Step 2c. The optimization problem (3.7) can be reformulated as follows. First, notice that $1_{A(s)} - \sum_{j=1}^d 1_{\{s_j\geq x_j\}} 1_{[x_j,\infty)} \leq 1_{B(x)}$ for all $s\in S$. Further, by the nonincreasingness of $\sum_{j=1}^d f_j$, the constraint

$$\sum_{j=1}^d f_j \leq 1_{B(x)} - \sum_{s\in S} a^s \big(1_{A(s)} - \sum_{j=1}^d 1_{\{s_j\geq x_j\}} 1_{[x_j,\infty)}\big)$$

is equivalent to $\sum_j f_j \leq (1-\sum_s a^s) 1_{B(x)}$. Therefore, we conclude that the optimal value of (3.7) is equal to

$$\begin{aligned} &\sup \sum_{j=1}^d \int_{\mathbb{R}} f_j \,\mathrm{d}F_j^* + \sum_{s\in S} a^s \big(\pi^s - \sum_{j=1}^d 1_{\{s_j\geq x_j\}} F_j^*(x_j-)\big) \\ &\text{such that } \left\{ \begin{array}{l} 0\leq \sum_{s\in S} a^s \leq 1, \\ \sum_{j=1}^d f_j \leq (1-\sum_{s\in S} a^s) 1_{B(x)}. \end{array}\right. \end{aligned} \tag{3.8}$$

Then, the optimal value of (3.8) is given by

$$\begin{aligned} &\Big(\sup_{\sum_{j=1}^d f_j \leq (1-\sum_{s\in S} a^s) 1_{B(x)}} \sum_{j=1}^d \int_{\mathbb{R}} f_j \,\mathrm{d}F_j^* \Big) \vee \sup_{s\in S} \Big\{ \pi^s - \sum_{j=1}^d 1_{\{s_j\geq x_j\}} F_j^*(x_j-) \Big\} \\ &= \max\Big\{1-d+\sum_{j=1}^d F_j^*(x_j-), 0\Big\} \vee \sup_{s\in S} \Big\{ \pi^s - \sum_{j=1}^d 1_{\{s_j\geq x_j\}} F_j^*(x_j-) \Big\}, \end{aligned} \tag{3.9}$$

where the last inequality follows from Step 2a. This shows that the optimal value of (3.4), which is equal to the value of (3.6) and (3.8), is given by the value in (3.9).

Step 3. It remains to show the equality (3.3) under the assumption that S is finite. We approximate $A(x)$ with open intervals from above. To that end, for every $n \in \mathbb{N}$, let $x^n = (x_1 + \frac{1}{n}, \dots, x_d + \frac{1}{n})$ so that $A(x) = \cap_{n\in\mathbb{N}} B(x^n)$. Then, by the previous Step 2, we obtain

$$
\begin{aligned}
&\inf_{F\in\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*,\dots,F_d^*)} F(x) = \inf_{F\in\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*,\dots,F_d^*)} \int_{\mathbb{R}^d} 1_{A(x)}\,\mathrm{d}F\\
&= \inf_{n\in\mathbb{N}} \inf_{F\in\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*,\dots,F_d^*)} \int_{\mathbb{R}^d} 1_{B(x^n)}\,\mathrm{d}F\\
&= \inf_{n\in\mathbb{N}} \Big(\max\Big\{1-d+\sum_{j=1}^d F_j^*(x_j^n-),0\Big\} \vee \max_{s\in S}\Big\{\pi^s - \sum_{j=1}^d 1_{\{s_j\geq x_j^n\}}\big(1-F_j^*(x_j^n-)\big)\Big\}\Big)\\
&= \max\Big\{1-d+\sum_{j=1}^d F_j^*(x_i),0\Big\} \vee \max_{s\in S}\Big\{\pi^s - \sum_{j=1}^d 1_{\{s_j> x_j\}}\big(1-F_j^*(x_j)\big)\Big\}.
\end{aligned}
$$

The infiumum and the maxima can be interchanged, since for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_j^*(x_j) + \varepsilon \geq F_j^*(x_j^n-)$ and $-1_{\{s_j>x_j\}}(1-F_j^*(x_j)) + \varepsilon \geq -1_{\{s_j\geq x_j^n\}}(1-F_j^*(x_j^n-))$ for all $s \in S$, $j = 1,\dots,d$, and $n \geq n_0$. ∎

4. Sharpness of the improved upper Fréchet–Hoeffding bound for the class $\mathcal{F}^{S,\pi}_{\preceq_0}$.

The aim of this section, is to show that the improved upper Fréchet–Hoeffding bound is pointwise sharp for the class $\mathcal{F}^{S,\pi}_{\preceq_0}$. In order to deduce this result, we will show that the primal[2] problem $\phi_0(f)$ admits an explicit solution for the function $f = 1_B$ for rectangular sets $B \subset \mathbb{R}^d$. The main result of this section is the following theorem, while the sharpness of the improved upper Fréchet–Hoeffding bound is a direct corollary thereof.

THEOREM 4.1. *Let $B \subset \mathbb{R}^d$ be a rectangular set. The following holds:*

$$\max_{\mu\in\mathcal{Q}_0} \mu(B) = \min_{i=1,\dots,d}\big\{\nu_i((-\infty,B_i])\big\} \wedge \min_{i\in I}\Big\{\bar{\pi}^i + \sum_{j=1}^d \nu_j((A_j^i,B_j])\Big\}.$$

Remark 4.2. In order to ease the presentation of the proof of this result, we consider in the following the case $d = 2$ for a box $B = (-\infty,B_1]\times(-\infty,B_2]$ and finite I, i.e., $I = \{1,\dots,n\}$. In other words, we will prove that

$$\max_{\mu\in\mathcal{Q}_0} \mu(B) = \min\big\{\nu_1((-\infty,B_1]),\nu_2((-\infty,B_2]),\min_{i\in I}\big\{\bar{\pi}^i+\nu_1((A_1^i,B_1]))+\nu_2((A_2^i,B_2])\big\}\big\}.$$

The proof for the higher-dimensional case ($d > 2$) can be obtained by analogous arguments.

[2]In some parts of the literature on optimal transportation this is called the *primal* problem (see, e.g., [33]), while in other parts this is called the *dual* problem (see, e.g., [19]).

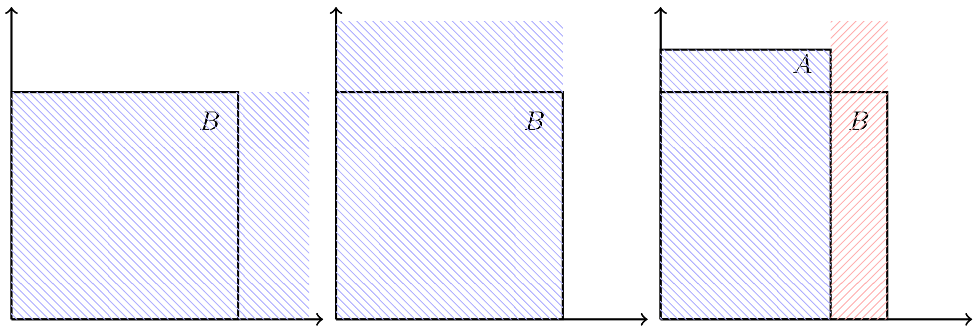

FIG. 2. *A graphical representation of Theorem* 4.1 *for* $d = 2$.

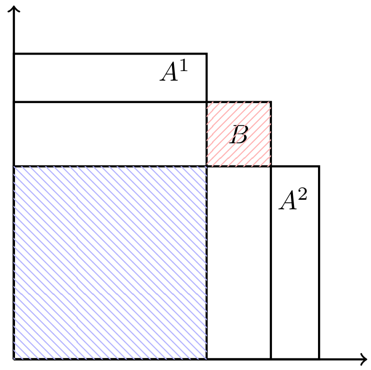

FIG. 3. *Nonoptimality of superhedging with two boxes.*

Figure 2 offers a graphical representation of Theorem 4.1 in the two-dimensional case. Let us call "box" a multiasset option with payoff 1_B with $B \subset \mathbb{R}^2$, i.e., an option that pays off one unit of currency in case all asset prices at maturity lie inside B. Moreover, let us call "strip" a single-asset option with payoff $1_{J\times\mathbb{R}}$ or $1_{\mathbb{R}\times J}$ with $J \subset \mathbb{R}$, i.e., an option that pays off one unit of currency in case the corresponding asset price at maturity lies inside J. Then, in the language of mathematical finance, this result states that there are three possible ways to superhedge the box B: either using a horizontal strip (left), or a vertical strip (middle), or another box A plus the horizontal and/or vertical strips adjacent to it (right).

Figure 3 offers an intuitive explanation of why it is not optimal to buy two boxes A^1 and A^2 in order to superhedge B, in the presence of shortselling constraints. Indeed, in case one buys both A^1 and A^2, then the shaded region is bought twice incurring unnecessary additional costs, while the shaded region is still not hedged. In order to hedge the latter, a further investment in horizontal and/or vertical strips is required, thus further increasing the cost of the hedging strategy.

Theorem 2.3 applied to $f = 1_B$ yields immediately that

$$\phi_0(1_B) = \max_{\mu\in\mathcal{Q}_0} \int 1_B \mathrm{d}\mu = \max_{\mu\in\mathcal{Q}_0} \mu(B);$$

hence we need to show that $\phi_0(1_B)$ admits the following representation:

$$\phi_0(1_B) = \min\left\{\nu_1((-\infty, B_1]), \nu_2((-\infty, B_2]), \min_{i\in I}\left\{\bar{\pi}^i + \nu_1((A_1^i, B_1])) + \nu_2((A_2^i, B_2])\right\}\right\}.$$

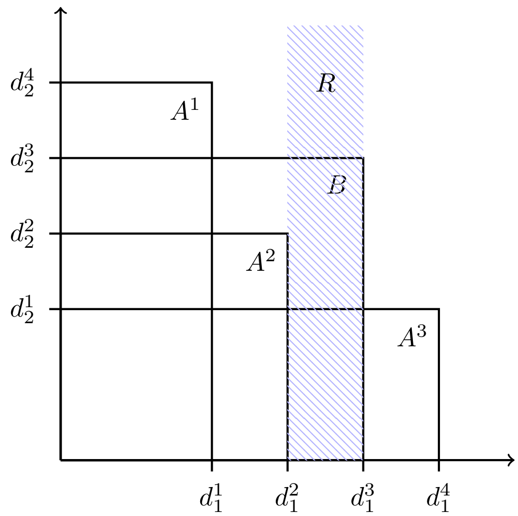

FIG. 4. *The main setting is illustrated in this figure, where* $d_1^3 = B_1$ *hence* $i_1 = 3$.

Let us introduce some notation now that will be used in the subsequent proofs; it is illustrated in Figure 4. Define $D_j := \{A_j^i : i \in I\} \cup \{B_j\}$ for $j = 1, 2$, and let $D_j = \{d_j^k : k = 1, \dots, m_j\}$ be an enumeration such that $d_j^1 < d_j^2 < \cdots < d_j^{m_j}$. Further, define $F_j^0 := (-\infty, d_j^1]$, $F_j^i := [d_j^i, d_j^{i+1})$ for $i = 1, \dots, m_j - 1$, and $F_j^{m_j} := (d_j^{m_j}, \infty)$ for $j = 1, 2$. Moreover, let i_1 be such that $d_1^{i_1} = B_1$. In a first step, notice that in the definition of $\phi_0(1_B)$ we can and will restrict ourselves, without loss of generality, to functions f_j of the form

$$f_j(x) := \sum_{i=1}^{m_j} f_j^i 1_{F_j^i}(x), \quad \text{where } f_j^i \text{ are positive constants.}$$

We will refer to the functions f_1 as "vertical marginals" and to the functions f_2 as "horizontal marginals."

LEMMA 4.3. *Let* $R := F_1^{i_1 - 1} \times \mathbb{R}$. *Then*

$$\phi_0(1_B) = \min_{s \in \{0,1\}} \big\{ s\nu_1(F_1^{i_1-1}) + s\phi_0(1_{B \setminus R}) + (1-s)\eta(1_B) \big\}, \tag{4.1}$$

where

$$\begin{aligned} \eta(1_B) &:= \inf \Big\{ \pi(0, f_2, a) : f_2(x_2) + \sum_{i \in I} a^i 1_{A^i}(x) = 1 \ \ \textit{for all } x \in B \textit{ and } f_2, a^i \geq 0 \Big\} \\ &= \min \Big\{ \nu_2((-\infty, B_2]), \min_{i \in I : B_1 \leq A_1^i} \big\{ \bar{\pi}^i + \nu_2((A_2^i, B_2]) \big\} \Big\}. \end{aligned} \tag{4.2}$$

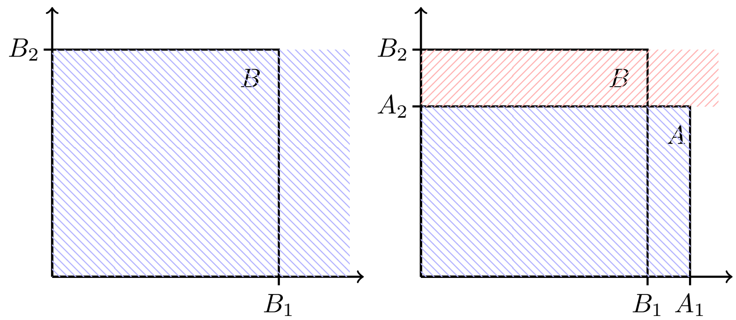

FIG. 5. *A graphical representation of the functional* $\eta(1_B)$.

The functional $\eta(1_B)$ is graphically illustrated in Figure 5 and states that there are two ways to superhedge the box B without using the vertical marginals: either using the horizontal strip $1_{\mathbb{R}\times(-\infty,B_2]}$ or using another box A with $A_1 \geq B_1$ and, in case $B_2 > A_2$, the horizontal strip "above" this box, i.e., $1_{\mathbb{R}\times(A_2,B_2]}$.

Proof. Initially, notice that all optimization problems appearing are finite-dimensional linear problems, so that minimizers always exist.

We start by proving (4.1) and first show that the LHS is smaller than the RHS. Indeed, in case $s = 0$, this reduces to the fact that obviously $\phi_0(1_B) \leq \eta(1_B)$, since $\phi_0(1_B)$ is defined as the infimum over a larger set. In case $s = 1$, let $(f_1, f_2, a) \in \Theta_0(1_{B\setminus R})$ be optimal—in the sense that $\pi(f_1, f_2, a) = \phi_0(1_{B\setminus R})$—and notice that one can assume without loss of generality that $f_1^{i_1-1} = 0$. Now define

$$\hat{f}_1^i := \begin{cases} f_1^i & \text{if } i \neq i_1 - 1, \\ 1 & \text{else,} \end{cases}$$

and it follows that $(\hat{f}_1, f_2, a) \in \Theta_0(1_B)$. By the definition of $\hat{f}_1^i$ it holds that

$$\pi(\hat{f}_1, f_2, a) = \nu_1(F_1^{i_1-1}) + \pi(f_1, f_2, a) = \nu_1(F_1^{i_1-1}) + \phi_0(1_{B\setminus R}),$$

which shows that $\phi_0(1_B) \leq \nu_1(F_1^{i_1-1}) + \phi_0(1_{B\setminus R})$.

In order to prove the reverse inequality, notice that by interchanging two minima it holds that

$$\phi_0(1_B) = \min_{s\in[0,1]} \big\{ s\nu_1(F_1^{i_1-1}) + \phi_0^{\setminus i_1-1}(1_B - s1_R) \big\}, \tag{4.3}$$

where

$$\phi_0^{\setminus i_1-1}(1_B - s1_R) := \inf \big\{ \pi(f_1, f_2, a) : (f_1, f_2, a) \in \Theta_0(1_B - s1_R) \text{ and } f_1^{i_1-1} = 0 \big\}.$$

Fix some optimal s in (4.3) and an optimal strategy (f_1, f_2, a) for $\phi_0^{\setminus i_1-1}(1_B - s1_R)$. Since $f_1^{i_1-1} = 0$, it follows that

$$f_2(x_2) + \sum_{i\in I} a^i 1_{A^i}(x) = f_2(x_2) + \sum_{i\in I : B_1 \leq A_1^i} a^i 1_{A^i}(x) \geq 1 - s \quad \text{for all } x \in B \cap R.$$

Let $t := \sum_{i\in I : B\subset A^i} a^i$. On the one hand, if $t \geq 1 - s$, set $\bar{a}^i := (1-s)a^i/t$ for every i such that $B \subset A^i$, $\bar{a}^i = 0$ else, and $\bar{f}_2 = 0$. Then $\sum_{i\in I} \bar{a}^i 1_{A^i}(x) = 1 - s$ for $x \in B$;

thus $(0,0,\bar a)$ is an admissible strategy for $\eta((1-s)1_B)=(1-s)\eta(1_B)$. Further define $\tilde a := a-\bar a \geq 0$. Then one can check that $(f_1,f_2,\tilde a)\in\Theta_0(s1_{B\setminus R})$. Therefore

$$
\begin{aligned}
\phi_0(1_B) &= s\nu_1(F_1^{i_1-1})+\pi(f_1,f_2,a)\\
&= s\nu_1(F_1^{i_1-1})+\pi(f_1,f_2,\tilde a)+\pi(0,0,\bar a)\\
&\geq \min_{s\in[0,1]}\big\{s\nu_1(F_1^{i_1-1})+s\phi_0(1_{B\setminus R})+(1-s)\eta(1_B)\big\}.
\end{aligned}
\tag{4.4}
$$

Moreover, since the last term is affine in s, it follows that the minimum over $s\in[0,1]$ yields the same value as the minimum over $s\in\{0,1\}$.

On the other hand, assume that $t<1-s$, and define $\bar a^i := a^i$ for all i such that $B\subset A^i$. For notational convenience we assume that $A_1^i \geq B_1$ exactly for $i=1,\dots,m$ and that $B_2 > A_2^1 > A_2^2 > \cdots > A_2^m$; the case where $A_2^i = A_2^j$ for some $i,j\leq m$ works in the same way but requires additional notation. Further denote by k_0 the index such that $d_2^{k_0}=B_2$ and by k_i the index such that $d_2^{k_i}=A_2^i$, for $i=1,\dots,m$. Then, for every $i=k_1,\dots,k_0-1$ it needs to hold that $f_2^i \geq \bar f_2^i := 1-s-t>0$. Moreover

$$
f_2^i + a^1 \geq 1-s-t \quad \text{for } k_2\leq i\leq k_1-1,
\tag{4.5}
$$

i.e., $f_2(x_2)+a^1\geq 1-s-t$ for all $x\in R$ with $x_2\in(A_2^1,B_2]$. Now, there are two possibilities:

- If $a^1 \geq \bar a^1 := 1-s-t$, then set $\bar f_2^i := 0$ for $i\leq k_1-1$ and $\bar a^i := 0$ for $i=2,\dots,m$. Then $(0,\bar f_2,\bar a)$ is an admissible strategy for $\eta(1_B)$ and $(f_1,\tilde f_2,\tilde a)\in\Theta_0(s1_{B\setminus R})$, where $\tilde f_2 := f_2-\bar f_2$ and $\tilde a := a-\bar a$. Hence, it follows from linearity of π, as in (4.4), that
$$\phi_0(1_B)\geq \min_{s\in[0,1]}\big\{s\nu_1(F_1^{i_1-1})+s\phi_0(1_{B\setminus R})+(1-s)\eta(1_B)\big\}.$$
- Otherwise, if $\bar a^1 := a^1 < 1-s-t$, define $\bar f_2^i := 1-s-t-a^1 \leq f_2^i$ for all $k_2\leq i\leq k_1-1$, and set $\tilde t := t+a^1$. Then
$$\bar f_2(x_2)+\sum_{i\in I: B_1\leq A_1^i}\bar a^i 1_{A^i}(x) = 1-s \quad \text{for } x\in B \text{ such that } A_2^1\leq x_2\leq B_2$$
and necessarily $f_2^i + a^2 \geq 1-s-\tilde t$ for $k_3\leq i\leq k_2-1$. This means that the situation is the same as in (4.5). Repeating this procedure at most m times, one finds an admissible strategy $(0,\bar f_2,\bar a)$ for $\eta(1_B)$. Since $(f_1,\tilde f_2,\tilde a)\in\Theta_0(s1_B)$, where $\tilde f_2 := f_2-\bar f_2\geq 0$ and $\tilde a := a-\bar a\geq 0$, it follows from the linearity of π that (4.4) holds true.

We proceed now with the proof of (4.2). First notice that for all i with $B\subset A^i$ it holds that
$$\eta(1_B)=\min_{a^i\in[0,1]}\big\{a^i\bar\pi^i+(1-a^i)\eta^{\setminus i}(1_B)\big\},$$
where $\eta^{\setminus i}$ is defined as η, with the additional requirement that $a^i=0$. Hence $a^i\in\{0,1\}$. If $a^i=1$ for some i with $B\subset A^i$, then the proof is complete. Otherwise denote by l an element in $\tilde I := \{i\in I : B_1\leq A_1^i \text{ and } A_2^i\leq B_2\}$ such that $A_2^i\leq A_2^l$ for all $i\in\tilde I$. Then $l=d_2^k$ for some k and it necessarily has to hold that $f_2=1$ on $(A_2^l,B_2]$. Thus
$$\eta(1_B)=\nu_2((A_2^l,B_2])+\eta(1_{B\setminus R}),$$
where $R:=\mathbb{R}\times(A_2^l,B_2]$. Since $B\setminus R$ is again a box, the claim now follows by induction. ∎

We are now ready to prove the main result of this section.

Proof of Theorem 4.1. Let $R := F_1^{i_1-1}\times\mathbb{R}$. If $s=0$ and $s=1$ are both optimizers in (4.1), we always chose $s=0$ in order to exclude many pathological cases (see the proof below).

Case 1*:* If $s=0$, this means that $\phi_0(1_B)=\eta(1_B)$. However, by (4.2) an optimal strategy for $\eta(1_B)$ consists of either the full horizontal marginal, i.e., $f_2 = 1_{(-\infty,B_2]}$ and $a=0$, or exactly one box A^i with $B_1 \leq A_1^i$ (i.e., $a^i=1$ and $a^j=0$ for $j\neq i$) and the horizontal marginal "above" this box, i.e., $f_2 = 1_{(A_2^i,B_2]}$; see again Figure 5. Since both strategies are elements of $\Theta_0(1_B)$, the proof is complete.

Case 2*:* If $s=1$, this means that an optimal strategy for $\phi_0(1_B)$ consists of $f_1^{i_1-1}=1$ plus an optimizer for $\phi_0(1_{B\setminus R})$. If $B\setminus R$ is empty, this means that the optimizer of $\phi_0(1_B)$ is the full vertical marginal, i.e., $f_1 = 1_{F_1^{i_1-1}} = 1_{(-\infty,B_1]}$. Otherwise notice that $\hat{B} := B\setminus R$ is again a (nonempty) box. Hence one can apply Lemma 4.3 again: Define $\hat{R} := F_1^{i_1-2}\times\mathbb{R}$ so that

$$\phi_0(1_{\hat{B}}) = \min_{s\in\{0,1\}}\left\{s\nu_1(F_1^{i_2-1}) + s\phi_0(1_{\hat{B}\setminus\hat{R}}) + \eta(1_{\hat{B}})\right\}.$$

Now, there are again two possibilities:

- If $s=0$, i.e., $\phi_0(1_{\hat{B}})=\eta(1_{\hat{B}})$, then an optimal strategy for $\eta(1_{\hat{B}})$ consists of either the full horizontal marginal $f_2 = 1_{(-\infty,\hat{B}_2]} = 1_{(-\infty,B_2]}$ only or exactly one box A^i with $\hat{B}_1 \leq A_1^i$ and the horizontal marginal above the box, i.e., $f_2 = 1_{(A_2^i,\hat{B}_2]} = 1_{(A_2^i,B_2]}$. We claim that the first case cannot happen, while in the second one it holds that $A_1^i=\hat{B}_1$. Indeed, if $f_2=1_{(-\infty,B_2]}$ is optimal, then $(0,f_2,0)\in\Theta_0(1_B)$. In particular the previous choice $f_1^{i_1-1}=1$ was not optimal. Similarly, it follows that in the second case $A_1^i=\hat{B}_1$.
- If $s=1$, then the optimal strategy for $\phi_0(1_{\hat{B}})$ consists of $f_1^{i_1-1}=f_1^{i_1-2}=1$ plus the optimal one for $\phi_0(1_{\hat{B}\setminus\hat{R}})$.

By induction, it follows that an optimal strategy for $\phi_0(1_B)$ can take one of the following forms:

$$\begin{aligned}
\text{either}\quad & f_1 = 1_{(\infty,B_1]}, f_2=0, \text{ and } a=0\\
\text{or}\quad & f_1 = 0, f_2 = 1_{(-\infty,B_2]}, \text{ and } a=0\\
\text{or}\quad & f_1 = 1_{(A_1^i,B_1]}, f_2 = 1_{(A_2^i,B_2]}, \text{ and } a^j=1 \text{ if } j=i \text{ and } a^j=0 \text{ else;}
\end{aligned}$$

compare again with Figure 2. This completes the proof. ∎

Now we are ready to provide the pointwise sharpness result for the improved upper Fréchet–Hoeffding bound for the class $\mathcal{F}^{S,\pi}_{\preceq_0}$.

COROLLARY 4.4. *The following holds, for every $x\in\mathbb{R}^d$:*

$$\max_{F\in\mathcal{F}^{S,\pi}_{\preceq_0}(F_1^*,\dots,F_d^*)} F(x) = \min_{i=1,\dots,d} F_i^*(x_i)\wedge\min\left\{\pi_s+\sum_{i=1}^d\left(F_i^*(x_i)-F_i^*(s_i)\right)^+ : s\in S\right\}.$$

Proof. This is a reformulation of Theorem 4.1 in the language of distribution functions. Indeed, the set $\mathcal{Q}_0$ contains all measures induced by the distribution functions in $\mathcal{F}^{S,\pi}_{\preceq_0}(F_1^*,\dots,F_d^*)$, and vice versa; see again Figure 1. ∎

Remark 4.5. Let us point out that an analogous result for the improved lower Fréchet–Hoeffding bound remains an open question for future research. The necessary duality result is straightforward in this case; however, the analogon of Theorem 4.1 cannot be proved with the techniques developed in the present article.

Appendix A. The improved upper Fréchet–Hoeffding bound is not pointwise sharp for the class $\mathcal{F}^{S,\pi}$.

The following counterexample—communicated to us by Stephan Eckstein—illustrates that the improved Fréchet–Hoeffding bounds (1.4) are in general not pointwise sharp for $\mathcal{F}^{S,\pi}(F_1^*, \dots, F_d^*)$, even in dimension $d = 2$.

Example A.1. The marginal cdfs are given by

$$F_1^* := F_2^* := 0.1 \cdot 1_{[0,1)} + 0.3 \cdot 1_{[1,2)} + 0.35 \cdot 1_{[2,3)} + 1_{[3,\infty)},$$

i.e., F_i^* are cdfs of the probability measure $0.1\delta_0 + 0.2\delta_1 + 0.05\delta_2 + 0.65\delta_3$. Consider the additional information

$$S = \{(0,0),(0,2),(2,0),(1,1)\} \quad \text{with} \quad \pi_{(0,0)} = 0 \text{ and } \pi_{(0,2)} = \pi_{(2,0)} = \pi_{(1,1)} = 0.1.$$

For the cdf $\hat{F}$ which corresponds to the probability measure $\sum_{x_1,x_2=0}^{3} c_{x_1,x_2} \delta_{(x_1,x_2)}$ with weights given by Table 1 one can verify that

$$\hat{F} \in \mathcal{F}^{S,\pi}(F_1^*, F_2^*) := \big\{F \in \mathcal{F}(F_1^*, F_2^*) \colon F(s) = \pi_s \text{ for all } s \in S\big\}.$$

This shows that $\mathcal{F}^{S,\pi}(F_1^*, F_2^*) \neq \emptyset$. Let $x = (x_1, x_2) := (0,1)$; then the improved upper Fréchet–Hoeffding bound is given by

$$\min\{F_1^*(0), F_2^*(1)\} \wedge \min\Big\{\pi_s + \sum_{j=1}^{2} (F_j^*(x_j) - F_j^*(s_j))^+ : s \in S\Big\} = 0.1,$$

whereas for $\varphi(u) = 1_{\{u \le x\}}$ it can easily be checked that

$$\sup_{F \in \mathcal{F}^{S,\pi}(F_1^*, F_2^*)} \int \varphi \,\mathrm{d}F = \sup_{F \in \mathcal{F}^{S,\pi}(F_1^*, F_2^*)} F(0,1) = \hat{F}(0,1) = 0.05.$$

Moreover, the improved lower Fréchet–Hoeffding bound is provided by

$$\big(F_1^*(0) + F_2^*(1) - 1\big)^+ \vee \max\Big\{\pi_s - \sum_{i=1}^{2} \big(F_i^*(s_i) - F_i^*(x_i)\big)^+ : s \in S\Big\} = 0,$$

while the infimum takes the value 0.05.

TABLE 1
Weights of the joint probability measure.

c_{x_1,x_2}	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$
$x_1 = 0$	0	0.05	0.05	0
$x_1 = 1$	0.05	0	0	0.15
$x_1 = 2$	0.05	0	0	0
$x_1 = 3$	0	0.15	0	0.5

Remark A.2. Tankov in [32] showed that in dimension 2, the upper improved Fréchet–Hoeffding bound is a copula in $\mathcal{F}^{S,\pi}$, and hence sharp, whenever the set S is decreasing, i.e., $(u_1 - v_1)(u_2 - v_2) \le 0$ for all $(u_1,u_2),(v_1,v_2) \in S$. Example A.1 is surprising in so far as it demonstrates that when S fails to be decreasing, then neither the copula property of the improved upper bound nor its pointwise sharpness over the set $\mathcal{F}^{S,\pi}$ is guaranteed. Note that the set $S = \{(0,0),(0,2),(2,0),(1,1)\}$ in the example is not decreasing, as for $(0,0),(1,1)$ we have $(1-0)(1-0)=1$.

Bernard, Jiang, and Vanduffel [3] extended Tankov's result and showed that the upper improved Fréchet–Hoeffding bound is a copula under weaker conditions, namely, it suffices that S is such that either the functions

$$\min\{u\colon (u,v)\in S\} \quad \text{and} \quad \max\{u\colon (u,v)\in S\}$$

are nonincreasing while $(\frac{u_1+u_2}{2}, v)\in S$ for all $(u_1,v),(u_2,v)\in S$ or the functions

$$\min\{v\colon (u,v)\in S\} \quad \text{and} \quad \max\{v\colon (u,v)\in S\}$$

are nonincreasing while $(u, \frac{v_1+v_2}{2})\in S$ for all $(u,v_1),(u,v_2)\in S$. Evidently, the set $S=\{(0,0),(0,2),(2,0),(1,1)\}$ fulfills neither of the two conditions, as $\min\{u\colon (u,0)\in S\} = 0 \le \min\{u\colon (u,1)\in S\} = 1$ and analogously for $\min\{v\colon (u,v)\in S\}$.

Appendix B. Derivation of the improved Fréchet–Hoeffding bounds for the classes $\mathcal{F}^{S,\pi}_{\preceq_0}$, $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$ and $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$.

In this appendix, we show that the improved upper Fréchet–Hoeffding bound is valid for the classes $\mathcal{F}^{S,\pi}_{\preceq_0}$ and $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}$, while the improved lower Fréchet–Hoeffding bound is valid for the class $\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}$. The derivation uses *simple* arguments borrowed from copula theory; see, e.g., [20]. Let us point out that the sharpness results in section 3 and section 4 allow us to recover the statements proved below.

Lemma B.1. *Let $G\in\mathcal{F}^{S,\pi}_{\preceq_0}(F_1^*,\dots,F_d^*)$, $H\in\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*,\dots,F_d^*)$, and $J\in\underline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*,\dots,F_d^*)$. Then, for all $(x_1,\dots,x_d)\in\mathbb{R}^d$, we have that*

$$G(x_1,\dots,x_d) \le \min_{i=1,\dots,d} F_i^*(x_i)\wedge \min\Big\{\pi_s+\sum_{i=1}^d \big(F_i^*(x_i)-F_i^*(s_i)\big)^+ : s\in S\Big\},$$
$$H(x_1,\dots,x_d) \le \min_{i=1,\dots,d} F_i^*(x_i)\wedge \min\{\pi_s : s\in S \text{ such that } x\le s\},$$
$$J(x_1,\dots,x_d) \ge \Big(\sum_{i=1}^d F^*(x_i)-d+1,\Big)^+ \vee \max_{s\in S}\Big\{\pi^s-\sum_{i=1}^d \big(1-F_i^*(x_i)\big)1_{\{s_i>x_i\}}\Big\}.$$

Proof. By the definition of the class $\mathcal{F}^{S,\pi}_{\preceq_0}(F_1^*,\dots,F_d^*)$, we have immediately that

$$G(x_1,\dots,x_d) = cF(x_1,\dots,x_d)\le c\min_{i=1,\dots,d}F_i(x_i) = \min_{i=1,\dots,d} cF_i(x_i)\le \min_{i=1,\dots,d}F_i^*(x_i). \tag{B.1}$$

Moreover, for any $x_i,s_i\in\mathbb{R}$, it holds that $F(x_1,\dots,x_i,\dots,x_d)-F(x_1,\dots,s_i,\dots,x_d)\le \big(F_i(x_i)-F_i(s_i)\big)^+$; hence, using a telescoping sum, we get that

$$F(x)-F(s)\le\sum_{i=1}^d\big(F_i(x_i)-F_i(s_i)\big)^+. \tag{B.2}$$

Therefore, using again the properties of this class, we arrive at

$$G(x)\le cF(s)+\sum_{i=1}^d\big(cF_i(x_i)-cF_i(s_i)\big)^+\le \pi_s+\sum_{i=1}^d\big(F_i^*(x_i)-F_i^*(s_i)\big)^+. \tag{B.3}$$

We conclude by minimizing over all $s \in S$ and combining the outcome with (B.1).

Now, let $H \in \overline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*, \ldots, F_d^*)$; then we get immediately that $H(x_1, \ldots, x_d) \leq \min_{i=1,\ldots,d} F_i^*(x_i)$. Moreover, the estimate in (B.2) is still valid; therefore from the definition of $\overline{\mathcal{F}}^{S,\pi}_{\preceq_1}(F_1^*, \ldots, F_d^*)$ we arrive at an estimate similar to the first inequality in (B.3) (with G replaced by H and $c = 1$). However, the information available on the marginals does not allow us to estimate the difference $F_i(x_i) - F_i(s_i)$, and the best we can say is that for $x \leq s$ this term collapses to zero. The statement follows once again by minimizing over all $s \in S$. Finally, the last part is similar to the other two, and the proof is omitted for the sake of brevity. ∎

Appendix C. Proof of duality results.

In this appendix, we provide sketches of the proofs of the duality results for the sake of brevity. Let us first introduce the functionals $\phi^*_{C_b}$ and $\phi^*_{U_b}$ which are defined as follows:

$$\phi^*_{C_b}(\mu) := \sup_{f \in C_b} \Big\{ \int f \mathrm{d}\mu - \phi(f) \Big\} \quad \text{and} \quad \phi^*_{U_b}(\mu) := \sup_{f \in U_b} \Big\{ \int f \mathrm{d}\mu - \phi(f) \Big\}, \tag{C.1}$$

where U_b denotes the set of all bounded upper semicontinuous functions $f : \mathbb{R}^d \to \mathbb{R}$, and C_b the set of all bounded continuous functions. Then, the representation result stated in the following paragraph holds true.

Let $\phi\colon U_b \to \mathbb{R}$ be a convex and increasing function, and assume that for every sequence (f^n) of continuous bounded functions such that f^n decreases pointwise to 0, it holds that $\phi(f^n) \downarrow \phi(0)$. Then, ϕ admits the following representation:

$$\phi(f) = \max_{\mu \in ca^+} \Big\{ \int f \mathrm{d}\mu - \phi^*_{C_b}(\mu) \Big\} \tag{C.2}$$

for all $f \in C_b$. Assume, in addition, that $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for any $\mu \in ca^+$; then $\phi(f) = \max_{\mu \in ca^+} \{\int f \mathrm{d}\mu - \phi^*_{U_b}(\mu)\}$ for all $f \in U_b$. The proof is similar to [2, Theorem 2.2]; see also [8, Theorem A.5].

Proof of Theorem 2.3. By definition, $\phi\colon U_b \to \mathbb{R} \cup \{-\infty\}$ is a sublinear and increasing function. By the exclusion of uniform strong arbitrage, it satisfies $\phi(m) = m$ for all $m \in \mathbb{R}$. Moreover, for every sequence (f^n) in C_b such that $f^n \downarrow 0$ pointwise, by tightness of the marginals $\nu_1, \ldots, \nu_d$, there exists a compact $K \subset \mathbb{R}^d$ such that $\phi(f^1 1_{K^c})$ is arbitrarily small, which by Dini's lemma ensures that $\phi(f^n) \downarrow 0$. Further, an explicit computation of the conjugates yields

$$\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) = \begin{cases} 0 & \text{if } \mu \in \mathcal{Q}, \\ +\infty & \text{otherwise,} \end{cases}$$

which by the above representation result implies (2.2) and that $\mathcal{Q}$ is nonempty. Conversely, if $\mathcal{Q}$ is not empty, it is straightforward to verify that there does not exist uniform strong arbitrage. The proof of (2.3) works analogously. ∎

Proof of Corollary 2.6. Define

$$\phi_{\max}(f) = \inf \Big\{ \sum_{j=1}^d \int_{\mathbb{R}} f_j \, \mathrm{d}\nu_j + \int_{\mathbb{R}} g \, \mathrm{d}\nu_{\max} : f_1, \ldots, f_d, g \Big\},$$

where $f_1, \ldots, f_d, g$ satisfy inequality (2.5). Verification shows that $\mathcal{Q}$ has the form given in (2.4), and $\phi(f) \geq \phi_{\max}(f) \geq \sup_{\mu \in \mathcal{Q}} \int_{\mathbb{R}^d} f \, \mathrm{d}\mu$. Finally, Theorem 2.3 yields that the previous inequalities are actually equalities. ∎

Proof of Theorem 2.7. First, on can check that $\phi_1\colon U_b \to \mathbb{R}$ is a sublinear and increasing functional which satisfies $\phi_1(m) = m$ for all $m \in \mathbb{R}$. Moreover, it holds that $\phi_1(f) \geq \int_{\mathbb{R}^d} f \, \mathrm{d}\mu$ for every $\mu \in ca_1^+(\mathbb{R}^d)$ such that $\underline{\nu}_j \preceq_1 \mu_j \preceq_1 \overline{\nu}_j$ for all $1 \leq j \leq d$ and $\underline{\pi}_i \leq \mu(A^i) \leq \overline{\pi}_i$ for all $i \in I$. Finally, let (f^n) be a sequence of bounded continuous functions which decreases pointwise to 0. For $\varepsilon > 0$, fix $m \in \mathbb{N}$ such that

$$\max\left\{\underline{\nu}_j((-\infty, -m+1]), \overline{\nu}_j([m-1, \infty))\right\} \leq \frac{\varepsilon}{cd}$$

for every $j = 1, \ldots, d$, where $c := \sup_{x \in \mathbb{R}^d} f^1(x)$, and define the nondecreasing functions $f_j(t) := c(1 + (0 \vee (t+1-m) \wedge 1))$ and $g_j(t) := c(0 \vee (t+m) \wedge 1)$. Then

$$f_j - g_j \geq \sup_{x \in \mathbb{R}^d} f_1(x) 1_{[-m,m]^c} \quad \text{and} \quad \int_{\mathbb{R}} f_j \, \mathrm{d}\overline{\nu}_j - \int_{\mathbb{R}} g_j \, \mathrm{d}\underline{\nu}_j \leq \frac{\varepsilon}{d},$$

which shows that $\phi_1(f^1 1_{K^c}) \leq \varepsilon$ for the compact $K = [-m, m]^d$. It follows from Dini's lemma that $\phi_1(f^n) \downarrow 0$. ∎

Appendix D. Proof of optimality results.

Proof of Proposition 2.8. We start with the proof for ϕ_0. By definition, there exists a sequence $(f_1^n, \ldots, f_d^n, a_n)_n$ in $\Theta_0(f)$ such that $\pi(f_1^n, \ldots, f_d^n, a_n)$ converges to $\phi_0(f)$. As all f_i^n are positive, an (iterated) application of Komlós's lemma (cf. [11, Lemma A.1]) implies the existence of forward convex combinations (denoted by $\tilde{f}_i^n$ for $i = 1, \ldots, d$) which have a μ_i-almost sure limit. Calling $(\tilde{a}_n)_n$ the sequence which emerges from $(a_n)_n$ by applying the same convex combinations as used for $\tilde{f}^n$, we find that

$$(\tilde{f}_1^n, \ldots, \tilde{f}_d^n, \tilde{a}_n) \in \Theta_0(f)$$

(as the constraint defining Θ_0 is linear). Moreover, convexity of π implies that $\phi_0(f) = \inf_n \pi(\tilde{f}_1^n, \ldots, \tilde{f}_d^n, \tilde{f}_n)$. In particular, for every $i \in I$ with $\pi_i > 0$, the sequence $(\tilde{a}_n^i)_n$ needs to be bounded. For every $i \in I$ with $\pi_i = 0$, set $\tilde{a}_n^i := \|f\|_\infty$. This does not change the admissibility, and this modified sequence is still a minimizing sequence.

Now we may pass to a (not relabeled) subsequence for which $\tilde{a}_n^i$ has a limit; denote the limit by a^i. Moreover, denote by $f_i := \limsup_n \tilde{f}_i^n$ for $i = 1, \ldots, d$. As the limsup is μ_i-almost surely an actual pointwise limit, Fatou's lemma implies

$$\pi(f_1, \ldots, f_d, a) \leq \liminf_n \pi(\tilde{f}_1^n, \ldots, \tilde{f}_d^n, \tilde{a}^n) = \phi_0(f).$$

Further, superlinearity of the limsup shows that $(f_1, \ldots, f_d, a) \in \Theta_0(f)$. In conclusion, $(f_1, \ldots, f_d, a)$ is the desired strategy.

We now come to the proof of the second part concerning ϕ. Denote by

$$\mathcal{T}(\cdot) := \inf\{\pi(f_1, \ldots, f_d, 0) : (f_1, \ldots, f_d, 0) \in \Theta(\cdot)\}$$

the value of the classical optimal transport problem. Then we can rewrite ϕ in the sense that

$$\phi(f) = \inf_{a \in \mathbb{R}^I} \Big\{ \mathcal{T}\big(f - \sum_{i \in I} a^i 1_{A^i}\big) + \pi(0, \ldots, 0, a_n) \Big\}. \tag{D.1}$$

Moreover, for any bounded measurable function g, we have $\mathcal{T}(g) = \sup_{\mu \in \mathrm{Cpl}} \int g \, \mathrm{d}\mu$, where Cpl denotes the set of all couplings between $\mu_1, \ldots, \mu_d$; see [19, Corollary 2.15]. Further, employing similar arguments as in the previous step shows that the infimum in the definition of $\mathcal{T}(g)$ is attained for every bounded function g; alternatively use [19, Theorem 2.21].

In conclusion, it remains to show that the infimum over $a \in \mathbb{R}^I$ in (D.1) is attained. To that end, it is enough to show that every minimizing sequence (a_n) needs to be bounded. Heading for a contradiction, assume otherwise; that is, after passing to a (not relabeled) subsequence, one has $|a_n| \to \infty$. Then, possibly passing to another subsequence, $a^n/|a^n|$ converges to some $\bar{a} \in \mathbb{R}^I$ with $|\bar{a}| = 1$. At this point we make use of our assumption that there is $\bar{\mu} \in \mathrm{Cpl}$ with $\bar{\varepsilon} := \pi(0, \ldots, 0, \bar{a}) - \sum_{i \in I} \bar{a}^i \bar{\mu}(A^i) > 0$. Since clearly $\mathcal{T}(\cdot) \geq \int \cdot \, \mathrm{d}\bar{\mu}$, we can estimate

$$\mathcal{T}\big(f - \sum_{i \in I} a_n^i 1_{A^i}\big) + \pi(0, \ldots, 0, a_n) \geq \pi(0, \ldots, 0, a_n) - \sum_{i \in I} a_n^i \bar{\mu}(A^i) - \|f\|_\infty.$$

As a_n behaves likes $|a_n|\bar{a}$ for large n, the last term above behaves like $|a_n|\bar{\varepsilon} \to \infty$ for large n. This gives the desired contradiction and thus completes the proof. ◻

Proof of Corollary 2.10. Assume that $\mu^\star \in \mathcal{Q}$ is optimal, and let $(f_1, \ldots, f_d, a) \in \Theta(f)$ be the optimizer obtained in Proposition 2.8. By definition of $\Theta(f)$, the LHS in (2.7) is larger than the RHS for every $x \in \mathbb{R}^d$. Employing that $(f_1, \ldots, f_d, a)$ attains the minimum in $\phi(f)$, we get

$$\int_{\mathbb{R}^d} f \, \mathrm{d}\mu^\star = \phi(f) = \int_{\mathbb{R}^d} \big\{ f_1(x_1) + \cdots + f_d(x_d) + \sum_{i \in I} a^i 1_{A^i}(x) \big\} \, \mu^\star(\mathrm{d}x).$$

This shows that the RHS in (2.7) needs to be larger than the LHS, at least for $\mu^\star$-almost every $x \in \mathbb{R}^d$.

To prove the reverse direction, let $(f_1, \ldots, f_d, a) \in \Theta(f)$ such that (2.7) holds true. By definition of ϕ and as $\mu^\star \in \mathcal{Q}$, we have

$$\int_{\mathbb{R}^d} f \, \mathrm{d}\mu^\star \leq \pi(f_1, \ldots, f_d, a) = \int_{\mathbb{R}^d} \big\{ f_1(x_1) + \cdots + f_d(x_d) + \sum_{i \in I} a^i 1_{A^i}(x) \big\} \, \mu^\star(\mathrm{d}x).$$

The last term equals $\int_{\mathbb{R}^d} f \, \mathrm{d}\mu^\star$ by assumption (2.7), which completes the proof. ◻

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