A Unified Approach towards Decomposition and Coordination for Multi-level Optimization

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PROEFSCHRIFT

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To my wife Julia

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Chapter 1

Introduction

1.1 Background

1.1.1 Hierarchical nature of today's complex structures

In today's complex structures, such as encountered in aerospace engineering, multiple levels of detail are distinguishable, that form a hierarchy of components that goes down to the material level. The origin of these levels is found in the growing complexity of structures, improvements in technology and higher demands on individual components. Tight interaction of components of the hierarchy with neighboring components is necessary such that better structural or material characteristics are exhibited that would not be possible when these components would be considered individually.

Because physicists have not come up with a single model that describes the entire universe in space and time, one has to rely on models that are only accurate within certain assumptions. Even if such an all-in-one model would be possible, it is unlikely that such an analysis model could be solved with sufficient resolution by a single person with or without the use of a computer. The amount of data to be processed and the complexity of the underlying mathematics would just be too large to handle at once.

Hence, the analysis of a complex structure involves accounting for physical phenomena that occur at various physical scales. Furthermore, in order to reduce the complexity of analyzing a structure's physical responses, it is necessary to identify levels in the structure that allow for a decomposed approach of the analysis. This decomposed analysis approach may involve models of physical scales, ever increasingly structural detail and/or a decomposition along disciplinary boundaries. These disciplines require specialist expertise in order to evaluate the physical responses.

The hierarchical nature of complex structures is also reflected in the design optimization process where design variables and design constraints are identified that are specific to an individual component or shared among various members of the hierarchy. The levels that are identified in the design optimization problem enable design optimization considerations specific to the performance of the entire structure to be separated from design optimization considerations that are specific to individual components.

Individual components that are optimized separately, hence without considering the entire hierarchy, may lead to designs that are optimal with respect to individual component demands. However, such an approach may lead to non-optimal structures that are a combination of such individually optimal components. Furthermore, not every desirable performance can be accomplished within a design optimization of an individual component. Multiple components combined covering various physical scales, structural detail and/or disciplinary expertise may be necessary to find a desirable performance. Hence, the design of complex structures benefits from a hierarchical design process that considers multiple physical scales, components and/or disciplines.

1.1.2 Analysis

Multi-scale analysis

In physics, engineering, and computer science, multi-scale analysis is the field of solving physical phenomena which have important features at multiple scales. The necessity to analyze such important features that are described by means of combining physical models, together with the huge amount of data and mathematics involved has resulted in the development of multi-scale analysis methods (Rudd and Broughton, 2000, Ghoniem *et al.*, 2003, Ladevèze, 2004).

The challenge of multi-scale analysis is to analyze physical phenomena that require detailed and expensive analysis on one scale, while depending on the physics at another scale. A hierarchy of many scales is obtained by combining these scale depended models, see Figure 1.1.

Multi-level analysis

A reason for identifying levels in the analysis of a complex structure is to apply a decomposition principle. Considering a structural analysis model utilizing model dimensions to capture general structural characteristics at a top-level, the individual components embedded in this structure are modeled at a second (lower) level. The latter consists of a detailed description of individual components where individual behavior of a component is captured. This behavior is coupled to the behavior of the larger component in the hierarchy and (possibly) mutually coupled with neighboring components.

This top-down process of zooming-in into the details continues until the smallest components which can be distinguished within the entire structure, see Figure 1.2. Likewise, a bottom-up approach starts at the lowest level of observance and at each

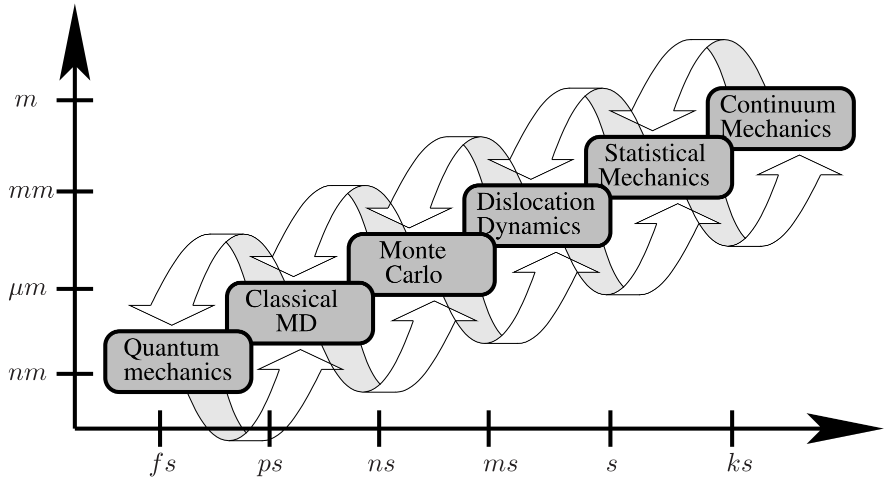

Figure 1.1: *Concurrent coupling of length and time scales in physics. Because no single model describing the entire universe has been derived yet, one has to rely on a number of theories, each applicable within a certain scale. Coupling these models results in multi-scale analysis models.*

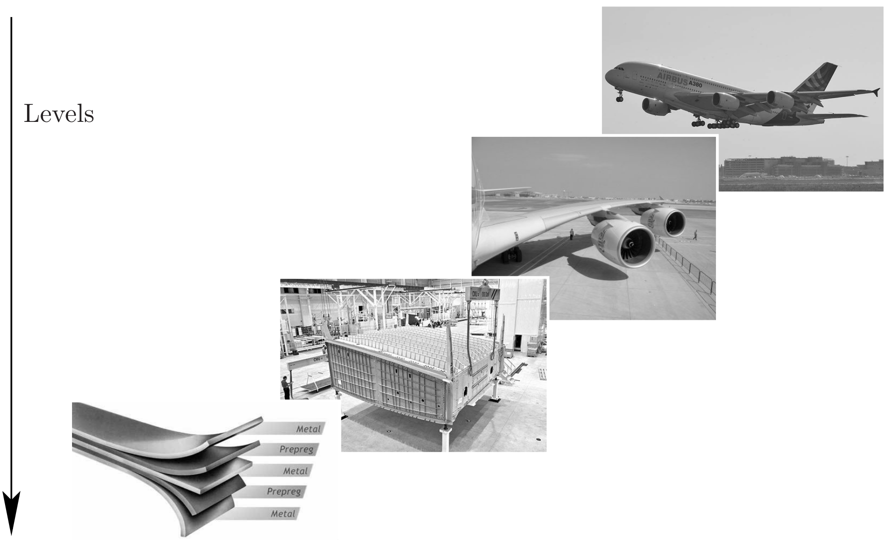

Figure 1.2: *In complex structures such as aircraft, multiple levels of structural detail are distinguishable. Traditionally, engineers have modeled these structures as individual components to analyze their individual behavior. In a later stage of the design these components are assembled to larger components where collective behavior of the components is analyzed while neglecting the individual component behavior.*

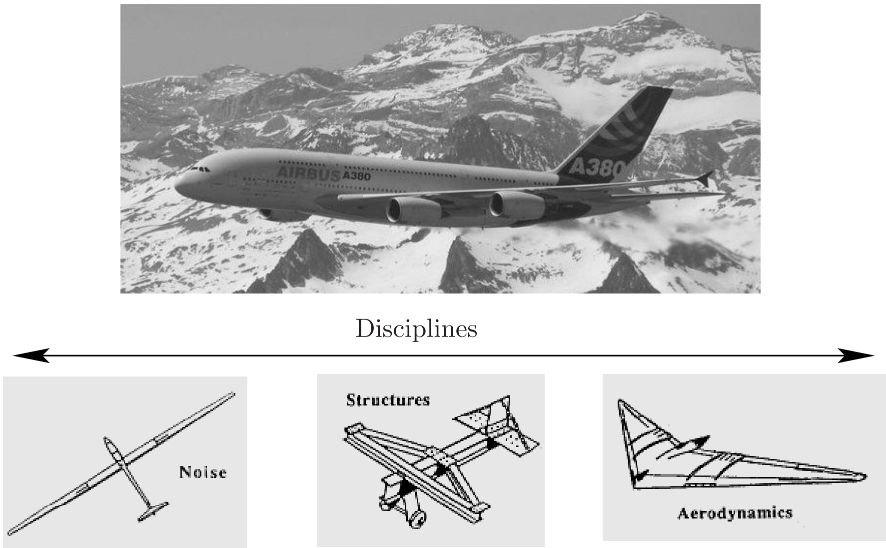

Figure 1.3: *The analysis of complex structures such as aircraft, requires the knowledge and expertise of various disciplines. Therefore, analyzing complex structures is nowadays divided into several, though coupled, disciplinary tasks.*

level larger components are distinguished that capture collective behavior of the lower-level components to which the behavior of the larger component is coupled. The systematic process of zooming-in (or zooming-out) is the area of multi-level analysis methods.

> In summary, multi-scale analysis methods capture physical phenomena at multiple scales via a hierarchy based on physical scales, whereas multi-level analysis methods capture structural characteristics at multiple levels via a hierarchy based on individual component size.

Multi-disciplinary analysis

Because areas of research in individual structural characteristics have become so complex, no single person can take on the task of analyzing an entire structure. Instead, the analysis is decomposed along the boundaries of individual disciplines for which specialized teams are responsible. The area of research that studies the interaction between these disciplines is called multi-disciplinary analysis.

Multi-disciplinary analysis emphasizes the need to account for different disciplines, see Figure 1.3. The focus of these disciplines has been traditionally on individual simulations of structural characteristics. However, because many structural characteristics can not be captured by one single discipline, interaction is necessary. Multi-disciplinary analysis tries to develop systematic approaches in order to handle these interactions.

Abstract elements

Recapitulating, multi-scale analysis, multi-level analysis and multi-disciplinary analysis are research fields that share a common interest. The development of analysis models that take advantage of the multiple scales, levels or disciplines present in many complex systems to balance between accuracy and efficiency of finding a solution. Essentially these three analysis methods consist of elements that are coupled. Therefore, in the remainder of this thesis reference to individual elements is made when discussing abstract characteristics of these three analysis methods. In cases where this is not possible, reference is made to the physical interpretation of the elements.

1.1.3 Optimization

Traditionally, the focus of engineering design optimization for complex structures has been on meeting certain requirements for individual components. Therefore, the focus of optimization has been on conducting a single analysis combined with an optimization problem formulated for that specific component. The lack of an abstract mathematical description for the coupling between elements causes many of the design decisions to be taken based on heuristics, experience of the designer and/or the art of engineering.

In contrast, optimization of complex structures with a distinguishable hierarchy seeks to minimize (or maximize) a certain objective function. This objective function can be assigned to an individual element and/or distributed over multiple elements of a hierarchy. Via changing sets of design variables that are distributed over individual elements of a hierarchy the objective function can be changed. Changes made to the objective function are permitted while satisfying sets of constraint functions specific to individual elements and/or shared over multiple elements. Finally, during the search for an optimal objective function the coupling between elements is accounted for. Hence, the optimization of complex structures focuses on explicitly changing design variables within individual elements of the hierarchy while accounting for the coupling between these elements.

Optimization of complex structures relies on the identification of a hierarchy, which is used to decompose the design optimization problem into more manageable less complex elements. The individual solutions of these elements require coordination, i.e. moving of data of coupled elements such that the optimum of the entire hierarchy is found. Both decomposition and coordination have to be done such that no changes to the design optimization problem are introduced. Hence, that no different optimal point is found as compared to an approach that does not involve decomposition and coordination of the optimization problem.

Decomposition and coordination depend heavily on each other and therefore computational frameworks are defined that cover both the decomposition of the complex structural optimization problem as well as the coordination of the individual optimization problems. Typically, these frameworks include a description on how to split an initially all-in-one optimization problem into an optimization covering multiple hierarchical levels. Alternatively, these frameworks describe a systematic approach towards formulating an optimization problem for a structure where a hierarchy is already present.

The complexity of optimizing complex structures that have a distinguishable hierarchy demands reproducible benchmark problems that are hierarchical. These benchmark problems are required in order to test and compare newly developed decomposition and/or coordination methods.

1.1.4 Current status

Optimization of complex structures consisting of a hierarchy of elements has four main areas of interest. Alexandrov (2005) has classified these areas for the field of multi-disciplinary design optimization and/or multi-level design optimization research. A similar distinction has been made for multi-scale analysis research (Fish and Shek, 2000). The main research areas are:

- the development of methods that **decompose** multi-disciplinary, multi-level and/or multi-scale problems;
- the development of **coordination** (optimization) algorithms;
- the development of **computational frameworks**;
- the development of multi-disciplinary, multi-level and/or multi-scale (optimization) **benchmark problems**.

Decomposition

In the early development of multi-disciplinary design methods much emphasis was put on problem decomposition. Some researchers argued that multi-disciplinary problems were decomposed by nature and hence the problem was not how to decompose, rather how to combine these individual disciplines into a problem that could be solved via optimization algorithms (e.g. Beers and Vanderplaats (1987), Sobieszczanski-Sobieski *et al.* (1987), Renaud and Gabriele (1993)). Other researchers argued that an optimization problem should be seen as a whole and therefore the development of decomposition procedures was necessary in order to assure that one was still solving the same optimization problem after it was decomposed into smaller problems (e.g. Alexandrov and Lewis (2002), Michelena *et al.* (2003), Lassiter *et al.* (2005)).

Whether the multi-disciplinary design problem is seen as a series of individual disciplines that are coupled or whether it is seen as an optimization problem that requires decomposition, research has shown (e.g. Bloebaum (1995), Tosserams *et al.* (2007)) that any decisions made on coupling between disciplines has a significant effect on computational efficiency and algorithmic behavior. Furthermore, as the number of disciplines increases, the algorithmic performance deteriorates.

Initial work on formulating multi-level optimization problems was conducted by Mesarović *et al.* (1970) who analyzed the decomposition of complex systems for which a hierarchy could be identified. Two decomposition techniques were identified. A model-oriented decomposition technique and a goal-oriented decomposition technique. The model-oriented decomposition technique prescribes the model parameters that couple the element to elements to which it is coupled. The goal-oriented decomposition technique prescribes the goal for an individual element. The goal resembles the model parameters that couple the element to elements to which it is coupled.

Likewise, Lasdon (1970) also focused on decomposition and introduced a number of decomposition techniques based on duality theory. The so-called angular and dual angular problem structure is explained. The angular problem structure indicates coupling between individual elements via coupling variables. The dual angular problem structure indicates coupling via optimization functions, i.e. objective and/or constraint functions depend on data of multiple individual elements. The angular and dual angular problem structure were used in a later paper by Barthelemy (1989) to classify multi-level methods and are still in use today as, for example, Tosserams *et al.* (2008a) demonstrate for the development of their extension to Analytical Target Cascading [1].

Multi-level optimization methods are closely related to the field of multi-disciplinary design optimization. Most researchers active in the field of multi-disciplinary design optimization are also active in the field of multi-level optimization. Hence, many of the decomposition methods that were developed initially for the design of multi-level systems, evolved into methods developed for multi-disciplinary design optimization and *vice versa*. Examples of such methods can be found in papers by Sobieszczanski-Sobieski *et al.* (1998) or Kim *et al.* (2003).

Multi-scale methods can be subdivided into multiple scale expansion methods and superposition based methods (Fish and Shek, 2000). Multiple scale expansion methods or homogenization methods consider large scale physical characteristics to remain constant at the lower scales. Furthermore, the small scale physical characteristics are considered to remain local. Finally, the small scale models (Representative Volume Elements) are periodically distributed over the large scale elements. Superposition based methods rely on a hierarchical decomposition of the solution space of the physical responses into large-scale and small-scale responses. Compatibility between the interfaces is prescribed via homogeneous boundary conditions on the interface between the top-level and the lower-level. Typically, these methods are applied to bridge material and structural scales.

Coordination

Once a hierarchy of coupled elements is present, data is send between elements to take into account the coupling that exists between the elements of a hierarchy. The process of regulating the exchange of data between elements is called coordination. Coordination is conducted, such that the individual coupled elements combined form a consistent system.

In this thesis the general approach to coordinating relevant data of individual elements is based on six main stream approaches that were distinguished in a literature

[1] Analytical Target Cascading is a multi-level optimization method

study, namely:

- Optimization by Linear Decomposition (Sobieszczanski-Sobieski *et al.*, 1985);
- Concurrent SubSpace Optimization (Sobieszczanski-Sobieski, 1988);
- Collaborative Optimization (Braun and Kroo, 1997);
- Bi-Level Integrated System Synthesis (Sobieszczanski-Sobieski *et al.*, 2003);
- Analytical Target Cascading (Kim *et al.*, 2003);
- Quasi-separable Subsystem Decomposition (Haftka and Watson, 2005).

These methods differ in how consistency between individual components of the hierarchy is maintained and how relevant data of the decoupled optimization problems is coordinated.

The six main stream approaches can be subdivided into two groups. Coordination techniques that coordinate model-based decomposed[2] elements and coordination techniques that coordinate goal-based decomposed[3] elements. Optimization by Linear Decomposition, Concurrent SubSpace Optimization, Bi-Level Integrated System Synthesis and Quasi-separable Subsystem Decomposition are coordination methods that coordinate model-based decomposed elements. Collaborative Optimization and Analytical Target Cascading are coordination methods that coordinate goal-based decomposed elements.

Coordination of elements that are decomposed via model-based decomposition initially focussed on linearizing the coupling between elements of a hierarchy. An element that represented the top-level prescribes the necessary coupling parameters to elements at a lower level. The lower elements provide the top-level with sensitivity data on their behavior under various coupling conditions. Multi-level optimization techniques that were developed from this principle were published by Sobieszczanski-Sobieski *et al.* (1985, 1987) and Beers and Vanderplaats (1987) amongst others. The need for optimum sensitivity analysis in order to provide additional information to other elements of the hierarchy was published by Barthelemey and Sobieszczanski-Sobieski (1983).

Research showed that not all complex systems could be modeled as a hierarchy. Therefore, a formulation that could handle non-hierarchic systems was developed by Sobieszczanski-Sobieski (1988) that relied on linearization techniques via calculation of the Global Sensitivity Equations (Sobieszczanski-Sobieski, 1990). Various alternative techniques were developed from this initial framework by Shankar *et al.* (1993), Renaud and Gabriele (1994) and Rodriguez *et al.*(1998) amongst others. These alternative techniques focussed on additional terms in the linearized coupling equations, construction of response surfaces to replace having to compute Global Sensitivity Equations and relaxation of the coupling via Augmented Lagrangian relaxation.

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[2]The model-oriented decomposition technique prescribes the model parameters that couple the element to elements to which it is coupled.

[3]The goal-oriented decomposition technique prescribes the goal for an individual element. The goal resembles the model parameters that couple the element to elements to which it is coupled.

To overcome difficulties in finding an optimum within each element optimization Sobieszczanski-Sobieski *et al.* (1998) formulated the Bi-Level Integrated System Synthesis method. This method was later modified (Sobieszczanski-Sobieski *et al.* (2003)) such that load balancing techniques were added. These load balancing techniques assign priority to individual elements and show similarities with techniques used to distribute the computational load in a multi-processor environment.

Work published by Alexandrov and Lewis (1999, 2002) initiated the search for convergence proofs of coordination techniques. Although, to the author's knowledge, no convergence proof was provided for Optimization by Linear Decomposition, a modified version not relying on optimal sensitivity analysis was shown to converge to the optimum under certain mathematical assumptions. This method is called Quasi-separable Subsystem Decomposition (Haftka and Watson, 2005) and relies on slack variables in the constraint functions.

Coordination of elements that are decomposed via goal-based decomposition, focussed on formulations that balance between coupled elements. Collaborative Optimization (Braun and Kroo, 1997) accomplished this balancing via consistency constraints that were formulated as quadratic functions. However, research has shown that these quadratic functions propose difficulties for the coordination near the optimum (DeMiguel and Murray, 2000). Therefore, alternative formulations to the quadratic functions were proposed by DeMiguel and Murray (2006), amongst others.

The search for a mathematically convergent coordination technique for goal-based decomposed elements resulted in the development of Analytical Target Cascading (Kim, 2001, Kim *et al.*, 2003). The method relies on penalty relaxation and balancing between element solutions via penalty parameters. Alternative formulations of the method that have been published and show better numerical performance include Lagrangian relaxation (Lassiter *et al.*, 2005, Kim *et al.*, 2006) and Augmented Lagrangian relaxation (Tosserams *et al.*, 2007).

Although some of the coordination techniques developed in literature are shown to converge to the optimum, computational efficiency of coordination techniques is still an issue. Therefore, DeMiguel and Nogales (2008) have studied alternative formulations of the coordination problem. These formulations use solution techniques that are developed to solve large systems of equations. The coordination problem formulated as such shows better numerical efficiency then previously developed methods. To the author's knowledge, no applications beyond academic examples have been published using this class of methods so far.

Computational framework

Computational frameworks have focused on formulating unifying approaches to handle the hierarchical design problem. A large effort has been made in this area in unifying the notation and formulating the individual optimization problems to handle the hierarchical nature of the design optimization.

Sobieszczanski-Sobieski *et al.* (1987) developed a unifying approach to formulating hierarchical problems of more then two levels. These hierarchical problems are decomposed via model-based decomposition. This hierarchical formulation is extended in later work (Sobieszczanski-Sobieski, 1988) to include non-hierarchical systems. Finally, a complete framework including distributed computing was proposed by Sobieszczanski-Sobieski *et al.* (2003). This framework is implemented into a commercial software package iSIGHT (Engineous Software, 2008).

A computational framework that focussed on formulating a hierarchy via relaxation-based decomposition was developed by Kim *et al.* (2003). Adjustments to notations were published later by Michalek and Papalambros (2005b) and modifications to include non-hierarchic systems were proposed by Tosserams *et al.* (2008a). A commercial software package that includes parts of this goal-oriented framework is HEEDS (Red Cedar Technology, 2008).

An effort to combine various decomposition and coordination techniques into a single software program is conducted by Martins and Marriage (2007, 2009). The authors propose an object-oriented framework that contains different solution techniques to solving the multi-level optimization problem.

Multi-scale design optimization in the context of adjusting small-scale parameters to obtain large-scale responses that are considered optimal is an area of research that is still in an early phase. Work addressing the necessity of robust design in multi-scale optimization was published by Allen *et al.* (2006). However, this paper does not propose an actual framework for the development of such techniques.

Development of benchmark problems

The development of design optimization problems that are multi-disciplinary, multi-level and/or multi-scale and can serve as a benchmark for testing and comparing multi-disciplinary, multi-level or multi-scale optimization methods has not received much attention. Efforts on defining such benchmark problems have been initiated by Padula *et al.* (1996).

Padula *et al.* (1996) addressed the need for reproducible benchmark problems and created a website on the internet where code for benchmark problems could be downloaded from and where new benchmark problems could be uploaded to. Unfortunately, to the author's knowledge the website is no longer accessible. However, the problems posted on the website are made available via the group of Prof. Bloebaum University of Buffalo (2009).

In literature a few benchmark problems have been developed and documented such that reproducible results can be obtained. The portal framework formulated by Sobieszczanski-Sobieski (1982) is used by various authors and a large effort in searching for local and global optima was conducted by Tosserams *et al.* (2008b). A second benchmark problem that is frequently used is that of a supersonic business jet formulated by Agte *et al.* (1999). A design search conducted by Tosserams *et al.* (2008b) showed various local optima. Furthermore, adjustments made to the problem formulation have led to different optimal configurations (Sobieszczanski-Sobieski *et al.*, 2003, Tosserams *et al.*, 2008b). More recently, Tosserams *et al.* (2009) developed a new benchmark problem of a micro-accelerometer where various disciplines are embedded in the optimization problem.

Examples that require solving analytical functions have extensively been used by

Kim *et al.* (2003), Li *et al.* (2008), Tosserams *et al.* (2008b) amongst others. An advantage of these examples is that the optimum can be found analytically and the computational costs of finding an optimum are small. However, these examples can be adjusted to fit the algorithm to be tested, whereas problems that rely on codes developed in industry are often limited in their means to access data. Furthermore, sensitivity information of analytical functions can be retrieved relatively straightforward, whereas industrial codes that are considered black-boxes do not allow for straightforward sensitivity calculations.

1.1.5 Problem Statement

The main question this thesis tries to answer is; How do we optimize complex structures that can be considered a hierarchy of coupled elements via multi-disciplinary design optimization and/or multi-level design optimization? Therefore, this thesis tries to propose a framework that covers all multi-disciplinary design optimization and/or multi-level design optimization methods.

To answer the main question we question:

- how accurate multi-level optimization and multi-disciplinary optimization methods defined in literature find the optimum value of an analytical example;
- what are the computational costs of multi-level optimization and multi-disciplinary optimization methods defined in literature to find an optimum;
- what efforts are necessary in order to implement the procedures defined in literature into a software code.

Furthermore, in order to define the framework a clear understanding of how multi-disciplinary design optimization and multi-level optimization methods handle coupling within the hierarchy as compared to multi-scale methods is necessary. Understanding how these couplings enter the optimization problem and how the relevant data of the individual optimization problems is exchanged is necessary. Consequently, it is necessary to answer the question how individual problem formulations can be generalized in such a form that every subproblem has the same format for the entire hierarchy, irrespective of the decoupling technique used. Likewise, it is questioned how the coordination process capable of coordinating individual subsystems independently from the chosen decoupling can be generalized. Finally, it is questioned how the generalized coupling and coordination can be transferred into a programming language and how this program can be tested to verify the implementation of the design optimization framework.

1.2 Objective and approach

1.2.1 Objective

The objective of this research is to formulate a design optimization framework that incorporates multi-disciplinary design optimization, multi-level design optimization and multi-scale design optimization for the design optimization of structures or systems that involve a hierarchy of multiple levels and/or disciplines. These levels may consist of increasingly structural detail, different disciplines or physical properties that involve multiple scales.

This research distinguishes itself from previous studies on multi-level optimization methods in that it proposes a framework that can use hierarchic and non-hierarchic decomposition methods. The decomposition can be formulated via exact and relaxed consistency constraints. Furthermore, different coordination methods can be used depending on the decomposition formulation. Finally, this research compares six main stream approaches found in literature and compares them on numerical performance, accuracy in finding the optimal solution and implementation effort.

1.2.2 Approach

In order to develop the multi-level design optimization framework a unifying notation of physical responses, physical coupling, design variables, design constraint functions and design objective functions is necessary, see Chapter 2. This unifying notation is applied to a number of main stream approaches in multi-disciplinary and multi-level optimization to uncover the general structure of the analysis and optimization of systems embedding a hierarchy. These main stream approaches were chosen after conducting a literature study on multi-disciplinary and multi-level optimization techniques. This literature study also explored how coupling of analysis models is managed in multi-level and multi-disciplinary design optimization methods as compared to multi-scale analysis models.

Decomposition

The structure in which analysis is embedded in multi-disciplinary design optimization and multi-level optimization is compared with that of multi-scale analysis methods, see Chapter 3. The research on multi-scale methods focuses on how the physical properties of a small scale problem are transferred to a larger scale and *vice versa*. The goal is to find an abstract approach to this coupling between scales that resembles to a great extent the coupling between disciplines and the coupling among levels in design optimization problems. Similarities between multi-disciplinary analysis, multi-level analysis and multi-scale analysis are combined into a general decomposition formulation.

How the coupling of analysis models is embedded in the optimization problem of complex systems consisting of a hierarchy is analyzed by means of the problem matrix. The problem matrix is a means of illustrating coupling between subsystems for an optimization problem formulation. This problem matrix has four characteristic patterns that are characteristic for optimization problems of structures with an embedded hierarchy.

The first pattern in the problem matrix is observed when the objective function depends on design variables and physical responses of the top-element and coupling variables that couple the top-element to elements identified at lower levels in the hierarchy. The design constraints of each individual element of the hierarchy depend on individual design variables, individual physical responses and coupling variables. The second pattern is observed when the individual elements contribute to the objective function of the top-level element. A third pattern is found when some of the individual elements contribute to design constraint functions of neighboring elements. Finally, a fourth pattern is observed when no hierarchy can be distinguished. The objective function and design constraint functions depend on design variables and physical responses from all neighboring elements. This problem matrix is used to categorize the main stream multi-level and multi-disciplinary design optimization approaches.

Coordination

Having developed an abstract formulation of the individual elements the coordination of the solutions of these individual elements is generalized, see Chapter 4. A number of different coordination strategies are taken from literature and formulated in such a format that they can be used on elements that are decomposed irrespective, up to some extent, of the decomposition process chosen, see Chapter 5.

The efficiency of these coordination methods is studied via a benchmark optimization problem that is similar for every coordination method. Furthermore, a study on the accuracy in finding a known solution is made of each of the methods. Finally, implementation efforts of each method are studied.

The abstract formulation of the individual elements allows for extension of the framework with additional coordination methods that are not commonly used for coordinating individual elements and delivers insight into directions of further development of coordination techniques, see Chapter 6.

Computational framework

From the theoretical framework an object-oriented framework is derived that is implemented in Java (Sun Microsystems, Inc., 2008a), see Chapter 7. In contrast to the object-oriented approach of Martins and Marriage (2007, 2009) that is implemented in Python, the objects of this framework consist of classes[4] that are equal for each multi-level, multi-disciplinary or multi-scale optimization method up to some extent, whereas their approach treats all implemented methods as separate classes. The benefit of such an approach is that each method can be implemented in separate routines that are linked via e.g. Python. Furthermore, such a framework focusses on individual routines and how these perform with respect to one another. However, such an approach does not take full advantage of object-oriented programming where equivalent behavior of the multi-level methods is combined into classes and hides the details of how this behavior is accomplished.

Objects are closely related to the hierarchical structure of a design optimization problem. The means by which interaction is managed through interfaces offers great flexibility for the individual implementation of components of the hierarchy and the

[4] A class is a blueprint from which objects are constructed and forms the basis for object-oriented programming.

implementation of new algorithms. A fixed approach of handling coupling and coordinating the entire multi-level optimization problem reduces the effort of applying multi-level optimization to design problems with an embedded hierarchy. Furthermore, the flexibility of the framework permits combining different techniques to develop new algorithms.

Benchmark problem

Finally, the proposed object-oriented framework is applied to a modified version of one of the main stream approaches (Analytical Target Cascading), which was found best suitable for the proposed framework in terms of implementation effort and numerical performance (de Wit and van Keulen, 2007). This implementation is tested on several benchmark examples taken from literature, see Chapter 8.

1.2.3 Constraints on the research

Throughout this thesis reference is made to a hierarchy of elements rather then multi-disciplinary optimization, multi-level optimization and/or to multi-scale analysis. The abstract form of coupling of the individual analysis problems, the definition of design variables, design objectives and design constraints per individual element allows for such a generalization. Hence, for the theoretical framework this thesis does not distinguish between multi-level optimization, multi-disciplinary design optimization or multi-scale optimization.

The research in this thesis is limited to linear elastic structural examples. These examples are small enough to allow for analytical treatment; however they are general enough to resemble real structural analysis problems. The multi-level optimization of a supersonic business jet is included to demonstrate the performance of the proposed object-oriented framework on one of the multi-level optimization techniques.

1.3 Outline of the thesis

This thesis outline is shown in Figure 1.4 and can be read as follows. In Chapter 2 the notation is introduced that is necessary to distinguish between the individual analysis and optimization problems embedded in the hierarchy. Furthermore, this chapter describes fundamental optimization techniques that are referred to in later chapters and are necessary to understand the development of the framework. Readers that are familiar with constraint minimization, as well as post optimum sensitivity analysis and response surface methods may skip the later sections of this chapter.

In Chapter 3 the decomposition process is described. Here the analysis of a structure is split and different decomposition techniques are classified. Furthermore, the problem matrix is introduced which indicates how decomposed analysis models enter the optimization problem.

The individual solutions require coordination. This is accomplished via coordination techniques which are introduced in Chapter 4. A distinction is made between

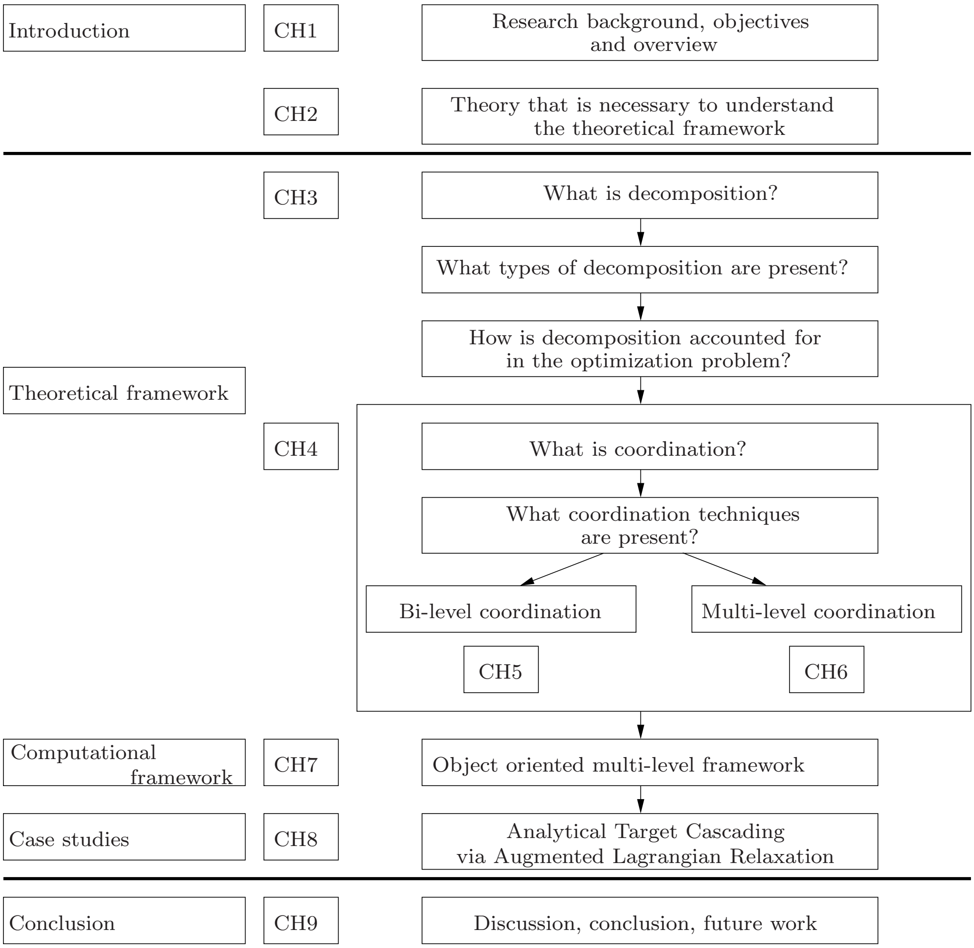


Figure 1.4: *Outline of the thesis, representing an overview of the background questions that are answered per chapter that are necessary to answer the main research question.*

bi-level coordination techniques which are discussed in Chapter 5, and multi-level coordination techniques which are the topic of Chapter 6.

An object-oriented framework is developed in Chapter 7 and a modified version of Analytical Target Cascading is implemented in the proposed framework. This framework is applied to the multi-level optimization of a two-bar truss, portal frame and supersonic business jet in Chapter 8.

Finally, Chapter 9 discusses the results and some conclusions are drawn from these. Furthermore, recommendations for future research are given.

Chapter 2

Theoretical framework

This chapter introduces preliminary knowledge necessary to understand the discussion of the multi-level design optimization framework in the following chapters.

The notation that is used throughout this thesis is introduced in Section 2.1. Necessary conditions for optimality are derived in Section 2.2 which are extended in Chapter 3 to coupled design optimization problems and form the basis of Chapter 6 for the development of multi-level coordination procedures. In Section 2.3 three definitions on measuring inconsistency between two individual hierarchical elements are defined. Furthermore, in Section 2.4 two techniques from the optimization field are introduced which are used for the development of the multi-level framework. Finally, two approximation techniques are introduced in Section 2.5 that approximate the behavior of the optimum of an individual component.

2.1 A unifying multi-level notation

A clear notation of the multi-level optimization problem will help to understand and compare the different multi-level methods studied in the present thesis. This thesis uses for each element the same symbols for design variables, responses and optimization functions, but with additional sub- and superscripts. In this section we shall elaborate on these notations and variables.

A multi-level problem typically has a hierarchical structure of individual elements as is illustrated in Figure 2.1(a). The top-level is denoted Level-0. At this level, the top-element or global design variables are distinguished, which are denoted ${}^{0}\mathbf{x}$. One level lower, i.e. at Level-1, the "children" of Level-0 ("parent") can be found. The elements at Level-1 are numbered in the left upper corner, i.e. ${}^{\cdot\cdot}\mathbf{x}$. At the second level, Level-2, the children of the Level-1 elements are found. They have two numbers in the upper-left corner, i.e. ${}^{\cdot\cdot,\cdot\cdot}\mathbf{x}$. The first reflects the parent in the hierarchy

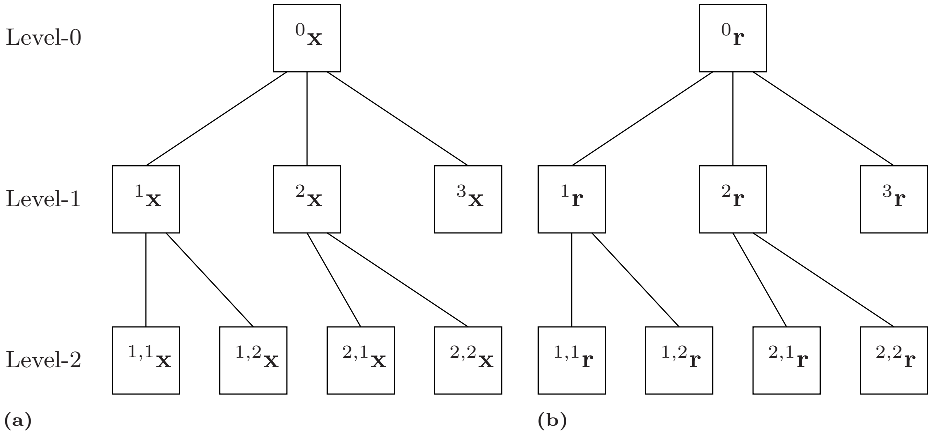


Figure 2.1: *Multi-level notation for a three-level hierarchy. (a) The design variables corresponding to the individual elements are illustrated. (b) The physical responses corresponding to each of the elements.*

and the second their place among the children of the parent element. In a similar fashion, more levels can be added to the hierarchy. Consequently, at every level the relative position can easily be seen from the superscript in front of a (design) variable. Similarly, the physical responses corresponding to the elements are identified. This is illustrated in Figure 2.1(b).

Although for each element a set of physical responses can be identified, these responses are typically interacting with responses in neighboring elements, i.e. parent, brothers/sisters and children. In other words, there are couplings which need to be taken into account as is illustrated in Figure 2.2. For this purpose, the operators $\mathcal{H}$ are introduced. These operators map the response from one element, onto another. In front of the operator in the top-left corner the origin of the information is indicated, i.e. ${}^{\cdot\cdot}\mathcal{H}$. In the bottom-left corner, the destination of the information is specified, i.e. ${}_{\cdot\cdot}\mathcal{H}$.

Design variables can be present that are shared among multiple elements. These design variables are considered separately from the individual design variables ${}^{\cdot\cdot}\mathbf{x}$ and these shared design variables are written as ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{z}$. The top-left index $({}^{\cdot\cdot})$ represents the element to which the design variable is assigned and the bottom-left index $({}_{\cdot\cdot})$ represents the element that shares that design variable. In case the same design variable is shared among the parent element and two or more child elements a dot $({}^{\cdot\cdot}_{\cdot\cdot\,.\,\cdot\cdot}\mathbf{z})$ separates the indices from each of the child elements. Shared design variables are shown in Figure 2.3, where a design variable is shared among three elements distributed over two levels of a hierarchy.

Responses that are mapped from one element onto another element are introduced via the operator $\mathcal{H}$. Once a response is mapped from one element onto another element it is written as ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$, see Figure 2.4. This variable is called a coupling variable and is defined as: ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h} = {}^{\cdot\cdot}_{\cdot\cdot}\mathcal{H}({}^{\cdot\cdot}\mathbf{r})$. This coupling variable can be used in one element as a

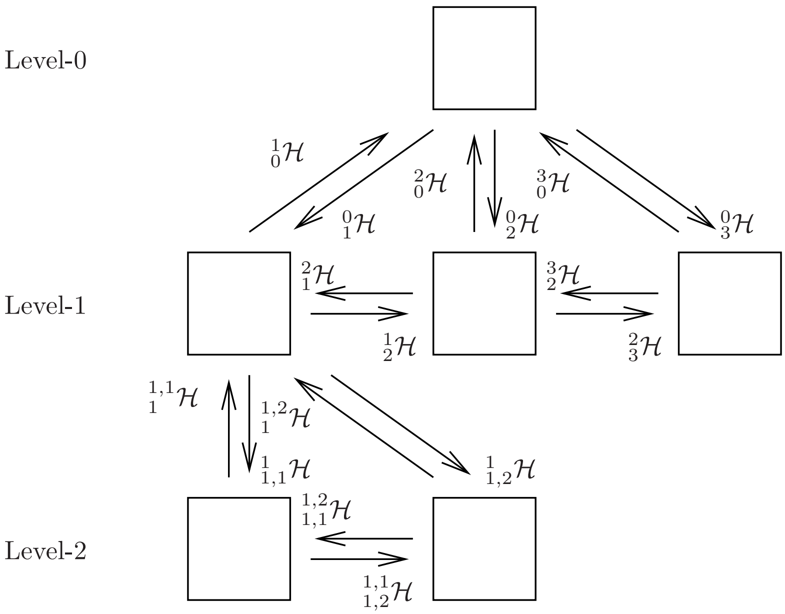


Figure 2.2: *Physical interactions between elements. The operators ${}^{\cdot\cdot}_{\cdot\cdot}\mathcal{H}$ map the physical responses of an element onto a neighboring element. Formally this can be a mapping taking place at the same level, though it may also map between different levels.*

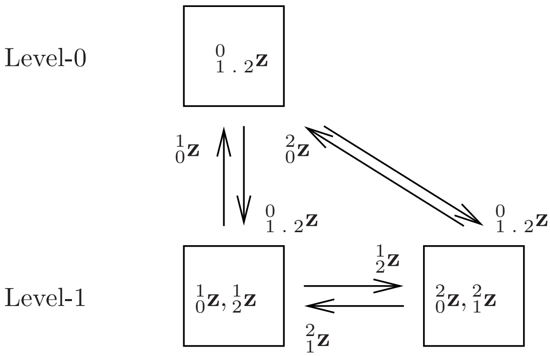


Figure 2.3: *Design variables that are shared among multiple elements. The top left index represents the element to which the design variable is associated and the bottom index represents the neighboring element(s) that share the design variable.*

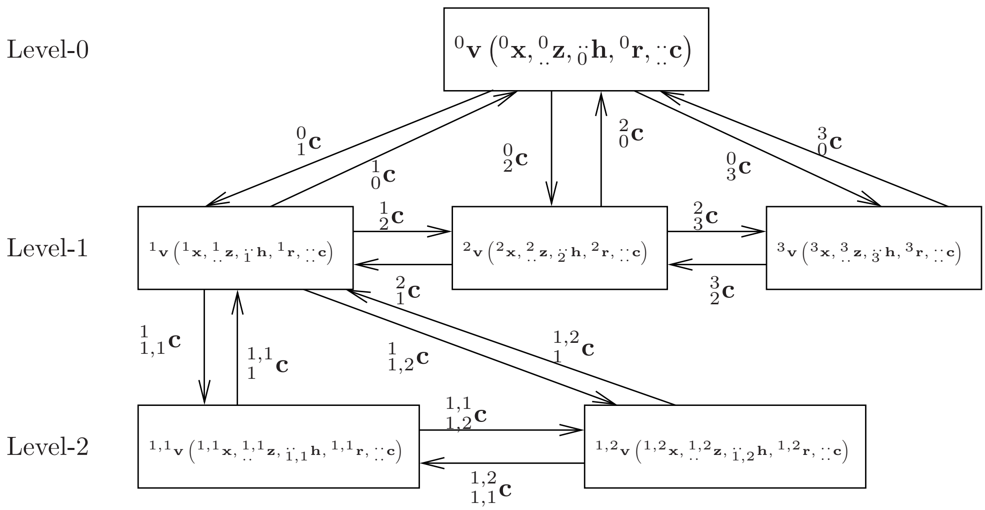


Figure 2.4: *Functions used in the optimization formulation of the hierarchy. These functions typically depend on design variables, the constraints and the physical responses of the corresponding element.*

desirable property of neighboring elements. Hence, instead of mapping the physical responses coming from a neighboring element onto the current element: ${}^{\cdot\cdot}_{\cdot\cdot}\mathcal{H}({}^{\cdot\cdot}\mathbf{r}) = {}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$, the value of the coupling variable (${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$) is chosen. The corresponding physical responses (${}^{\cdot\cdot}\mathbf{r}$) are determined by the neighboring element.

Consistency constraints are introduced in order to temporarily decouple the elements from their surroundings, such that each of them is solved without interacting with other elements. These consistency constraints are mathematically defined as: ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{c} = {}^{\cdot\cdot}_{\cdot\cdot}\mathcal{H}({}^{\cdot\cdot}\mathbf{r}) - {}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$. A consistency constraint that is assigned to an element enables the element to change ${}^{\cdot\cdot}\mathbf{r}$ or when applicable ${}^{\cdot\cdot}\mathbf{h}$ without communicating directly to the element to which it is coupled. These consistency constraints are written ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{c}$, where the upper left subscript ${}^{\cdot\cdot}\mathbf{c}$ indicates the origin of the mapped responses and/or shared design variables and the lower left subscript ${}_{\cdot\cdot}\mathbf{c}$ indicates the destination, see Figure 2.4. This notation is in accordance with the notation for coupling (hence operators $\mathcal{H}$) or shared design variables ($\mathbf{z}$) between two individual elements. Decoupled hierarchical elements depend on consistency constraints, therefore the consistency constraints are added to the optimization functions inside the hierarchical elements.

The final step in setting up an optimization problem is to formulate objective and constraint functions (${}^{\cdot\cdot}\mathbf{v}$) that can be minimized (maximized) by the optimizer. These functions depend on local design variables (${}^{\cdot\cdot}\mathbf{x}$), local responses (${}^{\cdot\cdot}\mathbf{r}$) and consistency conditions (${}^{\cdot\cdot}\mathbf{c}$) as well as the coupling (${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$ and ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{z}$) between two individual elements. These design functions are illustrated in Figure 2.4.

The hierarchy described in this section is an abstract representation of a complex system. It can be interpreted in three different ways:

- The physical behavior of the system can be fully described via a huge amount of parameters. However, the size of the model poses problems on the computer infrastructure on which the model should be evaluated. Therefore, a hierarchy is defined that splits the optimization problem into smaller coupled problems that can be evaluated on the available computer infrastructure. Behavior of individual components that reflect the smaller coupled problems is computed via a small number of parameters. Coupling is present between the individual components to translate properties of elements higher in the hierarchy into properties of elements lower in the hierarchy and *vice versa*. This coupling is covered via the coupling between levels.

- The physical behavior cannot be described by means of a single disciplinary model. Coupling is present, however there is no single model that can replace the two individual models. This type of coupling is present in the mapping of physical responses from one element onto a neighboring element. This coupling is covered via the coupling between elements on the same level.

- The physical behavior covers multiple scales, the numerical computations of the underlying theory are too costly to accurately model certain physical behavior. Therefore, a mapping from a small-scale model onto a larger scale model is used and *vice versa*. Hence, no all-in-one formulation of the problem is present. This coupling is covered via the coupling between levels.

The analysis of large structures typically involves the above three categories. Optimization of a structure that embeds one or more of the above categories can be accomplished via an all-in-one optimization given enough resources to solve the problem. Hence, a single optimization problem is formulated where the responses are computed at the different elements of the hierarchy they occur. Consistency between individual elements is to be maintained between the physical analysis models throughout every iteration step of the optimization process. In contrast, a multi-level formulation takes the consistency constraints into account as part of the optimization problem formulation such that part of the optimization process can be conducted at the individual elements.

2.2 Conditions for optimality

Multi-level optimization of complex systems involves minimization or maximization of an objective function that mathematically represents the performance of the complex system. In the present section the focus lies on decomposing a single problem into two or more individual problems that are coupled. Optimality criteria for the all-in-one problem are compared with optimality criteria for the associated decoupled problem. Because the focus lies on the effect of introducing consistency constraints into the decoupled problem, for simplicity no design constraints are considered and the all-in-one unconstrained optimization problem is expressed as:

$$\min_{\mathbf{x}} \quad v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) \quad . \tag{2.1}$$

For the all-in-one problem, the objective function $v_f(\mathbf{x}, \mathbf{r}(\mathbf{x}))$ is minimized with respect to the design variables $\mathbf{x}$.

Local optimality for the AiO problem requires first-order and second-order optimality conditions to be satisfied. For an unconstrained convex optimization problem the minimum is found if

$$\nabla_{\mathbf{x}} v_f(\mathbf{x}, \mathbf{r}(\mathbf{x}))\partial\mathbf{x} \geq \mathbf{0}, \tag{2.2}$$

where $\nabla_{\mathbf{x}} v_f = \left[\frac{\partial v_f}{\partial x_1}, \ldots, \frac{\partial v_f}{\partial x_n}\right]^T$ and $\partial\mathbf{x}$ represents a vector of arbitrary perturbations of the design variables $\mathbf{x}$.

In case v_f is non-convex the first-order condition (2.2) is only a necessary condition. At maxima and saddle points the gradient of the function v_f is zero as well and, therefore, second-order information is required to inspect the very nature of the stationary point.

The second-order condition can mathematically be expressed as:

$$\partial v_f = \partial\mathbf{x}^T \nabla^2_{\mathbf{x},\mathbf{x}} v_f \partial\mathbf{x}, \tag{2.3}$$

where $\nabla^2_{\mathbf{x},\mathbf{x}} v_f = \left[\frac{\partial \nabla_{\mathbf{x}} v_f}{\partial x_1}, \ldots, \frac{\partial \nabla_{\mathbf{x}} v_f}{\partial x_n}\right]^T$. When Equation 2.3 is positive for all arbitrary perturbations $\partial\mathbf{x}$, then the stationary point is a local minimum. Thus, $\nabla^2_{\mathbf{x},\mathbf{x}} v_f$ must be positive definite. This is a sufficient condition for local optimality of the unconstrained AiO problem.

Multi-level optimization involves identifying a hierarchy in the design optimization problem that allows for a decomposition. Hence, the AiO optimization problem defined in Equation 2.1 is decomposed into multiple, coupled, optimization problems. The decomposition is accomplished via consistency constraints which are added to the decoupled optimization problem. These consistency constraints reflect the coupling between physical quantities or responses and/or shared design variables.

Determining local optimality of a decoupled optimization problem implies again that the first order optimality conditions are satisfied. For this purpose, the following optimization problem with consistency constraints is considered:

$$\begin{array}{lll} \min\limits_{\mathbf{x}} & v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) & \\ \text{s.t.} & \mathbf{c}(\mathbf{x}, \mathbf{r}(\mathbf{x})) & = \mathbf{0} \end{array} . \tag{2.4}$$

The constraint functions $\mathbf{c}$ provide the coupling reflecting the physics of the coupling and shared design variables.

A first step in deriving the first-order optimality conditions is to formulate the Lagrangian. The Lagrangian is defined as:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) \quad = \quad v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) + \boldsymbol{\lambda}^T \mathbf{c}(\mathbf{x}, \mathbf{r}(\mathbf{x})). \tag{2.5}$$

The elements of vector $\boldsymbol{\lambda}$ are unknown Lagrange multipliers. Necessary conditions for a stationary point are:

$$\begin{array}{lll} \nabla_{\mathbf{x}}\mathcal{L} = \nabla_{\mathbf{x}} v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) + \boldsymbol{\lambda}^T \nabla_{\mathbf{x}} \mathbf{c}(\mathbf{x}, \mathbf{r}(\mathbf{x})) & = & \mathbf{0} \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L} = \mathbf{c}(\mathbf{x}, \mathbf{r}(\mathbf{x})) & = & \mathbf{0} \end{array} ; \tag{2.6}$$

When the problem is non-convex, second-order information is required as well. For this purpose, we study the second-order derivatives $\nabla^2\mathcal{L}$ which read

$$\begin{array}{l} \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L} = \nabla^2_{\mathbf{x},\mathbf{x}} v_f(\mathbf{x},\mathbf{r}(\mathbf{x})) + \boldsymbol{\lambda}^T \nabla^2_{\mathbf{x},\mathbf{x}}\mathbf{c}(\mathbf{x},\mathbf{r}(\mathbf{x})) \\ \nabla^2_{\boldsymbol{\lambda},\mathbf{x}}\mathcal{L} = \nabla_{\mathbf{x}}\mathbf{c}(\mathbf{x},\mathbf{r}(\mathbf{x})) \\ \nabla^2_{\mathbf{x},\boldsymbol{\lambda}}\mathcal{L} = \nabla_{\mathbf{x}}\mathbf{c}(\mathbf{x},\mathbf{r}(\mathbf{x})) \\ \nabla^2_{\boldsymbol{\lambda},\boldsymbol{\lambda}}\mathcal{L} = 0 \end{array} . \tag{2.7}$$

or in matrix notation

$$\nabla^2\mathcal{L} = \begin{bmatrix} \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L} & \nabla^2_{\boldsymbol{\lambda},\mathbf{x}}\mathcal{L} \\ \nabla^2_{\mathbf{x},\boldsymbol{\lambda}}\mathcal{L} & \mathbf{0} \end{bmatrix}, \tag{2.8}$$

which is the Hessian matrix. A sufficient condition for local optimality is that:

$$\partial\mathbf{x}^T \left(\nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\right) \partial\mathbf{x} > 0, \tag{2.9}$$

for all directions $\partial\mathbf{x}$ for which $\partial\mathbf{x}^T\nabla_{\mathbf{x}}\mathbf{c} = 0$ hold. This condition reflects that all feasible perturbations lead to an increase of the objective function.

First and second order optimality conditions provide necessary and sufficient conditions for an optimal point of both constrained and unconstrained optimization problems. To determine whether a decomposition technique introduces changes in the design problem, the first and second order optimality equations should not return different solutions after decomposition. Thus, the first order conditions require that the consistency constraints $\mathbf{c}$ have the property that:

$$\begin{array}{lcl} \mathbf{c}(\mathbf{x},\mathbf{r}(\mathbf{x})) & = & \mathbf{0} \\ \boldsymbol{\lambda}^T\nabla_{\mathbf{x}}\mathbf{c}(\mathbf{x},\mathbf{r}(\mathbf{x})) & = & \mathbf{0} \end{array}, \tag{2.10}$$

and the second order conditions require that the consistency constraints $\mathbf{c}$ have the property that:

$$\boldsymbol{\lambda}^T\nabla^2_{\mathbf{x},\mathbf{x}}\mathbf{c}(\mathbf{x},\mathbf{r}(\mathbf{x})) \geq \mathbf{0}. \tag{2.11}$$

If these conditions are not met at the same optimal point for the problem defined by Equation 2.1 and Equation 2.4, then the decomposition has changed the design problem.

Additional requirements on the decomposition yield: no extra optimal points are introduced, therefore condition 2.6 should not hold for points other than the optimal points determined for Equation 2.1.

2.3 Norms for measuring inconsistency

Two elements that are coupled and that require individual solving are temporarily decoupled via consistency constraints during the individual solution processes. However, these consistency constraints are not necessarily satisfied and thus inconsistencies between levels, i.e. $\mathbf{c}$, are present. These inconsistencies are measured as the distance between the mapped responses $\mathcal{H}(\mathbf{r})$ and the expected or prescribed responses $\mathbf{h}$, i.e.

$$\mathbf{c} = \mathcal{H}(\mathbf{r}) - \mathbf{h}. \tag{2.12}$$

The distance between mapped responses and expected responses that occur during the solution of the individual design optimization problems is defined as the length of a vector. For example the length of $\mathbf{c} = [c_1, \ldots, c_n]$ is captured by the expression:

$$||\mathbf{c}||_2 = \sqrt{c_1^2 + \ldots + c_n^2}. \tag{2.13}$$

This measure is the *Euclidean* norm or l^2-norm. Other norms that are frequently used to measure the inconsistency as a means of performance measures for iterative procedures are:
the l^1-norm:

$$||\mathbf{c}||_1 = \sum_{i=1}^{n} |c_i|. \tag{2.14}$$

The $|\ldots|$ indicate absolute values. The infinity norm (or maximum norm) is mathematically expressed as:

$$||\mathbf{c}||_\infty = \max\{|c_1|, \ldots, |c_n|\} \tag{2.15}$$

The norm of an inconsistency is used in multi-level techniques that require relaxation. This relaxation is discussed in more detail in Chapter 3.

2.4 Optimization techniques

2.4.1 Penalty function methods

Characteristic for multi-level optimization are equality constraints on the physical responses that interact between two individual elements and/or by equality constraints on shared design variables. These equality constraint functions limit the search domain in which the objective function is minimized. One remedy to milden this effect is to penalize the constraints.

Penalization permits the optimizer to allow for small violations of the constraints. One common approach is to multiply the square of the constraint violation with a penalty parameter which has to be high enough such that the violations remain small. The following example illustrates this for a constrained optimization problem:

$$\begin{array}{lll} \min\limits_{\mathbf{x}} & v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) & \\ \text{s.t.} & c(\mathbf{x}, \mathbf{r}(\mathbf{x})) & = 0. \end{array} \tag{2.16}$$

The constrained minimization problem is replaced with an unconstrained minimization problem and becomes:

$$\min_{\mathbf{x}} \quad v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) + s\left(c(\mathbf{x}, \mathbf{r}(\mathbf{x}))\right)^2 \tag{2.17}$$

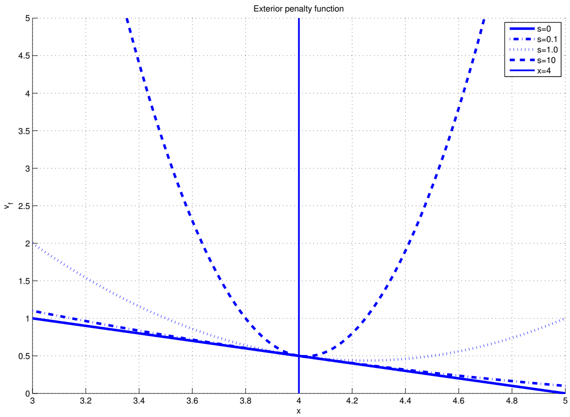


Figure 2.5: *Example of relaxation through an exterior penalty function. The constraint function: $v_h = x - 4 = 0$ is relaxed and added to the objective function to obtain: $v_f = 2.5 - 0.5x + s\,(x-4)^2$. A penalty function relaxes the constraints such that it is easier for the optimizer to satisfy the constraints. The penalty parameter s is incremented such that the steep slope of the penalty function does not pose difficulties for the minimization in the beginning of the search for the constraint minimum.*

where s is the penalty parameter.

The penalty parameter is increased by small increments letting the optimizer gradually approach the constraint. Minimizing a penalty function is thus an iterative process, because the curvature of the penalty function is very sharp for high penalty parameters. This is illustrated in Figure 2.5 for a constrained optimization problem. Furthermore, the high penalty parameters introduce ill-conditioning making it difficult to solve the optimization problem. This ill-conditioning can be avoided if the penalty function is added to the Lagrangian of the optimization problem (see 2.2) instead of the objective function. This alternative formulation shifts the origin of the penalty function such that the minimum of the objective function is found without the penalty term going to large values.

In multi-level optimization the exterior penalty function allows for small deviations on the consistency constraints. Therefore the optimization of the individual element can concentrate on minimizing the objective function in the initial stage of the optimization process. Later during the solution stage, the consistencies become more important and the penalties are increased. The advantage of this approach is that initially little effort is spend on satisfying the constraints for configurations that are not optimal. However, during the optimization process the combined hierarchy of elements is infeasible since the consistency constraints are violated. Optimization techniques that adopt penalty functions are discussed in Chapter 5.

2.4.2 Constraint minimization

Multi-level optimization is characterized by equality constraints on the physical responses that interact between two individual elements and/or by equality constraints on shared design variables. These constraints are added to the design optimization problem in order to temporarily decouple the optimization problem associated with an individual element from it's surroundings.

In general it is undesirable to increase the number of constraints, because constraints involve function calls that can be expensive in terms of computational cost. However, in case of multi-level optimization typically a huge number of equality constraints is present in every individual optimization problem due to decomposition combined with the constraints on the individual design of the element.

For problems with a large number of inequality constraints, it is possible to replace them by one equivalent constraint. Likewise, for problems consisting of a large number of equality constraints a single equivalent constraint can be formulated. Advantage is that several constraints of various origin can be combined. A disadvantage is that non-feasible designs are obtained because the equivalent is not an exact representation of the combination of all the individual constraints. An example of such a cumulative equivalent constraint is the Kreisselmeier-Steinhauser(KS) function. This KS-function was initially introduced to multi-level optimization problems by Sobieszczanski-Sobieski (1992).

Considering a single optimization problem, the KS-function is a differentiable envelope function for a set of inequalities of the form $v = v_i(\mathbf{x}_i) \leq 0$. It is assumed that each function is continuous in $\mathbf{x}_i$ but does not necessarily has derivatives that are continuous. The KS-function is expressed as:

$$KS(\mathbf{v}) = \frac{1}{\rho} \ln \left[\sum_{i=1}^{n} e^{(\rho v_i)} \right]. \tag{2.18}$$

$$\tag{2.19}$$

where $i = 1..n$ represents each individual function v_i that is present in the optimization problem.

If the KS function generates too large values of the exponential function an alternative expression is used:

$$KS(\mathbf{v}) = v_{\max} + \frac{1}{\rho} \ln \left[\sum_{i=1}^{n} e^{(\rho(v_i - v_{\max}))} \right]. \tag{2.20}$$

A property of the KS-function is that:

$$v_{\max} \leq KS(v_i) \leq v_{\max} + \frac{\ln(n)}{\rho}. \tag{2.21}$$

The parameter ρ is set by the user in order to control the accuracy of the KS-function in following the piecewise envelope of the set of inequalities $\mathbf{v}$, see Figure 2.6(a).

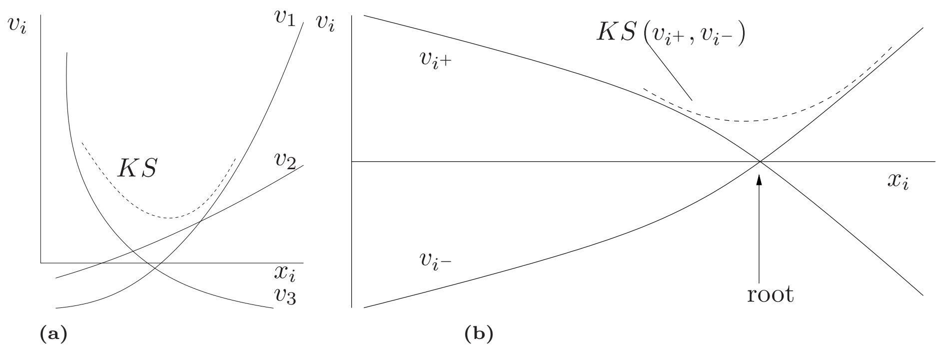


Figure 2.6: *(a)KS function used as an envelope function of a set of functions (v_i). The function acquires a minimum at the point where the combined functions (v_i) obtain a minimum. (b) The minimum of the KS function coincides with the root of a function (v_i), hence the KS function can be used as a tool for root finding.*

Equality constraints are formulated as inequality constraints such that the KS-function can be used as a tool for root finding. Hence, the equality constraint $v_i = 0$ is split into a positive part v_{i+} and a negative part v_{i-}. Both parts are rewritten as inequality constraints and inserted into the KS-function ($KS(v_{i+}, v_{i-})$). The KS-function has the property that the minimum lies at the point where the root of the function v_i is located, see Figure 2.6(b).

The KS function performs as an extended-interior penalty function (see for a definition of penalty functions Section 2.4.1) for the multilevel optimization because it is defined throughout the infeasible and the feasible domains. In literature other envelope functions can be found, e.g. the empirical function (Sobieszczanski-Sobieski, 1993).

2.5 Optimum approximation techniques

2.5.1 Optimum sensitivity derivatives

Every element of the hierarchy that consists of an individual optimization problem searches for the optimal values of the design variables, physical responses, and objective and constraint functions for certain fixed parameter settings. In a multi-level hierarchy typically many of these temporarily fixed parameters will be changed after the optimization has finished due to changes elsewhere in the hierarchy. In order to analyze the influence such changes have on the local optimum and in order to approximate such changes in other parts of the hierarchy an Optimum Sensitivity Analysis can be performed (a complete derivation is discussed by Barthelemy and Sobieszczanski-Sobieski (1983)).

Optimum Sensitivity Analysis analyses changes in objective function, constraint functions and optimal parameters due to changes in parameters that were initially kept fixed during the optimization. To distinguish between derivatives necessary to minimize objective and constraint functions (behavior derivatives) and derivatives computed in this section the latter are called optimum sensitivity derivatives.

A constrained optimization problem defined for a single element is considered in order to derive the necessary optimum sensitivity derivatives, hence:

$$\begin{array}{lll} \min\limits_{\mathbf{x}} & v_f(\mathbf{x}, \mathbf{r}(\mathbf{x}), \mathbf{h}) & \\ \text{s.t.} & \mathbf{c}(\mathbf{x}, \mathbf{r}(\mathbf{x}), \mathbf{h}) & = \mathbf{0} \end{array} \tag{2.22}$$

where $\mathbf{h}$ are initially fixed parameters that are prescribed via elements elsewhere in the hierarchy. Stationary conditions were defined in Section 2.2 and for the current problem these read:

$$\begin{array}{lll} \nabla_{\mathbf{x}}\mathcal{L} & = \nabla_{\mathbf{x}} v_f(\mathbf{x}, \mathbf{r}(\mathbf{x}), \mathbf{h}) + \boldsymbol{\lambda}^T \nabla_{\mathbf{x}} \mathbf{c}(\mathbf{x}, \mathbf{r}(\mathbf{x}), \mathbf{h}) & = 0 \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L} & = \mathbf{c}(\mathbf{x}, \mathbf{r}(\mathbf{x}), \mathbf{h}) & = 0 \end{array}. \tag{2.23}$$

However, these stationary conditions depend on the parameters $\mathbf{h}$ that are put into the optimization problem. In other words, $\mathbf{x} = \mathbf{x}(\mathbf{h})$ and $v_f = v_f(\mathbf{x}(\mathbf{h}), \mathbf{r}(\mathbf{x}, \mathbf{h}), \mathbf{h})$. Sensitivity of these stationary conditions with respect to $\mathbf{h}$ is obtained by applying the chain rule ($\mathbf{D_h}(\ldots) = \nabla_{\mathbf{h}}(\ldots) + \nabla_{\mathbf{x}}(\ldots)^T \mathbf{D_h}(\mathbf{x})$) to Equation 2.23, where the total derivatives are defined as $\mathbf{D_h}.. = [\frac{\mathrm{d}..}{\mathrm{d}h_1}, \ldots, \frac{\mathrm{d}..}{\mathrm{d}h_n}]^T$. This results in

$$\begin{array}{rcl} \nabla_{\mathbf{h}}\nabla_{\mathbf{x}} v_f + \nabla^2_{\mathbf{x},\mathbf{x}} v_f \mathbf{D_h}(\mathbf{x}) + \boldsymbol{\lambda}^T \left(\nabla_{\mathbf{h}}\nabla_{\mathbf{x}}\mathbf{c} + \nabla^2_{\mathbf{x},\mathbf{x}}\mathbf{c}\mathbf{D_h}(\mathbf{x})\right) + \mathbf{D_h}(\boldsymbol{\lambda}^T)\nabla_{\mathbf{x}}\mathbf{c} & = & \mathbf{0}, \\ \nabla_{\mathbf{h}}\mathbf{c} + \nabla_{\mathbf{x}}\mathbf{c}^T \mathbf{D_h}(\mathbf{x}) & = & \mathbf{0}, \end{array} \tag{2.24}$$

or in matrix notation

$$\begin{bmatrix} \nabla^2_{\mathbf{x},\mathbf{x}} v_f + \boldsymbol{\lambda}^T \nabla^2_{\mathbf{x},\mathbf{x}}\mathbf{c} & \nabla_{\mathbf{x}}\mathbf{c} \\ \nabla_{\mathbf{x}}\mathbf{c}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D_h}(\mathbf{x}) \\ \mathbf{D_h}(\boldsymbol{\lambda}^T) \end{bmatrix} + \begin{bmatrix} \nabla_{\mathbf{h}}\nabla_{\mathbf{x}} v_f + \boldsymbol{\lambda}^T \nabla_{\mathbf{h}}\nabla_{\mathbf{x}}\mathbf{c} \\ \nabla_{\mathbf{h}}\mathbf{c} \end{bmatrix} = \mathbf{0}. \tag{2.25}$$

At the constraint optimal point the above equations are evaluated to find the unknown derivatives of the optimum solution $\mathbf{D_h}(\mathbf{x})$ and $\mathbf{D_h}(\boldsymbol{\lambda}^T)$. Once $\mathbf{D_h}(\mathbf{x})$ are found, the optimum sensitivity derivative of the objective function v_f can be computed. The total derivative of v_f is obtained via:

$$\mathbf{D_h}(v_f) \quad = \quad \nabla_{\mathbf{h}} v_f + \nabla_{\mathbf{x}} v_f \mathbf{D_h}(\mathbf{x}) \tag{2.26}$$

The total derivatives of each individual element can be used in other elements of a hierarchy in order to estimate the changes in neighboring elements. Chapter 4 discusses usage of these optimum sensitivity derivatives in the multi-level optimization hierarchy.

2.5.2 Response surface approximations

In a multi-level hierarchy coupling among the elements is present. The optimal solution of each of the individual elements depends on data that comes from neighboring elements. Because it is computationally expensive to find the optimum of these elements, techniques that approximate the behavior of these elements become useful. In the previous section a similar approach, denoted Optimum Sensitivity Analysis, was introduced which gives additional insight into the behavior of the optimum of an element in the close neighborhood of the optimal point. Not only the optimum value, but also the direction to which this optimum shifts under changes in other elements is computed. However, computing derivatives might not always be feasible and the optimum sensitivity derivatives are only useful for small changes in the multi-level hierarchy. Therefore, often Response Surface Methods are used. See for a thorough discussion on this topic, e.g., Myers and Montgomery (1995).

Response surface methods rely on a combination of mathematical and statistical techniques. In principle one is interested in the quantitative relationship between the response of a black box (here the optimization of an individual element) to changes made to the input parameters. The latter should in the current context be associated with the parameters $\mathbf{h}$ and $\mathbf{z}$, respectively. The systematic process of choosing input parameters and determining the resulting output is called Design of Experiments(DOE).

A DOE involves choosing points within a subdomain of the space of the input parameters, which will reveal as much of the elements behavior within that subdomain of the entire space. For this purpose use is made of techniques that select points within the space, for example, a Latin Hypercube Sampling(LHS) (McKay *et al.*, 1979). LHS chooses random points that are guaranteed to be uniformly distributed over the space. For each of these initial points the computed responses are normalized. After all points of the LHS sample have been evaluated the resulting normalized responses combined are fitted with a polynomial.

For brevity of notations the objective function ${}^{n}v_f$ is written v, the coupling parameters $\underset{..}{\mathbf{h}}$ are written $\mathbf{h}$ and shared design variables $\mathbf{z}$ are not considered in the current discussion. Shared design variables are treated similarly in the response surface construction as the coupling variables. A linear response surface model for the objective function of Element-n in the hierarchy is:

$$v(\mathbf{h}) \quad = \quad a_0 + \sum_{l=1}^{m} a_l h_l. \tag{2.27}$$

The coefficients $a_0, .., a_m$ are unknown coefficients. These unknown coefficients are determined as follows. Changing the value of h_l via, e.g. a Latin Hypercube Sampling of a subdomain of the space, different values of $v(\mathbf{h})$ are observed. These observations are indicated via subscript $_o$ and each observation v_0 is written as:

$$v_o(\mathbf{h}_o) \quad = \quad a_0 + \sum_{l=1}^{m} a_l h_{l_o} + \varepsilon_{res_o} \tag{2.28}$$

where ε_{res_o} is called the residual. The residual is defined as the difference between the response value obtained from evaluating the element's objective function v for parameters (h_l) chosen by the LHS and the fitted value from the response surface function for those same parameters. Hence:

$$\varepsilon_{res_o} = v_o - v_{RSM_o}. \tag{2.29}$$

One approach to choose the coefficients ($a_{..}$) is the method of least squares, which chooses the coefficients ($a_{..}$) such that the sum of the squares of the errors (ε_{res_o}) are minimized. The least squares function is mathematically expressed as:

$$L = \sum_{o=1}^{p} \varepsilon_{res_o}^2 \tag{2.30}$$

$$= \sum_{o=1}^{p} \left(v_o(\mathbf{h}) - a_0 - \left(\sum_{l=1}^{m} a_l h_{l_o} \right) \right)^2. \tag{2.31}$$

The observations (v_o) can also be written in matrix vector notation. In this case Equation 2.28 is rewritten and becomes:

$$\mathbf{v} = \mathbf{Ha} + \boldsymbol{\varepsilon_{res}}. \tag{2.32}$$

The least squares function simplifies to:

$$L = \boldsymbol{\varepsilon}_{res}^T \boldsymbol{\varepsilon}_{res} = (\mathbf{v} - \mathbf{Ha})^T (\mathbf{v} - \mathbf{Ha}). \tag{2.33}$$

The least squares estimator is minimized with respect to the coefficients ($a_{..}$). Therefore, the least squares estimator must satisfy:

$$\nabla_{\mathbf{a}} L = -2\mathbf{H}^T\mathbf{v} + 2\mathbf{H}^T\mathbf{Ha} = 0. \tag{2.34}$$

From Equation 2.34 the coefficients ($\mathbf{a}$) can be evaluated, hence:

$$\mathbf{a} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1} \mathbf{H}^T\mathbf{v}. \tag{2.35}$$

The fitted response surface function is mathematically expressed as:

$$\widehat{\mathbf{v}} = \mathbf{Ha}. \tag{2.36}$$

Difference between the observations $\mathbf{v}$ and the fitted values $\widehat{\mathbf{v}}$ is measured via:

$$\mathbf{e} = \mathbf{v} - \widehat{\mathbf{v}}. \tag{2.37}$$

These residuals are important in order to determine model adequacy. However, scaled residuals often convey more information than these ordinary residuals. Two important scaled residual measures are the F-ratio (signal to noise):

$$F_0 = \frac{\sum_{o=1}^{p} \left(\widehat{v}_o - \frac{1}{p} \sum_{o=1}^{p} v_o \right)^2 / m}{\mathbf{e}^T\mathbf{e} / (p - m - 1)}, \tag{2.38}$$

which measures if one of the variables (h_l) contributes significantly to the model. Typically (Myers and Montgomery, 1995), one tests if F_0 exceeds $F_{0.05,m,p-m-1}$. Upper critical values of the F-distribution can be found in books on statistics, e.g. Ross (2009).

The second scaled residual is the coefficient of determination R^2, mathematically expressed as:

$$R^2 = 1 - \frac{\mathbf{e}^T\mathbf{e}}{\sum_{o=1}^{p}\left(\widehat{v}_o - \frac{1}{p}\sum_{o=1}^{p} v_o\right)^2} \tag{2.39}$$

which determines the amount of reduction in the variability of $\widehat{\mathbf{v}}$ (the diversity of computed values $\widehat{\mathbf{v}}$) by using the h_l variables in the model. Values for R are $0 \leq R^2 \leq 1$. A higher value of R does not imply that the accuracy of the model is better because adding data points will always increase R even if they do not significantly contribute to the model.

Based upon these two statistical data measures the response surface is accepted or modified. Modifications yield evaluating additional points inside the subset of the design space, changing the degree of the fitted polynomial or increasing or decreasing the size of the subset of the design space.

Chapter 3

Decomposition methods

The previous chapter introduced a unifying notation for the development of the multi-level design optimization framework. Furthermore, a few necessary techniques from the field of optimization were introduced that are applied in this chapter to decomposition techniques.

The concept of identifying coupled elements is introduced in Section 3.1. This coupling is represented by an abstract coupling circle, which is discussed in Section 3.2 along with decomposition formulations.

Following the discussion on decomposition techniques, Section 3.3 introduces the problem matrix. The problem matrix is a means of illustrating coupling between elements in optimization problems.

The discussion on the problem matrix is then extended to the multi-level case for systems consisting of many levels in Section 3.4. Furthermore, the necessary conditions for optimality of an optimization problem with embedded hierarchy are derived in Section 3.5.

Finally, the various steps in the decomposition process are illustrated via decomposition of a two-bar structural optimization problem in Section 3.6 and in Section 3.7 a discussion on the presented decomposition methods concludes the chapter.

3.1 Identifying coupled problems

Complex systems, such as those encountered in aerospace engineering, can typically be considered as a hierarchy of individual elements. This hierarchy is reflected in the analysis techniques that are used to analyze the physical characteristics of the system as a whole. These analysis techniques rely on a hierarchy of models, each accounting for different length scales, components or disciplines.

Many complex systems can be considered a single component when viewed from far and a collection of components when closely observed. An aircraft can be seen as a single element when viewed from far. Zooming in, the aircraft consists of wing, fuselage, vertical tail, engines etc. Each element is built from smaller components, e.g. the wing consists of a skin, torsion box, flaps, etc. Each of these small elements is built from material that can be considered a single component when viewed as an element, however when closely observing the material, the material is built up of several components, e.g. fibers in composites, material matrix, grain boundaries and at the smallest scales individual atoms. Therefore, analysis models have been developed that describe a structure at the level of observation. Explicit links between the multiple levels are necessary to develop an efficient multi-level description of a structure's behavior.

Provided the scales can be decoupled, for that reason they must differ substantially, the multi-level analysis of elements at different length or time scales is typically conducted independently. That is, the output from one analysis forms the input to the neighboring element. When elements are part of a hierarchy and the higher elements in the hierarchy pass output to elements lower in the hierarchy, approaches are called hierarchical top-down approaches. The analysis covering general characteristics of the structure is conducted and results are passed on to the lower levels in the hierarchy. Likewise, a hierarchic bottom-up approach is used that starts at the smallest scale of observation and passes the results to higher levels in the hierarchy.

In cases where independent analysis of phenomena at different levels is not possible, coupled approaches are required. Modern techniques that focus on this coupling are multi-scale analysis techniques and domain decomposition techniques. Multi-scale techniques analyze physical responses that cover multiple length and time scales and domain decomposition techniques typically couple the domains of different media (e.g. fluid structure interaction) or partition a domain in smaller segments (e.g. sub-structuring in case non-overlapping sub-domains are used).

Multi-scale computational techniques can be subdivided into multiple scale expansion methods and superposition based methods (Fish and Shek, 2000). Typically these methods are applied to bridge material and structural scales.

Multiple scale expansion methods or homogenization techniques rely on three assumptions. (1) The physical characteristics observed at the top-level are constant at the lower levels. (2) The physical characteristics that are observed within the lower level element are local. (3) The top-level physical characteristics are formed by a spatial periodicity of the lower level elements (Representative Volume Elements(RVE)). These assumptions are valid as long as the scale of the lower level element (RVE) physical characteristics is much smaller then the scale of the top-level element physical responses.

Superposition methods rely on a hierarchical decomposition of the solution space of the physical responses r into: top-level (large scale) responses ${}^{0}r$ and lower level (small scale) responses ${}^{1}r, \ldots, {}^{1,\ldots,i}r$. Hence, $r = {}^{0}r + \ldots + {}^{1,\ldots,i}r$, the compatibility between the physical responses is prescribed by homogeneous boundary conditions on ${}^{0}r$ at the interface between Level-0 and Level-1. Superposition methods are characterized via three aspects: approximation of physical responses of individual elements of the hierarchy; mathematical formulation of the interface between hierarchical elements, in the present thesis these interfaces are expressed as ${}^{i}_{j}\mathcal{H}({}^{i}r)-{}^{i}_{j}h$ or ${}^{j}_{i}\mathcal{H}({}^{j}r)-{}^{j}_{i}h$; solution methods of the analysis problem, hence numerical or exact solution techniques.

The design of a structure involves the combined performance of various disciplines. Analysis of the aerodynamics, hence fluid computations, the analysis of structural performance, hence computational solid mechanics, the analysis of power consumption, the analysis of operational costs and business models are examples of such disciplines. Explicit links between these models are essential for a designer to consider effects on other disciplines while making changes within an individual disciplinary model. Typically such computations are done in sequence, data is passed to the next element until all elements have been evaluated. A coupled approach iterates between the disciplines such that changes in one discipline are accounted for in all the disciplines.

Likewise, the designer may decompose the design problem over various levels of a hierarchy. Elements of the hierarchy are formulated as individual optimization problems that are coupled to individual optimization problems elsewhere in the hierarchy. The designer can focus on changing smaller more manageable parts of the design without neglecting the combined performance of the entire hierarchy. Such an approach is successful if many design variables and/or constraint functions and/or objectives can be assigned to individual hierarchical elements. Typically, the design optimization is started at the top or the bottom of the hierarchy and level by level the individual optimizations are conducted. Cycling over the levels is necessary if the design of a single element largely effects the optimal design of neighboring elements.

Lastly, a hierarchy can be created because the model is too large to solve or it is more convenient to solve it in parts. Such techniques search for weak links between the analysis functions and/or optimization functions that allow for a decomposition. Examples of such approaches are found in the work of Bloebaum (1995). Bloebaum developed a procedure to quantify the strength of coupling. A hierarchy might not seem present at first, but due to analyzing the properties of the underlying problem, blocks of local analysis equations and/or optimization functions can be distinguished such that decomposition can take place.

Essential in the identification of elements, whether they are naturally present or artificially identified, is that physical responses and/or design problems can be computed locally. These responses and/or design problems might weakly or strongly depend on responses and/or design problems elsewhere, however their main computational effort is local. The influence of computed local responses and/or design problems are communicated to neighboring elements through boundary conditions. Hence, the physical responses are mapped onto the neighboring domain and *vice versa*. Formulating such boundary conditions between coupled elements is discussed in the next section.

3.2 Separating coupled problems

There are two distinct approaches to obtain a multi-level design hierarchy as introduced in Chapter 2. The first is that one combines various elements together and in this manner a complete structure is formed. Here the hierarchy is automatically em-

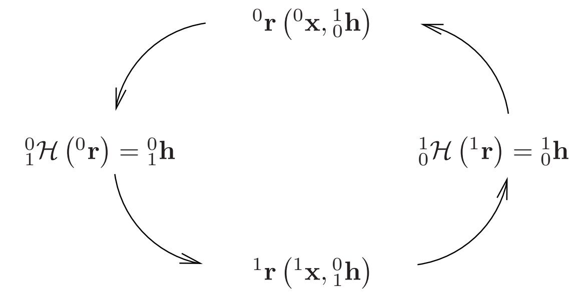


Figure 3.1: *Physical response interaction between two elements. The responses* ${}^{0}\mathbf{r}$ *computed at the parent element are transformed by the operator* ${}^{0}_{1}\mathcal{H}$ *into information that influences the responses of the child element* ${}^{1}\mathbf{r}$*. Likewise, the responses from the child element are transformed by the operator* ${}^{1}_{0}\mathcal{H}$ *into information that influences the responses at the parent element.*

bedded in the resulting design problem. The second approach is that a design problem is too large to solve by a single approach and is distributed over multiple elements. The hierarchy obtained in either of these two approaches has similar characteristics. The main characteristic being the interaction between two elements which will be the focus of this section.

The interaction between two elements can be described by the interactions between a parent element and a child element, see Figure 3.1. In this figure a representation of the physical coupling is visible.

The physical responses of a single element can be temporarily isolated via decomposition. Decomposition refers to the process by which the links between the coupled elements are (temporarily) broken down such that both elements can be considered individual, making the process of finding a solution to the coupled problem more manageable.

Decomposition is accomplished via the introduction of consistency constraints:

$$ {}^{i}_{j}\mathbf{c} \;=\; {}^{i}_{j}\mathcal{H}\left({}^{i}\mathbf{r}\right) - {}^{i}_{j}\mathbf{h}, \tag{3.1} $$

where for the present coupling circle $i \neq j$ and $i, j = 0, 1$. There are two formulations for the consistency constraints that are used frequently in order to maintain consistency for two coupled elements. Exact formulations expressed as equality constraints and inexact formulations where the consistency is relaxed:

1. equality constraints/conditions:

$$ \begin{aligned} {}^{0}_{1}\mathbf{c} &= {}^{0}_{1}\mathcal{H}\left({}^{0}\mathbf{r}\right) - {}^{0}_{1}\mathbf{h} &= \mathbf{0}; \\ {}^{1}_{0}\mathbf{c} &= {}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h} &= \mathbf{0}. \end{aligned} \tag{3.2} $$

2. relaxation, via, e.g., Lagrange multipliers:

$$ \begin{aligned} {}^{0}_{1}\boldsymbol{\lambda}^{T}\,{}^{0}_{1}\mathbf{c} &= {}^{0}_{1}\boldsymbol{\lambda}^{T}\left({}^{0}_{1}\mathcal{H}\left({}^{0}\mathbf{r}\right) - {}^{0}_{1}\mathbf{h}\right); \\ {}^{1}_{0}\boldsymbol{\lambda}^{T}\,{}^{1}_{0}\mathbf{c} &= {}^{1}_{0}\boldsymbol{\lambda}^{T}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right). \end{aligned} \tag{3.3} $$

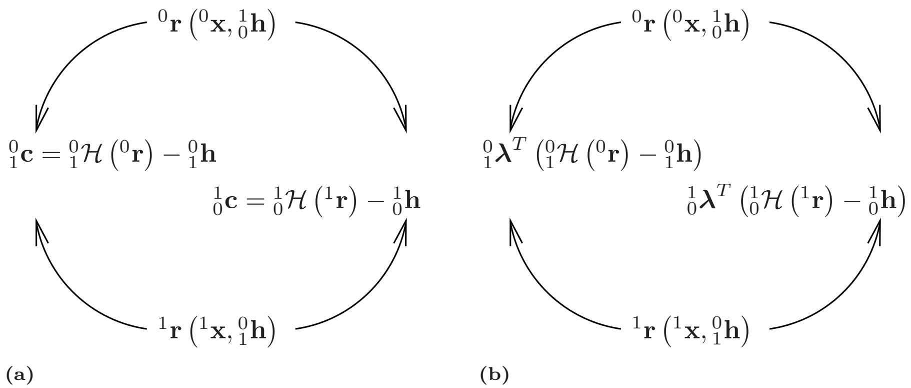


Figure 3.2: *(a) Consistency between the coupled elements is maintained through interface compatibility or equilibrium. (b) Consistency can be relaxed via relaxation. In the current figure Lagrange multipliers are applied, however Penalty function methods or Augmented Lagrangian relaxation can be used similarly. Externally the relaxation parameters (in the current figure the Lagrange multipliers) are controlled, such that the inconsistencies vanish at compatibility or equilibrium of the interface.*

Relaxation-based decomposition is accomplished via relaxation of the consistency constraints. There are three typical approaches that relax these constraints: Lagrangian relaxation, Penalty function methods and Augmented Lagrangian relaxation.

The effect of decomposition on the coupling circle shown in Figure 3.1 is shown in Figure 3.2(a) in case equality consistency constraints are formulated and in Figure 3.2(b) in case relaxation of the consistency constraints is applied. In the current figure, Lagrangian decomposition is shown, however Penalty function relaxation or Augmented Lagrangian relaxation can be used similarly.

For each of the categories of the previously mentioned consistency formulations, two different formulations can be distinguished. These are:

1. hierarchic decomposition, which is subdivided into top-down or/and bottom-up formulations.

2. non-hierarchic decomposition, which treats all elements equal.

Hierarchic decomposition is the result of identifying elements in the hierarchy that dictate the output from other elements. This is illustrated in Figure 3.3(a) in case of a top-down decomposition formulation with equality consistency constraints. The top, or Level-0 element prescribes the necessary output from the Level-1 element. A similar but opposite approach is possible, where the Level-1 element prescribes the necessary output of the Level-0 element. The latter approach is shown in Figure 3.3(b) and is called a bottom-up decomposition.

In case the consistency constraints are relaxed via, e.g., Lagrange multipliers, the resulting top-down decomposition is shown in Figure 3.4(a) and the bottom-up decomposition in Figure 3.4(b). The relaxed constraints are added to the Level-1

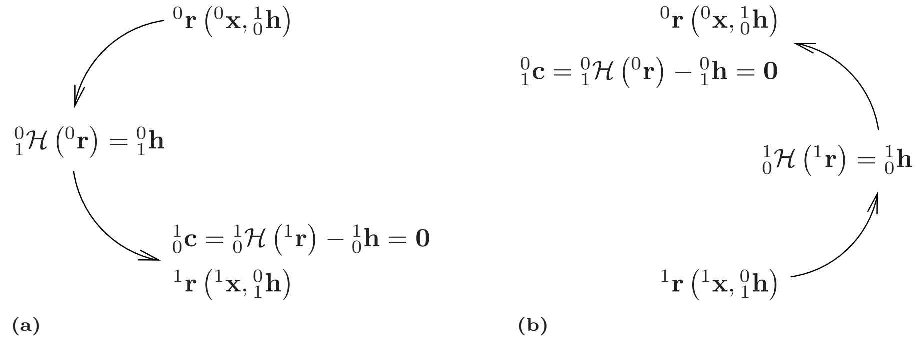


Figure 3.3: *(a) The Level-0 element prescribes the necessary responses for the Level-1 element by means of equality consistency constraints. This is called top-down decomposition. (b) Level-1 prescribes the responses of the Level-0 element, hence a bottom-up decomposition by means of equality consistency constraints.*

element objective function causing a violation of the consistency constraint to have a negative effect on the measured performance of that individual element.

Figure 3.3 and Figure 3.4 show hierarchic decomposition of the coupling circle shown in Figure 3.1. The coupling in one direction, left or right depending on respectively top-down or bottom-up decomposition is replaced via consistency constraints. The opposite direction is still intact meaning that coupling is still present making one element the "leader"and the other element the "follower".

Non-hierarchic decomposition of the coupling circle shown in Figure 3.1 involves replacing the links in both directions of the coupling circle. The decomposition involves elements that do not prescribe output from each other. Instead, output is estimated and after element solutions are found, the information is exchanged among the elements via a coordination method (discussed in Chapter 4) that updates the coupling parameters and confirms if the consistency constraints are satisfied.

In case decomposition is accomplished via equality constraints the non-hierarchic decomposition is shown in Figure 3.5. Additional information is required to take into account changes in the interaction. This is accomplished via derivation of the Global Sensitivity Equations (Sobieszczanski-Sobieski, 1990):

$$\begin{aligned}
\mathbf{D}_{{}^0\mathbf{x}}({}^0_1\mathcal{H}) &= \nabla_{{}^0\mathbf{x}}({}^0_1\mathcal{H}) + \nabla_{{}^1_0\mathbf{h}}({}^0_1\mathcal{H})^T \mathbf{D}_{{}^0\mathbf{x}}({}^1_0\mathcal{H}) \\
\mathbf{D}_{{}^1\mathbf{x}}({}^0_1\mathcal{H}) &= \nabla_{{}^1\mathbf{x}}({}^0_1\mathcal{H}) + \nabla_{{}^1_0\mathbf{h}}({}^0_1\mathcal{H})^T \mathbf{D}_{{}^1\mathbf{x}}({}^1_0\mathcal{H}) \\
\mathbf{D}_{{}^0\mathbf{x}}({}^1_0\mathcal{H}) &= \nabla_{{}^0\mathbf{x}}({}^1_0\mathcal{H}) + \nabla_{{}^0_1\mathbf{h}}({}^1_0\mathcal{H})^T \mathbf{D}_{{}^0\mathbf{x}}({}^0_1\mathcal{H}) \\
\mathbf{D}_{{}^1\mathbf{x}}({}^1_0\mathcal{H}) &= \nabla_{{}^1\mathbf{x}}({}^1_0\mathcal{H}) + \nabla_{{}^0_1\mathbf{h}}({}^1_0\mathcal{H})^T \mathbf{D}_{{}^1\mathbf{x}}({}^0_1\mathcal{H})
\end{aligned} \tag{3.4}$$

The necessary sensitivity information is the solution to the system of equations:

$$\begin{bmatrix} \mathbf{I} & -\nabla_{{}^1_0\mathbf{h}}({}^0_1\mathcal{H})^T \\ -\nabla_{{}^0_1\mathbf{h}}({}^1_0\mathcal{H})^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{{}^0\mathbf{x}}({}^0_1\mathcal{H}) & \mathbf{D}_{{}^1\mathbf{x}}({}^0_1\mathcal{H}) \\ \mathbf{D}_{{}^0\mathbf{x}}({}^1_0\mathcal{H}) & \mathbf{D}_{{}^1\mathbf{x}}({}^1_0\mathcal{H}) \end{bmatrix} = \dots$$
$$\begin{bmatrix} \nabla_{{}^0\mathbf{x}}({}^0_1\mathcal{H}) & \mathbf{0} \\ \mathbf{0} & \nabla_{{}^1\mathbf{x}}({}^1_0\mathcal{H}) \end{bmatrix} \tag{3.5}$$

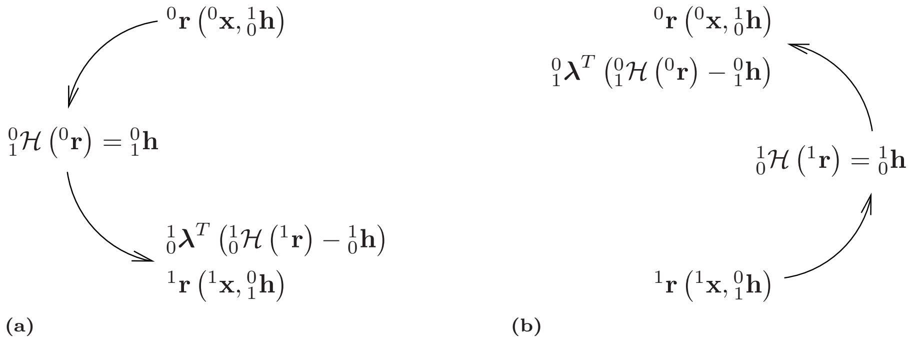


Figure 3.4: *(a) Top-down decomposition by means of relaxation of the consistency constraint and assigning the Lagrangian multipliers to the Level-1 element. (b) Bottom-up decomposition by means of relaxation of the consistency constraint and assigning the Lagrangian multipliers to the Level-0 element.*

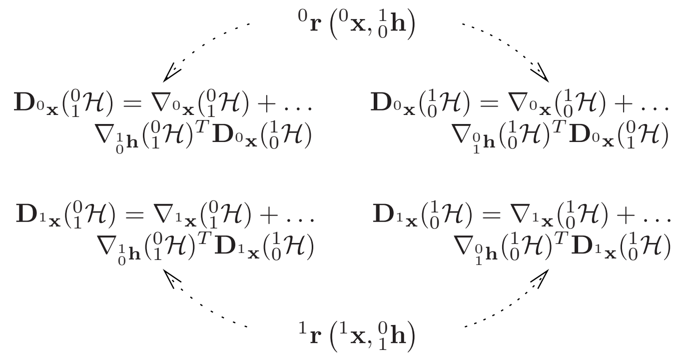


Figure 3.5: *None of the elements prescribes the output of the neighboring element, thus a non-hierarchic decomposition. Equality consistency constraints are assumed ($\because\mathbf{c} = 0$) and the interaction is approximated by means of computing the Global Sensitivity Equations.*

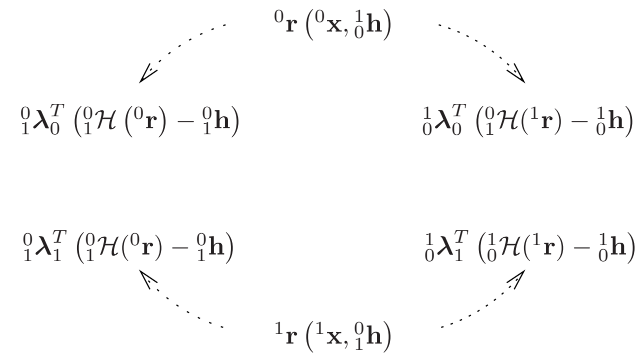


Figure 3.6: *None of the elements prescribes the output of a neighboring element, thus a non-hierarchic decomposition. The consistency between the two elements is relaxed on both sides of the coupling circle, via e.g. Lagrange multipliers. Via coordination the Lagrange multipliers are updated such that the consistency is restored. The relaxation parameters applied to the relaxed consistency constraints in each individual element are not necessarily equal. Therefore, an additional index distinguishes the relaxation parameters, here* $^{\cdot\cdot}_{\cdot\cdot}\boldsymbol{\lambda}_0$ *and* $^{\cdot\cdot}_{\cdot\cdot}\boldsymbol{\lambda}_1$.

Another frequently used technique to account for interactions between the elements is the use of Response Surface(RS) methods. An RS method seeks to fit a function through the responses of an element which are changing due to different input parameters for the element. The basic principles of RS techniques were discussed in Chapter 2.

A non-hierarchic decomposition by means of relaxation of the consistency constraints via e.g. introducing Lagrange multipliers is shown in Figure 3.6. A coordination method is necessary to update the Lagrange multipliers.

Figure 3.7 presents an overview of the two-decomposition techniques for coupled elements: equality based decomposition; and relaxation based decomposition. Relaxation based decomposition is accomplished via relaxation of the consistency constraints. There are three typical approaches that relax these constraints, e.g. Lagrangian relaxation, Penalty function relaxation and Augmented Lagrangian relaxation. Both decomposition techniques can be subdivided into hierarchic and non-hierarchic formulations.

Thus far the decomposition of physical properties of elements is discussed, how this decomposition enters the optimization problem is characterized through the problem matrix, which will be discussed in the next section.

3.3 Illustrating coupling via the problem matrix

The decomposition process of optimization problems involves identifying relationships between the design variables, physical responses and objectives/constraints that permit us to separate them into elements that are connected. Optimization problems for

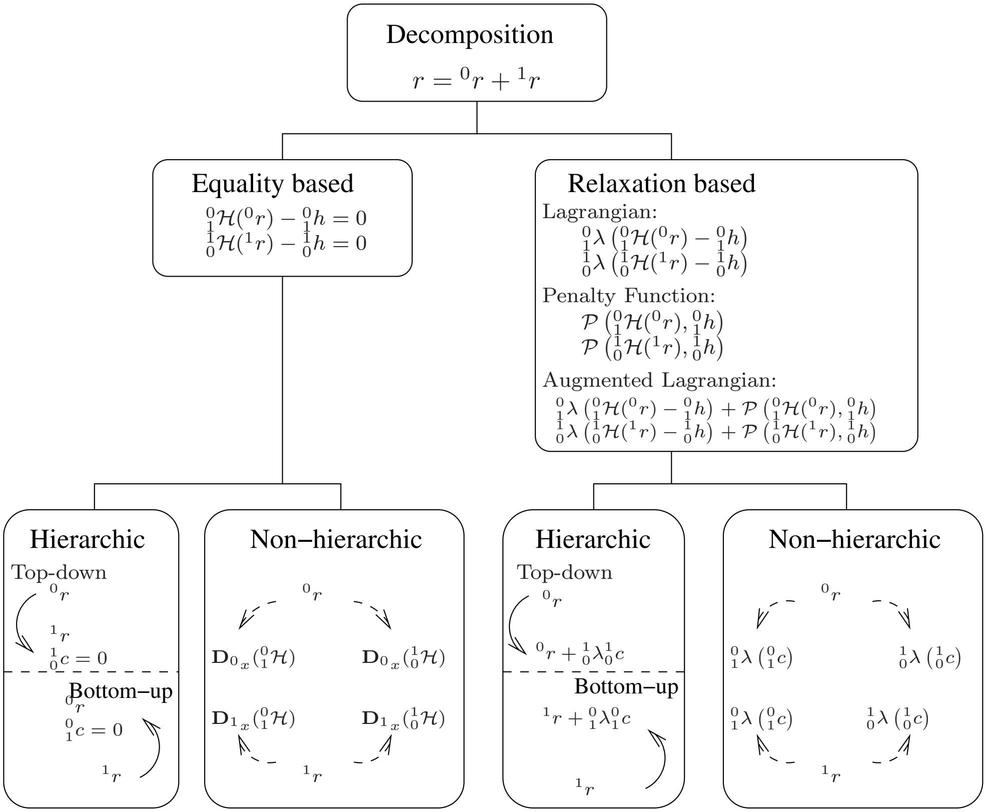


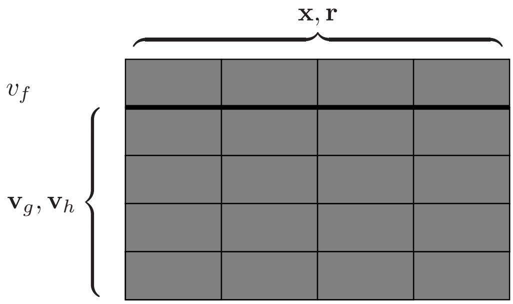
Figure 3.7: *Summary of the decomposition process. Two choices are available for decomposing coupled problems. These are equality based decomposition and relaxation based decomposition. Furthermore, these approaches are subdivided in hierarchic top-down or bottom-up formulations and non-hierarchic formulations.*

which no elements have been identified yet are defined as:

$$\begin{aligned} \min_{\mathbf{x}} \quad & v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) \\ \text{s.t.} \quad & \mathbf{v}_g(\mathbf{x}, \mathbf{r}(\mathbf{x})) \leq \mathbf{0} \\ & \mathbf{v}_h(\mathbf{x}, \mathbf{r}(\mathbf{x})) = \mathbf{0} \end{aligned} \tag{3.6}$$

where $v_f(\ldots)$ is the objective function, $\mathbf{v}_g(\ldots)$ are inequality constraints and $\mathbf{v}_h(\ldots)$ are equality constraints. The relationship between variables ($\mathbf{x}$), responses ($\mathbf{r}$) and functions (v_f, $\mathbf{v}_g$ and $\mathbf{v}_h$) is illustrated amongst others by Barthelemy (1989) with the *problem matrix* (also known as *Functional Dependence Table* (Wagner, 1993)).

When the problem matrix is full, all functions depend on all the variables and responses. This is illustrated in the problem matrix of Figure 3.8, where the dependence of $\mathbf{r}$ on the design variables $\mathbf{x}$ is dropped for brevity of notations. Above the



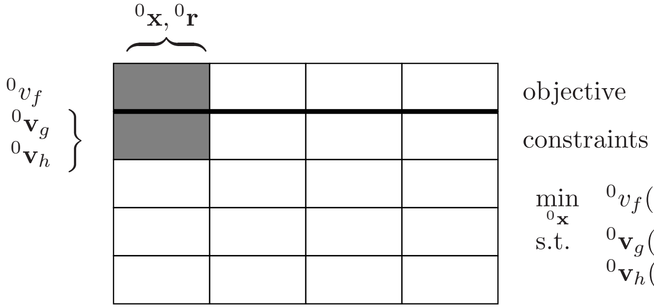
$$\begin{array}{ll} \min\limits_{\mathbf{x}} & v_f(\mathbf{x}, \mathbf{r}) \\ \text{s.t.} & \mathbf{v}_g(\mathbf{x}, \mathbf{r}) \leq \mathbf{0} \\ & \mathbf{v}_h(\mathbf{x}, \mathbf{r}) = \mathbf{0} \end{array}$$

Figure 3.8: *Problem matrix, illustrating the relationship between variables and functions. The problem matrix is full, meaning that objective and constraint functions depend on all the design variables.*

thick black line the objective function v_f is reflected and the blocks of variables $\mathbf{x}$ and responses $\mathbf{r}$ on which this objective function depends are marked. Likewise, the design constraints $\mathbf{v}_g$ and $\mathbf{v}_h$ are listed below the thick black line and the blocks of variables and responses to which these functions correspond are marked.

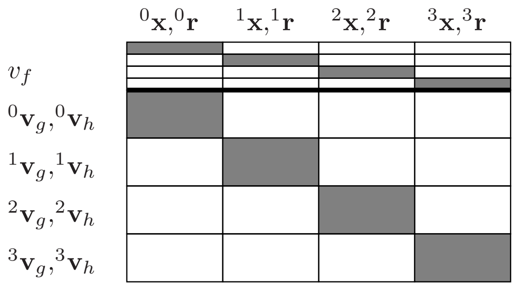
A single individual element depends on element design variables and element physical responses. Such an element is shown in Figure 3.9. The objective function ${}^0v$ depends on the element design variables ${}^0\mathbf{x}$ and element physical responses ${}^0\mathbf{r}$. Below the thick black line the element design constraints ${}^0\mathbf{v}_g$ and ${}^0\mathbf{v}_h$ are listed.

For a problem consisting of many elements, the simplest case is when there are only *uncoupled* problems (Kirsch and Moses, 1979). The problem matrix resembles that of Figure 3.10. A hierarchy of three individual elements is shown that are uncoupled. The dependencies of optimization functions (horizontal) on design variables and physical responses (vertical) form a diagonal pattern in the problem matrix. This is called a block diagonal problem matrix. The constraints are completely separable and the objective function is additively separable, meaning that the objective function consists of a summation of element objectives that are uncoupled. Combined these element objectives form the entire objective function. The element objectives are minimized



$$\begin{array}{ll} \min\limits_{{}^0\mathbf{x}} & {}^0v_f({}^0\mathbf{x}, {}^0\mathbf{r}) \\ \text{s.t.} & {}^0\mathbf{v}_g({}^0\mathbf{x}, {}^0\mathbf{r}) \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^0\mathbf{x}, {}^0\mathbf{r}) = \mathbf{0} \end{array}$$

Figure 3.9: *Problem matrix, illustrating the relationship between variables and functions. A gray block indicates that the functions (horizontal) depend on the design variables and responses (vertical).*



$$
\begin{array}{lrl}
\min\limits_{\mathbf{x}} & v_f = & \sum_0^3 {}^i v_f({}^i\mathbf{x}, {}^i\mathbf{r}) \\
\text{s.t.} & {}^0\mathbf{v}_g = & {}^0\mathbf{v}_g({}^0\mathbf{x}, {}^0\mathbf{r}) \\
 & {}^0\mathbf{v}_h = & {}^0\mathbf{v}_h({}^0\mathbf{x}, {}^0\mathbf{r}) \\
 & {}^1\mathbf{v}_g = & {}^1\mathbf{v}_g({}^1\mathbf{x}, {}^1\mathbf{r}) \\
 & \vdots & \\
 & {}^3\mathbf{v}_h = & {}^3\mathbf{v}_h({}^3\mathbf{x}, {}^3\mathbf{r})
\end{array}
$$

Figure 3.10: *Block diagonal matrix, showing the relationships between variables and fully uncoupled functions. In this case there is no need for coordinating the solution process to reach the optimum of the multi-level optimization problem.*

separately. Uncoupled problems can be solved independently and therefore there is no coordination (see Chapter 4) necessary.

In order to make the problem more manageable, one searches for interaction (or *coupling*) between groups of variables and/or responses such that individual elements are formed or identified. These interactions are subdivided into:

1. non existing, hence elements that are formed or identified are uncoupled.

2. weak, if the number of coupling variables is substantially less than the total number of variables associated with each of the individual elements and none of the coupling variables that are identified has a significant impact on the solution of the individual element.

3. strong, if most of the coupling variables are shared among the elements and a diagonal on the problem matrix cannot be identified.

In real life problems, uncoupled elements do not exist. But often a distinction between *weak* and *strong* dependencies of the elements can be made. In Figure 3.11 such a distinction is illustrated. The light gray blocks indicate weak dependencies and the dark gray blocks strong dependencies.

Preferably one has a *Block Diagonal* problem matrix such as illustrated in Figure 3.10 since the individual elements are then uncoupled in the constraints and additively separable in the objective function. However, more frequently encountered problems involve coupling. In that case, the subproblems are connected through design ${}^i_j\mathbf{z}$ and/or coupling variables ${}^i_j\mathbf{h}$. Shared design variables ${}^i_j\mathbf{z}$ are design variables that are present in multiple hierarchical elements. Coupling variables ${}^i_j\mathbf{h}$ represent the mapped physical responses $({}^i_j\mathcal{H}({}^i\mathbf{r}))$ from one element onto a neighboring element to which it is coupled, see Section 3.2. An example of a problem consisting of shared design variables and coupled responses is the minimization of an additively separable

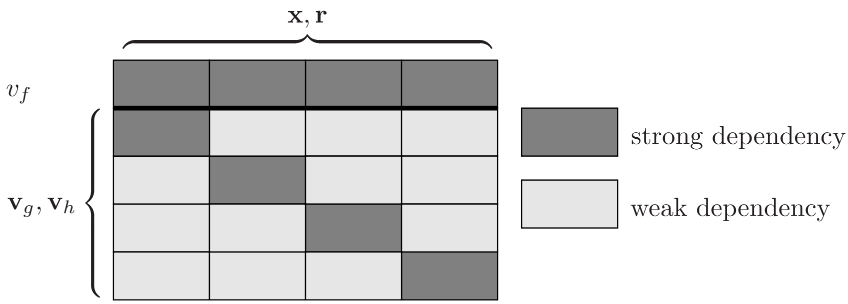


Figure 3.11: *Problem matrix, illustrating the weak (off-diagonal) and strong (main-diagonal) dependencies between individual elements. The objective is a function of all the design variables and depends on all the elements. However, the constraints can be organized in strong and weak dependencies associated with each individual element.*

objective function subjected to constraints:

$$
\begin{array}{llcll}
\min\limits_{\mathbf{x}} & {}^{0}v_f\left({}^{0}_{i}\mathbf{z}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}\left({}^{0}_{i}\mathbf{z}, {}^{0}\mathbf{x}, {}^{i}_{0}\mathbf{h}\right)\right) + \sum\limits_{i=1}^{3} {}^{i}v_f\left({}^{i}_{0}\mathbf{z}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\left({}^{i}_{0}\mathbf{z}, {}^{i}\mathbf{x}, {}^{0}_{i}\mathbf{h}\right)\right) & & & \\
\text{s.t.} & {}^{0}\mathbf{v}_g\left({}^{0}_{i}\mathbf{z}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}\left({}^{0}_{i}\mathbf{z}, {}^{0}\mathbf{x}, {}^{i}_{0}\mathbf{h}\right)\right) & \leq & \mathbf{0} & \\
 & {}^{i}\mathbf{v}_g\left({}^{i}_{0}\mathbf{z}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\left({}^{i}_{0}\mathbf{z}, {}^{i}\mathbf{x}, {}^{0}_{i}\mathbf{h}\right)\right) & \leq & \mathbf{0} & i = 1, \ldots, 3 \\
 & {}^{0}\mathbf{v}_h\left({}^{0}_{i}\mathbf{z}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}\left({}^{0}_{i}\mathbf{z}, {}^{0}\mathbf{x}, {}^{i}_{0}\mathbf{h}\right)\right) & = & \mathbf{0} & \\
 & {}^{i}\mathbf{v}_h\left({}^{i}_{0}\mathbf{z}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\left({}^{i}_{0}\mathbf{z}, {}^{i}\mathbf{x}, {}^{0}_{i}\mathbf{h}\right)\right) & = & \mathbf{0} & \\
\text{where} & {}^{0}\mathbf{x}, {}^{0}\mathbf{r} & & \text{top-element level variables} & \\
 & {}^{i}\mathbf{x}, {}^{i}\mathbf{r} & & \text{element level variables} & \\
 & {}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h} & & \text{coupling variables} & \\
 & {}^{\cdot\cdot}_{\cdot\cdot}\mathbf{z} & & \text{shared design variables} & \\
 & \mathbf{x} = \left[{}^{0}\mathbf{x}, {}^{0}_{i}\mathbf{z}, {}^{i}\mathbf{x}, {}^{i}_{0}\mathbf{z}\right] & & &
\end{array}
\qquad (3.7)
$$

Notice that one may now distinguish *global* variables ${}^{0}\mathbf{x}$ and ${}^{0}\mathbf{r}$ related to the Level-0 problem, *local* variables ${}^{i}\mathbf{x}$ and ${}^{i}\mathbf{r}$, which are related to Level-1, coupling variables ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$ connecting the two levels via physical coupling and shared design variables that are shared among individual elements (${}^{i}_{j}\mathbf{z} = {}^{j}_{i}\mathbf{z}$). The current example involves four individual elements that are coupled and divided over two levels.

The problem matrix that illustrates the relations between the variables of Equation 3.7 is called an *Angular* problem matrix. This problem matrix is illustrated in Figure 3.12 for the problem of Equation 3.7.

In the previous section two decoupling formulations were introduced, hence equality based decoupling and relaxation based decoupling. Furthermore, two decomposition approaches for each of the decoupling formulations were discussed, hence hierarchical and non-hierarchical decomposition. In the next section first the equality based decoupling for hierarchical decomposition is introduced followed by the non-hierarchical equality based decomposition. After the discussion on equality based decoupling the relaxation based decoupling is illustrated with the problem matrix for hierarchic and non-hierarchical decomposition.

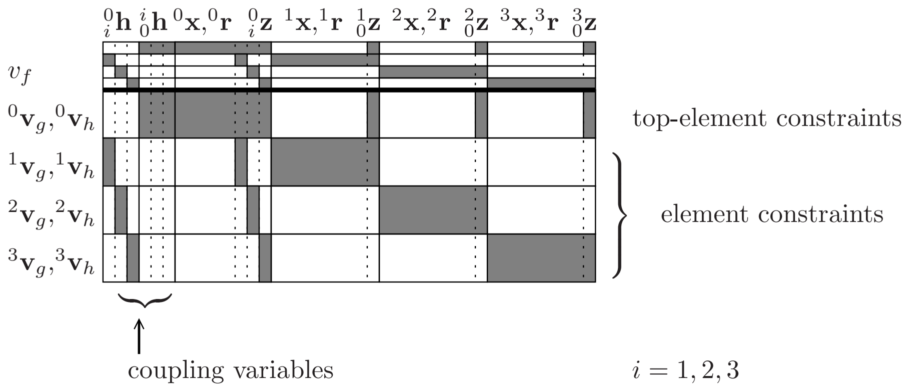


Figure 3.12: *Angular problem matrix, coordination between levels is necessary due to the coupling variables ${}^0_i\mathbf{h}$ and ${}^i_0\mathbf{h}$. Each element receives a copy of the variable ${}^i_0\mathbf{h}$ or ${}^0_i\mathbf{h}$ and coordination is necessary to introduce these copies into the other element.*

3.3.1 Equality based decoupling

In case of hierarchic decomposition, the coupling circle shown in Figure 3.13(a) is decomposed and consistency is maintained between Level-0 and Level-1 elements through consistency constraints, e.g., top-down hierarchic decomposition shown in Figure 3.13(b). The consistency constraints are assigned either to Level-0 or to Level-1, recall Section 3.2. In case consistency is maintained via equality constraints at Level-1, the coupling variables ${}^i_0\mathbf{h}$ become additional design variables for the Level-0 optimization problem. The physical responses ${}^0_i\mathcal{H}\left({}^0\mathbf{r}\right) = {}^0_i\mathbf{h}$ that are mapped from Level-0 to Level-1 are not added as design variables but accounted for in the analysis of the Level-1 element. The coupling variables are constant in the Level-1 optimization problems and are therefore omitted for brevity of notations as well as the dependence of $\mathbf{r}$ on the design variables $\mathbf{x}$.

An additively separable objective function is considered and a hierarchy is present that consists of a single element at Level-0 and three individual elements are located at Level-1. The Level-0 problem, after hierarchic top-down equality based decomposition of the Level-1 elements is:

$$\begin{array}{lll} \min\limits_{{}^i_0\mathbf{h},{}^0\mathbf{x}} & {}^0v_f({}^i_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^i_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^i_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = \mathbf{0} \\ \text{where} & i = 1,2,3 & \end{array} . \tag{3.8}$$

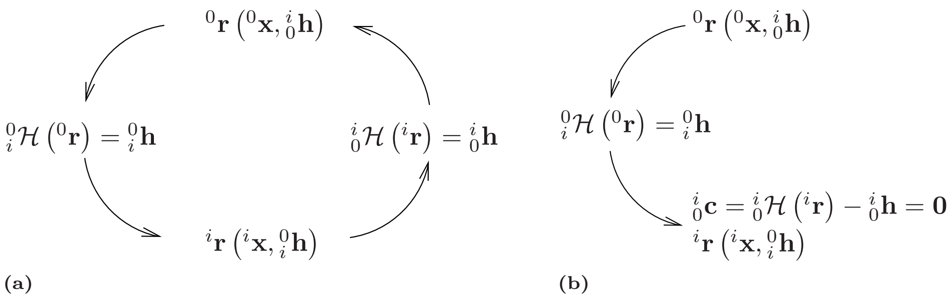


Figure 3.13: *(a) Physical response interaction between a single Level-0 element and the i^{th} element present at Level-1, where $i = 1, \dots, n$ n being the amount of elements present at Level-1. The responses $^0\mathbf{r}$ computed at the parent element are transformed by the operator ${}^0_i\mathcal{H}$ into information that influences the responses of the child element $^i\mathbf{r}$. Likewise, the responses from the child element are transformed by the operator ${}^i_0\mathcal{H}$ into information that influences the responses at the parent element. (b) The Level-0 element prescribes the necessary responses for the Level-1 element by means of equality consistency constraints. This is called top-down decomposition.*

The optimization problem for the Level-1 elements now reads:

$$\begin{array}{lll} \min\limits_{^i\mathbf{x}} & {}^iv_f({}^i\mathbf{x}, {}^i\mathbf{r}) & \\ \text{s.t.} & {}^i\mathbf{v}_g({}^i\mathbf{x}, {}^i\mathbf{r}) & \leq \mathbf{0} \\ & {}^i\mathbf{v}_h({}^i\mathbf{x}, {}^i\mathbf{r}) & = \mathbf{0} \\ & {}^i\mathbf{v}_{{}^i_0c}({}^i_0\mathbf{c}({}^i\mathbf{x}, {}^i\mathbf{r})) & = {}^i_0\mathcal{H}({}^i\mathbf{r}) - {}^i_0\mathbf{h} = \mathbf{0} \\ \text{where} & i = 1, 2, 3 & \end{array} . \tag{3.9}$$

Thus, an additional set of constraints is added to the lower level subproblems ${}^i\mathbf{v}_{{}^i_0c}({}^i_0c(..))$. These constraints are the result of temporary decoupling of the physical responses, see Figure 3.13 or for a more detailed discussion Section 3.2. These constraints depend on ${}^i_0c$ which is a function of the actual response of the element and the expected response.

The resulting problem matrix is shown in Figure 3.14. The consistency is maintained at the lower level elements.

Similar to the previous discussion, a bottom-up formulation can be constructed where the consistency is maintained at Level-0. The problem matrix in this case has the form shown in Figure 3.15. Such a formulation uses for the Level-0 problem:

$$\begin{array}{lll} \min\limits_{^0\mathbf{x}} & {}^0v_f({}^0\mathbf{x}, {}^0\mathbf{r}) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^0\mathbf{x}, {}^0\mathbf{r}) & \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^0\mathbf{x}, {}^0\mathbf{r}) & = \mathbf{0} \\ & {}^0\mathbf{v}_{{}^0_i\mathbf{c}}({}^0_i\mathbf{c}({}^0\mathbf{x}, {}^0\mathbf{r})) & = {}^0_i\mathcal{H}({}^0\mathbf{r}) - {}^0_i\mathbf{h} = \mathbf{0} \\ \text{where} & i = 1, 2, 3 & \end{array} . \tag{3.10}$$

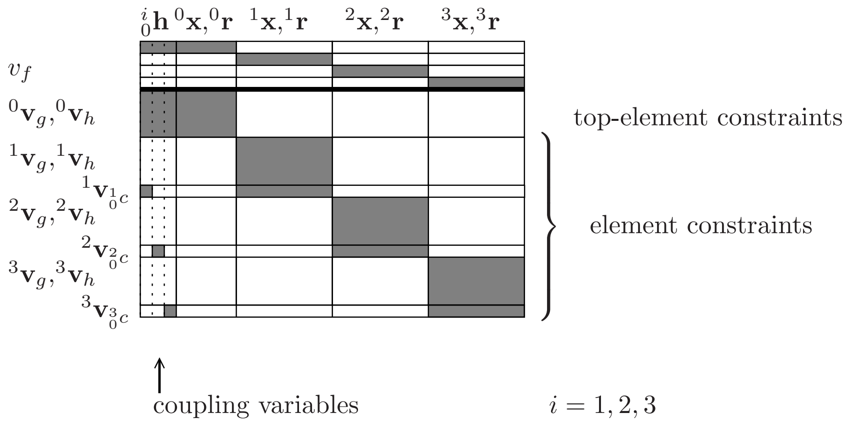


Figure 3.14: *Problem matrix showing a top-down decomposition approach. Consistency is maintained at Level-1 by means of additional constraints and coupling variables* ${}^{i}_{0}\mathbf{h}$ *are added to the Level-0 optimization problem.*

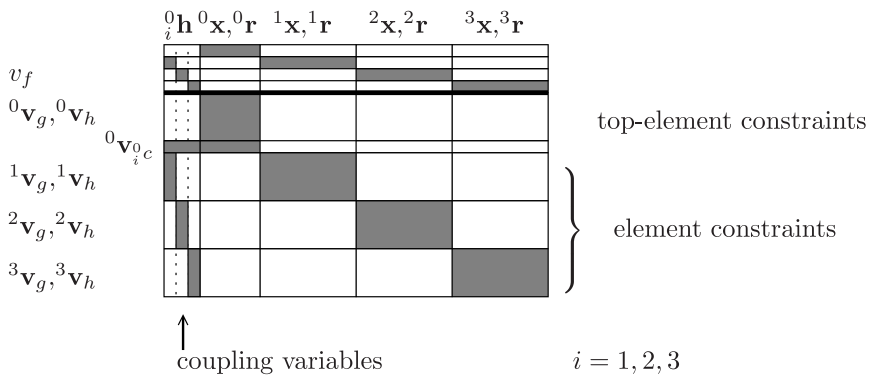


Figure 3.15: *Problem matrix showing a bottom-up decomposition approach. Consistency is maintained at Level-0 by means of additional constraints and coupling variables* ${}^{0}_{i}\mathbf{h}$ *are added to the Level-1 optimization problems as design variables.*

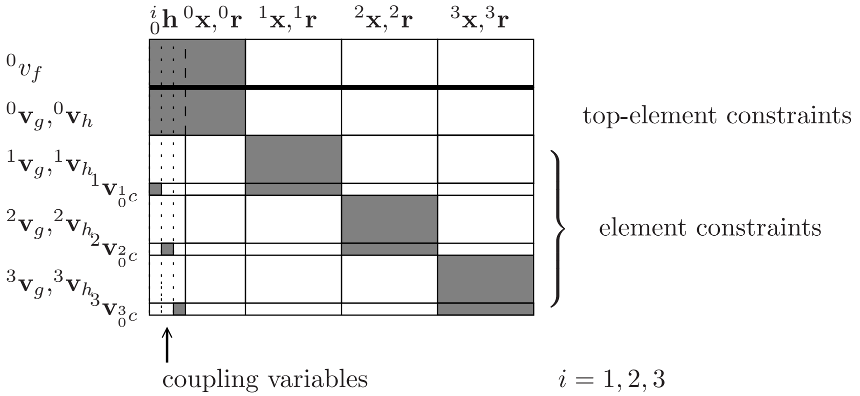


Figure 3.16: *Problem matrix where only coupling variables are connecting the elements. For this case, the number of constraints of each Level-1 element is reduced using a cumulative constraint formulation. The inequality and equality constraints are embedded in the objective function of the Level-1 problems. Furthermore, coupling variables* ${}^{i}_{0}\mathbf{h}$ *are added tot the Level-0 optimization problem.*

The resulting problem formulation for Level-1 becomes then:

$$
\begin{array}{lll}
\min\limits_{{}^{i}\mathbf{x},{}^{0}_{i}\mathbf{h}} & {}^{i}v_f({}^{0}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}) & \\
\text{s.t.} & {}^{i}\mathbf{v}_g({}^{0}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}) & \leq \mathbf{0} \\
 & {}^{i}\mathbf{v}_h({}^{0}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}) & = \mathbf{0} \\
\text{where} & i = 1, 2, 3 &
\end{array} . \tag{3.11}
$$

If the overall objective function is only a function of the coupling variables ${}^{i}_{0}\mathbf{h}$, then the Level-1 inequality and equality constraints may be replaced by a single envelope function ${}^{i}v_e(..)$ (e.g. KS-function (Sobieszczanski-Sobieski, 1992)). The inequality (${}^{i}\mathbf{v}_g$) and equality (${}^{i}\mathbf{v}_h$) constraints are embedded in the envelope function. The equality constraints are split into a positive part ${}^{i}\mathbf{v}_{h^+}$ and a negative part ${}^{i}\mathbf{v}_{h^-}$ that allows them to be written as two sets of inequality constraints that can be inserted into the envelope function. The problem matrix will then have a form as shown in Figure 3.16.

In this case there is no contribution to the Level-0 objective function through the Level-1 design variables. Therefore, Equation 3.7 reduces to:

$$
\begin{array}{lll}
\min\limits_{{}^{i}_{0}\mathbf{h},{}^{0}_{i}\mathbf{h},\mathbf{x}} & {}^{0}v_f({}^{i}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & \\
\text{s.t.} & {}^{0}\mathbf{v}_g({}^{i}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) \leq \mathbf{0} \quad ; \quad {}^{0}\mathbf{v}_h({}^{i}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & = \mathbf{0} \\
 & {}^{i}v_e({}^{i}\mathbf{v}_g({}^{0}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}), {}^{i}\mathbf{v}_{h^+}({}^{0}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}), {}^{i}\mathbf{v}_{h^-}({}^{0}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r})) & \leq 0 \\
\text{where} & {}^{i}v_e(\ldots) = \text{envelope function} & \\
 & {}^{i}\mathbf{v}_{h^+}, {}^{i}\mathbf{v}_{h^-} = \text{positive and negative part of equality constraint} & \\
 & i = 1, 2, 3 &
\end{array} \tag{3.12}
$$

The problem is split into a Level-0 optimization problem and Level-1 optimization

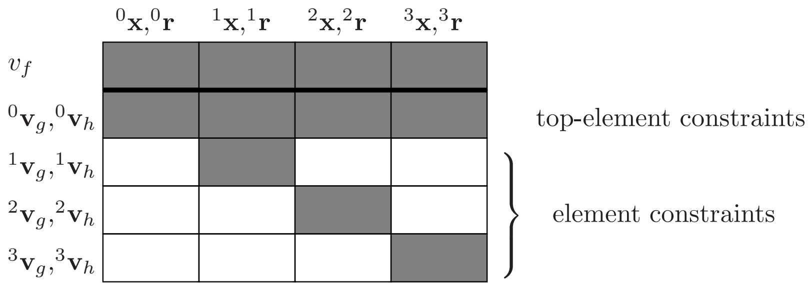


Figure 3.17: *Typical problem matrix in case of shared design variable vectors. Hence, design variables that are used on Level-0 are also used in the Level-1 elements.*

problems. The Level-1 optimization problems are formulated as constraint minimization problems. The Level-1 element constraints are embedded in the envelope function and this function is formulated as the objective function of the Level-1 elements. The envelope function embeds inequality constraints and therefore the Level-1 optimization problem is to push the design point of the individual elements into the feasible domain. This allows room for improvement in these elements since the real constraints of the Level-1 elements are not active. Therefore, the objective function of the Level-1 elements is added to the Level-0 problem to take into account the influence design changes in Level-0 have on the Level-1 elements (on the design constraints embedded in the envelope function).

With ${}^i v_e({}^i\mathbf{v}_{..}(..))$, the optimum objective of the i^{th} subproblem. The Level-0 optimization problem becomes:

$$\begin{array}{lll} \min\limits_{{}^i_0\mathbf{h}, {}^0\mathbf{x}} & {}^0 v_f({}^i_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^i_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) & \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^i_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) & = \mathbf{0} \\ & {}^i v_e^*({}^i\mathbf{v}_{..}(..)) & \leq \mathbf{0} \\ \text{where} & i = 1, 2, 3 & \end{array} \tag{3.13}$$

Where the coupling variables ${}^i_0\mathbf{h}$ are assigned to the Level-0 design variables and the coupling variables ${}^0_i\mathbf{h}$ in the Level-1 problems are naturally accounted for by the mapping of Level-0 onto Level-1, hence ${}^0_i\mathcal{H}({}^i\mathbf{r}) - {}^0_i\mathbf{h}$. The Level-1 problem becomes:

$$\begin{array}{ll} \min\limits_{{}^i\mathbf{x}} & {}^i v_e({}^i\mathbf{v}_g({}^i\mathbf{x}, {}^i\mathbf{r}), {}^i\mathbf{v}_{h^+}({}^i\mathbf{x}, {}^i\mathbf{r}), {}^i\mathbf{v}_{h^-}({}^i\mathbf{x}, {}^i\mathbf{r})) \\ \text{s.t.} & {}^i\mathbf{v}_{{}^i_0 c}({}^i_0\mathbf{c}({}^i\mathbf{x}, {}^i\mathbf{r})) = {}^i_0\mathcal{H}({}^i\mathbf{r}) - {}^i_0\mathbf{h} \\ \text{where} & i = 1, 2, 3 \end{array} \quad , \tag{3.14}$$

where consistency with the Level-0 element is maintained via the consistency constraint ${}^i\mathbf{v}_{{}^i_0 c}$.

In case the coupling originates from design variables that are shared over multiple levels, the problem matrix has a form as shown in Figure 3.17. The optimization

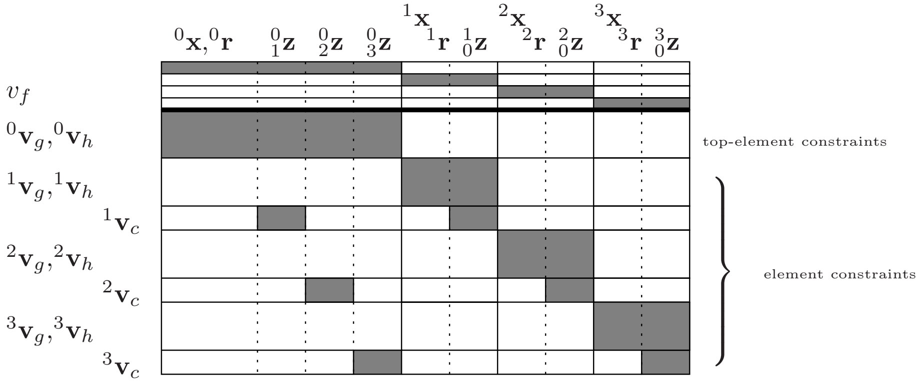


Figure 3.18: *Decomposed problem matrix in case of shared design variables. The design variables* ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{z}$ *are shared design variables between elements and therefore:* ${}^0_i\mathbf{z} = {}^i_0\mathbf{z}$*. Consistency is maintained at the lower levels, hence a top-down decomposition.*

problem in this case has the form:

$$\begin{array}{lllll} \min\limits_{\mathbf{x}} & v_f(\mathbf{x},\mathbf{r}) & & & \\ \text{s.t.} & {}^0\mathbf{v}_g(\mathbf{x},\mathbf{r}) \le \mathbf{0} & ; & {}^i\mathbf{v}_g({}^i\mathbf{x},{}^i\mathbf{r}) \le \mathbf{0} & \\ & {}^0\mathbf{v}_h(\mathbf{x},\mathbf{r}) = \mathbf{0} & ; & {}^i\mathbf{v}_h({}^i\mathbf{x},{}^i\mathbf{r}) = \mathbf{0} & , \\ \text{where} & \mathbf{x} = {}^0\mathbf{x}, {}^i\mathbf{x} & & & \\ & i = 1,2,3 & & & \end{array} \qquad (3.15)$$

and the objective function v_f is considered additively separable.

The optimization problem is decomposed introducing consistency constraints. These constraints force variables that are shared among Level-0 and Level-1 elements to equal value. These consistency constraints are assigned to either the lower level, see Figure 3.18, or to the upper level, see Figure 3.19, in case of a bottom up approach. To distinguish between design variables that belong to individual elements and to design variables that are shared among elements the shared design variables are written ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{z}$. For a top-down equality based decomposition approach the shared design variables read ${}^0_i\mathbf{z}$ and the Level-0 part of Equation 3.15 becomes:

$$\begin{array}{lll} \min\limits_{{}^0\mathbf{x}, {}^0_i\mathbf{z}} & {}^0v_f({}^0_i\mathbf{z}, {}^0\mathbf{x}, {}^0\mathbf{r}) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^0_i\mathbf{z}, {}^0\mathbf{x}, {}^0\mathbf{r}) \le \mathbf{0} & . \\ & {}^0\mathbf{v}_h({}^0_i\mathbf{z}, {}^0\mathbf{x}, {}^0\mathbf{r}) = \mathbf{0} & \\ \text{where} & i = 1,2,3 & \end{array} \qquad (3.16)$$

The Level-1 problem formulation for each element is:

$$\begin{array}{lll} \min\limits_{{}^i\mathbf{x}, {}^i_0\mathbf{z}} & {}^iv_f({}^i_0\mathbf{z}, {}^i\mathbf{x}, {}^i\mathbf{r}) & \\ \text{s.t.} & {}^i\mathbf{v}_g({}^i_0\mathbf{z}, {}^i\mathbf{x}, {}^i\mathbf{r}) \le \mathbf{0} & \\ & {}^i\mathbf{v}_h({}^i_0\mathbf{z}, {}^i\mathbf{x}, {}^i\mathbf{r}) = \mathbf{0} & , \\ & {}^i\mathbf{v}_{{}^i_0c_z}({}^i_0\mathbf{c}_z({}^i_0\mathbf{z})) = {}^0_i\mathbf{z} - {}^i_0\mathbf{z} = \mathbf{0} & \\ \text{where} & i = 1,2,3 & \end{array} \qquad (3.17)$$

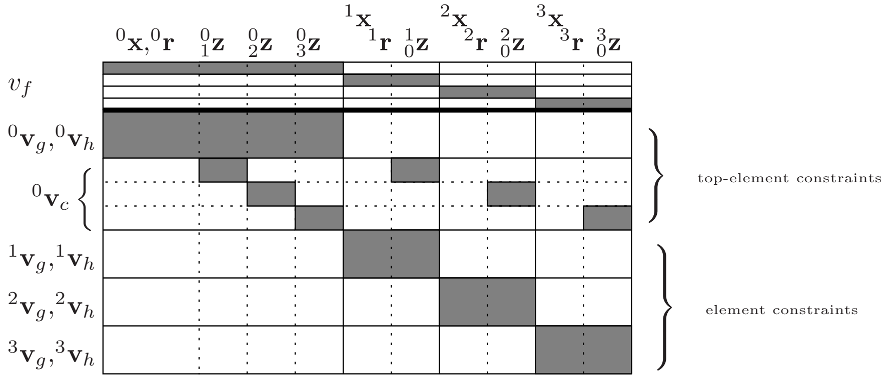


Figure 3.19: *Problem matrix with shared design variables $^{\cdot\cdot}_{\cdot\cdot}\mathbf{z}$. Consistency is maintained at the upper level, hence a bottom-up decomposition approach.*

where the shared design variables associated to Level-1 read $^i_0\mathbf{z}$. The index $_z$ is added to distinguish between consistency expressions related to physical coupling and consistency expressions related to shared design variables.

The bottom-up approach results in a similar problem description, where the consistency between the variable sets is maintained at Level-0. The Level-0 problem formulation of Equation 3.15 is in this case:

$$\begin{array}{lll} \min\limits_{^0\mathbf{x}, ^0_i\mathbf{z}} & ^0v_f\left(^0_i\mathbf{z}, ^0\mathbf{x}, ^0\mathbf{r}\right) & \\ \text{s.t.} & ^0\mathbf{v}_g\left(^0_i\mathbf{z}, ^0\mathbf{x}, ^0\mathbf{r}\right) & \leq \mathbf{0} \\ & ^0\mathbf{v}_h\left(^0_i\mathbf{z}, ^0\mathbf{x}, ^0\mathbf{r}\right) & = \mathbf{0} \\ & ^0\mathbf{v}_{^0_i c_z}\left(^0_i\mathbf{c}_z\left(^0_i\mathbf{z}\right)\right) & = {^0_i\mathbf{z}} - {^i_0\mathbf{z}} = \mathbf{0} \\ \text{where} & i = 1, 2, 3 & \end{array}, \tag{3.18}$$

and the Level-1 formulation resembles:

$$\begin{array}{lll} \min\limits_{^i\mathbf{x}, ^i_0\mathbf{z}} & ^iv_f\left(^i_0\mathbf{z}, ^i\mathbf{x}, ^i\mathbf{r}\right) & \\ \text{s.t.} & ^i\mathbf{v}_g\left(^i_0\mathbf{z}, ^i\mathbf{x}, ^i\mathbf{r}\right) & \leq \mathbf{0} \\ & ^i\mathbf{v}_h\left(^i_0\mathbf{z}, ^i\mathbf{x}, ^i\mathbf{r}\right) & = \mathbf{0} \\ \text{where} & i = 1, 2, 3 & \end{array}. \tag{3.19}$$

The problem matrix of the bottom-up approach is shown in Figure 3.19.

So far the discussion on coupling between elements is limited to coupling between a Level-0 element and Level-1 elements, thus coupling between levels. Whenever a problem involves coupling of two elements on the same level as shown in Figure 3.20 or even if there is a fully populated problem matrix, such as shown in Figure 3.8 the problem can still be decomposed. Such a decomposition is called a non-hierarchic decomposition(Sobieszczanski-Sobieski, 1988) or system-oriented decomposition (Kirsch, 1993). The previous discussed decomposition methods all fall within the category of hierarchic decomposition or process-oriented decomposition.

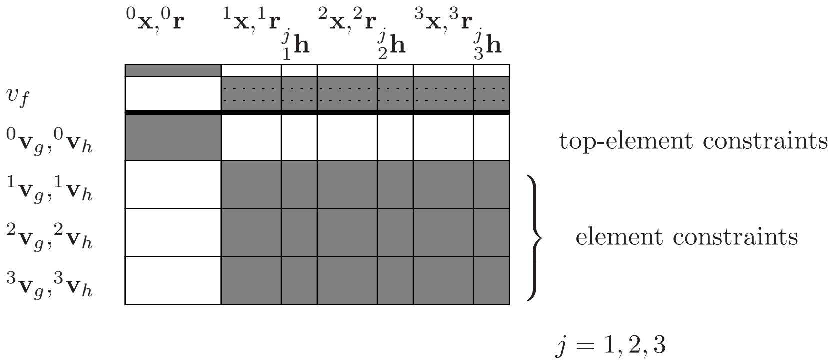


Figure 3.20: *Coupling between elements on Level-1 via coupling variables $^j_i\mathbf{h}$. This will result in a non-hierarchic decomposition, hence decoupling of the elements is necessary in two-directions.*

The essential difference between hierarchic decomposition and non-hierarchic decomposition is that the first decomposes one direction of the coupling circle and the latter decomposes both directions of the coupling circle. One element prescribes the coupling variables for the neighboring element. In non-hierarchic decomposition both directions of the coupling circle are decomposed and both elements are required to satisfy the consistency constraints.

Elements that are coupled on the same level (Level-1) are shown in Figure 3.20. The initial optimization problem before decomposition reads:

$$
\begin{array}{llrcl}
\min\limits_{\mathbf{x}} & v_f = {}^0v_f({}^0\mathbf{x}, {}^0\mathbf{r}\left({}^0\mathbf{x}\right)) + \sum\limits_{i=1}^{3} {}^iv_f\left({}^i\mathbf{x}, {}^i\mathbf{r}\left({}^i\mathbf{x}, {}^j_i\mathbf{h}\right)\right) & & & \\
\text{s.t.} & {}^0\mathbf{v}_g({}^0\mathbf{x}, {}^0\mathbf{r}\left({}^0\mathbf{x}\right)) \leq \mathbf{0}; \quad {}^0\mathbf{v}_h({}^0\mathbf{x}, {}^0\mathbf{r}\left({}^0\mathbf{x}\right)) & = & \mathbf{0} & \\
 & {}^i\mathbf{v}_g({}^i\mathbf{x}, {}^i\mathbf{r}\left({}^i\mathbf{x}, {}^j_i\mathbf{h}\right)) \leq \mathbf{0}; \quad {}^i\mathbf{v}_h({}^i\mathbf{x}, {}^i\mathbf{r}\left({}^i\mathbf{x}, {}^j_i\mathbf{h}\right)) & = & \mathbf{0} & . \\
\text{where} & {}^0\mathbf{x}, {}^0\mathbf{r} \quad \text{top-element level variables} & & & \\
 & {}^{\cdot}\mathbf{x}, {}^{\cdot}\mathbf{r} \quad \text{element level variables} & & & \\
 & {}^{\cdot}_{\cdot}\mathbf{h} \quad \text{coupling variables} & & & \\
 & i, j = 1, 2, 3 \quad i \neq j & & &
\end{array}
\tag{3.20}
$$

The elements on Level-1 are coupled to each other and the Level-0 element is uncoupled from the Level-1 elements.

The elements on Level-1 are decomposed via consistency constraints between the elements. To take into account changes in the neighboring element the coupling variable vector $^i_j\mathbf{h}$ is introduced as a design variable vector for the Level-1 elements. For brevity of notation the dependence of the response vector $\mathbf{r}$ on the design variables is omitted. Interaction via mapping of physical responses $^i_j\mathcal{H}\left(^i\mathbf{r}\right) = {}^i_j\mathbf{h}$ is approximated by means of computing the Global Sensitivity Equations(GSE), see Section

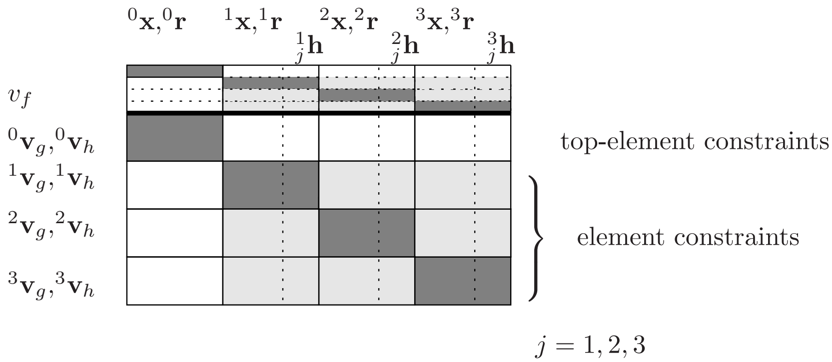


Figure 3.21: *Decomposed problem after introducing the physical responses as design variables into the Level-1 optimization problems. The changes in the physical coupling are accounted for by means of computing the Global Sensitivity Equations. Therefore, the light shaded blocks indicate the approximated coupling.*

3.2. Hence, on Level-1 the individual problems to optimize become:

$$\begin{array}{lll} \min\limits_{{}^i_j\mathbf{h},{}^i\mathbf{x}} & {}^iv_f({}^i_j\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}) & \\ \text{s.t.} & {}^i\mathbf{v}_g({}^i_j\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}) \leq \mathbf{0} & . \\ & {}^i\mathbf{v}_h({}^i_j\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}) = \mathbf{0} & \\ \text{where} & i,j = 1,2,3 \quad i \neq j & \end{array} \tag{3.21}$$

The Level-0 optimization is independent of the Level-1 optimizations and reads:

$$\begin{array}{lll} \min\limits_{{}^i\mathbf{x}} & {}^iv_f({}^i\mathbf{x},{}^i\mathbf{r}) & \\ \text{s.t.} & {}^i\mathbf{v}_g({}^i\mathbf{x},{}^i\mathbf{r}) \leq \mathbf{0} & . \\ & {}^i\mathbf{v}_h({}^i\mathbf{x},{}^i\mathbf{r}) = \mathbf{0} & \\ \text{where} & i = 1,2,3 & \end{array} \tag{3.22}$$

The problem matrix for the non-hierarchic decomposed problem is shown in Figure 3.21. The influence of the Global Sensitivity Equations (GSE) is shown via lighter shaded blocks in the problem matrix.

Three characteristic cases of coupling have been described by means of comparing problem matrices. These cases were physical coupling between levels, coupling by means of shared design variables and coupling on the same level. All coupled problems were decomposed using equality consistency constraint formulation. In the next section the consistency constraints are relaxed and the changes involving this relaxation on the three coupling cases are discussed.

3.3.2 Relaxation based decoupling

Similar to the equality consistency constraint formulation, decoupling through relaxation of the consistency constraints is done via hierarchic or non-hierarchic decom-

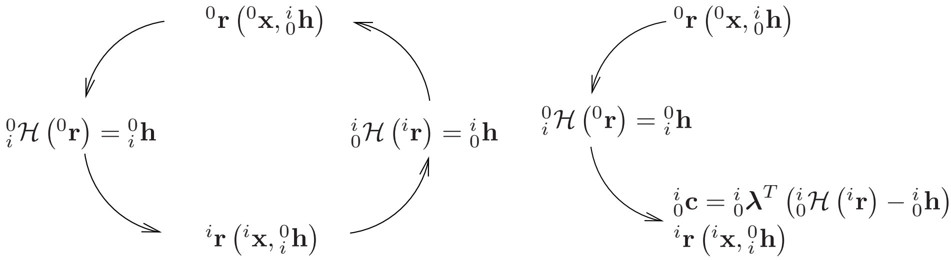


Figure 3.22: *(a) Physical response interaction between two elements. The responses ${}^0\mathbf{r}$ computed at the parent element are transformed by the operator ${}^0_i\mathcal{H}$ into information that influences the responses of the child element ${}^i\mathbf{r}$. Likewise, the responses from the child element are transformed by the operator ${}^i_0\mathcal{H}$ into information that influences the responses at the parent element. (b) Top-down decomposition by means of relaxation of the consistency constraint via, e.g., Lagrange multipliers and assigning the Lagrangian multipliers to the Level-1 element.*

position. The hierarchic decomposition approach is discussed first and is sub-divided into a top-down and a bottom-up approach.

First the coupling circle, Figure 3.22(a), is relaxed via a top-down decomposition, see Figure 3.22(b). Hence, the relaxed consistency constraints using, e.g., Lagrange multipliers read:

$$ {}^{\cdot\cdot}\mathbf{v}_{{}^i_0 c}({}^i_0\mathbf{c}) \quad = \quad {}^i_0\boldsymbol{\lambda}^T\,{}^i_0\mathbf{c} \quad = \quad {}^i_0\boldsymbol{\lambda}^T\left({}^i_0\mathcal{H}({}^i\mathbf{r}) - {}^i_0\mathbf{h}\right) \ . \tag{3.23}$$

Because the constraints are relaxed, a procedure to reduce the inconsistency between the via Level-0 prescribed response $({}^i_0\mathbf{h})$ and the actual response computed at Level-1 $({}^i_0\mathcal{H}({}^i\mathbf{r}))$ is necessary. For the present problem, Lagrangian relaxation, this is accomplished via the so-called dual problem [1]. While keeping the design variables fixed, the minimum of the Lagrangian function of the relaxed optimization problem is maximized. The dual problem for a two-level hierarchy consisting of a single element at Level-0, three individual elements at Level-1 and an additively separable objective function reads:

$$\begin{array}{lll}
\max\limits_{{}^i_0\boldsymbol{\lambda}^T} \ \min\limits_{{}^i_0\mathbf{h}, {}^0\mathbf{x}, {}^i\mathbf{x}} & {}^0v_f({}^i_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) + \sum\limits_{i=1}^{3} {}^iv_f({}^0_i\mathbf{h}, {}^i\mathbf{x}, {}^i\mathbf{r}) + \sum\limits_{i=1}^{3} {}^i\mathbf{v}_{{}^i_0 c}({}^i_0\mathbf{c}({}^i\mathbf{r}, {}^i_0\mathbf{h})) & \\
\text{s.t.} & {}^0\mathbf{v}_g({}^i_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) \quad \leq \mathbf{0} & \\
 & {}^0\mathbf{v}_h({}^i_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) \quad = \mathbf{0} & \\
 & {}^i\mathbf{v}_g({}^i\mathbf{x}, {}^i\mathbf{r}) \quad \leq \mathbf{0} & . \\
 & {}^i\mathbf{v}_h({}^i\mathbf{x}, {}^i\mathbf{r}) \quad = \mathbf{0} & \\
\text{where} & {}^i\mathbf{v}_{{}^i_0 c}({}^i_0\mathbf{c}({}^i\mathbf{r}, {}^i_0\mathbf{h})) = {}^i_0\boldsymbol{\lambda}^T\left({}^i_0\mathcal{H}\left({}^i\mathbf{r}\right) - {}^i_0\mathbf{h}\right) & \\
 & i = 1, 2, 3 &
\end{array} \tag{3.24}$$

The problem of finding relaxation parameters (here optimal Lagrange multipliers) is a coordination problem. This coordination problem is discussed in chapters 4 and

[1] The dual problem is defined in Proposition 5.1.1 through 5.1.6 in the book of Bertsekas (1995)

5. The focus of this chapter is individual element optimization problems. Therefore, relaxation parameters are considered fixed parameters in the examples discussed and are adjusted via the coordinator problem (Chapter 4 and 5).

The inconsistency is added to the objective of the Level-1 elements. An increase of inconsistency has a negative effect on the performance (the objective function) of the individual element.

The Level-0 problem, after decomposition of the Level-1 elements reads:

$$\begin{array}{lll} \min\limits_{{}^i_0\mathbf{h},{}^0\mathbf{x}} & {}^0v_f({}^i_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^i_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^i_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = \mathbf{0} \\ \text{where} & i = 1,2,3 & \end{array} . \tag{3.25}$$

Again, index ^{i} indicates the i^{th} lower level element. The individual problem formulation for the Level-1 elements of Equation 3.7 becomes:

$$\begin{array}{lll} \min\limits_{{}^i\mathbf{x}} & {}^iv_f({}^i\mathbf{x},{}^i\mathbf{r}) + {}^i\mathbf{v}_{{}^i_0c}({}^i_0c({}^i\mathbf{r})) & \\ \text{s.t.} & {}^i\mathbf{v}_g({}^i\mathbf{x},{}^i\mathbf{r}) & \leq \mathbf{0} \\ & {}^i\mathbf{v}_h({}^i\mathbf{x},{}^i\mathbf{r}) & = \mathbf{0} \\ \text{where} & {}^i\mathbf{v}_{{}^i_0c}({}^i_0\mathbf{c}({}^i\mathbf{r})) = {}^i_0\boldsymbol{\lambda}^T\left({}^i_0\boldsymbol{\mathcal{H}}({}^i\mathbf{r}) - {}^i_0\mathbf{h}\right) & \\ & i = 1,2,3 & \end{array} . \tag{3.26}$$

The Lagrange multipliers ${}^i_0\boldsymbol{\lambda}$ present in the Level-1 optimization problems form part of the optimal solution that is determined via an external coordination routine (Chapter 5).

Similar to the top-down approach, the equations of the bottom-up approach can be obtained via relaxation of the consistency constraints, e.g., via Lagrange multipliers. The Level-0 system becomes:

$$\begin{array}{lll} \min\limits_{{}^0\mathbf{x}} & {}^0v_f({}^0\mathbf{x},{}^0\mathbf{r}) + \sum\limits_{i=1}^{3} {}^0\mathbf{v}_{{}^0_ic}({}^0_ic({}^0\mathbf{r})) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^0\mathbf{x},{}^0\mathbf{r}) & \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^0\mathbf{x},{}^0\mathbf{r}) & = \mathbf{0} \\ \text{where} & {}^0\mathbf{v}_{{}^0_ic}({}^0_ic({}^0\mathbf{r})) = {}^0_i\boldsymbol{\lambda}^T\left({}^0_i\boldsymbol{\mathcal{H}}({}^0\mathbf{r}) - {}^0_i\mathbf{h}\right) & \\ & i = 1,2,3 & \end{array} , \tag{3.27}$$

and the problem formulation for the Level-1 elements becomes:

$$\begin{array}{lll} \min\limits_{{}^i\mathbf{x},{}^0_i\mathbf{h}} & {}^iv_f({}^0_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}) & \\ \text{s.t.} & {}^i\mathbf{v}_g({}^0_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}) & \leq \mathbf{0} \\ & {}^i\mathbf{v}_h({}^0_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}) & = \mathbf{0} \\ \text{where} & i = 1,2,3 & \end{array} . \tag{3.28}$$

Similar to the top-down decomposition approach the Lagrange multipliers form part of the solution that is determined via a coordination approach (Chapter 5).

When coupling exists between elements due to shared design variables, the relaxed formulation of the consistency constraints is similar to that of the coupled physical responses. The total optimization problem before decomposition is:

$$\begin{array}{lllll}
\min\limits_{\mathbf{x}} & v_f(\mathbf{x},\mathbf{r}) & & & \\
\text{s.t.} & {}^0\mathbf{v}_g(\mathbf{x},\mathbf{r}) \leq \mathbf{0} & ; & {}^i\mathbf{v}_g({}^i\mathbf{x},{}^i\mathbf{r}) \leq \mathbf{0} & \\
 & {}^0\mathbf{v}_h(\mathbf{x},\mathbf{r}) = \mathbf{0} & ; & {}^i\mathbf{v}_h({}^i\mathbf{x},{}^i\mathbf{r}) = \mathbf{0} & . \\
\text{where} & \mathbf{x} = {}^0\mathbf{x}, {}^i\mathbf{x} & & & \\
 & i = 1,2,3 & & &
\end{array} \tag{3.29}$$

Design variables that are shared between multiple elements are expressed as ${}^{:}_{:}\mathbf{z}$ and consistency constraints between these variables are introduced. Relaxing the consistency constraints between elements the Level-0 optimization problem becomes:

$$\begin{array}{ll}
\min\limits_{{}^0_i\mathbf{z},{}^0\mathbf{x}} & {}^0v_f({}^0_i\mathbf{z}, {}^0\mathbf{x}, {}^0\mathbf{r}) \\
\text{s.t.} & {}^0\mathbf{v}_g({}^0_i\mathbf{z}, {}^0\mathbf{x}, {}^0\mathbf{r}) \leq \mathbf{0} \\
 & {}^0\mathbf{v}_h({}^0_i\mathbf{z}, {}^0\mathbf{x}, {}^0\mathbf{r}) = \mathbf{0} \\
\text{where} & i = 1,2,3
\end{array} , \tag{3.30}$$

and the relaxed Level-1 optimization problems become:

$$\begin{array}{ll}
\min\limits_{{}^i_0\mathbf{z},{}^i\mathbf{x}} & {}^iv_f({}^i_0\mathbf{z}, {}^i\mathbf{x}, {}^i\mathbf{r}) + {}^i\mathbf{v}_{{}^i_0c_\mathbf{z}}\left({}^i_0\mathbf{c}_\mathbf{z}\left({}^i_0\mathbf{z}\right)\right) \\
\text{s.t.} & {}^i\mathbf{v}_g({}^i_0\mathbf{z}, {}^i\mathbf{x}, {}^i\mathbf{r}) \leq \mathbf{0} \\
 & {}^i\mathbf{v}_h({}^i_0\mathbf{z}, {}^i\mathbf{x}, {}^i\mathbf{r}) = \mathbf{0} \\
\text{where} & {}^i\mathbf{v}_{{}^i_0c_\mathbf{z}}({}^i_0\mathbf{c}_\mathbf{z}({}^i_0\mathbf{z})) = {}^i_0\boldsymbol{\lambda}^T\left({}^0_i\mathbf{z} - {}^i_0\mathbf{z}\right) \\
 & i = 1,2,3
\end{array} . \tag{3.31}$$

In the case of a fully populated problem matrix, such as illustrated in Figure 3.8, or in case coupling is only present between elements on the same level, as illustrated in Figure 3.20, a non-hierarchic decomposition approach is applied. For simplicity, a multi-level problem is considered where the Level-0 element is uncoupled from the Level-1 elements and coupling is only present between the elements on Level-1.

The resulting problem formulation for the individual elements of Level-1 have the form:

$$\begin{array}{ll}
\min\limits_{{}^i\mathbf{x},{}^j_i\mathbf{h}} & {}^iv_f({}^j_i\mathbf{h}, {}^i\mathbf{x}, {}^i\mathbf{r}) + {}^i\mathbf{v}_{{}^j_ic}({}^j_i\mathbf{c}({}^j_i\mathbf{h})) + {}^i\mathbf{v}_{{}^i_jc}({}^i_j\mathbf{c}({}^i_j\mathcal{H}({}^i\mathbf{r}))) \\
\text{s.t.} & {}^i\mathbf{v}_g({}^j_i\mathbf{h}, {}^i\mathbf{x}, {}^i\mathbf{r}) \leq \mathbf{0} \\
 & {}^i\mathbf{v}_h({}^j_i\mathbf{h}, {}^i\mathbf{x}, {}^i\mathbf{r}) = \mathbf{0} \\
\text{where} & {}^i\mathbf{v}_{{}^j_ic}({}^j_i\mathbf{c}({}^j_i\mathbf{h})) = {}^j_i\boldsymbol{\lambda}_i^T\left({}^j_i\boldsymbol{\mathcal{H}}({}^j\mathbf{r}) - {}^j_i\mathbf{h}\right) \\
 & {}^i\mathbf{v}_{{}^i_jc}({}^i_j\mathbf{c}({}^i_j\boldsymbol{\mathcal{H}}({}^i\mathbf{r}))) = {}^i_j\boldsymbol{\lambda}_i^T\left({}^i_j\boldsymbol{\mathcal{H}}({}^i\mathbf{r}) - {}^i_j\mathbf{h}\right)
\end{array} . \tag{3.32}$$

The Level-0 is not coupled to the Level-1 elements and writes:

$$\begin{array}{ll}
\min\limits_{{}^0\mathbf{x}} & {}^0v_f({}^0\mathbf{x}, {}^0\mathbf{r}) \\
\text{s.t.} & {}^0\mathbf{v}_g({}^0\mathbf{x}, {}^0\mathbf{r}) \leq \mathbf{0} \\
 & {}^0\mathbf{v}_h({}^0\mathbf{x}, {}^0\mathbf{r}) = \mathbf{0}
\end{array} . \tag{3.33}$$

The Lagrange multipliers necessary for the Level-1 elements are determined via coordination approaches which are discussed in Chapter 5.

Three characteristic cases of coupling between elements were discussed in accordance with the discussion on equality-based decomposition. The fourth case, where the objective function only depends on the top-level element does not change the decomposition approach in case relaxation based decomposition is applied. A summary of these characteristic cases is shown in Figure 3.23 via typical patterns of the problem matrix. The first pattern (a) illustrates a problem matrix where the objective function only depends on the top-level element. This resembles a special case in equality-based decomposition approaches where use is made of constraint minimization in the lower level elements. Pattern (b) illustrates the case where a small number of coupling variables couples the objective and constraint functions of all the levels. Pattern (c) illustrates a problem where the design variables and/or physical responses are shared over multiple levels. And in pattern (d) the objective function and the constraints depend on all the design variables and physical responses of all the elements. Hence, no hierarchy can be distinguished and use is made of Global Sensitivity Equations in case of equality based decomposition and of relaxation in both directions of the coupling circle in case of relaxation based decomposition.

In the next section the results of both the equality-based decomposition and the relaxation-based decomposition are extended to a general multi-level optimization problem.

3.4 From two-level to multi-level methods

In the previous section a two-level decomposition was considered. In this section the decomposition principles applied to a two-level hierarchy are extended to a multiple level hierarchy.

A multi-level formulation is characterized by a top-level, n intermediate levels and a bottom level. Therefore, a 3-level description is sufficient to cover the essence of a multi-level description. A three-level hierarchy is considered, Level-0 consisting of a single element and the two lower levels consisting of two individual elements. The

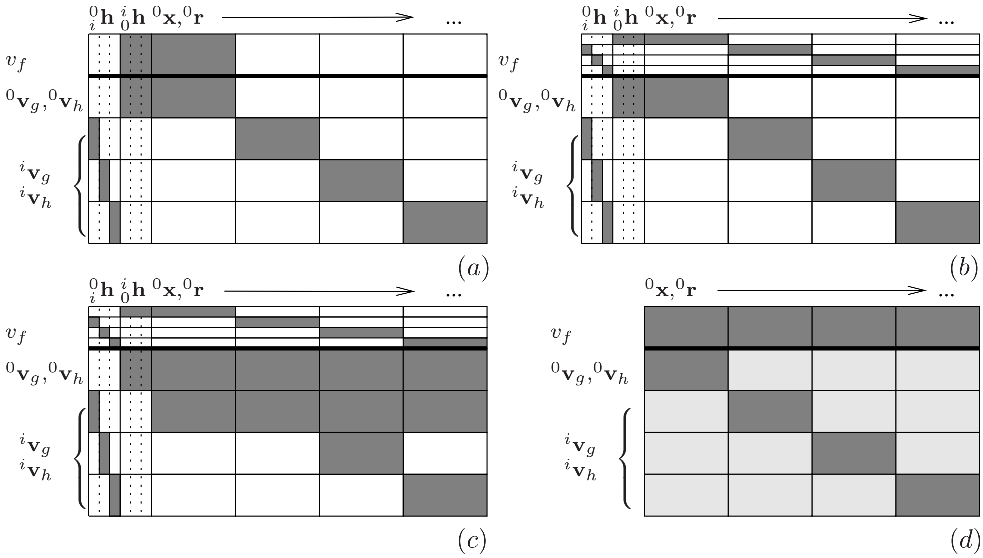


Figure 3.23: *Four typical patterns can be distinguished in the problem matrix. (a) A hierarchy where the objective function depends on the top-level design variables* $^0\mathbf{x}$ *and physical responses* $^0\mathbf{r}$ *and on the coupling variables* $^i_0\mathbf{h}$. *The lower level elements do not have an individual objective function. (b) A hierarchy where the objective function and constraint functions share a small number of coupling variables. Level-0 as well as elements present at Level-1 contribute to the minimization of the additively separable objective function* v_f. *(c) A hierarchy where the design variables and physical responses of different elements contribute to the constraint functions of neighboring elements. Some of the elements in lower levels depend on physical responses and/or design variables from neighboring elements. (d) No clear hierarchy can be distinguished. The objective function as well as the constraint functions depend on the physical responses and design variables from all the elements. Either Global Sensitivity Equations (GSE) are required to separate the elements or relaxation of the consistency constraints between the elements is necessary.*

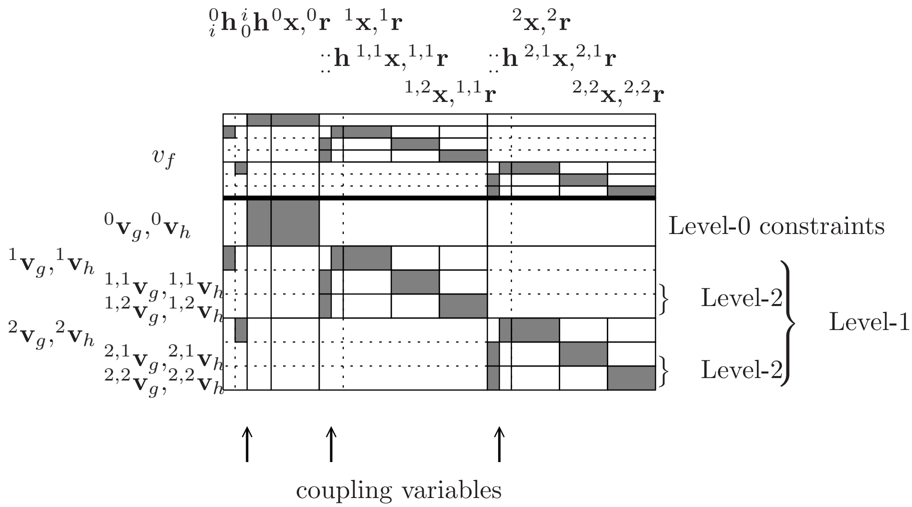


Figure 3.24: *Problem matrix for a three level problem with coupling indicated by coupling variables between the levels. The hierarchy consists of a single element at Level-0, two elements at Level-1 and each element present at Level-1 is coupled to two individual elements on Level-2.*

coupled optimization problem for a 3-level optimization problem is

$$
\begin{array}{lrcl}
\min\limits_{\mathbf{x}} & \multicolumn{3}{l}{{}^{0}v_f({}^{0}\mathbf{x},{}^{0}\mathbf{r}({}^{0}\mathbf{x},{}^{i}_{0}\mathbf{h}))+} \\
& \multicolumn{3}{l}{\sum\limits_{i=1}^{2}\left({}^{i}v_f({}^{i}\mathbf{x},{}^{i}\mathbf{r}({}^{i}\mathbf{x},{}^{0}_{i}\mathbf{h},{}^{i,j}_{i}\mathbf{h}))+\sum\limits_{j=1}^{2}\left({}^{i,j}v_f({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}({}^{i,j}\mathbf{x},{}^{i}_{i,j}\mathbf{h}))\right)\right)} \\
\text{s.t.} & {}^{0}\mathbf{v}_g({}^{0}\mathbf{x},{}^{0}\mathbf{r}({}^{0}\mathbf{x},{}^{i}_{0}\mathbf{h})) & \leq & \mathbf{0} \\
& {}^{i}\mathbf{v}_g({}^{i}\mathbf{x},{}^{i}\mathbf{r}({}^{i}\mathbf{x},{}^{0}_{i}\mathbf{h},{}^{i,j}_{i}\mathbf{h})) & \leq & \mathbf{0} \\
& {}^{i,j}\mathbf{v}_g({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}({}^{i,j}\mathbf{x},{}^{i}_{i,j}\mathbf{h})) & \leq & \mathbf{0} \\
& {}^{0}\mathbf{v}_h({}^{0}\mathbf{x},{}^{0}\mathbf{r}({}^{0}\mathbf{x},{}^{i}_{0}\mathbf{h})) & = & \mathbf{0} \\
& {}^{i}\mathbf{v}_h({}^{i}\mathbf{x},{}^{i}\mathbf{r}({}^{i}\mathbf{x},{}^{0}_{i}\mathbf{h},{}^{i,j}_{i}\mathbf{h})) & = & \mathbf{0} \\
& {}^{i,j}\mathbf{v}_h({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}({}^{i,j}\mathbf{x},{}^{i}_{i,j}\mathbf{h})) & = & \mathbf{0} \\
\text{where} & {}^{0}\mathbf{x},{}^{0}\mathbf{r} & & \text{Level-0 variables} \\
& {}^{i}\mathbf{x},{}^{i}\mathbf{r} & & \text{Level-1 variables} \\
& {}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r} & & \text{Level-2 variables} \\
& {}^{:}_{:}\mathbf{h} & & \text{coupling variables} \\
& \multicolumn{3}{l}{i,j=1,2}
\end{array}
\quad . \qquad (3.34)
$$

For simplicity, only physical coupling between levels is present. Hence, no coupling between elements on the same level is considered. Furthermore, shared design variables are not considered. Shared design variables are treated similar as the physical coupling between elements. The problem matrix showing the coupling between the three levels and the individual elements is shown in Figure 3.24.

In order to decouple the optimization problem, the coupling variables **h** are assigned to the higher level elements (meaning Level-0 and Level-1) in case of a top-down decomposition approach. In case of a bottom-up decomposition approach they are assigned to the lower element levels (meaning Level-1 and Level-2). A top-down decomposition is considered here, thus the Level-0 optimization problem includes the physical responses of the intermediate level as design variables. The Level-0 optimization problem becomes:

$$\begin{array}{lll} \min\limits_{{}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x}} & {}^{0}v_f({}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) & \\ \text{s.t.} & {}^{0}\mathbf{v}_g({}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) & \leq \mathbf{0} \\ & {}^{0}\mathbf{v}_h({}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) & = \mathbf{0} \\ \text{where} & i = 1,2 & \end{array} . \tag{3.35}$$

This formulation is similar to that of the two-level problem discussed in the previous section.

The biggest changes in problem formulation are at the intermediate level, which is connected to both the Level-0 and Level-2 elements. The physical responses which connect the intermediate level to the lowest level are added as design variables ${}^{j}_{i}\mathbf{h}$. Consistency with the Level-0 level is maintained via additional consistency constraints. The intermediate problem writes:

$$\begin{array}{lll} \min\limits_{{}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x}} & {}^{i}v_f({}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}) & \\ \text{s.t.} & {}^{i}\mathbf{v}_g({}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}) & \leq \mathbf{0} \\ & {}^{i}\mathbf{v}_h({}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}) & = \mathbf{0} \\ \text{where} & {}^{i}\mathbf{v}_{{}^{i}_{0}c}({}^{i}_{0}\mathbf{c}({}^{i}\mathbf{r})) = {}^{i}_{0}\boldsymbol{\mathcal{H}}({}^{i}\mathbf{r}) - {}^{i}_{0}\mathbf{h} = \mathbf{0} & \\ & i,j = 1,2 & \end{array} . \tag{3.36}$$

Finally, the Level-2 problem formulation writes:

$$\begin{array}{lll} \min\limits_{{}^{i,j}\mathbf{x}} & {}^{i,j}v_f({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}) & \\ \text{s.t.} & {}^{i,j}\mathbf{v}_g({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}) & \leq \mathbf{0} \\ & {}^{i,j}\mathbf{v}_h({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}) & = \mathbf{0} \\ \text{where} & {}^{i,j}\mathbf{v}_{{}^{i,j}_{i}c}({}^{i,j}_{i}\mathbf{c}({}^{i,j}\mathbf{r})) = {}^{i,j}_{i}\boldsymbol{\mathcal{H}}({}^{i,j}\mathbf{r}) - {}^{i,j}_{i}\mathbf{h} = \mathbf{0} & \\ & i,j = 1,2 & \end{array} . \tag{3.37}$$

The problem formulation is similar to that of the Level-1 problem of the previous section. The relation between the position in the hierarchy and the index notation was introduced in Section 2.1.

A similar formulation holds for the case of a coupled design variable vector. The case of elements on the same level was already discussed in the previous section for three connected elements and does not differ for a multi-level hierarchy. In order to complete the discussion, the changes due to relaxation of the consistency constraints are now discussed.

The three level optimization problem (3.34) is considered and a top-down hierarchic decomposition is assumed. The consistency constraints between levels are relaxed, for the present example, via Lagrangian relaxation. Therefore, an additional objective is added to the individual optimization problems of Level-1 and Level-2. This additional objective has a negative contribution to the objective function if the consistency between elements is violated. The Level-0 optimization problem becomes:

$$\begin{array}{lll} \min\limits_{{}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x},} & {}^{0}v_f({}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) & \\ \text{s.t.} & {}^{0}\mathbf{v}_g({}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) & \leq \mathbf{0} \\ & {}^{0}\mathbf{v}_h({}^{i}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) & = \mathbf{0} \\ \text{where} & i = 1,2 & \end{array} . \tag{3.38}$$

The intermediate level is new compared to the two-level hierarchy. It includes a contribution to the objective function that takes into account a violation of the consistency between Level-0 and Level-1 and between Level-1 and Level-2. The intermediate level, i.e. the Level-1 optimization problem becomes:

$$\begin{array}{lll} \min\limits_{{}^{i,j}_{i}\mathbf{h},{}^{i}\mathbf{x}} & {}^{i}v_f({}^{i,j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}) + {}^{i}\mathbf{v}_{{}^{i}_{0}c}({}^{i}_{0}\mathbf{c}({}^{i}\mathbf{r})) & \\ \text{s.t.} & {}^{i}\mathbf{v}_g({}^{i,j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}) & \leq \mathbf{0} \\ & {}^{i}\mathbf{v}_h({}^{i,j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}) & = \mathbf{0} \\ \text{where} & {}^{i}\mathbf{v}_{{}^{i}_{0}c}({}^{i}_{0}\mathbf{c}({}^{i}\mathbf{r})) = {}^{i}_{0}\boldsymbol{\lambda}^T \left({}^{i}_{0}\boldsymbol{\mathcal{H}}({}^{i}\mathbf{r}) - {}^{i}_{0}\mathbf{h}\right) & \\ & i,j = 1,2 & \end{array} . \tag{3.39}$$

Two additional objectives are present in the objective function. These involve the relaxed consistency between Level-0 and Level-1 and the relaxed consistency between Level-2 and Level-1. The optimal Lagrange multipliers ${}^{\cdot\cdot}_{\cdot\cdot}\boldsymbol{\lambda}_{\cdot\cdot}$ are determined as part of the coordination problem that is discussed in chapters 4 and 5.

The lowest level optimization problems are similar to those of the two-level problem formulation. For the current three-level hierarchy the lowest-level optimization problems become, i.e. Level-2:

$$\begin{array}{lll} \min\limits_{{}^{i,j}\mathbf{x}} & {}^{i,j}v_f({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}) + {}^{i,j}\mathbf{v}_{{}^{i,j}_{i}c}({}^{i,j}_{i}\mathbf{c}({}^{i,j}\mathbf{r})) & \\ \text{s.t.} & {}^{i,j}\mathbf{v}_g({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}) & \leq \mathbf{0} \\ & {}^{i,j}\mathbf{v}_h({}^{i,j}\mathbf{x},{}^{i,j}\mathbf{r}) & = \mathbf{0} \\ \text{where} & {}^{i,j}\mathbf{v}_{{}^{i,j}_{i}c}({}^{i,j}_{i}\mathbf{c}({}^{i,j}\mathbf{r})) = {}^{i,j}_{i}\boldsymbol{\lambda}^T \left({}^{i,j}_{i}\boldsymbol{\mathcal{H}}({}^{i,j}\mathbf{r}) - {}^{i,j}_{i}\mathbf{h}\right) & \\ & i,j = 1,2 & \end{array} . \tag{3.40}$$

The inconsistency between the physical response of the Level-2 elements and the expected response of these elements at Level-1 is accounted for by an additional term in the objective function.

The problem description of a shared design variable vector is similar to the decoupling of physical responses and the implications on the individual problem descriptions was discussed in the previous section. Therefore, the problem description of problems involving shared design variable vectors will not be discussed here. In case elements on the same level are coupled, the equations follow from the discussion in the previous section as well. The multi-level setting does not change these formulations and the reader is directed to the two-level problem description.

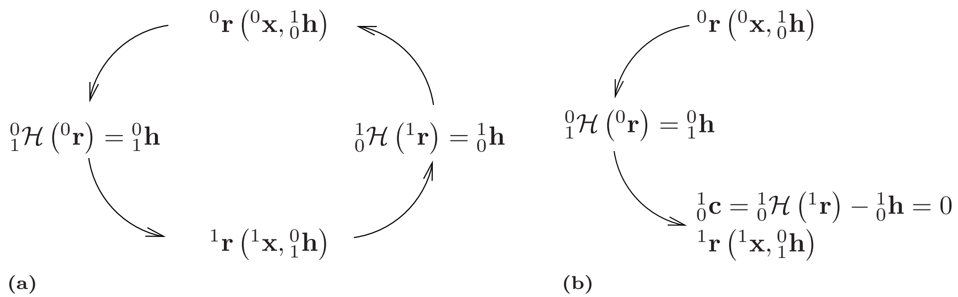


Figure 3.25: *(a) Physical response interaction between two elements. The responses* $^0\mathbf{r}$ *computed at the parent element are transformed by the operator* ${}^0_1\mathcal{H}$ *into information that influences the responses of the child element* $^1\mathbf{r}$*. Likewise, the responses from the child element are transformed by the operator* ${}^1_0\mathcal{H}$ *into information that influences the responses at the parent element. (b) The Level-0 element prescribes the necessary responses for the Level-1 element by means of equality consistency constraints. This is called top-down decomposition.*

3.5 Optimality conditions for coupled problems

In this section the optimality criteria for optimization problems with embedded coupling are derived. The optimization problem formulation that describes the entire optimization problem before identifying a hierarchy is written as:

$$\min_{\mathbf{x}} \quad v_f \quad = \quad v_f(\mathbf{x}, \mathbf{r}(\mathbf{x})) \quad . \tag{3.41}$$

An unconstrained optimization problem is considered for simplicity. However, the derivation is similarly for constrained problems. Equation 3.41 may reflect an optimization formulation for a complex problem. A hierarchy of two elements that are coupled is considered for the present unconstrained optimization problem.

The physical response $\mathbf{r}$ is separated into a Level-0 component $^0\mathbf{r}$ and a Level-1 component $^1\mathbf{r}$ that are coupled. Changes in $^0\mathbf{r}$ effect $^1\mathbf{r}$ and *vice versa*. Furthermore, the design variable vector $\mathbf{x}$ is considered to consist of Level-0 design variables $^0\mathbf{x}$ and Level-1 design variables $^1\mathbf{x}$. The problem after identifying a two level hierarchy is written as:

$$\min_{{}^0\mathbf{x}, {}^1\mathbf{x}} \quad v_f = v_f({}^0\mathbf{x}, {}^1\mathbf{x}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h}), {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) \quad . \tag{3.42}$$

A choice is made on decoupling the coupling circle, see Figure 3.25(a). Hence, in case of top-down hierarchic decomposition (see Figure 3.25(b)) of the physical responses the all-in-one optimization problem with a consistency constraint on the responses that are mapped from the element at Level-1 onto the element at Level-0 is written as:

$$\begin{array}{ll} \min\limits_{{}^0\mathbf{x}, {}^1\mathbf{x}, {}^1_0\mathbf{h}} & v_f = v_f\left({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^1\mathbf{x}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h}), {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})\right) \\ \text{s.t.} & {}^1_0\mathbf{c}({}^1_0\mathbf{h}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} = 0 \end{array} \quad . \tag{3.43}$$

The Lagrangian is defined as:

$$\begin{aligned}\mathcal{L}({}^0\mathbf{x}, {}^1\mathbf{x}, {}^1_0\mathbf{h}, {}^1_0\boldsymbol{\lambda}) \;\; = \;\; & v_f\left({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^1\mathbf{x}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h}), {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})\right) + \\ & {}^1_0\boldsymbol{\lambda}^T\, {}^1_0\mathbf{c}({}^1_0\mathbf{h}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})).\end{aligned} \tag{3.44}$$

The elements of vector ${}^1_0\boldsymbol{\lambda}$ are unknown Lagrange multipliers. Necessary conditions for a stationary point are:

$$\begin{array}{lcl}\nabla_{{}^0\mathbf{x}}\mathcal{L} = \nabla_{{}^0\mathbf{x}} v_f + {}^1_0\boldsymbol{\lambda}^T \nabla_{{}^0\mathbf{x}}{}^1_0\mathbf{c} & = & 0 \\ \nabla_{{}^1\mathbf{x}}\mathcal{L} = \nabla_{{}^1\mathbf{x}} v_f + {}^1_0\boldsymbol{\lambda}^T \nabla_{{}^1\mathbf{x}}{}^1_0\mathbf{c} & = & 0 \\ \nabla_{{}^1_0\mathbf{h}}\mathcal{L} = \nabla_{{}^1_0\mathbf{h}} v_f + {}^1_0\boldsymbol{\lambda}^T \nabla_{{}^1_0\mathbf{h}}{}^1_0\mathbf{c} & = & 0 \\ \nabla_{{}^1_0\boldsymbol{\lambda}}\mathcal{L} = {}^1_0\mathbf{c}\left({}^1_0\mathbf{h}, {}^1\mathbf{r}\right) & = & 0\end{array} . \tag{3.45}$$

where the dependence of the function on coupling variables, design variables and responses are omitted for brevity of notation. According to Section 2.2, the first order derivatives of the Lagrangian is a necessary, however not sufficient condition for optimality. Therefore, the second order conditions are required to determine optimality.

In matrix notation the second-order condition is:

$$\nabla^2\mathcal{L} \;\; = \;\; \begin{bmatrix} \nabla^2_{{}^0\mathbf{x},{}^0\mathbf{x}}\mathcal{L} & \nabla^2_{{}^1\mathbf{x},{}^0\mathbf{x}}\mathcal{L} & \nabla^2_{{}^1_0\mathbf{h},{}^0\mathbf{x}}\mathcal{L} & \nabla^2_{{}^1_0\boldsymbol{\lambda},{}^0\mathbf{x}}\mathcal{L} \\ \nabla^2_{{}^0\mathbf{x},{}^1\mathbf{x}}\mathcal{L} & \nabla^2_{{}^1\mathbf{x},{}^1\mathbf{x}}\mathcal{L} & \nabla^2_{{}^1_0\mathbf{h},{}^1\mathbf{x}}\mathcal{L} & \nabla^2_{{}^1_0\boldsymbol{\lambda},{}^1\mathbf{x}}\mathcal{L} \\ \nabla^2_{{}^0\mathbf{x},{}^1_0\mathbf{h}}\mathcal{L} & \nabla^2_{{}^1\mathbf{x},{}^1_0\mathbf{h}}\mathcal{L} & \nabla^2_{{}^1_0\mathbf{h},{}^1_0\mathbf{h}}\mathcal{L} & \nabla^2_{{}^1_0\boldsymbol{\lambda},{}^1_0\mathbf{h}}\mathcal{L} \\ \nabla^2_{{}^0\mathbf{x},{}^1_0\boldsymbol{\lambda}}\mathcal{L} & \nabla^2_{{}^1\mathbf{x},{}^1_0\boldsymbol{\lambda}}\mathcal{L} & \nabla^2_{{}^1_0\mathbf{h},{}^1_0\boldsymbol{\lambda}}\mathcal{L} & \mathbf{0} \end{bmatrix}, \tag{3.46}$$

which is called the Hessian matrix. A sufficient condition for optimality is that:

$$\partial\mathbf{x}^T\left(\nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\right)\partial\mathbf{x} > 0. \tag{3.47}$$

for all directions $\partial\mathbf{x}$ for which $\partial\mathbf{x}^T\nabla_{\mathbf{x}}\mathbf{c} = 0$ holds and $\mathbf{x} = [{}^0\mathbf{x}, {}^1\mathbf{x}, {}^1_0\mathbf{h}]$.

In case of a bottom-up decomposition the first- and second-order conditions are derived similar. A decomposition in two-directions, hence non-hierarchic decomposition, leads to consistency constraints in both directions of the coupling circle:

$$\begin{array}{ll}\min\limits_{{}^0\mathbf{x},{}^1\mathbf{x},{}^0_1\mathbf{h},{}^1_0\mathbf{h}} & v_f = v_f\left({}^0_1\mathbf{h}, {}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^1\mathbf{x}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h}), {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})\right) \\ \text{s.t.} & {}^0_1\mathbf{c}({}^0_1\mathbf{h}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h})) = {}^0_1\mathcal{H}({}^0\mathbf{r}) - {}^0_1\mathbf{h} = \mathbf{0} \\ & {}^1_0\mathbf{c}({}^1_0\mathbf{h}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} = \mathbf{0}\end{array} , \tag{3.48}$$

and the corresponding Lagrangian function is:

$$\begin{aligned}\mathcal{L}({}^0\mathbf{x}, {}^1\mathbf{x}, {}^1_0\mathbf{h}, {}^1_0\boldsymbol{\lambda}, {}^0_1\boldsymbol{\lambda}) \;\; = \;\; & v_f\left({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^1\mathbf{x}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h}), {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})\right) + \\ & {}^0_1\boldsymbol{\lambda}^T\, {}^0_1\mathbf{c}({}^0_1\mathbf{h}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h})) + {}^1_0\boldsymbol{\lambda}^T\, {}^1_0\mathbf{c}({}^1_0\mathbf{h}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})),\end{aligned} \tag{3.49}$$

after differentiating once the first-order conditions are obtained. And for the second-order conditions the first-order conditions are differentiated once more.

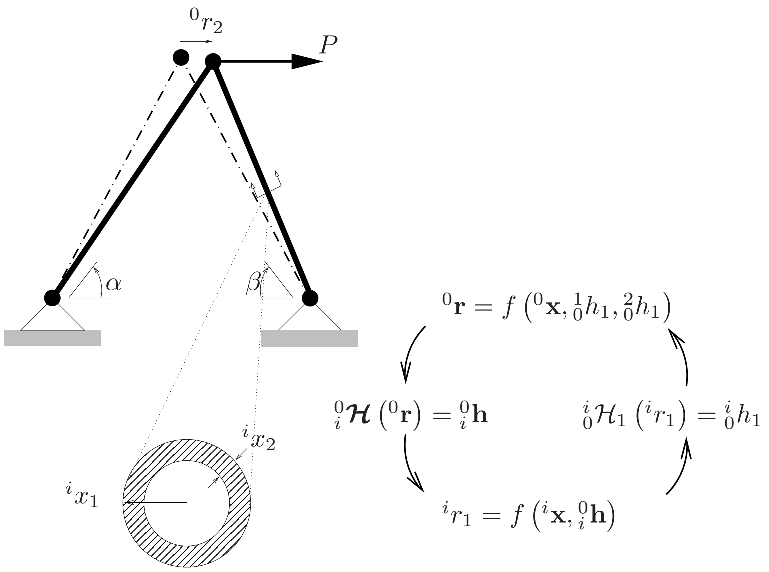


Figure 3.26: *Two-bar truss structure with embedded hierarchy. Level-0 responses are the displacement and the total mass of the structure. These responses depend on cross-sectional details of the two bar members. The detailed cross-section is described via Level-1 parameters.*

Typically the Hessian matrix (3.46) is sparse, because not all the functions depend on all the design variables and coupling variables. Furthermore, due to partitioning of the physical responses the optimization problem has become a saddle point problem. Therefore, techniques that efficiently solve such problems can be used as a guideline to develop coordination techniques that coordinate the data between the individual elements of the hierarchy. In chapters 4, 5 and 6 techniques that take advantage of sparse matrices and that efficiently solve saddle point problems are discussed in the context of multi-level optimization.

3.6 Example: Two-bar truss

The process of formulating a multi-level structural optimization problem is demonstrated on the basis of a two-bar truss design optimization problem. The two-bar truss problem is shown in Figure 3.26. Two levels can be distinguished, namely, the general lay-out of the structure consisting of two bar elements (Level-0) and the detailed cross-sectional area described by the diameter and the thickness of the wall (Level-1).

As general responses of the structure we study total structural mass $^0r_1$ and the horizontal displacement $^0r_2$, both depending on the cross-sectional areas that are considered Level-0 properties. The detailed cross-sectional geometry formulation of the bar elements are considered Level-1 properties. The two-bar truss design problem is shown in Figure 3.27.

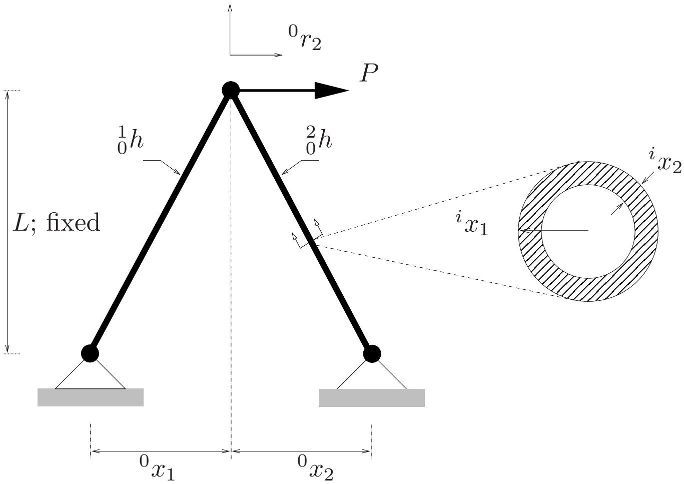


Figure 3.27: *Two-bar truss structure with embedded hierarchy. The general lay-out of the structure is described via design parameters* ${}^0\mathbf{x}$ *and cross-sectional areas of the elements. The cross-sectional area is represented via coupling variables* ${}^1_0h, {}^2_0h$*. The element cross-sectional areas are described in detail via design variables* ${}^ix_1, {}^ix_2$*.*

The optimization functions associated with the two-bar truss design problem are mathematically expressed as:

$$
\begin{array}{rcl}
{}^0v_f & = & \frac{{}^0r_1({}^0\mathbf{x},{}^1\mathbf{x},{}^2\mathbf{x})}{{}^0r_{mmax}} \\
{}^0v_g & = & \frac{{}^0r_2({}^0\mathbf{x},{}^1\mathbf{x},{}^2\mathbf{x})}{{}^0r_{umax}} - 1 \\
{}^1v_g & = & \frac{{}^1r_1({}^1\mathbf{x},{}^0\mathbf{r})}{0.9\,{}^1r_{cr}} - 1 \\
{}^2v_g & = & \left(2\frac{{}^2r_1({}^2\mathbf{x},{}^0\mathbf{r})}{{}^2r_{Euler}({}^0x_2,{}^2x_1)}\right)^2 - 1
\end{array}, \qquad (3.50)
$$

and the parameters determining the structural lay-out are listed in Table 3.1.

The problem matrix illustrating the dependencies of the optimization functions of Equation 3.50 on the design variables and the physical responses is shown in Figure 3.28. The objective function and constraint function of the Level-0 optimization problem depend on the design variables of Level-0 and Level-1 elements.

Coupling variables are added in Figure 3.29 to show the coupling between the two levels. Individual objectives for the Level-1 elements are not present and contributions of the Level-1 elements to the Level-0 objective function are accounted for by the coupling parameters ${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$.

Two decomposition formulations of the coupling circle (Figure 3.26) are considered: hierarchic top-down equality based decomposition (Figure 3.30(a)), and hierarchic top-down relaxation based decomposition (Figure 3.30(b)).

The Level-1 elements have no individual objective function. The problem matrix is shown in Figure 3.31. Because after the decomposition via equality constraints no individual objective functions are present in the Level-1 elements, constraint mini-

Table 3.1: *Definition of design variables, physical responses, physical interactions between subsystems and optimization parameters for the two-bar truss.*

design	description	physical response	description	interaction	description
${}^0x_1$	support 1	${}^0r_1$	total mass ${}^0r_1 = f({}^0\mathbf{x}, {}^1_0h_1, {}^2_0h_1)$	${}^0_1\mathbf{h}_1$	nodal forces 1
${}^0x_2$	support 2	${}^0r_2$	horizontal displacement ${}^0r_2 = f({}^0\mathbf{x}, {}^1_0h_1, {}^2_0h_1)$	${}^0_2\mathbf{h}_1$	nodal forces 2
${}^1x_1$	radius element 1	${}^1r_1$	stress left member	${}^1_0h_1$	area element 1
${}^1x_2$	thickness element 1		${}^1r_1 = \frac{{}^0_1h_{1_2} - {}^0_1h_{1_1}}{2\pi\,{}^1x_1\,{}^1x_2}$		
${}^2x_1$	radius element 2	${}^2r_1$	stress right member	${}^2_0h_1$	area element 2
${}^2x_2$	thickness element 2		${}^2r_1 = \frac{{}^0_2h_{1_2} - {}^0_2h_{1_1}}{2\pi^2x_1\,{}^2x_2}$		

optimization	description	
${}^0r_{mmax}$	reference mass	
${}^0r_{umax}$	allowed displacement	$r_{Euler} = \frac{\pi^2({}^2x_1)^2E}{2\,{}^0_2h_2}$
${}^1r_{cr}$	maximum allowed stress	E = Young's modulus
${}^2r_{Euler}$	Euler buckling stress*	${}^0_2h_2$ element length

* Not part of the physical coupling between levels, but a restriction on the subsystem physical response.

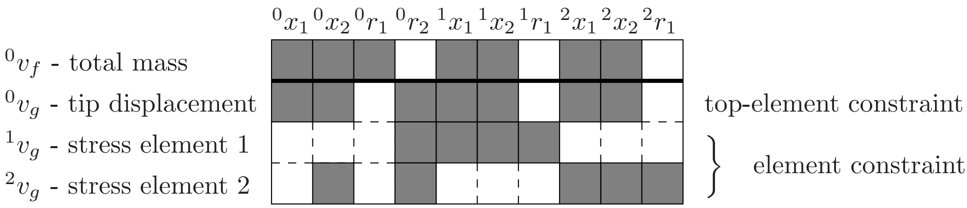


Figure 3.28: *Coupling matrix Two bar truss optimization problem before decomposition. On the left the optimization problem functions are listed and on the top the design variables and physical responses. A function that depends on a specific design variable or physical response is shaded.*

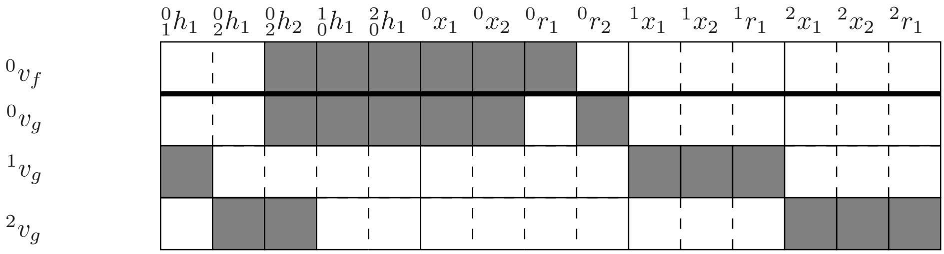


Figure 3.29: *Coupling parameters are introduced into the problem matrix, which illustrates the data that is required between the Level-0 element and the two Level-1 elements.*

mization is applied. The minimization of constraints is accomplished via an envelope function, here the KS-function is applied. The properties of the KS-function were discussed in Chapter 2.

The equality based consistency constraints between Level-0 and Level-1 are written as:

$$\begin{aligned} {}^1_0c &= {}^1_0\mathcal{H}_1\left({}^1r_1\right) - {}^1_0h_1 = 0; \\ {}^2_0c &= {}^2_0\mathcal{H}_1\left({}^2r_1\right) - {}^2_0h_1 = 0. \end{aligned} \tag{3.51}$$

These constraints are added to the Level-1 elements as equality constraints ${}^1v_c$ and ${}^2v_c$. The coupling variables ${}^1_0h$ and ${}^2_0h$ are introduced as design variables in the Level-0 optimization problem. The optimization problem for Level-0 now becomes:

$$\begin{array}{llcll} \min\limits_{{}^0\mathbf{x},{}^1_0h,{}^2_0h} & {}^0v_f & = & \dfrac{{}^0r_1\left({}^0\mathbf{x},{}^1_0h,{}^2_0h\right)}{{}^0r_{mmax}} & \\ \text{s.t.} & {}^0v_g & = & \dfrac{{}^0r_2\left({}^0\mathbf{x},{}^1_0h_1,{}^2_0h_1\right)}{{}^0r_{umax}} - 1 & \leq 0 \\ & {}^0v_{1a} & = & {}^1v_e\left({}^1v_g({}^1_0h)\right) & \leq 0 \\ & {}^0v_{2a} & = & {}^2v_e\left({}^2v_g({}^2_0h)\right) & \leq 0 \\ & & & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & \end{array}, \tag{3.52}$$

where the functions ${}^0v_{1a}$, ${}^0v_{2a}$ take into account that the Level-1 optima change due to changes in the Level-0 optimization problem. For the Level-1 elements the individual optimization problems become:

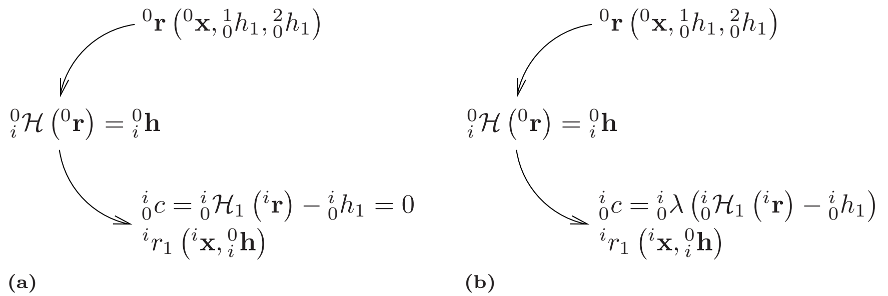


Figure 3.30: *(a)The Level-0 element prescribes the necessary responses for the Level-1 element by means of equality consistency constraints. This is called hierarchical top-down decomposition via equality constraints. (b) Top-down hierarchical relaxation based decomposition via, e.g. relaxation of the consistency constraint with Lagrangian multipliers assigned to the Level-1 element.*

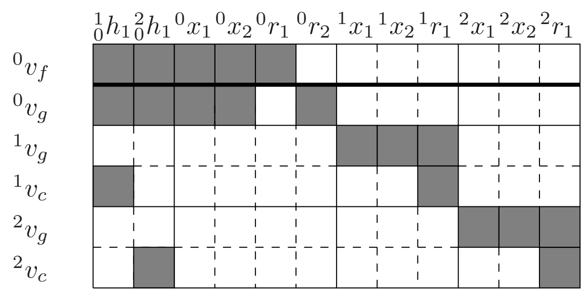


Figure 3.31: *Coupling matrix Two bar truss after decomposition. A top-down decomposition scheme is used, where the consistency constraints are added to the Level-1 elements.*

Element 1:

$$\begin{array}{llll}\min\limits_{{}^1x_1,{}^1x_2} & {}^1v_e\left({}^1v_g\right) & = & \frac{1}{\rho}\ln\left(e^{\rho\left(\frac{{}^1r_1}{0.9{}^1r_{cr}}-1\right)}\right) \\ \text{s.t.} & {}^1v_{{}^1_0c}({}^1_0c) & = & {}^1_0\mathcal{H}_1({}^1r_1)-{}^1_0h_1 \quad = 0 \\ & & & {}^1\underline{\mathbf{x}}\leq{}^1\mathbf{x}\leq{}^1\overline{\mathbf{x}}\end{array}. \tag{3.53}$$

Element 2:

$$\begin{array}{llll}\min\limits_{{}^2x_1,{}^2x_2} & {}^2v_e\left({}^2v_g\right) & = & \frac{1}{\rho}\ln\left(e^{\rho\left(\left(2\frac{{}^2r_1}{{}^2r_{Euler}}\right)^2-1\right)}\right) \\ \text{s.t.} & {}^2v_{{}^2_0c}({}^2_0c) & = & {}^2_0\mathcal{H}_1({}^2r_1)-{}^2_0h_1 \quad = 0 \\ & & & {}^2\underline{\mathbf{x}}\leq{}^2\mathbf{x}\leq{}^2\overline{\mathbf{x}}\end{array}. \tag{3.54}$$

The consistency constraint equations ${}^1v_{{}^1_0c}$, ${}^2v_{{}^2_0c}$ temporarily decouple the consistency constraints such that three individual optimization problems are solved.

In case relaxation is applied to the consistency constraints via, e.g., Lagrange multipliers the consistency constraints read:

$$\begin{array}{lll}{}^1_0\lambda^1_0c & = & {}^1_0\lambda\left({}^1_0\mathcal{H}_1\left({}^1r_1\right)-{}^1_0h_1\right); \\ {}^2_0\lambda^2_0c & = & {}^2_0\lambda\left({}^2_0\mathcal{H}_1\left({}^2r_1\right)-{}^2_0h_1\right).\end{array} \tag{3.55}$$

Relaxation of the consistency constraints means that an inconsistency is present because the constraints are not exactly satisfied. An additional term is added to each individual optimization problem present at Level-1 that adds a negative contribution to the objective function that takes into account the inconsistency. The optimization problem for the Level-0 element is:

$$\begin{array}{llll}\min\limits_{{}^0x_1,{}^0x_2,{}^1_0h_1,{}^2_0h_1} & {}^0v_f & = & \frac{{}^0r_1\left({}^0\mathbf{x},{}^1_0h_1,{}^2_0h_1\right)}{{}^0r_{mmax}} \\ \text{s.t.} & {}^0v_g & = & \frac{{}^0r_2\left({}^0\mathbf{x},{}^1_0h_1,{}^2_0h_1\right)}{{}^0r_{umax}}-1 \quad \leq 0 \\ & & & {}^0\underline{\mathbf{x}}\leq{}^0\mathbf{x}\leq{}^0\overline{\mathbf{x}}\end{array}, \tag{3.56}$$

and for the Level-1 elements the individual optimization problems write:
Element 1:

$$\begin{array}{llll}\min\limits_{{}^1x_1,{}^1x_2} & {}^1v_f & = & {}^1v_c({}^1_0c) \\ \text{s.t.} & {}^1v_g & = & \frac{{}^1r_1}{0.9{}^1r_{cr}}-1 \quad \leq 0 \\ & & & {}^1\underline{\mathbf{x}}\leq{}^1\mathbf{x}\leq{}^1\overline{\mathbf{x}} \\ \text{where} & \multicolumn{3}{l}{{}^1v_c({}^1_0c)={}^1_0\lambda\left({}^1_0\mathcal{H}_1\left({}^1r_1\right)-{}^1_0h_1\right)}\end{array}. \tag{3.57}$$

Element 2:

$$\begin{array}{llll}\min\limits_{{}^2x_1,{}^2x_2} & {}^2v_f & = & {}^2v_c({}^2_0c) \\ \text{s.t.} & {}^2v_g & = & \left(2\frac{{}^2r_1}{{}^2r_{Euler}}\right)^2-1 \quad \leq 0 \\ & & & {}^2\underline{\mathbf{x}}\leq{}^2\mathbf{x}\leq{}^2\overline{\mathbf{x}} \\ \text{where} & \multicolumn{3}{l}{{}^2v_c({}^2_0c)={}^2_0\lambda\left({}^2_0\mathcal{H}_1\left({}^2r_1\right)-{}^2_0h_1\right)}\end{array}. \tag{3.58}$$

Two different decompositions were shown for a two-bar truss optimization problem. Equality-based decomposition where the Level-1 optimization problems have no objective function. Instead these Level-1 optimization problems are given the task of minimizing the element design constraint while satisfying the consistency constraint. Relaxation-based decomposition distributes the task of satisfying the consistencies of both levels over all the elements, while each element has to minimize element objective and satisfy element constraint functions.

3.7 Discussion

This chapter discussed the decoupling of coupled problems and its implications on optimization problems. Three types of decomposition were identified. Furthermore, by means of a problem matrix, four characteristic coupling scenarios where treated and an extension to general multi-level optimization problems was made. Finally, necessary and sufficient conditions for finding an optimum were derived for unconstrained optimization problems with embedded hierarchy. The next chapter deals with the coordination of the results of the decoupled elements.

Chapter 4

Coordination

The previous chapter discussed techniques that temporarily decouple the analysis and/or optimization of complex structures into individual elements and how this temporarily decoupling is embedded in the design optimization problem. Because the individual analyzes and/or optimizations of these individual elements are coupled, their individual solutions require coordination. Coordination should combine the intermediate solutions of the individual elements such that the solution to the entire coupled optimization problem is found.

First, this chapter introduces the concept of coordination for equality-based decomposed problems and for relaxation-based decomposed problems in Section 4.1. Followed by an introduction to bi-level coordinating methods in Section 4.2 and an introduction to multi-level coordinating methods in Section 4.3. Methods that implement the bi-level coordination or multi-level coordination are discussed in more detail in Chapter 5 and Chapter 6, respectively.

4.1 Coordinating individual problems

A complex structure that is decomposed into a hierarchy of individual elements that are coupled requires coordination of the solutions of the individual elements. This coordination can be subdivided in two distinct formulations. These are:

- equality-based coordination or model-based coordination;
- relaxation-based coordination or goal-oriented coordination.

Both coordination techniques are the direct consequence of choices made on formulating consistency constraints between coupled elements.

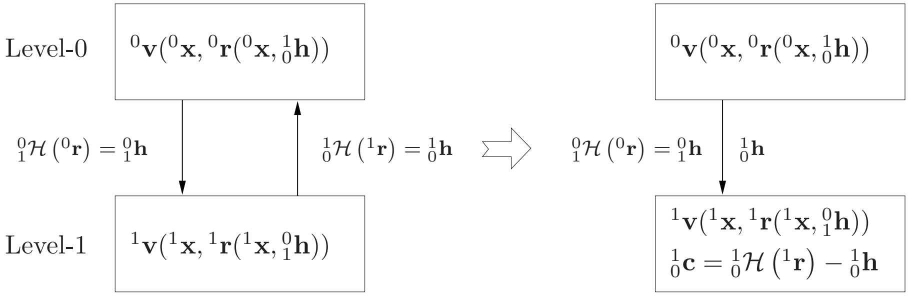


Figure 4.1: *On the left two elements that are coupled and require optimization are shown. After a top-down hierarchic decomposition of the physical responses that are mapped between the two elements the remaining coupling is shown on the right. Elements that are decoupled require coordination. The coordination should be done such that the solution of the coupled individual elements is the same as the all-in-one solution.*

4.1.1 Equality-based or model-based coordination

Equality-based coordination or model-based coordination uses physical responses from the model to steer the coordination process. To illustrate the coordination process, consider the consistency equations that model the interaction between two elements of a hierarchy:

$$\begin{aligned} {^0_1\mathbf{c}} &= {^0_1\mathcal{H}}\left(^0\mathbf{r}\right) - {^0_1\mathbf{h}} = \mathbf{0}; \\ {^1_0\mathbf{c}} &= {^1_0\mathcal{H}}\left(^1\mathbf{r}\right) - {^1_0\mathbf{h}} = \mathbf{0}. \end{aligned} \tag{4.1}$$

The two coupled elements are illustrated in Figure 4.1(left). In the previous chapter three decomposition procedures were introduced. In order to illustrate the coordination process a hierarchic top-down decomposition is considered, depicted in Figure 4.1(right). Hence, the mapping of Level-0 onto Level-1 is explicitly satisfied and the mapping of Level-1 onto Level-0 is decoupled and replaced via consistency constraints ${^1_0\mathbf{c}}$. The coupling variables ${^1_0\mathbf{h}}$ representing the mapped physical responses of Level-1 onto Level-0 are added to the Level-0 optimization problem as design variables and the consistency constraints ${^1_0\mathbf{c}}$ are added to the Level-1 optimization problem. The Level-0 optimization problem conducts an optimization returning an optimal $^0\mathbf{r}$ that is mapped via ${^0_1\mathcal{H}}(^0\mathbf{r}) = {^0_1\mathbf{h}}$ onto the Level-1 optimization problem. Furthermore, optimal ${^1_0\mathbf{h}}$ are determined that prescribe the expected mapped response coming from the Level-1 element optimization. Hence, optimal values ${^0_1\mathbf{h}}, {^1_0\mathbf{h}}$ are determined during the optimization of the Level-0 element and send to the Level-1 element.

Model-based coordination uses the values of physical responses to steer the coordination process. The Level-0 optimization problem determines optimal values for the physical responses ($^0\mathbf{r}$) and coupling variables (${^1_0\mathbf{h}}$). The optimal physical response is mapped via ${^0_1\mathcal{H}}(^0\mathbf{r}) = {^0_1\mathbf{h}}$ and then send together with the optimal coupling variables (${^1_0\mathbf{h}}$) to the Level-1 optimization problem. At Level-1, the mapped responses and optimal coupling parameters are kept fixed during the optimization of the Level-1 element as shown in Figure 4.2.

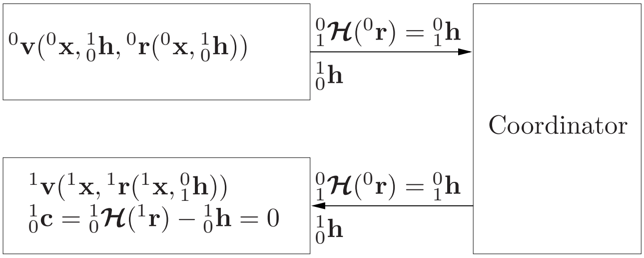


Figure 4.2: *Equality-based or model-based coordination, the optimal physical coupling between elements is determined in the parent element and send to the neighboring element as fixed parameters. The neighboring element keeps these variables fixed while performing it's individual design optimization.*

The coupling of Level-0 onto Level-1 is explicitly satisfied via ${}^{0}_{1}\mathcal{H}\left({}^{0}\mathbf{r}\right) = {}^{0}_{1}\mathbf{h}$. Consistency constraints ${}^{1}_{0}\mathbf{c}$ are satisfied via considering ${}^{1}_{0}\mathbf{h}$ a fixed set of parameters during optimization of the Level-1 element. An optimal value ${}^{1}\mathbf{r}$ is searched such that ${}^{1}_{0}\mathbf{c} = {}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h} = \mathbf{0}$ is satisfied. After the lowest level in the hierarchy is solved, the optimization of the entire hierarchy is finished. Coordinating the solution of the hierarchy is thus accomplished via the top-elements in the hierarchy that steer the lower level elements and no communication is present from the lower elements to the higher elements. Coordination via the multi-level hierarchy is called single-level coordination.

Equality-based or model-based coordination can also be applied in case a bottom-up decomposition or a non-hierarchic decomposition is applied. In case of a bottom-up decomposition, the top-level is the last level to be optimized before optimization is finished. When non-hierarchic decomposition is applied, physical responses are exchanged in both directions.

Sensitivity information may be added to the data exchange to improve the process of finding a solution to the coordination problem. In summary, when equality-based decomposition is applied the coordination involves sending physical responses and/or sensitivities of these responses (hence, model data) to neighboring elements.

The coupling due to shared design variables is treated similar as the physical coupling during the coordination process. Instead of requiring mapping of the physical responses the design variables can directly be send to neighboring elements. In the neighboring element the design variables are treated as fixed parameters such that, e.g. in case of top-down hierarchic equality-based decomposition, ${}^{0}_{1}\mathbf{z} = {}^{1}_{0}\mathbf{z}$ holds for the Level-1 element.

4.1.2 Relaxation-based or goal-oriented coordination

Relaxation-based coordination or goal-oriented coordination uses relaxation parameters to coordinate the individual element optimization problems. To illustrate the relaxation-based coordination process, consider the physical coupling between a two-

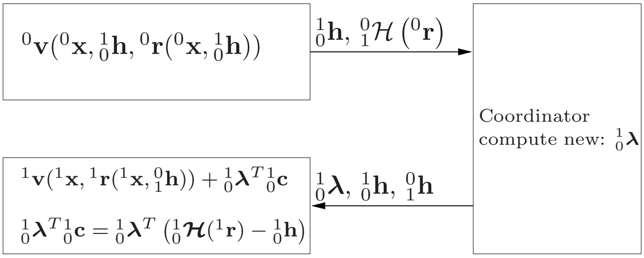


Figure 4.3: *Relaxation-based or goal-oriented coordination, to coordinate the convergence of the hierarchy, relaxation parameters* ${}^{1}_{0}\boldsymbol{\lambda}$ *are updated via the coordinator. The element present at Level-1 tries to reduce the contribution of the inconsistencies to the Level-1 objective function.*

level hierarchy consisting of a single element per level:

$$\begin{aligned} {}^{0}_{1}\mathcal{H}\left({}^{0}\mathbf{r}\right) &= {}^{0}_{1}\mathbf{h}; \\ {}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) &= {}^{1}_{0}\mathbf{h}. \end{aligned} \tag{4.2}$$

Similar to the equality-based coordination approach a hierarchic top-down decomposition is considered during the present discussion on relaxation-based coordination. Hence, the consistency between the mapping of physical responses from the Level-0 element onto the Level-1 element follows explicitly from ${}^{0}_{1}\mathcal{H}\left({}^{0}\mathbf{r}\right) = {}^{0}_{1}\mathbf{h}$ and the mapping of physical responses from Level-1 onto Level-0 is replaced via consistency constraints:

$${}^{1}_{0}\mathbf{c} = {}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}. \tag{4.3}$$

The consistency constraints are relaxed via, e.g., Lagrange multipliers. The relaxed consistency constraints applying Lagrange multipliers yield:

$${}^{1}_{0}\boldsymbol{\lambda}^{T}\,{}^{1}_{0}\mathbf{c} = {}^{1}_{0}\boldsymbol{\lambda}^{T}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right). \tag{4.4}$$

The interaction is relaxed and data is communicated between the two elements to update the interaction variables, see Figure 4.3. The relaxation parameters ${}^{1}_{0}\boldsymbol{\lambda}$ are updated by the coordinator in order to balance the deviation between physical responses ${}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right)$ and coupling variables ${}^{1}_{0}\mathbf{h}$. Techniques to update the relaxation parameters are discussed in Chapter 5.

The relaxation parameters ${}^{1}_{0}\boldsymbol{\lambda}$ pull the Level-1 element solution towards finding a physical response that reduces the consistency violation. Determining the optimal relaxation parameters that pull the Level-1 element solution towards a consistent solution with the neighboring Level-0 element requires a solution process. This solution process requires iterations that update the relaxation parameters until an optimum is found. The consistencies are only satisfied at convergence of the entire hierarchy, meaning that re-optimizing each individual element is necessary as long as the consistency constraints are not satisfied within allowable tolerances.

In case the decomposition is bottom-up the consistency constraints ${}^{0}_{1}\mathbf{c}$ are relaxed and coupling data is send between the levels via the coordinator together with relax-

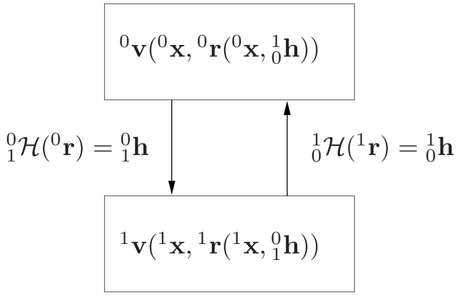


Figure 4.4: *Information exchange between two individual coupled optimization problems. Coordination of the data that is send between the two elements is necessary.*

ation parameters that are updated by the coordinator. Likewise, non-hierarchic decomposed elements are relaxed via relaxation parameters and coupling data together with relaxation parameters are send back to the individual elements via the coordinator. The coordinator updates the relaxation parameters for both the elements. Because the coordinator performs an additional solution step, i.e. computing optimal relaxation parameters, this type of coordination is called bi-level coordination.

Shared design variables are treated similarly as the coupled physical responses. Consistency constraints are introduced that require consistency between the design variables that are shared in both elements. Relaxation of these consistency constraints is done similar to the relaxation of physical coupling and the relaxation parameters are updated via the coordinator.

In summary, the coordination process can thus be steered via model parameters or via relaxation parameters. In case model parameters are used and no additional computations are performed via the coordinator the coordination is called single level coordination. In case additional computations are performed on the model data or relaxation parameters are computed via the coordinator, the coordination is Bi-level. Bi-level coordination is discussed in the next section and for systems with many elements the bi-level coordination is extended to multi-level coordination, which is discussed in Section 4.3.

4.2 Bi-level coordination

Coordination of interaction between elements involves decision making. Between two individual optimization problems data is transferred, see Figure 4.4. This data cannot be arbitrarily send from one element to the neighboring element. Therefore, decisions on which data to send and in what order this is done are necessary.

According to Shupe *et al.* (1987) there are three types of decisions that can be distinguished:

1. separable decisions;

2. inseparable decisions;

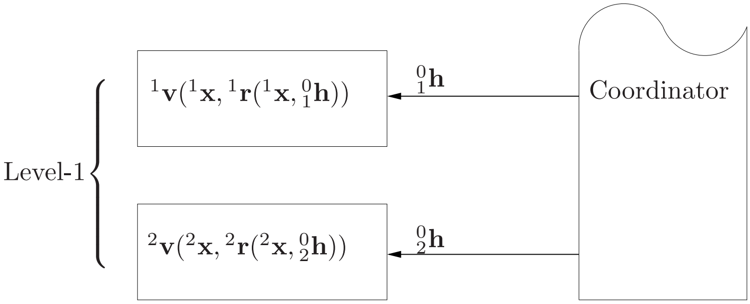


Figure 4.5: *Separable decisions, the Level-1 optimization problems of element 1 and element 2 are not coupled to eachother. Hence, the data that is used and the optimal data computed in element 1 at Level-1 does not change the outcome of the optimization of the neighboring element 2 at Level-1 and vice versa.*

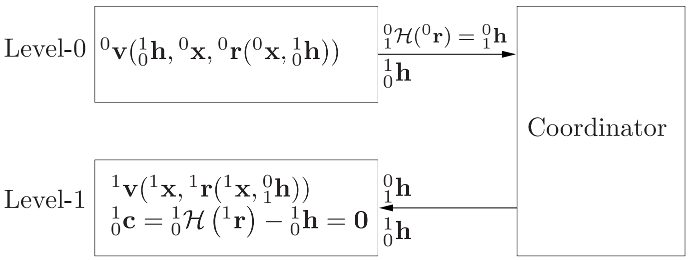


Figure 4.6: *Inseparable decisions, the optimization problems are coupled and data is send from one element to the neighboring element.*

3. inseparable and coupled decisions.

The first type of decisions, the separable decisions are illustrated in Figure 4.5. These type of decisions occur in a multi-level hierarchy where the update of information of one element does not depend on the data that is send to other neighboring elements. This is typically the case for elements on the same level that are not coupled to each other. Hence, after solving the parent element (Level-0), the updates of child elements (Level-1) are independent from one-another.

The second type of decisions are inseparable decisions, which are illustrated in Figure 4.6 for, e.g., equality-based coordination. The output of one element is send to the input of the neighboring element. These decisions are done sequentially and are typically the result of top-down or bottom-up decomposition schemes, recall Chapter 3 for a definition of these decomposition procedures. The sequential process moves from level to level and is finished when it reaches the bottom (top-down) or the top-level (bottom-up).

Inseparable decisions require no operations inside the coordinator block and no changes are made to the Level-0 optimization problem after it has finished. Various researchers (e.g. Vanderplaats *et al.* (1990), Kirsch (1997)) reported that this type

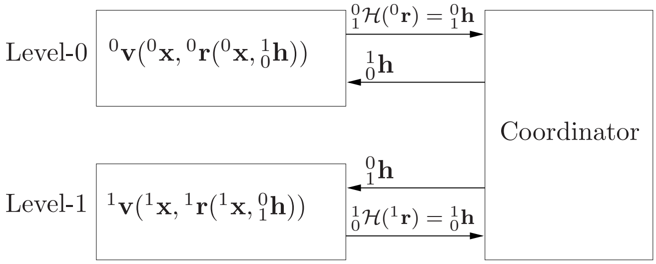


Figure 4.7: *Inseparable coupled decisions require iteration. The solution of both elements depends on the solution of the neighboring elements.*

of coordination fails in cases where the Level-0 element prescribes physical responses (i.e. ${}^{1}_{0}\mathbf{h}$) or shared design variable vectors (i.e. ${}^{0}_{1}\mathbf{z} = {}^{1}_{0}\mathbf{z}$) that the Level-1 element is unable to meet. Either due to restrictions on design variables (i.e. lower bounds $\underline{\mathbf{x}}$ and/or upper bounds $\overline{\mathbf{x}}$) and/or physical restrictions and/or design restrictions on the Level-1 element. Hence, no feed-back of the optimization results is present. Therefore, additional information on the Level-1 optimization problem is required by the Level-0 element. Procedures to overcome this drawback are discussed in Chapter 5.

The third type of decisions are the inseparable and coupled decisions. The output of one element is the input of the neighboring element and *vice versa*. The decision process is illustrated in Figure 4.7. These decisions are done sequentially or done concurrently via iterations between elements or via optimization (e.g. determining optimal Lagrange multipliers). Examples of such approaches are introduced in chapters 5 and 6.

Inseparable decisions that require additional information to be added to the higher elements in the hierarchy and inseparable and coupled decisions belong to bi-level coordination methods. Single level coordination involves sending model data between elements. If additional computations are required on the model data, sensitivity information is required or, e.g., relaxation parameters are evaluated, the coordinator involves a second level in which these computations are conducted. Therefore, these type of coordination procedures are called bi-level coordination. Bi-level coordination typically focuses on the coupling between two individual elements. A hierarchy of multiple levels and elements is coordinated stepwise by focussing on the coupling between two levels or two individually coupled elements.

4.3 Multi-level coordination

In case the number of elements is large or the coupling among elements covers multiple levels, bi-level approaches become slow in terms of convergence characteristics, see e.g. DeMiguel and Nogales (2008). Therefore, additional levels can be added to the coordinator that allow for a pre-coordination of the data, see Figure 4.8.

Multi-level coordination is shown in Figure 4.8 for a hierarchic top-down decomposition. Instead of having data send from level to level, the data is directly send

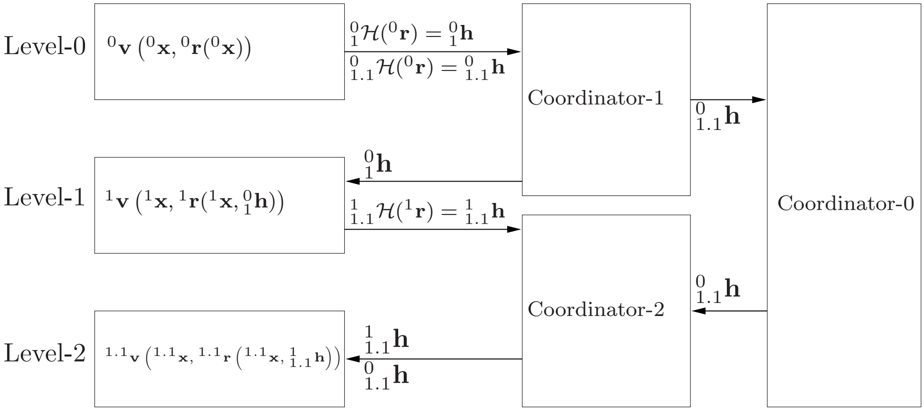


Figure 4.8: *Multi-level coordination is necessary if the amount of elements in the hierarchy becomes too large and a bi-level coordination involves too much copying of data between levels. The simplest form is to directly send data from an element to the neighboring element that depends on the data. However, in more complex situations levels in the coordination are used to approximate the interactions between elements to steer the coordination before actual data is send to the elements for evaluation of these elements.*

between the elements that depend on the data. In this example the coordinating levels are somewhat trivial, however in a more complex setting a similar hierarchy in the coordination is present.

Solving large optimization problems

Essential to the coordination process is to find an optimal solution for the consistency constraints. Additionally, the optimal solution of the individual elements is important, however these optimal solutions typically depend on the consistency constraints. Therefore, it is beneficial to first try to find an approximation to the optimal setting for the consistency constraints before trying to find an optimal setting for the individual elements. The procedure shows large analogies with those found in domain decomposition methods and/or multi-grid methods for optimization, see, e.g. Dreyer *et al.* (2000).

A necessary requirement for a multi-level coordinator that solves the consistency constraints is that a solution to the necessary conditions for the all-in-one problem is found. These conditions were derived in Chapter 3 for a two-level hierarchical optimization problem. The optimization problem considered is:

$$\begin{array}{ll} \min\limits_{{}^{0}\mathbf{x}, {}^{1}\mathbf{x}, {}^{1}_{0}\mathbf{h}} & v_f = v_f\left({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{1}\mathbf{x}, {}^{0}\mathbf{r}({}^{0}\mathbf{x}, {}^{1}_{0}\mathbf{h}), {}^{1}\mathbf{r}({}^{1}\mathbf{x}, {}^{0}_{1}\mathbf{h})\right) \\ \text{s.t.} & {}^{1}_{0}\mathbf{c}({}^{1}_{0}\mathbf{h}, {}^{1}\mathbf{r}({}^{1}\mathbf{x}, {}^{0}_{1}\mathbf{h})) = {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h} = 0 \end{array} . \qquad (4.5)$$

Necessary conditions for this optimization problem are:

$$\begin{array}{lcl}
\nabla_{^0\mathbf{x}}\mathcal{L} = \nabla_{^0\mathbf{x}} v_f + {}^1_0\boldsymbol{\lambda}^T \nabla_{^0\mathbf{x}} {}^1_0\mathbf{c} & = & 0 \\
\nabla_{^1_0\mathbf{h}}\mathcal{L} = \nabla_{^1_0\mathbf{h}} v_f + {}^1_0\boldsymbol{\lambda}^T \nabla_{^1_0\mathbf{h}} {}^1_0\mathbf{c} & = & 0 \\
\nabla_{^1\mathbf{x}}\mathcal{L} = \nabla_{^1\mathbf{x}} v_f + {}^1_0\boldsymbol{\lambda}^T \nabla_{^1\mathbf{x}} {}^1_0\mathbf{c} & = & 0 \\
\nabla_{^1_0\boldsymbol{\lambda}}\mathcal{L} = {}^1_0\mathbf{c} & = & 0
\end{array} \tag{4.6}$$

where the dependence of v_f and ${}^1_0\mathbf{c}$ on coupling variables, design variables and responses are omitted for brevity of notations. Furthermore, coupling variables ${}^1_0\mathbf{h}$ are assigned as design variables to the Level-0 element and a hierarchic top-down decomposition is considered. Therefore, the consistency constraints ${}^1_0\mathbf{c}$ are added to the Level-1 optimization problem.

To search for a solution of Equation 4.5 in the local neighborhood of the current configuration a Newton process can be used. The effect of a small perturbation of the design variables, coupling variables and Lagrange multipliers is approximated via a first-order Taylor series expansion that yields:

$$\nabla\mathcal{L}(\mathbf{x}+\Delta\mathbf{x}, {}^1_0\boldsymbol{\lambda}+\Delta{}^1_0\boldsymbol{\lambda}) \quad = \quad \nabla\mathcal{L} + \nabla\left(\nabla\mathcal{L}\right)\left[\Delta\mathbf{x}, \Delta{}^1_0\boldsymbol{\lambda}\right]^T \ , \tag{4.7}$$

where $\mathbf{x} = \left[{}^0\mathbf{x}, {}^1_0\mathbf{h}, {}^1\mathbf{x}\right]^T$. A stationary point to these equations is found setting the left-hand-side equal to zero. Then Equation 4.7 becomes:

$$\nabla^2\mathcal{L}\left[\Delta\mathbf{x}, \Delta{}^1_0\boldsymbol{\lambda}\right]^T = -\nabla\mathcal{L}. \tag{4.8}$$

The iteration matrix $\nabla^2\mathcal{L}$ is used to find a search direction towards a local stationary point. In matrix notation the iteration matrix is expressed as:

$$\nabla^2\mathcal{L} \quad = \quad \begin{bmatrix}
\nabla^2_{^0\mathbf{x},^0\mathbf{x}}\mathcal{L} & \nabla^2_{^1_0\mathbf{h},^0\mathbf{x}}\mathcal{L} & \nabla^2_{^1\mathbf{x},^0\mathbf{x}}\mathcal{L} & \nabla^2_{^1_0\boldsymbol{\lambda},^0\mathbf{x}}\mathcal{L} \\
\nabla^2_{^0\mathbf{x},^1_0\mathbf{h}}\mathcal{L} & \nabla^2_{^1_0\mathbf{h},^1_0\mathbf{h}}\mathcal{L} & \nabla^2_{^1\mathbf{x},^1_0\mathbf{h}}\mathcal{L} & \nabla^2_{^1_0\boldsymbol{\lambda},^1_0\mathbf{h}}\mathcal{L} \\
\nabla^2_{^0\mathbf{x},^1\mathbf{x}}\mathcal{L} & \nabla^2_{^1_0\mathbf{h},^1\mathbf{x}}\mathcal{L} & \nabla^2_{^1\mathbf{x},^1\mathbf{x}}\mathcal{L} & \nabla^2_{^1_0\boldsymbol{\lambda},^1\mathbf{x}}\mathcal{L} \\
\nabla^2_{^0\mathbf{x},^1_0\boldsymbol{\lambda}}\mathcal{L} & \nabla^2_{^1_0\mathbf{h},^1_0\boldsymbol{\lambda}}\mathcal{L} & \nabla^2_{^1\mathbf{x},^1_0\boldsymbol{\lambda}}\mathcal{L} & \mathbf{0}
\end{bmatrix}. \tag{4.9}$$

A sufficient condition for local optimality is:

$$\Delta\mathbf{x}^T\left(\nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\right)\Delta\mathbf{x} > \mathbf{0}. \tag{4.10}$$

for all directions $\Delta\mathbf{x}$ for which $\Delta\mathbf{x}^T\nabla_{\mathbf{x}}\mathbf{c} = 0$ hold. This condition reflects that all feasible perturbations lead to an increase of the objective function.

A problem that immediately comes to mind is that not all information required for Equation 4.8 can be computed when considering a multi-level hierarchy. A multi-level hierarchy of two-individual elements and a top-down hierarchic decomposition typically involves:
Level-0:

$$\min_{^1_0\mathbf{h},^0\mathbf{x}} \ {}^0v_f({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}({}^1_0\mathbf{h}, {}^0\mathbf{x})) \tag{4.11}$$

Level-1:

$$\begin{array}{ll} \min\limits_{{}^1\mathbf{x}} & {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x})) \\ \text{s.t.} & {}^1_0\mathbf{c}({}^1\mathbf{r}) = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} \end{array}, \tag{4.12}$$

where the objective function v_f is considered additively separable: $v_f = {}^0v_f + {}^1v_f$.

The first- and second-derivatives necessary to construct Equation 4.8 can only be found if, e.g., Equation 4.11 finds the optimal variables ${}^0\mathbf{x}, {}^1_0\mathbf{h}$ of Equation 4.8. If that is the case, then the data that is mapped from Level-0 onto Level-1 assures that the Level-1 optimization problem (Equation 4.12) can construct the correct first-order and second-order derivatives that are necessary for Equation 4.8. In general this will not be the case, however if the optimum found via Equation 4.11 is sufficiently close to the optimum of the two elements (Equation 4.8) then via Optimum Sensitivity Analysis it can be shown that the necessary first and second order information is closely approximated. This is shown by DeMiguel and Nogales (2008).

Solving large systems of equations via, e.g., the Null-space method

A means of searching for a solution of Equation 4.5 is to split the optimization problem into a part that tries to satisfy the constraints ${}^1_0\mathbf{c}$ and a part that tries to minimize the objective function value v_f. In linear algebra this approach is known as the Null-space method. The Null-space method assumes that the following is available:

- a particular solution $\Delta\hat{\mathbf{x}} = \left[\Delta^0\mathbf{x},\ \Delta^1_0\mathbf{h},\ \Delta^1\mathbf{x}\right]^T$ for which: $\nabla_\mathbf{x}\left({}^1_0\mathbf{c}\right)\Delta\hat{\mathbf{x}} = -{}^1_0\mathbf{c}$. The gradient of the constraints is defined as $\nabla_\mathbf{x}\left({}^1_0\mathbf{c}\right) = \left[\nabla_{{}^0\mathbf{x}}\left({}^1_0\mathbf{c}\right),\ \nabla_{{}^1_0\mathbf{h}}\left({}^1_0\mathbf{c}\right),\ \nabla_{{}^1\mathbf{x}}\left({}^1_0\mathbf{c}\right)\right]$.

- there is a matrix $\mathbf{T}$, such that: $\left[\nabla_{{}^0\mathbf{x}}{}^1_0\mathbf{c},\ \nabla_{{}^1_0\mathbf{h}}{}^1_0\mathbf{c},\ \nabla_{{}^1\mathbf{x}}{}^1_0\mathbf{c}\right][\mathbf{T}] = [\mathbf{0}]$ (The columns of $\mathbf{T}$ form a complete basis for the null space of $\nabla_\mathbf{x}{}^1_0\mathbf{c}$).

If $\Delta\hat{\mathbf{x}}$ is a solution of $\nabla_\mathbf{x}\left({}^1_0\mathbf{c}\right)\Delta\mathbf{x} = -{}^1_0\mathbf{c}$, then other solutions that satisfy this equation can be found. Hence, $\Delta\mathbf{x} = \mathbf{T}\Delta\mathbf{y} + \Delta\hat{\mathbf{x}}$ are solutions as well, where $\Delta\mathbf{y}$ can be chosen arbitrarily.

$\Delta\mathbf{x} = \mathbf{T}\Delta\mathbf{y} + \Delta\hat{\mathbf{x}}$ is substituted into Equation 4.8. The matrix vector calculation and rearrangement of terms yields:

$$\nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\left(\mathbf{T}\Delta\mathbf{y} + \Delta\hat{\mathbf{x}}\right) = -\nabla_\mathbf{x}\mathcal{L} - \nabla_\mathbf{x}\left({}^1_0\mathbf{c}\right)^T\Delta^1_0\boldsymbol{\lambda}, \tag{4.13}$$

$$\nabla_\mathbf{x}\left({}^1_0\mathbf{c}\right)\left(\mathbf{T}\Delta\mathbf{y} + \Delta\hat{\mathbf{x}}\right) = -\nabla_{{}^1_0\boldsymbol{\lambda}}\mathcal{L}, \tag{4.14}$$

where $\nabla^2_{\mathbf{x},{}^1_0\boldsymbol{\lambda}}\mathcal{L} = \nabla^2_{{}^1_0\boldsymbol{\lambda},\mathbf{x}}\mathcal{L}^T$, $\nabla^2_{\mathbf{x},{}^1_0\boldsymbol{\lambda}}\mathcal{L} = \nabla_\mathbf{x}\left({}^1_0\mathbf{c}\right)^T$ and $\mathbf{x} = [{}^0\mathbf{x},\ {}^1_0\mathbf{h},\ {}^1\mathbf{x}]^T$.

Rearranging terms of Equation 4.13 and pre-multiplying by the null-space $\mathbf{T}^T$ gives:

$$\mathbf{T}^T\nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\mathbf{T}\Delta\mathbf{y} = \mathbf{T}^T\left(-\nabla_\mathbf{x}\mathcal{L} - \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\Delta\hat{\mathbf{x}}\right), \tag{4.15}$$

where use is made of the fact that $\mathbf{T}^T \nabla_{\mathbf{x}} \left({}^1_0\mathbf{c}\right)^T = \mathbf{0}$.

The dimension of Equation 4.15 is smaller (reduced in size with the number of constraints ${}^1_0\mathbf{c}$) then the dimension of Equation 4.8 and involves solving an unconstrained optimization problem.

Once $\Delta\mathbf{y}$ is found, the update of the design variable vector $\Delta\mathbf{x}$ can be computed via:

$$\Delta\mathbf{x} = \mathbf{T}\Delta\mathbf{y} + \Delta\hat{\mathbf{x}}. \tag{4.16}$$

The update of the Lagrange multipliers is then found via substitution of $\Delta\mathbf{x}$ into:

$$\nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\Delta\mathbf{x} \quad = \quad -\nabla_{\mathbf{x}}\mathcal{L} - \nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)^T \Delta^1_0\boldsymbol{\lambda}. \tag{4.17}$$

After reordering terms and pre-multiplying both sides with $\nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)$ the system to solve is:

$$\nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)\nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)^T \Delta^1_0\boldsymbol{\lambda} \quad = \quad \nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)\left(-\nabla_{\mathbf{x}}\mathcal{L} - \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\Delta\mathbf{x}\right). \tag{4.18}$$

Equation 4.18 may also be regarded as the normal equations for the over-determined system $\nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)^T \Delta^1_0\boldsymbol{\lambda} = \left(-\nabla_{\mathbf{x}}\mathcal{L} - \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\Delta\mathbf{x}\right)$. Which can be mathematically expressed as:

$$\min_{\Delta^1_0\boldsymbol{\lambda}} ||(-\nabla_{\mathbf{x}}\mathcal{L} - \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\Delta\mathbf{x}) - \nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)^T \Delta^1_0\boldsymbol{\lambda}||_2. \tag{4.19}$$

Equation 4.19 is the coordination problem that computes updates of the Lagrange multipliers given the current design configuration at time ${}^{(t)}$. If $\Delta^1_0\boldsymbol{\lambda}$ is replaced in Equation 4.8 via $\Delta^1_0\boldsymbol{\lambda} = {}^1_0\boldsymbol{\lambda}^{(t+1)} - {}^1_0\boldsymbol{\lambda}^{(t)}$ then Equation 4.19 is rewritten slightly such that it directly computes the new Lagrange multipliers:

$$\min_{{}^1_0\boldsymbol{\lambda}^{(t+1)}} ||(-\nabla_{\mathbf{x}}v_f - \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\Delta\mathbf{x}) - \nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)^T {}^1_0\boldsymbol{\lambda}^{(t+1)}||_2. \tag{4.20}$$

and Equation 4.15 is then rewritten to yield:

$$\mathbf{T}^T\nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\mathbf{T}\Delta\mathbf{y} = \mathbf{T}^T\left(-\nabla_{\mathbf{x}}v_f - \nabla^2_{\mathbf{x},\mathbf{x}}\mathcal{L}\Delta\hat{\mathbf{x}}\right). \tag{4.21}$$

Individual element optimization problems

In multi-level optimization the null-space is computed via the individual elements. The consistency constraints are assigned to individual optimization problems of the multi-level hierarchy. In order to find $\hat{\mathbf{x}}$, a first problem to solve is:

$$\nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)\Delta\mathbf{x} = -{}^1_0\mathbf{c}, \tag{4.22}$$

while $\nabla_{\mathbf{x}}\left({}^1_0\mathbf{c}\right)\ \mathbf{T} = \mathbf{0}$.

The consistency constraints are defined as:

$${}^1_0\mathbf{c} \quad = \quad {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}, \tag{4.23}$$

and in the current example these are assigned to the Level-1 problem.

To find a solution to Equation 4.22 a first step is found, e.g., via minimization of the Gauss-Newton system of Equation 4.22 in terms of Level-1 variables:

$$\begin{array}{ll} \min\limits_{{}^1\mathbf{x}} & \frac{1}{2}\Delta^1\mathbf{x}^T\nabla_{{}^1\mathbf{x}}\left({}^1_0\mathbf{c}\right)\Delta^1\mathbf{x} + {}^1_0\mathbf{c}^T\Delta^1\mathbf{x} \\ \text{s.t.} & \nabla_{{}^1\mathbf{x}}\left({}^1_0\mathbf{c}\right)\ \mathbf{T}\Delta^1\mathbf{x} = \mathbf{0} \end{array} \tag{4.24}$$

It is not necessary to find the exact solution to Equation 4.22 immediately, because Equation 4.22 depends on non-optimal data from Level-0. A step into the direction of the constraints is sufficient. In the work of, e.g., Alexandrov (1998) techniques are shown that monitor the direction of the step-size. A derivation of such monitoring procedures lies outside the scope of the present thesis.

Typically, the objective function v_f is additively separable for a multi-level hierarchy. Therefore, part of the objective function v_f, namely ${}^1v_f$, can be added to Equation 4.24. The Level-1 optimization problem including the Level-1 objective function and additional terms to compute a step into the direction of the constraints becomes:

$$\begin{array}{ll} \min\limits_{{}^1\mathbf{x}} & {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}) + \frac{1}{2}\Delta^1\mathbf{x}^T\nabla_{{}^1\mathbf{x}}\left({}^1_0\mathbf{c}\right)\Delta^1\mathbf{x} + {}^1_0\mathbf{c}^T\Delta^1\mathbf{x} \\ \text{s.t.} & \nabla_{{}^1\mathbf{x}}\left({}^1_0\mathbf{c}\right)\ \mathbf{T}\Delta^1\mathbf{x} = \mathbf{0} \end{array} \tag{4.25}$$

A second step in finding a solution to the multi-level optimization problem involves a search into the direction that minimizes the Level-0 optimization problem (recall Equation 4.15):

$$\min_{{}^0\mathbf{x}^*} \quad \mathbf{T}^T\nabla^2_{{}^0\mathbf{x}^*,{}^0\mathbf{x}^*}\mathcal{L}\mathbf{T}\Delta^0\mathbf{y} + \mathbf{T}^T\left(\nabla_{{}^0\mathbf{x}^*}\mathcal{L} + \nabla^2_{{}^0\mathbf{x}^*,{}^0\mathbf{x}^*}\mathcal{L}\Delta^0\hat{\mathbf{x}^*}\right), \tag{4.26}$$

where ${}^0\mathbf{x}^* = \left[{}^1_0\mathbf{h},\ {}^0\mathbf{x}\right]$; $\nabla^2_{{}^0\mathbf{x}^*,{}^0\mathbf{x}^*}\mathcal{L} = \nabla^2_{{}^0\mathbf{x}^*,{}^0\mathbf{x}^*}{}^0v_f + {}^1_0\boldsymbol{\lambda}^T\nabla^2_{{}^0\mathbf{x}^*,{}^0\mathbf{x}^*}{}^1_0\mathbf{c}$; $\nabla_{{}^0\mathbf{x}^*}\mathcal{L} = \nabla_{{}^0\mathbf{x}^*}v_f + {}^1_0\boldsymbol{\lambda}^T\nabla_{{}^0\mathbf{x}^*}{}^1_0\mathbf{c}$. The coordination involves updating of the Lagrange multipliers via Equation 4.19 or Equation 4.20 depending on the formulation chosen.

For each element that is introduced the null space ($\mathbf{T}$) decreases in size by the size of the consistency constraints between the introduced element and the neighboring elements to which this element is coupled. In practice, elements are chosen such that the number of coupling variables and/or shared design variables is much smaller than the amount of individual element design variables ${}^{\cdot\cdot}\mathbf{x}$. Therefore, the projected Hessian matrix is of much smaller size then the Hessian matrix. The projected Hessian matrix can be used to search for the objective function that describes the performance of the entire hierarchy, i.e. v_f, or the single objective function of Level-0 when the objective function is additively separable over multiple elements.

When the total optimization problem of the entire hierarchy becomes too large the projected Hessian matrix will also become too large to solve as a single problem. Therefore, additional levels in the coordination can be introduced that construct a search direction towards the solution of the entire optimization problem. These levels act as approximations to the distribution of coupling data that is send between the individual elements. In the work of Smith *et al.* (1996) multi-level coordinators for domain decomposition methods are derived in the context of pre-conditioners for solving large systems of equations. Such an approach allows for development of pre-conditioners that efficiently coordinate the solution of the system of equations. To show the coordination process in the context of multi-level optimization as a preconditioning approach for large systems of equations lies outside the scope of the present thesis.

A multi-level coordinator based on pre-conditioning techniques is constructed via restriction and extension vectors (**R**). The total system of equations to solve is a summation of each level of the coordinator. Let **B** represent the projected Hessian matrix, then the multi-level coordinator is written as:

$$\mathbf{B} = \sum_{i=1}^{n} \left(\mathbf{R}^i\right)^T \nabla^2 \mathcal{L}^{i-1} \left(\mathbf{R}^i\right), \tag{4.27}$$

where n is the number of coordinating levels.

Expressions for the restriction and extension vectors are well established in domain decomposition or multi-grid methods, see e.g. Smith *et al.* (1996), Quarteroni and Valli (1999). In the field of optimization the multilevel coordinator is applied to PDE-constrained optimization by e.g. Biros and Ghattas (2005) and in multi-grid optimization by e.g. Dreyer *et al.* (2000).

To the author's knowledge, restriction and extension vectors in the context of multi-level coordination of multi-level optimization problems are not derived yet. However, techniques that seem promising into this direction are introduced, e.g., by Alexandrov (1998) and DeMiguel and Nogales (2008). These multi-level coordination techniques are discussed in Chapter 6.

Multi-level coordination can easily be mistaken for single level coordination embedded in a multi-level hierarchy of a decomposed problem. However, a multi-level coordination hierarchy focuses on the decomposition of a solution algorithm, while decomposition that was considered in Chapter 3 deals with finding links between elements in the physical and/or optimization problem that allow for decomposition.

4.4 Discussion

This chapter introduced the basic concepts of coordination and discussed the origin of levels in the coordination that make a large coordination problem easier to solve. In Chapters 5 and 6 methods are introduced that can be categorized in either Bi-level coordination or Multi-level coordination schemes.

Chapter 5

Bi-level coordination methods

Bi-level methods are most commonly used in multi-level optimization. The individual optimization results of the elements are send to a coordinator which then distributes the data over the individual elements. Within the bi-level coordination methods two approaches are distinguished: equality-based coordination which is discussed in Section 5.1; relaxation-based coordination which is discussed in Section 5.2. Both approaches are subdivided into hierarchic approaches and non-hierarchic approaches. Each section concludes with a discussion of a few of the main stream coordination techniques that are applied to a multi-level formulation of a two-bar truss problem. In Section 5.3, performance indicator methods are discussed which give insight into an algorithm's convergence characteristics. Finally, in Section 5.4 the bi-level coordination methods are summarized.

5.1 Equality-based coordination

Equality-based coordination techniques are subdivided in two categories:

- hierarchic coordination,
- non-hierarchic coordination.

Furthermore, the hierarchic coordination techniques can be subdivided into:

- techniques that take into account neighboring elements via linearizing the coupling,
- techniques that take into account neighboring elements via creating design freedom for neighboring elements.

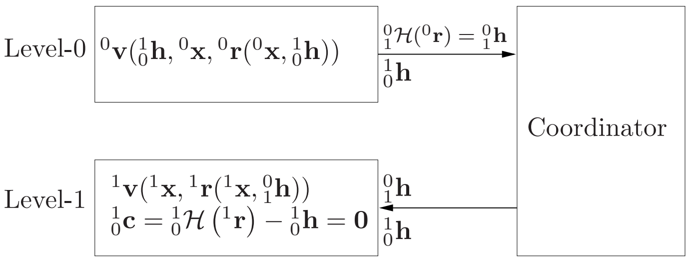


Figure 5.1: *Inseparable decisions, the optimization problems are coupled and data is send from one element to the neighboring element.*

In the present section the hierarchic coordination techniques are discussed first, followed by a discussion on non-hierarchic coordination techniques. For each category two main stream approaches are considered in detail and the section is concluded with a multi-level optimization of a two-bar truss problem. Each section considers a hierarchy of two levels containing a single element per level. A two-level problem consisting of a single element per level is sufficient to show the coordination between coupled elements and simplifies the formulation of the optimization problems. The discussion focuses on physical coupling, however coupling via shared design variables is treated similarly.

5.1.1 Hierarchic linearized coordination

Optimization by Linear Decomposition

In Chapter 4 three types of decisions were distinguished. These were: separable decisions; inseparable decisions; and inseparable and coupled decisions. In the present section the inseparable decisions are considered. In Chapter 4 a drawback of this type of decisions was mentioned which is repeated in the present section and a procedure to overcome this drawback is introduced called Optimization by Linear Decomposition (Sobieszczanski-Sobieski *et al.*, 1985, 1987).

Inseparable decisions are present when a Level-1 optimization problem receives information from the Level-0 element that prescribes the mapped responses (${}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})$) and/or shared design variables (${}^{1}_{0}\mathbf{z}$) that the Level-1 optimization has to satisfy. As an example, prescribing mapped responses is shown in Figure 5.1. Shared design variables are treated similarly.

Various researchers (e.g. Kirsch (1997), Vanderplaats *et al.* (1990)) reported that this type of coordination fails in cases where the Level-0 element prescribes physical responses (i.e. ${}^{1}_{0}\mathbf{h}$) or shared design variable vectors (i.e. ${}^{0}_{1}\mathbf{z} = {}^{1}_{0}\mathbf{z}$) that the Level-1 element is unable to meet. Either due to restrictions on design variables (i.e. lower bounds $\underline{\mathbf{x}}$ and/or upper bounds $\overline{\mathbf{x}}$) and/or physical restrictions on the Level-1 element. Hence, additional information on the Level-1 optimization problem is required by the Level-0 element.

Optimization by Linear Decomposition was derived to add information of the

Level-1 element to the Level-0 element such that the Level-0 element does not prescribe physical responses and/or design variables for the Level-1 problem that cannot be satisfied. The derivation of Optimization by Linear Decomposition initially involved problems where the Level-1 element does not share design variables and/or physical responses with the objective function (recall Figure 3.16 showing this particular problem matrix). In the present thesis the procedure is generalized to arbitrary multi-level problems. Furthermore, the bundling of constraints is omitted in the discussion and is treated in Subsection 5.1.2 to compare cumulative constraints with a slack variable approach.

Adding additional information of the Level-1 element to the Level-0 element is illustrated according to an equality-based top-down decomposition scheme. This type of decomposition was introduced in Section 3.2. The element of Level-1 is considered first and the optimization problem for Level-1 yields:

$$\begin{array}{lll} \min\limits_{{}^1\mathbf{x}} & {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}) & \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x}, {}^1\mathbf{r}) & \leq \mathbf{0} \\ & {}^1\mathbf{v}_h({}^1\mathbf{x}, {}^1\mathbf{r}) & = \mathbf{0} \\ & {}^1\mathbf{v}_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) & = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} = \mathbf{0} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \end{array} \tag{5.1}$$

Lower bounds (${}^1\underline{\mathbf{x}}$) and upper bounds (${}^1\overline{\mathbf{x}}$) on the design variables of Level-1 are present as well as inequality constraints ${}^1\mathbf{v}_g$, equality constraints ${}^1\mathbf{v}_h$ and consistency constraints ${}^1\mathbf{v}_{{}^1_0c}$ that couple the Level-1 optimization problem to the Level-0 optimization problem, where ${}^1_0\mathbf{h}$ is a vector of fixed parameters during optimization. These parameters can change via changes in the Level-0 element that are send to the Level-1 element. However, during optimization these parameters do not change.

An optimum for Equation 5.1 is found for a certain value of the coupling variables (${}^1_0\mathbf{h}, {}^0_1\mathbf{h}$). Coupling variables ${}^1_0\mathbf{h}$ originate from Level-0 where they were used as design variables. ${}^0_1\mathbf{h}$ are coupling variables that are directly mapped onto Level-1 (${}^0_1\mathcal{H}({}^0\mathbf{r}) = {}^0_1\mathbf{h}$). Hence, there is an effect of Level-0 via the parameters ${}^1_0\mathbf{h}$ and ${}^0_1\mathbf{h}$, that is, the optimum found for Equation 5.1 changes when ${}^0_1\mathbf{h}$ and ${}^1\mathbf{v}_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) = \mathbf{0}$ are changed.

For certain values of the consistency constraints (${}^1\mathbf{v}_{{}^1_0c}(\ldots)$) there might not be a feasible solution possible. Either due to bounds on the design variables or due to equality or inequality constraints. The Level-0 optimization problem thus, must not prescribe ${}^1_0\mathbf{h}$ that do not lead to a feasible solution of the Level-1 element. Therefore, information is added to the Level-0 element by means of exploring the optimum of the Level-1 problem.

Optimization of Level-1 requires a feasible solution. This optimum is then differentiated with respect to the coupling parameters ${}^0_1\mathbf{h}$ and ${}^1_0\mathbf{h}$ via a technique called Optimum Sensitivity Analysis. A sensitivity analysis of the Level-1 element is carried out and the effect of changes in the physical coupling (${}^0_1\mathbf{h}$ and ${}^1_0\mathbf{h}$) and/or shared design variables (${}^0_1\mathbf{z}$) is approximated. This sensitivity analysis procedure was introduced by Barthelemy and Sobieszczanski-Sobieski (1983) who named it Optimum Sensitivity Analysis. This method has been introduced in Chapter 2 and for the current multi-level problem the optimum sensitivity derivatives are found as follows.

Stationary conditions (defined in Section 2.2) for Equation 5.1 read:

$$
\begin{array}{lcll}
\nabla_{{}^1\mathbf{x}}\mathcal{L} & = & \nabla_{{}^1\mathbf{x}}{}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) + \ldots & \\
 & & {}^1_0\boldsymbol{\lambda}_c^T \nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_{{}^1_0c}({}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h}), {}^1_0\mathbf{h}) + \ldots & \\
 & & {}^1\boldsymbol{\lambda}_g^T \nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_g({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) + \ldots & \\
 & & {}^1\boldsymbol{\lambda}_h^T \nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_h({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) & = \mathbf{0} \\
\nabla_{{}^1_0\boldsymbol{\lambda}_c}\mathcal{L} & = & {}^1\mathbf{v}_{{}^1_0c}({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) & = \mathbf{0} \\
\nabla_{{}^1\boldsymbol{\lambda}_g}\mathcal{L} & = & {}^1\mathbf{v}_g({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) - \boldsymbol{\mu}^2 & = \mathbf{0} \\
\nabla_{{}^1\boldsymbol{\lambda}_h}\mathcal{L} & = & {}^1\mathbf{v}_h({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) & = \mathbf{0} \\
\nabla_{\boldsymbol{\mu}}\mathcal{L} & = & -2{}^1\boldsymbol{\lambda}_g^T \boldsymbol{\mu} & = \mathbf{0}
\end{array}, \qquad (5.2)
$$

where the inequality constraints ${}^1\mathbf{v}_g$ are transformed into equality constraints via addition of slack variables $\boldsymbol{\mu}$.

These stationary conditions depend on the parameters that are put into the optimization problem. In other words, ${}^1\mathbf{x} = {}^1\mathbf{x}({}^0_1\mathbf{h}, {}^1_0\mathbf{h})$, ${}^1v_f = {}^1v_f({}^1\mathbf{x}({}^0_1\mathbf{h}, {}^1_0\mathbf{h}), {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h}, {}^1_0\mathbf{h}), {}^0_1\mathbf{h}, {}^1_0\mathbf{h})$ and ${}^1\mathbf{v}_g(..)$, ${}^1\mathbf{v}_h(..)$, ${}^1\mathbf{v}_c(..)$, $\boldsymbol{\mu}$ similarly. Sensitivity of these stationary conditions is obtained via applying the chain rule ($\mathbf{D}_{{}^i_j\mathbf{h}}(\ldots) = \nabla_{{}^i_j\mathbf{h}}(\ldots) + \nabla_{{}^1\mathbf{x}}(\ldots)^T \mathbf{D}_{{}^i_j\mathbf{h}}({}^1\mathbf{x})$) for $i, j = 0, 1$ and $i \neq j$ to Equation 5.2, where the total derivatives are defined as $\mathbf{D}_{{}^i_j\mathbf{h}}.. = [\frac{\mathrm{d}..}{\mathrm{d}{}^i_jh_1}, \ldots, \frac{\mathrm{d}..}{\mathrm{d}{}^i_jh_n}]$. This results in an expression for the sensitivity of the stationary point, which yields in matrix notation:

$$
\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{0} & \mathbf{C}^T & \mathbf{B}^T & \mathbf{0} & \mathbf{C}^T \\ \mathbf{0} & \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}^0\mathbf{y} \\ {}^1\mathbf{y} \end{bmatrix} + \begin{bmatrix} {}^0\mathbf{w}_1 \\ {}^0\mathbf{w}_2 \end{bmatrix} + \begin{bmatrix} {}^1\mathbf{w}_1 \\ {}^1\mathbf{w}_2 \end{bmatrix} = \mathbf{0} \qquad (5.3)
$$

where:

$$
\begin{array}{rcl}
\mathbf{A} & = & \left[\ \nabla^2_{{}^1\mathbf{x},{}^1\mathbf{x}}{}^1v_f + {}^1_0\boldsymbol{\lambda}_c^T \nabla^2_{{}^1\mathbf{x},{}^1\mathbf{x}}{}^1\mathbf{v}_{{}^1_0c} + {}^1\boldsymbol{\lambda}_g^T \nabla^2_{{}^1\mathbf{x},{}^1\mathbf{x}}{}^1\mathbf{v}_g + {}^1\boldsymbol{\lambda}_h^T \nabla^2_{{}^1\mathbf{x},{}^1\mathbf{x}}{}^1\mathbf{v}_h\ \right]; \\
\mathbf{B} & = & \left[\ \nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_{{}^1_0c} \quad \nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_g \quad \nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_h\ \right]; \\
\mathbf{C} & = & \left[\ \mathbf{0} \quad -\mathbf{I} \quad \mathbf{0}\ \right]; \\
{}^i\mathbf{y} & = & \left[\ \mathbf{D}_{{}^i_j\mathbf{h}}({}^1\mathbf{x}) \quad \mathbf{D}_{{}^i_j\mathbf{h}}({}^1_0\boldsymbol{\lambda}_c^T) \quad \mathbf{D}_{{}^i_j\mathbf{h}}({}^1\boldsymbol{\lambda}_g^T) \quad \mathbf{D}_{{}^i_j\mathbf{h}}({}^1\boldsymbol{\lambda}_h^T) \quad \mathbf{D}_{{}^i_j\mathbf{h}}(\boldsymbol{\mu})\ \right]^T; \\
{}^i\mathbf{w}_1 & = & \left[\ \nabla_{{}^i_j\mathbf{h}}\nabla_{{}^1\mathbf{x}}{}^1v_f + {}^1_0\boldsymbol{\lambda}_c^T \nabla_{{}^i_j\mathbf{h}}\nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_{{}^1_0c} + {}^1\boldsymbol{\lambda}_g^T \nabla_{{}^i_j\mathbf{h}}\nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_g + {}^1\boldsymbol{\lambda}_h^T \nabla_{{}^i_j\mathbf{h}}\nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_h\ \right]; \\
{}^i\mathbf{w}_2 & = & \left[\ \nabla_{{}^i_j\mathbf{h}}{}^1\mathbf{v}_{{}^1_0c} \quad \nabla_{{}^i_j\mathbf{h}}{}^1\mathbf{v}_g \quad \nabla_{{}^i_j\mathbf{h}}{}^1\mathbf{v}_h \quad \mathbf{0}\ \right]^T,
\end{array}
$$

where $i, j = 0, 1$ and $i \neq j$.

At the constraint optimal point Equation 5.3 is evaluated to find the unknown derivatives of the optimum solution $\mathbf{D}_{{}^i_j\mathbf{h}}({}^1\mathbf{x})$ and $\mathbf{D}_{{}^i_j\mathbf{h}}({}^1_0\boldsymbol{\lambda}_c^T)$, $\mathbf{D}_{{}^i_j\mathbf{h}}({}^1\boldsymbol{\lambda}_g^T)$, $\mathbf{D}_{{}^i_j\mathbf{h}}({}^1\boldsymbol{\lambda}_h^T)$, $\mathbf{D}_{{}^i_j\mathbf{h}}(\boldsymbol{\mu}^T)$. Once $\mathbf{D}_{{}^i_j\mathbf{h}}({}^1\mathbf{x})$ are found, the optimum sensitivity derivative of the objective function ${}^1v_f$ and/or constraint functions ${}^1\mathbf{v}_g$, ${}^1\mathbf{v}_h$ can be computed. The total derivatives of, e.g., ${}^1\mathbf{v}_g$ are obtained via:

$$
\mathbf{D}_{{}^i_j\mathbf{h}}({}^1\mathbf{v}_g) \quad = \quad \nabla_{{}^i_j\mathbf{h}}{}^1\mathbf{v}_g + \nabla_{{}^1\mathbf{x}}{}^1\mathbf{v}_g^T \mathbf{D}_{{}^i_j\mathbf{h}}({}^1\mathbf{x}) \qquad (5.4)
$$

The total derivatives are used to linearize the Level-1 constraints with respect to the coupling variables (${}^0_1\mathbf{h}$ and ${}^1_0\mathbf{h}$) and the Level-1 extrapolated optimum is added to the Level-0 optimization problem to yield:

$$
\begin{array}{lllll}
\min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h}} & {}^0v_f\left({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}\right) & & & \\
\text{s.t.} & {}^0\mathbf{v}_g({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \leq & \mathbf{0} & \\
& {}^0\mathbf{v}_h({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & \mathbf{0} & \\
& {}^0\mathbf{v}_{{}^1a_g}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & {}^1\mathbf{v}_g+\mathbf{D}_{{}^1_0\mathbf{h}}\left({}^1\mathbf{v}_g\right)\Delta{}^1_0\mathbf{h}+\mathbf{D}_{{}^0_1\mathbf{h}}({}^1\mathbf{v}_g)\left({}^0_1\mathcal{H}({}^0\mathbf{r})-{}^0_1\mathbf{h}\right)\leq\mathbf{0} & \\
& {}^0\mathbf{v}_{{}^1a_h}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & {}^1\mathbf{v}_h+\mathbf{D}_{{}^1_0\mathbf{h}}\left({}^1\mathbf{v}_h\right)\Delta{}^1_0\mathbf{h}+\mathbf{D}_{{}^0_1\mathbf{h}}({}^1\mathbf{v}_h)\left({}^0_1\mathcal{H}({}^0\mathbf{r})-{}^0_1\mathbf{h}\right)=\mathbf{0} & , \\
& {}^0\mathbf{v}_{{}^1a_{\underline{\mathbf{x}}}}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & {}^1\underline{\mathbf{x}}-{}^1\mathbf{x}-\mathbf{D}_{{}^1_0\mathbf{h}}({}^1\mathbf{x})\Delta{}^1_0\mathbf{h}-\mathbf{D}_{{}^0_1\mathbf{h}}({}^1\mathbf{x})\left({}^0_1\mathcal{H}({}^0\mathbf{r})-{}^0_1\mathbf{h}\right)\leq\mathbf{0} & \\
& {}^0\mathbf{v}_{{}^1a_{\overline{\mathbf{x}}}}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & -{}^1\overline{\mathbf{x}}+{}^1\mathbf{x}+\mathbf{D}_{{}^1_0\mathbf{h}}({}^1\mathbf{x})\Delta{}^1_0\mathbf{h}+\mathbf{D}_{{}^0_1\mathbf{h}}({}^1\mathbf{x})\left({}^0_1\mathcal{H}({}^0\mathbf{r})-{}^0_1\mathbf{h}\right)\leq\mathbf{0} & \\
& {}^0\underline{\mathbf{x}}\leq{}^0\mathbf{x}\leq{}^0\overline{\mathbf{x}} & & &
\end{array}
\tag{5.5}
$$

where ${}^0_1\mathbf{h}$ are values of the mapped responses that were used to find the constraint optimal point of the Level-1 element for which the optimum sensitivities have been computed. The linearized constraints added to Level-0 are relatively easier to solve then the actual constraints present at Level-1.

A significant amount of linearized constraints are added to the Level-0 optimization problem. In the present example a single element is considered at Level-1. In case multiple elements are present at Level-1 the amount of linearized constraints increases in the Level-0 optimization problem for every element that is coupled to the Level-0 optimization problem. In order to reduce the number of additional linearized constraints, Sobieszczanski-Sobieski (1992) proposed the use of an envelope function (recall Chapter 2) in which all the constraints of the Level-1 optimization problem are combined into a single function. In this case only a single envelope function needs to be extrapolated to the Level-0 optimization problem.

Due to linearizing the Level-1 constraints, the results of the Level-0 optimization are only accurate in the neighborhood of the initial coupling variables. Therefore, large changes in these values may require a re-optimization of the Level-1 elements and a new sensitivity analysis of these optima.

The coordination procedure of Optimization by Linear Decomposition is illustrated in Figure 5.2. There are two levels of optimization, Level-0 and Level-1 and the solution process also involves two levels. The first yields the individual optimization of the Level-0 and Level-1 problems and the second the sensitivity analysis. Here, the coefficients necessary for the linearized constraints are computed.

Linearized Multi-level Optimization

A different technique that linearizes the coupling between levels was introduced by Vanderplaats *et al.* (1990). The authors pointed out two drawbacks of Optimization by Linear Decomposition. The first being the fact that nonlinear equality constraints (${}^1\mathbf{v}_{{}^1_0c}({}^1_0\mathbf{c})$) are introduced as consistency constraints at Level-1. The second drawback arises when use is made of envelope functions (introduced in Chapter 2) and is twofold, potentially highly nonlinear cumulative constraints[1] are send to Level-0 and these

[1] A cumulative constraint is a constraint that combines a set of constraints into a single constraint called an envelope function. An example of such an envelope function is the KS-function.

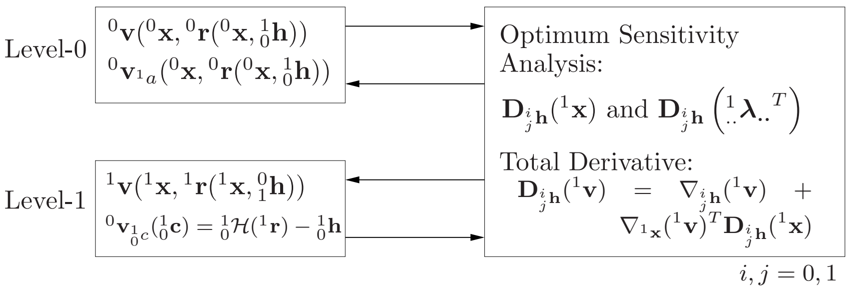


Figure 5.2: *Coordination via Optimum Sensitivity Analysis. A sensitivity of the Level-1 optimization is conducted to compute the total derivatives ($\mathbf{D}_{{}^i_j\mathbf{h}}({}^1\mathbf{v})$). These total derivatives are used to construct a linearization of the constraint equations with respect to the coupling variables and these linearized constraints are then added as constraints to the Level-0 optimization problem. The Lagrange multipliers ${}^1_{..}\boldsymbol{\lambda}_{..}$ are the multipliers associated with the constraints of the Level-1 optimization problem.*

cumulative constraints require optimum sensitivities. These optimum sensitivities can be discontinuous which requires special attention.

The basic idea of the method of Vanderplaats *et al.* is that individual design constraints are linearized with respect to coupling variables. These linearized constraints are added to neighboring elements to which the present element is coupled. The effect is that neighboring elements avoid designs that might result in an infeasible design of the present element to which the neighboring elements are coupled.

Similar to Optimization by Linear Decomposition, first the Level-1 optimization problem is discussed. The linearized Level-1 optimization problem yields:

$$\begin{array}{lll} \min\limits_{{}^1\mathbf{x}} & {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}) & \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x}, {}^1\mathbf{r}) & \leq \mathbf{0} \\ & {}^1\mathbf{v}_h({}^1\mathbf{x}, {}^1\mathbf{r}) & = \mathbf{0} \\ & {}^1\mathbf{v}_{{}^0a_g}({}^1\mathbf{x}, {}^1\mathbf{r}) & = {}^0\mathbf{v}_g + \mathbf{D}_{{}^1_0\mathbf{h}}\left({}^0\mathbf{v}_g\right)\left({}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right) \leq \mathbf{0} \\ & {}^1\mathbf{v}_{{}^0a_h}({}^1\mathbf{x}, {}^1\mathbf{r}) & = {}^0\mathbf{v}_h + \mathbf{D}_{{}^1_0\mathbf{h}}\left({}^0\mathbf{v}_h\right)\left({}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right) = \mathbf{0} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \end{array} . \tag{5.6}$$

Notice the difference with Equation 5.1, the current problem no longer has a constraint to remain consistency. The additional constraints ${}^1\mathbf{v}_{{}^0a_g}$ and ${}^1\mathbf{v}_{{}^0a_h}$ represent the linearized equality and inequality constraints of the neighboring element (Level-0). The Level-0 optimization problem with linearized constraints from Level-1 yields:

$$\begin{array}{lll} \min\limits_{{}^0\mathbf{x}} & {}^0v_f\left({}^0\mathbf{x}, {}^0\mathbf{r}\right) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^0\mathbf{x}, {}^0\mathbf{r}) & \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^0\mathbf{x}, {}^0\mathbf{r}) & = \mathbf{0} \\ & {}^0\mathbf{v}_{{}^1a_g}({}^0\mathbf{x}, {}^0\mathbf{r}) & = {}^1\mathbf{v}_g + \mathbf{D}_{{}^0_1\mathbf{h}}\left({}^1\mathbf{v}_g\right)\left({}^0_1\mathcal{H}({}^0\mathbf{r}) - {}^0_1\mathbf{h}\right) \leq \mathbf{0} \\ & {}^0\mathbf{v}_{{}^1a_h}({}^0\mathbf{x}, {}^0\mathbf{r}) & = {}^1\mathbf{v}_h + \mathbf{D}_{{}^0_1\mathbf{h}}\left({}^1\mathbf{v}_h\right)\left({}^0_1\mathcal{H}({}^0\mathbf{r}) - {}^0_1\mathbf{h}\right) = \mathbf{0} \\ & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & \end{array} . \tag{5.7}$$

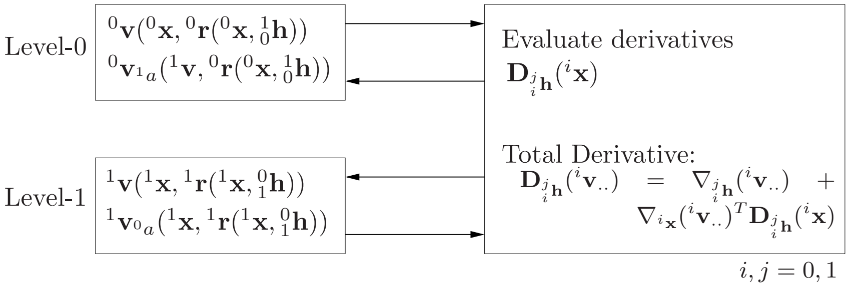


Figure 5.3: *Coordination by means of linearization of the coupling. The total derivatives ($\mathbf{D}_{^j_i\mathbf{h}}(^i\mathbf{v})$) are computed via the coordinator. These total derivatives are used to construct a linearization of the constraint equations with respect to the coupling variables. These linearized constraints are then added as constraints to the neighboring optimization problem.*

The advantage of linearizing the coupling via the above procedure is that the degree of nonlinearity is reduced with respect to the approach of Sobieski *et al.* (1985). It does not require optimum sensitivities and the Level-1 optimization problem has more design freedom as compared to Optimization by Linear Decomposition due to the absence of consistency constraints. Furthermore, it eliminates the need for nonlinear consistency constraints.

A disadvantage of linearizing the coupling is that the coupling between elements is not dealt with in a precise mathematical manner.

Figure 5.3 shows the main characteristics of the linearized multi-level optimization method. Both elements are solved independently and the derivatives necessary for the linearization of the coupling are evaluated by the coordinator block.

5.1.2 Hierarchic constraint margin approach

Optimization by Linear Decomposition

A special case in decomposition are the problems where the overall objective function is only a function of the coupling variables between elements and Level-0 design variables. Recall Figure 3.16 where the problem matrix for such an optimization problem is shown. Sobieszczanski-Sobieski *et al.* (1985) formulated Optimization by Linear Decomposition to solve such multi-level optimization problems.

After decomposition the Level-0 optimization problem yields:

$$\begin{array}{lll} \min\limits_{^0\mathbf{x}, ^1_0\mathbf{h}} & {^0v_f}\left({^1_0\mathbf{h}}, {^0\mathbf{x}}, {^0\mathbf{r}}\right) & \\ \text{s.t.} & {^0\mathbf{v}_g}({^1_0\mathbf{h}}, {^0\mathbf{x}}, {^0\mathbf{r}}) & \leq \mathbf{0} \\ & {^0\mathbf{v}_h}({^1_0\mathbf{h}}, {^0\mathbf{x}}, {^0\mathbf{r}}) & = \mathbf{0} \\ & {^0\underline{\mathbf{x}}} \leq {^0\mathbf{x}} \leq {^0\overline{\mathbf{x}}} & \end{array} \tag{5.8}$$

and the Level-1 optimization yields:

$$\begin{array}{lll} \min\limits_{^1\mathbf{x}} & & \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x},{}^1\mathbf{r}) & \leq \mathbf{0} \\ & {}^1\mathbf{v}_h({}^1\mathbf{x},{}^1\mathbf{r}) & = \mathbf{0} \\ & {}^1\mathbf{v}_{{}^1_0 c}({}^1_0\mathbf{c}({}^1\mathbf{r})) & = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} = \mathbf{0} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \end{array} . \tag{5.9}$$

The Level-1 optimization problem does not have an objective that can be minimized/maximized. Instead, the design constraints of the Level-1 elements become the objective of interest.

In order to combine the design constraints (${}^1\mathbf{v}_g$, ${}^1\mathbf{v}_h$) into a single objective function Sobieszczanski-Sobieski *et al.* (1985) used a KS-function. For Level-1 this KS-function yields:

$$ {}^1v_e({}^1\mathbf{v}_g, {}^1\mathbf{v}_{h^+}, {}^1\mathbf{v}_{h^-}) = \frac{1}{\rho} \ln \left[\sum e^{\rho\, {}^1v_{g_k}} + \sum e^{\rho\, {}^1v_{h_l^+}} + \sum e^{\rho\, {}^1v_{h_m^-}} \right]. \tag{5.10}$$

This function is called an envelope function and the properties of this function were discussed in Chapter 2. The envelope function combines the constraints into a single function. This function is added to the objective function of Level-1. The effect is that the Level-1 design point is pushed as far as possible into the feasible domain away from the constraints.

Incorporating the envelope function into the Optimization by Linear Decomposition formulation, the Level-0 optimization problem yields:

$$\begin{array}{llll} \min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h}} & {}^0v_f\,({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \leq \mathbf{0} & \\ & {}^0\mathbf{v}_h({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = \mathbf{0} & \\ & {}^0\mathbf{v}_{{}^1a_e}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & {}^1v_e + \mathbf{D}_{{}^1_0\mathbf{h}}\,({}^1v_e)\,\Delta^1_0\mathbf{h} + \mathbf{D}_{{}^0_1\mathbf{h}}({}^1v_e)\,({}^0_1\mathcal{H}({}^0\mathbf{r}) - {}^0_1\mathbf{h}) \leq \mathbf{0} \, , \\ & {}^0\mathbf{v}_{{}^1a_{\underline{\mathbf{x}}}}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & {}^1\underline{\mathbf{x}} - {}^1\mathbf{x} - \mathbf{D}_{{}^1_0\mathbf{h}}({}^1\mathbf{x})\Delta^1_0\mathbf{h} - \mathbf{D}_{{}^0_1\mathbf{h}}({}^1\mathbf{x})\,({}^0_1\mathcal{H}({}^0\mathbf{r}) - {}^0_1\mathbf{h}) \leq \mathbf{0} \\ & {}^0\mathbf{v}_{{}^1a_{\overline{\mathbf{x}}}}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = & -{}^1\overline{\mathbf{x}} + {}^1\mathbf{x} + \mathbf{D}_{{}^1_0\mathbf{h}}({}^1\mathbf{x})\Delta^1_0\mathbf{h} + \mathbf{D}_{{}^0_1\mathbf{h}}({}^1\mathbf{x})\,({}^0_1\mathcal{H}({}^0\mathbf{r}) - {}^0_1\mathbf{h}) \leq \mathbf{0} \\ & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & & \end{array} \tag{5.11}$$

which is similar to that of Equation 5.5. However, in the above problem formulation the constraints are clustered into one envelope function (${}^1v_e(\ldots)$) which is extrapolated. Necessary sensitivity information of the optimum is found via Optimum Sensitivity Analysis (recall Chapter 2 and Section 5.1.1).

The Level-1 optimization problem yields:

$$\begin{array}{lll} \min\limits_{{}^1\mathbf{x}} & {}^1\mathbf{v}_f({}^1\mathbf{x},{}^1\mathbf{r}) & = {}^1v_e({}^1\mathbf{v}_g({}^1\mathbf{x},{}^1\mathbf{r}), {}^1\mathbf{v}_{h^+}({}^1\mathbf{x},{}^1\mathbf{r}), {}^1\mathbf{v}_{h^-}({}^1\mathbf{x},{}^1\mathbf{r})) \\ \text{s.t.} & {}^1\mathbf{v}_{{}^1_0 c}({}^1_0\mathbf{c}({}^1\mathbf{r})) & = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} = \mathbf{0} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \end{array} , \tag{5.12}$$

with ${}^1v_e({}^1\mathbf{v}_{..}(..))$ an envelope function (cumulative constraint function) encompassing all the subproblem design constraints. Balling and Sobieszczanski-Sobieski (1995) proposed a number of different envelope functions that minimize all the constraints at once by means of minimizing just one envelope function.

Quasi-separable Subsystem Decomposition

Haftka and Watson (2005) introduced a different approach for which a mathematical justification of the method was derived for convex optimization problems. In that approach, named Quasi-separable Subsystem Decomposition, the authors consider the same decomposed multi-level optimization problem as Optimization by Linear Decomposition. However, instead of applying a cumulative constraint to minimize the constraint violation in the Level-1 elements, a slack variable is introduced. The modified Level-1 optimization problem yields:

$$\begin{array}{lll} \min\limits_{{}^1\mathbf{x},{}^1\boldsymbol{\mu}} & {}^1v_f & = -\sum\limits_{n=1}^{m} {}^1\mu_n \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x},{}^1\mathbf{r}) & \leq -{}^1\boldsymbol{\mu} \\ & {}^1\mathbf{v}_h({}^1\mathbf{x},{}^1\mathbf{r}) & = \mathbf{0} \\ & {}^1\mathbf{v}_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) & = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} = \mathbf{0} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \end{array} . \tag{5.13}$$

The additional design variable vector ${}^1\boldsymbol{\mu}$ (slack variables) measures the distance between the current value of the constraints and the position where these constraints become active. This margin is maximized in the Level-1 elements.

Equation 5.13 is solved a number of times for different fixed values of the mapped responses (${}^0_1\mathcal{H}({}^0\mathbf{r}) = {}^0_1\mathbf{h}$) and coupling variables (${}^1_0\mathbf{h}$). The values of mapped responses and coupling variables that are used to solve Equation 5.13 are chosen via a Latin Hypercube Sampling (see Chapter 2). Hence, for various settings of the coupling variables (${}^0_1\mathcal{H}({}^0\mathbf{r}) = {}^0_1\mathbf{h}$ and ${}^1_0\mathbf{h}$) the Level-1 optimizations are repeated. The Level-1 results (${}^1v_f$) are utilized to construct a Response Surface (${}^1v_{RSM}$) that is fitted through the optimal values ${}^1v_f$ found. The response function that is constructed approximates the design freedom (shift in optimum) of the Level-1 element with respect to changes in the coupling.

The Response Surface approximation is added to the Level-0 optimization problem as additional constraints that approximate the behavior of the vector ${}^1\boldsymbol{\mu}$ during the optimization of the Level-0 element. This technique resembles the extrapolation of constraints from Level-1 to Level-0 as was previously shown (see Section 5.1.1).

The Level-0 optimization problem now includes information on the Level-1 problems. The effect is similar to that of linearizing the constraints. Hence, the Level-0 optimization problem tries to avoid designs which are infeasible in the Level-1 problems. The Level-0 problem including the additional Response Surface constraint yields:

$$\begin{array}{lll} \min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h}} & {}^0v_f({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \\ \text{s.t.} & {}^0v_g({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \leq 0 \\ & {}^0v_h({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = 0 \\ & {}^0v_{{}^1a}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = {}^1v_{f_{RSM}}({}^1\boldsymbol{\mu}({}^0\mathbf{r},{}^1_0\mathbf{h})) \leq 0 \\ & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & \end{array} . \tag{5.14}$$

The Response Surface covers a larger part of the domain of the Level-1 optimization problems as opposed to linearizing the constraints. Linearization is only

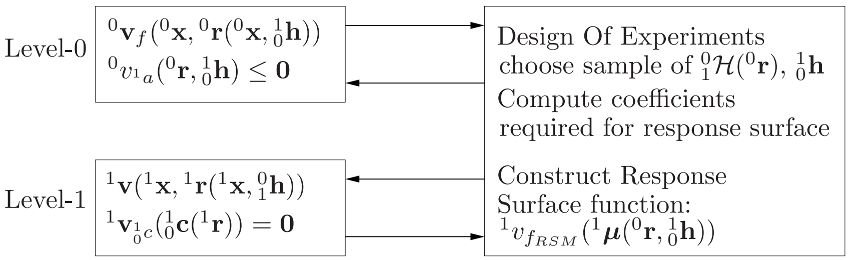


Figure 5.4: *Coordination by means of constructing a Response Surface of the constraint margins. The Level-1 optimization is repeated for a number of different coupling variables* (${}^0_1\mathcal{H}({}^0\mathbf{r})$, ${}^0_1\mathbf{h}$)*. The optima* (${}^1v_f$) *that are obtained for these different coupling variables are then fitted by means of a response surface method and added as additional constraints to the Level-*0 *optimization problem.*

valid in the neighborhood of the design point, i.e. for small changes in the consistency constraints. Therefore, the Response Surface approach is expected to require less cycling between the Level-1 and Level-0 optimization problems as compared to techniques that require optimum sensitivity information. Furthermore, the response surface technique is still applicable if the Level-1 optimum has discontinuous derivatives.

The Response Surface coordination method is illustrated in Figure 5.4, which is similar to Figure 5.2. The major difference is that the coordinator involves constructing a Response Surface of the Level-1 problems instead of performing a sensitivity analysis.

5.1.3 Non-hierarchic linearized coordination

The previous discussed coordination techniques solve hierarchic top-down and/or bottom-up decomposition schemes based on equality consistency constraints. A third coordination scheme that is suitable for non-hierarchic equality-based decomposition schemes has not yet been discussed.

A non-hierarchic decomposition scheme differs from a hierarchic scheme in the sense that all the individual elements are treated equally. The data of these individual elements is then send to the coordinator, which re-distributes the coupling data and sends back updates of this data to the elements.

Because all levels and/or elements of the hierarchy are treated equally, some sort of balancing between the individual elements is necessary. Various solution techniques for this balancing can be found in literature. The main stream procedures are discussed in this section.

Concurrent SubSpace Optimization

Concurrent SubSpace Optimization (CSSO) was first introduced by Sobieszczanski-Sobieski (1988) as an alternative to Optimization by Linear Decomposition (OLD).

OLD did not satisfy the observation that in some systems no clear hierarchy can be distinguished. When no hierarchy is present a non-hierarchic decomposition is applied (see Chapter 3). The decoupling is accomplished in two-directions as a result of coupling between elements on the same level or as a choice of the designer.

The non-hierarchic decomposition leads to decoupling in both directions and no additional consistency constraints between levels are required. Instead, the changes in the coupling are accounted for via sensitivities which are computed via the Global Sensitivity Equations (GSE) (see Chapter 3, Equation 3.5). A check on consistency is required after data between elements is exchanged.

In the following discussion a decomposition into two elements is considered. The individual optimization problem after non-hierarchic decomposition is for the Level-0 element:

$$\begin{array}{lll} \min\limits_{{}^1_0\mathbf{h},{}^0\mathbf{x}} & {}^0v_f({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = \mathbf{0} \\ & {}^0\mathbf{v}_{g_s}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = \mathbf{v}_{g_s}(\mathbf{x},\mathbf{r}) \leq \quad \mathbf{0} \\ & {}^0\mathbf{v}_{h_s}({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & = \mathbf{v}_{h_s}(\mathbf{x},\mathbf{r}) = \quad \mathbf{0} \\ & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & \end{array} \tag{5.15}$$

and for the Level-1 element:

$$\begin{array}{lll} \min\limits_{{}^0_1\mathbf{h},{}^1\mathbf{x}} & {}^1v_f({}^0_1\mathbf{h},{}^1\mathbf{x},{}^1\mathbf{r}) & \\ \text{s.t.} & {}^1\mathbf{v}_g({}^0_1\mathbf{h},{}^1\mathbf{x},{}^1\mathbf{r}) & \leq \mathbf{0} \\ & {}^1\mathbf{v}_h({}^0_1\mathbf{h},{}^1\mathbf{x},{}^1\mathbf{r}) & = \mathbf{0} \\ & {}^1\mathbf{v}_{g_s}({}^0_1\mathbf{h},{}^1\mathbf{x},{}^1\mathbf{r}) & = \mathbf{v}_{g_s}(\mathbf{x},\mathbf{r}) \leq \quad \mathbf{0} \\ & {}^1\mathbf{v}_{h_s}({}^0_1\mathbf{h},{}^1\mathbf{x},{}^1\mathbf{r}) & = \mathbf{v}_{h_s}(\mathbf{x},\mathbf{r}) = \quad \mathbf{0} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \end{array} , \tag{5.16}$$

where $\mathbf{x} = \left[{}^0\mathbf{x}, {}^1\mathbf{x}\right]$, ${}^0\mathbf{r}({}^0\mathbf{x}, {}^0_1\mathbf{h})$, ${}^1\mathbf{r}({}^1\mathbf{x}, {}^1_0\mathbf{h})$ and $\mathbf{v}_{g_s}$, $\mathbf{v}_{h_s}$ are constraints that are present in both elements due to the presence of both ${}^0\mathbf{x}$ and ${}^1\mathbf{x}$ in these constraint equations.

The possible problem that arises in the above formulation is that both levels are trying to satisfy the same constraints ($\mathbf{v}_{g_s}$, $\mathbf{v}_{h_s}$). Which may be successful in one element and fail in the other element. The elements combined might result in a feasible design. However, since one element did not converge to a feasible design the data from that optimization is questionable.

In order to allow for unfeasible designs (additional design freedom) in one element, another element has to "over-satisfy" those constraints that are not satisfied. This is accomplished via trade-off and responsibility factors. Hence, the responsibility of one element to reduce the violation of a (cumulative) constraint[2] is indicated by a weighting vector ${}^i_j\mathbf{s}_r$, where ^{i} represents the position in the hierarchy (see Chapter 2) and ${}_j$ the position of elements that share the constraint(s). These weights are defined such that the sum of all the weights equals one.

[2]Sobieszczanski-Sobieski applied cumulative constraints to CSSO so to reduce the number of constraints. However, without the use of cumulative constraints the discussion is still valid and the essential steps of the method are clearer.

Furthermore, the objective function may be reduced even further if (cumulative) constraints are allowed to be violated in one element, provided that they are over-satisfied in another element. Therefore, a weight-vector ${}^{i}_{j}\mathbf{s}_t$ is introduced under the condition that the sum of the weights per (cumulative) constraint equals zero.

Summarizing, the responsibility weights (${}^{i}_{j}\mathbf{s}_r$) are necessary in case of a violation of a constraint. The trade-off weights (${}^{i}_{j}\mathbf{s}_t$) are necessary in case constraints become critical. Therefore, only one of the weight-vectors is necessary at a time. A third weight-vector ${}^{i}_{j}\mathbf{s}_s$ is introduced to switch between the two cases. If a (cumulative) constraint is violated the weight ${}^{i}_{j}s_s$ is set to one (which activates ${}^{i}_{j}s_r$). Once a (cumulative) constraint becomes critical during the optimization, the weight ${}^{i}_{j}s_s$ is set to zero for that particular constraint (which activates ${}^{i}_{j}\mathbf{s}_t$). The weight ${}^{i}_{j}s_s$ stays at zero until the optimization terminates. Multiplying the natural constraints by a factor $\max\{{}^{i}v_{g_m}, 0\}$ or $\max\{{}^{i}v_{h_n}, 0\}$ (where $i = 0, 1$ and m or n the constraint that is considered), has the effect that constraints that are already satisfied are not considered in the trade-off and responsibility.

Incorporating the above reasoning into the individual element optimization problems transforms Equation 5.15 into:

$$
\begin{array}{lllll}
\min\limits_{{}^{1}_{0}\mathbf{h},{}^{0}\mathbf{x}} & {}^{0}v_f({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & & & \\
\text{s.t.} & {}^{0}\mathbf{v}_g({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & \leq & {}^{0}_{1}\mathbf{s}_s \circ (1 - {}^{0}_{1}\mathbf{s}_r) \circ \max\{{}^{0}\mathbf{v}_g, \mathbf{0}\} & + \quad (1 - {}^{0}_{1}\mathbf{s}_s) \circ {}^{0}_{1}\mathbf{s}_t \\
 & {}^{0}\mathbf{v}_h({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & = & {}^{0}_{1}\mathbf{s}_s \circ (1 - {}^{0}_{1}\mathbf{s}_r) \circ \max\{{}^{0}\mathbf{v}_h, \mathbf{0}\} & + \quad (1 - {}^{0}_{1}\mathbf{s}_s) \circ {}^{0}_{1}\mathbf{s}_t \\
 & {}^{0}\mathbf{v}_{g_s}({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & \leq & {}^{0}_{1}\mathbf{s}_s \circ (1 - {}^{0}_{1}\mathbf{s}_r) \circ \mathbf{v}_{g_s}(\mathbf{x}, \mathbf{r}) & + \quad (1 - {}^{0}_{1}\mathbf{s}_s) \circ {}^{0}_{1}\mathbf{s}_t \\
 & {}^{0}\mathbf{v}_{h_s}({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & = & {}^{0}_{1}\mathbf{s}_s \circ (1 - {}^{0}_{1}\mathbf{s}_r) \circ \mathbf{v}_{h_s}(\mathbf{x}, \mathbf{r}) & + \quad (1 - {}^{0}_{1}\mathbf{s}_s) \circ {}^{0}_{1}\mathbf{s}_t \\
 & {}^{0}\underline{\mathbf{x}} \leq {}^{0}\mathbf{x} \leq {}^{0}\overline{\mathbf{x}} & & &
\end{array}, \tag{5.17}
$$

and Equation 5.16 into:

$$
\begin{array}{lllll}
\min\limits_{{}^{0}_{1}\mathbf{h},{}^{1}\mathbf{x}} & {}^{1}v_f({}^{0}_{1}\mathbf{h}, {}^{1}\mathbf{x}, {}^{1}\mathbf{r}) & & & \\
\text{s.t.} & {}^{1}\mathbf{v}_g({}^{0}_{1}\mathbf{h}, {}^{1}\mathbf{x}, {}^{1}\mathbf{r}) & \leq & {}^{1}_{0}\mathbf{s}_s \circ (1 - {}^{1}_{0}\mathbf{s}_r) \circ \max\{{}^{1}\mathbf{v}_g, \mathbf{0}\} & + \quad (1 - {}^{1}_{0}\mathbf{s}_s) \circ {}^{1}_{0}\mathbf{s}_t \\
 & {}^{1}\mathbf{v}_h({}^{0}_{1}\mathbf{h}, {}^{1}\mathbf{x}, {}^{1}\mathbf{r}) & = & {}^{1}_{0}\mathbf{s}_s \circ (1 - {}^{1}_{0}\mathbf{s}_r) \circ \max\{{}^{1}\mathbf{v}_h, \mathbf{0}\} & + \quad (1 - {}^{1}_{0}\mathbf{s}_s) \circ {}^{1}_{0}\mathbf{s}_t \\
 & {}^{1}\mathbf{v}_{g_s}({}^{0}_{1}\mathbf{h}, {}^{1}\mathbf{x}, {}^{1}\mathbf{r}) & \leq & {}^{1}_{0}\mathbf{s}_s \circ (1 - {}^{1}_{0}\mathbf{s}_r) \circ \mathbf{v}_{g_s}(\mathbf{x}, \mathbf{r}) & + \quad (1 - {}^{1}_{0}\mathbf{s}_s) \circ {}^{1}_{0}\mathbf{s}_t \\
 & {}^{1}\mathbf{v}_{h_s}({}^{0}_{1}\mathbf{h}, {}^{1}\mathbf{x}, {}^{1}\mathbf{r}) & = & {}^{1}_{0}\mathbf{s}_s \circ (1 - {}^{1}_{0}\mathbf{s}_r) \circ \mathbf{v}_{h_s}(\mathbf{x}, \mathbf{r}) & + \quad (1 - {}^{1}_{0}\mathbf{s}_s) \circ {}^{1}_{0}\mathbf{s}_t \\
 & {}^{1}\underline{\mathbf{x}} \leq {}^{1}\mathbf{x} \leq {}^{1}\overline{\mathbf{x}} & & &
\end{array}, \tag{5.18}
$$

where $\max\{{}^{1}\mathbf{v}_{..}, \mathbf{0}\}$ means that constraints that are already satisfied are no longer considered in the trade-off and responsibility. Hence, they are removed from the individual element optimization problem that satisfies the shared constraint.

The two levels are solved in parallel and a coordinator is necessary to determine which element requires additional design freedom via the trade-off and responsibility factors. The coordinator determines the new trade-off and responsibility factors for each element by means of solving a minimization problem. The constraint minimum of both levels ($v_f = f({}^{0}v_f, {}^{1}v_f)$), obtained from updating all the parameters from each element optimization is a function of the weights ${}^{i}_{j}\mathbf{s}_t$ and ${}^{i}_{j}\mathbf{s}_r$. Using gradient information of the objective and (cumulative) constraint functions, a linear approximation of v_f is obtained. This linear approximation is minimized during the coordinator step.

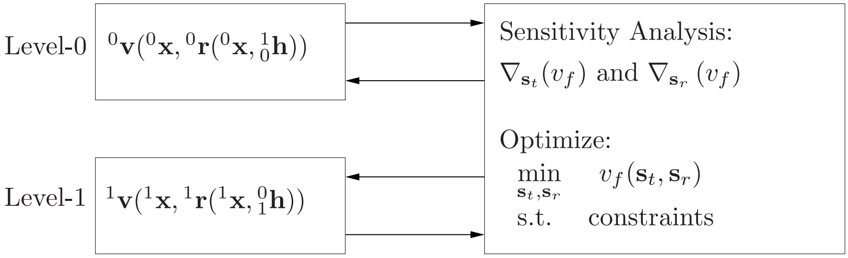


Figure 5.5: *Coordination by means of solving a linear programming problem for the trade-off and responsibility factors is a two step process. In the first step the Level-1 optimization and Level-0 optimization are perturbed in order to obtain sensitivity information with respect to the trade-off and responsibility factors. The second step is that an optimization problem is solved, which delivers new trade-off and responsibility factors for both levels.*

Move limits on the weights ${}^{i}_{j}\mathbf{s}_t$ and ${}^{i}_{j}\mathbf{s}_r$ are necessary to prevent large changes in these coefficients caused by nonlinearity of the initial problem.

Incorporating the above reasoning into the coordinator problem leads to a linear programming problem which yields:

$$
\begin{array}{ll}
\min\limits_{\mathbf{s}_t,\mathbf{s}_r} & v_f + \nabla_{\mathbf{s}_t} v_f \cdot \Delta\mathbf{s}_t + \nabla_{\mathbf{s}_r} v_f \cdot \Delta\mathbf{s}_r \\
\text{s.t.} & {}^{0}_{1}\mathbf{s}_t + {}^{1}_{0}\mathbf{s}_t - \mathbf{I} = \mathbf{0} \\
& {}^{0}_{1}\mathbf{s}_r + {}^{1}_{0}\mathbf{s}_r = \mathbf{0} \\
& \mathbf{0} \quad \leq \quad [{}^{0}_{1}\mathbf{s}_t, {}^{1}_{0}\mathbf{s}_t] \quad \leq \quad \mathbf{I} \\
& [{}^{0}_{1}\underline{\mathbf{s}}_t, {}^{1}_{0}\underline{\mathbf{s}}_t]^T \quad \leq \quad [{}^{0}_{1}\mathbf{s}_t, {}^{1}_{0}\mathbf{s}_t]^T \quad \leq \quad [{}^{0}_{1}\overline{\mathbf{s}}_t, {}^{1}_{0}\overline{\mathbf{s}}_t]^T \\
& [{}^{0}_{1}\underline{\mathbf{s}}_r, {}^{1}_{0}\underline{\mathbf{s}}_r]^T \quad \leq \quad [{}^{0}_{1}\mathbf{s}_r, {}^{1}_{0}\mathbf{s}_r]^T \quad \leq \quad [{}^{0}_{1}\overline{\mathbf{s}}_r, {}^{1}_{0}\overline{\mathbf{s}}_r]^T
\end{array} . \tag{5.19}
$$

The coordination procedure is shown in Figure 5.5 for a two level hierarchy consisting of two individual elements.

Shankar *et al.* (1993) and de Wit and van Keulen (2007) conducted a numerical study of the CSSO method. A few drawbacks of the method were observed. Constraint switching was pointed out as having a negative impact on overall algorithm performance. This occurs when constraints that are shared among elements are switched on by the responsibility factors. Furthermore, it was observed that the coordinator problem keeps sending new trade-off and responsibility factors even after a feasible optimum is found. Later designs might even become infeasible, however the coordinator is not aware of this. Furthermore, the number of design variables grows with the number of elements due to the weight vectors included in each element. This generates a tremendous amount of data to be send to the coordinator in addition to the physical coupling and optimization data.

Concurrent SubSpace Optimization has been further developed to include higher-order sensitivity information (Renaud and Gabriele, 1993, 1994), dealing with discrete design variables (Sellar *et al.*, 1994), Response Surface approximations instead of a linear programming problem coordinating the individual elements (Sellar *et al.*, 1996) and the inclusion of Augmented Lagrangian Relaxation of the consistency constraints

(Rodríguez *et al.*, 1998). In the latter approach consistency constraints between the elements are introduced and these consistency constraints are relaxed via an Augmented Lagrangian method.

Bi-Level Integrated System Synthesis

Sobieszczanski-Sobieski *et al.* (1998) proposed a method to overcome the constraint switching that occurs in the elements when Concurrent SubSpace Optimization is conducted. Instead of introducing trade-off and responsibility factors, a linearization of optimization functions is constructed.

Sobieszczanski-Sobieski *et al.* consider an objective function present at Level-0 that depends on design variables (${}^{0}\mathbf{x}$), physical responses (${}^{0}_{..}\mathcal{H}({}^{0}\mathbf{r}) = {}^{0}_{..}\mathbf{h}$) and/or shared design variables (${}^{0}_{..}\mathbf{z}$). For simplicity, shared design variables are not considered in the present derivation. Shared design variables are treated similar as physical responses. Furthermore, a two-level system is considered with a single element per level. Finally, in the current derivation the method of Sobieszczanski-Sobieski *et al.* is generalized to arbitrary optimization functions assigned to every element present in the hierarchy.

An optimization function shared over elements in the hierarchy yields:

$$v_{..s} \quad = \quad v_{..s}\left(\mathbf{x}, \mathbf{r}\right). \tag{5.20}$$

And a step that increases or decreases this function yields:

$$v_{..s}^{(t+1)} \quad = \quad v_{..s}^{(t)} + \Delta v_{..s}. \tag{5.21}$$

An expression for $\Delta v_{..s}$ is found via a linear Taylor series expansion of Equation 5.20 at the current point $\mathbf{x}^{*}, \mathbf{r}^{*}$:

$$v_{..s}(\mathbf{x}, \mathbf{r}) \quad = \quad v_{..s}(\mathbf{x}^{*}, \mathbf{r}^{*}) + \mathbf{D}_{{}^{0}\mathbf{x}}\left(v_{..s}\right)\Delta^{0}\mathbf{x} + \mathbf{D}_{{}^{1}\mathbf{x}}\left(v_{..s}\right)\Delta^{1}\mathbf{x}. \tag{5.22}$$

The total derivatives $\mathbf{D}_{{}^{0}\mathbf{x}}\left(v_{..}\right)$ and $\mathbf{D}_{{}^{1}\mathbf{x}}\left(v_{..}\right)$ are found computing the Global Sensetivity Equations (GSE), see Chapter 3 (Equation 3.4).

An increase or decrease of the optimization function yields:

$$\Delta v_{..s} \quad = \quad \mathbf{D}_{{}^{0}\mathbf{x}}\left(v_{..s}\right)\Delta^{0}\mathbf{x} + \mathbf{D}_{{}^{1}\mathbf{x}}\left(v_{..s}\right)\Delta^{1}\mathbf{x}. \tag{5.23}$$

This expression shows contributions from Element-0 at Level-0 and contributions from Element-1 at Level-1. Therefore, Equation 5.23 can be seen as a composite function with contributions from both elements. These contributions may be computed separately and therefore can be added to the individual element optimization problems.

Each individual element contributes to the optimization via:

$$\begin{array}{lllll}
\min\limits_{{}^{i}\mathbf{x}} & {}^{i}v_{f}\left({}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) + v_{f_s}\left(\mathbf{x}^{*}, \mathbf{r}^{*}\right) + \mathbf{D}_{{}^{i}\mathbf{x}}\left(v_{f_s}\right)\Delta^{i}\mathbf{x} & & \\
\textbf{s.t.} & {}^{i}\mathbf{v}_{g_s}\left({}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & = & \mathbf{v}_{g_s}\left(\mathbf{x}^{*}, \mathbf{r}^{*}\right) + \mathbf{D}_{{}^{i}\mathbf{x}}\left(\mathbf{v}_{g_s}\right)\Delta^{i}\mathbf{x} \leq \mathbf{0} \\
& {}^{i}\mathbf{v}_{h_s}\left({}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & = & \mathbf{v}_{h_s}\left(\mathbf{x}^{*}, \mathbf{r}^{*}\right) + \mathbf{D}_{{}^{i}\mathbf{x}}\left(\mathbf{v}_{h_s}\right)\Delta^{i}\mathbf{x} = \mathbf{0} \\
& {}^{i}\mathbf{v}_{g}\left({}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & \leq & \mathbf{0} \\
& {}^{i}\mathbf{v}_{h}\left({}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & = & \mathbf{0} \\
& & & {}^{i}\underline{\mathbf{x}} \leq {}^{i}\mathbf{x} \leq {}^{i}\overline{\mathbf{x}}
\end{array} \tag{5.24}$$

Individual optimization functions $({}^{i}v_f, {}^{i}\mathbf{v}_g, {}^{i}\mathbf{v}_h)$ are present and optimization functions that are shared between elements $(v_{f_s}, \mathbf{v}_{g_s}, \mathbf{v}_{h_s})$ are added to the individual optimization problems. Coupling data, ${}^{0}_{1}\mathbf{h}, {}^{1}_{0}\mathbf{h}$, is kept fixed during the individual element optimizations.

Coupling data is transferred via a coordinator method. Sobieszczanski-Sobieski *et al.* propose different methods to construct the sensitivity of the shared functions with respect to changes in the coupling variables. In the present thesis the coordinator method conducts an Optimum Sensitivity Analysis (OSA), see Chapter 2, of the individual elements with respect to changes in the coupling data $({}^{0}_{1}\mathbf{h}, {}^{1}_{0}\mathbf{h})$. The coordinator problem can improve the optimization functions $(v_{f_s}, \mathbf{v}_{g_s}, \mathbf{v}_{h_s})$ via an optimization problem that considers the coupling variables as design variables. The coordinator problem yields:

$$\begin{array}{lll} \min\limits_{{}^{i}_{j}\mathbf{h}} & v_{f_s}(\mathbf{x}^*, \mathbf{r}^*) + \mathbf{D}_{{}^{i}_{j}\mathbf{h}}(v_{f_s})\, \Delta^{i}_{j}\mathbf{h} & \\ \textbf{s.t.} & \mathbf{v}_{g_s}(\mathbf{x}^*, \mathbf{r}^*) + \mathbf{D}_{{}^{i}_{j}\mathbf{h}}(\mathbf{v}_{g_s})\, \Delta^{i}_{j}\mathbf{h} & \leq \mathbf{0} \\ & \mathbf{v}_{h_s}(\mathbf{x}^*, \mathbf{r}^*) + \mathbf{D}_{{}^{i}_{j}\mathbf{h}}(\mathbf{v}_{h_s})\, \Delta^{i}_{j}\mathbf{h} & = \mathbf{0} \\ & {}^{i}_{j}\underline{\mathbf{h}} \leq {}^{i}_{j}\mathbf{h} \leq {}^{i}_{j}\overline{\mathbf{h}} & \end{array} \tag{5.25}$$

where the total derivatives are computed via the OSA. This method is called Bi-Level Integrated System Synthesis (BLISS).

Modified Bi-Level Integrated System Synthesis

A more convenient means of distributing the optimization problem resulted in a redefinition of the method by Sobieszczanski-Sobieski *et al.* (2003). The modified BLISS method distributes optimization of the multi-level problem as follows.

An optimization function shared over elements in the hierarchy yields:

$$v_{..s} = v_{..s}(\mathbf{x}, \mathbf{r}). \tag{5.26}$$

This function depends on design variables and physical responses from individual elements. The physical responses $\mathbf{r}$ are a function of design variables $\mathbf{x}$ and mapped physical responses ${}^{i}_{j}\mathcal{H}({}^{i}\mathbf{r}) = {}^{i}_{j}\mathbf{h}$. Optimization of the shared functions is done first at the individual elements. Therefore each individual element optimizes the shared function with respect to design variables present in the element.

The individual element optimizations yield:
Level-0:

$$\begin{array}{llll} \min\limits_{{}^{0}\mathbf{x}} & {}^{0}v_f({}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & = & {}^{0}v_f({}^{0}\mathbf{x}, {}^{0}\mathbf{r}) + {}^{0}v_{f_s}({}^{0}\mathbf{x}, {}^{0}\mathbf{r}) \\ \text{s.t.} & {}^{0}\mathbf{v}_{g_s}\left({}^{0}\mathbf{x}, {}^{0}\mathbf{r}\right) & = & \mathbf{v}_{g_s}(\mathbf{x}, \mathbf{r}) \leq \mathbf{0} \\ & {}^{0}\mathbf{v}_{h_s}\left({}^{0}\mathbf{x}, {}^{0}\mathbf{r}\right) & = & \mathbf{v}_{h_s}(\mathbf{x}, \mathbf{r}) = \mathbf{0} \\ & {}^{0}\mathbf{v}_g({}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & \leq & \mathbf{0} \\ & {}^{0}\mathbf{v}_h({}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & = & \mathbf{0} \\ & {}^{0}\underline{\mathbf{x}} \leq {}^{0}\mathbf{x} \leq {}^{0}\overline{\mathbf{x}} & & \\ \text{where} & {}^{0}v_{f_s}({}^{0}\mathbf{x}, {}^{0}\mathbf{r}) & = & {}^{0}_{1}\mathbf{s}^T\, {}^{0}_{1}\mathcal{H}({}^{0}\mathbf{r}) \end{array} . \tag{5.27}$$

Level-1:

$$
\begin{array}{lllll}
\min\limits_{{}^1\mathbf{x}} & {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}) & = & {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}) + {}^1v_{f_s}({}^1\mathbf{x}, {}^1\mathbf{r}) & \\
\text{s.t.} & {}^1\mathbf{v}_{g_s}\left({}^1\mathbf{x}, {}^1\mathbf{r}\right) & = & \mathbf{v}_{g_s}(\mathbf{x}, \mathbf{r}) \leq \mathbf{0} & \\
 & {}^1\mathbf{v}_{h_s}\left({}^1\mathbf{x}, {}^1\mathbf{r}\right) & = & \mathbf{v}_{h_s}(\mathbf{x}, \mathbf{r}) = \mathbf{0} & \\
 & {}^1\mathbf{v}_g({}^1\mathbf{x}, {}^1\mathbf{r}) & \leq & \mathbf{0} & . \\
 & {}^1\mathbf{v}_h({}^1\mathbf{x}, {}^1\mathbf{r}) & = & \mathbf{0} & \\
 & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & & & \\
\text{where} & {}^1v_{f_s}({}^1\mathbf{x}, {}^1\mathbf{r}) & = & {}^1_0\mathbf{s}^T\, {}^1_0\mathcal{H}({}^1\mathbf{r}) &
\end{array}
\tag{5.28}
$$

Instead of directly mapping the physical responses onto the neighboring element that depends on these responses (${}^0_1\mathcal{H}({}^0\mathbf{r}), {}^1_0\mathcal{H}({}^1\mathbf{r})$), the mapped responses are send to the coordinator. Additional coefficients (${}^0_1\mathbf{s}, {}^1_0\mathbf{s}$) are introduced that are set via the coordinator and are kept fixed during the individual element optimizations. These coefficients can be positive or negative depending on whether a certain mapped physical response needs to be increased or decreased.

Individual elements depend on physical responses from neighboring elements. Therefore, different physical responses of the individual elements can be found if the mapped responses coming from neighboring elements are changed. To study the input-output relation of an individual element, a Design of Experiments (DoE) can be conducted. This DoE explores the change in physical responses in one element with respect to changes in the mapped responses coming from neighboring elements.

Each individual component of the vector of mapped responses ${}^i_j\mathcal{H}({}^i\mathbf{r}) = {}^i_j\mathbf{h}$ is approximated via a response surface. This response surface is constructed via, e.g., a Least Squares method (see Chapter 2). A vector of coefficients ($\mathbf{a}$) is determined that can be used to construct for each individual component of ${}^i_j\mathcal{H}({}^i\mathbf{r}) = {}^i_j\mathbf{h}$ a polynomial function ${}^iv_{f_{RSM}}$. These polynomial functions approximate the behavior of the mapped physical responses of the individual element onto neighboring elements.

The shared design function, $v_{.._s}$, depends on mapped physical responses (${}^i_j\mathbf{h}$) from the individual elements. These responses are approximated via the response surfaces (${}^iv_{f_{RSM}}$) that are constructed for the mapped physical responses of each individual element. The coordinator optimizes the shared optimization functions via changing coupling variables (mapped physical responses) that are assigned as design variables. Consistency constraints ($\mathbf{v}_{{}^0_1c}$, $\mathbf{v}_{{}^1_0c}$) are added that require that the approximated mapped responses (${}^iv_{f_{RSM}}$) are equal to expected coupling variables (${}^i_j\mathbf{h}$).

The coordinator problem yields:

$$
\begin{array}{llll}
\min\limits_{{}^i_j\mathbf{h}, {}^i_j\mathbf{s}} & v_{f_s}\left({}^0\mathbf{v}_{f_{RSM}}({}^1_0\mathbf{h}, {}^1_0\mathbf{s}), {}^1\mathbf{v}_{f_{RSM}}({}^0_1\mathbf{h}, {}^0_1\mathbf{s})\right) & & \\
\text{s.t.} & \mathbf{v}_{g_s}\left({}^0\mathbf{v}_{f_{RSM}}({}^1_0\mathbf{h}, {}^1_0\mathbf{s}), {}^1\mathbf{v}_{f_{RSM}}({}^0_1\mathbf{h}, {}^0_1\mathbf{s})\right) & \leq & \mathbf{0} \\
 & \mathbf{v}_{h_s}\left({}^0\mathbf{v}_{f_{RSM}}({}^1_0\mathbf{h}, {}^1_0\mathbf{s}), {}^1\mathbf{v}_{f_{RSM}}({}^0_1\mathbf{h}, {}^0_1\mathbf{s})\right) & = & \mathbf{0} \\
 & \mathbf{v}_{{}^0_1c}({}^0_1\mathbf{h}, {}^1_0\mathbf{h}, {}^1_0\mathbf{s}) = {}^0\mathbf{v}_{f_{RSM}}({}^1_0\mathbf{h}, {}^1_0\mathbf{s}) - {}^0_1\mathbf{h} & = & \mathbf{0} \; . \\
 & \mathbf{v}_{{}^1_0c}({}^0_1\mathbf{h}, {}^1_0\mathbf{h}, {}^0_1\mathbf{s}) = {}^1\mathbf{v}_{f_{RSM}}({}^0_1\mathbf{h}, {}^0_1\mathbf{s}) - {}^1_0\mathbf{h} & = & \mathbf{0} \\
 & {}^i_j\underline{\mathbf{h}} \leq {}^i_j\mathbf{h} \leq {}^i_j\overline{\mathbf{h}} & & \\
 & {}^i_j\underline{\mathbf{s}} \leq {}^i_j\mathbf{s} \leq {}^i_j\overline{\mathbf{s}} & &
\end{array}
\tag{5.29}
$$

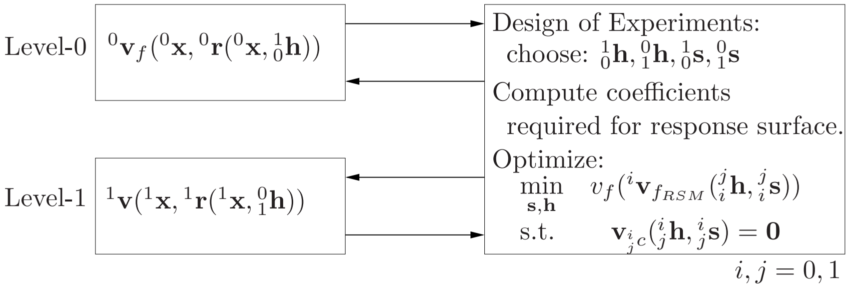


Figure 5.6: *First, a response surface is constructed for the output of the two levels. Then, a Coordination via a linear programming problem for the vectors* $\mathbf{s}$ *and coupling variables* $\mathbf{h}$ *is conducted.*

The vectors ${}^i_j\mathbf{s}$ determine whether components of these vectors require an increase or decrease during the individual element optimizations.

The entire coordination process is shown in Figure 5.6. Two individual element optimizations are conducted for a number of different mapped physical responses and coefficient vectors $\mathbf{s}$. After enough experiments have been conducted a fit through each of the results is computed and the resulting fitted functions are used to minimize the approximated objective function that models the performance of the entire hierarchy.

5.1.4 Design of a two-bar truss structure

In Chapter 3 the decomposition process was applied to a two-bar truss optimization problem. In this section results for equality-based coordination techniques applied to a multi-level two-bar truss optimization are summarized. The equality-based methods tested are:

- Hierarchic linearized coordination: Optimization by Linear Decomposition (OLD),
- Hierarchic constraint margin approach: Quasi-separable Subsystem Decomposition (QSD),
- Non-hierarchic linearized coordination: Concurrent SubSpace Optimization (CSSO) and modified Bi-Level Integrated System Synthesis (BLISS).

The multi-level optimization was started from six different initial designs and the reference optimum was found solving the All-in-One optimization problem. The total number of individual element function calls were counted for one single run, as well as the total number of optimization iterations and the number of hierarchical updates.

The total number of individual element function calls is counted as follows: each time an analysis model is evaluated during the optimization process to compute new values ${}^i\mathbf{r}$, a counter is increased by one. The function calls of each individual element are then added to one another to give the total number of individual element function calls.

The total number of optimization iterations is counted as follows: each time a new search direction is computed by the optimizer of the individual element a counter is increased by one. The optimization iterations of each individual element are then added to one another to give the total number of optimization iterations.

The total number of hierarchical updates is counted as follows: each time an individual element receives updated coupling data (mapped physical responses and/or shared design variables) and starts a new optimization (in case of the All-in-One approach constraint and objective function evaluations) of the updated model, a counter is increased by one.

Costs associated with the coordination problem are listed separately. These costs include constructing response surfaces (RSM), sensitivity analysis (SA), optimum sensitivity analysis (OSA) or solving an optimization problem (OP) during the coordination face. Furthermore, additional computations that are necessary to evaluate the final design after the optimization is finished, e.g in case of response surface methods, are not accounted for.

Because the coupling variables ${}^1_0h$ and ${}^2_0h$ associated with Level-0 cannot unambiguously be expressed in Level-1 design variables, the Level-0 design variables (${}^0\mathbf{x}$) as well as the coupling variables (${}^1_0h, {}^2_0h$) are utilized to compare the different optima found. The solution error is measured subtracting the optimum design variables computed at Level-0 via one of the multi-level optimization methods from the reference optimum found via All-in-One and divided component wise by the All-in-One solution. Taking the infinity-norm of the vector representing the component wise difference between the optimum and reference optimum gives the solution error. Finally, the objective function value is compared to the objective function value found via an All-in-One approach.

The results for the five performance criteria are listed in Table 5.1. In addition, Table 5.2 lists the results of the consistency constraints ${}^1_0c$, ${}^2_0c$ and the individual element design constraints. At the reference optimum found via an All-in-One optimization approach all individual element design constraints are active. Furthermore, a lower bound on one of the design constraints is active ${}^2\underline{x}_2$. In case no value is listed for the equality constraints in Table 5.2, these constraints are not added to the optimization problem and are automatically satisfied via a direct mapping of one individual element onto a neighboring element. Design constraints that are not active are represented via their function value.

It is clear from the results of Table 5.1 that the equality-based multi-level optimization approaches require significantly more element function evaluations as compared to the All-in-One approach. However, the amount of hierarchical updates are less for most of the coordination methods. Which means that communication between elements is reduced. This is due to the approximation techniques that approximate the behavior of the neighboring elements.

Optimization by Linear Decomposition and Quasi-Separable Subsystem Decomposition without a Response Surface approximation converge to the same optimal point as was found via AiO. Bi-Level Integrated System Synthesis converged close to the same point, although the lower bound on the Level-1 design variable was not active. Since approximations are only accurate within a certain margin, none of the response

Table 5.1: *Numerical comparison of multi-level coordination techniques for equality-based decomposition. Total number of function evaluations, total number of optimization iterations and the total number of hierarchical updates are listed. Furthermore, the difference between the reference optimal design variables and those found via the multi-level optimization procedure are listed as well as the optimum found with respect to the reference optimum.*

Method	Func.eval.	Opt.Iter	Hierar.upd.	Sol.Error	Obj./Obj.AiO
AiO	285		95	0.0	1.00
OLD	1045	192	12	7.24×10^{-6}	1.00
OLD OSA*1	1035	331			
QSD*2	51836	9498	590	7.17×10^{-7}	1.00
QSD	50	9	1	5.38×10^{-2}	1.02
QSD RSM*3	1369	272			
CSSO	1104	152	8	2.36×10^{-1}	1.03
CSSO GSE*4	48				
CSSO C*5	380	12			
BLISS*2	12085	3484	226	2.67×10^{-7}	1.00
BLISS	328	60	5	1.47×10^{-1}	1.01
BLISS RSM*3	1907	451			

*1 Costs related to the Optimum Sensitivity Analysis (OSA) of the Level-1 elements.
*2 Without the use of a response surface for the Level-1 elements.
*3 Costs associated with applying a response surface method (RSM).
*4 Costs associated with computing Global Sensitivity Equations (GSE).
*5 Costs associated with solving the coordination problem (Equation 5.19).

Table 5.2: *Consistency constraint values (^i_0c) and design constraint values (^iv_g, $^2x_2$) of the equality-based coordination methods. The reference design (AiO) has four active design constraints. Constraints that are not listed are not present in the multi-level optimization formulation and constraints that are not active are represented by their function value.*

Method	^1_0c	^2_0c	^0v_g	^1v_g	^2v_g	$^2\underline{x}_2$
AiO			active	active	active	active
OLD	active	active	active	active	active	active
QSD*1	active	active	active	active	active	active
QSD*2	active	active	active	-4.2×10^{-3}	-1.6×10^{-1}	active
CSSO			active	-4.0×10^{-2}	-9.4×10^{-1}	-3.4×10^{-6}
BLISS*1			active	active	active	-7.1×10^{-4}
BLISS*2			-1.2×10^{-2}	active	active	active

*1 Without the use of a response surface for the Level-1 elements.
*2 Constraint values evaluated for the real model instead of the response surface model.

surface based coordination methods finds exactly the same optimal point. Concurrent SubSpace Optimization did not converge to the optimal point. The Level-0 optimization constraint becomes active, however in the Level-1 optimization problems there is still room for improvement. Due to the non-active Level-1 constraints the objective function value found is slightly higher then that found via AiO.

During the multi-level optimization tests the following observations were made:

- Hierarchic linearized coordination (OLD): No difficulties with the minimization of the envelope function or calculation of sensitivities were observed. These sensitivities were computed via a finite difference technique for which the costs are listed separately in Table 5.1. The tolerance on the objective function value and constraint function value was set to $1 \cdot 10^{-9}$ during the multi-level optimization process. To compute the optimum sensitivities via finite differences the tolerance on objective and constraint functions was increased to $1 \cdot 10^{-15}$ during the perturbed optimization runs. The perturbation step necessary to calculate the finite differences was set to $1 \cdot 10^{-10}$, which was found to produce accurate optimum sensitivity values.

- Hierarchic constraint margin approach (QSD): The optimizer had difficulties finding feasible solutions for the Level-1 elements when the necessary data from these elements was directly evaluated. Likewise, approximating the Level-1 elements via a response surface did not succeed in many cases due to infeasible Level-1 element optimizations that were observed while conducting a Design of Experiments necessary to construct the response surface. Therefore, the design space that was used to choose points via a Latin Hypercube Sampling was reduced manually until only feasible lower level element optimizations were observed. The costs of finding a design space that would only result in feasible designs is not accounted for in Table 5.1, because no automated process was defined for this process. Through the results of the Level-1 feasible optima a response surface was fitted, which was used during the Level-0 optimization to obtain the final optimal design.

- Non-hierarchic linearized coordination (CSSO): Tight move limits in the coordinator problem were applied as a means of damping the cycling that was observed in de Wit and van Keulen (2007). These move limits are mathematically expressed as:

$$0.8\mathbf{s}_r^{(t-1)} \leq \mathbf{s}_r^{(t)} \leq 1.2\mathbf{s}_r^{(t-1)} + 0.01. \tag{5.30}$$

  The result is that the individual element optimization problems do not receive large changes in the responsibility coefficients after sufficient optimization runs are computed and therefore the solution found in these individual elements does not change. The costs of sensitivity analysis representing the coupling between elements and the costs for the optimization conducted for the coordination are listed separately in Table 5.1. The sensitivity calculations involve evaluating the analysis models and the coordination calculation involves linear functions and therefore the latter does not significantly contribute to the computational costs.

- Non-hierarchic linearized coordination (BLISS): Response surfaces that were computed for the BLISS method showed no difficulties, hence a Latin Hypercube Sampling covering the entire design space was used and all the Level-1 element optimizations necessary for the response surface converged. However, due to the size of the initial design space the BLISS method does require a few updates in which the design space is reduced and a new response surface is constructed before the optimum is found.

An advantage of the equality based multi-level optimization methods is that the user receives information on how the individual optimization problems interact and behave under changes in neighboring elements. Furthermore, the intermediate designs are all consistent and may only be inconsistent in design constraints that can be individual to a single element or shared over many elements. These advantages are also the disadvantage of these methods. Knowledge about the model is required in order to compute interaction sensitivities (GSE), accurate response surfaces or optimum sensitivities. Before starting any equality based multi-level optimization method, additional knowledge is required about the model that is not required by the relaxation based methods as is shown in the next section.

The response surface technique used for the multi-level optimization methods is discussed in Chapter 2. The multi-level optimization methods that use a response surface require fewer Level-0 computations then the multi-level optimization methods that rely on sensitivity analysis. The Level-1 computational cost is comparable for each of the multi-level optimization methods. More robust response surface methods may improve the results obtained for the two-bar truss structure, similarly the optimum sensitivity analysis can be conducted via less expensive approaches then finite differences. However, the focus in this thesis is on capturing the generic steps of each multi-level method and not on the approximation methods.

5.2 Relaxation-based coordination

Relaxation based coordination techniques are subdivided into two categories:

- hierarchic coordination,
- non-hierarchic coordination.

Furthermore, the hierarchic coordination techniques can be subdivided into:

- techniques that directly evaluate relaxation parameters,
- techniques that approximate relaxation parameters.

Methods that decompose a complex system via relaxation obtain their feasible or consistent design only at convergence of the algorithm. As opposed to equality-based decomposition schemes, which require feasibility or consistency at each iteration of the hierarchy. In order to obtain feasibility an additional relaxation parameter update procedure is necessary. This section describes different approaches to update these parameters depending on the type of decomposition formulation (see Chapter 3) that is used. For each coordination method a hierarchy of two levels containing a single element per level is considered. A two-level problem consisting of a single element per level is sufficient to show the coordination between coupled elements and simplifies the formulation of the optimization problems. The discussion focuses on physical coupling, however coupling via shared design variables is treated similarly.

5.2.1 Hierarchic relaxation methods

Collaborative Optimization

In Section 5.1 equality-based coordination procedures were introduced that can handle inseparable decisions (see Chapter 4). Equality-based coordination procedures add additional information to neighboring elements to overcome problems with inseparable decisions. Relaxation-based coordination methods add information on neighboring elements to the current element via copies of the relaxed constraints. The inconsistency is partly taken into account in the current element and partly taken into account at the neighboring element. The coordinator balances via updates of the relaxation parameters between the two individual elements. A first method that introduced this type of approach was Collaborative Optimization (CO).

Collaborative Optimization was first introduced by Braun and Kroo (1997). The method utilizes quadratic functions for the consistency constraints. Although the method was successfully applied to various large optimization problems (Braun *et al.*, 1997, Budianto and Olds, 2004), research conducted by Alexandrov and Lewis (2002) demonstrated that for convex problems it was very unlikely that CO would converge to the optimum.

A two-level problem of two individual elements is considered. The elements are decomposed via a top-down hierarchic relaxation-based decomposition scheme. The consistency constraints are relaxed via a quadratic function. According to Chapter 3, relaxed consistency constraint formulations minimize the consistency constraints at both levels. However, in the case of CO the requirement is that the consistency constraints are satisfied at Level-0 and minimized at Level-1. Thus, the consistency constraints are not added to the objective function of the Level-0 optimization problem. Instead, they are introduced as additional equality constraints. Summarizing, the Level-0 Optimization problem becomes:

$$
\begin{array}{lllll}
\min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h}} & {}^0v_f({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) & & & \\
\text{s.t.} & {}^0\mathbf{v}_g({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) & \leq & \mathbf{0} & \\
 & {}^0\mathbf{v}_h({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) & = & \mathbf{0} & \\
 & {}^0v_{{}^1_0c}({}^1_0\mathbf{c}({}^1_0\mathbf{h})) & = & \sum\limits_n \left({}^1_0\mathcal{H}_n({}^1\mathbf{r}) - {}^1_0h_n\right)^2 = 0 & . \\
 & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & & & \\
 & {}^1_0\underline{\mathbf{h}} \leq {}^1_0\mathbf{h} \leq {}^1_0\overline{\mathbf{h}} & & &
\end{array}
\qquad (5.31)
$$

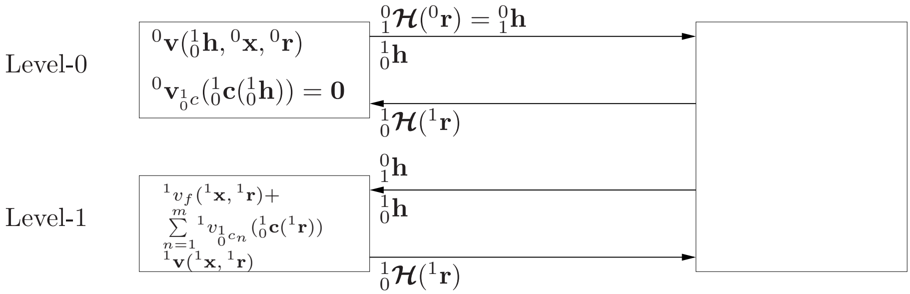


Figure 5.7: *The process of coordinating element data between the levels in case of Collaborative Optimization. The coordinator does not operate on the data and sends it directly to the other element.*

The requirement that the quadratic function of the consistency constraints should be zero ($^0v_{^1_0c}(^1_0\mathbf{c}(^1_0\mathbf{h}))$) is explicitly stated by the authors, see (Braun and Kroo, 1997).

The Level-1 optimization problem tries to find an optimum value while keeping the deviation between the prescribed physical responses ($^1_0\mathbf{h}$) and the actual physical responses ($^1_0\mathcal{H}(^1\mathbf{r})$) as small as possible:

$$\begin{array}{lll} \min\limits_{^1\mathbf{x}} & ^1v_f(^1\mathbf{x}, {}^1\mathbf{r}) + \sum\limits_{n=1}^{m} {}^1v_{^1_0c_n}(^1_0\mathbf{c}(^1\mathbf{r})) & \\ \text{s.t.} & ^1\mathbf{v}_g(^1\mathbf{x}, {}^1\mathbf{r}) & \leq \mathbf{0} \\ & ^1\mathbf{v}_h(^1\mathbf{x}, {}^1\mathbf{r}) & = \mathbf{0} \\ & ^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \\ \text{where} & ^1\mathbf{v}_{^1_0c}(^1_0\mathbf{c}(^1\mathbf{r})) & = \left(^1_0\mathcal{H}(^1\mathbf{r}) - {}^1_0\mathbf{h}\right)^2 \end{array} . \tag{5.32}$$

Satisfying the quadratic consistency constraint functions has shown to be difficult because the gradients of these constraint functions vanish near the optimum (see e.g. DeMiguel and Murray (2000)). Vanishing of the gradients near the optimum means that the Linear Independent Constraint Qualification[3] is not satisfied and therefore the coordination method does not converge towards the optimum as found via an All-in-One approach. The process that coordinates the data between the two levels is illustrated in Figure 5.7. The coordinator step involves sending data from Level-1 to Level-0 and *vice versa*. No additional operations are necessary and the method is actually a single level coordination method.

CO has many characteristic features of a relaxation-based decomposition technique. However, the original paper written by Braun and Kroo (1997) requires the consistency constraints to be satisfied at every Level-0 optimization (in case a hierarchic top-down decomposition scheme is used). According to Kodiyalam (1998), in practice these consistency constraints are relaxed as follows:

$$^0\mathbf{v}_{^1_0c}(^1_0\mathbf{c}(^1_0\mathbf{h})) \quad = \quad \left(^1_0\mathcal{H}(^1\mathbf{r}) - {}^1_0\mathbf{h}\right)^2 \leq \boldsymbol{\varepsilon}, \tag{5.33}$$

where $\boldsymbol{\varepsilon}$ is chosen by the multi-level optimization specialist.

[3]See DeMiguel and Murray (2000) for a definition.

Equation 5.33 allows for small numerical errors such that the Level-0 problem remains feasible, i.e. feasible in the context of Equation 5.33. For this reason, the algorithm is considered a penalty function method and not one of the equality-based approaches. Furthermore, the requirement that the constraints are minimized at Level-1 and implemented as equality constraints at Level-0 is similar to introducing a quadratic penalty function on the consistency constraints:

$$ {}^{0}\mathbf{v}_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}({}^{1}_{0}\mathbf{h})) \quad = \quad {}^{1}_{0}\mathbf{s}^{T}_{..} \left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right)^{2} \tag{5.34} $$

The penalty weights (${}^{1}_{0}\mathbf{s}_{..}$) on the consistency constraints are set to one in the Level-1 elements and set to infinity in the Level-0 element (e.g. DeMiguel and Murray (2006) discuss this issue).

Collaborative Optimization via inexact penalty function

Recent work carried out by DeMiguel and Murray (2006) on reformulating the inconsistencies via a penalty function relaxation and externally updated weights has led to a mathematical validation of the method. DeMiguel and Murray propose two different relaxation procedures; an inexact penalty function and an exact penalty function formulation leading to different coordination procedures. The inexact penalty decomposition scheme requires minor changes to the original CO formulation and is discussed first.

The two-level optimization problem, Equations 5.31 and 5.32, is considered. Hence, a top-down hierarchic decomposition is applied and the consistency constraints are relaxed via an inexact penalty function:

$$ {}^{i}v_{{}^{1}_{0}c}\left({}^{1}_{0}\mathbf{c}\right) \quad = \quad {}^{1}_{0}s||{}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}||_{2}^{2}, \tag{5.35} $$

the penalty parameter ${}^{1}_{0}s$ is a fixed parameter during the individual element optimizations and is incremented in the coordinator problem. The penalty function is called inexact because the All-in-One solution is not found, rather an approximated solution is found.

When a top-down hierarchic decomposed element is considered the relaxed consistency constraint (Equation 5.35) is assigned to the Level-0 element and the Level-1 element. The consistency gap is minimized at both Level-0 and Level-1. The Level-0 optimization problem is mathematically expressed as:

$$ \begin{array}{lll} \min\limits_{{}^{0}\mathbf{x},{}^{1}_{0}\mathbf{h}} & {}^{0}v_{f}({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) + {}^{0}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}({}^{1}_{0}\mathbf{h})) & \\ \text{s.t.} & {}^{0}\mathbf{v}_{g}({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) \quad \leq \quad 0 & \\ & {}^{0}\mathbf{v}_{h}({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{0}\mathbf{r}) \quad = \quad 0 & \\ & {}^{0}\underline{\mathbf{x}} \leq {}^{0}\mathbf{x} \leq {}^{0}\overline{\mathbf{x}} & \\ & {}^{1}_{0}\underline{\mathbf{h}} \leq {}^{1}_{0}\mathbf{h} \leq {}^{1}_{0}\overline{\mathbf{h}} & \\ \text{where} & {}^{0}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}({}^{1}_{0}\mathbf{h})) \quad = & {}^{1}_{0}s\left|\left|{}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right|\right|_{2}^{2} \end{array} . \tag{5.36} $$

In the above problem formulation Level-0 determines the optimal values of the coupling variables. The Level-1 optimization problem minimizes the consistency gap and yields:

$$\begin{array}{lll} \min\limits_{{}^1\mathbf{x}} & {}^1v_f({}^1\mathbf{x},{}^1\mathbf{r}) + {}^1v_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) & \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x},{}^1\mathbf{r}) & \leq \mathbf{0} \\ & {}^1\mathbf{v}_h({}^1\mathbf{x},{}^1\mathbf{r}) & = \mathbf{0} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \\ \text{where} & {}^1v_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) & = {}^1_0s \left|\left|{}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right|\right|_2^2 \end{array} . \tag{5.37}$$

The penalty parameter ${}^1_0s$ is equal in both elements.

DeMiguel and Murray state the mathematical conditions (second-order optimality conditions, linear independent constraint qualification, smoothness of the Level-1 optima, see DeMiguel and Murray (2006)) under which the computed design variables will approach the optimum values ($\mathbf{x}^*$) of the AiO problem if:

$$\mathbf{x}^* = \lim_{{}^1_0s\to\infty} \mathbf{x}({}^1_0s). \tag{5.38}$$

Thus, the computed optimal design variables depend on the weights. The weights ${}^1_0s$ are incrementally updated to obtain a sufficiently accurate solution, conform Equation 5.38.

Collaborative Optimization via exact penalty function

A second approach suggested by DeMiguel and Murray (2006) is the use of an exact penalty function to decompose the multi-level optimization problem. An optimization problem decomposed via an exact penalty function finds the same optimum as an All-in-One approach for finite values of the penalty parameters ${}^1_0\mathbf{s}$ rather then an approximated solution. The same two-level hierarchic top-down optimization problem is considered as previously, i.e. Equations 5.31 and 5.32.

An exact penalty function is used to relax the consistency constraints:

$${}^iv_{{}^1_0c}\left({}^1_0\mathbf{c}\right) \quad = \quad ||{}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}||_1 = \sum_n |{}^1_0\mathcal{H}_n({}^1\mathbf{r}) - {}^1_0h_n|. \tag{5.39}$$

The l^1-norm is not a continuous function and therefore adding this contribution to the Level-1 objective function has the effect that the changing optimum value of Level-1 under changing coupling variables does not represent a continuous function. To avoid non-smoothness of the absolute value function in Equation 5.39 two vectors of elastic variables (${}^1_0\boldsymbol{\mu}, {}^1_0\boldsymbol{\nu}$) are introduced. The penalty function in terms of elastic variables yields:

$${}^iv_{{}^1_0c}\left({}^1_0\mathbf{c}\right) \quad = \quad \sum_n |{}^1_0\mathcal{H}_n({}^1\mathbf{r}) - {}^1_0h_n| = \sum_n \left({}^1_0\mu_n + {}^1_0\nu_n\right). \tag{5.40}$$

A requirement for the elastic variables is that: ${}^1_0\boldsymbol{\mu}, {}^1_0\boldsymbol{\nu} \geq \mathbf{0}$.

The two-level optimization problem is now written as:

Level-0:

$$\begin{array}{lll} \min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h}} & {}^0v_f({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) \leq \mathbf{0} & \\ & {}^0\mathbf{v}_h({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) = \mathbf{0} & \\ & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & \\ & {}^1_0\underline{\mathbf{h}} \leq {}^1_0\mathbf{h} \leq {}^1_0\overline{\mathbf{h}} & \end{array} \tag{5.41}$$

Level-1:

$$\begin{array}{lll} \min\limits_{{}^1\mathbf{x},{}^1_0\boldsymbol{\mu},{}^1_0\boldsymbol{\nu}} & {}^1v_f({}^1\mathbf{x},{}^1\mathbf{r}) + {}^1_0\mathbf{s}_1^T\left({}^1_0\boldsymbol{\mu}+{}^1_0\boldsymbol{\nu}\right) & \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x},{}^1\mathbf{r}) \leq \mathbf{0} & \\ & {}^1\mathbf{v}_h({}^1\mathbf{x},{}^1\mathbf{r}) = \mathbf{0} & \\ & {}^1\mathbf{v}_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) = {}^1_0\mathcal{H}({}^1\mathbf{r}) + {}^1_0\boldsymbol{\mu} - {}^1_0\boldsymbol{\nu} = {}^1_0\mathbf{h} & . \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \\ & {}^1_0\boldsymbol{\mu}, {}^1_0\boldsymbol{\nu} \geq \mathbf{0} & \end{array} \tag{5.42}$$

DeMiguel and Murray state that in general the gradients of the constraints of Equation 5.42 are linear dependent, because most engineering optimization problems are non-convex. Therefore, the output of Level-1 that forms the input for the Level-0 optimization problem is non-smooth. Meaning that it is difficult to find an optimum for the Level-0 problem if the mapped responses of the Level-1 element cannot be described as a smooth function.

To overcome the difficulty of linear dependent gradients of the constraints barrier terms are introduced. These barrier terms remove the inequality constraints on the elastic variables. The problem statement via exact penalty function relaxation with elastic variables and additional barrier terms for the Level-0 optimization is:

$$\begin{array}{ll} \min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h}} & {}^0v_f({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) + {}^1_0\mathbf{s}_1^T\left({}^1_0\boldsymbol{\mu}+{}^1_0\boldsymbol{\nu}\right) + {}^1_0s_2\sum\limits_{m=1}^{n}\left(\log{}^1_0\mu_m + \log{}^1_0\nu_m\right) \\ \text{s.t.} & {}^0\mathbf{v}_g({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) \leq \mathbf{0} \\ & {}^0\mathbf{v}_h({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) = \mathbf{0} \\ & {}^0\mathbf{v}_{{}^1_0c}({}^1_0\mathbf{c}({}^1_0\mathbf{h})) = {}^1_0\mathcal{H}({}^1\mathbf{r}) + {}^1_0\boldsymbol{\mu} - {}^1_0\boldsymbol{\nu} = {}^1_0\mathbf{h} \\ & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} \\ & {}^1_0\underline{\mathbf{h}} \leq {}^1_0\mathbf{h} \leq {}^1_0\overline{\mathbf{h}} \end{array} \tag{5.43}$$

and the Level-1 optimization problem yields:

$$\begin{array}{ll} \min\limits_{{}^1\mathbf{x},{}^1_0\boldsymbol{\mu},{}^1_0\boldsymbol{\nu}} & {}^1v_f({}^1\mathbf{x},{}^1\mathbf{r}) + {}^1_0\mathbf{s}_1^T\left({}^1_0\boldsymbol{\mu}+{}^1_0\boldsymbol{\nu}\right) + {}^1_0s_2\sum\limits_{m=1}^{n}\left(\log{}^1_0\mu_m + \log{}^1_0\nu_m\right) \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x},{}^1\mathbf{r}) \leq \mathbf{0} \\ & {}^1\mathbf{v}_h({}^1\mathbf{x},{}^1\mathbf{r}) = \mathbf{0} \qquad . \\ & {}^1\mathbf{v}_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) = {}^1_0\mathcal{H}({}^1\mathbf{r}) + {}^1_0\boldsymbol{\mu} - {}^1_0\boldsymbol{\nu} = {}^1_0\mathbf{h} \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} \end{array} \tag{5.44}$$

Instead of increasing the penalty weight to a sufficiently large number as was the case for the inexact penalty decomposition method (5.38) a barrier term ${}^1_0s_2$ (recall Section

2.4.1 for an explanation) is decreased via the coordinator to zero via:

$$ {}^1_0s_2^{(t+1)} = \beta \cdot {}^1_0s_2^{(t)}, \tag{5.45} $$

for a fixed large value of the weight vector ${}^1_0\mathbf{s}_1$. Furthermore, two additional slack variables $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are added to the Level-1 optimization problem. If the penalty weight vector ${}^1_0\mathbf{s}_1$ is chosen large enough, these two parameters are a measure of the gap size of the inconsistency:

$$ \left|\left|{}^1_0\mathbf{c}\right|\right|_1 = \left|\left|{}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right|\right|_1 = \sum_{m=1}^{n} {}^1_0\mu_m + {}^1_0\nu_m \ . \tag{5.46} $$

For sufficient conditions (second order sufficient conditions, linear independent constraint qualification (DeMiguel and Murray, 2006)) this approach converges to the optimum.

Analytical Target Cascading

Another improvement over the Collaborative Optimization procedure was introduced by Kim *et al.* (2003). Analytical Target Cascading (ATC) overcomes some of the drawbacks of Collaborative Optimization. The procedure was derived for bottom-up hierarchic decomposition schemes (top-down decomposition was shown by Allison *et al.* (2005)) via relaxation through penalty functions multiplied with an iteratively increasing weight. Extensive research on generalizing the method was conducted by Michalek and Papalambros (2005b) and mathematical justification of the method under convexity assumptions was shown by Michelena *et al.* (2003). This mathematical justification was questioned however by Lassiter *et al.* (2005), which will be touched upon briefly during the discussion on incremental weight update techniques hereafter.

A top-down hierarchic decomposition scheme is considered. The constraints are relaxed by taking the l^2-norm of the quadratic of the inconsistency in contrast to Collaborative Optimization which uses a quadratic function of the consistency constraints. Weights ${}^1_0\mathbf{s}$ are introduced that are updated via a coordinator scheme that pushes the consistency constraints to zero. The Level-0 problem is mathematically expressed as:

$$ \begin{array}{lll} \min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h}} & {}^0v_f({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) + {}^0v_{{}^1_0c}({}^1_0\mathbf{c}({}^1_0\mathbf{h})) & \\ \text{s.t.} & {}^0\mathbf{v}_g({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) \ \leq \ \mathbf{0} & \\ & {}^0\mathbf{v}_h({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^0\mathbf{r}) \ = \ \mathbf{0} & \\ & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & \\ & {}^1_0\underline{\mathbf{h}} \leq {}^1_0\mathbf{h} \leq {}^1_0\overline{\mathbf{h}} & \\ \text{where} & {}^0v_{{}^1_0c}({}^1_0\mathbf{c}({}^1_0\mathbf{h})) \ = \ \left|\left|{}^1_0\mathbf{s}_1 \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right)\right|\right|_2^2 & \end{array} \ . \tag{5.47} $$

The weights ${}^1_0\mathbf{s}_1$ are adjusted externally via the coordinator. The coordinator effectively assigns priority between minimizing the objective function ${}^0v_f({}^0\mathbf{x}, {}^0\mathbf{r})$ and/or reducing the consistency violation ${}^1_0\mathbf{c}({}^1_0\mathbf{h})$. The consistency constraint is included at the Level-0 problem as well as the Level-1 problem, which is possible due to the relaxation of the constraint. Both levels are required to reduce any inconsistency between the two levels.

The Level-1 optimization problem yields:

$$\begin{array}{lll} \min\limits_{{}^1\mathbf{x}} & {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}) + {}^1v_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) & \\ \text{s.t.} & {}^1\mathbf{v}_g({}^1\mathbf{x}, {}^1\mathbf{r}) \quad \leq \quad \mathbf{0} & \\ & {}^1\mathbf{v}_h({}^1\mathbf{x}, {}^1\mathbf{r}) \quad = \quad \mathbf{0} & . \\ & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \\ \text{where} & {}^1v_{{}^1_0c}({}^1_0\mathbf{c}({}^1\mathbf{r})) \quad = \quad \left|\left|{}^1_0\mathbf{s}_2 \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right)\right|\right|_2^2 & \end{array} \tag{5.48}$$

The weight vectors ${}^1_0\mathbf{s}_1$ and ${}^1_0\mathbf{s}_2$ remain equal per component of the vector according to Kim *et al.* (2003), Michalek and Papalambros (2005b) and Tosserams *et al.* (2008a). However, to the authors knowledge this does not need to be the case and numerical experiments conducted by the author suggest that the method converges faster if these vectors are not kept equal. Therefore, numerical results listed in Section 5.2.4 and Chapter 8 are based on relaxation via parameters within individual elements that are not kept equal between elements that are coupled.

The convergence of the algorithm is proven via the observation that the decomposed optimization problem is a saddle point problem such as discussed in the work of Benzi *et al.* (2005). Then finding the solution to the decomposed optimization problem is based upon the observation that:

$$\mathbf{x}^* = \lim_{{}^1_0\mathbf{s} \to \infty} \mathbf{x}({}^1_0\mathbf{s}), \tag{5.49}$$

where $\mathbf{x}^*$ is the unique solution for the problem solved all-in-one. Hence, via iterative increments after each element optimization the coordinator step involves:

$${}^1_0\mathbf{s}^{(t+1)} = \beta \cdot {}^1_0\mathbf{s}^{(t)}. \tag{5.50}$$

For a bounded sequence of increments of ${}^1_0\mathbf{s}$, a convex optimization problem, strict linear independent constraint qualification and second order sufficient conditions these weight increments will approach the optimal weights that are required to solve the saddle point type of problem originating from decomposing the all-in-one problem.

Lassiter *et al.* (2005) argue that there is no procedure or mathematical proof that optimal weights can be computed. The optimum is not found exactly and thus the procedure cannot be stated as such. However, the penalty method is a well-established approach for solving saddle point problems approximately.

The only means of finding the exact solution utilizing a penalty method is via a procedure suggested by (Benzi *et al.*, 2005) that employs a direct solver. A direct solver suggests that the subproblems are solved via the same solution technique. ATC does not have this restriction. The method allows for individual optimization techniques for each element allowing more freedom to the designer.

The total procedure for ATC is sketched in Figure 5.8. Besides sending data from one level to the next, the coordinator has to increment the weights ${}^1_0\mathbf{s}$ and send the updated weights to both elements.

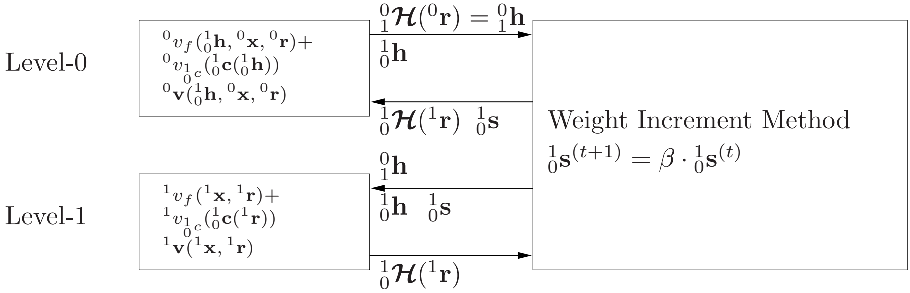


Figure 5.8: *ATC coordination through incremental weight updates. By means of increasing the penalty weight both levels are forced in each new step to assign greater priority to closing the consistency gap.*

Analytical Target Cascading via Lagrangian relaxation

Recently, various alternative constraint relaxation and coordination techniques have been introduced. Lassiter *et al.* (2005) and Kim *et al.* (2006) use Lagrange multipliers in order to relax the consistency constraints. In case of a top-down hierarchic decomposition the relaxed consistency constraints for the two-level problem yield:

$$ {}^{i}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}) \quad = \quad {}^{1}_{0}\boldsymbol{\lambda}^{T}\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right). \tag{5.51} $$

The individual element optimization problems after decomposition were discussed in Chapter 3 and are not repeated here.

The coordinator problem involves solving the so-called dual problem[4], see Appendix A. Hence, the coordinator solves an optimization problem in which a function representing the minimization of both the Level-0 and Level-1 optimization problems is maximized keeping the design variables fixed and changing the Lagrange multipliers:

$$ \max_{{}^{1}_{0}\boldsymbol{\lambda}} \quad v_f({}^{1}_{0}\boldsymbol{\lambda}) \tag{5.52} $$

where v_f represents the combined individual optimization problems of the hierarchy with relaxed consistency constraints. Neglecting terms that do not depend on the Lagrange multipliers associated with the relaxation of the consistency constraints, the dual problem solved via the coordinator is simplified to:

$$ \max_{{}^{1}_{0}\boldsymbol{\lambda}} \quad v_f({}^{1}_{0}\boldsymbol{\lambda}) = {}^{1}_{0}\boldsymbol{\lambda}^{T}\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right) \ . \tag{5.53} $$

The solution to the dual problem is found via stepwise increments of the Lagrange multipliers computed as:

$$ {}^{1}_{0}\boldsymbol{\lambda}^{(t+1)} = {}^{1}_{0}\boldsymbol{\lambda}^{(t)} + \beta \cdot {}^{1}_{0}\mathbf{c}^{(t)}, \tag{5.54} $$

[4] The dual problem is defined in the book of Bertsekas (1995), Propositions 5.1.1 through 5.1.6.

where β is the step size which is used here as a numerical damping parameter and ${}^1_0\mathbf{c}$ the size of the inconsistency.

After each increment of the Lagrange multipliers the individual optimizations are repeated with the updated multipliers. New increments of the Lagrange multipliers are computed after these optimizations have finished and coupling data is exchanged again.

If each element is assigned individual Lagrange multipliers then the coordination problem of finding optimal Lagrange multipliers is split and Equation 5.53 becomes:

$$\max_{{}^1_0\boldsymbol{\lambda}_1} \quad v_f({}^1_0\boldsymbol{\lambda}_1) = {}^1_0\boldsymbol{\lambda}_1^T \left({}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right) \; ; \tag{5.55}$$

$$\max_{{}^1_0\boldsymbol{\lambda}_2} \quad v_f({}^1_0\boldsymbol{\lambda}_2) = {}^1_0\boldsymbol{\lambda}_2^T \left({}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right) \; ; \tag{5.56}$$

where the Lagrange multipliers are assigned to the individual elements.

The procedures proposed by Lassiter *et al.* (2005) and Kim *et al.* (2006) are almost identical. The difference is that Lassiter *et al.* add ${}^1_0\boldsymbol{\lambda}^T(-{}^1_0\mathbf{h})$ to the Level-0 objective and ${}^1_0\boldsymbol{\lambda}^T({}^1_0\mathcal{H}({}^1\mathbf{r}))$ to the Level-1 element objective instead of the complete expression of the relaxed consistency constraints (Equation 5.51), while Kim *et al.* add ${}^1_0\boldsymbol{\lambda}^T({}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h})$ to each individual element objective (hence Equation 5.51). Furthermore, Lassiter *et al.* solve each individual element optimization concurrent while Kim *et al.* solve the hierarchy level by level.

These formulations that coordinate relaxed consistency constraints are applicable to hierarchic top-down decomposition schemes using relaxation. The bottom-up procedures are similar and the non-hierarchic variants of these coordination approaches are discussed in Section 5.2.3.

Analytical Target Cascading via Augmented Lagrangian relaxation

Tosserams *et al.* (2006, 2007, 2008a) use an augmented Lagrangian relaxation of the consistency constraints. A common procedure to modify the penalty relaxation method in order to avoid ill-conditioning due to large weights ${}^1_0\mathbf{s}$. A second benefit of the method is that combining Lagrange multipliers and penalty weights results in finding the exact solution for convex optimization problems rather then an approximated one. The method is called method of multipliers, which was according to Benzi *et al.* (2005) independently derived by various researchers.

Incorporating augmented Lagrangian relaxation into a top-down hierarchic decomposition scheme is accomplished via the mathematical expression:

$${}^i v_{{}^1_0 c}({}^1_0\mathbf{c}) \quad = \quad {}^1_0\boldsymbol{\lambda}^T \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right) + \left|\left|{}^1_0\mathbf{s} \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right)\right|\right|_2^2 , \tag{5.57}$$

added to the Level-0 and Level-1 objective function. Updates of ${}^1_0\boldsymbol{\lambda}$ and of ${}^1_0\mathbf{s}$ are controlled via the coordinator. After solving the individual element optimizations the

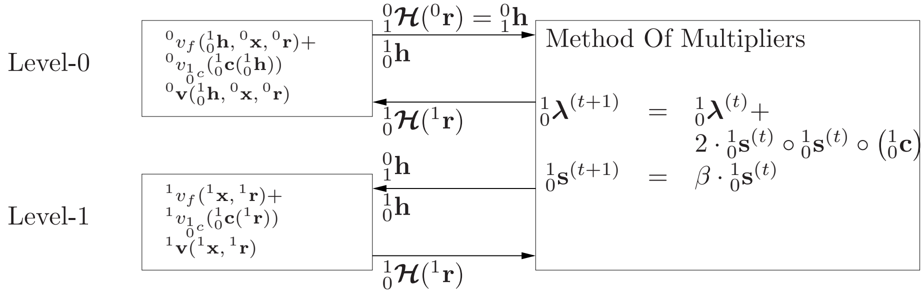


Figure 5.9: *Coordination for elements where the consistency constraints are relaxed with an augmented Lagrangian function. The coordinator redistributes the coupling data and computes new relaxation parameters for the augmented Lagrangian function.*

parameters are updated via:

$$\begin{aligned} {}^1_0\boldsymbol{\lambda}^{(t+1)} &= {}^1_0\boldsymbol{\lambda}^{(t)} + 2\,{}^1_0\mathbf{s}^{(t)} \circ {}^1_0\mathbf{s}^{(t)} \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right) \\ {}^1_0\mathbf{s}^{(t+1)} &= \begin{cases} {}^1_0\mathbf{s}^{(t)} & \text{if } {}^1_0\mathbf{c}^{(t)} \leq \gamma {}^1_0\mathbf{c}^{(t-1)} \\ \beta \cdot {}^1_0\mathbf{s}^{(t)} & \text{if } {}^1_0\mathbf{c}^{(t)} > \gamma {}^1_0\mathbf{c}^{(t-1)} \end{cases} \\ & \beta > 1 \text{ and } 0 < \gamma < 1 \end{aligned} \tag{5.58}$$

where ${}^1_0\mathbf{c}^{(t)} = \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right)^{(t)}$, the inconsistency at iteration $^{(t)}$. According to Tosserams *et al.* (2008a) the parameter β is generally chosen between $2 < \beta < 3$ and $\gamma = 0.25$. For mathematical justification of the method, Tosserams *et al.* refer to the work of Bertsekas and Tsitsiklis (1989).

In order to reduce computational cost, Tosserams *et al.* (2008a) propose the use of the Alternating Direction Method of Multipliers (see Bertsekas and Tsitsiklis (1989)). Instead of iterating between levels until the solution of the objective function stabilizes, a pass through all the Levels is made once, after which the penalty parameters are updated.

The entire procedure for Augmented Lagrangian Relaxation coordination is shown in Figure 5.9. The coordinator is involved with distributing the coupling data as well as updating the relaxation parameters.

5.2.2 Hierarchic approximation methods

Analytical Target Cascading via Approximated Weight Update Method

The previously discussed coordination methods for relaxed decomposition schemes all require iteratively updated weights, where the size of β is determined by experience. An alternative was introduced by Michalek and Papalambros (2005a), who derived a technique to approximate the weights necessary to drive the inconsistencies to zero. Their approach is derived in the present thesis for a top-down hierarchic decomposition with consistency constraints relaxed with a quadratic norm, however this derivation can easily be extended to any of the other relaxation-based decomposition approaches.

The approximation technique introduced by Michalek and Papalambros (2005a) is illustrated for the two-level case. Consider the Level-0 optimization problem defined previously for ATC:

$$\begin{array}{lll} \min\limits_{{}^{0}\mathbf{x},{}_{0}^{1}\mathbf{h}} & {}^{0}v_f({}_{0}^{1}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) + {}^{0}v_{{}_{0}^{1}c}({}_{0}^{1}\mathbf{c}({}_{0}^{1}\mathbf{h})) & \\ \text{s.t.} & {}^{0}\mathbf{v}_g({}_{0}^{1}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) \leq \mathbf{0} & \\ & {}^{0}\mathbf{v}_h({}_{0}^{1}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) = \mathbf{0} & \\ & {}^{0}\underline{\mathbf{x}} \leq {}^{0}\mathbf{x} \leq {}^{0}\overline{\mathbf{x}} & \\ & {}_{0}^{1}\underline{\mathbf{h}} \leq {}_{0}^{1}\mathbf{h} \leq {}_{0}^{1}\overline{\mathbf{h}} & \\ \text{where} & {}^{0}v_{{}_{0}^{1}c}({}_{0}^{1}\mathbf{c}({}_{0}^{1}\mathbf{h})) = \left|\left|{}_{0}^{1}\mathbf{s}_1 \circ \left({}_{0}^{1}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}_{0}^{1}\mathbf{h}\right)\right|\right|_2^2 & \end{array} \quad . \tag{5.59}$$

The Lagrangian of Equation 5.59 yields:

$$\mathcal{L} = {}^{0}v_f({}_{0}^{1}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) + \left|\left|{}_{0}^{1}\mathbf{s}_1 \circ \left({}_{0}^{1}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}_{0}^{1}\mathbf{h}\right)\right|\right|_2^2 + {}^{0}\boldsymbol{\lambda}_g^{T\,0}\mathbf{v}_g + {}^{0}\boldsymbol{\lambda}_h^{T\,0}\mathbf{v}_h \tag{5.60}$$

The first order necessary conditions for an optimal point state that the derivative of the Lagrangian with respect to any design variable or Lagrange multiplier equals zero. Hence, for the Level-0 optimization problem (Equation 5.59) differentiated with respect to the coupling variables ${}_{0}^{1}\mathbf{h}$ this condition yields:

$$\begin{aligned} \nabla_{{}_{0}^{1}\mathbf{h}}\mathcal{L} &= \nabla_{{}_{0}^{1}\mathbf{h}}({}^{0}v_f) + 2({}_{0}^{1}\mathbf{s})^2\left({}_{0}^{1}\mathcal{H}({}^{0}\mathbf{r}) - {}_{0}^{1}\mathbf{h}\right)\nabla_{{}_{0}^{1}\mathbf{h}}({}_{0}^{1}\mathbf{h}) \\ &\quad + {}^{0}\boldsymbol{\lambda}_g^T\nabla_{{}_{0}^{1}\mathbf{h}}({}^{0}\mathbf{v}_g) + {}^{0}\boldsymbol{\lambda}_h^T\nabla_{{}_{0}^{1}\mathbf{h}}({}^{0}\mathbf{v}_h) = 0, \end{aligned} \tag{5.61}$$

where ${}^{0}\boldsymbol{\lambda}_g$ are the lagrange multipliers associated with the inequality constraints and ${}^{0}\boldsymbol{\lambda}_h$ are the lagrange multipliers associated with the equality constraints. From Equation 5.61 the weights, necessary for satisfying the consistency constraints are computed as:

$$\begin{aligned} \left({}_{0}^{1}\mathbf{s}\right)^2 = \frac{1}{2}\frac{1}{({}_{0}^{1}\mathbf{h} - {}_{0}^{1}\mathcal{H}({}^{0}\mathbf{r}))}\nabla_{{}_{0}^{1}\mathbf{h}}({}^{0}v_f) + \frac{1}{2}\frac{{}^{0}\boldsymbol{\lambda}_g}{({}_{0}^{1}\mathbf{h} - {}_{0}^{1}\mathcal{H}({}^{0}\mathbf{r}))}\nabla_{{}_{0}^{1}\mathbf{h}}({}^{0}\mathbf{v}_g) + \\ \frac{1}{2}\frac{{}^{0}\boldsymbol{\lambda}_h}{({}_{0}^{1}\mathbf{h} - {}_{0}^{1}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}))}\nabla_{{}_{0}^{1}\mathbf{h}}({}^{0}\mathbf{v}_h) \end{aligned} \tag{5.62}$$

This approach has shown to reduce the cycling between elements and/or levels compared to iteratively increasing weights as is done in Equation 5.50, see de Wit and van Keulen (2007).

The Weight Update Method is shown in Figure 5.10. After a solution of the individual elements is found a sensitivity analysis of the top-level element is conducted. The sensitivity information that is obtained is used to compute new penalty weights.

Analytical Target Cascading via Diagonal Quadratic Approximation Method

A different approximation method was proposed by Li *et al.* (2008). These authors propose an approximation to the quadratic norm of the consistency constraints that

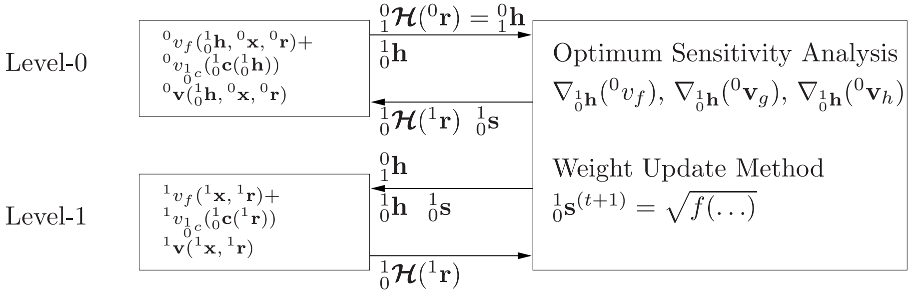


Figure 5.10: *ATC coordination by means of approximating the weights necessary to reduce the gap in the inconsistency. First, an optimum sensitivity analysis is conducted. Second, with the gradient information from the optimum sensitivity analysis the weights necessary to push the consistencies to some predetermined small value are computed.*

are used for the Analytical Target Cascading (ATC) method. The modified ATC method is called Diagonal Quadratic Approximation Method. By means of a Taylor series expansion an approximation to the inseparable coupled expressions of the quadratic norm used to relax the consistency constraints is found.

For a top-down hierarchic decomposition problem consisting of two-levels of each one element, the consistency constraints relaxed via a quadratic l_2-norm of the consistencies multiplied by a weight coefficient yield:

$$ {}^{1}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}) = ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right)||_2^2. \tag{5.63} $$

Rewriting the relaxed consistency constraint into equivalent l_1-norm yields:

$$ \begin{array}{l} ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right)||_2^2 = \ldots \\ \left|\left|{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) \circ {}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) + {}^{1}_{0}\mathbf{h} \circ {}^{1}_{0}\mathbf{h} - 2\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) \circ {}^{1}_{0}\mathbf{h}\right)\right)\right|\right|_1 \end{array} . \tag{5.64} $$

This expression is additively separable except for the last part ${}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) \circ {}^{1}_{0}\mathbf{h}$. In order to separate this last part a linear Taylor series expansion is applied.

If ${}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)}$ and ${}^{1}_{0}\mathbf{h}^{(t)}$ are the values for the mapped physical responses and the coupling variables of the current solution of the elements at iteration $^{(t)}$, then a linear Taylor series expansion up to first order of the coupling terms is:

$$ \begin{array}{rcl} {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) \circ {}^{1}_{0}\mathbf{h} & \cong & {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)} \circ {}^{1}_{0}\mathbf{h}^{(t)} + {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)} \circ \left({}^{1}_{0}\mathbf{h}^{(t+1)} - {}^{1}_{0}\mathbf{h}^{(t)}\right) + \ldots \\ & & {}^{1}_{0}\mathbf{h}^{(t)} \circ \left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t+1)} - {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)}\right) \\ & = & {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)} \circ {}^{1}_{0}\mathbf{h}^{(t+1)} - {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)} \circ {}^{1}_{0}\mathbf{h}^{(t)} + {}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t+1)} \circ {}^{1}_{0}\mathbf{h}^{(t)} \end{array} \tag{5.65} $$

Inserting Equation 5.65 into Equation 5.64 leads to:

$$ \begin{array}{l} ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right)||_2^2 \cong \ldots \\ ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) \circ {}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) + {}^{1}_{0}\mathbf{h} \circ {}^{1}_{0}\mathbf{h} - 2\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) \circ {}^{1}_{0}\mathbf{h}\right)\right)||_1 = \ldots \\ ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t)} \circ {}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t)} + {}^{1}_{0}\mathbf{h}^{(t+1)} \circ {}^{1}_{0}\mathbf{h}^{(t+1)} - 2\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)} \circ {}^{1}_{0}\mathbf{h}^{(t+1)}\right)\right)||_1 + \ldots \\ ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t+1)} \circ {}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t+1)} + {}^{1}_{0}\mathbf{h}^{(t)} \circ {}^{1}_{0}\mathbf{h}^{(t)} - 2\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t+1)} \circ {}^{1}_{0}\mathbf{h}^{(t)}\right)\right)||_1 + \ldots \\ ||{}^{1}_{0}\mathbf{s} \circ \left(2\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r})^{(t)} \circ {}^{1}_{0}\mathbf{h}^{(t)}\right) - {}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t)} \circ {}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t)} - {}^{1}_{0}\mathbf{h}^{(t)} \circ {}^{1}_{0}\mathbf{h}^{(t)}\right)||_1 = \ldots \\ ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t)} - {}^{1}_{0}\mathbf{h}^{(t+1)}\right)||_2^2 + ||{}^{1}_{0}\mathbf{s} \circ \left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r})^{(t+1)} - {}^{1}_{0}\mathbf{h}^{(t)}\right)||_2^2 + \text{Constant} \end{array} $$

(5.66)

The final expression depends on the new values of the coupling variables ${}^1_0\mathbf{h}^{(t+1)}$, the new values of the mapped physical responses ${}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r})^{(t+1)}$ and constant terms, because the values calculated at iteration ${}^{(t)}$ are kept constant during iteration ${}^{(t+1)}$.

The approximation to the quadratic norm of the consistency constraints:

$$\left|\left|{}^1_0\mathbf{s} \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r})^{(t)} - {}^1_0\mathbf{h}^{(t+1)}\right)\right|\right|_2^2 + \left|\left|{}^1_0\mathbf{s} \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r})^{(t+1)} - {}^1_0\mathbf{h}^{(t)}\right)\right|\right|_2^2 \tag{5.67}$$

is additively separable and can be solved at both levels simultaneously. Hence, at Level-0:

$$\left|\left|{}^1_0\mathbf{s} \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r})^{(t)} - {}^1_0\mathbf{h}^{(t+1)}\right)\right|\right|_2^2 \tag{5.68}$$

is solved, because ${}^1_0\mathbf{h}$ are considered design variables at Level-0. At Level-1:

$$\left|\left|{}^1_0\mathbf{s} \circ \left({}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{r})^{(t+1)} - {}^1_0\mathbf{h}^{(t)}\right)\right|\right|_2^2 \tag{5.69}$$

is solved, because new physical responses ${}^1\mathbf{r}$ are computed at Level-1.

Essentially this is an application of a block Jacobi instead of a block Gauss-Seidel method for solving a large system of equations. Lassiter *et al.* (2005) use a similar technique for their Lagrangian coordination method because the Lagrangian function (Equation 5.51):

$$ {}^i v_{{}^1_0 c}({}^1_0\mathbf{c}) \quad = \quad {}^1_0\boldsymbol{\lambda}^T \left({}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right), \tag{5.70}$$

depends on a value ${}^1_0\mathcal{H}({}^1\mathbf{r})$ that is evaluated during the Level-1 optimization, while keeping ${}^1_0\mathbf{h}$ constant and *vice versa* in case of the Level-0 optimization problem. The updating of the information from one element onto the other element is done parallel (simultaneously) instead of sequential (first Level-0 is optimized followed by Level-1 with updated ${}^1_0\mathbf{h}$).

To reduce overall cost of the Diagonal Quadratic Approximation method Li *et al.* (2008) propose to use an Alternating Direction method such as suggested by Tosserams *et al.* (2008a) for his Augmented Lagrangian coordination method. Hence, only one pass through all the individual element optimizations is made after which the penalty weights are updated and coupling variables and mapped response values are exchanged.

5.2.3 Non-hierarchic relaxed coordination

Analytical Target Cascading via Lagrangian relaxation

Coordination methods for hierarchic top-down and/or bottom-up relaxation can also be applied to non-hierarchic problems. In Chapter 3 the decomposition of physical coupling was shown and for the non-hierarchic case this required the relaxation of the consistency constraints in two directions. A two level hierarchy consisting of one element per level is considered and the relaxation by means of Lagrange multipliers yields:

$$\begin{aligned} {}^{i}v_{{}^{0}_{1}c}({}^{0}_{1}\mathbf{c}) &= {}^{0}_{1}\boldsymbol{\lambda}\left({}^{0}_{1}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) - {}^{0}_{1}\mathbf{h}\right); \\ {}^{i}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}) &= {}^{1}_{0}\boldsymbol{\lambda}\left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right). \end{aligned} \tag{5.71}$$

Compared to the hierarchic case an additional set of multipliers ${}^{0}_{1}\boldsymbol{\lambda}$ and consistency constraints ${}^{0}_{1}\mathbf{c}$ are present. The Level-0 element optimization problem yields:

$$\begin{array}{lll} \min\limits_{{}^{0}\mathbf{x},{}^{1}_{0}\mathbf{h}} & {}^{0}v_f({}^{1}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) + {}^{0}v_{{}^{0}_{1}c}({}^{0}_{1}\mathbf{c}) + {}^{0}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}) & \\ \text{s.t.} & {}^{0}\mathbf{v}_g({}^{1}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) \leq \mathbf{0} & \\ & {}^{0}\mathbf{v}_h({}^{1}_{0}\mathbf{h},{}^{0}\mathbf{x},{}^{0}\mathbf{r}) = \mathbf{0} & \\ & {}^{0}\underline{\mathbf{x}} \leq {}^{0}\mathbf{x} \leq {}^{0}\overline{\mathbf{x}} & , \\ & {}^{1}_{0}\underline{\mathbf{h}} \leq {}^{1}_{0}\mathbf{h} \leq {}^{1}_{0}\overline{\mathbf{h}} & \\ \text{where} & {}^{0}v_{{}^{0}_{1}c}({}^{0}_{1}\mathbf{c}({}^{0}\mathbf{r})) = {}^{0}_{1}\boldsymbol{\lambda}^T\left({}^{0}_{1}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) - {}^{0}_{1}\mathbf{h}\right) & \\ & {}^{0}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}({}^{1}_{0}\mathbf{h})) = {}^{1}_{0}\boldsymbol{\lambda}^T\left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right) & \end{array} \tag{5.72}$$

and the Level-1 element optimization problem yields:

$$\begin{array}{lll} \min\limits_{{}^{1}\mathbf{x},{}^{0}_{1}\mathbf{h}} & {}^{1}v_f({}^{0}_{1}\mathbf{h},{}^{1}\mathbf{x},{}^{1}\mathbf{r}) + {}^{1}v_{{}^{0}_{1}c}({}^{0}_{1}\mathbf{c}) + {}^{1}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}) & \\ \text{s.t.} & {}^{1}\mathbf{v}_g({}^{0}_{1}\mathbf{h},{}^{1}\mathbf{x},{}^{1}\mathbf{r}) \leq \mathbf{0} & \\ & {}^{1}\mathbf{v}_h({}^{0}_{1}\mathbf{h},{}^{1}\mathbf{x},{}^{1}\mathbf{r}) = \mathbf{0} & . \\ & {}^{1}\underline{\mathbf{x}} \leq {}^{1}\mathbf{x} \leq {}^{1}\overline{\mathbf{x}} & \\ \text{where} & {}^{1}v_{{}^{0}_{1}c}({}^{0}_{1}\mathbf{c}({}^{0}_{1}\mathbf{h})) = {}^{0}_{1}\boldsymbol{\lambda}^T\left({}^{0}_{1}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) - {}^{0}_{1}\mathbf{h}\right) & \\ & {}^{1}v_{{}^{1}_{0}c}({}^{1}_{0}\mathbf{c}({}^{1}\mathbf{r})) = {}^{1}_{0}\boldsymbol{\lambda}^T\left({}^{1}_{0}\boldsymbol{\mathcal{H}}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right) & \end{array} \tag{5.73}$$

The relaxed consistency constraints are added to both elements because the mapped responses $({}^{i}_{j}\boldsymbol{\mathcal{H}}({}^{i}\mathbf{r}))$ are computed in Element-i, while coupling variables $({}^{i}_{j}\mathbf{h})$ are considered design variables in Element-j. Either mapped responses or coupling variables are kept fixed during the individual optimization. Because inconsistency between mapped physical responses and coupling variables has only a negative impact on the objective, both elements try to contribute to keeping the negative impact as low as possible.

The coordinator problem is to maximize the individual optimization problems of the entire hierarchy for fixed design variables via changing the Lagrange multipliers:

$$\max_{{}^{0}_{1}\boldsymbol{\lambda},{}^{1}_{0}\boldsymbol{\lambda}} \quad v_f({}^{0}_{1}\boldsymbol{\lambda},{}^{1}_{0}\boldsymbol{\lambda}), \tag{5.74}$$

where v_f represents the combined individual hierarchical element optimization problems. Neglecting terms that do not depend on the Lagrange multipliers associated with the relaxation of the consistency constraints, the coordination problem is simplified and yields:

$$\max_{{}^{0}_{1}\boldsymbol{\lambda},{}^{1}_{0}\boldsymbol{\lambda}} \quad v_f({}^{0}_{1}\boldsymbol{\lambda},{}^{1}_{0}\boldsymbol{\lambda}) = {}^{0}_{1}\boldsymbol{\lambda}^T\left({}^{0}_{1}\mathcal{H}({}^{0}\mathbf{r}) - {}^{0}_{1}\mathbf{h}\right) + {}^{1}_{0}\boldsymbol{\lambda}^T\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right) \tag{5.75}$$

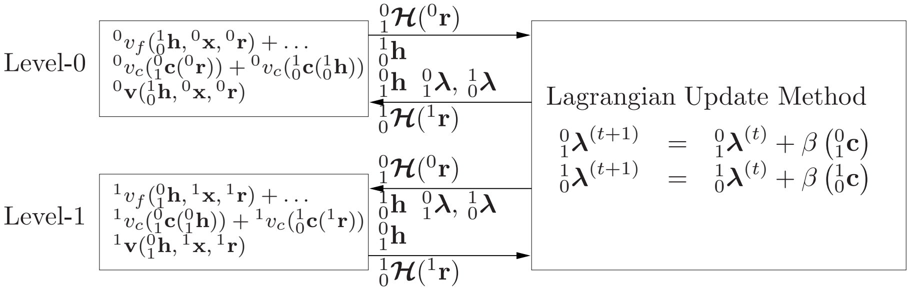


Figure 5.11: *Coordination in case of a non-hierarchic decomposition with consistency constraints that are relaxed with Lagrange multipliers. The coordinator redistributes the coupling data and computes new relaxation parameters for the Lagrangian function.*

Similar to the hierarchical case, the consistency constraints on both levels can be assigned individual Lagrange multipliers. In that case Equation 5.75 is split into two coordination problems:

$$\max_{^0_1\boldsymbol{\lambda}_1, {}^1_0\boldsymbol{\lambda}_1} \quad v_f(^0_1\boldsymbol{\lambda}_1, {}^1_0\boldsymbol{\lambda}_1) = {}^0_1\boldsymbol{\lambda}_1^T \left(^0_1\mathcal{H}(^0\mathbf{r}) - {}^0_1\mathbf{h}\right) + {}^1_0\boldsymbol{\lambda}_1^T \left(^1_0\mathcal{H}(^1\mathbf{r}) - {}^1_0\mathbf{h}\right) \tag{5.76}$$

$$\max_{^0_1\boldsymbol{\lambda}_2, {}^1_0\boldsymbol{\lambda}_2} \quad v_f(^0_1\boldsymbol{\lambda}_2, {}^1_0\boldsymbol{\lambda}_2) = {}^0_1\boldsymbol{\lambda}_2^T \left(^0_1\mathcal{H}(^0\mathbf{r}) - {}^0_1\mathbf{h}\right) + {}^1_0\boldsymbol{\lambda}_2^T \left(^1_0\mathcal{H}(^1\mathbf{r}) - {}^1_0\mathbf{h}\right) \tag{5.77}$$

If the consistency constraints are decomposed via a quadratic norm or an augmented Lagrangian function the relaxation parameters are updated with the same procedures introduced for the hierarchic top-down decomposition case. Therefore, these methods are not repeated here and the reader is referred to Section 5.2.1 and to Section 5.2.2. The methods are adjusted for non-hierarchic problems via an additional constraint that takes into account the coupling in the opposite direction.

The non-hierarchic relaxed coordination method is shown in Figure 5.11. Notice that in addition to the additional multipliers, additional coupling data is send from the Level-1 element to the Level-0 element to account for the second consistency constraint.

5.2.4 Design of a two-bar truss structure

In Chapter 3 the decomposition process was applied to a two-bar truss optimization problem. In this section results for relaxation-based coordination techniques on the two-bar truss optimization are presented. The relaxation based methods that were applied are:

- Hierarchic relaxed coordination: Collaborative Optimization (CO), Collaborative Optimization with Inexact Penalty Decomposition (CO IPD), Analytical Target Cascading (ATC);
- Hierarchic approximation methods: Analytical Target Cascading via Approximated Weight Update Method (ATC WUM);

Table 5.3: *Additional settings used for the multi-level coordination methods. The convergence parameters ε_{v_f} and ε_{v_c} are discussed in Section 5.3*

Method	${}^{1}_{0}s_{initial}$	${}^{2}_{0}s_{initial}$		ε_{v_f}	ε_{v_c}
CO				1×10^{-4}	1×10^{-4}
CO IPD			$\beta = 1.1$, $\gamma_{init.} = 1 \times 10^{-4}$	1×10^{-5}	1×10^{-5}
ATC	1×10^{-2}	1×10^{-2}	$\beta = 1.4$	1×10^{-6}	1×10^{-6}
ATC WUM	1×10^{-3}	1×10^{-3}	$\frac{2\times10^{-4}}{(t)}$ *	1×10^{-5}	1×10^{-5}
ATC AL	5×10^{-3}	5×10^{-3}	$\beta = 2.0$, $\gamma = 0.4$	5×10^{-6}	5×10^{-6}

* The allowed inconsistency is gradually reduced via dividing a measure of inconsistency by the iteration number of the hierarchical updates.

Table 5.4: *Numerical comparison of multi-level coordination techniques for relaxation-based hierarchic top-down decomposition. Total number of function evaluations, total number of optimization iterations and the total number of hierarchical updates are listed. Furthermore, the difference between the reference optimal design variables and those found via the multi-level optimization procedure are listed as well as the optimum found with respect to the reference optimum.*

Method	Func.eval.	Opt.iter	Hier.upd.	Sol.error	Obj./Obj.AiO
AiO	285		95	0.0	1.00
CO	214	42	2	6.44×10^{-1}	3.19
CO IPD	14568	3386	97	2.10×10^{-1}	1.01
ATC	2800	506	24	2.11×10^{-1}	1.01
ATC WUM	23847	2236	14	2.12×10^{-1}	1.02
ATC AL	7326	1674	47	2.11×10^{-1}	1.01

- Non-hierarchic relaxed coordination: Analytical Target Cascading via Augmented Lagrangian relaxation (ATC AL).

The optimization was performed on six different initial designs and the reference optimum was found solving the optimization problem All-in-One, i.e. without utilizing a multi-level optimization technique. Necessary parameters that are required by the multi-level optimization techniques to start the optimization are listed in Table 5.3.

The numerical costs were counted as described in Section 5.1.4 and are listed in Table 5.4. In addition, Table 5.5 lists the results of the consistency constraints ${}^{1}_{0}c$, ${}^{2}_{0}c$ (normalized with respect to reference coupling values ${}^{1}_{0}h$, ${}^{2}_{0}h$ respectively) and the individual element design constraints. At the reference optimum found via All-in-One all individual element design constraints are active. In addition, a lower bound on one of the design constraints is active ${}^{2}\underline{x}_{2}$. In case design constraints are not active the function value of the constraint is listed.

It is clear from Table 5.4 that the relaxation-based multi-level methods require significantly more function evaluations then the All-in-One optimization method. However, the amount of hierarchical updates is significantly reduced utilizing an Analytical Target Cascading (ATC) based approach. Collaborative Optimization (CO)

Table 5.5: *Consistency constraint values and design constraint values of the relaxation-based coordination methods. The reference design (AiO) has four active design constraints.*

Method	${}^1_0c$	${}^2_0c$	${}^0v_g$	${}^1v_g$	${}^2v_g$	${}^2\underline{x}_2$
AiO			active	active	active	active
CO	active	active	active	-6.8×10^{-1}	-3.5×10^{-2}	-2.4×10^{-4}
CO IPD	-2.6×10^{-4}	-2.7×10^{-4}	active	active	active	active
ATC	-2.0×10^{-6}	-6.7×10^{-8}	active	active	active	active
ATC WUM	-2.4×10^{-6}	7.9×10^{-9}	active	active	-2.3×10^{-2}	active
ATC AL	-7.1×10^{-7}	5.5×10^{-9}	active	-2.4×10^{-9}	active	active

prematurely converged to a non-optimal solution and Collaborative Optimization via Inexact Penalty Decomposition (CO IPD) still required many hierarchical updates before finally converging towards the constrained optimum.

Analytical Target Cascading (ATC) converged towards the same constraint optimum at less computational costs then the Inexact Penalty Decomposition (CO IPD) method. Analytical Target Cascading utilizing a weight update method (ATC WUM) converged to a point where the constraint at Level-0 and the constraint in Element-1 at Level-1 was active and the lower bound on the design variable in Element-2 was active. Analytical Target Cascading via Augmented Lagrangian relaxation (ATC AL) almost converged to the same constrained optimal point as Analytical Target Cascading (ATC), however the constraint in Element-1 at Level-1 was not active. Furthermore, the numerical costs were higher then those observed for Analytical Target Cascading (ATC).

During the multi-level optimization tests the following observations were made:

- Hierarchic relaxation-based coordination (CO): Although Collaborative Optimization is straightforward to implement the procedure prematurely converges to a non-optimal value making it less attractive for optimization.

- Hierarchic relaxation-based coordination (CO IPD): The inexact penalty decomposition approach solves the problem of premature convergence of the CO method, however the increments in the penalty parameter have to remain small not to prematurely converge. The result is that the method is relatively expensive. Furthermore, the reference optimum is not found exactly and the normalized inconsistency size is relatively large compared to the Analytical Target Cascading approaches indicating that a less consistent design is obtained.

- Hierarchic relaxation-based coordination (ATC): Analytical Target Cascading is straightforward to implement, however the computational cost compared to AiO is significant and similar to the Collaborative Optimization approaches the reference optimum is not found exact. Furthermore, some tuning of parameters ε_{v_c}, ε_{v_f}, β and ${}^i_0s_{initial}$ is required to find a configuration close to the reference optimum.

- Hierarchic approximation-based coordination (ATC WUM): The Weight Update Method effectively reduces the amount of hierarchical updates as required by the ATC method. No difficulties were observed calculating the necessary sensitivities for the method. These sensitivities were calculated via finite differences. If the target inconsistency size is set too small to compute the approximated weights these approximated weights become too large and pose problems for the individual optimization problems to converge to feasible configurations. Therefore, instead of setting the target inconsistency at once this target inconsistency is gradually reduced via:

$$\max\left\{\frac{2\times 10^{-4}}{(t)}, \varepsilon_{v_c}\right\} \tag{5.78}$$

  The effect is that the approximated weights gradually increase after each cycle through all the individual element optimizations instead of immediately imposing large weights on the consistency constraints after one single pass though the individual element optimizations. The reference optimum is not found, but rather a close approximation of the optimum. All the constraints are active except for the design constraint ${}^2v_g$ in Element-2 at Level-1. Furthermore, the gap in consistency is of equal magnitude as that of Analytical Target Cascading.

- Non-hierarchic relaxation-based coordination (ATC AL): The method does not find the reference optimum and significant computational costs are observed as compared to the All-in-One approach and Analytical Target Cascading. Some tuning of the parameters β, γ, ${}^i_0s_{initial}$, ε_{v_f} and ε_{v_c} is necessary in order to find a solution that is close to the optimal reference solution.

In general the relaxation-based methods are easier to implement then equality-based methods, because no additional gradient information is required by the individual elements from the neighboring elements which is essential in the case of equality based coordination methods. A drawback of the relaxation is that during the optimization no feasible solution is available, only at convergence of the method the inconsistencies have vanished and a consistent design is obtained. Furthermore, except for the hierarchic approximation method (ATC WUM) all knowledge on individual design problem changes due to changes in neighboring elements is lost.

5.3 Indicators for performance

Performance of a multi-level optimization is measured inside the individual elements and at the coordinator. The performance of individual elements is defined as local performance, while the performance of the coordinator is defined as global performance. A third means of measuring performance is the ability of finding an optimal point. These three performance measurements are discussed in more detail in the following sections.

5.3.1 Local performance of elements

Local performance of individual elements is expressed as the total amount of function evaluations and the total amount of optimization iterations for the individual elements. The total amount of function evaluations is measured as the number of times the physical responses ($\mathbf{r}$) are evaluated and the total number of optimization iterations is measured as the number of times the functions associated with the optimization problem ($\mathbf{v}$) are evaluated.

The individual element optimizations are repeated until local convergence, which is mathematically expressed as:

$$ {}^{i}\varepsilon_{v_f} = ||{}^{i}v_f^{(t+1)} - {}^{i}v_f^{(t)}||_\infty, \tag{5.79} $$

The objective function values are compared between two successive completed optimizations of the element optimization problem. This objective function value changes between two successive element optimizations due to updated coupling variables (${}^{j}_{i}\mathbf{h}$) and/or updated shared design variables (${}^{j}\mathbf{z}$) and/or changes due to increments of the relaxation parameters in case relaxation of the consistency constraints is applied. Hence, the cost of optimizing the individual elements lies in the optimization algorithm chosen and updates of coupling variables, shared design variables and/or changes in the relaxation parameters.

5.3.2 Global performance of complete hierarchy

Global performance of the entire hierarchy is expressed as the total amount of hierarchical updates. Hence, the number of times data is exchanged between individual elements. Exchange of data can become a time consuming process when individual elements are evaluated on different machines connected via a network or when processes that are related to the multi-level optimization have to be stopped in order to make the data exchange possible.

The data exchange or updating of coupling variables is mathematically expressed as:

$$ {}^{i}_{j}\mathbf{h}^{(t+1)} = {}^{i}_{j}\mathbf{h}^{(t)} + \tau^{(t)}\left({}^{i}_{j}\mathbf{h}^{(t+1)} - {}^{i}_{j}\mathbf{h}^{(t)}\right); \tag{5.80} $$

$$ {}^{j}_{i}\mathbf{h}^{(t+1)} = {}^{j}_{i}\mathbf{h}^{(t)} + \tau^{(t)}\left({}^{j}_{i}\mathbf{h}^{(t+1)} - {}^{j}_{i}\mathbf{h}^{(t)}\right); \tag{5.81} $$

$$ {}^{i}\mathbf{x}^{(t+1)} = {}^{i}\mathbf{x}^{(t)} + \tau^{(t)}\left({}^{j}\mathbf{x}^{(t+1)} - {}^{j}\mathbf{x}^{(t)}\right); \tag{5.82} $$

where $\tau^{(t)}$ is an adjustable step-size.

The costs of exchanging data is related to the size of inconsistencies and rate of change in the inconsistencies. Therefore, convergence of the coordination problem is evaluated via measuring the inconsistency size and rate of change of the inconsistency, mathematically expressed as:

$$ \varepsilon_{v_c} = \max\left\{||{}^{i}\mathbf{v}_c^{(t+1)}||_\infty, ||{}^{i}\mathbf{v}_\mathbf{c}^{(t+1)} - {}^{i}\mathbf{v}_c^{(t)}||_\infty\right\}. \tag{5.83} $$

In some cases (e.g. the Null-Space method (Chapter 6)) Equation 5.83 is replaced via a merit function. The merit function provides an alternative means of measuring convergence of the coordination algorithm. Typically merit functions are used for multi-level coordination methods. The mathematical form of the merit function depends on the type of coordinator chosen.

Performance of the coordinator is mathematically expressed as the rate of convergence (α) of the coupling variables (${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}$) and/or shared design variables (${}^{\cdot\cdot}\mathbf{x}$) towards their optimal values (${}^{\cdot\cdot}_{\cdot\cdot}\mathbf{h}_{opt}$ and ${}^{i}\mathbf{x}_{opt}$:

$$\left.\begin{aligned} ||{}^{i}_{j}\mathbf{h}^{(t+1)} - {}^{i}_{j}\mathbf{h}_{opt.}||_2 &\leq \alpha||{}^{i}_{j}\mathbf{h}^{(t)} - {}^{i}_{j}\mathbf{h}_{opt.}||_2 \\ ||{}^{j}_{i}\mathbf{h}^{(t+1)} - {}^{j}_{i}\mathbf{h}_{opt.}||_2 &\leq \alpha||{}^{j}_{i}\mathbf{h}^{(t)} - {}^{j}_{i}\mathbf{h}_{opt.}||_2 \\ ||{}^{i}\mathbf{x}^{(t+1)} - {}^{i}\mathbf{x}_{opt.}||_2 &\leq \alpha||{}^{i}\mathbf{x}^{(t)} - {}^{i}\mathbf{x}_{opt.}||_2 \end{aligned}\right\} \quad 0 < \alpha < 1 \quad \text{linear} \tag{5.84}$$

$$\left.\begin{aligned} ||{}^{i}_{j}\mathbf{h}^{(t+1)} - {}^{i}_{j}\mathbf{h}_{opt.}||_2 &\leq \alpha^{(t)}||{}^{i}_{j}\mathbf{h}^{(t)} - {}^{i}_{j}\mathbf{h}_{opt.}||_2 \\ ||{}^{j}_{i}\mathbf{h}^{(t+1)} - {}^{j}_{i}\mathbf{h}_{opt.}||_2 &\leq \alpha^{(t)}||{}^{j}_{i}\mathbf{h}^{(t)} - {}^{j}_{i}\mathbf{h}_{opt.}||_2 \\ ||{}^{i}\mathbf{x}^{(t+1)} - {}^{i}\mathbf{x}_{opt.}||_2 &\leq \alpha^{(t)}||{}^{i}\mathbf{x}^{(t)} - {}^{i}\mathbf{x}_{opt.}||_2 \end{aligned}\right\} \quad \alpha^{(t)} \rightarrow 0 \quad \text{superlinear} \tag{5.85}$$

$$\left.\begin{aligned} ||{}^{i}_{j}\mathbf{h}^{(t+1)} - {}^{i}_{j}\mathbf{h}_{opt.}||_2 &\leq \alpha||{}^{i}_{j}\mathbf{h}^{(t)} - {}^{i}_{j}\mathbf{h}_{opt.}||_2^2 \\ ||{}^{j}_{i}\mathbf{h}^{(t+1)} - {}^{j}_{i}\mathbf{h}_{opt.}||_2 &\leq \alpha||{}^{j}_{i}\mathbf{h}^{(t)} - {}^{j}_{i}\mathbf{h}_{opt.}||_2^2 \\ ||{}^{i}\mathbf{x}^{(t+1)} - {}^{i}\mathbf{x}_{opt.}||_2 &\leq \alpha||{}^{i}\mathbf{x}^{(t)} - {}^{i}\mathbf{x}_{opt.}||_2^2 \end{aligned}\right\} \quad \alpha \epsilon \Re \quad \text{quadratic} \tag{5.86}$$

These performance measures measure the rate of convergence of the coordinator problem and not that of the individual optimization problems.

Figure 5.12 shows typical convergence rates of relaxation-based bi-level coordination algorithms. Initial convergence rates are due to the optimizer algorithm used for the individual element optimizations. After a few iterations in which data between elements is exchanged the bi-level coordination algorithms exhibit linear convergence rate. The linear convergence rate is due to the fixed point iteration step embedded in the bi-level coordination algorithms. A solution is found via iterating between:

$$ {}^{i}_{j}\boldsymbol{\mathcal{H}}^{(t+1)}({}^{i}\mathbf{r}^{(t+1)}) \Leftrightarrow {}^{j}v_f^{(t)}({}^{i}_{j}\mathbf{h}^{(t)}), \tag{5.87}$$

until a stable solution is found. The coupling variables computed for element ${}^{i}\ldots$ depend on the outcome of the optimization of element ${}^{j}\ldots$. However, the optimization of element ${}^{j}\ldots$ depends on the outcome of the optimization of element ${}^{i}\ldots$.

The convergence rate of a fixed point iteration process can be improved via e.g. Newton's method. The basic idea is to add gradient information to the exchanged coupling data. Equality-based multi-level methods rely on gradient information and/or other means of approximations, however none of the relaxation-based techniques has this property. Multi-level coordination methods are promising in this sense because the coordination of coupling data takes into account the characteristics of the underlying coordination problem.

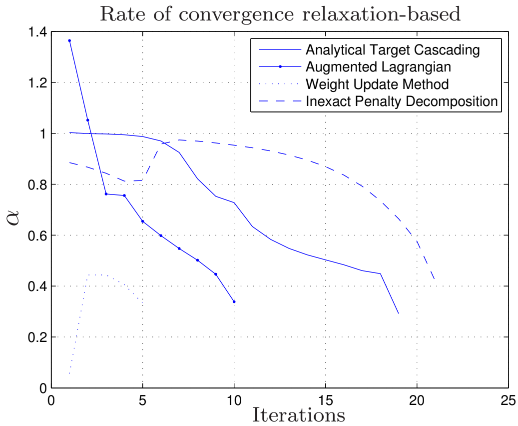


Figure 5.12: *Rate of convergence α for relaxation-based bi-level coordination methods. Initial convergence rates are due to the optimization algorithm used for the individual element optimizations. After a few iterations in which data is exchanged between elements all methods show linear convergence rate.*

5.3.3 Ability of finding the optimal point

The ability of a multi-level optimization algorithm to find the optimal point is a difficult measurement since the optimal point should be know beforehand. However, it is often used as an argument to choose one multi-level method over another multi-level method. Especially if a mathematical validation of the method is present this argument is explicitly pointed out in favor over other methods for which no mathematical justification is known.

The ability of an algorithm to find the optimal point is in the present thesis measured via the mathematical expression:

$$\left\| \frac{\mathbf{x}}{\mathbf{x}_{opt.}} - 1 \right\|_{\infty}. \tag{5.88}$$

Typically the optimal point is found performing an All-in-One optimization starting from a large number of initial design points that cover the entire design domain.

In practice the local and global performance rates are just as important measurements since these give insight into the time it will take to find a feasible design. An algorithm that can find the optimum at a very low convergence rate α where the intermediate designs are infeasible may be less attractive then a prematurely converting algorithm that has a higher convergence rate and produces a non-optimal design that is still better then the initial design.

5.4 Discussion

In this chapter the bi-level coordination methods were classified into two groups: equality-based coordination methods and relaxation-based coordination methods. Each group was subdivided into three smaller categories: hierarchic coordination methods, hierarchic approximation based methods and non-hierarchic methods.

To summarize the multi-level methods that were discussed, Figure 5.13 presents an overview of each method classified into each individual group. The methods differ in individual problem definition, however, in general two distinct formulations can be identified for equality-based and relaxation-based problems that require coordination.

How specific data is coordinated between the individual element optimizations is characteristic for each type of method. For equality-based coordination three types of coordination were distinguished. Coordination by means of calculating sensitivity data, coordinating by means of distributing design freedom and non-hierarchic coordination based on sensitivity analysis. Relaxation based coordination methods were also characterized into three different approaches. Coordination by assigning increments in relaxation parameters, coordination by means of approximations and non-hierarchic coordination by means of increments in relaxation parameters.

From each group a few methods were tested on a two-bar truss design problem. The equality-based coordination methods, except for Concurrent SubSpace Optimization, were able to find the reference optimum. Using approximations via Response Surfaces reduced the computational costs, however the design obtained had a higher objective function value then the configuration found without utilizing a Response Surface. It was observed that constructing a Response Surface using a Bi-Level Integrated System Synthesis approach was easier then constructing a Response Surface for a Quasi-separable Subsystem Decomposition approach. The relaxation-based coordination methods, except for Collaborative Optimization, were able to get close to the reference optimal design. However, none of these approaches found the reference optimal point exact.

The computational costs were compared showing that in terms of function evaluations and optimization iterations the costs were of the same order when comparing equality-based and relaxation-based coordination methods. Best results in terms of function evaluations, optimization iterations and optimal design were observed for Optimization by Linear Decomposition. In terms of hierarchical updates, a means of measuring the communication between individual elements, fewer hierarchical updates were observed especially when utilizing approximations. However, the difference in amount of hierarchical updates is small between the equality-based coordination approaches and the relaxation-based coordination approaches.

In the next chapter two multi-level coordination methods are discussed which have more robust convergence behavior then the bi-level methods discussed in this chapter. From the classification of the methods in this chapter and Chapter 6 a framework for multi-level optimization is presented in Chapter 7.

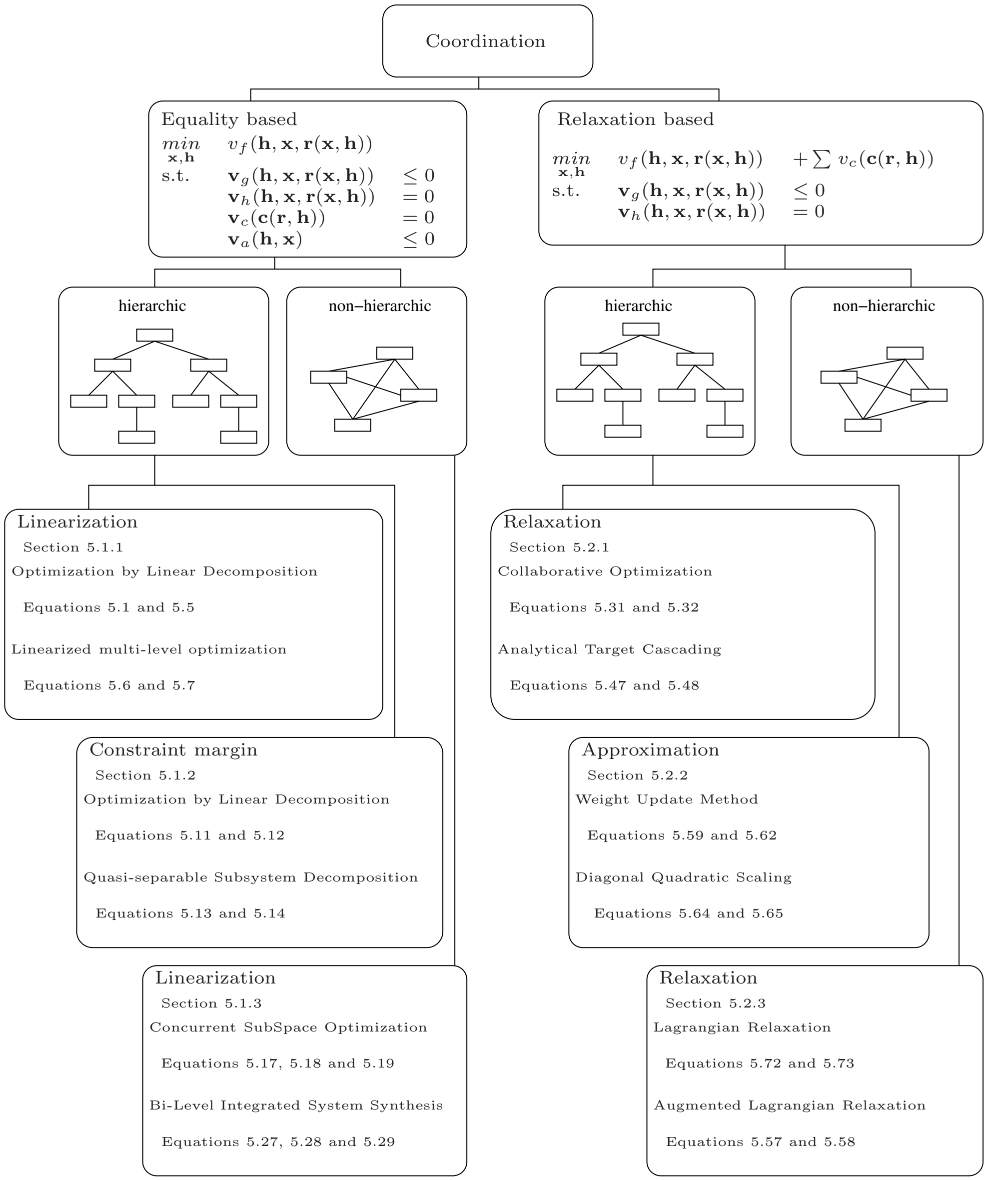


Figure 5.13: *Overview of coordination approaches classified into two main stream approaches. The first (left): methods that coordinate individual elements where the decomposition is based on equality consistency constraints. The second (right): methods that coordinate individual elements where the decomposition is based on relaxed consistency constraints. Each main stream approach is subdivided into three classifications. The equality-based coordination into: linearization, approximation and non-hierarchic methods. The relaxation-based coordination into: relaxation, approximation and non-hierarchic relaxation. Each bi-level optimization method discussed in this thesis falls into one of these categories.*

Chapter 6

Outlook: Multi-level coordination methods

The previous chapter discussed Bi-level coordination techniques. If the number of individual elements in the hierarchy becomes large, Bi-level coordination techniques become less attractive in terms of computational costs. A solution is to extend the Bi-level coordination technique to a multi-level coordination technique.

In this chapter an outlook to multi-level coordination methods is presented. The derivation of the multi-level coordination methods is developed from optimality criteria for coupled problems (see Chapter 3) and the introductory chapter on coordination (see Chapter 4). Two multi-level coordination methods are presented: a Null-space method which is discussed in Section 6.2 and a Schur-complement reduction method which is discussed in Section 6.3.

6.1 Introduction

Multi-level coordination methods are derived from techniques that are developed for solving large systems of equations. If multi-level optimization problems could be written as an all-in-one problem, then a large system of equations would need to be solved. In practice such an approach may not be feasible. However, assuming that all the equations could be solved as a single large system. What are the necessary steps to split the large problem into smaller problems? Can individual element optimization problems be identified from these smaller problems? What coordination techniques are then necessary to coordinate the individual elements to a solution of the entire hierarchy?

In the present thesis the large system of equations to solve is identified as the equations that find a solution to the first-order conditions of a consistency constraint optimization problem (Chapter 3). In the present chapter methods are discussed that solve a large system of equations via a technique based on a Null-space method and via an approach based on Schur-complement reduction. To introduce the necessary equations first a two-level hierarchy is considered consisting of a single element per level. Subsequently, the equations derived from this two-level hierarchy are extended to multiple levels.

6.2 Null-space method

The basic idea of a null-space method is to reduce the dimension of the problem by first computing a step towards the solution of the equality constraints and then minimizing the objective restricted to the null-space of the linearized constraints. The resulting minimization problem is smaller then the original one. Alexandrov and Dennis (1994) (see also, (Alexandrov, 1998, Alexandrov *et al.*, 1998)) adopted a null-space method called tangent space approach to formulate a multi-level optimization method combined with a trust region approach.

Alexandrov and Dennis assume that the multi-level optimization problem is of the form:

$$\begin{array}{llll} \min\limits_{{}^0\mathbf{x},{}^j\mathbf{x},{}^j_i\mathbf{h}} & v_f & = & {}^0v_f\left({}^j_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}({}^j_0\mathbf{h},{}^0\mathbf{x})\right) \\ \text{s.t.} & & & {}^j_i\mathbf{c}({}^j_i\mathbf{h},{}^j\mathbf{r}({}^i_j\mathbf{h},{}^j\mathbf{x}))=0 \end{array}, \tag{6.1}$$

where $i = 0, 1, 2, .., 1.1, ..; j = 1, 2, .., 1.1, ..; i \neq j$ and i representing a higher level then j. Hence, a top-down hierarchic decomposition using equality consistency constraints is considered and the objective function depends solely on Level-0 design variables and physical responses. Shared design variables ${}^i_j\mathbf{z}$ are omitted in the current derivation because these are treated similarly as the coupled physics for the null-space method.

To simplify the derivation, a two-level hierarchy of a single individual element per level is considered, hence $i = 0$ and $j = 1$. For brevity of notation the dependence of the physical responses $\mathbf{r}$ on the design variables $\mathbf{x}$ is omitted. A first step in solving the first order conditions is to formulate the Lagrangian function of the AiO optimization problem:

$$\mathcal{L}({}^0\mathbf{x},{}^1\mathbf{x},{}^1_0\mathbf{h},{}^1_0\boldsymbol{\lambda}) \quad = \quad {}^0v_f({}^1_0\mathbf{h},{}^0\mathbf{x},{}^0\mathbf{r}) + {}^1_0\boldsymbol{\lambda}^T{}^1_0\mathbf{c}({}^1_0\mathbf{h},{}^1\mathbf{r}) \tag{6.2}$$

To simplify the notation, the vector $\mathbf{y} = \left[\begin{array}{ccc} {}^0\mathbf{x} & {}^1_0\mathbf{h} & {}^1\mathbf{x} \end{array}\right]^T$ is introduced as well as $v = {}^0v_f$, $\boldsymbol{\lambda} = {}^1_0\boldsymbol{\lambda}$ and $\mathbf{c} = {}^1_0\mathbf{c}$. Necessary conditions for a stationary point of Equation 6.2 are:

$$\begin{array}{lll} \nabla_{\mathbf{y}}\mathcal{L} = \nabla_{\mathbf{y}}v_f + \boldsymbol{\lambda}^T\nabla_{\mathbf{y}}\mathbf{c} & = & 0 \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L} = \mathbf{c} & = & 0 \end{array}. \tag{6.3}$$

A linear Taylor series expansion up to first order of Equation 6.3 yields:

$$\nabla\mathcal{L}(\mathbf{y}+\Delta\mathbf{y},\boldsymbol{\lambda}+\Delta\boldsymbol{\lambda}) \quad = \quad \nabla\mathcal{L} + \nabla\left(\nabla\mathcal{L}\right)\left[\Delta\mathbf{y},\Delta\boldsymbol{\lambda}\right]^T . \tag{6.4}$$

A stationary point to Equation 6.2 is found setting the left hand side of Equation 6.4 to zero. Introducing $\Delta\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(t+1)} - \boldsymbol{\lambda}^{(t)}$, Equation 6.4 can be rewritten as[1]:

$$\begin{array}{lcl} \nabla^2_{\mathbf{y},\mathbf{y}}\mathcal{L}\Delta\mathbf{y} + \nabla_{\mathbf{y}}\mathbf{c}\boldsymbol{\lambda}^{(t+1)} + \nabla_{\mathbf{y}}v_f & \approx & 0 \\ \nabla_{\mathbf{y}}\mathbf{c}^T\Delta\mathbf{y} + \mathbf{c} & \approx & 0 \end{array}. \tag{6.5}$$

These equations may be viewed as the first order conditions for a quadratic model where the design variables are $\mathbf{y}$ and the multipliers $\boldsymbol{\lambda}^{(t+1)}$.

A common approach to solve the quadratic model of which Equation 6.5 are the first order conditions is via Sequential Quadratic Programming (Haftka and Gürdal, 1999). An iterative procedure that solves at each step $^{(t)}$ the quadratic problem:

$$\begin{array}{lcl} \min\limits_{\Delta\mathbf{y}} & \frac{1}{2}\Delta\mathbf{y}^T\nabla^2_{\mathbf{y},\mathbf{y}}\mathcal{L}\Delta\mathbf{y} + \nabla_{\mathbf{y}}\mathcal{L}\Delta\mathbf{y} + v_f & \\ \text{s.t.} & \nabla_{\mathbf{y}}\mathbf{c}^T\Delta\mathbf{y} + \mathbf{c} & = 0 \\ & ||\Delta\mathbf{y}||_2 \leq \delta & \end{array} \tag{6.6}$$

where δ is the radius of the trust region, $\Delta\mathbf{y}$ is a vector that in the current derivation contains $\Delta^0\mathbf{x}$, $\Delta^1_0\mathbf{h}$ and $\Delta^1\mathbf{x}$ (hence, $\Delta\mathbf{y} = \left[\begin{array}{ccc} \Delta^0\mathbf{x} & \Delta^1_0\mathbf{h} & \Delta^1\mathbf{x} \end{array}\right]^T$), the multipliers of the quadratic problem are $\Delta\boldsymbol{\lambda}$, which relate to the multipliers ($\boldsymbol{\lambda}^{(t+1)}$) of Equation 6.5 as $\Delta\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(t+1)} - \boldsymbol{\lambda}^{(t)}$ and the term $\nabla_{\mathbf{y}}\mathcal{L}$ is defined as $\nabla_{\mathbf{y}}\mathcal{L} = \nabla_{\mathbf{y}}v_f + {\boldsymbol{\lambda}^{(t)}}^T\nabla_{\mathbf{y}}\mathbf{c}$, with $\mathbf{y} = \left[\begin{array}{ccc} {}^0\mathbf{x} & {}^1_0\mathbf{h} & {}^1\mathbf{x} \end{array}\right]$ and $\boldsymbol{\lambda}^{(t)}$ is known at the current iterate $^{(t)}$.

In Equation 6.6 the linearized constraint and the trust-region constraint may conflict. Hence, the quadratic problem does not have a solution. One approach to overcome this problem is the Tangent-Space approach (Dennis *et al.*, 1997). This approach relies on a splitting of the search for an optimal solution into two (or more) directions. Alexandrov (1998) identifies the splitting of the search for an optimal solution into two (or more) directions as solving a multi-level optimization or multi-disciplinary design optimization problem. In the context of multi-level optimization or multi-level design optimization the search directions represent the individual element optimization problems (elements such as used throughout the present thesis).

6.2.1 Extending to multiple levels

Multi-level optimization problems are characterized via a hierarchy that is present. Due to the decomposition of the optimization problem into separate levels and individual elements a number of consistency constraints is introduced. To derive the Tangent-Space approach, see Dennis *et al.* (1997), Equation 6.6 is extended to an optimization problem covering multiple levels as follows.

The Lagrangian function is formulated for the optimization problem covering the entire hierarchy as was done in Equation 6.2. This Lagrangian function is differentiated with respect to the design variables (${}^i\mathbf{x}$), coupling variables (${}^i_j\mathbf{h}$) and Lagrange multipliers (${}^i_j\boldsymbol{\lambda}$). The resulting expressions are set to zero (first order conditions).

[1] $\nabla^2_{\mathbf{y},\mathbf{y}}\mathcal{L}\Delta\mathbf{y} + \nabla_{\mathbf{y}}\mathbf{c}\Delta\boldsymbol{\lambda} = -\nabla_{\mathbf{y}}v_f - \nabla_{\mathbf{y}}\mathbf{c}\boldsymbol{\lambda}^{(t)}$ introducing $\Delta\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(t+1)} - \boldsymbol{\lambda}^{(t)}$ gives $\nabla^2_{\mathbf{y},\mathbf{y}}\mathcal{L}\Delta\mathbf{y} + \nabla_{\mathbf{y}}\mathbf{c}\boldsymbol{\lambda}^{(t+1)} = -\nabla_{\mathbf{y}}v_f$

Vector $\mathbf{y}$ is introduced combining all design variables and coupling variables into a single vector. The first order conditions are then linearized via a first order Taylor series expansion (similar to Equation 6.4). Introducing $\Delta\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(t+1)} - \boldsymbol{\lambda}^{(t)}$ yields Equation 6.5 for a problem of multiple levels and elements. In matrix-vector format:

$$\begin{bmatrix} \nabla^2_{\mathbf{y},\mathbf{y}}\mathcal{L} & \nabla_{\mathbf{y}}\mathbf{c} \\ \nabla_{\mathbf{y}}\mathbf{c}^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{y} \\ \boldsymbol{\lambda}^{(t+1)} \end{bmatrix} = - \begin{bmatrix} \nabla_{\mathbf{y}}v \\ \mathbf{c} \end{bmatrix} \tag{6.7}$$

where $\nabla_{\mathbf{y}}\mathbf{c}$ is a block diagonal matrix:

$$\begin{bmatrix} \nabla_{\mathbf{y}_0^1}\mathbf{c} & & \\ & \ddots & \\ & & \nabla_{\mathbf{y}_i^j}\mathbf{c} \end{bmatrix} \tag{6.8}$$

For the Tangent-Space approach no special structure of the matrix $\nabla_{\mathbf{y}}\mathbf{c}$ is required. The Tangent-Space approach can be applied to find a solution to Equation 6.7.

6.2.2 Sub step conditions

The tangent space approach splits the trial step $\Delta\mathbf{y}$ into a normal component $\Delta\mathbf{y}_\mathbf{n}$ and a tangential component $\Delta\mathbf{y}_\mathbf{t}$. Therefore, the trial step is written as;

$$\Delta\mathbf{y} = \Delta\mathbf{y}_\mathbf{n} + \Delta\mathbf{y}_\mathbf{t} \tag{6.9}$$

where $\Delta\mathbf{y}_\mathbf{n}$ is the normal component which is inside the trust-region and the normal direction to the null-space of the constraint Jacobian, thus $\mathcal{N}(\nabla_{\mathbf{y}}\mathbf{c}^T)$. The vector $\Delta\mathbf{y}_\mathbf{t}$ is the component of the step in the tangent space of the constraints and $\Delta\mathbf{y}_\mathbf{t}$ is defined[2] as $\Delta\mathbf{y}_\mathbf{t} = \mathbf{T}\Delta\widehat{\mathbf{y}}_\mathbf{t}$. Where $\mathbf{T}$ is a matrix whose columns form a basis for $\mathcal{N}(\nabla_{\mathbf{y}}\mathbf{c}^T)$.

The objective function relies on the Level-0 design variables increment $\Delta^0\mathbf{x}$ and coupling variables increment $\Delta_0^i\mathbf{h}$ and the consistency constraint functions rely on the lower level design variables increment $\Delta^1\mathbf{x}, \ldots, \Delta^j\mathbf{x}$ and coupling variables increment $\Delta_i^j\mathbf{h}$. Therefore, the splitting of the design vector increment into a normal direction to the null-space of the constraint Jacobian can be seen as the Element-j component of the design problem and the step into the tangent space of the constraints can be seen as the Level-0 component of the design problem. Hence, Equation 6.9 is rewritten as:

$$\Delta\mathbf{y} = \underbrace{\Delta^1\mathbf{y} + \ldots + \Delta^n\mathbf{y}}_{\textbf{normal component}} + \underbrace{\Delta^0\mathbf{y}}_{\textbf{tangential component}} \tag{6.10}$$

for a hierarchy of $n+1$ elements.

A complete derivation of the individual element optimization problems is beyond the scope of the present thesis. Therefore, the interested reader is referred to Dennis *et al.* (1997) for a complete derivation of the tangent space method. Essential is that in each individual optimization problem a direction is computed that

[2]Recall Chapter 4 for an introduction to the null-space method.

satisfies criteria that make sure that each calculated step is into the direction of the consistency constraints. Furthermore, procedures are defined that make sure that each total step of the entire hierarchy is into the direction that minimizes the Level-0 objective function ${}^{0}v_f$.

6.3 Schur-complement reduction

DeMiguel and Nogales (2008) discussed the poor convergence characteristics of the well established Bi-Level algorithms in multi-level optimization. The authors propose the use of interior point methods which are well established solution techniques for large optimization problems. Especially the super linear or quadratic convergence properties of these methods outperform the Bi-level procedures. However, the mathematical assumptions on which these algorithms rely are more strict then that of Bi-level algorithms. In their work, DeMiguel and Nogales study a Schur interior point method that can only be applied to problems that satisfy the so-called Strict Linear Independent Constraint Qualification (SLICQ), see DeMiguel and Nogales (2008) for a definition. In contrast, Bi-level optimization methods can be applied to problems that satisfy the less restrictive so-called Linear Independent Constraint Qualification. Furthermore, Newton approximations rely on the use of a single optimization approach, whereas Bi-level algorithms do not require the use of the same optimization algorithm in each subsystem. DeMiguel and Nogales show a number of steps to overcome these difficulties such that interior point methods can be applied as a coordination method for multi-level optimization problems. These steps are discussed in the following sections.

6.3.1 The Schur-complement

DeMiguel and Nogales consider multi-level optimization problems of the form:

$$\begin{array}{ll} \min\limits_{{}^{0}\mathbf{x},{}^{j}\mathbf{x},{}^{j}_{i}\mathbf{h}} & v_f = v_f\left({}^{0}\mathbf{x}, {}^{j}\mathbf{x}, {}^{j}_{i}\mathbf{h}, \mathbf{r}({}^{0}\mathbf{x}, {}^{j}\mathbf{x}, {}^{j}_{i}\mathbf{h})\right) \\ \text{s.t.} & {}^{j}_{i}\mathbf{c}({}^{j}_{i}\mathbf{h}, {}^{j}\mathbf{r}({}^{j}\mathbf{x}, {}^{j}_{i}\mathbf{h})) = \mathbf{0} \end{array} , \tag{6.11}$$

where $i = 0, 1, 2, .., 1.1, ..$; $j = 1, 2, .., 1.1, ..$; $i \neq j$ and i representing a higher level then j. Shared design variables ${}^{j}_{i}\mathbf{z}$ are not considered in the present derivation because the consistency constraints for shared design variables are treated similarly as the consistency constraints for the coupled physical responses. Furthermore, for the current derivation the multi-level hierarchy is reduced to a two-level hierarchy consisting of a single individual element per level. Therefore, the multi-level optimization problem that is considered yields:

$$\begin{array}{ll} \min\limits_{{}^{0}\mathbf{x},{}^{1}\mathbf{x},{}^{1}_{0}\mathbf{h}} & v_f = v_f\left({}^{1}_{0}\mathbf{h}, {}^{0}\mathbf{x}, {}^{1}\mathbf{x}, {}^{0}\mathbf{r}({}^{0}\mathbf{x}, {}^{1}_{0}\mathbf{h}), {}^{1}\mathbf{r}({}^{1}\mathbf{x}, {}^{0}_{1}\mathbf{h})\right) \\ \text{s.t.} & {}^{1}_{0}\mathbf{c}({}^{1}_{0}\mathbf{h}, {}^{1}\mathbf{r}({}^{1}\mathbf{x}, {}^{0}_{1}\mathbf{h})) = \mathbf{0} \end{array} \tag{6.12}$$

Hence, a top-down hierarchic decomposition using equality consistency constraints is considered.

For brevity of notations the dependence of the physical responses $\mathbf{r}$ on the design variables $\mathbf{x}$ and mapped physical responses $\mathbf{h}$ is omitted. To introduce the Schur-complement first the first-order conditions for the Lagrangian function:

$$\mathcal{L}({}^0\mathbf{x}, {}^1\mathbf{x}, {}^1_0\mathbf{h}, {}^1_0\boldsymbol{\lambda}) = v_f({}^1_0\mathbf{h}, {}^0\mathbf{x}, {}^1\mathbf{x}, {}^0\mathbf{r}, {}^1\mathbf{r}) + {}^1_0\boldsymbol{\lambda}^T {}^1_0\mathbf{c}({}^1_0\mathbf{h}, {}^1\mathbf{r}) \tag{6.13}$$

of the problem defined in Equation 6.12 are formulated. Necessary conditions for a stationary point of Equation 6.13 are:

$$\begin{array}{lll} \nabla_{\mathbf{y}}\mathcal{L} = \nabla_{\mathbf{y}} v_f + {}^1_0\boldsymbol{\lambda}^T \nabla_{\mathbf{y}} {}^1_0\mathbf{c} & = & 0 \\ \nabla_{{}^1_0\boldsymbol{\lambda}}\mathcal{L} = {}^1_0\mathbf{c} & = & 0 \end{array}, \tag{6.14}$$

where $\left[\begin{array}{cccc} {}^0\mathbf{x} & {}^1_0\mathbf{h} & {}^1\mathbf{x} & {}^1_0\boldsymbol{\lambda} \end{array}\right] = \mathbf{y}$.

A Taylor series expansion up to first-order that is evaluated at the current step yields:

$$\left[\begin{array}{cc} \mathbf{A} & -\mathbf{C} \\ -\mathbf{C}^T & \mathbf{M} \end{array}\right] \left[\begin{array}{c} \Delta^0\mathbf{y} \\ \Delta^1\mathbf{y} \end{array}\right] = - \left[\begin{array}{c} {}^0\mathbf{w} \\ {}^1\mathbf{w} \end{array}\right] \tag{6.15}$$

where

$$\begin{array}{lll} \Delta^0\mathbf{y} & = & \left[\begin{array}{cc} \Delta^0\mathbf{x} & \Delta^1_0\mathbf{h} \end{array}\right]^T \\ \Delta^1\mathbf{y} & = & \left[\begin{array}{cc} \Delta^1\mathbf{x} & \Delta^1_0\boldsymbol{\lambda} \end{array}\right]^T \\ {}^0\mathbf{w} & = & \left[\nabla_{{}^0\mathbf{y}}\mathcal{L}\right]^T \\ {}^1\mathbf{w} & = & \left[\begin{array}{cc} \nabla_{{}^1\mathbf{y}}\mathcal{L} & {}^1_0\mathbf{c} \end{array}\right]^T \end{array} \tag{6.16}$$

which means that $\mathbf{A} = \nabla_{{}^0\mathbf{y}} {}^0\mathbf{w}$, $\mathbf{C} = -\nabla_{{}^0\mathbf{y}} {}^1\mathbf{w} = -\left(\nabla_{{}^1\mathbf{y}} {}^0\mathbf{w}\right)^T$ and $\mathbf{M} = \nabla_{{}^1\mathbf{y}} {}^1\mathbf{w}$.

One approach to solve Equation 6.15 is via Schur-complement reduction. The Schur-complement approach is a well-established approach within domain decomposition methods. The method relies on a splitting of the system of equations into internal degrees of freedom of the individually connected domains and the degrees of freedom on the interfaces that connect the individual domains. The degrees of freedom of the internal domains are eliminated and the resulting Schur-complement matrix is used to solve the interfaces. This allows for parallelization of computations in the sense that computations on internal degrees of freedom can be done independently from neighboring domains after the interfaces are evaluated.

In the context of multi-level optimization the interfaces associated with the Schur-complement approach are the gradients of the consistency constraints. The individual connected domains associated with the Schur-complement approach are the individual element optimization problems that enter the iteration matrix of the interior point method via Equation 6.13 into Equation 6.15.

The Schur-complement of the iteration matrix in Equation 6.15 is defined:

$$\mathbf{S} = \mathbf{A} - \mathbf{C}^T \left(\mathbf{M}^{-1}\right) \mathbf{C} \tag{6.17}$$

where it is assumed that matrix $\mathbf{M}$ is invertible.

DeMiguel and Nogales discuss the necessary conditions for $\mathbf{M}$ and $\mathbf{S}$ to be invertible. Linear independence constraint qualification and second-order sufficient conditions for Level-1 imply that matrix $\mathbf{M}$ is invertible. Linear independence constraint qualification means that the gradients of the constraints $\nabla \mathbf{c}$ that are active are linear independent. Consistency constraints can only be inactive if there is no coupling between individual elements and therefore it is unlikely that such consistency constraints are present.

If the Schur-complement matrix $\mathbf{S}$ is also invertible the Level-0 part of the search direction is found as:

$$\mathbf{S}\Delta^0\mathbf{y} = -\left({}^0\mathbf{w} - \mathbf{C}^T\left(\mathbf{M}^{-1}\right){}^1\mathbf{w}\right) \tag{6.18}$$

and the Level-1 part of the search direction is found as:

$$\mathbf{M}\Delta^1\mathbf{y} = -\left({}^1\mathbf{w} - \mathbf{C}\Delta^0\mathbf{y}\right) \tag{6.19}$$

DeMiguel and Nogales show that under the Strong Linear Independence Constraint Qualification (for any solution to the Level-0 optimization problem the Level-1 optimization has an optimum at which all the consistency constraints are active) the Schur-complement approach converges super linearly and in some cases quadratically. Furthermore, DeMiguel and Nogales discuss the fact that the gradients that are necessary to evaluate the interior point matrix cannot be calculated from standard sensitivity analysis of the individual element optimizations. Because the individual optimization problems are coupled, sensitivities of, e.g. Level-1, are computed with optimal parameters coming from Level-0. However, optimal parameters from Level-0 assume that the optimum of Level-1 is found beforehand and that coupling between elements is consistent. Because the consistent optimal solution of both individual elements is not known beforehand the sensitivities of each individual element are not calculated exact. The authors show how approximations to these gradients can be constructed under certain mathematical assumptions. A complete derivation of these approximations lies outside the scope of the present thesis. However, one of their main assumptions is that optima of the individual elements behave as a continuous function under changes in the coupling variables. This assumption is not valid for most multi-level optimization formulations that are discussed in Chapter 5. Therefore, care has to be taken in formulating the individual element optimization problems such that their optima do behave as a continuous function in order to use the Schur-complement approach.

6.3.2 Extending to multiple levels

In case of a top-down hierarchic decomposition with $n+1$ individual optimization problems the Newton search direction has the characteristic form:

$$\begin{bmatrix} \mathbf{A} & {}^0\mathbf{C} & & & & \\ {}^0\mathbf{C}^T & {}^1\mathbf{B} & {}^1\mathbf{C} & & & \\ & {}^1\mathbf{C}^T & \ddots & \ddots & & \\ & & \ddots & \ddots & & \\ & & & {}^{n-1}\mathbf{B} & {}^{n-1}\mathbf{C} \\ & & & {}^{n-1}\mathbf{C}^T & \mathbf{M} \end{bmatrix} \begin{bmatrix} \Delta^0\mathbf{y} \\ \Delta^1\mathbf{y} \\ \\ \vdots \\ \\ \Delta^n\mathbf{y} \end{bmatrix} = - \begin{bmatrix} {}^0\mathbf{w} \\ {}^1\mathbf{w} \\ \\ \vdots \\ \\ {}^n\mathbf{w} \end{bmatrix} \tag{6.20}$$

where $\mathbf{A} = \begin{bmatrix} \nabla^2_{{}^0\mathbf{x}}\mathcal{L} & \nabla_{{}^i_0\mathbf{h}}\nabla_{{}^0\mathbf{x}}\mathcal{L} \\ \nabla_{{}^0\mathbf{x}}\nabla_{{}^i_0\mathbf{h}}\mathcal{L} & \nabla^2_{{}^i_0\mathbf{h}}\mathcal{L} \end{bmatrix}$; ${}^0\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \nabla_{{}^i\mathbf{x}}\nabla_{{}^i_0\mathbf{h}}\mathcal{L} & \nabla_{{}^i_0\mathbf{h}}{}^i_0\mathbf{c} \end{bmatrix}$;

${}^{\cdot\cdot}\mathbf{B} = \begin{bmatrix} \nabla^2_{{}^j\mathbf{x},{}^j\mathbf{x}}\mathcal{L} & \nabla_{{}^j\mathbf{x}}{}^j_i\mathbf{c} & \nabla_{{}^k_j\mathbf{h}}\nabla_{{}^j\mathbf{x}}\mathcal{L} \\ \nabla_{{}^j\mathbf{x}}{}^j_i\mathbf{c}^T & \mathbf{0} & \mathbf{0} \\ \nabla_{{}^j\mathbf{x}}\nabla_{{}^k_j\mathbf{h}}\mathcal{L} & \mathbf{0} & \nabla^2_{{}^k_j\mathbf{h},{}^k_j\mathbf{h}}\mathcal{L} \end{bmatrix}$; ${}^{\cdot\cdot}\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \nabla_{{}^k\mathbf{x}}\nabla_{{}^k_j\mathbf{h}}\mathcal{L} & \nabla_{{}^k\mathbf{h}}{}^k_j\mathbf{c} \end{bmatrix}$;

and

$$\mathbf{M} \quad = \quad \begin{bmatrix} \nabla^2_{{}^m\mathbf{x}}\mathcal{L} & \nabla_{{}^m\mathbf{x}}{}^m_l\mathbf{c} \\ \nabla_{{}^m\mathbf{x}}{}^m_l\mathbf{c}^T & \mathbf{0} \end{bmatrix}. \tag{6.21}$$

where indices i, j, k, l and m denote the position in the hierarchy. Furthermore, the right hand side is constructed as:

$${}^0\mathbf{w} = \begin{bmatrix} \nabla_{{}^0\mathbf{y}}\mathcal{L} \end{bmatrix}^T \qquad {}^{1..n}\mathbf{w} = \begin{bmatrix} \nabla_{{}^k\mathbf{y}}\mathcal{L} & {}^k_j\mathbf{c} \end{bmatrix}^T; \tag{6.22}$$

and the vector $\Delta\mathbf{y}$ is defined as:

$$\Delta^0\mathbf{y} = \begin{bmatrix} {}^0\mathbf{x} & {}^i_0\mathbf{h} \end{bmatrix}^T \qquad \Delta^{1..n}\mathbf{y} = \begin{bmatrix} {}^k\mathbf{x} & {}^i_j\boldsymbol{\lambda} & {}^k_j\mathbf{h} \end{bmatrix}^T \tag{6.23}$$

The Schur-complement for individual optimization problems becomes then for problems on the first level:

$${}^1\mathbf{S} = {}^0\mathbf{A} - {}^1\mathbf{C}^{T\,1}\mathbf{B}^{-1\,1}\mathbf{C} \tag{6.24}$$

and for the bottom level:

$${}^{\cdot\cdot}\mathbf{S} = {}^{\cdot\cdot}\mathbf{B} - {}^{\cdot\cdot}\mathbf{C}^{T\,\cdot\cdot}\mathbf{M}^{-1\,\cdot\cdot}\mathbf{C}. \tag{6.25}$$

For the intermediate levels the Schur-complement writes:

$${}^n\mathbf{S} = {}^{n-2}\mathbf{B} - {}^{n-1}\mathbf{C}^{T\,n-1}\mathbf{B}^{-1\,n-1}\mathbf{C}. \tag{6.26}$$

The search direction for Level-0 is computed according to Equation 6.18 and the search direction for the individual elements of the lower levels are evaluated according to Equation 6.19, where the matrices have to be replaced according to the level that is evaluated. This procedure holds for an hierarchic top-down decomposition procedure without coupling between elements on the same level. Decomposition where coupling among elements on the same level is present and non-hierarchic decomposition are derived similarly. The interested reader is referred to Smith *et al.* (1996) for a derivation of the Schur-complement when such couplings are present.

6.4 Discussion

The null-space method derived by Alexandrov and Lewis who denote the method Trust Region Model Management has been widely accepted in the field of multidisciplinary design optimization. An overview is presented by (Rodríguez *et al.*, 2000). The method has been implemented in the DAKOTA (Giunta and Eldred, 2000) optimization package and various successful applications of the method are recorded in literature (e.g. Campana *et al.* (2006), Hoyle *et al.* (2006)).

To the author's knowledge Schur-complement reduction has not been applied to multi-level optimization problems yet. However, it is expected that the large computational costs associated with solving bi-level optimization problems will stimulate the search for coordination algorithms that show better convergence characteristics such as the null-space method and Schur-interior point iteration presented in this chapter.

Chapter 7

Object-oriented multi-level optimization framework

The previous chapters lay-out the theoretical foundation for the framework that is developed in this chapter. The framework is described along the concept of object-oriented programming. Object-oriented programming focuses on the actual data (the objects) and the procedures that belong to this data rather then procedures or algorithms that can change data. Implementation details on how behavior of an object is accomplished are hidden inside the object. The benefit is an easier to maintain software framework. This chapter focuses on the structure of the multi-level optimization framework and the interfaces of each of the components of the framework. Implementation details are not provided in the text of the present thesis.

In Section 7.1 an introduction into the concept of object-oriented programming is provided. Followed by a discussion of the object-oriented framework for multi-level optimization in Section 7.2.

7.1 Object-oriented programming

Object-oriented programming is a methodology to develop a software program via focussing on objects (the data) and the corresponding methods (procedures) that belong to this data, rather then focussing on algorithms that are provided with data that is changed. Objects have two characteristic features; objects have *state* and exhibit *behavior*. The state of an object is stored in *fields* also known as *attributes*. The state of the object can be changed via accessing the object's *methods*. Methods operate on an object to change it's state which is the primary mechanism for object-to-object communication.

Access to the internal state (the data) of an object can be hidden from other parts of the program via exclusively providing access to the state through accessor methods. This is known as *data encapsulation* and is a basic principle of object-oriented programming.

Defining states of an object and providing methods to give access to and to change these states, the object remains in control of how other objects or programs may use the attributes of the object. This has a number of advantages:

- Modularity: Every object's source code is independently written from other objects. Therefore, changes to the internal source code of the object do not influence other objects. An object can be reused, once defined it can be easily used in other parts of the program.

- Information-hiding: Providing access to the object's state only through it's methods hides the internal implementation of the object to other objects and/or programs.

- Code re-use: Already existing objects can be easily inserted into another program. Therefore, complex or task specific code can be tested separately by others and used without considering the object's implementation.

- Plug-in and debugging ease: Objects that cause difficulties can easily be removed and replaced with a different object.

A *class* is a blueprint from which an object can be created. The class describes the state and behavior of an object. All code in an object-oriented program is written inside a class. An *instance* of a class is the actual object at runtime.

Inheritance is one of the powerful advantages of object-oriented programming. Classes may inherit state and behavior from super classes, such classes that inherit the super class's behavior represent a more complex version of the superclass that has additional attributes and/or methods and/or overwrites method's behavior. In such a case a blueprint of more complex behavior is created, while inheriting the attributes and methods of the super class. If more complex structures are necessary, a subclass can be created that inherits attributes and methods from the class and super class. This is called multiple inheritance. Inheritance is a natural way to organize and structure the code.

An *interface* is a contract between a class that implements the interface and other components of the program. The class that implements a specific interface is more formal towards the outside world on the behavior it promises to have. A class that implements an interface has to provide all the behavior that the interface has published. An interface itself has no attributes and objects cannot be created from it.

A *package* collects classes and interfaces that belong together into a single namespace. Packages enable the programmer to manage large software programs easier and to test parts of the code separately. Packages also provide a convenient means of extending a program with basic routines that any program may require at specific time. These routines are accessible through the Application Programming Interface (API) which is a collection of packages provided with most (object-oriented) programming languages.

7.2 Framework components

The multi-level framework is modeled using an Integrated Development Environment (IDE). This is a software application that provides tools for computer programmers to develop or maintain software. In general an IDE consists of a source editor, a compiler and/or software code interpreter, automation tools for building the code and a debugger. Furthermore, the IDE may have a procedure for version control (CVS) (The CVS Team, 2008) and in some cases a drawing tool to draw the basic structure of the program according to the Unified Modeling Language (UML) specification (Object Management Group, 2008).

There are many IDEs available, however in the present thesis the Netbeans (Sun Microsystems, Inc., 2008b) IDE is used. It is free (and open-source), has an active developer community and it provides all the necessary tools required for the development of software, including a UML editor[1].

The Unified Modeling Language (UML) is a standardized visual specification language for object modeling. The language includes a graphical notation used to create an abstract model of a system, called UML model and can be used for any object-oriented programming language.

The Netbeans environment is used to draw and program the framework and the target language for implementation is the Java (Sun Microsystems, Inc., 2008a) programming language. However, no implementation specific details are given therefore any object-oriented language can be applied to implement the present multi-level framework.

Java is an object-oriented language developed by Sun Microsystems that has most of its syntax derived from older languages such as C and C++. In contrast to languages such as C or C++ which need to be compiled on each individual computer architecture, Java is compiled to bytecode[2] that can run on any Java virtual machine independently of the computer architecture.

7.2.1 The *mlprogram* package

The multi-level framework resides in the *mlprogram* name-space (directory or package) separated from the user supplied classes that are required to model the analysis problem and to model the optimization problem, see Figure 7.1. The framework consists of five major components:

- The *subsystem* package, holding the individual element analysis, optimization problem data and the procedures to operate on this data.

[1]The UML specification used in Netbeans is 1.4 extended with an early version of the UML2.0 specification. The Netbeans editor and therefore the UML diagrams in the present thesis follow the 1.4 specification of the language and not the 2.0 specification.

[2]Bytecode is a term which is used to denote various forms of instruction sets designed for efficient execution by a software interpreter.

- The *coordination* package holding the procedures to coordinate the individual elements, hence the objects instantiated from the classes inside the *subsystem* package.
- The *middlelevel* package holding the coupling data that is shared among individual elements (the subsystems) and it provides procedures mapping the data from one element onto a neighboring element.
- The *postprocess* package holding the procedures operating on the data after the multi-level optimization has finished, such as writing results to file, etc.
- The *userfiles* package holding user supplied files.

The main components of the *mlprogram* package are shown in Figure 7.1 together with the *userfiles* package that contains user supplied files. Each folder in the figure represents a name-space (package) holding the classes specific to that name-space (package).

7.2.2 The *subsystem* package

The *subsystem* package holds the individual element optimization problem formulation, see Figure 7.2. Mathematically the problem that is embedded in the *subsystem* package has one of two formats:

$$\begin{array}{lll}
\multicolumn{3}{l}{\text{Equality-based:}} \\
\min\limits_{{}^i\mathbf{x},{}^j_i\mathbf{h}} & {}^iv_f({}^j_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}({}^i\mathbf{x},{}^j_i\mathbf{h})) & \\
\text{s.t.} & {}^i\mathbf{v}_g({}^j_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}({}^i\mathbf{x},{}^j_i\mathbf{h})) & \leq 0 \\
 & {}^i\mathbf{v}_h({}^j_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}({}^i\mathbf{x},{}^j_i\mathbf{h})) & = 0 \;; \\
 & {}^i\mathbf{v}_c({}^i_j\mathbf{c}({}^i\mathbf{r},{}^i_j\mathbf{h})) & = 0 \\
 & {}^i\mathbf{v}_c({}^j_i\mathbf{c}({}^j\mathbf{r},{}^j_i\mathbf{h})) & = 0 \\
 & {}^i\mathbf{v}_a({}^i_j\mathbf{h},{}^i\mathbf{x}) & \leq 0 \\
 & \text{where } i \neq j &
\end{array} \tag{7.1}$$

$$\begin{array}{lll}
\multicolumn{3}{l}{\text{Relaxation-based:}} \\
\min\limits_{{}^i\mathbf{x},{}^j_i\mathbf{h}} & {}^iv_f({}^j_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}({}^i\mathbf{x},{}^j_i\mathbf{h})) + \sum \left({}^iv_c({}^i_j\mathbf{c}({}^i\mathbf{r},{}^i_j\mathbf{h})) + {}^iv_c({}^j_i\mathbf{c}({}^j\mathbf{r},{}^j_i\mathbf{h}))\right) & \\
\text{s.t.} & {}^i\mathbf{v}_g({}^j_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}({}^i\mathbf{x},{}^j_i\mathbf{h})) & \leq \mathbf{0} \\
 & {}^i\mathbf{v}_h({}^j_i\mathbf{h},{}^i\mathbf{x},{}^i\mathbf{r}({}^i\mathbf{x},{}^j_i\mathbf{h})) & = \mathbf{0} \\
 & \text{where } i \neq j &
\end{array} .$$

In contrast to an optimization problem formulation that is not coupled, mathematically expressed as:

$$\begin{array}{lll}
\min\limits_{\mathbf{x}} & v_f(\mathbf{x},\mathbf{r}(\mathbf{x})) & \\
\text{s.t.} & \mathbf{v}_g(\mathbf{x},\mathbf{r}(\mathbf{x})) & \leq 0 \;\cdot \\
 & \mathbf{v}_h(\mathbf{x},\mathbf{r}(\mathbf{x})) & = 0
\end{array} \tag{7.2}$$

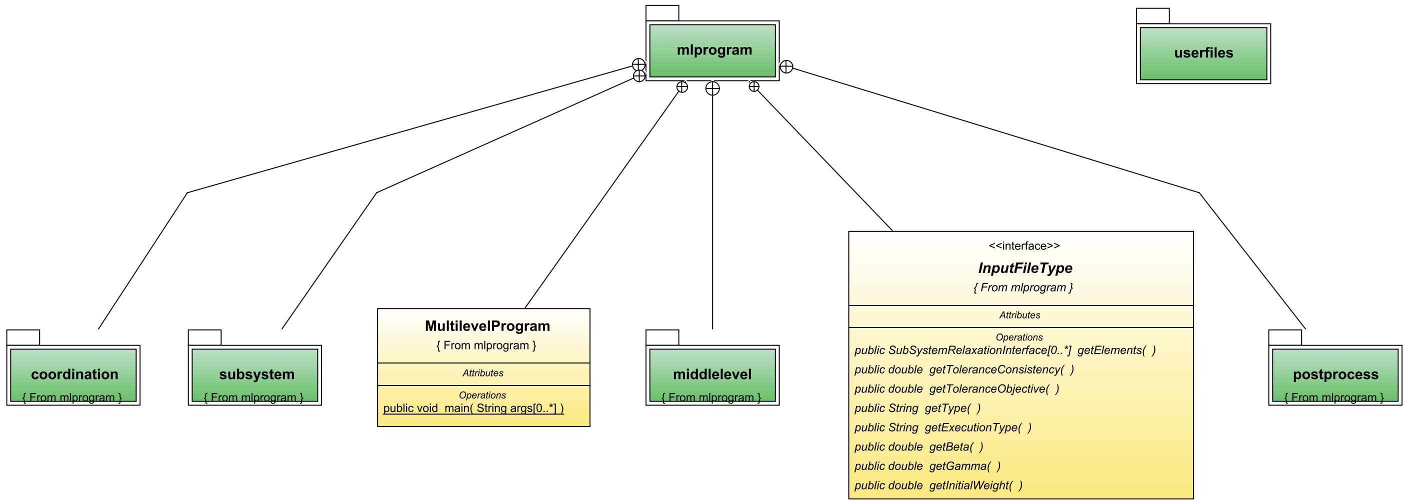


Figure 7.1: *The multi-level program* mlprogram *consists of four packages, an interface, a class and a package for user supplied program files. The* subsystem *package containing all the program code for the methods operating on the elements of the hierarchy and the storage of the data. The* middlelevel *package, which acts as an interface between elements that are coupled. The* coordinator *package that provides the computational algorithms that guide the individual elements towards the solution. Finally, the* postprocess *package that handles storage and plotting of the results. Furthermore, a compulsory main method is present in the* mlprogram *class for executing the program and an interface that defines the necessary methods for classes used as input files. The program files associated with the analysis and optimization of the hierarchy are stored inside a separate* userfiles *package.*

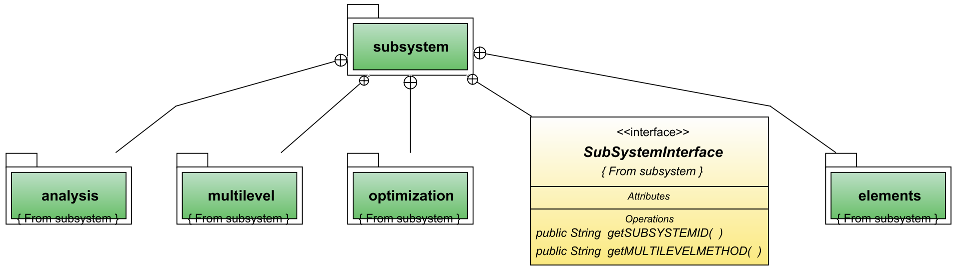


Figure 7.2: *The* subsystem *package consists of four main components. The* elements *package, which contains the classes that provide access to the element data. The* analysis *package that contains interfaces which describe the methods to operate on the element data and how communication with analysis code is to be handled. Furthermore, it contains the classes necessary to execute the analysis code. The* Optimization *package that contains the interfaces that execute the optimization algorithm. The* multilevel *package, which contains additional classes to account for the multi-level nature of the element's optimization problem. Furthermore, the* subsystem *package contains an interface* SubSystemInterface *that defines two methods that every element implementing the interface should have.*

The multi-level formulation for individual problems (Equation 7.1) has additional terms. In case of equality-based decomposition, the additional terms are present as constraints (${}^{i}\mathbf{v}_c$ and ${}^{i}\mathbf{v}_a$). In case of relaxation-based decomposition, the additional terms (${}^{i}\mathbf{v}_c$) are present as additional terms in the objective function. In both cases additional coupling variables are present.

Object-oriented programming focuses on the data first and secondly on the operations on this data. Therefore, the *subsystem* package is split into four individual packages:

- the *elements* package providing the bookkeeping for the element data such as: design variables (${}^{i}\mathbf{x}$); physical responses (${}^{i}\mathbf{r}$) and optimization functions (${}^{i}\mathbf{v}$) amongst others and provides accessor methods allowing access to the element data;

- the *analysis* package providing methods that evaluate the physical responses of an individual element (${}^{i}\mathbf{r}$);

- the *optimization* package providing methods that evaluate the objective function (${}^{i}v_f$) and/or constraint functions (${}^{i}\mathbf{v}_h$, ${}^{i}\mathbf{v}_g$) of each individual element;

- the *multilevel* package providing methods that account for the multi-level nature of the individual analysis and/or optimization problem. In case of equality-based decomposition methods, additional constraints (${}^{i}\mathbf{v}_c$ and ${}^{i}\mathbf{v}_a$) are evaluated. In case of relaxation-based decomposition methods, additional terms (${}^{i}\mathbf{v}_c$) that are added to the objective function are evaluated.

The class that provides access to individual optimization problem data (located in the *elements* package) is constructed from interfaces. These interfaces are defined inside the *analysis* package, inside the *optimization* package and inside the *multilevel* package. Figure 7.3 shows the inheritance of interfaces and their equivalent mathematical expression.

The hierarchical pattern of accessor methods for the elements in the *element* package is also applied to the methods operating on the element classes. In Figure 7.4 the interfaces that define the behavior that analysis programs and optimization programs should implement are shown.

The *RunAnalysisInterface* prescribes the methods that are necessary to compute physical responses. Furthermore, methods required for mapping of responses onto neighboring elements are prescribed via the *RunAnalysisInterface*. Responses are required for the optimization algorithm. The optimization algorithm is not an extension of the analysis program. Analysis and optimization are considered individual algorithms that require different accessor methods.

The *RunOptimizationInterface* prescribes methods conducting optimization applied to elements that implement the *RunAnalysisInterface*. Because individual optimization problems can be coupled to other elements (multi-level optimization) an extension of the interface is present. In Figure 7.4 a relaxation based interface is shown. Furthermore, differences in mathematical expression between coupled optimization problem formulation and optimization problem formulation without coupling are shown.

Finally, the *subsystem* package contains an interface for the subsystems (Figure 7.2). Each subsystem implements the *SubSystemInterface* and therefore provides methods to access the subsystem's identifier and type of decomposition that is used for decomposition.

The content of each package is discussed separately and modeled via class diagrams. Attributes and methods that do not directly contribute to the discussion are omitted for brevity of the class diagrams.

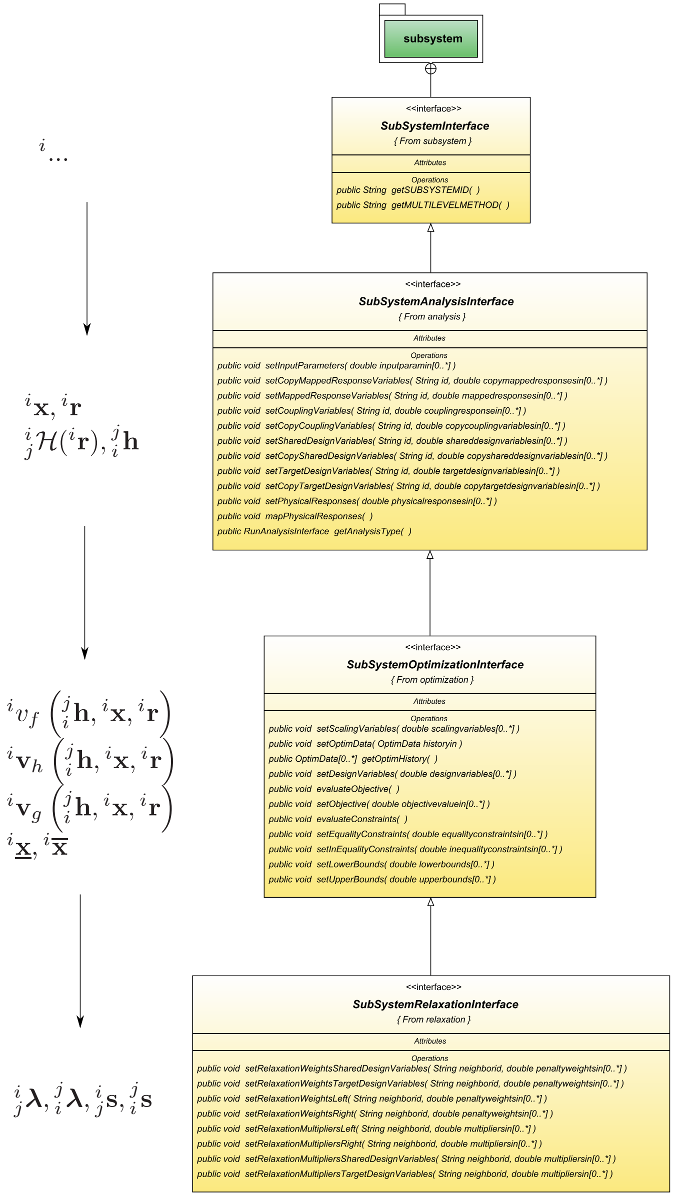
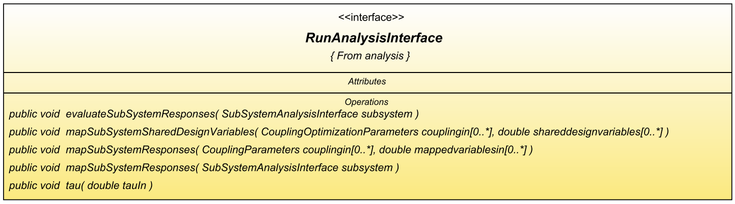


Figure 7.3: *Interfaces describing the structure of a typical individual optimization problem. The top interface(the* SubSystemInterface*) provides a contract for methods allowing access to the identity and type of the multilevel method. One level lower, the interface (the* SubSystemAnalysisInterface*) defines methods that provide access to input variables and physical responses related to the analysis. One level lower the interface (*SubSystemOptimizationInterface*) defines attributes and methods providing access to the optimization data. The bottom level, the interface (the* SubSystemRelaxationInterface*) describes the methods related to the multi-level optimization method. In the present figure a relaxation based method is shown.*

$$ {}^{i}\mathbf{r}\left({}^{j}_{i}\mathbf{h}, {}^{i}\mathbf{x}\right) $$
$$ {}^{i}_{j}\boldsymbol{\mathcal{H}}\left({}^{i}\mathbf{r}\right) $$



$$
\begin{array}{lll}
\min\limits_{{}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x}} & {}^{i}v_f\left({}^{j}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & \\
\text{s.t.} & {}^{i}\mathbf{v}_h\left({}^{j}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & = \mathbf{0} \\
& {}^{i}\mathbf{v}_g\left({}^{j}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & \leq \mathbf{0} \\
& {}^{i}\underline{\mathbf{x}} \leq {}^{i}\mathbf{x} \leq {}^{i}\overline{\mathbf{x}} &
\end{array}
$$

$$ \downarrow $$

$$
\begin{array}{lll}
\min\limits_{{}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x}} & {}^{i}v_f\left({}^{j}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) + {}^{i}_{j}\boldsymbol{\lambda}^{T}\,{}^{i}_{j}\mathbf{c} + ||{}^{i}_{j}\mathbf{s} \circ {}^{i}_{j}\mathbf{c}||_2^2 & \\
\text{s.t.} & {}^{i}\mathbf{v}_h\left({}^{j}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & = \mathbf{0} \\
& {}^{i}\mathbf{v}_g\left({}^{j}_{i}\mathbf{h}, {}^{i}\mathbf{x}, {}^{i}\mathbf{r}\right) & \leq \mathbf{0} \\
& {}^{i}\underline{\mathbf{x}} \leq {}^{i}\mathbf{x} \leq {}^{i}\overline{\mathbf{x}} &
\end{array}
$$

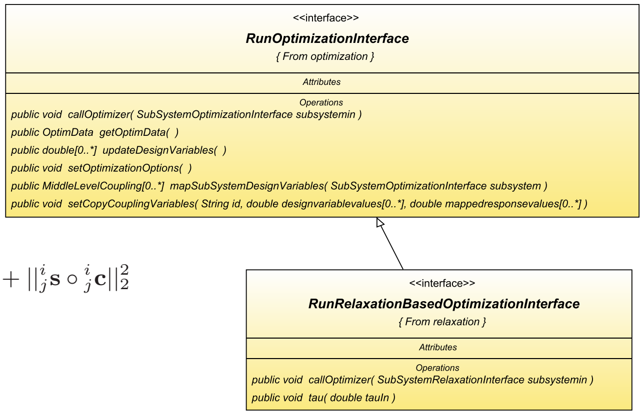


Figure 7.4: *Interfaces describing the structure of a typical individual optimization problem. The* RunAnalysis *interface prescribes methods that analysis algorithms have to implement in order to be able to communicate with the elements. Methods that compute the physical responses, as well as, methods to compute the mapping of these physical responses to neighboring elements should be implemented. The* RunOptimizationInterface *interface prescribes the necessary methods that optimization algorithms should implement. This interface covers the basic methods associated with executing optimization algorithms. The interface is extended via* RunRelaxationBasedOptimizationInterface *in case relaxation-based decomposition is applied to decompose consistency constraints. The mathematically equivalent expression of the interface shows an Augmented Lagrangian decomposition. The added functionality of the* RunRelaxationBasedOptimizationInterface *is shown compared to the* RunOptimizationInterface.

The *elements* package

The *elements* package (Figure 7.2) contains the subsystem classes. These classes are defined reflecting the decomposition method chosen (Chapter 3 and coordination method (see chapters 5 and 6). An example of a class that implements, e.g. a relaxation based method, is a relaxation based class shown in Figure 7.5. This *SubSystemAugmentedLagrangian* class provides methods to access, and fields to store, the data associated with an individual element. This individual element is, for the present class, decomposed from the multi-level optimization hierarchy via a relaxation based decomposition (here: Augmented Lagrangian decomposition, see Chapter 5). Hence, the *SubSystemAugmentedLagrangian* class provides methods to access data associated with:

$$\begin{array}{lll}\min\limits_{{}^{i}\mathbf{x},{}^{j}_{i}\mathbf{h}} & {}^{i}v_f({}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}({}^{i}\mathbf{x},{}^{j}_{i}\mathbf{h})) + \sum\left({}^{j}_{i}\boldsymbol{\lambda}^T\cdot({}^{j}_{i}\mathbf{c}({}^{i}\mathbf{r},{}^{j}_{i}\mathbf{h})) + \|{}^{j}_{i}\mathbf{s}\circ{}^{j}_{i}\mathbf{c}({}^{i}\mathbf{r},{}^{j}_{i}\mathbf{h})\|_2^2\right)\dots & \\ & \qquad + \sum\left({}^{i}_{j}\boldsymbol{\lambda}^T\cdot({}^{i}_{j}\mathbf{c}({}^{j}\mathbf{r},{}^{i}_{j}\mathbf{h})) + \|{}^{i}_{j}\mathbf{s}\circ{}^{i}_{j}\mathbf{c}({}^{j}\mathbf{r},{}^{i}_{j}\mathbf{h})\|_2^2\right) & \\ \text{s.t.} & {}^{i}\mathbf{v}_g({}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}({}^{i}\mathbf{x},{}^{j}_{i}\mathbf{h})) & \leq 0 \\ & {}^{i}\mathbf{v}_h({}^{j}_{i}\mathbf{h},{}^{i}\mathbf{x},{}^{i}\mathbf{r}({}^{i}\mathbf{x},{}^{j}_{i}\mathbf{h})) & = 0\end{array} \tag{7.3}$$

Furthermore, the class provides accessor methods that provide access to attributes (states) stored inside the fields of the class. Element classes themselves have no methods to operate on the data other then creating, replacing and removing the data. Hence, no function evaluations are done inside any of the classes in the *elements* package. Additionally, the element classes contain methods that call methods from other objects. These methods from other objects can evaluate physical responses or conduct optimization from the data offered via the element's methods.

The *analysis* package

The *analysis* package, shown in Figure 7.6, contains two interfaces that are required to communicate between a *subsystem* and an external program or class that can evaluate the responses (${}^{i}\mathbf{r}$). The two interfaces are:

- the *SubSystemAnalysisInterface* that defines methods every element implements in order to provide access to attributes from the element class.
- the *RunAnalysisInterface* defines methods programs or classes implement in order to evaluate responses (${}^{i}\mathbf{r}$) from each individual element.

Thus, the *SubSystemAnalysisInterface* is implemented by elements to allow the outside world to access the attributes required for analysis. Typically, these attributes are constant values required to set algorithm specific parameters. Furthermore, these attributes consist of design variables (${}^{i}\mathbf{x}$) and/or shared design variables (${}^{i}_{j}\mathbf{z}$) that are input parameters to the analysis. Finally, these attributes consist of coupling variables (${}^{j}_{i}\mathbf{h}$) that are input parameters to the analysis. Therefore, the *SubSystemAnalysisInterface* provides a formal contract to access these attributes. Additionally, a method executing analysis is provided without specifying the type of analysis.

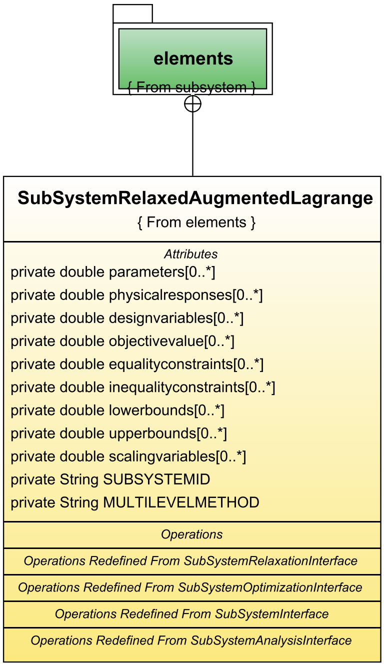


Figure 7.5: *The* element *package contains elements that implement the subsystem interface and interfaces that extend the subsystem interface with multi-level optimization method specific attributes and methods.* SubsystemAugmentedLagrangian *is an element that expects an Augmented Lagrangian decomposition to decouple the physical model and associated optimization problem.*

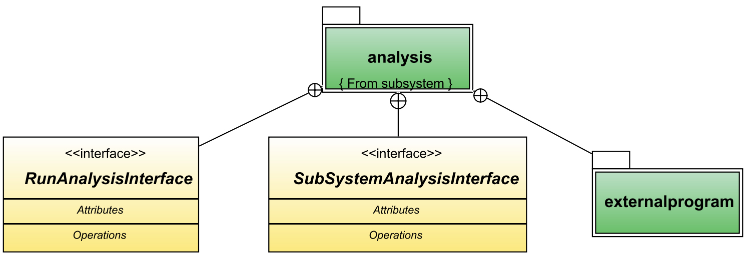


Figure 7.6: *The* analysis *package consists of two interfaces. The first interface describes the methods that every analysis program has to implement that will be used by the multi-level program. The second interface describes the methods that every element has to implement in order to allow access to the analysis variables.*

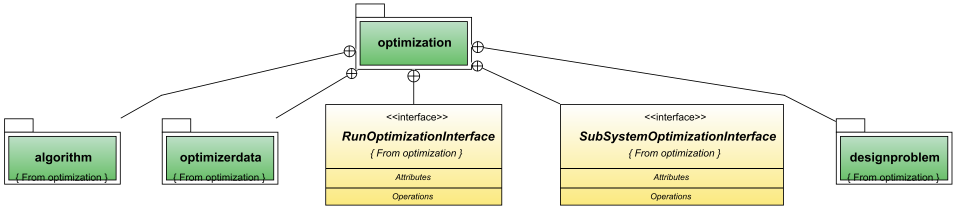


Figure 7.7: *The* optimization *package consists of a package containing the optimization algorithm in the* algorithm *package. A package is present that is involved with storing the optimization results. Furthermore, a package is present that contains the design problem (objective and constraint functions) and finally two additional interfaces are present. The* SubSystemOptimizationInterface *interface defines methods providing access to data stored in the element for optimization purpose. The* RunOptimizationInterface *is implemented for each optimizer algorithm to communicate with the multi-level program.*

A requirement for the analysis class is that it implements the *RunAnalysisInterface*. This interface specifies a formal contract to return ${}^{i}\mathbf{r}$ and ${}^{i}_{j}\boldsymbol{\mathcal{H}}({}^{i}\mathbf{r})$ to other parts of the object oriented program. Therefore, classes that implement the *RunAnalysisInterface* provide a method to evaluate the physical responses and mapping of physical responses onto neighboring elements.

Finally, the *externalprogram* package can be used to attach an external analysis program to the Java framework.

The *optimization* package

The *optimization* package, see Figure 7.7, contains classes required for optimization of a *subsystem* object. There are two interfaces that require implementation:

- the *SubSystemOptimizationInterface* defines methods for classes of the *SubSystem* package allowing access to attributes associated with the optimization problem.
- the *RunOptimizationInterface* defines methods every optimization algorithm implements to access *SubSystem* attributes and methods necessary for optimization.

The *SubSystemOptimizationInterface* requires methods accessing the design variables present in the element. Furthermore, it requires a method calling an optimization routine that carries out the optimization, methods that access the scaling variables to properly scale the design problem, methods accessing the objective function value, methods accessing the values of the constraint functions and methods accessing the value of the upper and lower bounds.

The *RunOptimizationInterface* requires methods executing the optimization routine, methods passing updated values of the design variables to the *SubSystem* object and methods mapping updated values of shared design variables (${}^{i}\mathbf{x}$) and/or coupling variables (${}^{j}_{i}\mathbf{h}$).

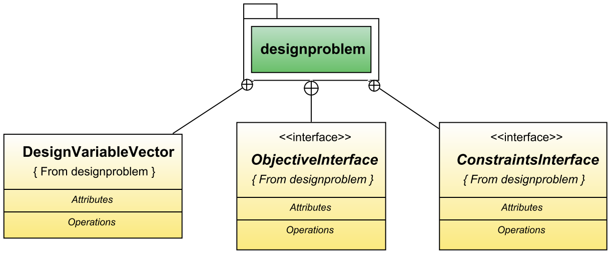


Figure 7.8: *The* designproblem *package consists of an interface which user defined objective function classes have to implement. An interface for the constraints of the design problem which does not include consistency constraints ($^i\mathbf{v}_c$) or additional information from neighboring elements ($^i\mathbf{v}_a$). This is dealt with elsewhere inside the* multilevel *package. A* DesignVariableVector *class, which rearranges coupling variables and/or shared design variables from the local design problem.*

The *designproblem* package

An *optimization* package expects in it's arguments an objective function, constraint functions and lower and upper bounds on the design variables. The lower and upper bounds are attributes that can be passed to the arguments directly, however the objective function and constraint functions are specified by the user. Therefore, separate classes that implement the objective function and the constraint function are required.

The *designproblem* package (Figure 7.8) contains interfaces to evaluate the objective function and the constraint functions. Furthermore, the *designproblem* package provides a class *DesignVariableVector* used to create a single design variable vector from local design variables, coupling variables and shared design variables. The user provides objective and constraint functions to the optimization problem via implementing the *ConstraintsInterface* and *ObjectiveInterface* for classes that model the design problem.

The *optimizerdata* package

The *optimizerdata* package attributes computed via the optimizer algorithm. These attributes are required for post-processing or for coordination of the individual elements. The package consists of the *OptimData* class that has two additional classes: the *Iterates* class holding numerical costs of the optimization; and the *Multipliers* class holding the Lagrange multipliers of the element design constraints.

The *multiLevel* package

Coupling between elements is separated from analysis and optimization via the *multilevel* package. Embedded in this package the classes are defined that store and provide access to coupling parameters and classes that provide methods to compute contributions of the coupling problem formulation to the optimization problem.

Each individual element can be coupled to a number of neighboring elements. The coupling between an element and a single neighboring element is stored inside the *CouplingParameters* class, see Figure 7.9 and Figure 7.10. In case shared design variables between elements and neighboring elements are present, the *CouplingParameters* class is extended via a *CouplingOptimizationParameters* class. Finally, if relaxation of the consistency constraints is present the latter class is extended to provide access to relaxation parameters via a *CouplingMultipliers* class.

The *multilevel* package stores attributes necessary to compute consistency equations, see Figure 7.10. The consistency equations relaxed via, e.g., an Augmented Lagrangian decomposition are mathematically expressed as:

$$ {}^{i}\mathbf{v}_{c} = {}^{i}_{j}\boldsymbol{\lambda}^{T}\left({}^{i}_{j}\mathbf{c}\right) + ||{}^{i}_{j}\mathbf{s} \circ {}^{i}_{j}\mathbf{c}||_{2}^{2}; \quad (7.4) $$

$$ \text{where } {}^{i}_{j}\mathbf{c} = {}^{i}_{j}\boldsymbol{\mathcal{H}}\left({}^{i}\mathbf{r}\right) - {}^{i}_{j}\mathbf{h}. \quad (7.5) $$

The *CouplingParameters* class holds the coupling variables (${}^{i}_{j}\mathbf{h}$) and the mapped response variables (${}^{i}_{j}\mathcal{H}({}^{i}\mathbf{r})$). In case design variables are shared a subclass of *CouplingParameters* is used (see Figure 7.10) extending attributes and methods available in the *CouplingParameters* class. This subclass is called the *CouplingOptimizationParameters* class and holds shared design variables (${}^{i}\mathbf{x}$).

Depending on the relaxation method chosen, additional parameters are present in the consistency formulation, e.g. Augmented Lagrangian decomposition (see Equation 7.4). The classes that provide access to these parameters are defined inside the *relaxation* package.

The *relaxation* package

The *relaxation* package, see Figure 7.11 contains packages associated with a specific relaxation method (see Chapter 3.2) for the consistency constraints. Each package provides means to extend the *CouplingOptimization* class accounting for relaxation parameters. Furthermore, each package provides means to add a contribution to the objective function due to the relaxation method chosen (see Equation 7.1).

Furthermore, the *relaxation* package consists of two interfaces. The *SubSystemRelaxationInterface* specifying methods giving access to relaxation parameters (Figure 7.3) and a *RunRelaxationBasedOptimizationInterface* defining methods adding relaxed consistency constraint functions to the element objective function (Figure 7.3).

The *equality* package

The *equality* package (see Figure 7.12) contains packages associated with specific equality-based multi-level optimization methods. The equality-based methods add additional constraints to the individual element optimization problem. These additional constraints may consist of consistency constraints (${}^{i}\mathbf{v}_{\mathbf{c}}$) and/or constraints that approximate behavior of neighboring elements ($\mathbf{v}_{a}$).

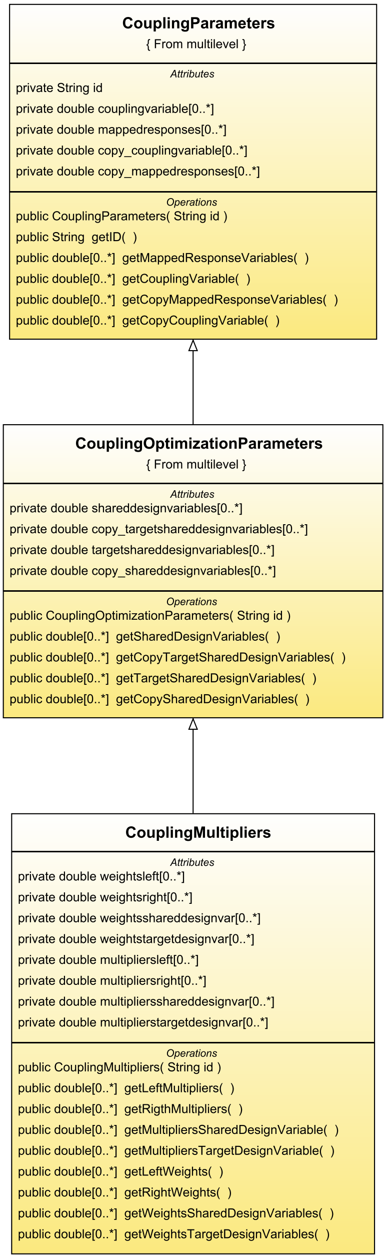


Figure 7.9: *Coupling between an element and a single neighboring element is stored inside the* CouplingParameters *class. In case shared design variables are present this class is extended via a* CouplingOptimizationParameters *class providing access to shared design variables. If consistency is relaxed a class extending the* CouplingOptimizationParameters *class is used providing access to relaxation parameters. In the present example, access is provided to Lagrange multipliers and penalty weights associated with Augmented Lagrangian decomposition.*

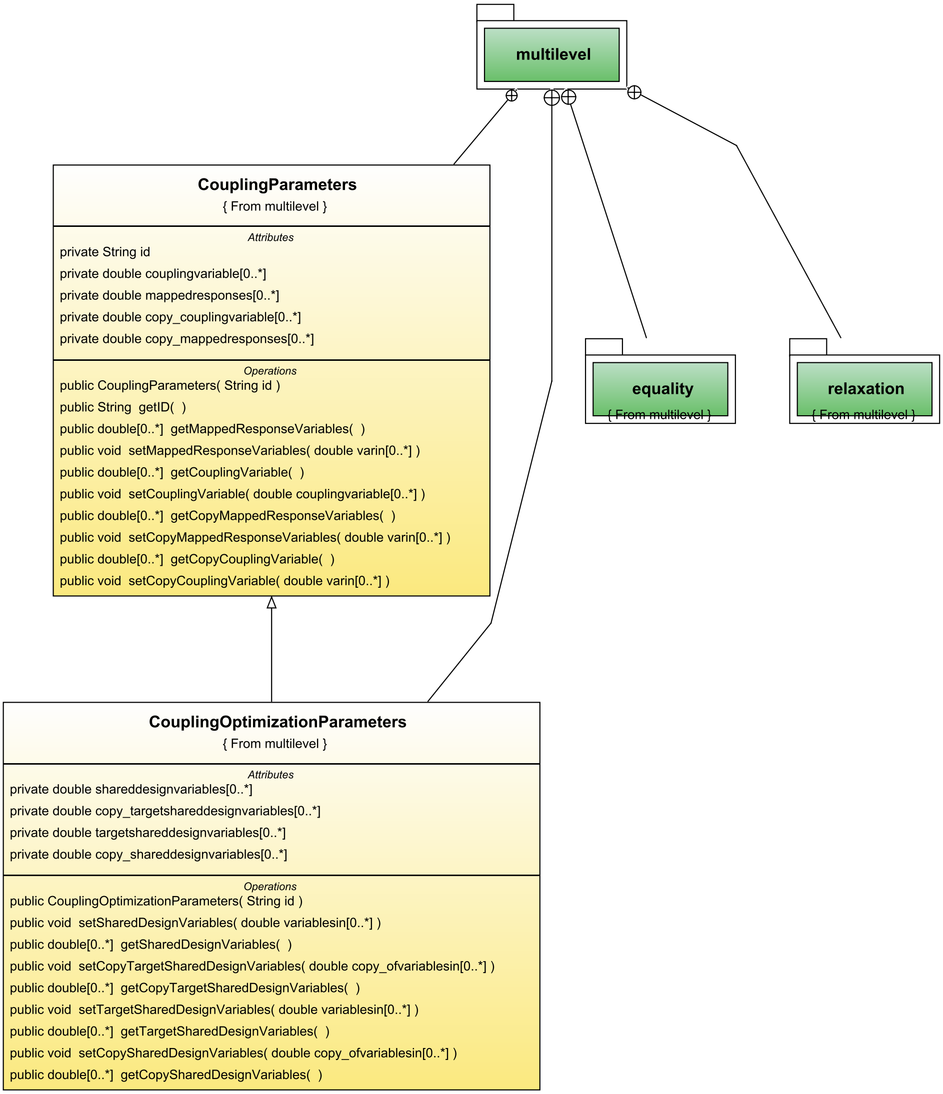


Figure 7.10: *The* multilevel *package provides two objects holding coupling data. Data that is mapped from one element onto a neighboring element in case of coupled analysis and shared design variables in case of coupled optimization functions. Copy variables represent values of neighboring elements for coupling variables used in the present element. For brevity of the class model the accessor methods to set the attributes of the classes are omitted.*

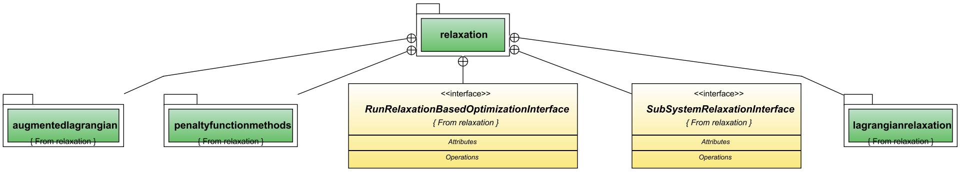


Figure 7.11: *The* relaxation *package contains objects that provide access to relaxation parameters. Three different relaxation packages are shown. Augmented Lagrangian Relaxation, Lagrangian Relaxation and Relaxation via a penalty function.*

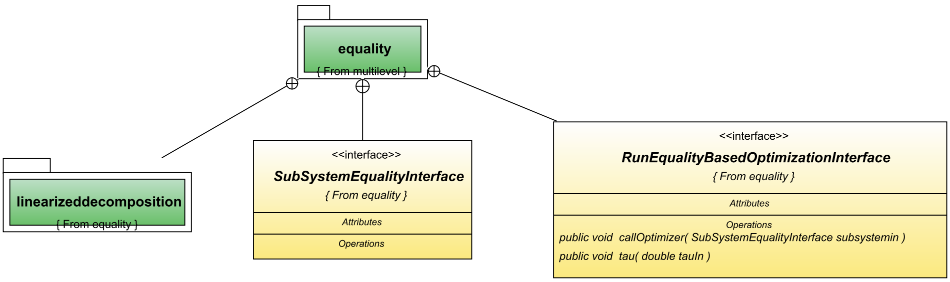


Figure 7.12: *The* equality *package contains interfaces describing the behavior of equality based multi-level optimization methods. The* SubSystemEqualityInterface *interface is implemented by classes adding access to additional approximation data stored in the attributes of the class. The* RunEqualityBasedOptimizationInterface *is implemented by every optimization algorithm that requires access to approximation data stored in the attributes of a class implementing the* SubSystemEqualityInterface*. Finally,* linearizeddecomposition *is a package that contains classes that implement both interfaces for multi-level methods based on linearized decomposition.*

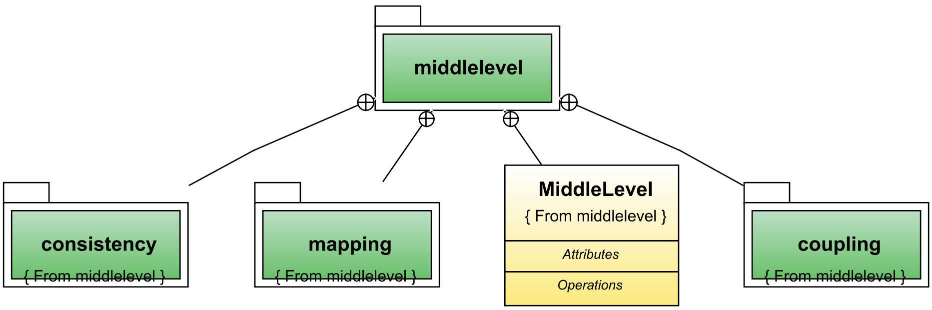


Figure 7.13: *The* middlelevel *package bridges between two elements that are coupled. It consists of a* mapping *package that maps the responses, coupling variables and coupled optimization data originating from the coupling between elements. Furthermore, the* middlelevel *package consists of a* consistency *package that computes the inconsistency size between the coupling variables and the mapped responses and between shared design variables. Finally, it acts as a buffer where the latest computed value is stored from the coupling parameters inside the* coupling *package. Hence, the individual elements do not directly communicate with each other, instead they communicate through the* middlelevel *package. The* middlelevel *package contains a* MiddleLevel *class from where* MiddleLevel *objects can be created. One object is created between each pair of elements that are coupled.*

7.2.3 The *middlelevel* package

The *middlelevel* package shown in Figure 7.13 contains the interface between two coupled subsystems. The package stores coupling data from one subsystem and provides methods to access this data to a coupled subsystem when requested. Both subsystems that are coupled retrieve coupling data via the classes specified in this package. Access to the class in the package is controlled via the *coordinator* package (Figure 7.1). The *coordinator* package allows access to update coupling data via the *mapping* package. Or the coordinator allows access to determine the gap in consistency between two coupling variables via the *consistency* package.

The *coupling* package

The *coupling* package shown in Figure 7.14 stores the coupling data from two elements. It stores mapped responses $({}^{i}_{j}\mathcal{H}\left({}^{i}\mathbf{r}\right))$ and coupling variables $({}^{j}_{i}\mathbf{h})$ from both elements, as well as, shared design variables $({}^{i}\mathbf{x})$. The *middleleveldatastorage* class stores identifiers of both elements. For each element a *MiddleLevelCoupling* class is instantiated that holds the coupling data. The *MiddleLevelDataStorage* class provides accessor methods to data and has a *Reentrantlock* procedure to ensure that no more then one element at a time tries to read and/or write to the *MiddleLevelCoupling* attributes.

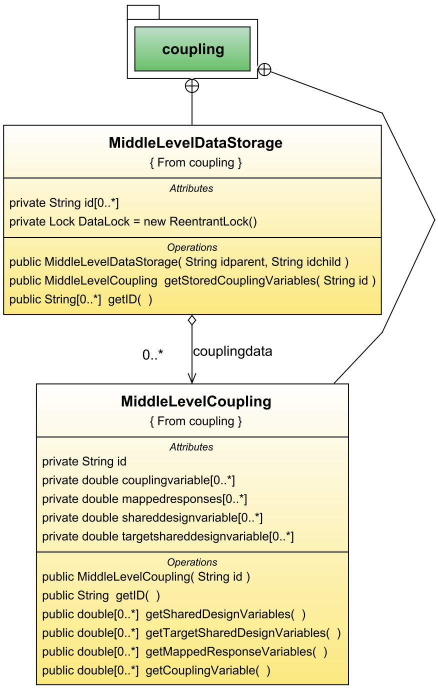


Figure 7.14: *The* coupling *package provides two objects:* MiddleLevelDataStorage *that provides methods to check which coupling data should be accessed if an element requests access to the coupling data;* MiddleLevelCoupling *stores the physical coupling data and/or the shared design variables.*

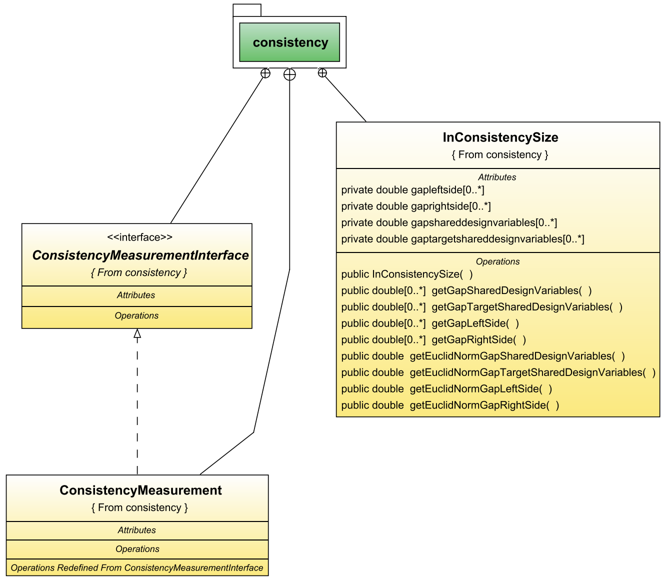


Figure 7.15: *The* consistency *package provides an interface for measuring the size of inconsistencies and a class that implements this interface. Furthermore, it contains a class that holds the size of the inconsistencies and provides methods to return various measures of the size of these inconsistencies.*

The *mapping* package

The *mapping* package holds a single class providing a single method *MapPhysicalResponses*. This method calls the *mapSubSystemResponses* method (see Figure 7.4) from the analysis algorithm (see Figure 7.6) that implements the *RunAnalysisInterface* (Figure 7.6) mapping physical responses from an element onto the middle-level.

The *consistency* package

The *consistency* package shown in Figure 7.15 consists of two classes and an interface. The interface is called *ConsistencyMeasurementInterface* and defines what behavior is expected from classes computing the size of inconsistency between two elements. The consistency gap is mathematically expressed as:

$$ {}^{i}_{j}\mathbf{c} = {}^{i}_{j}\boldsymbol{\mathcal{H}}({}^{i}\mathbf{r}) - {}^{i}_{j}\mathbf{h} \quad \text{left side;} \tag{7.6} $$

$$ {}^{j}_{i}\mathbf{c} = {}^{j}_{i}\boldsymbol{\mathcal{H}}({}^{j}\mathbf{r}) - {}^{j}_{i}\mathbf{h} \quad \text{right side;} \tag{7.7} $$

$$ {}^{i}_{j}\mathbf{c} = {}^{i}\mathbf{x} - {}^{j}\mathbf{x} \quad \text{left side;} \tag{7.8} $$

$$ {}^{j}_{i}\mathbf{c} = {}^{j}\mathbf{x} - {}^{i}\mathbf{x} \quad \text{right side.} \tag{7.9} $$

The left or right side reflects sides of the coupling circle (see Chapter 3).

ConsistencyMeasurement is a class implementing the *ConsistencyMeasurementInterface* interface and computes the inconsistencies and uses the *InConsistencySize* class to store the size of the inconsistency. The *InConsistencySize* class's attributes store the inconsistencies between mapped responses and shared design variables separately. Furthermore, different norms of these inconsistencies are returned depending on the method called, e.g. the Euclidean norm:

$$\text{GapLeftSide} = ||{}^i_j\mathbf{c}|| = ||{}^i_j\boldsymbol{\mathcal{H}}({}^i\mathbf{r}) - {}^i_j\mathbf{h}||_2. \tag{7.10}$$

7.2.4 The *coordination* package

The *coordination* package is shown in Figure 7.16. The package consists of a *coordinator* class that coordinates the steps required to conduct a multi-level optimization. The coordinator initiates for every element of the hierarchy objects of the *HierarchicElement* class. These objects have access to methods of the *elements*. These methods consist of methods to evaluate physical responses and/or evaluate the individual optimizations. Furthermore, access to the *middlelevel* is provided allowing mapping of the responses, exchange of shared design variables and exchange of coupling variables onto the *middlelevel*. Finally, methods can be accessed to retrieve copies of mapped responses, coupling variables and shared design variables from the *middlelevel*.

The *CoordinatorInterface* is an interface that defines methods executing individual element optimizations. There are three classes that each implement the interface in a different manner.

- *FullySequential*, the elements of the hierarchy are evaluated sequentially. The Level-0 element is evaluated first and the computed data is mapped onto the middle-level. The Level-1 elements are evaluated one-by-one and data is mapped onto the middle-levels after a solution is found. This process is continued until all elements located at the lowest level have been evaluated. The process is repeated from top to bottom until convergence.

- *FullyParallel*, the elements of the hierarchy are evaluated in parallel. As soon as an individual element is finished it maps the computed data onto the middle-level and waits until it is signalled to retrieve updated copies from neighboring elements from the middle-level. After retrieving the updated coupling data the element optimization is repeated.

- *LevelbyLevel*, the elements are executed sequentially over the levels and parallel per level.

Elements are executed as separate threads[3]. In order to run a thread a class has to provide a "run" method. All methods called inside this "run" method can be executed in the current threat. Therefore, the *HierarchicElement* class provides this run method in which computations for the element are executed.

[3] A thread in computer science is short for a thread of execution. Threads are a way for a program to fork (or split) itself into two or more simultaneously (or pseudo-simultaneously) running tasks.

The *coordination* package consists of three packages:

- the *approximationmodel* package providing additional methods to compute approximations for coordination purpose;
- the *convergence* package providing methods determining local convergence of individual elements and global convergence of the entire hierarchy;
- The *multilevelmethod* package providing multi-level method specific coordination methods.

The *multilevelmethod* package

The *multilevelmethod* package is shown in Figure 7.17. It consists of an interface *DecouplingConditionInterface* and two packages *equalitymethod* and *relaxationmethod*. Each package provides methods that are required to coordinate the decomposition method chosen.

The *equalitymethod* package provides methods to coordinate elements that are decomposed according to an equality-based decomposition. Each coordination method should implement the *EqualityMethodInterface* describing communication with objects instantiated from the *HierarchicElement* class (see Figure 7.16).

The *EqualityMethodInterface* provides a contract for coefficients ($\mathbf{a}_{..}$). These coefficients are present in the equations that model the behavior of neighboring elements (${}^{i}v_{a}$, see Equation 7.1), mathematically expressed as a p^{th}-order polynomial function:

$$ {}^{i}\mathbf{v}_{a}({}^{i}_{j}\mathbf{h}, {}^{j}_{i}\mathbf{h}) \leq \mathbf{a}_0 + \mathbf{a}_1 f({}^{i}_{j}\mathbf{h}) + \mathbf{a}_2 f({}^{j}_{i}\mathbf{h}) + \ldots + \mathbf{a}_n f^p({}^{i}_{j}\mathbf{h}) + a_{n+1} f^p({}^{j}_{i}\mathbf{h}). \qquad (7.11) $$

The function $f(\ldots)$ depends on the multi-level method chosen and is defined inside the *equality* package (Figure 7.12).

The *relaxationmethod* package provides methods necessary to coordinate the elements that are decomposed via relaxation of the consistency constraints. Typically these elements require updating of relaxation parameters. The *RelaxationMethodInterface* dictates means of passing new values of the relaxation parameters to the individual elements. For example, in Figure 7.17 an *AugmentedLagrangianMethod* class is shown implementing the *RelaxationMethodInterface*.

The *AugmentedLagrangianMethod* class implements methods that compute updates of the Lagrange multipliers and the penalty weights according to:

$$ \begin{aligned} \boldsymbol{\lambda}^{(t+1)} &= \boldsymbol{\lambda}^{(t)} + 2\mathbf{s}^{(t)} \circ \mathbf{s}^{(t)} \circ (\boldsymbol{\mathcal{H}}(\mathbf{r}) - \mathbf{h})\,; \\ \mathbf{s}^{(t+1)} &= \begin{cases} \mathbf{s}^{(t)} & \text{if} \quad \mathbf{c}^{(t+1)} \leq \gamma \mathbf{c}^{(t)} \\ \beta \mathbf{s}^{(t)} & \text{if} \quad \mathbf{c}^{(t+1)} > \gamma \mathbf{c}^{(t)} \end{cases}\,; \\ & \beta > 0 \text{ and } 0 < \gamma < 1; \end{aligned} \qquad (7.12) $$

where indices have been omitted for brevity of notations.

The *approximationmodel* package

Depending on decomposition and coordination method different behavior of elements is approximated. However, techniques that compute these approximations do not change and therefore approximation methods are combined into a single *approximationmodel* package.

The *approximationmodel* package shown in Figure 7.18 provides methods to compute coefficients $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ defined in Equation 7.11. These coefficients are obtained via analysis of the behavior of individual elements for changing values of the coupling variables. If constraints are relaxed the package provides approximated weights, e.g. the Weight Update Method (Chapter 5).

In the present thesis two approximation models are defined: response surface methods; and linear approximations that are constructed via sensitivity analysis. Therefore, the *approximationmodel* package consists of two packages and an interface:

- the *responsesurfacemethod* package providing methods to compute coefficients for a polynomial function;
- the *sensitivityanalysis* package providing methods to compute sensitivity information;
- the *ApproximationMethodInterface* providing an interface for the coordinator classes to communicate with the approximation models.

The *ApproximationMethodInterface* is extended inside the two packages via additional methods that implement specific approximation methods.

The *convergence* package

The *convergence* package shown in Figure 7.19 provides a *ConvergenceInterface* interface prescribing methods required to make decisions on evaluating individual elements. Type and implementation of these convergence measures may have different convergence criteria. Therefore, the convergence indicator is modeled as an interface. The *ConvergenceIndicator* class implements the *ConvergenceInterface* interface and implements the methods to retrieve information on individual element convergence (see Chapter 5, Section 5.3), hence:

$$\varepsilon_{v_f}^{(t)} = \max\left\{ ||{}^i v_f^{(t)} - {}^i v_f^{(t-1)}||_\infty \right\} \tag{7.13}$$

where (t) indicates the current iteration number. Convergence of the entire multi-level hierarchy is measured via the consistency constraints:

$$\varepsilon_{v_c}^{(t)} = \max\left\{ ||{}^i \mathbf{v}_c^{(t)}||_\infty, ||{}^i \mathbf{v}_c^{(t)} - {}^i \mathbf{v}_c^{(t-1)}||_\infty \right\}. \tag{7.14}$$

Hence, Equation 7.14 measures convergence according to the value of the inconsistency size and the rate of change in the consistency size reduction. However, other formulations may apply depending on the decomposition and coordination algorithm chosen.

7.2.5 The *postprocess* package

The *postprocess* package contains classes that perform postprocessing operations on the computed data. The package holds two classes. The first class *CheckOS* changes the file writing process according to the operating system the multi-level program is executed on. The second class *writeToFile* creates output files and writes data to these files.

7.2.6 The *userfiles* package

The *userfiles* package shown in Figure 7.20 contains the user supplied multi-level optimization problem. For each element of the hierarchy a package is created. The name of each package is *subsystem* followed by a number representing the position in the hierarchy (recall Chapter 2). Every package should contain a class implementing the *ObjectiveInterface*. The name of the class implementing the *ObjectiveInterface* is *Objective* followed by the package number. Similarly, a class that implements the *ConstraintsInterface* should be implemented. The class name is *Constraints* followed by the package number.

The *Objective* class evaluates the individual element specific objective function. No additional terms need to be evaluated inside this class that are related to the multi-level nature of the optimization problem.

Likewise, the *Constraints* class evaluates individual element specific constraint functions. No additional constraints are added that take into account the multi-level nature of the optimization problem. The multi-level nature of the optimization problem is accounted for inside the *mlprogram* program and requires no implementations from the user.

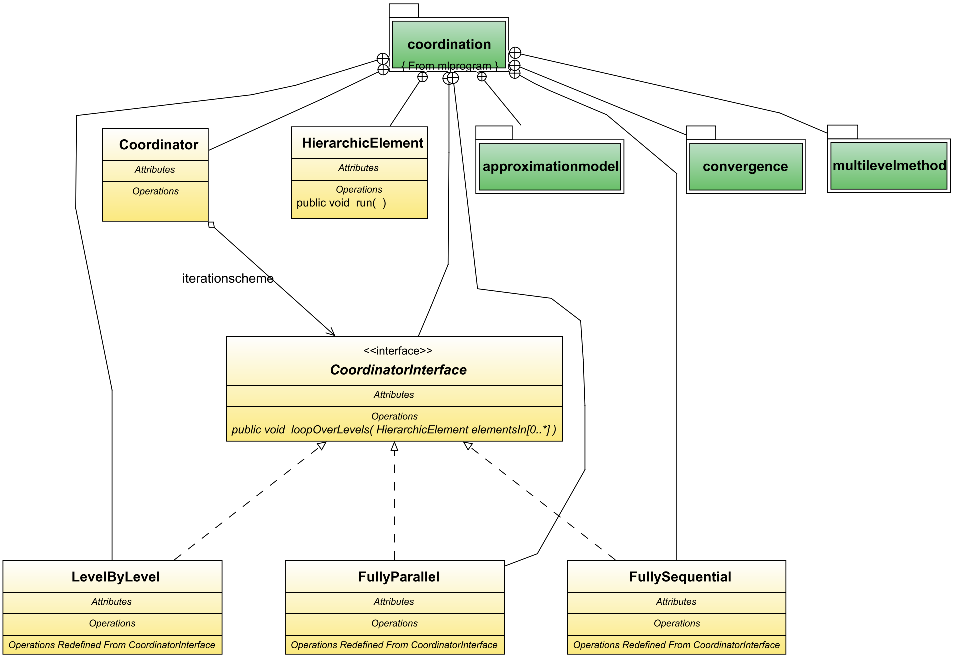


Figure 7.16: *The* coordination *package coordinates the order in which elements are solved and calls routines when additional coordination data is required. The package consists of a Coordinator that communicates via a coordination interface that determines the order in which elements are solved. Three objects that implement this interface are:* FullyParallel *computing all the elements at once;* LevelByLevel *solving the elements starting at the top level and then solving each level sequentially; and* LevelByLevel *solving the levels sequentially and the elements on the same level in parallel. The* HierarchicElement *class contains a subsystem object in it's attributes and implements the* Runnable *interface of the Java API allowing the subsystem optimization and/or analysis to be executed as a separate threat. Furthermore, a local convergence check of the element and the global convergence of the element inconsistency convergence criteria is done inside the methods of the* HierarchicElement *class. The* coordination *package consists of an* approximationmodel *package where approximation methods are placed, a* multilevelmethod *package consisting of multi-level optimization method specific classes and a* convergence *package is present to evaluate convergence of the multi-level algorithms.*

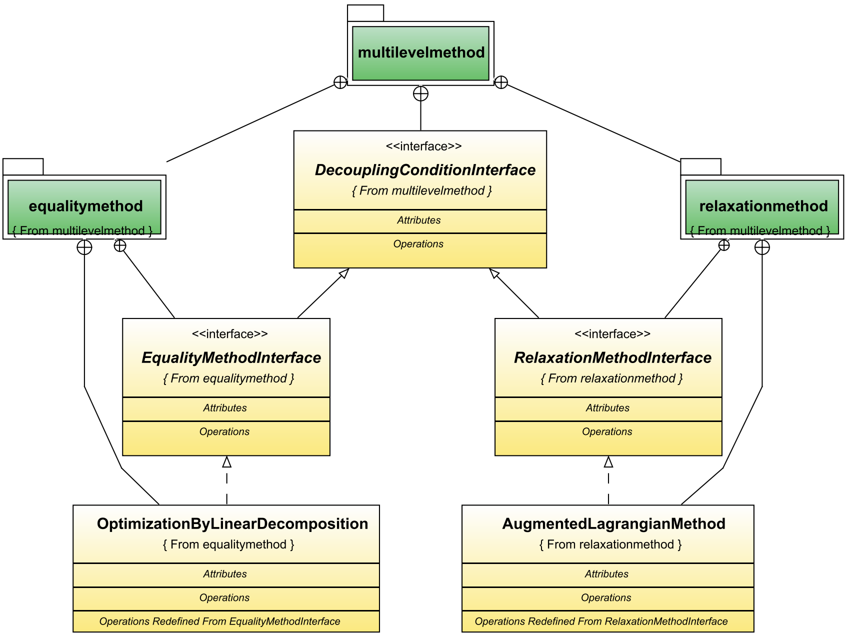


Figure 7.17: *The* multilevelmethod *package consists of a* DecouplingConditionInterface *that describes the behavior of the multi-level coordination methods. Furthermore, two packages are present: the* equalitymethod *package and the* relaxationmethod *package. These packages provide methods specific for a coordinator method. In case equality based decomposition is applied the* EqualityMethodInterface *should be implemented and in case relaxation based decomposition is applied the* RelaxationMethodInterface *is implemented. In the present figure an implementation of the* EqualityMethodInterface *is shown showing* OptimizationByLinearDecomposition *and an implementation of the* RelaxationMethodInterface *is shown showing the* AugmentedLagrangianMethod *class.*

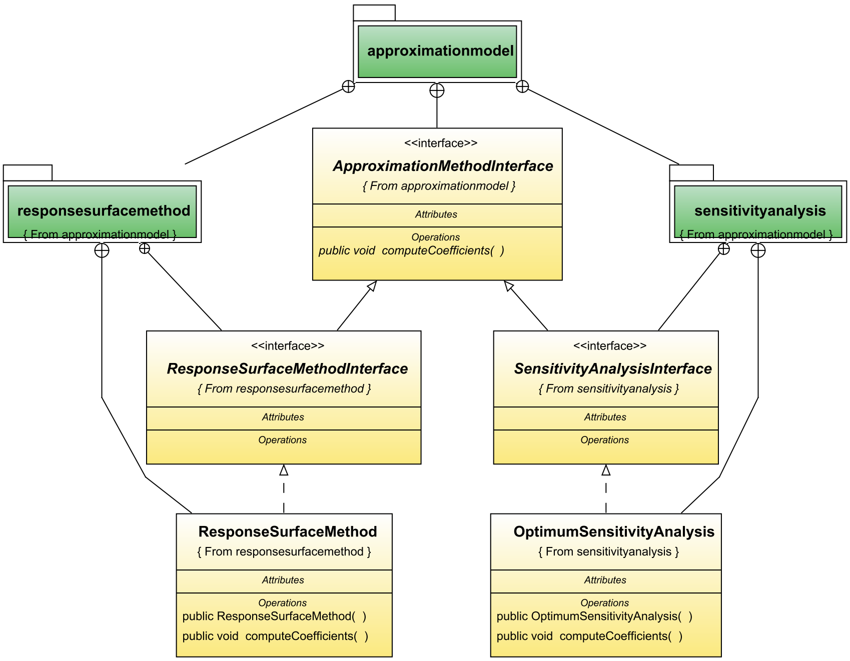


Figure 7.18: *The* approximationmodel *package consists of an* ApproximationMethodInterface *interface. Furthermore, the package contains two packages: the first package* sensitivityanalysis *consisting of a single interface for sensitivity analysis and implemented via the* OptimumSensitivityAnalysis *class; the second package* ResponseSurfaceMethod *consists of a* ResponseSurfaceMethodInterface *that is implemented via the* ResponseSurfaceMethod *class.*

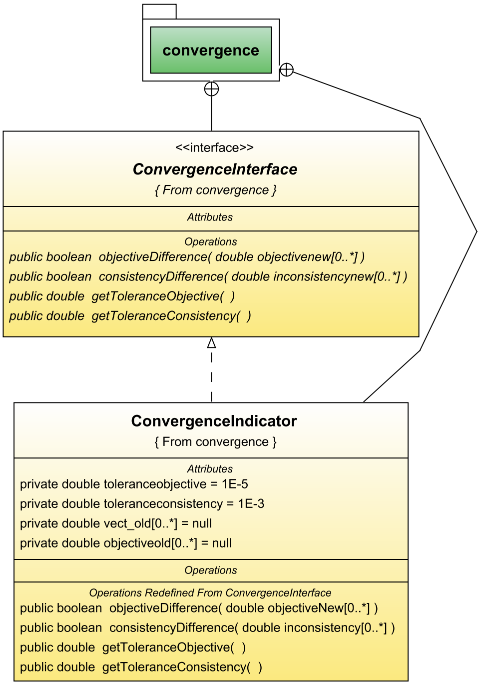


Figure 7.19: *The* convergence *package consists of a* ConvergenceInterface *interface that describes the communication with classes providing methods to measure convergence. Furthermore, the* ConvergenceIndicator *class implements the* ConvergenceInterface *interface and consists of a number of procedures to determine local and global convergence.*

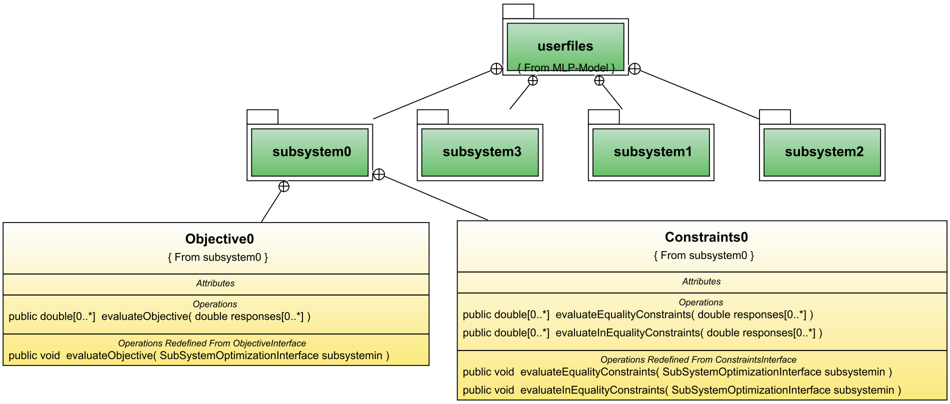


Figure 7.20: *The* userfiles *package consists of packages defining the multi-level optimization problem. Each* package *consists of an individual element optimization problem formulation. Packages are named* subsystem *followed by a number indicating the position in the hierarchy. Each package consists of a class file* Objective *followed by the package number and a class* Constraints *followed by the package number. The* Objective *class implements the* ObjectiveInterface *and the* Constraints *class implements the* ConstraintsInterface*. The classes provide methods to compute individual objective function and constraint function values. These classes do not consider objective and/or constraint formulations that are associated with the multi-level nature of the design problem.*

Chapter 8

Applying the framework on relaxation-based decomposition methods

In this chapter the object oriented framework described in the previous chapter is applied to four examples of which three are taken from the literature. In Section 8.1 the two bar truss example is analyzed, which has been used throughout this thesis indicating the steps required to decompose and coordinate a multi-level optimization. In Section 8.2 two examples are presented considering multi-level optimization of a portal frame according to the problem definition by Tosserams *et al.* (2008b) and an alternative problem definition by Sobieszczanski-Sobieski *et al.* (1985). Finally, in Section 8.3 a model of a supersonic business jet is optimized taken from (Agte *et al.*, 1999). This model includes multiple disciplines in the optimization problem.

8.1 Two-bar truss

8.1.1 Multi-level optimization problem

The process of formulating a multi-level structural optimization problem is demonstrated on the basis of a two-bar truss design optimization problem. The two-bar truss design problem is shown in Figure 8.1 and the parameters determining the structural lay-out are listed in Table 8.1. Typical objectives in structural optimization are minimal mass or displacement (maximum stiffness).

For this problem, the objective $({}^{0}v_f)$ is to minimize the total mass $({}^{0}r_1)$. The problem is subjected to a constraint $({}^{0}v_g)$ on the horizontal displacement $({}^{0}r_2)$, a

Table 8.1: *Definition of design variables, physical responses, physical interactions between subsystems and optimization parameters for the two-bar truss.*

design	description	physical response	description	interaction	description
${}^0x_1$	location support 1	${}^0r_1$	total mass ${}^0r_1 = f({}^0\mathbf{x}, {}^1_0h_1, {}^2_0h_1)$	${}^0_1\mathbf{h}_1$	nodal forces 1
${}^0x_2$	location support 2	${}^0r_2$	horizontal displacement ${}^0r_2 = f({}^0\mathbf{x}, {}^1_0h_1, {}^2_0h_1)$	${}^0_2\mathbf{h}_1$	nodal forces 2
${}^1x_1$	radius Element-1	${}^1r_1$	stress left member	${}^1_0h_1$	area Element-1
${}^1x_2$	thickness Element-1		${}^1r_1 = \frac{{}^0_1h_{1_2} - {}^0_1h_{1_1}}{2\pi {}^1x_1 {}^1x_2}$		
${}^2x_1$	radius Element-2	${}^2r_1$	stress right member	${}^2_0h_1$	area Element-2
${}^2x_2$	thickness Element-2		${}^2r_1 = \frac{{}^0_2h_{1_2} - {}^0_2h_{1_1}}{2\pi {}^2x_1 {}^2x_2}$		

optimization	description	
${}^0r_{mmax}$	mass norm	
${}^0r_{umax}$	allowed displacement	$r_{Euler} = \frac{\pi^2 \left({}^2x_1\right)^2 E}{{}^0_2h_2}$
${}^1r_{cr}$	maximum allowed stress	E = Young's modulus
${}^2r_{Euler}$	Euler buckling stress*	${}^0_2h_2$ element length

* Not part of the physical coupling between levels, but a restriction on the subsystem physical response.

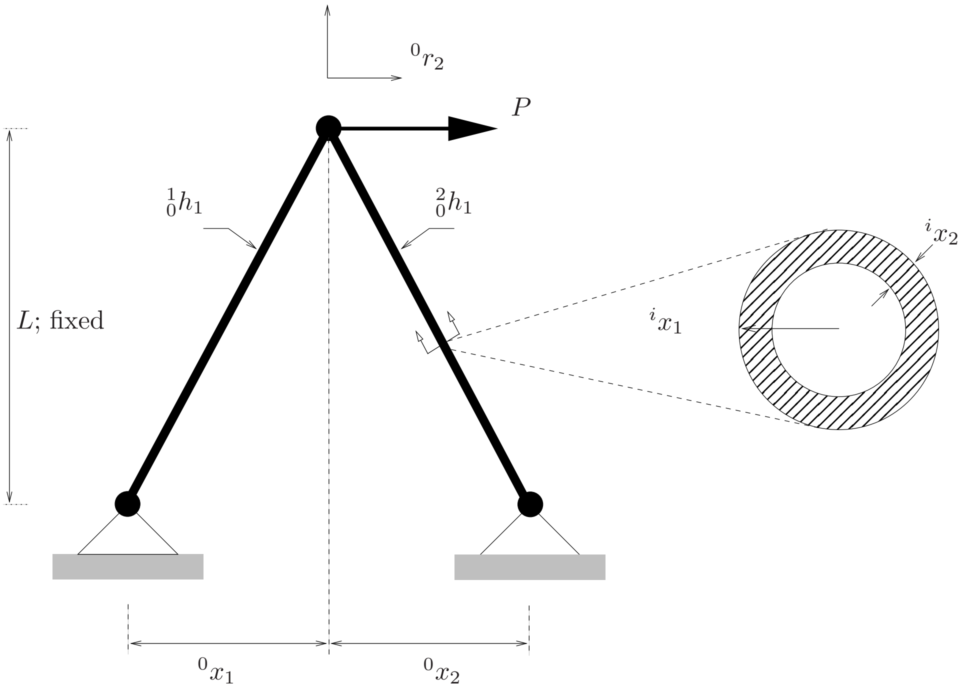


Figure 8.1: *Two-bar truss structure with embedded hierarchy. The general lay-out of the structure is described via design parameters* $^0x_1, {}^0x_2$ *and cross-sectional areas. The element's cross-section is described in detail via design variables* $^ix_1, {}^ix_2$. *These cross-sections are present as coupling variables* $^1_0h_1, {}^2_0h_1$.

constraint (^1v_g) on stress ($^1r_1$) in the left member and a constraint (^2v_g) considering Euler buckling ($^2r_1$) for the right member. Design variables are: $^0x_1, {}^0x_2$, which determine the location of the supports w.r.t. the centerline, $^{\cdot\cdot}x_1$ the radius and $^{\cdot\cdot}x_2$ the thickness of each individual truss member. Furthermore, lower($^{\cdot\cdot}\underline{\mathbf{x}}$) and upper($^{\cdot\cdot}\overline{\mathbf{x}}$) bounds on the design variables are present, see Table 8.2.

Two levels can be distinguished in the design problem. The top level (Level-0) involving minimization of the total mass ($^0r_1$), while satisfying constraints on horizontal displacement ($^0r_2$). And the bottom level (Level-1) that involves satisfying a constraint on stress ($^1r_1$) for the left member and a constraint considering Euler buckling ($^2r_1$) for the right member. Two individual hierarchical elements are identified at Level-1 covering the analysis and optimization of each individual bar column.

The Euler buckling stress ($^2r_{Euler}$) is a Level-1 geometrical and material characteristic and is mathematically expressed as:

$$^2r_{Euler} = \frac{\pi^2 \left({}^2x_1\right)^2 E}{2l_2}. \tag{8.1}$$

Because the length of the truss member is computed from Level-0 design parameters,

Table 8.2: *Scaling parameters and upper and lower bounds on the design variables for the two-bar truss problem.*

variable	lowerbound	upperbound	scaling parameter
${}^{0}x_1$	2.000×10^{-1}	1.000	1.000
${}^{0}x_2$	1.000×10^{-1}	1.100	1.100
${}^{1}x_1$	1.000×10^{-3}	1.000×10^{-1}	1.000×10^{-1}
${}^{1}x_2$	1.590×10^{-4}	1.000×10^{-2}	1.000×10^{-2}
${}^{2}x_1$	1.000×10^{-3}	1.000×10^{-1}	1.000×10^{-1}
${}^{2}x_2$	1.590×10^{-4}	1.000×10^{-2}	1.000×10^{-2}
${}^{0}_{1}\mathbf{h}_1$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{2}\mathbf{h}_1$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{2}h_2$	1.005	1.487	1.000
${}^{1}_{0}h_1$	1.590×10^{-7}	1.000×10^{-4}	1.000×10^{-3}
${}^{1}_{0}h_2$	1.590×10^{-7}	1.000×10^{-4}	1.000×10^{-3}

Table 8.3: *Constant values used for the two-bar truss multi-level design optimization problem*

constant	value	constant	value	constant	value
E	7.300×10^{10}	${}^{0}r_{mmax}$	1.700×10^{-1}	F	1.000×10^{4}
ρ	2.800×10^{3}	${}^{0}r_{umax}$	1.000×10^{-2}		
L	1.000	${}^{1}r_{cr}$	4.800×10^{8}		

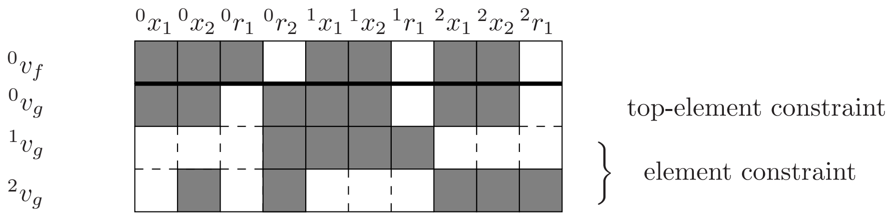


Figure 8.2: *Coupling matrix two bar truss optimization problem before decomposition. On the left the optimization problem functions are listed and on the top the design variables and physical responses. A function that depends on a specific design variable or physical response is represented via a shaded block.*

an additional mapping is required for Element-2:

$$l_2 = {}^0_2\mathcal{H}_2({}^0x_2) = \sqrt{(L)^2 + ({}^0x_2)^2} = {}^0_2h_2. \tag{8.2}$$

The optimization functions associated with the two-bar truss design problem are mathematically expressed as:

$$\begin{aligned} {}^0v_f &= \frac{{}^0r_1({}^0\mathbf{x}, {}^1\mathbf{x}, {}^2\mathbf{x})}{{}^0r_{mmax}}; \\ {}^0v_g &= \frac{{}^0r_2({}^0\mathbf{x}, {}^1\mathbf{x}, {}^2\mathbf{x})}{{}^0r_{umax}} - 1; \\ {}^1v_g &= \frac{{}^1r_1({}^1\mathbf{x}, {}^0\mathbf{r})}{0.9\,{}^1r_{cr}} - 1; \\ {}^2v_g &= \left(2\frac{{}^2r_1({}^2\mathbf{x}, {}^0\mathbf{r})}{{}^2r_{Euler}({}^0x_2, {}^2x_1)}\right)^2 - 1. \end{aligned} \tag{8.3}$$

The additional parameters ${}^0r_{mmax}$ and ${}^0r_{umax}$ represent respectively maximum allowed mass and displacement. Furthermore, the stress in Element-1 (${}^1r_1$) is not allowed to exceed 90% of the critical stress (${}^1r_{cr}$). Finally, the quadratic stress in Element-2 (${}^2r_1$) should not exceed 50% of the quadratic Euler buckling load (${}^2r_{Euler}$).

The problem matrix showing the dependencies of the optimization functions of Figure 8.3 on the design variables and the physical responses is shown in Figure 8.2. The objective function and constraint function of the Level-0 optimization problem depend on the design variables of Level-0 and Level-1 elements. Coupling variables are added in Figure 8.3 to show the coupling between the two levels.

Hierarchic decomposition

A top-down hierarchic decomposition of the two-bar truss design problem is considered. Therefore, the mapping of geometrical properties of Level-1 onto Level-0 is rewritten as consistency constraints:

$$\begin{aligned} {}^1_0c_1 &= {}^1_0\mathcal{H}_1\left({}^1\mathbf{x}\right) - {}^1_0h_1; \\ {}^2_0c_1 &= {}^2_0\mathcal{H}_1\left({}^2\mathbf{x}\right) - {}^2_0h_1. \end{aligned} \tag{8.4}$$

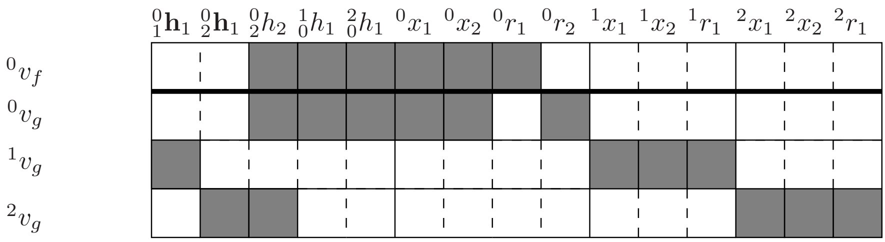


Figure 8.3: *Coupling parameters are introduced into the problem matrix that show the coupling between the Level-0 element and the two Level-1 elements.*

These consistency constraints are relaxed in the present thesis applying Augmented Lagrangian relaxation. Hence, the consistency constraint formulation becomes:

$$\begin{array}{l} {}^1_0\lambda_{..}\left({}^1_0c_1\right) + ||{}^1_0s_{..} \circ {}^1_0c_1||_2^2; \\ {}^2_0\lambda_{..}\left({}^2_0c_1\right) + ||{}^2_0s_{..} \circ {}^2_0c_1||_2^2. \end{array} \tag{8.5}$$

where ${}^{..}_0\lambda_{..}$ are the Lagrange multipliers and ${}^{..}_0s_{..}$ are the penalty weights. The Lagrange multipliers and penalty weigths are assigned to each individual element. Therefore, these relaxation parameters do not have to be equal between two coupled elements (see Chapter 5).

Relaxation of the consistency constraints yields that Equation 8.5 is added to the individual element optimization problems and therefore the multi-level optimization problem relaxed via Augmented Lagrangian relaxation yields:

Level-0, Element 0:

$$\begin{array}{llcll} \underset{{}^0x_1,{}^0x_2,{}^1_0h_1,{}^2_0h_1}{\min} & {}^0v_f & = & \frac{{}^0r_1({}^0\mathbf{x},{}^1_0h_1,{}^2_0h_1)}{{}^0r_{mmax}} + {}^0v_{{}^1_0c} + {}^0v_{{}^2_0c} & \\ \text{s.t.} & {}^0v_g & = & \frac{{}^0r_2({}^0\mathbf{x},{}^1_0h_1,{}^2_0h_1)}{{}^0r_{ymax}} - 1 & \leq 0 \\ & & & {}^0\underline{\mathbf{x}} \leq {}^0\mathbf{x} \leq {}^0\overline{\mathbf{x}} & \\ & & & {}^1_0\underline{h}_1 \leq {}^1_0h_1 \leq {}^1_0\overline{h}_1 & \\ & & & {}^2_0\underline{h}_1 \leq {}^2_0h_1 \leq {}^2_0\overline{h}_1 & \\ \text{where} & {}^0v_{{}^1_0c} & = & {}^1_0\lambda_1\left({}^1_0c_1\right) + ||{}^1_0s_1 \circ {}^1_0c_1||_2^2 & \\ & {}^0v_{{}^2_0c} & = & {}^2_0\lambda_1\left({}^2_0c_1\right) + ||{}^2_0s_1 \circ {}^2_0c_1||_2^2 & \end{array} \tag{8.6}$$

Level-1, Element 1:

$$\begin{array}{llcll} \underset{{}^1x_1,{}^1x_2}{\min} & {}^1v_f & = & {}^1v_{{}^1_0c} & \\ \text{s.t.} & {}^1v_g & = & \frac{{}^1r_1({}^1\mathbf{x})}{0.9\,{}^1r_{cr}} - 1 & \leq 0 \\ & & & {}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}} & \\ \text{where} & {}^1v_{{}^1_0c} & = & {}^1_0\lambda_2\left({}^1_0c_1\right) + ||{}^1_0s_2 \circ {}^1_0c_1||_2^2 & \end{array} \tag{8.7}$$

Level-1, Element 2:

$$\begin{array}{llll} \min\limits_{^2x_1,^2x_2} & ^2v_f & = & ^2v_{^2_0c} \\ \text{s.t.} & ^2v_g & = & \left(2\frac{^2r_1(^2\mathbf{x})}{^2r_{Euler}}\right)^2 - 1 \quad \leq 0 \\ & & & ^2\underline{\mathbf{x}} \leq {}^2\mathbf{x} \leq {}^2\overline{\mathbf{x}} \\ \text{where} & ^2v_{^2_0c} & = & {}^2_0\lambda_2\left({}^2_0c_1\right) + ||{}^2_0s_2 \circ {}^2_0c_1||^2_2 \end{array} \tag{8.8}$$

Coupling variables ${}^1_0h_1$ and ${}^2_0h_1$ are added to the Level-0 optimization problem as design variables. These have to match the mapped geometrical characteristics ${}^1_0\mathcal{H}_1({}^1\mathbf{x})$ and ${}^2_0\mathcal{H}_1({}^2\mathbf{x})$, respectively, that are evaluated during the Level-1 optimization.

Non-hierarchic decomposition

A non-hierarchic decomposition of the problem involves a decomposition of the coupling in both directions. The derivation is similar to that of hierarchical decomposition. In addition to the consistency constraints on the mapping of physical responses from Level-1 onto Level-0 (Equation 8.4), the responses (the displacements) are mapped from Level-0 onto Level-1 (the nodal forces) and are replaced via consistency constraints. These consistency constraints are relaxed via an Augmented Lagrangian function and added to the individual optimization problems.

8.1.2 Physical model

The two-bar truss problem is shown in Figure 8.4. Two levels can be distinguished, the general lay-out of the structure consists of two bar elements and the detailed cross-sectional area described by the diameter and the thickness of the wall. The expressions necessary to evaluate physical responses are derived in accordance with the coupling circle shown in Figure 8.4. For the current two-bar truss problem these expressions can be derived simpler. However, the current derivation is ment to demonstrate the coupling circle on a mechanical structure. The physical responses presented in the present format emphasize coupling as shown via the coupling circle.

The general responses of the structure are total structural mass ${}^0r_1$ and the horizontal displacement ${}^0r_2$, both depend on the cross-sectional areas of the bar elements. The horizontal displacement (${}^0r_2$) of the structure is found via evaluating the displacements of the structure. These displacements are found as a solution to the mathematical expression:

$$\mathbf{Ku} = \mathbf{P}, \tag{8.9}$$

where $\mathbf{K}$ is the stiffness matrix of the entire structure, $\mathbf{u}$ being the total displacement and $\mathbf{P}$ being the external loading. The stiffness matrix of the individual columns is mathematically expressed as:

$$\begin{aligned} \mathbf{K} = & \begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix} = \dots \\ & E\begin{bmatrix} \frac{{}^1_0h_1}{l_1}\cos^2\alpha + \frac{{}^2_0h_1}{l_2}\cos^2\beta & \frac{{}^1_0h_1}{l_1}\sin\alpha\cos\alpha + \frac{{}^2_0h_1}{l_2}\sin\beta\cos\beta \\ \frac{{}^1_0h_1}{l_1}\sin\alpha\cos\alpha + \frac{{}^2_0h_1}{l_2}\sin\beta\cos\beta & \frac{{}^1_0h_1}{l_1}\sin^2\alpha + \frac{{}^2_0h_1}{l_2}\sin^2\beta \end{bmatrix}; \end{aligned} \tag{8.10}$$

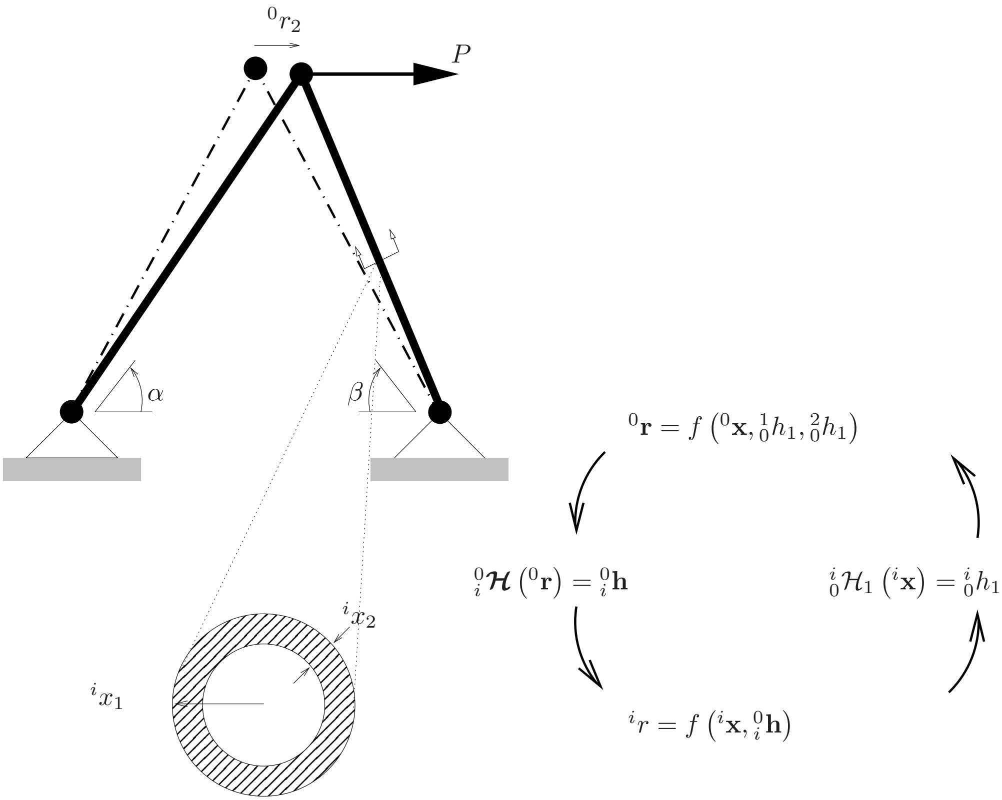


Figure 8.4: *Two-bar truss structure with an embedded hierarchy. The displacements ($^0\mathbf{r}$) identified at Level-0 are mapped onto the individual elements present at Level-1 as nodal force vectors (${}^0_i\mathbf{h}$). The stresses (^ir) identified as Level-1 properties determine the geometrical layout of the individual elements. The geometrical characteristics are mapped from Level-1 onto Level-0 as cross-sectional areas (${}^i_0h_1$).*

where the cross-sectional area of each individual bar column (^{i}) is expressed as ${}^i_0h_1$ and the length of these columns as l_i.

The horizontal displacement ($^0r_2$) is mathematically expressed as:

$$^0r_2 = \frac{1}{(k_{1,1} - \frac{k_{1,2}^2}{k_{2,2}})}F. \tag{8.11}$$

The stiffness matrix $\mathbf{K}$ depends on the cross-sectional area of the two bars. This cross-sectional area is represented via the coupling variables ${}^1_0h_1$ and ${}^2_0h_1$ for each individual column. The coupling variables are mathematically expressed as:

$$\begin{array}{rcccl} {}^1_0\mathcal{H}_1\left(^1x_1, {}^1x_2\right) & = & 2\pi^1x_1{}^1x_2 & = & {}^1_0h_1; \\ {}^2_0\mathcal{H}_1\left(^2x_1, {}^2x_2\right) & = & 2\pi^2x_1{}^2x_2 & = & {}^2_0h_1. \end{array} \tag{8.12}$$

Hence, the mapping of Level-1 onto Level-0 involves the mapping of geometrical properties. With the mapped geometrical properties the total mass of the structure can now be computed according to:

$$ {}^0r_1 \quad = \quad {}^1_0h_1 \rho l_1 + {}^2_0h_1 \rho l_2. \tag{8.13} $$

where ρ being the density of the material and the length l_i of both trusses is mathematically expressed as:

$$ l_1 = \sqrt{L^2 + ({}^0x_1)^2} \quad \text{and} \quad l_2 = \sqrt{L^2 + ({}^0x_2)^2}. \tag{8.14} $$

After the displacements are solved the internal forces can be computed. These internal forces are required by the individual elements present at Level-1 to analyze the strains (ϵ_i). The strain is obtained for each individual element from the column displacements.
Element 1:

$$ \mathbf{u}_{e_1} = [\mathbf{T}] \begin{bmatrix} 0 \\ 0 \\ {}^0r_2 \\ {}^0r_3 \end{bmatrix} \quad \Rightarrow \quad \epsilon_1 = \mathbf{B}\mathbf{u}_{e_1} = \ldots $$
$$ \tfrac{1}{l_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_{e_{1_x}} \\ u_{e_{1_y}} \end{bmatrix} = \tfrac{{}^0r_2 \cos\alpha + {}^0r_3 \sin\alpha}{l_1}. \tag{8.15} $$

Element 2:

$$ \mathbf{u}_{e_2} = [\mathbf{T}] \begin{bmatrix} {}^0r_2 \\ {}^0r_3 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \epsilon_2 = \mathbf{B}\mathbf{u}_{e_2} = \ldots $$
$$ \tfrac{1}{l_2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_{e_{2_x}} \\ u_{e_{2_y}} \end{bmatrix} = -\tfrac{{}^0r_2 \cos\beta - {}^0r_3 \sin\beta}{l_2}. \tag{8.16} $$

Secondly, the internal nodal forces are defined as:
Element 1:

$$ \mathbf{f}_1^{int} = \int_{x=0}^{l_1} \frac{1}{l_1} \mathbf{B}^T \mathbf{D} \epsilon_1 A_1 dx, \tag{8.17} $$

Element 2:

$$ \mathbf{f}_2^{int} = \int_{x=0}^{l_2} \frac{1}{l_2} \mathbf{B}^T \mathbf{D} \epsilon_2 A_2 dx, \tag{8.18} $$

where $A_1 = {}^1_0h_1$ and $A_2 = {}^2_0h_2$.

The mapping of displacements (${}^0r_2$ horizontal and ${}^0r_3$ vertical displacement) in order to obtain the internal nodal forces is mathematically expressed in multi-level notation as:

$$ {}^0_1\boldsymbol{\mathcal{H}}_1\left({}^0\mathbf{r}\right) = E^1_0 h_1 \begin{bmatrix} -\frac{{}^0r_2\cos\alpha + {}^0r_3\sin\alpha}{l_1} & \frac{{}^0r_2\cos\alpha + {}^0r_3\sin\alpha}{l_1} \end{bmatrix}^T = {}^0_1\mathbf{h}_1; \qquad (8.19) $$

$$ {}^0_2\boldsymbol{\mathcal{H}}_1\left({}^0\mathbf{r}\right) = E^2_0 h_1 \begin{bmatrix} \frac{{}^0r_2\cos\beta - {}^0r_3\sin\beta}{l_2} & -\frac{{}^0r_2\cos\beta - {}^0r_3\sin\beta}{l_2} \end{bmatrix}^T = {}^0_2\mathbf{h}_1. \qquad (8.20) $$

Nominal stress is defined as: $\sigma = \frac{p}{A}$. The load carried by the column is p and the cross-sectional area is A. Therefore, the nominal stress in each column member is:

$$ \begin{aligned} {}^1r_1 &= \frac{{}^0_1h_{1_2} - {}^0_1h_{1_1}}{2\pi\,{}^1x_1\,{}^1x_2}; \\ {}^2r_1 &= \frac{{}^0_2h_{1_2} - {}^0_2h_{1_1}}{2\pi\,{}^2x_1\,{}^2x_2}. \end{aligned} \qquad (8.21) $$

In this section the expressions necessary to evaluate physical responses were derived in accordance with the coupling circle shown in Figure 8.4. In the next section the numerical results obtained for the multi-level optimization are presented.

8.1.3 Numerical results

Two different coordination methods are applied to the two-bar truss multi-level optimization problem and the method of multipliers (Bertsekas and Tsitsiklis, 1989) is used to update the relaxation parameters[1].

All optimization starting points were chosen via random initial design variables and the coupling variables were initiated with random values that laid between feasible bounds (see Table 8.2). Therefore, the individual elements were not consistent with each other, meaning that an infeasible design point was used in all cases and no initial computations were done in order to have a consistent starting design configuration.

The optimal values of the reference solution found via an All-in-One optimization for the design variables are listed in Table 8.4. Furthermore, the value of the objective function and the optimal value of the coupling variables are listed. The design constraint functions present in each element of the hierarchy are active.

Optimization history

The history of the objective and constraint function of Element-0 is shown in Figure 8.5(a). The history is plotted for a top-down hierarchic decomposition of the two-bar truss multi-level optimization problem. Convergence settings for the inconsistencies (ε_{v_c}) and objective function (ε_{v_f}) correspond to the Alternating Descent method. Initially, the optimization problem of Element-0 focusses on finding an optimum of the local objective function. During the multi-level optimization process the consistency between Element-0 and elements 1 and 2 becomes increasingly important and the optimum of Element-0 shifts to a higher value that is optimal for the entire hierarchy. Likewise, the optimal design variables and coupling variables shift to values that are optimal for the entire hierarchy, see Figure 8.5(b).

[1] See Chapter 5 for an explanation of relaxation based coordination and the method of multipliers.

Table 8.4: *Optimal solution of the three hierarchical elements of the two bar truss in terms of design variables, coupling variables and optimization functions. The optimum is found via an All-in-One optimization and therefore consistency is satisfied exactly via direct mapping of physical responses.*

parameter Element-0	Element-1	Element-2	description
${}^0v_f = 1.06 \times 10^{+0}$			hor. displ.
${}^0v_g = active$			tot. mass
	${}^1v_g = active$		max. stress el. 1
		${}^2v_g = active$	max stress el. 2
		${}^2\underline{x}_2 = active$	lower bound ${}^2x_2$
	${}^0_1h_{1_1} = -8.64 \times 10^{+3}$		nod. force el. 1
	${}^0_1h_{1_2} = 8.64 \times 10^{+3}$		
		${}^0_2h_{1_1} = 7.67 \times 10^{+3}$	nod. force el. 2
		${}^0_2h_{1_2} = -7.67 \times 10^{+3}$	
		${}^0_2h_2 = 1.20 \times 10^{+0}$	length el. 2
	${}^1_0h_1 = 2.00 \times 10^{-5}$		cr.-sec. area 1
		${}^2_0h_1 = 3.12 \times 10^{-5}$	cr.-sec. area 2
${}^0x_1 = 9.03 \times 10^{-1}$			left att. point
${}^0x_2 = 6.55 \times 10^{-1}$			right att. point
	${}^1x_1 = 6.22 \times 10^{-3}$	${}^2x_1 = 3.12 \times 10^{-2}$	radius
	${}^1x_2 = 5.12 \times 10^{-4}$	${}^2x_2 = 1.59 \times 10^{-4}$	thickness

The increase in the objective function value of Element-0 (Figure 8.5(a)) can be explained via Figures 8.6, 8.7 and 8.8. These figures show the history of the inconsistency between Level-0 and Level-1 and the optimization process of Element-1 and Element-2, respectively.

Figure 8.6 shows the size of the inconsistencies (${}^1v_{{}^1_0c}$, ${}^2v_{{}^2_0c}$) between Level-0 and Level-1. Initially, the consistency is reduced via data exchange and no difficulties are observed for Element-1 and Element-2 in finding a feasible optimum. This can be seen in Figure 8.7(a) where the design constraint ${}^1v_g$ initially has a negative value and in Figure 8.8(a) where the design constraint ${}^2v_g$ initially has a negative value.

After the first decrease in inconsistency of Element-2 (after 18 hierarchical updates), see Figure 8.6, the inconsistency stays at a constant value. Element-2 is not able to find a different feasible optimum and therefore cannot reduce the inconsistency further. The design constraint present in Element-2 (${}^2v_g$) is active and remains active, see Figure 8.8(a). Design variable ${}^1x_1$ shifts after 18 hierarchical updates to a new value and the lower bound on design variable ${}^2x_2$ remains active throughout the entire optimization process.

After 30 hierarchical updates the sign of the size of the inconsistency between Element-0 and Element-1 changes, see Figure 8.6. The design constraint function present in Element-1 becomes active, see Figure 8.7. However, after a small number of hierarchical element updates the optimal design variables change to different values (Figure 8.7(b)) and the design constraint ${}^1v_g$ is negative again.

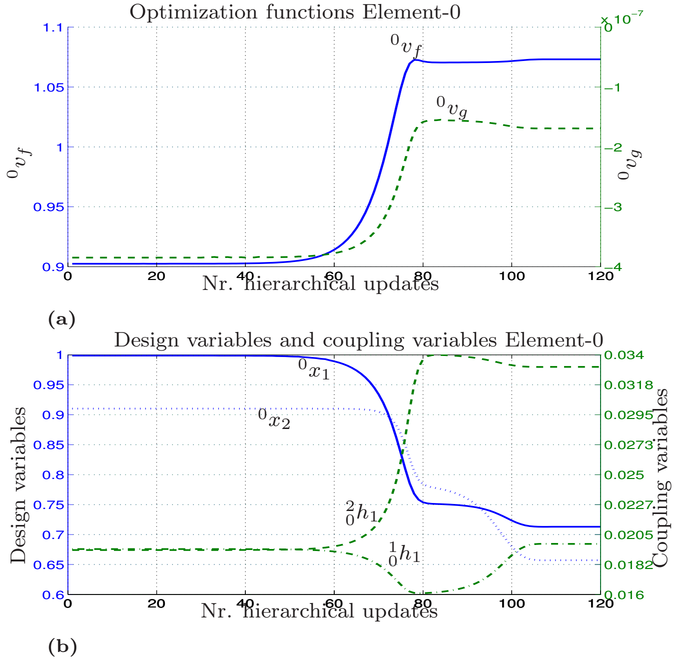


Figure 8.5: *History of a top-down hierarchic decomposition of the two-bar truss multi-level optimization problem. Convergence settings for inconsistencies (ϵ_{v_c}) and objective function (ϵ_{v_f}) correspond to the Alternating Descent method. Furthermore, a sequential coordination was applied to the iteration process. (a) Objective function (^0v_f) and constraint function (^0v_g) Element-0. (b) Optimal design variables ($^0x_1$, $^0x_2$) and optimal coupling variables ($^1_0h_1$, $^2_0h_1$) Element-0.*

Because the penalty weight embedded in the Augmented Lagrangian function is increasing, the objective function value of Element-0 (^0v_f), Element-1 (^1v_f) and of Element-2 (^2v_f) increases, see Figures 8.5(a), 8.7(a) and 8.8(a). To decrease this negative contribution to the objective function of Element-0, Element-1 and Element-2, the coupling variables and design variables in Element-0 are rearranged (see Figure 8.5(b)) and the objective of Element-2 decreases to zero after 80 hierarchical updates (see Figure 8.8(a)). The objective of Element-1 follows after a 105 hierarchical updates. The final optimum for the entire hierarchy is reached after 120 hierarchical updates. All design constraints ^0v_g, ^1v_g and ^2v_g are active and the inconsistencies are smaller then the pre-determined tolerance $^i\epsilon_{v_c}$, see Figure 8.6.

Results

Results for different decompositions, coordinations and algorithm settings for the two-bar truss multi-level optimization are listed in Table 8.5. Two decomposition methods are applied: hierarchic top-down decomposition and non-hierarchic decom-

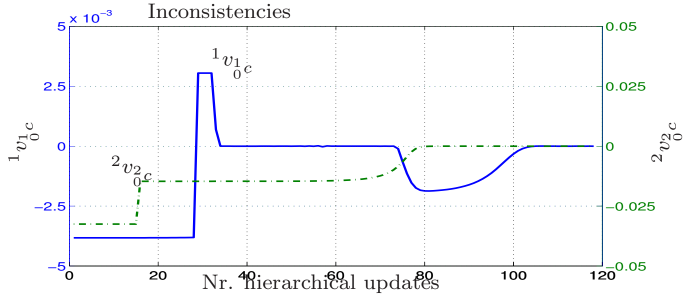


Figure 8.6: *History of a top-down hierarchic decomposition of the two-bar truss multi-level optimization problem. Convergence settings for inconsistencies (ϵ_{v_c}) and objective function (ϵ_{v_f}) correspond to the Alternating Descent method. Furthermore, a sequential coordination was applied to the iteration process. The figure shows the size of consistency violation between Element-0 and Element-1 (${}^1v_{{}^1_0c}$) and Element-0 and Element-2 (${}^2v_{{}^2_0c}$) during the multi-level optimization process.*

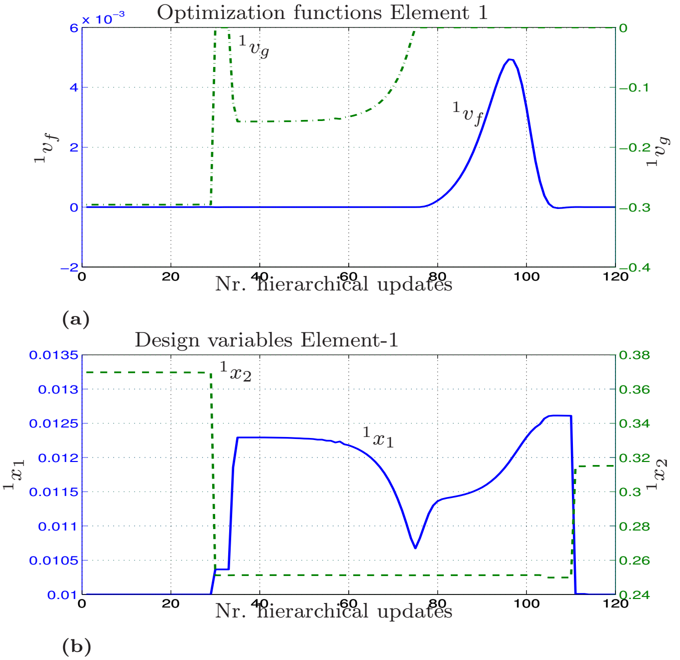


Figure 8.7: *History of a top-down hierarchic decomposition of the two-bar truss multi-level optimization problem. Convergence settings for inconsistencies (ϵ_{v_c}) and objective function (ϵ_{v_f}) correspond to the Alternating Descent method. Furthermore, a sequential coordination was applied to the iteration process. (a) Objective function (${}^1v_f$) and constraint function (${}^1v_g$) Element-1 (b) Optimal design variables (${}^1x_1$, ${}^1x_2$) Element-1.*

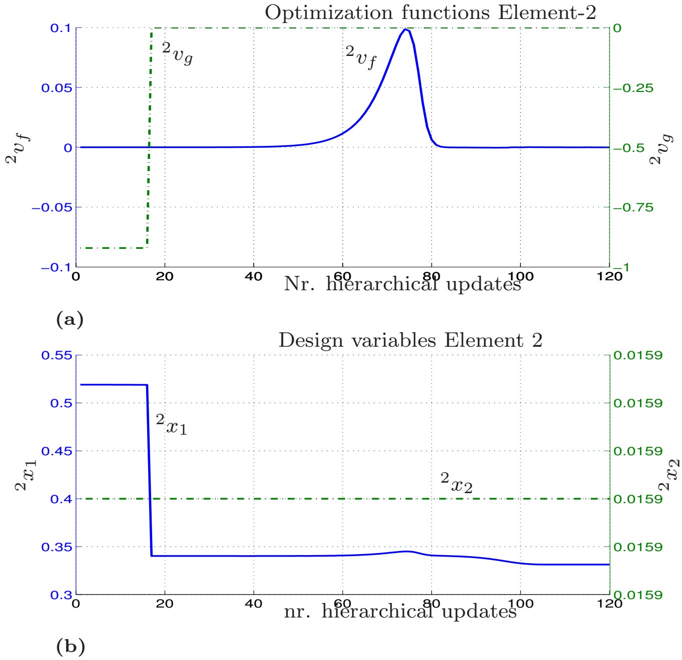


Figure 8.8: *History of a top-down hierarchic decomposition of the two-bar truss multi-level optimization problem. Convergence settings for inconsistencies (ϵ_{v_c}) and objective function (ϵ_{v_f}) correspond to the Alternating Descent method. Furthermore, a sequential coordination was applied to the iteration process. (a) Objective function (${}^2v_f$) and constraint function (${}^2v_g$) Element-2 (b) Optimal design variables (${}^2x_1$, ${}^2x_2$) Element-2.*

position. Furthermore, results are compared for three different convergence criteria and corresponding settings of β and γ combined with a sequential coordination process. These criteria correspond to three methods (see Section 5.3) that update the coupling and relaxation parameters:

- Block Coordinate Descent (BCD),
- Inexact (InE),
- Alternating Descent (AD).

Finally, a sequential and parallel coordination of the individual elements is compared. The best coupling and relaxation settings that worked are used to show the effect of parallel coordination with respect to sequential coordination. The parallel coordination is accomplished via an additional damping parameter τ, see Section 5.3.2.

Furthermore, in Table 8.6 the constraints are listed for the two-bar truss optimization problem. The All-in-One (AiO) solution is compared with respect to the multi-level optimization formulations. Constraints that are active in the AiO formula-

Table 8.5: *Computational costs of different convergence criteria considering the two-bar truss optimization problem with an Augmented Lagrangian relaxation technique. Order of magnitude of function evaluations and optimization iterations are presented and the number of hierarchical updates. Both hierarchic and non-hierarchic decompositions are considered. Furthermore, results for a sequential and parallel solution strategy are presented.*

decomp. coord.	func.eval.	opt.iter.	hier.upd.	conv. crit.
top-down sequential				
BCD: $\beta = 2.2$, $\gamma = 0.4$	8×10^3	1×10^3	67	$\epsilon_{v_c} \leq 1 \times 10^{-3}$ $\epsilon_{v_f} \leq 1 \times 10^{-5}$
InE: $\beta = 2.0$, $\gamma = 0.5$	5×10^3	7×10^2	44	$\epsilon_{v_c} \leq 1 \times 10^{-3}$ $\epsilon_{v_f} \leq 1 \times 10^{-3}$
AD: $\beta = 1.1$, $\gamma = 0.90$	2×10^4	2×10^3	120	$\epsilon_{v_c} \leq 1 \times 10^{-3}$ $\epsilon_{v_f} \leq +inf$
top-down parallel				
AD: $\beta = 1.05$, $\gamma = 0.80$, $\tau = 0.9$	3×10^4	6×10^3	222	$\epsilon_{v_c} \leq 1 \times 10^{-3}$ $\epsilon_{v_f} \leq +inf$
non-hier. sequential				
BCD: $\beta = 2.2$, $\gamma = 0.4$	5×10^4	5×10^3	382	$\epsilon_{v_c} \leq 1 \times 10^{-3}$ $\epsilon_{v_f} \leq 1 \times 10^{-5}$
InE: $\beta = 2.0$, $\gamma = 0.5$	9×10^3	1×10^3	58	$\epsilon_{v_c} \leq 1 \times 10^{-4}$ $\epsilon_{v_f} \leq 1 \times 10^{-4}$
AD: $\beta = 1.1$, $\gamma = 0.95$	1×10^4	1×10^3	149	$\epsilon_{v_c} \leq 1 \times 10^{-4}$ $\epsilon_{v_f} \leq +inf$
non-hier. parallel				
BCD: $\beta = 1.1$, $\gamma = 0.90$, $\tau = 0.9$	1×10^5	2×10^4	1026	$\epsilon_{v_c} \leq 1 \times 10^{-4}$ $\epsilon_{v_f} \leq 1 \times 10^{-5}$

tion are listed as well as constraints that become active in the multi-level optimization problem formulation.

In case a constraint is active in the AiO formulation and not at the optimum found via multi-level optimization the design constraint value is used. In case lower or upper bounds become active that are not active during the AiO optimization these are also listed.

Table 8.6: *Consistency constraint values and design constraint values of the relaxation-based coordination methods. The reference design (AiO) has four active design constraints.*

Method	${}^0v_g$	${}^1_0c$	${}^2_0c$	${}^0v_g$	${}^1v_g$	${}^2v_g$	${}^1\underline{x}_1$	${}^1\underline{x}_2$	${}^2\underline{x}_2$
AiO	1.06			act.	act.	act.			act.
BC	1.07	-8×10^{-9}	5×10^{-9}	act.	act.	act.		act.	act.
InE	1.07	-8×10^{-9}	-2×10^{-8}	act.	act.	act.		act.	act.
AD	1.07	2×10^{-8}	-1×10^{-9}	act.	act.	act.	act.		act.
AD Par.	1.07	-1×10^{-9}	1×10^{-8}	act.	act.	act.			act.
BC n.-h.*	1.06	-3×10^{-4}	1×10^{-6}	-0.06	act.	act.			act.
BC n.-h. Par*	1.08	-4×10^{-7}	-8×10^{-6}	-0.07	act.	act.			act.
InE n.-h.*	1.08	-3×10^{-9}	6×10^{-6}	-0.08	act.	act.	act.		act.
AD n.-h.*	1.07	1×10^{-6}	-3×10^{-8}	-0.11	act.	act.	act.	act.	act.

* Largest value of the consistency constraints is listed.

Discussion

Results in Table 8.5 and Table 8.6 show that for the two-bar truss multi-level optimization problem considered in the present thesis the java framework is able to carry out a distributed optimization of the two-bar truss structure. As expected from numerical results published in literature, (Tosserams *et al.*, 2008b, Li *et al.*, 2008, de Wit and van Keulen, 2008) the computational cost of the Analytical Target Cascading method combined with an Augmented Lagrangian relaxation is considerable. Furthermore, it was observed that in some cases the individual design optimization problems had difficulties to converge depending on the γ and β settings chosen and the convergence parameter settings. Therefore, the convergence parameters were adjusted such that the closest achievable point to the known optimum was found within reasonable computational time.

Comparing the hierarchic top-down decomposition with the non-hierarchic decomposition the results show that independent of the algorithm settings the top-down hierarchic decomposition converges faster in terms of actual wall-clock-time and finds a better optimal point then the non-hierarchic decomposition approach. All non-hierarchic multi-level optimization attempts converged to a non-optimal point where the Level-0 design constraint was not active.

Changing the convergence parameters ϵ_{v_c} and ϵ_{v_f} corresponding to Block Coordinate Descent (BCD), Inexact (InE) and Alternating Directions (AD) showed the effect on the individual optimization costs as well as the frequency in which information is send between elements. Alternating Directions converged the fastest of all algorithm settings. However, the cost of frequently updating the information between elements is considerable. If information exchange should be kept to a minimum Block Coordinate Descent (BCD) or Inexact (InE) convergence parameters perform well.

Sequential coordination compared to parallel coordination showed that for the current example the computational costs increase almost four times for the parallel coordination. Therefore, the total execution time is longer than that of a sequential coordination process. However, it is expected that for problems where the optimization of individual elements requires significant execution time it will reduce the total execution time. A sequential process requires each level and/or element to wait until a previous level/element is finished. A parallel coordination still waits for the element with the longest execution time, however this will also be the total waiting time for a single hierarchical update, whereas for a sequential process the waiting time is equal to the execution time of each individual element. The parallel coordination will not reduce the computational costs because of the damping parameter τ that reduces the step-size in which coupling data is updated between elements. Furthermore, the coupling data that is used to evaluate a single element is based on the previous iteration where a sequential process uses coupling data based on the current iteration.

The results show that even for a small problem the computational costs become very large. This is because initially, a significant amount of iterations are required before the penalty parameters start to work. Furthermore, when the multi-level optimization is almost finished a number of hierarchical updates is required before the process converges. Changing the penalty parameters to higher initial values does not solve the problem. A high penalty parameter in the Augmented Lagrangian function creates difficulties for the individual elements to reduce the negative contribution to the objective function value.

Because this example is small, the costs of starting an optimization and stopping an individual element optimization after it has finished dominates the algorithms execution time. It is expected that for examples with more complex element optimization problems this effect reduces or vanishes and the individual element optimizations become the dominant computational cost factor.

8.2 Portal frame

8.2.1 Multi-level optimization problem

Problem formulation

A portal frame is considered for multi-level design optimization. The portal frame design problem is shown in Figure 8.9 and the parameters determining the structural lay-out are listed in Table 8.7. There are eighteen design variables, six for each element represented by the vectors ${}^1\mathbf{x}$, ${}^2\mathbf{x}$, ${}^3\mathbf{x}$ that describe the cross-sectional area of the individual elements, see Figure 8.9.

For this problem, the objective (${}^0v_f$) is to minimize the horizontal displacement (${}^0r_1$). The portal frame design is subjected to constraints ${}^0v_{g_1}$ on the total volume (${}^0r_2$) of the structure and the rotation (${}^0r_3$) of the right upper corner of the structure. Furthermore, there are several constraints on the individual members of the structure. Each member has a maximum allowable normal stress (${}^iv_{g_1}$), (${}^iv_{g_2}$), (${}^iv_{g_3}$), (${}^iv_{g_4}$) which is computed at the top and bottom of both ends of the column. The maximum allowable normal stress holds for both compression and extension of the columns. Furthermore, each member has a maximum allowable shear stress (${}^iv_{g_5}$), (${}^iv_{g_6}$) which is computed at both ends of the columns at the axis running through the center of the cross-section. To prevent slender structures that are likely to buckle additional constraints on the geometry are added (${}^iv_{g_7}$), (${}^iv_{g_8}$), (${}^iv_{g_9}$), (${}^iv_{g_{10}}$) to prevent flange and web buckling. Finally, lower bounds (${}^{\cdot\cdot}\underline{\mathbf{x}}$) and upper bounds (${}^{\cdot\cdot}\overline{\mathbf{x}}$) on the design variables are present, see Table 8.8.

Two levels are distinguished in the design problem, see Figure 8.9. The top level (Level-0) consisting of a single hierarchical element and involving minimization of the horizontal displacement (${}^0r_1$), while satisfying constraints on the total volume (${}^0r_2$) and the rotation of the right upper corner of the structure (${}^0r_3$). The bottom level (Level-1) consisting of three individual elements involving constraints on maximum normal stress (${}^ir_1$,${}^ir_2$), maximum shear stress (${}^ir_3$,${}^ir_4$) and geometrical constraints to prevent buckling.

The optimization functions associated with the portal frame design problem are

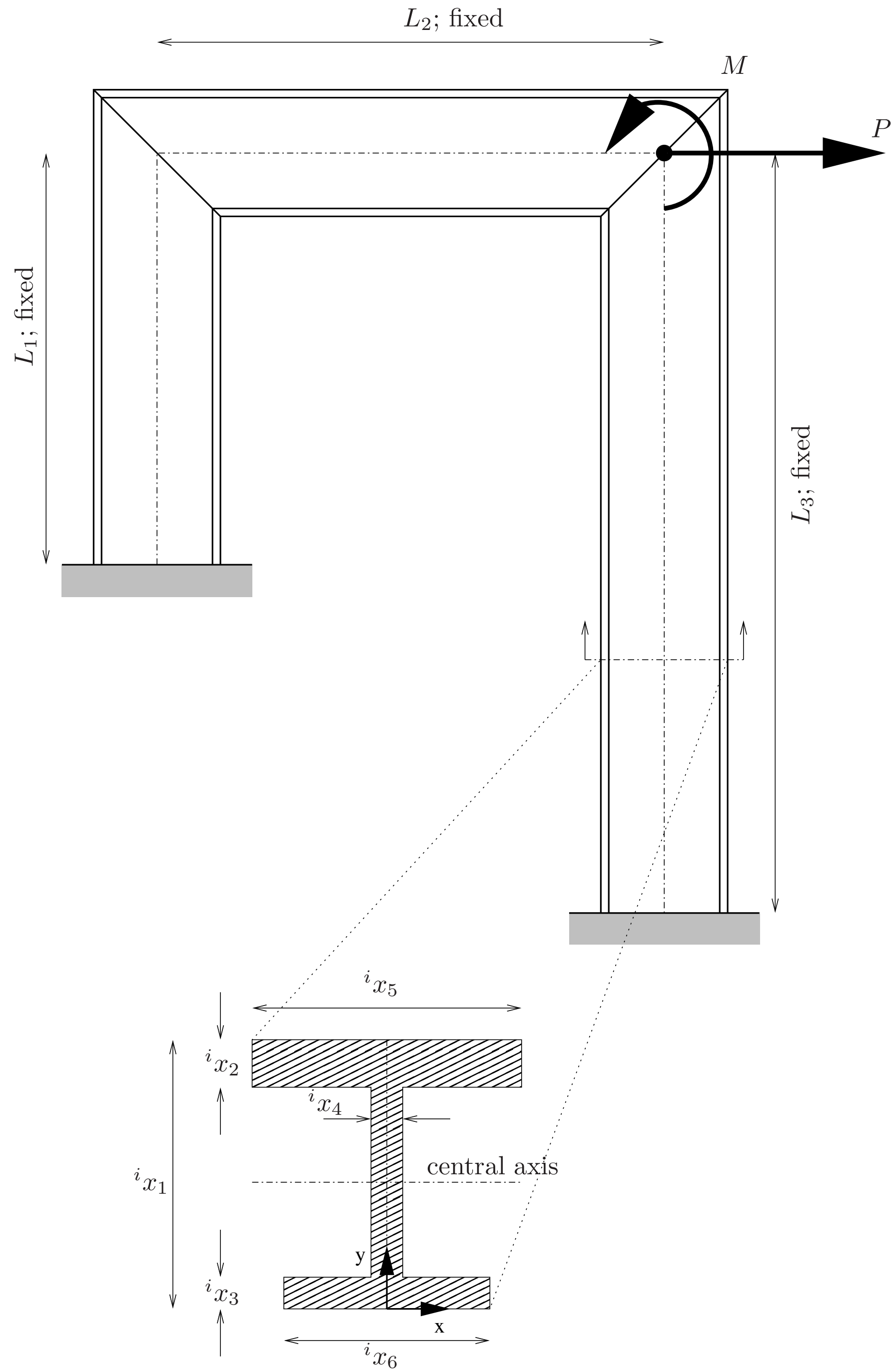


Figure 8.9: *Portal framework with embedded hierarchy. The general lay-out of the structure is described via fixed lengths of the columns and the cross-sectional areas of the columns. The elements are described in detail via design variable vectors* ${}^1\mathbf{x}, {}^2\mathbf{x}, {}^3\mathbf{x}$.

Table 8.7: *Definition of design variables, physical responses, physical interactions between subsystems and optimization parameters for the portal frame.*

design	description	physical response	description	interaction	description
		${}^{0}r_1$	displacement left corner	${}^{0}_{1}\mathbf{h}$	nodal forces Element-1
		${}^{0}r_2$	total volume	${}^{0}_{2}\mathbf{h}$	nodal forces Element-2
		${}^{0}r_3$	rotation left corner	${}^{0}_{3}\mathbf{h}$	nodal forces Element-3
${}^{i}x_1$	total height	${}^{i}r_1$	normal stress front	${}^{i}_{0}h_1$	cross-sectional area
${}^{i}x_2$	thickness top	${}^{i}r_2$	normal stress back	${}^{i}_{0}h_2$	moment of inertia
${}^{i}x_3$	thickness bottom	${}^{i}r_3$	shear stress front		
${}^{i}x_4$	thickness middle	${}^{i}r_4$	shear stress back		
${}^{i}x_5$	top width				
${}^{i}x_6$	bottom width				

optimization	description	
${}^{0}r_{V_{max}}$	reference volume	
${}^{0}r_{\theta_{max}}$	allowed rotation	
${}^{i}r_{\sigma_{cr}}$	maximum allowed normal stress	E = Young's modulus
${}^{i}r_{\tau_{cr}}$	maximum allowed shear stress	ρ = density

Table 8.8: *Scaling parameters and upper and lower bounds on the design variables for the portal frame problem, where* $i = 1, 2, 3$.

variable	lowerbound	upperbound	scaling parameter
${}^{i}x_1$	2.000×10^{-1}	20.00×10^{-1}	1.000×10^{-1}
${}^{i}x_2$	0.750×10^{-2}	7.500×10^{-2}	1.000×10^{-2}
${}^{i}x_3$	0.750×10^{-2}	7.500×10^{-2}	1.000×10^{-2}
${}^{i}x_4$	1.000×10^{-2}	10.00×10^{-2}	1.000×10^{-2}
${}^{i}x_5$	1.500×10^{-1}	15.00×10^{-1}	1.000×10^{-1}
${}^{i}x_6$	1.500×10^{-1}	15.00×10^{-1}	1.000×10^{-1}
${}^{0}_{i}h_1$	0.410×10^{-2}	41.00×10^{-2}	1.000×10^{-2}
${}^{0}_{i}h_2$	0.261×10^{-4}	2613×10^{-4}	1.000×10^{-4}
${}^{i}_{0}h_1$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{i}_{0}h_2$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_3$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_4$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_5$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_6$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}

Table 8.9: *Constant values necessary to evaluate the portal frame multi-level design optimization problem.*

constant	value	constant	value	constant	value
E	7.06×10^{10}	L_1	5	${}^{0}r_{V_{max}}$	3.00×10^{-1}
ρ	1.00×10^{0}	L_2	10	${}^{0}r_{\theta_{max}}$	1.50×10^{-2}
P	5.00×10^{4}	L_3	10	${}^{i}r_{\sigma_{cr}}$	2.00×10^{8}
M	2.00×10^{5}			${}^{i}r_{\tau_{cr}}$	1.16×10^{8}

mathematically expressed as:

$$\begin{array}{lllllll}
{}^{0}v_f & = & \dfrac{{}^{0}r_1\left({}^{1}\mathbf{x},{}^{2}\mathbf{x},{}^{3}\mathbf{x}\right)}{{}^{0}\mathbf{r}_{u_{max}}} & ; & {}^{i}v_{g4} & = & -\dfrac{{}^{i}r_2({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1; \\
{}^{0}v_{g_1} & = & \dfrac{{}^{0}r_2({}^{1}\mathbf{x},{}^{2}\mathbf{x},{}^{3}\mathbf{x})}{{}^{0}r_{V_{max}}} - 1 & ; & {}^{i}v_{g_5} & = & \dfrac{|{}^{i}r_3({}^{i}\mathbf{x},{}^{0}\mathbf{r})|}{{}^{i}r_{\tau_{cr}}} - 1; \\
{}^{0}v_{g_2} & = & \dfrac{{}^{0}r_3({}^{1}\mathbf{x},{}^{2}\mathbf{x},{}^{3}\mathbf{x})}{{}^{0}r_{\theta_{max}}} - 1 & ; & {}^{i}v_{g_6} & = & \dfrac{|{}^{i}r_4({}^{i}\mathbf{x},{}^{0}\mathbf{r})|}{{}^{i}r_{\tau_{cr}}} - 1; \\
{}^{i}v_{g_1} & = & \dfrac{{}^{i}r_1({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1 & ; & {}^{i}v_{g_7} & = & \dfrac{\left({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3\right)}{35\,{}^{i}x_4} - 1; \\
{}^{i}v_{g_2} & = & -\dfrac{{}^{i}r_1({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1 & ; & {}^{i}v_{g_8} & = & \dfrac{{}^{i}x_5}{20\,{}^{i}x_2} - 1; \\
{}^{i}v_{g_3} & = & \dfrac{{}^{i}r_2({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1 & ; & {}^{i}v_{g_9} & = & \dfrac{{}^{i}x_6}{20\,{}^{i}x_3} - 1; \\
{}^{i}v_{g_{10}} & = & \multicolumn{5}{l}{1 - \dfrac{5\left({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3\right){}^{i}x_4}{({}^{i}x_5)({}^{i}x_2) + ({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3){}^{i}x_4 + ({}^{i}x_6)({}^{i}x_3)}.}
\end{array} \tag{8.22}$$

The problem matrix showing the dependencies of the optimization functions of Equation 8.22 on the design variables and the physical responses is shown in Figure

8.10. The objective function and the constraint functions of Level-0 depend on the design variables of the individual elements identified at Level-1. Likewise, the physical responses at Level-1 depend on physical responses computed at Level-0. The coupling between the two levels is shown via coupling variables in the problem matrix, see Figure 8.11.

Hierarchic decomposition

A top-down hierarchic decomposition of the portal frame optimization problem is considered:
Level-0, Element 0:

$$
\begin{array}{llll}
\min\limits_{{}^1_0\mathbf{h},{}^2_0\mathbf{h},{}^3_0\mathbf{h}} & {}^0v_f & = & \dfrac{{}^0r_1\left({}^1_0\mathbf{h},{}^2_0\mathbf{h},{}^3_0\mathbf{h}\right)}{{}^0\mathbf{r}_{u_{max}}} + \sum_{i=1}^{3} {}^0v_{{}^i_0c} \\
\text{s.t.} & {}^0v_{g_1} & = & \dfrac{{}^0r_2({}^1_0\mathbf{h},{}^2_0\mathbf{h},{}^3_0\mathbf{h})}{{}^0r_{V_{max}}} - 1 \qquad \leq 0 \\
 & {}^0v_{g_2} & = & \dfrac{{}^0r_3({}^1_0\mathbf{h},{}^2_0\mathbf{h},{}^3_0\mathbf{h})}{{}^0r_{\theta_{max}}} - 1 \qquad \leq 0 \\
 & \multicolumn{3}{l}{{}^1_0\underline{\mathbf{h}} \leq {}^1_0\mathbf{h} \leq {}^1_0\overline{\mathbf{h}}, \quad {}^2_0\underline{\mathbf{h}} \leq {}^2_0\mathbf{h} \leq {}^2_0\overline{\mathbf{h}}, \quad {}^3_0\underline{\mathbf{h}} \leq {}^3_0\mathbf{h} \leq {}^3_0\overline{\mathbf{h}}\ ;} \\
\text{where} & {}^0v_{{}^1_0c} & = & {}^1_0\boldsymbol{\lambda}_1^T\left({}^1_0\mathbf{c}\right) + ||{}^1_0\mathbf{s}_1 \circ {}^1_0\mathbf{c}||_2^2 \\
 & {}^0v_{{}^2_0c} & = & {}^2_0\boldsymbol{\lambda}_1^T\left({}^2_0\mathbf{c}\right) + ||{}^2_0\mathbf{s}_1 \circ {}^2_0\mathbf{c}||_2^2 \\
 & {}^0v_{{}^3_0c} & = & {}^3_0\boldsymbol{\lambda}_1^T\left({}^3_0\mathbf{c}\right) + ||{}^3_0\mathbf{s}_1 \circ {}^3_0\mathbf{c}||_2^2
\end{array}
\tag{8.23}
$$

Level-1, Element i:

$$
\begin{array}{lllllll}
\min\limits_{{}^i\mathbf{x}} & {}^iv_f & = & {}^iv_{{}^i_0c}; & & & \\
\text{s.t.} & {}^iv_{g_1} & = & \dfrac{{}^ir_1({}^i\mathbf{x})}{{}^ir_{\sigma_{cr}}} - 1 \leq 0 & ; & {}^iv_{g_6} & = \dfrac{|{}^ir_4({}^i\mathbf{x})|}{{}^ir_{\tau_{cr}}} - 1 \leq 0; \\
 & {}^iv_{g_2} & = & -\dfrac{{}^ir_1({}^i\mathbf{x})}{{}^ir_{\sigma_{cr}}} - 1 \leq 0 & ; & {}^iv_{g_7} & = \dfrac{\left({}^ix_1 - {}^ix_2 - {}^ix_3\right)}{35\,{}^ix_4} - 1 \leq 0; \\
 & {}^iv_{g_3} & = & \dfrac{{}^ir_2({}^i\mathbf{x})}{{}^ir_{\sigma_{cr}}} - 1 \leq 0 & ; & {}^iv_{g_8} & = \dfrac{{}^ix_5}{20\,{}^ix_2} - 1 \leq 0; \\
 & {}^iv_{g_4} & = & -\dfrac{{}^ir_2({}^i\mathbf{x})}{{}^ir_{\sigma_{cr}}} - 1 \leq 0 & ; & {}^iv_{g_9} & = \dfrac{{}^ix_6}{20\,{}^ix_3} - 1 \leq 0; \\
 & {}^iv_{g_5} & = & \dfrac{|{}^ir_3({}^i\mathbf{x})|}{{}^ir_{\tau_{cr}}} - 1 \leq 0; & & & \\
 & \multicolumn{6}{l}{{}^iv_{g_{10}} = 1 - \dfrac{5\left({}^ix_1 - {}^ix_2 - {}^ix_3\right){}^ix_4}{({}^ix_5)({}^ix_2) + ({}^ix_1 - {}^ix_2 - {}^ix_3){}^ix_4 + ({}^ix_6)({}^ix_3)} \leq 0;} \\
 & \multicolumn{6}{l}{{}^i\underline{\mathbf{x}} \leq {}^i\mathbf{x} \leq {}^i\overline{\mathbf{x}}} \\
\text{where} & {}^iv_{{}^i_0c} & = & {}^i_0\boldsymbol{\lambda}_2^T\left({}^i_0\mathbf{c}\right) + ||{}^i_0\mathbf{s}_2 \circ {}^i_0\mathbf{c}||_2^2; & & &
\end{array}
\tag{8.24}
$$

Coupling variable vectors ${}^1_0\mathbf{h}$, ${}^2_0\mathbf{h}$ and ${}^3_0\mathbf{h}$ are added to the Level-0 optimization problem as design variables. These variables must match the mapped geometrical characteristics ${}^1_0\boldsymbol{\mathcal{H}}({}^1\mathbf{x})$, ${}^2_0\boldsymbol{\mathcal{H}}({}^2\mathbf{x})$ and ${}^3_0\boldsymbol{\mathcal{H}}({}^3\mathbf{x})$ respectively. The latter are evaluated during the Level-1 optimization.

Non-hierarchic decomposition

A non-hierarchic decomposition of the problem involves a decomposition of the coupling in both directions. Mapping of the displacements calculated at Level-0 onto

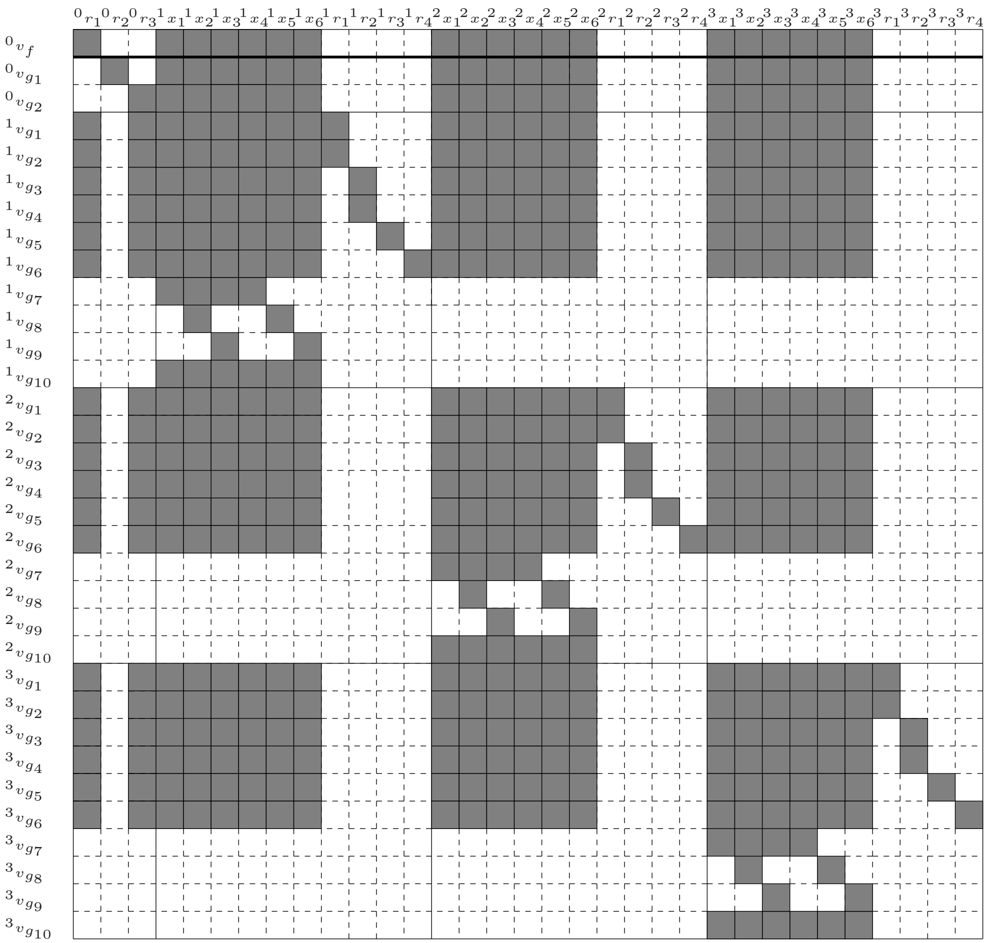


Figure 8.10: *Problem matrix portal frame optimization problem before decomposition. On the left the optimization problem functions are listed and on the top the design variables and physical responses. A function that depends on a specific design variable or physical response is represented by a shaded block.*

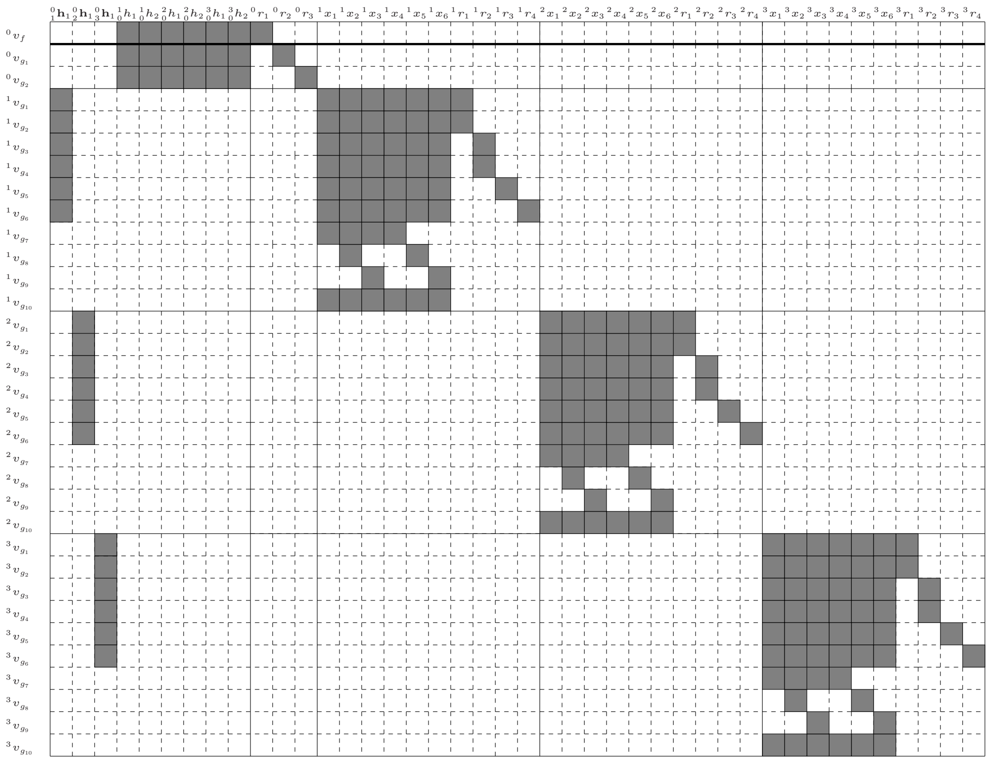


Figure 8.11: *Coupling parameters are introduced into the problem matrix, showing the data that is shared between the Level-0 element and the three Level-1 elements.*

the Level-1 elements as nodal forces is replaced via consistency constraints. These constraints are relaxed and added to the individual element objective function of each element.

8.2.2 Physical model

The initial shape of the portal frame and it's deformed state under external loading are shown in Figure 8.12. Two levels can be distinguished, the general lay-out of the portal frame structure consisting of three columns and the detailed cross-sectional area, which is described by the thicknesses and the widths of three plates.

General responses of the structure are the horizontal displacement (${}^0r_1$) at the point where the external forces are applied, the total volume of the structure (${}^0r_2$) and the angular displacement (${}^0r_3$) of the point where the external forces are applied. These responses depend on the cross-sectional areas and the moments of inertia of the columns which are considered Level-1 properties.

The general responses evaluated at Level-0 are computed by means of solving a

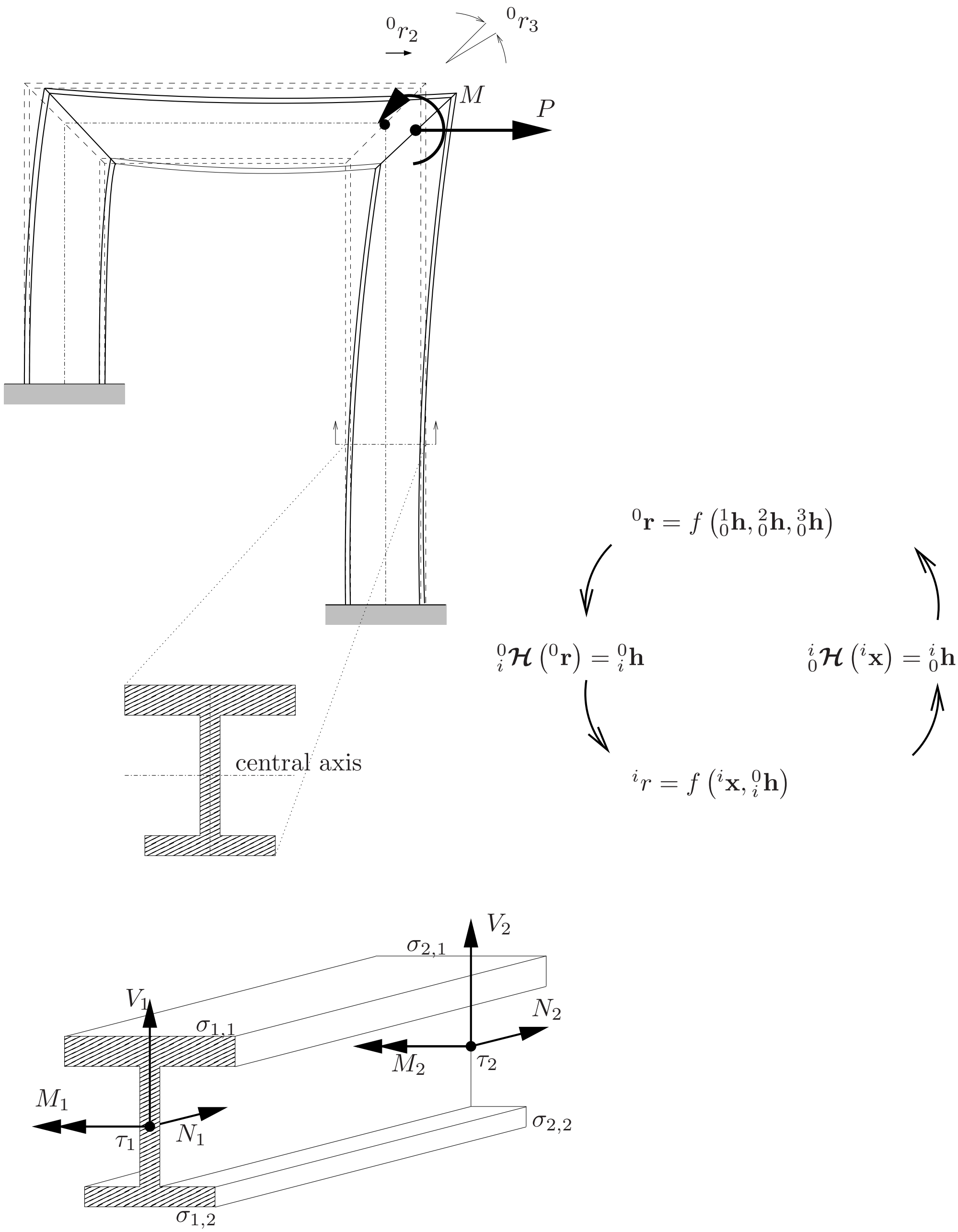


Figure 8.12: *Portal frame with embedded hierarchy. The general lay-out consists of three columns representing Level-0. The detailed cross-sectional area is identified at Level-1. The stress in each element is computed at the top ($\sigma_{1,1}$, $\sigma_{2,1}$), at the bottom ($\sigma_{1,2}$, $\sigma_{2,2}$) and at the axis running through the middle of the cross-section (τ_1, τ_2) on both ends of the column.*

finite element model. The displacements of the structure are computed via:

$$\mathbf{Ku} = \mathbf{P}, \tag{8.25}$$

where $\mathbf{K}$ is the stiffness matrix of the entire structure. The columns are modeled as beam elements and the flexural stiffness matrix of these beam elements is mathematically expressed as:

$$\mathbf{K}^e = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \tag{8.26}$$

A complete derivation of the necessary equations required to compute the displacements is presented in Appendix B. After assembly of the element stiffness matrices the total structural stiffness matrix is found. The displacements at the top right corner where the external loading is applied are of special interest. Therefore, the horizontal displacement at this node is written: ${}^{0}r_1$; and the angular rotation of the top right corner is written: ${}^{0}r_3$.

The stiffness matrix ($\mathbf{K}$) is considered a Level-0 property and depends on the cross-sectional area of the columns which is considered a Level-1 property. The mapping of the cross-sectional area from Level-1 onto Level-0 is mathematically expressed as:

$${}^{i}_{0}\mathcal{H}_1\left({}^{i}x_1, {}^{i}x_2, {}^{i}x_3, {}^{i}x_4, {}^{i}x_5, {}^{i}x_6\right) = {}^{i}_{0}h_1, \tag{8.27}$$

where $i = 1, 2, 3$. Furthermore, the stiffness matrix depends on the moment of inertia. The mapping of the moment of inertia is mathematically expressed as:

$${}^{i}_{0}\mathcal{H}_2\left({}^{i}x_1, {}^{i}x_2, {}^{i}x_3, {}^{i}x_4, {}^{i}x_5, {}^{i}x_6\right) = {}^{i}_{0}h_2, \tag{8.28}$$

where $i = 1, 2, 3$.

The total volume of the structure is defined as:

$${}^{0}r_2 = {}^{1}_{0}h_1 l_1 + {}^{2}_{0}h_1 l_2 + {}^{3}_{0}h_1 l_3, \tag{8.29}$$

where the lengths l_i are given parameters.

After the displacements of the structure are found, the internal forces can be computed. The internal forces are required by the Level-1 analysis of the individual elements. First the strains are computed for each individual element from the structural displacements. Secondly, the internal nodal forces are computed:
Element 1:

$$\mathbf{f}_1^{int} = \int_V \mathbf{B}^T \mathbf{D} \epsilon_1 dV \quad \Rightarrow \quad {}^{0}_{1}\mathbf{h} = {}^{0}_{1}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) \tag{8.30}$$

Element 2:

$$\mathbf{f}_2^{int} = \int_V \mathbf{B}^T \mathbf{D} \epsilon_2 dV \quad \Rightarrow \quad {}^{0}_{2}\mathbf{h} = {}^{0}_{2}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) \tag{8.31}$$

Element 3:

$$\mathbf{f}_3^{int} = \int_V \mathbf{B}^T \mathbf{D} \epsilon_3 dV \quad \Rightarrow \quad {}^{0}_{3}\mathbf{h} = {}^{0}_{3}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) \tag{8.32}$$

Axial stress is defined as $\sigma_a = \frac{p}{A}$. The axial load carried by an element is p and the cross-sectional area is A. Hence, the axial stress represented in multi-level notation is:

$$\sigma_a = \frac{{}^{0}_{i}h_{1_4} - {}^{0}_{i}h_{1_1}}{{}^{i}_{0}h}, \tag{8.33}$$

where ${}^{0}_{i}h_{1_1}$ and ${}^{0}_{i}h_{1_4}$ are the axial forces applied on each column. The bending stress is defined as $\sigma_b = \frac{dM}{I}$. The bending moment carried by the element is M, the distance d is the free surface distance and the moment of inertia of the element is I. The bending stress represented in multi-level notation is:

$$\sigma_{b_1} = \frac{d{}^{0}_{i}h_{1_3}}{{}^{i}_{0}h_2} \quad \text{and} \quad \sigma_{b_2} = \frac{d{}^{0}_{i}h_{1_6}}{{}^{i}_{0}h_2}, \tag{8.34}$$

where ${}^{0}_{i}h_{1_3}$ and ${}^{0}_{i}h_{1_6}$ are the bending moments applied on each end of the columns and d depends on the geometrical lay-out of the cross-section of the column. A mathematical expression for d is presented in Appendix B. The normal stress in the elements is a summation of axial and bending stresses:

$$ {}^{i}r_1 = \sigma_a + \sigma_{b_1} \qquad {}^{i}r_2 = \sigma_a + \sigma_{b_2}. \tag{8.35}$$

Finally, the shear stress τ is defined as $\tau = \frac{VQ}{Ib}$. The first moment of inertia is Q, the downward nodal force is V and the thickness of the vertical plate of the column is b. In multi-level notation the shear stress τ is represented as:

$$ {}^{i}r_3 = \frac{{}^{0}_{i}h_{1_2}Q}{{}^{i}_{0}h_2{}^{i}x_4} \quad \text{and} \quad {}^{i}r_4 = \frac{{}^{0}_{i}h_{1_5}Q}{{}^{i}_{0}h_2{}^{i}x_4} \tag{8.36}$$

where ${}^{0}_{i}h_{1_2}$ and ${}^{0}_{i}h_{1_5}$ are the downward applied loads on both ends of the column and Q depends on the geometrical lay-out of the cross-section of the column. A mathematical expression for Q is presented in Appendix B.

8.2.3 Numerical results

Two different coordination methods are applied to the portal frame multi-level optimization problem: Sequential and Parallel coordination. The relaxation parameters are updated via the method of multipliers (Bertsekas and Tsitsiklis, 1989).

All optimization starting points were chosen via random initial design variables. Furthermore, the coupling variables were initiated with variables that laid between feasible bounds, see Table 8.8. Therefore, the individual elements were not consistent with each other meaning that an infeasible design point was used in all cases and no initial computations were done in order to have a consistent starting design configuration.

The global optimal values for the design variables of the portal frame optimization problem are listed in Table 8.10. There is a second local optimum that can be found via an All-in-One optimization process that has a higher objective function value,

Table 8.10: *Optimal solution of the four elements of the portal frame in terms of design variables. The optimal solution was found via an All-in-One optimization. Therefore, the consistency constraints are satisfied exact.*

parameter			description
${}^{0}v_f = 9.2281 \times 10^{-3}$			horizontal displacement
${}^{1}_{0}h_1 = 3.1169 \times 10^{-2}$			cross-sectional area 1
${}^{1}_{0}h_2 = 1.4040 \times 10^{-4}$			moment of inertia 1
${}^{2}_{0}h_1 = 2.8001 \times 10^{-2}$			cross-sectional area 2
${}^{2}_{0}h_2 = 4.5000 \times 10^{-4}$			moment of inertia 2
${}^{3}_{0}h_1 = 1.1615 \times 10^{-2}$			cross-sectional area 3
${}^{3}_{0}h_2 = 4.2848 \times 10^{-4}$			moment of inertia 3
${}^{1}x_1 = 5.0000 \times 10^{-1}$	${}^{2}x_1 = 1.0004 \times 10^{-2}$	${}^{3}x_1 = 5.0000 \times 10^{-1}$	height
${}^{1}x_2 = 4.1559 \times 10^{-2}$	${}^{2}x_2 = 1.0000 \times 10^{-2}$	${}^{3}x_2 = 1.1300 \times 10^{-1}$	top thickness
${}^{1}x_3 = 4.1559 \times 10^{-2}$	${}^{2}x_3 = 1.0000 \times 10^{-2}$	${}^{3}x_3 = 1.1337 \times 10^{-2}$	bottom thickness
${}^{1}x_4 = 1.4953 \times 10^{-2}$	${}^{2}x_4 = 1.0000 \times 10^{-2}$	${}^{3}x_4 = 1.3639 \times 10^{-2}$	flange width
${}^{1}x_5 = 3.0000 \times 10^{-1}$	${}^{2}x_5 = 1.0000 \times 10^{-2}$	${}^{3}x_5 = 2.2600 \times 10^{-2}$	top width
${}^{1}x_6 = 3.0000 \times 10^{-1}$	${}^{2}x_6 = 1.0000 \times 10^{-2}$	${}^{3}x_6 = 2.2504 \times 10^{-1}$	bottom width

see Tosserams (2008). However, none of the multi-level optimizations converged to this second optimum. The value of the objective function at the global optimal point and the optimal value of the coupling variables are listed in Table 8.10. The design constraint functions present in each element of the hierarchy are active.

Results

Results for the portal frame multi-level optimizations that converged to an objective value of 9.2×10^{-3} are listed in Table 8.11. Two decomposition methods are applied: hierarchic top-down and non-hierarchic. Furthermore, results are compared for three different convergence criteria and corresponding settings of β and γ combined with a sequential coordination process. These criteria correspond to three methods that update the coupling and relaxation parameters:

- Block Coordinate Descent (BCD),
- Inexact (InE),
- Alternating Descent (AD).

Finally, a sequential and parallel coordination of the individual elements is compared. The best coupling and relaxation settings that worked are used to show the effect of parallel coordination with respect to sequential coordination. The parallel coordination is accomplished via an additional damping parameter τ, see Chapter 5.

In Table 8.11 the consistency tolerances (ϵ_{v_c}) are changed from 1×10^{-3} to 1×10^{-4} for some algorithm settings. Changing the convergence tolerance was necessary for non-hierarchic decompositions that otherwise would not converge to the optimal point.

Table 8.11: *Computational cost of each solution strategy for solving the portal framework via an Augmented Lagrangian relaxation technique. Order of magnitude for number of function evaluations and the number of optimization iterations are presented. Both hierarchic and non-hierarchic decompositions are considered. Furthermore, results for a sequential and parallel solution strategy are presented.*

decomp. coord.	func.eval.	opt.iter.	hier.upd.	conv. crit.
top-down sequential				$\epsilon_{v_c} \leq 5 \times 10^{-3}$
BCD: $\beta = 2.2$, $\gamma = 0.4$	2×10^4	2×10^3	595	$\epsilon_{v_f} \leq 1 \times 10^{-5}$
				$\epsilon_{v_c} \leq 1 \times 10^{-3}$
InE: $\beta = 2.0$, $\gamma = 0.5$	1×10^6	2×10^5	456	$\epsilon_{v_f} \leq 1 \times 10^{-3}$
				$\epsilon_{v_c} \leq 1 \times 10^{-3}$
AD: $\beta = 1.1$, $\gamma = 0.9$	2×10^4	2×10^3	97	$\epsilon_{v_f} \leq +inf$
top-down parallel				$\epsilon_{v_c} \leq 1 \times 10^{-3}$
InE: $\beta = 1.1$, $\gamma = 0.9$, $\tau = 0.9$	7×10^5	8×10^4	922	$\epsilon_{v_f} \leq 1 \times 10^{-3}$
non-hier. sequential				$\epsilon_{v_c} \leq 1 \times 10^{-3}$
BCD: $\beta = 2.2$, $\gamma = 0.4$	2×10^5	1×10^4	308	$\epsilon_{v_f} \leq 1 \times 10^{-5}$
				$\epsilon_{v_c} \leq 1 \times 10^{-3}$
InE: $\beta = 2.0$, $\gamma = 0.5$	7×10^4	5×10^3	109	$\epsilon_{v_f} \leq 1 \times 10^{-3}$
				$\epsilon_{v_c} \leq 1 \times 10^{-4}$
AD: $\beta = 1.1$, $\gamma = 0.9$	8×10^4	6×10^3	119	$\epsilon_{v_f} \leq +inf$
non-hier. parallel				$\epsilon_{v_c} \leq 1 \times 10^{-4}$
InE: $\beta = 1.1$, $\gamma = 0.9$, $\tau = 0.9$	1×10^6	8×10^4	712	$\epsilon_{v_f} \leq 1 \times 10^{-4}$

Table 8.12 lists the size of the largest inconsistencies between each element. Hence, all optimal points that were found via the different approaches tested resulted in consistent optimal designs.

Table 8.12: *Consistency constraint values for optimal values of the portal framework found via various settings of the relaxation-based coordination methods.*

Method	${}^1_0c$	${}^2_0c$	${}^3_0c$
BCD*	2×10^{-8}	6×10^{-10}	-2×10^{-8}
InE*	-1×10^{-8}	-2×10^{-8}	-2×10^{-8}
InE Parallel	3×10^{-6}	2×10^{-8}	3×10^{-7}
AD*	1×10^{-5}	-3×10^{-5}	6×10^{-5}
BCD non-hier. *	-7×10^{-5}	1×10^{-5}	-1×10^{-5}
InE non-hier. *	-1×10^{-4}	-2×10^{-4}	-1×10^{-3}
InE n.-h. Par.*	1×10^{-4}	-4×10^{-5}	-8×10^{-5}
AD non-hier. *	-2×10^{-6}	-4×10^{-5}	-5×10^{-5}

* Largest value of the consistency constraints is listed.

The optimal cross sectional areas found are listed in Table 8.13. These belong to the indicated costs listed in Table 8.15. From the table it is clear that all top-down hierarchic decomposition approaches and non-hierarchic decomposition approaches that were sequentially coordinated converge to the same optimal point. However, the parallel coordinated approaches converged to a different optimal point where the objective function value was approximately the same.

Table 8.13: *Objective function value and optimal cross-sectional areas for each individual column found via different multi-level optimization strategies.*

Method	${}^{0}v_f$	${}^{1}_{0}h$	${}^{2}_{0}h$	${}^{3}_{0}h$
BCD	9.229×10^{-3}	3.117×10^{-2}	2.801×10^{-2}	1.162×10^{-2}
InE	9.234×10^{-3}	3.116×10^{-2}	2.805×10^{-2}	1.162×10^{-2}
InE Parallel	9.222×10^{-3}	2.832×10^{-2}	4.910×10^{-2}	1.073×10^{-2}
AD	9.228×10^{-3}	3.117×10^{-2}	2.800×10^{-2}	1.162×10^{-2}
BCD non-hier.	9.229×10^{-3}	3.117×10^{-2}	2.800×10^{-2}	1.161×10^{-2}
InE non-hier.	9.228×10^{-3}	3.116×10^{-2}	2.800×10^{-2}	1.162×10^{-2}
InE n.-h. Par	9.228×10^{-3}	2.836×10^{-2}	4.846×10^{-2}	1.077×10^{-2}
AD non-hier.	9.229×10^{-3}	3.117×10^{-2}	2.800×10^{-2}	1.162×10^{-2}

Discussion

The results of Table 8.11 show the effect of adding additional coupling between Level-0 and Level-1 as compared to the two-bar truss example. The computational costs have increased, partly due to additional design constraints which increase the computational cost of individual element optimizations and partly due to the additional coupling parameters (moment of inertia and cross-sectional area in case top-down and moment of inertia, cross-sectional area and the internal forces of the beam elements in case non-hierarchic).

Comparing the hierarchic top-down decomposition with respect to a non-hierarchic decomposition results vary for different γ, β and convergence parameter settings. The hierarchic decomposition approach requires less computational effort if individual element optimizations are calculated via Block Coordinate Descent (BCD) or via Alternating Descent (AD). The non-hierarchic decomposition requires less computational effort when individual element optimizations are calculated via Inexact (InE). Overall, the amount of hierarchical updates required by the non-hierarchic decomposition approach is less then observed for the hierarchic decomposition approach.

Sequential coordination with respect to parallel coordination shows similar results as compared to the two-bar truss example problem. The amount of computational effort doubles in case a hierarchic decomposition is used. In case a non-hierarchic decomposition is used, the convergence tolerance is increased from 1×10^{-3} to 1×10^{-4} to converge to the optimal point. Therefore, the computational effort for the non-hierarchic decomposed and parallel coordinated approach is seven times higher then the sequentially coordinated approach. Hence, for the current example only top-down hierarchic decomposition benefits from parallel execution of the multi-level optimization approach.

The choice between Block Coordinate Descent (BCD), Inexact (InE) or Alternating Descent (AD) has a large effect on the numerical costs. Table 8.11 shows that

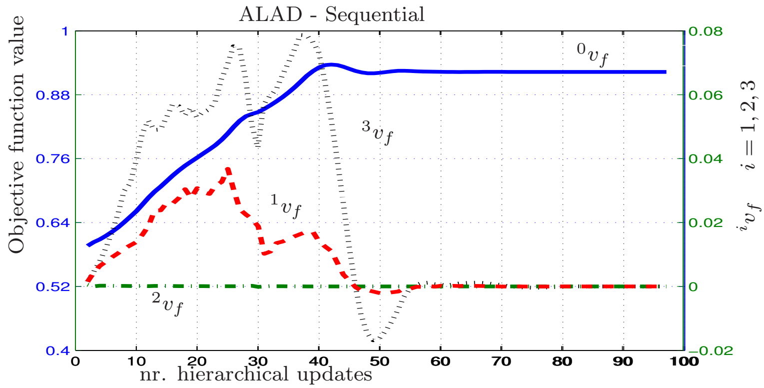


Figure 8.13: *Optimization history of the portal frame using a hierarchic top-down decomposition. Convergence settings for inconsistencies (ϵ_{v_c}) and objective function (ϵ_{v_f}) correspond to the Alternating Descent method. Furthermore, a sequential coordination was applied to the iteration process. ($^{0}v_f$) Element-0 objective function value versus the number of hierarchical updates. ($^{1}v_f$) Element-1 objective function value versus number of hierarchical updates. ($^{2}v_f$) Element-2, objective function value versus number of hierarchical updates. ($^{3}v_f$) Element-3, objective function value versus number of hierarchical updates.*

Alternating Descent requires less computational effort then the other approaches in most cases. Furthermore, it does not require more computational effort of the individual optimization and function evaluations and in some cases even reduces these costs.

In Figure 8.13 the optimization history is plotted for the portal frame multi-level optimization using a hierarchic top-down relaxation, alternating descent convergence settings and sequential coordination. A similar shift in optimal value of Element-0 is seen in the optimization history as compared to the optimization history of the two-bar truss multi-level optimization problem. The objective function value history of the Level-1 elements is less smooth than that of the Level-1 elements of the two-bar truss example. This is because more consistency constraints are present in the current example.

8.2.4 Alternative Portal Frame design problem

Multi-level design problem

A second portal frame design optimization problem is taken from Sobieszczanski-Sobieski *et al.*(1985). Sobieszczanski-Sobieski *et al.* solved the portal frame optimization problem via the Optimization by Linear Decomposition method (see Chapter 5). The decomposition and coordination of the model is identical to that of the previous portal frame problem, see Section 8.2. However, the design optimization functions of

Table 8.14: *Scaling parameters and upper and lower bounds on the design variables for the portal frame problem, where $i = 1, 2, 3$.*

variable	lowerbound	upperbound	scaling parameter
${}^{i}x_1$	1.000×10^{-1}	5.000×10^{-1}	1.000×10^{-1}
${}^{i}x_2$	1.000×10^{-2}	5.000×10^{-2}	1.000×10^{-2}
${}^{i}x_3$	1.000×10^{-2}	5.000×10^{-2}	1.000×10^{-2}
${}^{i}x_4$	1.000×10^{-2}	5.000×10^{-2}	1.000×10^{-2}
${}^{i}x_5$	1.000×10^{-1}	3.000×10^{-1}	1.000×10^{-1}
${}^{i}x_6$	1.000×10^{-1}	3.000×10^{-1}	1.000×10^{-1}
${}^{0}_{i}h_1$	2.800×10^{-3}	5.000×10^{-2}	1.000×10^{-2}
${}^{0}_{i}h_2$	4.500×10^{-6}	1.800×10^{-3}	1.000×10^{-4}
${}^{i}_{0}h_1$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{i}_{0}h_2$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_3$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_4$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_5$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}
${}^{0}_{i}h_6$	-1.000×10^{6}	1.000×10^{6}	1.000×10^{4}

the current optimization problem are slightly modified to yield:

$$
\begin{array}{lllllll}
{}^{0}v_f & = & \frac{{}^{0}r_2({}^{1}\mathbf{x},{}^{2}\mathbf{x},{}^{3}\mathbf{x})}{{}^{0}r_{V_{max}}} & ; & {}^{i}v_{g_5} & = & \frac{|{}^{i}r_3({}^{i}\mathbf{x},{}^{0}\mathbf{r})|}{{}^{i}r_{\tau_{cr}}} - 1; \\
{}^{i}v_{g_1} & = & \frac{{}^{i}r_1({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1 & ; & {}^{i}v_{g_6} & = & \frac{|{}^{i}r_4({}^{i}\mathbf{x},{}^{0}\mathbf{r})|}{{}^{i}r_{\tau_{cr}}} - 1; \\
{}^{i}v_{g_2} & = & -\frac{{}^{i}r_1({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1 & ; & {}^{i}v_{g_7} & = & \frac{\left({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3\right)}{35\,{}^{i}x_4} - 1; \\
{}^{i}v_{g_3} & = & \frac{{}^{i}r_2({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1 & ; & {}^{i}v_{g_8} & = & \frac{{}^{i}x_5}{20\,{}^{i}x_2} - 1; \\
{}^{i}v_{g_4} & = & -\frac{{}^{i}r_2({}^{i}\mathbf{x},{}^{0}\mathbf{r})}{{}^{i}r_{\sigma_{cr}}} - 1 & ; & {}^{i}v_{g_9} & = & \frac{{}^{i}x_6}{20\,{}^{i}x_3} - 1; \\
{}^{i}v_{g_{10}} & = & \multicolumn{5}{l}{1 - \frac{5\left({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3\right){}^{i}x_4}{({}^{i}x_5)({}^{i}x_2) + ({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3){}^{i}x_4 + ({}^{i}x_6)({}^{i}x_3)};} \\
{}^{i}v_{g_{11}} & = & \multicolumn{5}{l}{\left(1 - \frac{{}^{i}x_2\,{}^{i}x_5}{2\,{}^{i}x_3\,{}^{i}x_6}\right)\left(1 - \frac{{}^{i}x_2\,{}^{i}x_5}{2\,{}^{i}x_3\,{}^{i}x_6}\right).}
\end{array}
\tag{8.37}
$$

The objective is changed from minimizing the horizontal tip displacement to minimization of the volume (mass) of the total structure. Constraints on portal frame displacements have been omitted and an additional constraint is added to the individual elements (${}^{i}v_{g_{11}}$). This constraint represents a fabrication constraint. Either the top or the bottom flange of the columns must be twice the size of the opposite side. This enables one to distinguish between top and bottom of the I-columns. Finally, lower and upper bounds of the design variables are changed, see Table 8.14.

The problem is extensively studied in the work of Tosserams (2008). Tosserams shows that a number of optimal configurations exist. In the present discussion the focus is on computational costs for various decomposition, coordination and convergence settings and therefore a discussion on the range of global and local optima is omitted.

Table 8.15: *Computational cost of different convergence criteria for the portal frame optimization problem with an Augmented Lagrangian relaxation technique. The results show the effect of hierarchic and non-hierarchic decompositions. Furthermore, the results show the effect of different convergence settings and corresponding choice of β and γ. Finally, the results show the effect of sequential versus parallel coordination.*

decomp.	coord.	func.eval.	opt.iter.	hier.upd.	conv. crit.
top-down	sequential				$\epsilon_{v_c} \leq 1 \cdot 10^{-4}$
BCD: $\beta = 2.2$, $\gamma = 0.4$		2×10^4	1×10^3	104	$\epsilon_{v_f} \leq 1 \cdot 10^{-5}$
					$\epsilon_{v_c} \leq 1 \cdot 10^{-4}$
InE: $\beta = 2.0$, $\gamma = 0.5$		2×10^4	2×10^3	178	$\epsilon_{v_f} \leq 1 \cdot 10^{-4}$
					$\epsilon_{v_c} \leq 1 \cdot 10^{-4}$
AD: $\beta = 1.1$, $\gamma = 0.9$		8×10^3	7×10^2	52	$\epsilon_{v_f} \leq +inf$
top-down	parallel				$\epsilon_{v_c} \leq 1 \cdot 10^{-4}$
InE: $\beta = 1.1$, $\gamma = 0.90$, $\tau = 0.9$		1×10^5	1×10^4	216	$\epsilon_{v_f} \leq 1 \cdot 10^{-4}$
non-hier.	sequential				$\epsilon_{v_c} \leq 1 \cdot 10^{-4}$
BCD: $\beta = 2.2$, $\gamma = 0.4$		9×10^4	7×10^3	359	$\epsilon_{v_f} \leq 1 \cdot 10^{-5}$
					$\epsilon_{v_c} \leq 1 \cdot 10^{-4}$
InE: $\beta = 2.0$, $\gamma = 0.5$		2×10^5	1×10^4	705	$\epsilon_{v_f} \leq 1 \cdot 10^{-4}$
					$\epsilon_{v_c} \leq 1 \cdot 10^{-6}$
AD: $\beta = 1.1$, $\gamma = 0.9$		8×10^4	4×10^3	131	$\epsilon_{v_f} \leq +inf$
non-hier.	parallel				$\epsilon_{v_c} \leq 1 \cdot 10^{-4}$
InE: $\beta = 1.05$, $\gamma = 0.95$, $\tau = 0.9$		1×10^6	8×10^4	783	$\epsilon_{v_f} \leq 1 \cdot 10^{-4}$

Numerical results

Two decomposition methods are applied: hierarchic top-down decomposition and non-hierarchic decomposition. Furthermore, results are compared for different convergence criteria corresponding to Block Coordinate Descent (BCD), Inexact (InE) and Alternating Descent (AD). In addition, β and γ are tuned for the algorithm to remain numerically stable. Finally, for convergence settings that resulted in least computational effort, sequential coordination is compared to parallel coordination of the individual elements. The computational costs listed in Table 8.15 are typical for the algorithm settings that were used to obtain the results. The corresponding optima for which these results were obtained are listed in Table 8.16.

The optimal cross sectional areas found are listed in Table 8.16 and correspond to local optima found by Tosserams (2008). The optima listed in Table 8.16 belong to the computational costs listed in Table 8.15. Global optimal points were found for a top-down hierarchic decomposition with Block Coordinate Descent (BCD) or Inexact (InE) convergence settings and a sequential coordination process. The worst local optimal point was found for a non-hierarchic decomposition with Inexact (InE) convergence settings and a sequential coordination process.

In Table 8.17 the consistency constraints are listed for the portal frame optimization problem.

Table 8.16: *Different optima were found for the multi-level optimization problem. The optimum changes for different initial starting points and with different algorithm settings. The objective function (^0v_f) and coupling variables ($^i_0\mathbf{h}$) are listed for the various algorithm settings. The coupling variables represent the optimal areas of the three columns.*

Method	^0v_f	^1_0h	^2_0h	^3_0h
BCD	0.1688	0.5150	0.5150	0.9153
InE	0.1684	0.5150	0.5150	0.9120
InE Parallel	0.1684	0.5095	0.5086	0.8600
AD	0.1688	0.5150	0.5150	0.9153
BCD non-hier.	0.1708	0.5914	0.5153	0.8966
InE non-hier.	0.1800	0.7022	0.6777	0.7711
InE n.-h. Par	0.1690	0.5128	0.5029	0.8614
AD non-hier.	0.1763	0.6285	0.5823	0.8664

Table 8.17: *Consistency constraint values of the relaxation-based coordination methods for the portal frame multi-level optimization problem. The consistency constraints listed are the largest inconsistencies measured between elements.*

Method	^1_0c	^2_0c	^3_0c
BCD	$-2 \cdot 10^{-9}$	$-2 \cdot 10^{-9}$	$-2 \cdot 10^{-8}$
InE	$-6 \cdot 10^{-9}$	$-6 \cdot 10^{-9}$	$-8 \cdot 10^{-8}$
InE Par.	$-2 \cdot 10^{-9}$	$-4 \cdot 10^{-9}$	$2 \cdot 10^{-8}$
AD	$-1 \cdot 10^{-5}$	$2 \cdot 10^{-5}$	$-4 \cdot 10^{-6}$
BCD n.-h.	$-2 \cdot 10^{-5}$	$-2 \cdot 10^{-6}$	$-2 \cdot 10^{-6}$
InE n.-h.	$-1 \cdot 10^{-5}$	$-1 \cdot 10^{-5}$	$1 \cdot 10^{-5}$
InE n.-h. Par	$-1 \cdot 10^{-5}$	$-2 \cdot 10^{-6}$	$-3 \cdot 10^{-5}$
AD n.-h.	$4 \cdot 10^{-7}$	$-1 \cdot 10^{-6}$	$2 \cdot 10^{-7}$

Discussion

A comparison between the results of Table 8.11 and Table 8.15 shows the effect of removing Level-0 design constraints. The computational costs have decreased for the top-down decomposed optimization problems and increased for the non-hierarchic decomposed optimization problems.

Comparing the hierarchic top-down decomposition with respect to a non-hierarchic decomposition the results show that a hierarchic top-down decomposition requires less computational effort for the current multi-level optimization problem. The amount of hierarchical updates is approximately three times higher and the number of optimization iterations and function evaluations are an order of magnitude higher for the non-hierarchic decomposition.

Comparing sequential versus parallel coordination shows that given the inexact (InE) convergence parameter settings the computational effort for parallel coordination is in terms of function evaluations and optimization iterations much higher then for sequential coordination. This is due to attempts of optimizing individual elements that did not converge and were restarted from slightly different initial design points to try to converge to feasible optimal points.

Various results were obtained for the three methods that correspond to different convergence criteria and relaxation parameter update settings. In general, Alternating Descent (AD) performed best in terms of computational effort. Block Coordinate Descent (BCD) performed better then the Inexact (InE) approach. The inexact convergence tolerance on the objective function of the individual elements did not decrease the computational effort.

In Figure 8.14 a plot of the optimization history of the objective function values of the four elements is presented. A sequential solution process is used and convergence settings correspond to Alternating Descent(AD). A similar trend as was observed in the plots of the two-bar truss multi-level optimization problem is seen. The optimal value of Element-0 shifts during the multi-level optimization process to a higher value that is optimal for the entire hierarchy. The objective function values of the Level-1 elements show a smooth increasing Augmented Lagrangian function of the inconsistencies between Level-0 and the individual Level-1 elements. After a certain value of the relaxation parameters is reached the optimization history shows a large shift in objective function values. At this point the individual optimization problems are pushed towards a different configuration and the multi-level optimization converges towards the optimum value.

8.3 A supersonic business jet

The supersonic business jet shown in Figure 8.15 was introduced by Agte *et al.* (1999) as a benchmark problem for multi-disciplinary design optimization approaches. In the present chapter a brief overview of the multi-disciplinary design optimization problem is presented. The interested reader is referred to Agte *et al.* (1999) for a detailed derivation of the expressions used to model the supersonic business jet.

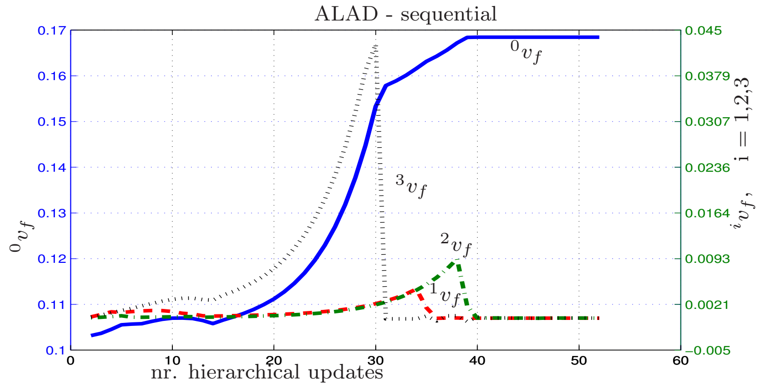


Figure 8.14: *Optimization history of the portal frame using a hierarchic top-down decomposition. Convergence settings for inconsistencies (ϵ_{v_c}) and objective function (ϵ_{v_f}) correspond to the Alternating Descent method. Furthermore, a sequential coordination was applied to the iteration process. (^0v_f) Element-0 objective function value versus the number of hierarchical updates. (^1v_f) Element-1 objective function value versus number of hierarchical updates. (^2v_f) Element-2, objective function value versus number of hierarchical updates. (^3v_f) Element-3, objective function value versus number of hierarchical updates.*

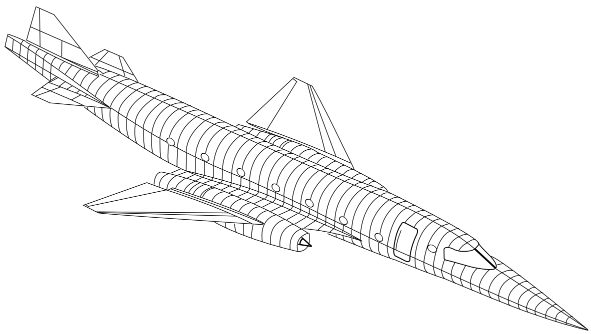


Figure 8.15: *Impression of a supersonic business jet, Sobieszczanski-Sobieski* et al. *(2003).*

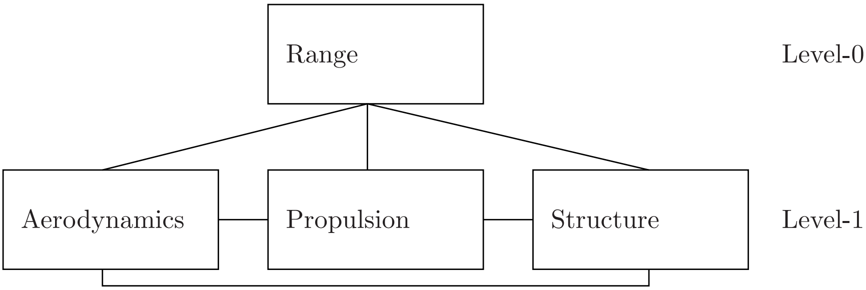


Figure 8.16: *Two levels are distinguished in the supersonic business jet multi-level optimization problem. Level-0 considering the range of the plane and Level-1 where 3 individual hierarchical elements are identified: Aerodynamics; Propulsion; and Structure. Each hierarchical element involves a discipline that computes physical characteristics required by neighboring elements to compute the physical responses.*

8.3.1 Multi-level optimization problem

The supersonic business jet design optimization hierarchy is illustrated in Figure 8.16 and the parameters determining the lay-out and performance of the supersonic business jet are listed in Table 8.18. There are 41 design variables distributed over four hierarchical elements. Eight of these design variables are present in more then one hierarchical element. The total structural weight is also listed twice, because this value is send to two different hierarchical elements.

For this problem, the objective (${}^0v_f$) is to maximize the range (${}^0r$) of the aircraft. The supersonic business jet design problem is subjected to constraints on adverse pressure gradient (${}^1v_{g_1}$) and constraints on the ratio between lift of the wing and lift of the horizontal tail wing (respectively ${}^1v_{g_2}$ and ${}^1v_{g_3}$). Furthermore, constraints on the maximum engine temperature (${}^2v_{g_1}$) and on the throttle setting (${}^2v_{g_2}$) are present. Finally, constraints on allowable stresses (${}^3v_{g_{1\ldots60}}$) and geometrical constraints (${}^3v_{g_{61\ldots72}}$) are considered in the design problem. Lower bounds $\ddot{\underline{\mathbf{x}}}$, $\dddot{\underline{\mathbf{z}}}$ and upper bounds $\ddot{\overline{\mathbf{x}}}$, $\dddot{\overline{\mathbf{z}}}$ are present on all the design variables, see Table 8.19.

Two levels are distinguished in the design problem, see Figure 8.16. The top level (Level-0) consisting of a single hierarchical element involved with maximizing the range (${}^0r_1$) of the business jet. No constraint functions are present in the Level-0 hierarchical element. The bottom level (Level-1) consisting of three individual elements involving constraints on aerodynamics (${}^1v_{g_1}$, ${}^1v_{g_2}$ and ${}^1v_{g_3}$), on propulsion (${}^2v_{g_1}$ and ${}^2v_{g_2}$) and constraints on structural characteristics (${}^3v_{g_{1\ldots60}}$ and ${}^3v_{g_{61\ldots72}}$).

The problem matrix showing the dependencies of the optimization functions on the design variables and physical responses is shown in Figure 8.17. The objective function depends on mapped responses from the Level-1 elements. Likewise, the constraint equations of the Level-1 hierarchical elements depend on physical responses that are mapped between the levels and between the individual hierarchical elements. The coupling between hierarchical elements is shown via coupling variables in the problem matrix, see Figure 8.18.

Table 8.18: *Definition of design variables, physical responses, physical interactions between subsystems and optimization parameters for the supersonic business jet.*

design	description	physical response	description	interaction	description
${}^{0}_{1..2}z_1$	Mach number	${}^{0}r_1$	range		
${}^{0}_{1..2}z_2$	altitude				
${}^{1}_{0}z_1$	Mach number	${}^{1}r_1$	adverse pressure gradient	${}^{1}_{0}h$	lift to drag ratio
${}^{1}_{0}z_2$	altitude	${}^{1}r_2$	Lift coefficient	${}^{1}_{2}h$	drag
${}^{1}_{3}z_1$	wing aspect ratio	${}^{1}r_3$	Drag coefficient	${}^{1}_{3}h$	lift
${}^{1}_{3}z_2$	wing surface area				
${}^{1}_{3}z_3$	tail aspect ratio				
${}^{1}_{3}z_4$	tail surface area				
${}^{1}_{3}z_5$	wing sweep				
${}^{1}_{3}z_6$	thickness/chord				
${}^{1}x_1$	tail sweep				
${}^{1}x_2$	wing lift				
${}^{1}x_3$	tail wing lift				
${}^{2}_{0}z_1$	Mach number	${}^{2}r_1$	engine temperature	${}^{2}_{0}h$	spec. fuel cons.
${}^{2}_{0}z_2$	altitude	${}^{2}r_2$	throttle setting	${}^{2}_{1}h$	engine scale fact.
${}^{2}x_1$	throttle			${}^{2}_{3}h$	engine weight
${}^{3}\mathbf{x}_{1...9}$	skin thickness	${}^{3}\mathbf{r}_{1...6}$	σ at $1 \ldots 6$	${}^{3}_{0}h_1$	fuel weight
${}^{3}\mathbf{x}_{10...18}$	sandwich thickness	${}^{3}\mathbf{r}_{7...12}$	τ at $1 \ldots 7$	${}^{3}_{0}h_2$	total weight
${}^{3}x_{19}$	taper ratio	${}^{3}r_{13}$	mass wing	${}^{3}_{1}h_1$	total weight
${}^{3}_{1}z_1$	wing aspect ratio	${}^{3}r_{14}$	mass tail wing	${}^{3}_{1}h_2$	wing twist
${}^{3}_{1}z_2$	wing surface area	${}^{3}r_{15}$	wing deformation		
${}^{3}_{1}z_3$	tail aspect ratio				
${}^{3}_{1}z_4$	tail surface area				
${}^{3}_{1}z_5$	wing sweep				
${}^{3}_{1}z_6$	thickness/chord				

Table 8.19: *Scaling parameters and lower and upper bounds for the supersonic business jet design variables and coupling variables.*

variable	lower bound	upper bound	scaling variable	variable	lower bound	upper bound	scaling variable
${}^{0}_{1\,.\,2}z_1$	1.4	1.8	1.6	${}^{2}_{0}z_1$	1.4	1.8	1.6
${}^{0}_{1\,.\,2}z_2$	30000.0	60000.0	45000.0	${}^{2}_{0}z_2$	30000.0	60000.0	45000.0
${}^{1}_{0}h$	0.1	10.0	5.0	${}^{2}x_1$	0.1	1.0	0.5
${}^{2}_{0}h$	1.0	4.0	2.5	${}^{1}_{2}h$	100.0	70000.0	35000.0
${}^{3}_{0}h_1$	0.1	0.9	0.5				
${}^{3}_{0}h_2$	5000.0	100000.0	45000.0				
${}^{1}_{0}z_1$	1.4	1.8	1.6	${}^{3}\mathbf{x}_{1\ldots9}$	0.1	4.0	2.0
${}^{1}_{0}z_2$	30000.0	60000.0	45000.0	${}^{3}\mathbf{x}_{10\ldots18}$	0.1	9.0	4.5
${}^{1}_{3}z_1$	2.5	8.5	5.5	${}^{3}x_{19}$	0.1	0.4	0.25
${}^{1}_{3}z_2$	200.0	800.0	500.0	${}^{3}_{1}z_1$	2.5	8.5	5.5
${}^{1}_{3}z_3$	2.5	8.5	5.5	${}^{3}_{1}z_2$	200.0	800.0	500.0
${}^{1}_{3}z_4$	50.0	148.9	100.0	${}^{3}_{1}z_3$	2.5	8.5	5.5
${}^{1}_{3}z_5$	40.0	70.0	55.0	${}^{3}_{1}z_4$	50.0	148.9	100.0
${}^{1}_{3}z_6$	0.01	0.09	0.05	${}^{3}_{1}z_5$	40.0	70.0	55.0
${}^{1}x_1$	40.0	70.0	55.0	${}^{3}_{1}z_6$	0.01	0.09	0.05
${}^{1}x_2$	0.01	0.2	0.1	${}^{1}_{3}h$	5000.0	100000.0	45000.0
${}^{1}x_3$	1.0	3.5	2.5	${}^{2}_{3}h$	100.0	30000.0	15000.0
${}^{2}_{1}h$	1.0	4.0	2.5				
${}^{3}_{1}h_1$	5000.0	100000.0	45000.0				
${}^{3}_{1}h_2$	0.2	50.0	25.0				

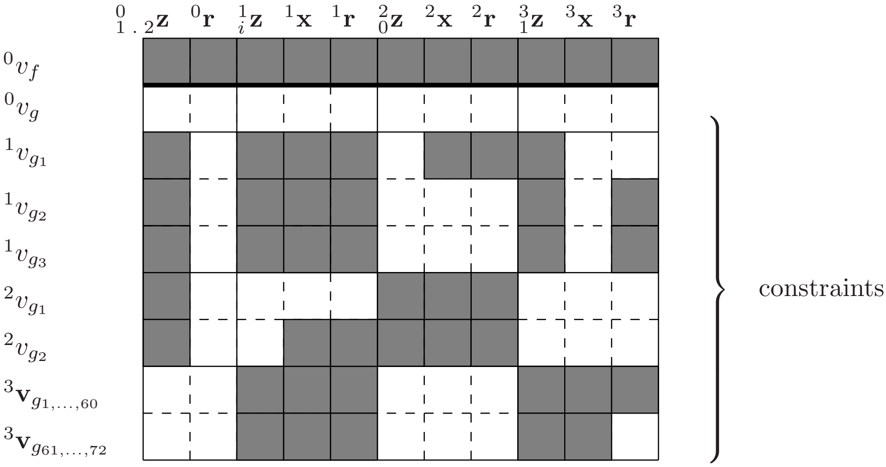


Figure 8.17: *Supersonic Business Jet problem matrix before decomposition. On the left the optimization problem functions are listed and on the top the design variables and physical responses. A function that depends on a specific design variable or physical response is represented via a shaded block. No constraints are present in the Level-0 element, hence ${}^{0}v_g$ is left blank. Furthermore, design variables are present in multiple hierarchical elements that are shared. For brevity of the problem matrix, the design variables, responses and design optimization functions are combined into vectors.*

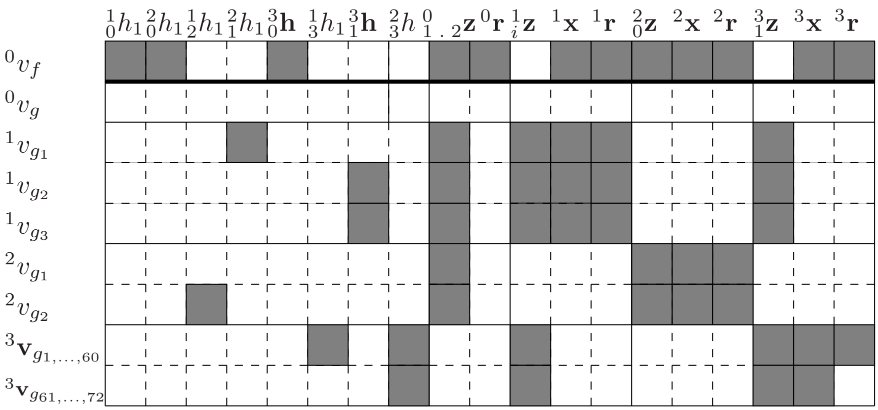


Figure 8.18: *Coupling parameters are introduced into the problem matrix, showing the data that is shared between the Level-0 element and the three Level-1 elements.*

Hierarchic decomposition

The optimization problem is decomposed via a top-down hierarchic decomposition. The consistency constraints between the individual elements are relaxed via Augmented Lagrangian relaxation. Therefore, the individual optimization problems become:

Element-0, Range:

$$
\begin{array}{lll}
\min\limits_{\substack{{}^{0}_{1.2}z_1,\,{}^{0}_{1.2}z_2,\\ {}^{1}_{0}h_1,\,{}^{2}_{0}h_1,\,{}^{3}_{0}h_1,\,{}^{3}_{0}h_2}} & {}^0v_f = & \frac{{}^0r_1}{{}^0r_{reference}} + {}^0v_{{}^1_0c} + {}^0v_{{}^2_0c} + {}^0v_{{}^3_0c} \\
\text{s.t.} & & \\
\multicolumn{3}{l}{{}^{0}_{1.2}\underline{\mathbf{z}} \leq {}^{0}_{1.2}\mathbf{z} \leq {}^{0}_{1.2}\overline{\mathbf{z}}\ ;\ {}^{1}_{0}\underline{h}_1 \leq {}^{1}_{0}h_1 \leq {}^{1}_{0}\overline{h}_1\ ;\ {}^{2}_{0}\underline{h}_1 \leq {}^{2}_{0}h_1 \leq {}^{2}_{0}\overline{h}_1\ ;\ {}^{3}_{0}\underline{\mathbf{h}} \leq {}^{3}_{0}\mathbf{h} \leq {}^{3}_{0}\overline{\mathbf{h}}} \\
\text{where} & {}^0v_{{}^1_0c} = & {}^1_0\lambda_1({}^1_0c) + ||{}^1_0s_1 \circ {}^1_0c||_2^2 \\
& {}^0v_{{}^2_0c} = & {}^2_0\lambda_1({}^2_0c) + ||{}^2_0s_1 \circ {}^2_0c||_2^2 \\
& {}^0v_{{}^3_0c} = & {}^3_0\boldsymbol{\lambda}_1^T({}^3_0\mathbf{c}) + ||{}^3_0\mathbf{s}_1 \circ {}^3_0\mathbf{c}||_2^2
\end{array}
\tag{8.38}
$$

Element-1, Aerodynamics:

$$
\begin{array}{llll}
\min\limits_{{}^2_1h,\,{}^3_1\mathbf{h},\,{}^1\mathbf{x},\,{}^1_3\mathbf{z}} & {}^1v_f = & {}^1v_{{}^1_0c} + {}^1v_{{}^2_1c} + {}^1v_{{}^1_2c} + {}^1v_{{}^3_1c} + {}^1v_{{}^1_3c} + {}^1v_{{}^1_3c_z} & \\
\text{s.t.} & {}^1v_{g_1} = & {}^1v_1\left({}^1r_1({}^1\mathbf{x}, {}^1_3\mathbf{z}, {}^2_1h)\right) & \leq 0 \\
& {}^1v_{g_2} = & {}^1v_2\left({}^1r_2({}^1\mathbf{x}, {}^1_3\mathbf{z}, {}^3_1\mathbf{h})\right) & \leq 0 \\
& {}^1v_{g_3} = & {}^1v_2\left({}^1r_2({}^1\mathbf{x}, {}^1_3\mathbf{z}, {}^3_1\mathbf{h})\right) & \leq 0 \\
\multicolumn{4}{l}{{}^1\underline{\mathbf{x}} \leq {}^1\mathbf{x} \leq {}^1\overline{\mathbf{x}}\ ;\ {}^1_3\underline{\mathbf{z}} \leq {}^1_3\mathbf{z} \leq {}^1_3\overline{\mathbf{z}}\ ;\ {}^2_1\underline{h} \leq {}^2_1h \leq {}^2_1\overline{h}\ ;\ {}^3_1\underline{\mathbf{h}} \leq {}^3_1\mathbf{h} \leq {}^3_1\overline{\mathbf{h}}} \\
\text{where} & {}^1v_{{}^1_0c} = & {}^1_0\lambda_2({}^1_0c) + ||{}^1_0s_2 \circ {}^1_0c||_2^2 & \\
& {}^1v_{{}^2_1c} = & {}^2_1\lambda_1({}^2_1c) + ||{}^2_1s_1 \circ {}^2_1c||_2^2 & \\
& {}^1v_{{}^1_2c} = & {}^1_2\lambda_2({}^1_2c) + ||{}^1_2s_2 \circ {}^1_2c||_2^2 & \\
& {}^1v_{{}^3_1c} = & {}^3_1\boldsymbol{\lambda}_1^T({}^3_1\mathbf{c}) + ||{}^3_1\mathbf{s}_1 \circ {}^3_1\mathbf{c}||_2^2 & \\
& {}^1v_{{}^1_3c} = & {}^1_3\lambda_2({}^1_3c) + ||{}^1_3s_2 \circ {}^1_3c||_2^2 & \\
& {}^1v_{{}^1_3c_z} = & {}^1_3\boldsymbol{\lambda}^T({}^1_3\mathbf{c}_z) + ||{}^1_3\mathbf{s}_z \circ {}^1_3\mathbf{c}_z||_2^2 &
\end{array}
\tag{8.39}
$$

Element-2, Propulsion:

$$
\begin{array}{llll}
\min\limits_{{}^1_2h,\,{}^2\mathbf{x}} & {}^2v_f = & {}^2v_{{}^2_0c} + {}^2v_{{}^1_2c} + {}^2v_{{}^2_1c} + {}^2v_{{}^2_3c} & \\
\text{s.t.} & {}^2v_{g_1} = & {}^2v_1\left({}^2r_1({}^2x, {}^2_0\mathbf{z})\right) & \leq 0 \\
& {}^2v_{g_2} = & {}^2v_2\left({}^2r_2({}^2x, {}^2_0\mathbf{z}, {}^1_2h)\right) & \leq 0 \\
& \multicolumn{3}{l}{{}^2\underline{x} \leq {}^2x \leq {}^2\overline{x}\ ;\ {}^1_2\underline{h} \leq {}^1_2h \leq {}^1_2\overline{h}} \\
\text{where} & {}^2v_{{}^2_0c} = & {}^2_0\lambda_2({}^2_0c) + ||{}^2_0s_2 \circ {}^2_0c||_2^2 & \\
& {}^2v_{{}^1_2c} = & {}^1_2\lambda_1({}^1_2c) + ||{}^1_2s_1 \circ {}^1_2c||_2^2 & \\
& {}^2v_{{}^2_1c} = & {}^2_1\lambda_2({}^2_1c) + ||{}^2_1s_2 \circ {}^2_1c||_2^2 & \\
& {}^2v_{{}^2_3c} = & {}^2_3\lambda_2({}^2_3c) + ||{}^2_3s_2 \circ {}^2_3c||_2^2 &
\end{array}
\tag{8.40}
$$

Element-3, Structures:

$$
\begin{array}{llll}
\min\limits_{{}^2_3h,{}^1_3h,{}^3\mathbf{x},{}^3_1\mathbf{z}} & {}^3v_f & = & {}^3v_{{}^3_0c} + {}^3v_{{}^3_1c_z} + {}^3v_{{}^2_3c} + {}^3v_{{}^1_3c} + {}^3v_{{}^3_1c} \\
\text{s.t.} & {}^3v_{g_1} & = & {}^3v_1\left({}^3r_1({}^3\mathbf{x},{}^3_1\mathbf{z},{}^i_3h)\right) \qquad \leq 0 \\
& & \vdots & \\
& {}^3v_{g_{60}} & = & {}^3v_{60}\left({}^3r_6({}^3\mathbf{x},{}^3_1\mathbf{z},{}^i_3h)\right) \qquad \leq 0 \\
& {}^3v_{g_{61}} & = & {}^3v_{61}\left({}^3\mathbf{x},{}^3_1\mathbf{z},{}^2_3h\right) \qquad \leq 0 \\
& & \vdots & \\
& {}^3v_{g_{70}} & = & {}^3v_{70}\left({}^3\mathbf{x},{}^3_1\mathbf{z},{}^2_3h\right) \qquad \leq 0 \\
\multicolumn{4}{l}{{}^3\underline{\mathbf{x}} \leq {}^3\mathbf{x} \leq {}^3\overline{\mathbf{x}}\ ;\ {}^3_1\underline{\mathbf{z}} \leq {}^3_1\mathbf{z} \leq {}^3_1\overline{\mathbf{z}}\ ;\ {}^1_3\underline{h} \leq {}^1_3h \leq {}^1_3\overline{h}\ ;\ {}^2_3\underline{h} \leq {}^2_3h \leq {}^2_3\overline{h}} \\
\text{where} & {}^3v_{{}^3_0c} & = & {}^3_0\boldsymbol{\lambda}_2^T({}^3_0\mathbf{c}) + ||{}^3_0\mathbf{s}_2 \circ {}^3_0\mathbf{c}||_2^2 \\
& {}^3v_{{}^2_3c} & = & {}^2_3\lambda_1({}^2_3c) + ||{}^2_3s_1 \circ {}^2_3c||_2^2 \\
& {}^3v_{{}^1_3c} & = & {}^1_3\lambda_1({}^1_3c) + ||{}^1_3s_1 \circ {}^1_3c||_2^2 \\
& {}^3v_{{}^3_1c} & = & {}^3_1\boldsymbol{\lambda}_2^T({}^3_1\mathbf{c}) + ||{}^3_1\mathbf{s}_2 \circ {}^3_1\mathbf{c}||_2^2 \\
& {}^3v_{{}^3_1c_z} & = & {}^3_1\boldsymbol{\lambda}^T({}^3_1\mathbf{c}_z) + ||{}^3_1\mathbf{s}_z \circ {}^3_1\mathbf{c}_z||_2^2
\end{array}
\tag{8.41}
$$

Coupling variables (${}^{\cdot\cdot}_{\cdot\cdot}h$) are added to the Level-0 optimization problem as design variables. These variables must match the mapped physical responses (${}^{\cdot\cdot}_{\cdot\cdot}\mathcal{H}({}^{\cdot\cdot}r)$). The latter are evaluated during the Level-1 optimizations.

The Aerodynamics, Propulsion and Structures hierarchical element are located on Level-1. Since these hierarchical elements are coupled on the same level the consistency constraints are relaxed in both directions. Therefore, coupling variables (${}^{\cdot\cdot}_{\cdot\cdot}h$) are added to the design variables of these hierarchical elements. These variables must match the mapped physical responses (${}^{\cdot\cdot}_{\cdot\cdot}\mathcal{H}(r)$) that are computed in the neighboring element.

In the present thesis a top-down hierarchic decomposition is considered with relaxed consistency constraints. This decomposition is different from that of Tosserams *et al.* (2008b). In their approach all the shared design variables are send to the Level-0 element. However, such an approach does not take advantage of the purpose of decomposition, which is to hide information from elements that do not depend on this information. Furthermore, additional data flow is generated since the data is transported along elements that do not depend on it. Finally, additional complexity is introduced to the individual optimization problems, because more design variables are present and additional terms are added to the objective functions.

Non-hierarchic decomposition

Agte *et al.* (1999) use a non-hierarchic decomposition of the supersonic business jet model. Hence, the consistency constraints between Level-0 and Level-1 are relaxed in both directions (from Level-0 onto Level-1 and *vice versa*). Therefore, in addition to the consistency constraints already present in the top-down hierarchic decomposition, the mapping of shared design variables from Level-0 onto Level-1 is expressed as:

Consistency of design variables between Range and Aerodynamics element:

$$\begin{array}{rcl} {}^{1}_{0}c_{z_1} & = & {}^{0}_{1\,.\,2}z_1 - {}^{1}_{0}z_1 \\ {}^{1}_{0}c_{z_2} & = & {}^{0}_{1\,.\,2}z_2 - {}^{1}_{0}z_2 \end{array} . \tag{8.42}$$

Consistency of design variables between Range and Propulsion element:

$$\begin{array}{rcl} {}^{2}_{0}c_{z_1} & = & {}^{0}_{1\,.\,2}z_1 - {}^{2}_{0}z_1 \\ {}^{2}_{0}c_{z_2} & = & {}^{0}_{1\,.\,2}z_2 - {}^{2}_{0}z_2 \end{array} . \tag{8.43}$$

8.3.2 Numerical results

Two different coordination methods are applied to the supersonic business jet multi-level optimization problem: sequential and parallel coordination. The relaxation parameters are updated via the method of multipliers (Bertsekas and Tsitsiklis, 1989).

Results

The optimal design variable values and coupling variable values for the best solution found via the multi-level optimization framework for the supersonic business jet multi-level optimization problem are listed in Table 8.20.

Results for different decomposition and coordination approaches applied to the supersonic business jet multi-level optimization are listed in Table 8.21. Two decomposition methods are applied: hierarchic top-down and non-hierarchic decomposition. Furthermore, results are compared for three different combinations of convergence criteria ϵ_{v_c} and ϵ_{v_f} that correspond to Block Coordinate Descent (BCD), Inexact (InE) and Alternating Descent (AD) methods. In addition, settings for β and γ are adjusted so that the algorithms converge. Finally, for settings that exhibited least computational cost via sequential coordination, the sequential coordination is compared with parallel coordination of the individual hierarchical elements.

The computational costs listed in Table 8.21 do not represent multi-level optimization runs that converged to the same optimum. The best optimum was obtained via non-hierarchic decomposition with convergence settings that correspond to Block Coordinate Descent (BCD) and a sequential solution process. The same optimal range was also found via parallel coordination with convergence settings that correspond to Inexact (InE). However, the sequential and parallel approach that converged to the best optimum were also the most expensive methods considering computational effort. An optimum value found close to the optimum value of 2626 was found via sequential coordination and convergence settings that correspond to Alternating Descent (AD) (2489) at significantly lower computational effort. The worst optimum (628) was found via non-hierarchic decomposition with convergence settings that corresponded to the Inexact method.

In general, settings for γ and β are chosen such that the relaxation parameters are updated frequently and with small increments. If for the current example, combinations of γ and β are chosen such that large increments in the relaxation parameters

Table 8.20: *Optimal solution of the supersonic business jet multi-level optimization problem for an optimal range:* ${}^{0}r_1 = 2626$. *This optimum was found for a non-hierarchic decomposition, sequential coordination process and convergence settings and choice of* β *and* γ *that correspond to Alternating Directions (AD) method.*

design	value	interaction	value
${}^{0}_{1\,.\,2}z_1 = 1.5013$	Mach number		
${}^{0}_{1\,.\,2}z_2 = 5.9996 \cdot 10^4$	altitude		
${}^{1}_{0}z_1 = 1.5013$	Mach number	${}^{1}_{0}h = 6.236$	lift to drag ratio
${}^{1}_{0}z_2 = 5.9996 \cdot 10^4$	altitude	${}^{1}_{2}h = 7.974 \cdot 10^3$	drag
${}^{1}_{3}z_1 = 2.500$	wing aspect ratio	${}^{1}_{3}h = 5.0368 \cdot 10^4$	lift
${}^{1}_{3}z_2 = 8.000 \cdot 10^2$	wing surface area		
${}^{1}_{3}z_3 = 2.500$	tail aspect ratio		
${}^{1}_{3}z_4 = 1.489 \cdot 10^2$	tail surface area		
${}^{1}_{3}z_5 = 6.995 \cdot 10^1$	wing sweep		
${}^{1}_{3}z_6 = 9.000 \cdot 10^{-2}$	thickness/chord		
${}^{1}x_1 = 7.000 \cdot 10^1$	tail sweep		
${}^{1}x_2 = 1.000 \cdot 10^{-2}$	wing lift		
${}^{1}x_3 = 3.500$	tail wing lift		
${}^{2}_{0}z_1 = 1.5013$	Mach number	${}^{2}_{0}h = 1.000$	spec. fuel cons.
${}^{2}_{0}z_2 = 5.9996 \cdot 10^4$	altitude	${}^{2}_{1}h = 1.4320$	engine scale fact.
${}^{2}x_1 = 1.7220 \cdot 10^{-1}$	throttle	${}^{2}_{3}h = 1.2713 \cdot 10^4$	engine weight
${}^{3}\mathbf{x}_{1\ldots9} = [1.1799, 0.59396, 0.23809, 4.0000, 4.0000, 1.1091, 1.1799, 0.59396, 0.23809]$ skin thickness			
${}^{3}\mathbf{x}_{10\ldots18} = [1.9348, 1.2952, 0.66949, 4.4000, 4.4000, 1.2200, 1.9348, 1.2952, 0.66949]$ sandwich thickness			
${}^{3}x_{19} = 1.000 \cdot 10^{-1}$	taper ratio	${}^{3}_{0}h_1 = 3.870 \cdot 10^{-1}$	fuel weight
${}^{3}_{1}z_1 = 2.500$	wing aspect ratio	${}^{3}_{0}h_2 = 5.037 \cdot 10^4$	total weight
${}^{3}_{1}z_2 = 8.000 \cdot 10^2$	wing surface area	${}^{3}_{1}h_1 = 5.037 \cdot 10^4$	total weight
${}^{3}_{1}z_3 = 2.500$	tail aspect ratio	${}^{3}_{1}h_2 = 1.115 \cdot 10^1$	wing twist
${}^{3}_{1}z_4 = 1.4890 \cdot 10^2$	tail surface area		
${}^{3}_{1}z_5 = 6.9953 \cdot 10^1$	wing sweep		
${}^{3}_{1}z_6 = 9.0000 \cdot 10^{-2}$	thickness/chord		

Table 8.21: *Computational cost of each solution strategy for solving the supersonic business jet. Settings that did not converge are indicated by nc.*

decomp. coord.	func.eval.	opt.iter.	hier.upd.	conv. crit.
top-down sequential				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
BCD: $\beta = 2.2$, $\gamma = 0.4$, $\tau = 0.9$	9×10^5	4×10^4	480	$\epsilon_{v_f} \leq 1 \cdot 10^{-4}$
				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
InE: $\beta = 1.1$, $\gamma = 0.9$, $\tau = 0.9$	6×10^5	3×10^4	1101	$\epsilon_{v_f} \leq 1 \cdot 10^{-3}$
				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
AD: $\beta = 1.1$, $\gamma = 0.90$, $\tau = 0.85$	nc	nc	nc	$\epsilon_{v_f} \leq +inf$
top-down parallel				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
InE: $\beta = 1.05$, $\gamma = 0.95$, $\tau = 0.9$	nc	nc	nc	$\epsilon_{v_f} \leq 1 \cdot 10^{-3}$
non-hier. sequential				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
BCD: $\beta = 1.1$, $\gamma = 0.9$	3×10^6	3×10^5	4790	$\epsilon_{v_f} \leq 1 \cdot 10^{-4}$
				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
InE: $\beta = 2.0$, $\gamma = 0.5$	5×10^5	3×10^4	560	$\epsilon_{v_f} \leq 1 \cdot 10^{-3}$
				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
AD: $\beta = 1.05$, $\gamma = 0.95$	1×10^5	7×10^3	174	$\epsilon_{v_f} \leq +inf$
non-hier. parallel				$\epsilon_{v_c} \leq 1 \cdot 10^{-3}$
InE: $\beta = 1.05$, $\gamma = 0.95$, $\tau = 0.9$	9×10^6	4×10^5	4450	$\epsilon_{v_f} \leq 1 \cdot 10^{-3}$

occur, the multi-level optimization process diverges and no solution is found. Furthermore, as a consequence of large increments in the relaxation parameters individual optimization problems have difficulties to converge to an individually feasible solution.

Parallel coordination requires small increments in both relaxation parameters and coupling variable updates. No convergence was observed for Alternating Descent (AD) convergence settings together with parallel coordination. However, inexact (InE) convergence settings and tuning of β and γ was sufficient to update coupling variables such that the coordination process was numerically stable.

In Figure 8.19 the history of the optimization process is plotted using different convergence settings for non-hierarchic decomposed and sequentially coordinated (Figures 8.19(a), 8.19(b) and 8.19(c)) or parallel coordinated (Figure 8.19(d)) multi-level optimization. During the shift from individual optimal to overall optimal design large jumps in the objective function values are observed. The computations that were conducted during this jump showed numerical difficulties with respect to convergence of the individual optimization problems and difficulties with finding feasible solutions.

The results for the supersonic business jet show that the Java framework is able to perform a multi-level optimization of a large design problem involving many design constraints and a significant amount of coupling. However, the computational effort required to converge to an optimum is considerable and the optima found show a large difference between best optimum and worst optimum found. Because the computational effort is considerable it might not be feasible in a commercial environment to rerun these optimizations frequently enough to assure that one finds a good optimal point. The convergence rate still poses a challenge before multi-level optimization becomes an alternative in a commercial environment.

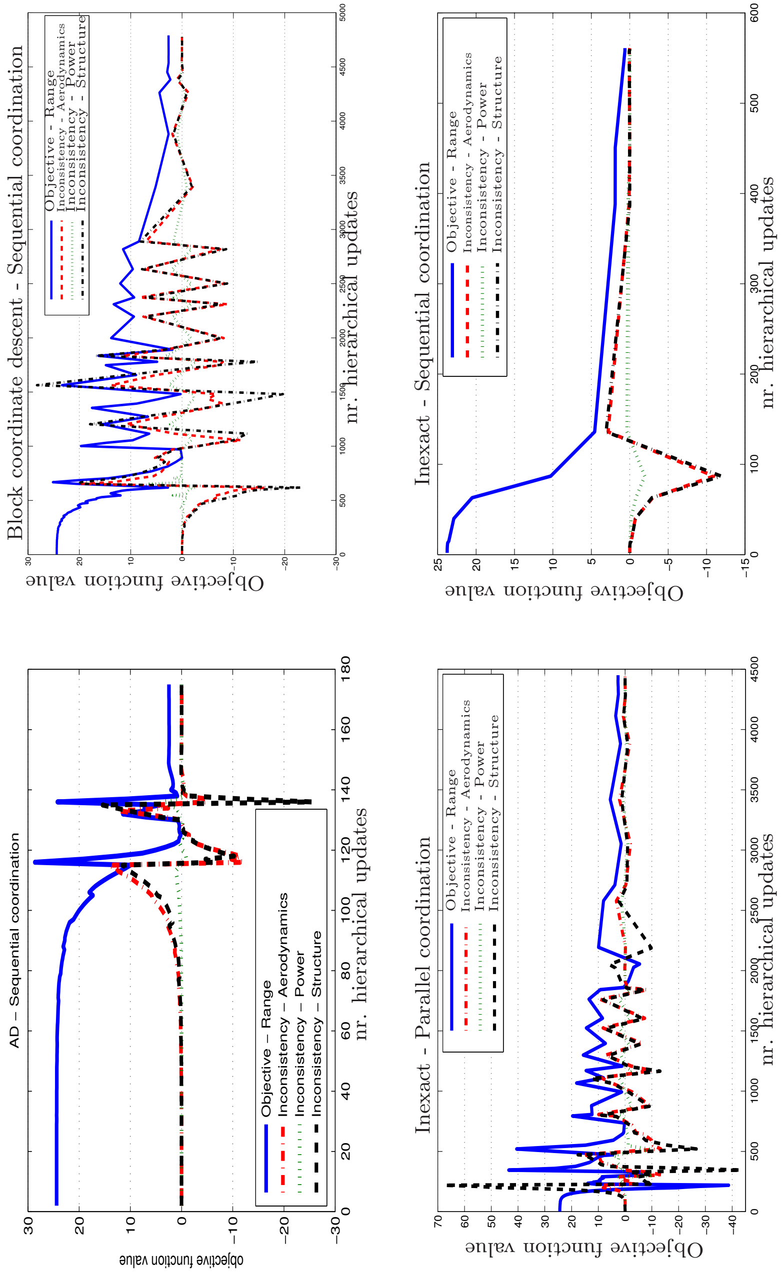


Figure 8.19: *Optimization history of the Supersonic business jet using a non-hierarchic decomposition with a sequential coordination process and (a) exact (BCD) convergence criteria. (b) inexact convergence criteria. (c) Alternating Descent (AD) convergence criteria. In (d) a non-hierarchic decomposition with a parallel coordination process is shown and inexact convergence criteria.*

Chapter 9

Conclusions and Recommendations

9.1 Conclusions

In this thesis a unified approach towards decomposition and coordination for multi-level optimization problems is presented. A multi-level notation has been introduced in order to compare different main-stream formulations in multi-level optimization. The notation clearly distinguishes between the aspects of optimization and analysis within the multi-level design. In addition, it emphasizes the handling of inconsistency between subsystems and clarifies where in the multi-level formulation these are solved. To study the implementation efforts and numerical costs of different main-stream formulations a numerical comparison of multi-level optimization and multi-disciplinary design optimization approaches was performed, motivated by a specific interest for structural multi-scale mechanics.

Each multi-level method was tested on the basis of a two bar truss example. Most of the multi-level methods approached the optimum that was found with an All-in-One method (AiO), which was used as a standard to compare the methods. The following observations were made:

- [Section 5.1.4] Constructing a response surface of individual elements optima according to Quasi-separable Subsystem Decomposition was more challenging then constructing a response surface from the mapped physical response data dictated via Bi-Level Integrated System Synthesis. Using optimal objective or constraint function data of neighboring elements can be challenging because this data cannot be fitted via a smooth function. As observed by other researchers this optimal data is often discontinuous. However, introducing, e.g., barrier functions into the objective function overcomes the problem of discontinuity of the optimal functions.

- [Section 5.1.4] Linearizing objective and/or constraints using sensitivity analysis requires tight move limits to maintain accurate sensitivity data. Furthermore, introducing trade-off and responsibility factors to switch on or switch off constraints that are shared introduces numerical difficulties.

- [Section 5.2.4] Coordination of parameters related to a quadratic of the l^2-norm (Analytical Target Cascading) of the inconsistencies requires the least computational effort. For Augmented Lagrangian relaxation and Inexact Penalty Decomposition the computational costs were of the same order of magnitude. Approximating the relaxation parameters via a Weight Update Method was more expensive in terms of computational effort then approaches that computed the relaxation parameters. Finally, relaxation via a quadratic penalty function and fixed weights did not converge to a known solution and therefore the computational costs are not representative.

- [Sections 5.1.4 and 5.2.4] Equality-based methods find the exact optimum whereas the relaxation-based methods converge to designs close to the optimal point found via an All-in-One optimization. Collaborative Optimization converges to a non-optimal point. This is due to the definition of the penalty function embedded in the Collaborative Optimization problem formulation. Properly introducing the correct penalty function (e.g., via exact penalty function decomposition) and associated update technique for the relaxation parameters converges towards the optimum.

- [Sections 5.1.4 and 5.2.4] The computational costs of equality-based coordination methods compared to relaxation-based coordination methods are of the same order of magnitude. Equality-based methods require more computational effort from the individual elements and relaxation-based methods require more data exchange between elements.

- [Chapter 6] To improve the convergence characteristics of coordination methods so-called multi-level coordination methods are promising. These methods are developed from methods that solve large systems of equations. A drawback of these methods is that the mathematical assumptions on which these methods rely are more stringent then the so-called bi-level coordination methods that are commonly used in multi-level optimization. This thesis showed that multi-level optimization problems that are formulated for bi-level optimization problems can be transformed with little effort into problems that can be solved via a multi-level coordination technique.

- [Sections 5.1.4 and 5.2.4] An advantage of the relaxation-based coordination methods with respect to the equality-based coordination methods is that they are relatively simple to implement. Equality-based coordination methods require additional techniques such as optimum sensitivity analysis or response surface methods in order to function properly. Therefore, in terms of implementation effort relaxation-based coordination methods are preferred.

By capturing the generic steps of multi-level optimization and multi-disciplinary design optimization the key components of a multi-level software program were defined. The flexibility of the framework was shown on typical multi-level optimization problems for which different decomposition and coordination schemes can easily be exchanged.

A relaxation based multi-level optimization technique (Analytical Target Cascading via Augmented Lagrangian relaxation) was implemented according to the multi-level optimization framework developed in this thesis and the following observations were done:

- [Chapter 8] The computational effort required to solve the top-down hierarchic decomposition and non-hierarchic decomposition varied between test cases. The smaller test cases showed faster convergence via hierarchic top-down decomposition and the test cases with many design constraints converged faster via non-hierarchic decomposition.
- [Chapter 8] Two coordination strategies were considered. A sequential coordination process and a parallel coordination process. The parallel process exhibited more computational effort then the sequential coordination process. However, it is expected that if the problem size increases the computational effort becomes close to that of sequential coordination. Therefore, the actual time it takes to execute an entire multi-level optimization in parallel is smaller as compared to the same problem executed sequentially.
- [Chapter 8] In all test cases considered, the method of alternating directions exhibited best computational characteristics as compared to inexact and block coordinate descent.

Findings in this study show that design of complex structures that exhibit multi-scale behavior in the view of multi-scale mechanics via methods that are proposed in literature is challenging.

- [Section 5.1] Equality-based methods suffer from the fact that for each parameter that is send from one element to a neighboring element either sensitivity information is required or a response surface is constructed. Constructing accurate sensitivity information is not straightforward and can become a numerically expensive procedure on its own. Fitting a response surface through elements that output more then ten parameters to a single neighboring element (a small number if a detailed local finite element model communicates physical responses and/or shared design variables with a global finite element model) is challenging.
- [Section 5.3] Relaxation-based methods suffer from poor numerical convergence characteristics. The linear convergence rate of these methods and the fact that intermediate designs during the multi-level optimization cannot be used makes these methods challenging to use in a commercial environment.

Multi-level optimization and multi-disciplinary design optimization methods are useful when the amount of data that is exchanged between individual elements remains small and models are weakly coupled. Weak coupling in the sense that large errors in the solution of a neighboring element do not occur when mapping the data from one element onto a neighboring element. The number of parameters that can be send from one element to a neighboring element is limited. Therefore, one may question the effort that is spend on finding an accurate solution within individual elements. Due to the limited amount of parameters that can be mapped accuracy of coupling may be lost.

9.2 Recommendations

Recommendations of this thesis are two-fold. First directions for the research conducted in this thesis are presented. Second, directions for further development of the framework are presented.

9.2.1 Validating framework

- In this thesis the multi-level framework is demonstrated via Augmented Lagrangian relaxation and coordination via the method of multipliers. To demonstrate the unified formulation of the framework an equality based method should be implemented as well. Equality based methods implemented according to the framework can be used to compare them with relaxation based methods implemented via this framework. A comparison between individual multi-level optimization methods implemented according to the multi-level framework is more objective because the same building blocks are used. In the present thesis the multi-level optimization methods are compared via method specific implementations of the multi-level optimization methods. An approach that lends itself best for a validation of the framework is Bi-Level Integrated System Synthesis because it requires various additional techniques that are specific to equality-based decomposition. Furthermore, it is derived for the same type of multi-level optimization problems as Analytical Target Cascading when looking at the problem matrix.

- The Bi-Level Integrated System Synthesis (BLISS) approach relies on global sensitivity equations or response surface modeling. Development of the global sensitivity module can be used to develop and study the effect of different sensitivity methods on coupled problems. The development of a module that constructs the response surfaces required for BLISS can be used to introduce response surfaces into the relaxation based methods. Because of the generalized formulation of the framework the effect of approximating responses of neighboring elements in the relaxation based methods can be studied.

- The current framework implements Augmented Lagrangian relaxation. Because Augmented Lagrangian relaxation consists of a penalty formulation combined with a Lagrangian relaxation the packages for penalty relaxation and Lagrangian relaxation are already present. Hence, the benchmark problems conducted in the present thesis can be repeated via Penalty Relaxation and Lagrangian relaxation. A comparison between the relaxation based approaches should show which approach is faster in what kind of multi-level optimization setting.

- Numerical tests conducted in the present thesis were limited to hierarchic top-down and non-hierarchic decomposition. For completeness and to demonstrate the flexibility of the framework a bottom-up hierarchic decomposition should be included in the examples as well.

- The current examples are limited to two-level examples. To show the flexibility of multi-level optimization methods a three-level example should be included. A three (or more) level example should show the effect of coordinating individual optimization problems in different sequences. Furthermore, a three (or more) level problem lends itself as an initial study towards multi-level coordination of a hierarchy.

- The present thesis suggests that multi-level coordination is more promising as compared to bi-level coordination in terms of computational effort. Two approaches were suggested that initiated work into the multi-level coordination direction. To show the numerical performance of such an approach with respect to bi-level coordination methods a null-space method according to Alexandrov's Trust Region Model Management strategy looks promising.

- A null-space method opens up possibilities to further extend the framework via methods that are common to solve large systems of equations. Examples are the Schur complement reduction method introduced in the present thesis and Schwarz-decomposition. Schur complement reduction involves solving the consistency constraints via a reduced problem that is much smaller in size then the original problem. Schwarz-decomposition should make it possible to avoid the use of consistency constraints and map computed data from one model directly onto a neighboring model and still consider both models independent.

9.2.2 Extending the framework

- The present framework focuses on decomposition of coupling. Although decomposition enables distribution of the optimization problem it also introduces communication between individual elements. To reduce this communication the effect of completely neglecting parts of the coupling should be studied. A promising approach into this direction is that of Bloebaum (1995) who defines a means of measuring coupling strength between elements. Based on coupling strength a choice which coupling to account for and which coupling to neglect can be made. Other techniques that are promising are diagonal approximations which neglect off-diagonal terms in the coupling matrix or methods that rely on reduction techniques coming from the theory of solving large systems of equations.

- Sensitivity calculations in the present thesis are restricted to calculating finite differences. In literature techniques can be found that reduce the computational costs of sensitivity calculations with several orders of magnitude. Sensitivities are present in the direct coupling of individual elements, hence the Global Sensitivity Equations, as well as, in optimum sensitivity calculations.

- The development of a multi-level coordination method should speed-up the multi-level optimization process with several orders of magnitude. However, a real multi-level coordination process in which separate levels are introduced that pre-coordinate the coordination of element data is challenging. Solution techniques that utilize such an approach are already present. An example is a hierarchy of ever increasing resolution of the same finite element model. The solution to this finite element model is found via first solving an inexpensive model with a coarse mesh. Each individual element in the mesh is then remeshed and solved individually until the finest mesh partitioning is reached. A similar technique should be applicable to the coordination process of multi-level optimization of complex structures.

- The development of a structural multi-level benchmark problem that includes geometric nonlinearity of individual elements in the hierarchy and/or nonlinear material behavior. Within such a model the effect of different decoupling strategies on nonlinearities can be studied and how these couplings should be constrained within the individual elements.

- The development of a structural multi-level benchmark problem that involves geometric and/or material nonlinearities that cover multiple elements. Hence, the nonlinear behavior cannot be isolated within a single element and coupling between individual elements is strong. This problem can be used to test the effect of coordination strategies on reducing the cycling between individual elements. Because coupling is strong in such problems a significant gain in computational effort is expected from approaches that rely on more advanced data exchange methods than fixed point iteration between the elements.

- Finally, the development of a structural multi-level benchmark problem in which geometric and/or material nonlinearities cover multiple levels in the hierarchy. In this test case small material parameters and/or geometrical parameters can be changed on the small scale that can be used to change responses at the large scale for the purpose of optimization. The decomposition of the physical model can be accomplished via, e.g., the Variational Multi-scale Method. Initially, the optimization problem should be constructed such that the multi-level behavior is present in the physical model and optimization problems at the different levels are coupled via the physics. However, eventually optimization problems should be formulated as an All-in-One problem and decomposed via the same mathematical procedure that forms the basis for the Variational Multi-scale Method.

This thesis proposes a framework for multi-level optimization and should function as a basis from which multi-level optimization approaches can be developed and/or applied to the optimization of complex structures. Therefore, the possibilities of validating and extending the framework are endless. We expect that the research conducted in the present thesis inspires the community to further develop multi-level optimization into the four areas that were covered in this thesis.

Bibliography

Agte, J., Sobieszczanski-Sobieski, J., and Jr., R. S. (1999). Supersonic business jet design through bi-level integrated system synthesis. In *Proceedings of the World Aviation Conference, San Francisco, MCB Press, SAE paper 1999-01-5622*.

Alexandrov, N. (1998). A Trust-Region Framework for Managing Approximations in Constrained Optimization and MDO Problems. In *ISSMO/NASA 1st Internet Conference on Approximations and Fast Reanalysis in Engineering Optimization, June 14-27*, pages 1–14.

Alexandrov, N. (2005). Editorial - multidisciplinary design optimization. *Optimization and Engineering*, 6:5–7.

Alexandrov, N. and Dennis, J. (1994). Multilevel algorithms for nonlinear optimization. In Bogged, J. and JohnGunzburger, M., editors, *Optimal design and control; Proceedings of the Workshop on Optimal Design and Control*.

Alexandrov, N., Dennis, J., Lewis, R., and Torczon, V. (1998). A trust-region framework for managing the use of approximation models in optimization. *Structural Optimization*, 15(1).

Alexandrov, N. and Lewis, R. (1999). Comparative properties of collaborative optimization and other approaches to mdo. Technical Report ICASE Report No. 99-24, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, Virigina.

Alexandrov, N. and Lewis, R. (2002). Analytical and computational aspects of collaborative optimization for multidisciplinary design. *AIAA Journal*, 40.

Allen, J., Seepersad, C., Choi, H., and Mistree, F. (2006). Robust design for multiscale and multidisciplinary applications. *Journal of Mechanical Design*, 128:832–843.

Allison, J., Kokkolaras, M., and Papalambros, P. (2005). On the impact of coupling strength on complex system optimization for single-level formulations. In *2005 ASME Design Engineering Technical Conferences*, number DETC2005-84790.

Balling, R. and Sobieszczanski-Sobieski, J. (1995). An algorithm for solving the system-level problem in multilevel optimization. *Structural Optimization*, 9:168–177.

Barthelemy, J.-F. (1989). Engineering design applications of multilevel optimization methods. In Brebbia, C. and Hernandez, S., editors, *Computer Aided Optimum Design of Structures: Applications*. Springer-Verlag.

Barthelemy, J.-F. M. and Sobieszczanski-Sobieski, J. (1983). Optimum sensitivity derivatives of objective functions in nonlinear programming. Technical notes, AIAA.

Beers, M. and Vanderplaats, G. (1987). A linearization method for multilevel optimization. In *Proceedings of the international conference on numerical methods in engineering: Theory and applications/SWANSEA*. Martinus Nijhoff Publishers.

Benzi, M., Golub, G. H., and Liesen, L. (2005). Numerical solution of saddle point problems. *Acta Numerica*, pages 1–137.

Bertsekas, D. (1982). *Constrained optimization and lagrange multiplier methods*. Academic press, inc.

Bertsekas, D. (1995). *Nonlinear Programming*. Athena Scientific, Belmont, Massachusetts.

Bertsekas, D. and Tsitsiklis, J. (1989). *Parallel and Distributed computation*, volume 1. Prentice Hall, Englewood Cliffs, New Jersey.

Biros, G. and Ghattas, O. (2005). Parallel lagrange–newton–krylov–schur methods for pde-constrained optimization. part i: The krylov–schur solver. *SIAM Journal on Scientific Computing*, 27(2):687–713.

Bloebaum, C. (1995). Coupling strenght-based system reduction for complex engineering design. *Structural Optimization*, 10:113–121.

Braun, R. and Kroo, I. (1997). Development and application of the collaborative optimization architecture in a multidisciplinary design environment. In Alexandrov, N. and Hussaini, M., editors, *Multidisciplinary Design Optimization, State of the Art*. Philadelphia: SIAM.

Braun, R., Moore, A., and Kroo, I. (1997). Collaborative approach to launch vehicle design. *Journal of Spacecraft and Rockets*, 34(4):478–486.

Budianto, I. and Olds, J. (2004). Design and deployment of a satellite constellation using collaborative optimization. *Journal of Spacecraft and Rockets*, 41(6):956–963.

Campana, E., Fasano, G., Peri, D., and Pinto, A. (2006). Convergence results for multidisciplinary design optimization (mdo), in ship design problems. In *Proceedings of the III European Conference on Computational Mechanics, Solids, Structures and Coupled Problems in Engineering, june 5-8, Lisbon, Portugal*.

de Wit, A. and van Keulen, F. (2007). Numerical Comparison of Multi-Level Optimization Techniques. In *3rd AIAA Multidisciplinary Design Optimization Specialist Conference, Waikiki in Honolulu, Hawaii, 23-26 April*.

de Wit, A. and van Keulen, F. (2008). Framework for multilevel optimization. In *CJK-OSM5, 5th China-Japan-Korea Joint Symposium on Optimization of Structural and Mechanical Systems, Jeju Island, South Korea, 16-19 April*.

DeMiguel, V. and Murray, W. (2000). An analysis of collaborative optimization methods. In *proceedings of the 8th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*.

DeMiguel, V. and Murray, W. (2006). A local convergence analysis of bilevel decomposition algorithms. *Optimization and Engineering*, 7:99–133.

DeMiguel, V. and Nogales, F. (2008). On Decomposition Methods for a Class of Partially Separable Nonlinear Programs. *Mathematics of Operations Research*, pages 119–139.

Dennis, J., El-Alem, M., and Maciel, M. (1997). A Global Convergence Theory for General Trust-Region-Based Algorithms for Equality Constrained Optimization. Technical report tr92-28, Department of Computational and Applied Mathematics, Rice university, Houston, Texas.

Dreyer, T., Maar, B., and Schulz, V. (2000). Multigrid optimization in applications. *Journal of Computational and Applied Mathematics*, 120:67–84.

Engineous Software (2008). iSIGHT.

Fish, J. and Shek, K. (2000). Multiscale analysis of large scale nonlinear structures and materials. *International Journal for Computational Civil and Structural Engineering*, 1:79–90.

Ghoniem, N., Busso, E., Kioussis, N., and Huang, H. (2003). Multiscale modelling of nanomechanics and micromechanics: an overview. *Philosophical Magazine*, 83:3475–3528.

Giunta, A. and Eldred, M. (2000). Implementation of a trust region model management strategy in the dakota optimization toolkit. In *Proceedings of the 8th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, Long Beach, CA, September 6-8, paper AIAA-2000-4935*.

Haftka, R. and Gürdal, Z. (1999). *Elements of Structural Optimization*, volume 11 of *Solid Mechanics and its Applications*. Kluwer Academic Publishers.

Haftka, R. and Watson, L. (2005). Multidisciplinary Design Optimization with Quasiseparable Subsystems. *Optimization and Engineering*, 6:9–20.

Hoyle, N., Bressloff, N., and Keane, A. (2006). Design optimization of a two-dimensional subsonic engine air intake. *American Institute of Aeronautics and Astronautics Journal*, 44(11):2672–2681.

Kim, H. (2001). *Target Cascading in Optimal System Design*. PhD thesis, The University of Michigan.

Kim, H., Chen, W., and Wiecek, M. (2006). Lagrangian coordination for enhancing the convergence of analytical target cascading. *American Institute of Aeronautics and Astronautics Journal*, 44(10):2197–2207.

Kim, H. M., Michelena, N. F., Papalambros, P. Y., and Jiang, T. (2003). Target cascading in optimal system design. *Journal of Mechanical Design*, 125:474–480.

Kirsch, U. (1993). *Structural Optimization. Fundamentals and Applications*. Springer-Verlag.

Kirsch, U. (1997). Two-level optimization of prestressed structures. *Engineering Structures*, 19(4):309–317.

Kirsch, U. and Moses, F. (1979). Decomposition in optimum structural design. *Journal of the structural division*, 105:85–100.

Kodiyalam, S. (1998). Evaluation of methods for multidisciplinary design optimization (mdo), phase i. Technical Report CR-1998-208716, NASA.

Ladevèze, P. (2004). Multiscale modelling and computational strategies for composites. *International Journal for Numerical Methods in Engineering*, 60:233–253.

Lasdon, L. (1970). *Optimization Theory for Large Systems*. The Macmillan Company.

Lassiter, J., Wiecek, M., and Andrighetti, K. (2005). Lagrangian coordination and analytical target cascading: Solving atc-decomposed problems with lagrangian duality. *Optimization and Engineering*, 6:361–381.

Li, Y., Lu, Z., and Michalek, J. (2008). Diagonal Quadratic Approximation for Parallelization of Analytical Target Cascading. *ASME Journal of Mechanical Design*, 130(5).

Martins, J. and Marriage, C. (2007). An object-oriented framework for multidisciplinary design optimization. In *3rd AIAA Multidisciplinary Design Optimization Specialist Conference, Waikiki in Honolulu, Hawaii, 23-26 April*.

Martins, J., Marriage, C., and Tedford, N. (2009). pymdo: An object-oriented framework for multidisciplinary design optimization. *ACM Transactions on Mathematical Software*, 36.

McKay, M., Beckman, R., and Conover, W. (1979). A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21:239–245.

Mesarović, M., Macko, D., and Takahara], Y. (1970). *Theory of Hierarchical, Multilevel, Systems*, volume 68 of *Mathematics in science and engineering*. Academic Press.

Michalek, J. and Papalambros, P. (2005a). An efficient weighting update method to achieve acceptable consistency deviation in analytical target cascading. *Journal of Mechanical Design*, 127:206–214.

Michalek, J. and Papalambros, P. (2005b). Weights, norms, and notation in analytical target cascading. *Journal of Mechanical Design*, 127:499–501.

Michelena, N., Park, H., and Papalambros, P. (2003). Convergence properties of analytical target cascading. *American Institute of Aeronautics and Astronautics Journal*, 41:897–905.

Myers, R. and Montgomery, D. (1995). *Response Surface Methodology, Process and Product Optimization Using Designed Experiments*. John Wiley & Sons, Inc.

Object Management Group (2008). Unified Modeling Language.

Padula, S., Alexandrov, N., and Green, L. (1996). MDO test suite at NASA Langley Research Center. In *AIAA, NASA, and ISSMO, Symposium on Multidisciplinary Analysis and Optimization, 6th, Bellevue, WA, Sept. 4-6*, pages 410–420.

Quarteroni, A. and Valli, A. (1999). *Domain Decomposition Methods for Partial Differential Equations*. Oxford University Press.

Red Cedar Technology (2008). HEEDS.

Renaud, J. and Gabriele, G. (1993). Improved Coordination in Nonhierarchic System Optimization. *American Institute of Aeronautics and Astronautics Journal*, 31:2367–2373.

Renaud, J. and Gabriele, G. (1994). Approximation in Nonhierarchic System Optimization. *American Institute of Aeronautics and Astronautics Journal*, 32:198–205.

Rodríguez, J., Renaud, J., and Watson, L. (1998). Convergence of trust region augmented Lagrangian methods using variable fidelity approximation data. *Structural Optimization*, 15:141–156.

Rodríguez, J., Renaud, J., Wujek, B., and Tappeta, R. (2000). Trust region model management in multidisciplinary design optimization. *Journal of Computational and Applied Mathematics*, 124:139–154.

Ross, S. M. (2009). *Introduction to probability and statistics for engineers and scientists*. Elsevier.

Rudd, R. and Broughton, J. (2000). Concurrent coupling of length scales. *physica status solidi (b)*, 217:251–291.

Sellar, R., Batill, S., and Renaud, J. (1994). Optimization of mixed discrete/continuous design variable systems using neural networks. *AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, Panama City, Florida*, (AIAA-94-4348).

Sellar, R., Stelmack, M., Batill, S., and Renaud, J. (1996). Response surface approximations for discipline coordination in multidisciplinary design optimization. *AIAA/ASME/ASCE/AHS/ASC 37th Structures, Structural Dynamics and Materials Conference, Salt Lake City, Utah*, (AIAA-96-1383):583–593.

Shankar, J., Ribbens, C., Haftka, R., and Watson, L. (1993). Computational study of a nonhierarchical decomposition algorithm. *Computational Optimization and Applications*, 2:273–293.

Shupe, J., Mistree, F., and Sobieszanski-Sobieski, J. (1987). Compromise: An effective approach for the hierarchical design of structural systems. *Computers and Structures*, 26:1027–1037.

Smith, B., Bjorstad, P., and Gropp, W. (1996). *Domain Decomposition, Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press.

Sobieszczanski-Sobieski, J. (1982). A linear decomposition method for large optimization problems. blueprint for development. Technical Report NASA-TM-83248, NASA.

Sobieszczanski-Sobieski, J. (1988). A step from hierarchic to non-hierarchic systems. In *2nd NASA Air Force symposium on recent advances in multidisciplinary analysis and optimization*.

Sobieszczanski-Sobieski, J. (1990). Sensitivity of complex, internally coupled systems. *AIAA Journal*, 28(1):153–160.

Sobieszczanski-Sobieski, J. (1992). A technique for locating function roots and satisfying equality constraints in optimization. *Structural Optimization*, 4:241–243.

Sobieszczanski-Sobieski, J. (1993). Two alternative ways for solving the coordination problem in multilevel optimization. *Structural Optimization*, 6:205–215.

Sobieszczanski-Sobieski, J., Agte, J., and Sandusky JR, R. (1998). Bi-level integrated system synthesis (bliss). In *AIAA-98-4916*.

Sobieszczanski-Sobieski, J., Altus, T., Phillips, M., and Sandusky, R. (2003). Bilevel integrated system synthesis for concurrent and distributed processing. *American Institute of Aeronautics and Astronautics Journal*, 41(10):1996–2003.

Sobieszczanski-Sobieski, J., James, B., and Dovi, A. (1985). Structural optimization by multilevel decomposition. *American Institute of Aeronautics and Astronautics Journal*, 23:124–142.

Sobieszczanski-Sobieski, J., James, B., and Riley, M. (1987). Structural sizing by generalized, multilevel optimization. *American Institute of Aeronautics and Astronautics Journal*, 25:139–145.

Sun Microsystems, Inc. (2008a). Java se 6.

Sun Microsystems, Inc. (2008b). Netbeans, version 6.1.

The CVS Team (2008). Concurrent versions system.

Tosserams, S. (2008). *Distributed optimization for systems design*. PhD thesis, Eindhoven University of Technology.

Tosserams, S., Etman, L., Papalambros, P., and Rooda, J. (2006). An augmented lagrangian relaxation for analytical target cascading using the alternating direction method of multipliers. *Structural and Multidisciplinary optimization*, 31:176–189.

Tosserams, S., Etman, L., and Rooda, J. (2007). An augmented lagrangian decomposition method for quasi-separable problems in mdo. *Structural and Multidisciplinary Optimization*, 34(3):211–227.

Tosserams, S., Etman, L., and Rooda, J. (2008a). Augmented lagrangian coordination for distributed optimal design in MDO. *International Journal for Numerical Methods in Engineering*, 73:1885–1910.

Tosserams, S., Etman, L., and Rooda, J. (2008b). Performance evaluation of augmented lagrangian coordination for distributed multidisciplinary design optimization. In *Proceedings of the 4th AIAA Multidisciplinary Design Optimization Specialist Conference, Schaumburg, IL, United States, 1-17*.

Tosserams, S., Hofkamp, A., Etman, L., and Rooda, J. (2009). A micro-accelerometer mdo benchmark problem. *Structural Multidisciplinary Optimization*, Accepted, to appear.

University of Buffalo (Accessed November 4, 2009). Multidisciplinary optimization and engineering laboratory. http://www.eng.buffalo.edu/Research/MODEL/.

Vanderplaats, G., Yang, Y., and Kim, D. (1990). Sequential linearization method for multilevel optimization, old. *American Institute of Aeronautics and Astronautics Journal*, 28:290–295.

Wagner, T. (1993). *A general decomposition methodology for optimal system design*. PhD thesis, The University of Michigan.

Appendix A

Significance of relaxation parameters within multi-level optimization

Relaxation based multi-level optimization methods rely on external updates of the relaxation parameters. The significance of these parameters with respect to the individual optimization problems and the combined optimization problem can be seen as follows.

Consider an optimization problem where the objective function is additively separable and coupling is present via physical responses between two individual elements. The consistency is maintained via a consistency constraint ${}^1_0\mathbf{c}$ that is taken into account in the optimization problem formulation. Mathematically the problem can be expressed as:

$$\begin{array}{ll} \min\limits_{{}^0\mathbf{x},{}^1_0\mathbf{h},{}^1\mathbf{x}} & v_f = {}^0v_f({}^0\mathbf{x}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h})) + {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) \\ \text{s.t.} & {}^1_0\mathbf{c} = {}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h} = 0 \end{array} . \tag{A.1}$$

The two individual elements that are embedded in the optimization problem are Element-0 and Element-1. Constructing individual coupled optimization problems from Equation A.1 in the context of multi-level optimization is accomplished via formulating the Lagrangian.

The Lagrangian of Equation A.1 is mathematically expressed as:

$$\mathcal{L}(\mathbf{y}, {}^1_0\boldsymbol{\lambda}) \;=\; {}^0v_f({}^0\mathbf{x}, {}^0\mathbf{r}({}^0\mathbf{x}, {}^1_0\mathbf{h})) + {}^1v_f({}^1\mathbf{x}, {}^1\mathbf{r}({}^1\mathbf{x}, {}^0_1\mathbf{h})) + {}^1_0\boldsymbol{\lambda}^T \left({}^1_0\mathcal{H}({}^1\mathbf{r}) - {}^1_0\mathbf{h}\right) \;, \tag{A.2}$$

where the Lagrange multipliers (${}^1_0\boldsymbol{\lambda}$) are additional parameters that relax the consistency constraints (${}^1_0\mathbf{c}$) and $\mathbf{y} = \left[\begin{array}{ccc} {}^0\mathbf{x} & {}^1\mathbf{x} & {}^1_0\mathbf{h} \end{array}\right]^T$.

For brevity of notation the vectors ${}^{0}\mathbf{y} = \left[\begin{array}{cc} {}^{0}\mathbf{x} & {}^{1}_{0}\mathbf{h} \end{array}\right]$ and ${}^{1}\mathbf{y} = \left[\begin{array}{c} {}^{1}\mathbf{x} \end{array}\right]$ are introduced. Necessary conditions for a stationary solution of Equation A.2 are:

$$\begin{array}{rcl} \nabla_{{}^{0}\mathbf{y}}{}^{0}v_f + {}^{1}_{0}\boldsymbol{\lambda}^T \nabla_{{}^{0}\mathbf{y}} \left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right) & = & 0 \\ \Rightarrow \left(\nabla_{{}^{0}\mathbf{y}}{}^{0}v_f\right)^T = -\nabla_{{}^{0}\mathbf{y}} \left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right)^T {}^{1}_{0}\boldsymbol{\lambda} & & \end{array} \tag{A.3}$$

$$\begin{array}{rcl} \nabla_{{}^{1}\mathbf{y}}{}^{1}v_f + {}^{1}_{0}\boldsymbol{\lambda}^T \nabla_{{}^{1}\mathbf{y}} \left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right) & = & 0 \\ \Rightarrow \left(\nabla_{{}^{1}\mathbf{y}}{}^{1}v_f\right)^T = -\nabla_{{}^{1}\mathbf{y}} \left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right)^T {}^{1}_{0}\boldsymbol{\lambda} & & \end{array} \tag{A.4}$$

$$\nabla_{{}^{1}_{0}\boldsymbol{\lambda}}\mathcal{L} = \left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right) = 0 \tag{A.5}$$

where $\nabla_{\cdot\cdot\mathbf{y}}.. = \left[\frac{\partial ..}{\partial\, {}^{\cdot\cdot}y_1}, \ldots, \frac{\partial ..}{\partial\, {}^{\cdot\cdot}y_n}\right]^T$ and $\nabla_{{}^{1}_{0}\boldsymbol{\lambda}}..$ defined similarly.

Equation A.3, A.4 and A.5 are stationary conditions for the coupled optimization problem expressed in Equation A.1. The total objective function depends on the individual objective function of Element-0 (${}^{0}v_f$) and the objective function of Element-1 (${}^{1}v_f$). A stationary solution to ${}^{0}v_f$ changes via changes in the design variables of Element-1. Sensitivity with respect to these changes is found via applying the chain rule ($\mathbf{D}_{{}^{i}\mathbf{y}}\left(\ldots\right) = \nabla_{{}^{i}\mathbf{y}}\left(\ldots\right) + \nabla_{{}^{j}\mathbf{y}}\left(\ldots\right)^T \mathbf{D}_{{}^{i}\mathbf{y}}\left({}^{j}\mathbf{y}\right)$) to ${}^{0}v_f$. $\mathbf{D}_{\cdot\cdot\mathbf{y}}\left(\ldots\right)$ are total derivatives and are defined as $\mathbf{D}_{\cdot\cdot\mathbf{y}}\left(\ldots\right) = \left[\frac{\mathrm{d}..}{\mathrm{d}\,{}^{\cdot\cdot}\mathbf{y}}, \ldots, \frac{\mathrm{d}..}{\mathrm{d}\,{}^{\cdot\cdot}\mathbf{y}}\right]^T$. Hence, sensitivity of a stationary solution of ${}^{0}\mathbf{v}_f$ that is found via Equation A.3 with respect to changes in the coupled neighboring element are mathematically expressed as:

$$\mathbf{D}_{{}^{1}\mathbf{y}}\left({}^{0}v_f\right) = \nabla_{{}^{1}\mathbf{y}}{}^{0}v_f + (\nabla_{{}^{0}\mathbf{y}}{}^{0}v_f)^T \mathbf{D}_{{}^{1}\mathbf{y}}\left({}^{0}\mathbf{y}\right) . \tag{A.6}$$

Substitution of Equation A.3 into Equation A.6 gives for Element-0:

$$\mathbf{D}_{{}^{1}\mathbf{y}}\left({}^{0}v_f\right) = \nabla_{{}^{1}\mathbf{y}}{}^{0}v_f - \nabla_{{}^{0}\mathbf{y}}\left({}^{1}_{0}\mathcal{H}({}^{1}\mathbf{r}) - {}^{1}_{0}\mathbf{h}\right)^T \mathbf{D}_{{}^{1}\mathbf{y}}\left({}^{0}\mathbf{y}\right) {}^{1}_{0}\boldsymbol{\lambda} . \tag{A.7}$$

Sensitivity of the consistency constraint in Element-0 with respect to changes within Element-1 is found via the chain rule and is mathematically expressed as:

$$\begin{array}{l} \nabla_{{}^{1}\mathbf{y}}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right) + \nabla_{{}^{0}\mathbf{y}}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right)^T \mathbf{D}_{{}^{1}\mathbf{y}}\left({}^{0}\mathbf{y}\right) = 0 \\ \Rightarrow \nabla_{{}^{1}\mathbf{y}}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right) = -\nabla_{{}^{0}\mathbf{y}}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right)^T \mathbf{D}_{{}^{1}\mathbf{y}}\left({}^{0}\mathbf{y}\right) \end{array} . \tag{A.8}$$

Substituting Equation A.8 into Equation A.7 we obtain:

$$\mathbf{D}_{{}^{1}\mathbf{y}}\left({}^{0}v_f\right) = \nabla_{{}^{1}\mathbf{y}}{}^{0}v_f + {}^{1}_{0}\boldsymbol{\lambda}^T \nabla_{{}^{1}\mathbf{y}}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right) . \tag{A.9}$$

A similar derivation for the objective of Element-1 (${}^{1}v_f$) for the sensitivity of the objective function value of ${}^{1}v_f$ with respect to changes in Element-0 leads to:

$$\mathbf{D}_{{}^{0}\mathbf{y}}\left({}^{1}v_f\right) = \nabla_{{}^{0}\mathbf{y}}{}^{1}v_f + {}^{1}_{0}\boldsymbol{\lambda}^T \nabla_{{}^{0}\mathbf{y}}\left({}^{1}_{0}\mathcal{H}\left({}^{1}\mathbf{r}\right) - {}^{1}_{0}\mathbf{h}\right) . \tag{A.10}$$

From Equation A.9 and Equation A.10 we see how the objective of Element-0 (${}^{0}v_f$) or Element-1 (${}^{1}v_f$) changes with respect to changes in the design variables in the neighboring element.

The two individual optimization problems are related to one another via Equation A.2. For the combined optimization problem to be a (local) minimum within the space of arbitrary perturbations $\mathbf{y}$ we must have:

$$\nabla_{\mathbf{y}}\mathcal{L}_{\lambda}\partial\mathbf{y} \quad \geq \quad \mathbf{0} \tag{A.11}$$

where $\mathcal{L}_{\lambda}$ is the Lagrangian function for fixed ${}^1_0\boldsymbol{\lambda}$ and perturbations must be feasible ($\partial\mathbf{y}^T\left({}^1_0\mathcal{H}\left({}^1\mathbf{r}\right) - {}^1_0\mathbf{h}\right) = \mathbf{0}$). Hence, a minimum of the combined Element-0 and Element-1 optimization problems is found via inserting Equation A.9 and Equation A.10 into Equation A.11 leading to[1]:

$$\begin{array}{rcll}\nabla_{\mathbf{y}}\mathcal{L}_{\lambda}\partial\mathbf{y} & = & \left(\mathbf{D}_{{}^1\mathbf{y}}({}^0v_f) + \mathbf{D}_{{}^0\mathbf{y}}({}^1v_f)\right)\partial\mathbf{y} & \geq \; 0 \\ & = & \nabla_{{}^1\mathbf{y}}{}^0v_f\partial^1\mathbf{y} + \nabla_{{}^0\mathbf{y}}{}^1v_f\partial^0\mathbf{y} + {}^1_0\boldsymbol{\lambda}^T\nabla_{\mathbf{y}}\left({}^1_0\mathcal{H}\left({}^1\mathbf{r}\right) - {}^1_0\mathbf{h}\right)\partial\mathbf{y} & \geq \; 0\end{array} \tag{A.12}$$

After rearrangements of terms Equation A.12 can be written as:

$$\nabla_{{}^1\mathbf{y}}{}^0v_f\partial^1\mathbf{y} + \nabla_{{}^0\mathbf{y}}{}^1v_f\partial^0\mathbf{y} \quad \geq \quad -{}^1_0\boldsymbol{\lambda}^T\left(\nabla_{\mathbf{y}}{}^1_0\mathcal{H}\left({}^1\mathbf{r}\right) - {}^1_0\mathbf{h}\right)\partial\mathbf{y} \; . \tag{A.13}$$

If $\left(\nabla_{\mathbf{y}}\left({}^1_0\mathcal{H}\left({}^1\mathbf{r}\right) - {}^1_0\mathbf{h}\right)\right)$ is invertible, then Equation A.13 can be further rearranged to:

$$-\left(\nabla_{{}^1\mathbf{y}}{}^0v_f\partial^1\mathbf{y} + \nabla_{{}^0\mathbf{y}}{}^1v_f\partial^0\mathbf{y}\right)\left(\nabla_{\mathbf{y}}\left({}^1_0\mathcal{H}\left({}^1\mathbf{r}\right) - {}^1_0\mathbf{h}\right)\right)^{-1} \quad \leq \quad {}^1_0\boldsymbol{\lambda}^T\partial\mathbf{y} \; . \tag{A.14}$$

Hence, according to Equation A.14 the Lagrange multipliers form an upper bound for arbitrary perturbations of the combined stationary point of the optimization problems expressed in Equation A.9 and Equation A.10.

We assume that $\left(\nabla_{{}^1\mathbf{y}}{}^0v_f + \nabla_{{}^0\mathbf{y}}{}^1v_f\right)$ is positive definite[2] and $\nabla_{\mathbf{y}}\left({}^1_0\mathcal{H}\left({}^1\mathbf{r}\right) - {}^1_0\mathbf{h}\right)$ has full rank, it follows that the left hand side of Equation A.14 is negative definite and therefore:

$$\mathcal{L}_{\lambda}\left({}^1_0\boldsymbol{\lambda}\right) = \min_{\mathbf{y}} \quad \mathcal{L}\left(\mathbf{y}, {}^1_0\boldsymbol{\lambda}\right) \tag{A.15}$$

is concave. Therefore, finding a stationary solution to Equation A.15 is obtained via:

$$\begin{array}{ll}\max\limits_{{}^1_0\boldsymbol{\lambda}} & \mathcal{L}_{\lambda}\left({}^1_0\boldsymbol{\lambda}\right) \\ \text{where} & \mathcal{L}_{\lambda}\left({}^1_0\boldsymbol{\lambda}\right) = \min\limits_{\mathbf{y}} \quad \mathcal{L}\left(\mathbf{y}, {}^1_0\boldsymbol{\lambda}\right)\end{array} \; . \tag{A.16}$$

Equation A.16 is also known as the Dual of Equation A.1. The latter defined as the primal. Under certain mathematical assumptions, see Proposition 5.1.1 to 5.1.6 of Bertsekas (1982) the solution to Equation A.16 is equal to the solution of Equation A.1.

---

[1] $\mathbf{D}_{{}^0\mathbf{y}}{}^0v_f = 0$ and $\mathbf{D}_{{}^1\mathbf{y}}{}^1v_f = 0$ due to optimality of the individual optimization problems

[2] With positive definite we assume that $\partial\mathbf{y}^T\left(\nabla_{{}^1\mathbf{y}}{}^0v_f + \nabla_{{}^0\mathbf{y}}{}^1v_f\right)\partial\mathbf{y} > 0$ for $\partial\mathbf{y} \neq 0$. This is true if the optima $({}^0v_f, {}^1v_f)$ for which the sensitivities are evaluated are additively separable. In that case we have $\partial\mathbf{y}^T\nabla^2_{\mathbf{y},\mathbf{y}}v_f\partial\mathbf{y} > 0$ and $\partial\mathbf{y}^T\nabla_{\mathbf{y}}{}^1_0\mathbf{c} = 0$ for the all-in-one problem.

Appendix B

Bar element and beam element analysis

The necessary equations for the analysis of the structural responses of the two-bar truss and the portal framework are derived in this chapter.

B.1 bar element

The displacement ($\mathbf{u}$) of the two-bar truss structure, see Figure B.1, is computed via:

$$\mathbf{K}\mathbf{u} = \mathbf{P}, \tag{B.1}$$

where $\mathbf{K}$ is the stiffness matrix of the entire structure and $\mathbf{P}$ the externally applied load. The bar element stiffness matrix of individual elements with respect to element coordinates is computed as:

$$\mathbf{K}^e = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV;$$

$$\mathbf{B} = \frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad \mathbf{D} = [E] \qquad \mathbf{K}^e = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The transformation matrix from element coordinates to structural coordinates is:

$$\mathbf{T} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \end{bmatrix},$$

where ϕ is defined in Figure B.2.

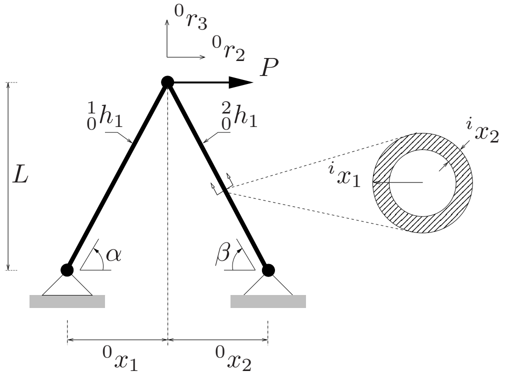


Figure B.1: *Two-bar truss structure with embedded hierarchy. The general lay-out of the structure is described via design parameters $^0x_1, {}^0x_2$ and cross-sectional areas. The element's cross-section is described in detail via design variables $^ix_1, {}^ix_2$. These cross-sections are present as coupling variables $^1_0h_1, {}^2_0h_1$.*

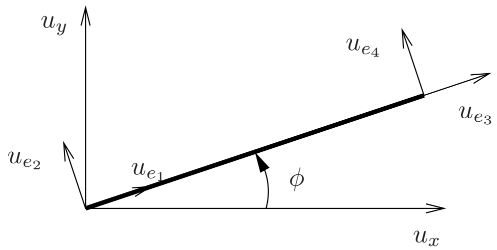


Figure B.2: *Transformation from element coordinates to structural coordinates.*

After assembly of the individual element stiffness matrices and elimination of boundary conditions the stiffness matrix necessary to compute the horizontal displacement is:

$$\mathbf{K} = \begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix} = \dots$$

$$E \begin{bmatrix} \frac{^1_0h}{l_1}\cos^2\alpha + \frac{^2_0h}{l_2}\cos^2\beta & \frac{^1_0h}{l_1}\sin\alpha\cos\alpha + \frac{^2_0h}{l_2}\sin\beta\cos\beta \\ \frac{^1_0h}{l_1}\sin\alpha\cos\alpha + \frac{^2_0h}{l_2}\sin\beta\cos\beta & \frac{^1_0h}{l_1}\sin^2\alpha + \frac{^2_0h}{l_2}\sin^2\beta \end{bmatrix},$$

where the cross-sectional area of each individual bar column (^{i}) is expressed as $^i_0h_1$ and the length of these columns ar l_i. The displacement vector ($\mathbf{u}$) and the applied force vector ($\mathbf{P}$) are given as:

$$\mathbf{u} = \begin{bmatrix} ^0r_2 \\ ^0r_3 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

where $^0r_2$ is the horizontal displacement in structural coordinates and $^0r_3$ the vertical displacement in structural coordinates.

The displacements are then obtained as follows

$$k_{1,1}{}^0r_2 + k_{1,2}{}^0r_3 = F \qquad k_{1,2}{}^0r_2 + k_{2,2}{}^0r_3 = 0$$
$$\text{where} \qquad k_{1,2} = k_{2,1}$$
$${}^0r_2 = \frac{1}{(k_{1,1} - \frac{k_{1,2}^2}{k_{2,2}})}F$$

The stiffness matrix **K** depends on the cross-sectional area of the two bars. A thin-walled bar element is considered and therefore the cross-section can be approximated as:

$$ {}^1_0\mathcal{H}\left({}^1x_1, {}^1x_2\right) = {}^1_0h_1 = 2\pi{}^1x_1{}^1x_2; \tag{B.2}$$
$$ {}^2_0\mathcal{H}\left({}^2x_1, {}^2x_2\right) = {}^2_0h_1 = 2\pi{}^2x_1{}^2x_2. \tag{B.3}$$

The total mass of the structure can now be computed according to:

$$ {}^0r_1 = {}^1_0h\rho l_1 + {}^2_0h\rho l_2 \tag{B.4}$$

where the length of both trusses is computed according to:

$$ l_1 = \sqrt{L^2 + ({}^0x_1)^2} \quad \text{and} \quad l_2 = \sqrt{L^2 + ({}^0x_2)^2}. \tag{B.5}$$

After solving for the displacements the internal forces can be computed. First, the strains are computed for each individual element from the structural displacements. Secondly, the internal nodal forces are computed:
Element 1:

$$\mathbf{u}_{e_1} = \mathbf{T}\begin{bmatrix} 0 \\ 0 \\ {}^0r_2 \\ {}^0r_3 \end{bmatrix} \quad \Rightarrow \quad \epsilon_1 = \mathbf{B}\mathbf{u}_e = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} u_{e_1} \\ u_{e_2} \end{bmatrix} = \frac{{}^0r_2\cos\alpha + {}^0r_3\sin\alpha}{l_1},$$

$$\mathbf{f}_1^{int} = \int_{x=0}^{l_1} \frac{1}{l_1}\mathbf{B}^T\mathbf{D}\epsilon_1 A_1 dx.$$

Element 2:

$$\mathbf{u}_{e_2} = \mathbf{T}\begin{bmatrix} {}^0r_2 \\ {}^0r_3 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \epsilon_2 = \mathbf{B}\mathbf{u}_e = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} u_{e_2} \\ u_{e_3} \end{bmatrix} = -\frac{{}^0r_2\cos\beta - {}^0r_3\sin\beta}{l_2},$$

$$\mathbf{f}_2^{int} = \int_{x=0}^{l_2} \frac{1}{l_2}\mathbf{B}^T\mathbf{D}\epsilon_2 A_2 dx.$$

Mapping the displacements ${}^0r_2$, ${}^0r_3$ gives the internal nodal forces:

$$ {}^0_1\mathcal{H}\left({}^0r\right) = {}^0_1\mathbf{h} = E{}^1_0h_1\begin{bmatrix} -\frac{{}^0r_2\cos\alpha + {}^0r_3\sin\alpha}{l_1} & \frac{{}^0r_2\cos\alpha + {}^0r_3\sin\alpha}{l_1} \end{bmatrix}; \tag{B.6}$$
$$ {}^0_2\mathcal{H}\left({}^0r\right) = {}^0_2\mathbf{h} = E{}^2_0h_1\begin{bmatrix} \frac{{}^0r_2\cos\beta - {}^0r_3\sin\beta}{l_2} & -\frac{{}^0r_2\cos\beta - {}^0r_3\sin\beta}{l_2} \end{bmatrix}. \tag{B.7}$$

Nominal stress is defined as: $\sigma = \frac{f}{A}$. The load carried by the element is f and the cross-sectional area is A. Therefore, the nominal stress in each member is:

$$^1r_1 = \frac{{}^0_1h_1}{2\pi {}^1x_1 {}^1x_2}; \tag{B.8}$$

$$^2r_1 = \frac{{}^0_2h_1}{2\pi {}^2x_1 {}^2x_2}. \tag{B.9}$$

B.2 beam element

The displacements of the portal frame structure, see Figure B.3 are computed via:

$$\mathbf{K}\mathbf{u} = \mathbf{P}, \tag{B.10}$$

where $\mathbf{K}$ is the stiffness matrix of the entire structure. The columns are modeled with beam elements. The degrees of freedom for the flexural stiffness matrix of these beam elements are defined in Figure B.4. The stiffness matrix is computed via:

$$\mathbf{K}^e = \int_V \mathbf{B}^T\mathbf{D}\mathbf{B}dV \quad \mathbf{D} = [EI]$$

$$\mathbf{B} = \begin{bmatrix} \frac{6}{l^2} - 12\frac{x}{l^3} & \frac{4}{l^2} - \frac{6x}{l^2} & -\frac{6}{l^2} + 12\frac{x}{l^3} & \frac{2}{l} - 6\frac{x}{l^2} \end{bmatrix}$$

$$\mathbf{K}^e_f = \begin{bmatrix} 12\frac{EI}{l^3} & 6\frac{EI}{l^2} & -12\frac{EI}{l^3} & 6\frac{EI}{l^2} \\ 6\frac{EI}{l^2} & 4\frac{EI}{l} & -6\frac{EI}{l^2} & 2\frac{EI}{l} \\ -12\frac{EI}{l^3} & -6\frac{EI}{l^2} & 12\frac{EI}{l^3} & -6\frac{EI}{l^2} \\ 6\frac{EI}{l^2} & 2\frac{EI}{l} & -6\frac{EI}{l^2} & 4\frac{EI}{l} \end{bmatrix}.$$

The axial stiffness matrix is equal to the bar element stiffness matrix in Section B.1:

$$\mathbf{K}^e_a = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The transformation matrix that maps element coordinates to structural coordinates is:

$$\mathbf{T} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{B.11}$$

The element stiffness matrix is rotated into global coordinates via:

$$K^e = \mathbf{T}^T\mathbf{K}^e\mathbf{T}; \tag{B.12}$$

after assembly of the element stiffness matrices the total structural stiffness matrix is found.

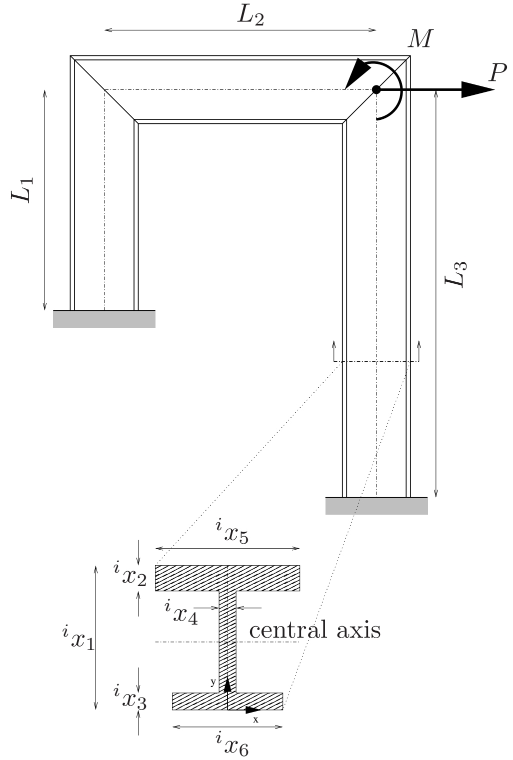


Figure B.3: *Portal framework with embedded hierarchy. The general lay-out of the structure is described via fixed lengths of the columns and the cross-sectional areas of the columns. The elements are described in detail via design variable vectors* $^1\mathbf{x}, {}^2\mathbf{x}, {}^3\mathbf{x}$.

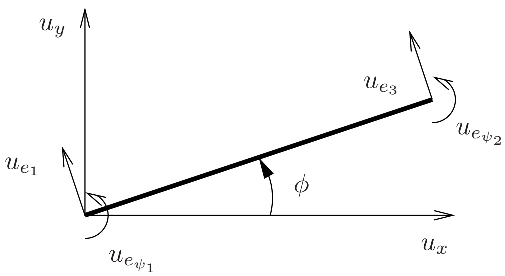


Figure B.4: *Transformation from element coordinates to structural coordinates.*

The stiffness matrix $\mathbf{K}$ depends on the cross-sectional area of the columns, the area is computed as:

$$
\begin{aligned}
&{}^{i}_{0}\mathcal{H}_1\left({}^{i}x_1, {}^{i}x_2, {}^{i}x_3, {}^{i}x_4, {}^{i}x_5, {}^{i}x_6\right) = \\
&({}^{i}x_5)({}^{i}x_2) + ({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3){}^{i}x_4 + ({}^{i}x_6)({}^{i}x_3).
\end{aligned} \tag{B.13}
$$

Furthermore, the stiffness matrix depends on the moment of inertia:

$$
\begin{aligned}
{}^{i}_{0}\mathcal{H}_2\left({}^{i}x_1, {}^{i}x_2, {}^{i}x_3, {}^{i}x_4, {}^{i}x_5, {}^{i}x_6\right) = \frac{{}^{i}x_5({}^{i}x_2)^3}{12} + ({}^{i}x_5)({}^{i}x_2)(y_c - ({}^{i}x_1 - \\
\frac{1}{2}{}^{i}x_2))^2 + \frac{{}^{i}x_4({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3)^3}{12} + {}^{i}x_4({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3)(y_c - \frac{1}{2}{}^{i}x_1)^2 + \\
\frac{{}^{i}x_6({}^{i}x_3)^3}{12} + ({}^{i}x_6)({}^{i}x_3)(y_c - \frac{1}{2}{}^{i}x_3)^2
\end{aligned} \tag{B.14}
$$

where the centroid y_c is given by:

$$
y_c = \frac{{}^{i}x_5{}^{i}x_2({}^{i}x_1 - \frac{1}{2}{}^{i}x_2) + \frac{1}{2}({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3)({}^{i}x_4)({}^{i}x_1) + \frac{1}{2}{}^{i}x_6({}^{i}x_3)^2}{{}^{i}x_5{}^{i}x_2 + {}^{i}x_4({}^{i}x_1 - {}^{i}x_2 - {}^{i}x_3) + ({}^{i}x_6)({}^{i}x_3)} \tag{B.15}
$$

The total volume of the structure can be computed as:

$$
{}^{0}r_2 = {}^{1}_{0}hl_1 + {}^{2}_{0}hl_2 + {}^{3}_{0}hl_3; \tag{B.16}
$$

where the lengths $l_{..}$ are fixed.

After solving the displacements of the structure the internal forces can be computed. First the strains are computed for each individual element from the structural displacements. Secondly, the internal nodal forces are computed:
Element 1:

$$
\mathbf{f}_1^{int} = \int_V \mathbf{B}^T\mathbf{D}\epsilon_1 dV \quad \Rightarrow \quad {}^{0}_{1}\mathbf{h} = {}^{0}_{1}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) \tag{B.17}
$$

Element 2:

$$
\mathbf{f}_2^{int} = \int_V \mathbf{B}^T\mathbf{D}\epsilon_2 dV \quad \Rightarrow \quad {}^{0}_{2}\mathbf{h} = {}^{0}_{2}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) \tag{B.18}
$$

Element 3:

$$
\mathbf{f}_3^{int} = \int_V \mathbf{B}^T\mathbf{D}\epsilon_3 dV \quad \Rightarrow \quad {}^{0}_{3}\mathbf{h} = {}^{0}_{3}\boldsymbol{\mathcal{H}}({}^{0}\mathbf{r}) \tag{B.19}
$$

Axial stress is defined as $\sigma_a = \frac{f}{A}$. The axial load carried by the element is f and the cross-sectional area is A. Hence, the axial stress is:

$$
\sigma_a = \frac{{}^{0}_{i}h_{1_4} - {}^{0}_{i}h_{1_1}}{{}^{i}_{0}h}, \tag{B.20}
$$

where ${}^0_ih_{1_1}$ and ${}^0_ih_{1_4}$ are the axial forces applied on each column. The bending stress is defined as $\sigma_b = \frac{dM}{I}$. The bending moment carried by the element is M, the distance d is the free surface distance and the moment of inertia of the element is I. The bending stress is:

$$\sigma_{b_1} = \frac{d{}^0_ih_{1_3}}{{}^i_0h_2} \quad \text{and} \quad \sigma_{b_2} = \frac{d{}^0_ih_{1_6}}{{}^i_0h_2}, \tag{B.21}$$

where ${}^0_ih_{1_3}$ and ${}^0_ih_{1_6}$ are the bending moments applied on each end of the columns and d depends on the geometrical lay-out of the cross-section of the column. The free surface distance d is mathematically expressed as:

$$d = \begin{cases} {}^ix_1 - y_c & \text{for } \sigma_{b_i} \text{ at the top of the flange} \\ y_c & \text{for } \sigma_{b_i} \text{ at the bottom of the flange} \end{cases} \tag{B.22}$$

The normal stress in the elements is a summation of axial and bending stresses:

$${}^ir_1 = \sigma_a + \sigma_{b_1} \qquad {}^ir_2 = \sigma_a + \sigma_{b_2}. \tag{B.23}$$

Finally, the shear stress τ is defined as $\tau = \frac{VQ}{Ib}$. The first moment of inertia is Q, the downward nodal force is V and the thickness of the vertical plate of the column is b. In multi-level notation the shear stress τ is represented as:

$${}^ir_3 = \frac{{}^0_ih_{1_2}Q}{{}^i_0h_2{}^ix_4} \quad \text{and} \quad {}^ir_4 = \frac{{}^0_ih_{1_5}Q}{{}^i_0h_2{}^ix_4} \tag{B.24}$$

where ${}^0_ih_{1_2}$ and ${}^0_ih_{1_5}$ are the downward applied loads on both ends of the column and Q depends on the geometrical lay-out of the cross-section of the column. The first moment of inertia is mathematically expressed as:

$$Q = {}^ix_4{}^ix_2({}^ix_1 - yc - \frac{1}{2}{}^ix_2) + \frac{1}{2}{}^ix_4({}^ix_1 - yc - {}^ix_2)^2; \tag{B.25}$$

Summary

A unified approach towards decomposition and coordination for multi-level optimization

Complex systems, such as those encountered in aerospace engineering, can typically be considered as a hierarchy of individual elements. This hierarchy is reflected in the analysis techniques that are used to analyze the characteristics of the system as a whole. Consequently, a hierarchy of models is to be used, thus each accounting for different physical scales, components or disciplines. This thesis presents a framework for the multi-level optimization of such complex systems.

Decomposition of physical models and/or optimization problems of large complex structures is important in order to make the problem more manageable and/or to account for various types of physics and/or disciplines in quantifying the performance of the structure. The decomposition process involves identifying a hierarchy in the analysis model and/or optimization problem such that part of the analysis and/or optimization of the structure can be conducted individually, while taking into account the coupling with neighboring elements in the hierarchy. The result is an optimum that is optimal for the entire hierarchy of the complex structure.

A multi-level optimization problem is characterized via a hierarchy that is present. This hierarchy can be present due to:

- a coupling of physical models that each analyze and/or optimize physical properties at different scales;
- levels in the structural details of an analysis and/or optimization model;
- or a coupling between disciplines in the analysis and/or optimization problem.

A hierarchy consists of individual elements that consist of analysis and/or optimization problems that can be isolated up to some extent and communicate with their surroundings via mapping of physical responses and/or exchanging design variables that are shared.

Decomposition is accomplished via consistency constraints that are introduced to the physical model and/or the optimization problem formulation. There are two types of consistency constraint formulations:

1. equality consistency constraints;
2. relaxed consistency constraints.

These two formulations are further subdivided into hierarchic and non-hierarchic formulations. Hierarchic decomposition is further subdivided into:

- top-down decomposition;
- bottom-up decomposition.

Top-down decomposition involves elements that are higher in the multi-level hierarchy that prescribe the physical responses and/or design variables that are desired from the lower elements in the hierarchy. Bottom-up decomposition involves elements that are lower in the hierarchy and prescribe the physical responses and/or design variables that are desired from the higher elements in the hierarchy.

To show the different decomposition approaches a coupling circle was introduced that shows the coupling between two individual elements. This coupling circle distinguishes between the physical responses that are computed individually for each element and the mapping of the computed physical responses onto the neighboring element. Furthermore, it shows where consistency constraints are placed between coupled elements. Therefore, the coupling circle clearly illustrates the consequences of decomposition approach chosen on the individual element problems.

The multi-level optimization methods studied in this thesis handle coupling via two distinct approaches:

1. via equality-based consistency constraints: Optimization by Linear Decomposition (OLD), Concurrent SubSpace Optimization (CSSO), Bi-Level Integrated System Synthesis (BLISS) and Quasi-separable Subsystem Decomposition (QSD) use equality constraints to decompose the coupling between elements. Optimization by Linear Decomposition and Quasi-separable Subsystem Decomposition consider a top-down decomposition of the hierarchy. Concurrent SubSpace Optimization and Bi-Level Integrated System Synthesis consider a non-hierarchic decomposition of the hierarchy.
2. via relaxation-based consistency constraints: Collaborative Optimization (CO) and Analytical Target Cascading (ATC) use relaxation of the consistency constraints to decompose the coupling between elements. Collaborative Optimization considers a top-down decomposition of the hierarchy. Analytical Target Cascading considers a bottom-up decomposition of the hierarchy.

Coupling enters the optimization problem via copies of design variables when design variables are shared or coupling enters the optimization problem via coupling variables. Coupling variables represent mapped physical responses from neighboring elements that are mapped onto the current element. Instead of receiving this mapped physical response, a design variable representing this response is introduced to the optimization problem of the current element.

A problem matrix is used to illustrate how the decomposition of coupled physical responses and coupling in the design problem enters the optimization problem. Typically four different patterns can be distinguished:

1. a problem matrix where the coupling is present in the design constraints;

2. a problem matrix where the coupling is present between the design constraints and coupling is present in the objective function;

3. a problem matrix where design variables and physical responses of various elements are present in a few design constraints;

4. a problem matrix where no hierarchy can be identified. All the design functions depend on responses and design variables of all the hierarchical elements.

Based on the graphical representation of the multi-level optimization problem, changes can be made that change the optimization problem formulation of the individual elements. These changes are necessary to change the multi-level optimization problem such that it can be solved via a multi-level optimization method. These changes are:

- reformulating the design constraints as objectives via, e.g., envelope functions;

- introducing copies of design variables that are shared among elements;

- or neglecting coupling such that individual elements can be distinguished.

Via the problem matrix the multi-level optimization methods have been categorized as:

- Coupling is present in the design constraints: Optimization by Linear Decomposition (OLD) and Quasi-separable Subsystem Decomposition (QSD) are methods developed for this type of multi-level optimization problems.

- Coupling is present between the design constraints and the objective function: Collaborative Optimization (CO) and Analytical Target Cascading (ATC) are methods developed to handle this type of multi-level optimization problems.

- Design variables and physical responses of more than a single element are present in a small number of design constraints: Analytical Target Cascading (ATC) and Bi-Level Integrated System Synthesis (BLISS) are developed for this type of multi-level optimization problems.

- No hierarchy can be identified: Concurrent SubSpace Optimization (CSSO).

The generalized notation introduced in the present thesis and the application of all multi-level optimization methods studied in the present thesis applied to the same benchmark problem suggests that the multi-level optimization methods can be used for problems having a different pattern in the problem matrix as well. However, the efficiency of a method may be effected if it is applied to a different type of problem.

Coordination of an optimization problem that is decomposed into a hierarchy of multiple levels and/or multiple elements per level is important to steer the individual optimal designs such that an overall optimal structure is obtained.

Coordination depends on the type of decomposition chosen. Therefore, two types of coordination are distinguished:

1. equality-based or model-based coordination;

2. relaxation-based or goal-based coordination.

Equality-based coordination involves coordination of model data between individual elements of the hierarchy. The coordinator provides additional behavioral properties to neighboring elements when required. Relaxation-based coordination involves load balancing via the coordinator. Relaxation parameters are updated for each individual element without involving individual elements with the behavior of neighboring elements. Hence, relaxation parameters are coordinated.

Three types of decision making are distinguished:

1. separable decisions;

2. inseparable decisions;

3. inseparable and coupled decisions.

Separable decisions do not impose problems to the coordinator. Because they are separable the decision to update coupling data or to evaluate an individual element does not influence neighboring elements. In case inseparable decisions are present, the order in which the decisions to update coupling data and/or to evaluate the individual elements are made becomes important. This is the case where the order in which elements are solved may result in infeasible elements elsewhere.

To prevent infeasible elements in the hierarchy, the coordinator provides additional data on neighboring elements that are coupled. Elements that are solved first in the hierarchy receive additional data on the behavior of coupled neighboring elements. This additional data allows the element to avoid solutions that result in infeasible solutions elsewhere in the hierarchy. The additional data is obtained via:

- optimum sensitivity analysis of elements;

- switching shared design constraints on or off;

- linearization of constraints in neighboring elements with respect to responses coming from the current element;

- response surfaces that model the behavior of neighboring elements under changes made in the current element;

- or via a copy of the relaxed consistency constraints in case a relaxation-based decomposition is present.

Inseparable and coupled decisions require iterations and are typically the result of relaxation-based coordination techniques. Because the relaxation parameters are used to balance between solving the individual optimization problem and reaching consistency with neighboring elements in the hierarchy.

Coordination of individual elements is based on the type of decomposition used. The multi-level optimization methods studied in the present thesis are subdivided into:

- Equality-based coordination, these methods include: Optimization by Linear Decomposition (OLD), Quasi-separable Subsystem Decomposition (QSD), Concurrent SubSystem Optimization (CSSO) and Bi-Level Integrated System Synthesis (BLISS);

- Relaxation-based coordination, these methods include: Collaborative Optimization (CO) and Analytical Target Cascading (ATC).

The type of data required by the equality-based coordination method is important. This data is retrieved via the coordinator from a single element and represents an approximation to the elements behavior. Various approaches exist to construct approximations of the behavior of individual elements. The main approaches are:

- using objective and constraint function values of neighboring elements via extrapolation;

- linearize constraints of neighboring elements;

- using mapped physical responses from neighboring elements.

Numerical tests on equality-based multi-level optimization methods showed that:

- constructing a response surface of individual elements optima required via Quasi-separable Subsystem Decomposition was more challenging then constructing a response surface from the mapped physical response data required via Bi-Level Integrated System Synthesis. Using optimal objective or constraint function data of neighboring elements can be challenging because this data cannot be fitted via a smooth function. As observed by other researchers this optimal data is often discontinuous. However, introducing, e.g., barrier functions into the objective function overcomes the problem of discontinuity of the optimal functions.

- linearizing objective and/or constraints using sensitivity analysis requires tight move limits to maintain accurate sensitivity data. Furthermore, introducing trade-off and responsibility factors to switch on or switch off constraints that are shared introduces numerical difficulties.

The type of data that is send between elements for relaxation-based coordination methods depends on the relaxation formulation chosen. In general three distinct approaches are present. These approaches are:

- relaxation via a penalty function;
- relaxation via a Lagrangian function;
- relaxation via an Augmented Lagrangian function.

Updating strategies of the relaxation parameters depend on the relaxation formulation chosen. The relaxation parameters are iteratively updated to approach the optimal relaxation parameters; computed via analytical expressions; or approximated via a linearization technique that requires optimum sensitivity analysis.

The ability of a multi-level optimization algorithm to find the optimal point is a difficult measurement since the optimal point should be know beforehand. However, it is often used as an argument to choose one multi-level method over another multi-level method. Especially if a mathematical validation of the method is present this argument is explicitly pointed out in favor over other methods for which no mathematical justification is present.

A numerical study on equality-based coordination methods and relaxation-based coordination methods showed that for the test case considered equality-based methods find the exact optimum whereas the relaxation-based methods converged to designs close to the optimal point. Collaborative Optimization converged to a non-optimal point. This is due to the definition of the penalty function embedded in the Collaborative Optimization problem formulation. Properly introducing the correct penalty function and associated update technique for the relaxation parameters converged towards the optimum.

Numerical tests conducted on relaxation-based coordination methods showed that convergence rates of these algorithms are slow and additional effort to improve these characteristics is required before these algorithms can be used in a competitive environment. To improve the convergence characteristics of coordination methods so-called multi-level coordination methods are promising. These methods are developed from methods that solve large systems of equations. A drawback of these methods is that the mathematical assumptions on which these methods rely are more stringent then the so-called bi-level coordination methods that are commonly used in multi-level optimization. This thesis showed that multi-level optimization problems that are formulated for bi-level optimization problems can be transformed with little effort into problems that can be solved via a multi-level coordination technique.

Relaxation-based coordination methods provide minimal insight into the behavior of individual optimization problems and the influence of coupling. Essentially, an initial design and a final result are meaningful designs. The intermediate steps cannot be used since they are infeasible.

The multi-level optimization methods that were studied are formulated for specific optimization problems. To deal with different multi-level optimization techniques in an efficient manner, a general framework was formulated that can handle the majority of multi-level optimization problems using a variety of multi-level optimization techniques. The proposed framework combines generic aspects of multi-level optimization methods into a single computational framework.

The multi-level framework was tested on the test problem developed in the present thesis and benchmark problems taken from literature via an Analytical Target Cascading with Augmented Lagrangian relaxation implementation. Flexibility of the framework was demonstrated utilizing two different decomposition formulations on each of the presented multi-level optimization problems. A top-down decomposition and a non-hierarchic decomposition of the multi-level optimization problems was constructed.

The computational effort required to solve the top-down hierarchic decomposition and non-hierarchic decomposition varied between the test cases. The smaller test cases showed faster convergence with hierarchic top-down decomposition and the test cases with many design constraints converged faster when using non-hierarchic decomposition.

Two coordination strategies were considered. A sequential coordination process and a parallel coordination process. The parallel process exhibited more computational effort then the sequential coordination process. However, it is expected that if the problem size increases the computational effort becomes close to that of sequential coordination. Therefore, the actual time it takes to execute an entire multi-level optimization in parallel is smaller as compared to the same problem executed sequentially.

Different convergence criteria were applied to test algorithm performance. The convergence criteria were changed together with parameters that coordinate the update of relaxation parameters according to Block Coordinate Descent, Inexact and Alternating Descent. A choice between one of the three approaches is based on placing computational effort at the individual elements or at the coordinator and the accuracy of the final solution. In all test cases considered, the Alternating Descent exhibited best computational characteristics.

Findings in this study show that design of complex structures that exhibit multi-scale behavior in the view of multi-scale mechanics via methods that are proposed in literature is challenging.

- Equality-based methods suffer from the fact that for each parameter that is send from one element to a neighboring element either sensitivity information is required or a response surface is constructed. Constructing accurate sensitivity information is not straightforward and can become a numerically expensive procedure on it's own. Fitting a response surface through elements that output more then ten parameters to a single neighboring element (a small number if a detailed local finite element model communicates physical responses and/or shared design variables with a global finite element model) is challenging.

- Relaxation-based methods suffer from poor numerical convergence characteristics. The linear convergence rate of these methods and the fact that intermediate designs during the multi-level optimization cannot be used makes these methods challenging to use in a commercial environment.

Multi-level optimization and multi-disciplinary design optimization methods are useful when the amount of data that is exchanged between individual elements remains small and models are weakly coupled. Weak coupling in the sense that large errors in the solution of a neighboring element do not occur when mapping the data from one element onto a neighboring element. The number of parameters that can be send from one element to a neighboring element is limited. Therefore, one may question the effort that is spend on finding an accurate solution within individual elements. Due to the limited amount of parameters that can be mapped accuracy of coupling may be lost.

Albert Jan de Wit

Samenvatting

Een uniforme werkwijze voor ontkoppelen en coördineren ten behoeve van multi-level optimalisatie

Complexe systemen, zoals men die tegenkomt in de luchtvaart -en ruimtevaarttechniek, kunnen worden beschouwd als een hiërarchie van individuele elementen. Deze hiërarchie uit zich in de analyse technieken die gebruikt worden om de eigenschappen van het systeem als geheel te bepalen. Een hiërarchie van modellen wordt gebruikt, waarbij elk model verschillende fysische schalen, componenten of disciplines beschouwd. Dit proefschrift presenteert een raamwerk voor de multi-level optimalisatie van dergelijke complexe systemen.

Het ontkoppelen van fysische modellen en/of optimalisatie problemen van grote complexe constructies is belangrijk, omdat het ontkoppelde probleem beter beheersbaar wordt, en/of omdat verschillende fysica en/of disciplines te beschouwen zijn die de prestaties van het totale systeem bepalen. Het ontkoppelingsproces bestaat uit het identificeren van een hiërarchie in het analyse model, en/of het optimalisatie probleem, zodat een deel van de analyse en/of optimalisatie van de constructie individueel uitgevoerd kan worden. Waarbij men tijdens de individuele uitvoering van de optimalisatie ook rekening houdt met de koppeling met naburige elementen. Het resultaat is een optimaal ontwerp, dat optimaal is voor de gehele hiërarchie van de complexe constructie.

Een multi-level optimalisatie probleem wordt gekenmerkt door een hiërarchie die aanwezig is. Deze hiërarchie bestaat uit:

- een koppeling van fysische modellen waarvan analyse, en/of optimalisatie, verschillende fysische eigenschappen op verschillende fysische schalen beschouwd;
- niveaus in componenten van een constructie via analyse, en/of optimalisatie,

modellen;

- een koppeling tussen disciplines in het analyse, en/of optimalisatie, probleem.

Een hiërarchie bevat individuele elementen bestaande uit analyse en/of optimalisatie problemen. Deze individuele elementen kunnen tot op zekere hoogte geïsoleerd worden en communiceren met de omgeving via het projecteren van fysische responsies, en/of het uitwisselen van gedeelde ontwerp variabelen.

Ontkoppelen wordt bereikt via consistentie restricties die geïntroduceerd worden in het fysische model, en/of het optimalisatie probleem. Er zijn twee typen consistentie restricties:

1. overeenkomst consistentie restricties;

2. relaxatie consistentie restricties.

Deze twee formuleringen worden vervolgens onderverdeeld in hiërarchische en niet-hiërarchische ontkoppelingsformuleringen. Hiërarchische ontkoppeling is verder onder te verdelen in:

1. top-down;

2. bottom-up.

Top-down ontkoppeling impliceert dat, elementen die boven aan de hiërarchie staan, de fysische responsies en/of gedeelde ontwerpvariabelen voorschrijven, welke verlangd worden van de lager in de hiërarchie gelegen elementen. Bottom-up ontkoppeling impliceert dat, elementen die onderaan de hiërarchie staan, de fysische responsies en/of gedeelde ontwerpvariabelen voorschrijven, welke verlangd worden van de hoger in de hiërarchie gelegen elementen.

Om de verschillende ontkoppelingstechnieken te illustreren is een koppelingscirkel geïntroduceerd die de koppeling tussen twee elementen laat zien. De koppelingscirkel maakt onderscheid tussen de fysische responsies die individueel uitgerekend worden, en responsies die geprojecteerd worden op naburige elementen. Ook maakt de koppelingscirkel zichtbaar waar consistentie restricties geplaatst worden. Hierdoor wordt via de koppelingscirkel duidelijk wat de consequenties zijn, van de gekozen ontkoppelingstechniek, op de formulering van analyse en/of optimalisatie problemen van individuele elementen van de hiërarchie.

De multi-level optimalisatie methoden die bestudeerd worden in dit proefschrift behandelen koppeling via twee verschillende methoden:

1. via overeenkomst consistentie restricties: “Optimization by Linear Decomposition” (OLD), “Concurrent SubSpace Optimization” (CSSO), “Bi-Level Integrated System Synthesis” (BLISS) en “Quasi-separable Subsystem Decomposition” (QSD) gebruiken overeenkomst consistentie restricties om de koppeling tussen elementen te ontkoppelen. OLD en QSD beschouwen een top-down ontkoppeling van de hiërarchie. CSSO en BLISS beschouwen een niet-hiërarchische ontkoppeling van de hiërarchie.

2. via relaxatie consistentie restricties: “Collaborative Optimization” (CO) en “Analytical Target Cascading” (ATC) gebruiken relaxatie consistentie restricties om de koppelingen tussen de elementen te ontkoppelen. CO beschouwt een top-down ontkoppeling van de hiërarchie. ATC beschouwt een bottom-up ontkoppeling van de hiërarchie.

Koppelingen worden in het optimalisatie probleem beschouwd via kopieën van ontwerp variabelen wanneer ontwerp variabelen gedeeld worden, of via koppelingsvariabelen. Koppelingsvariabelen representeren geprojecteerde fysische responsies van naburige elementen op het huidige element. In plaats van het direct ontvangen van deze geprojecteerde responsies, wordt een ontwerpvariabele geïntroduceerd die de geprojecteerde responsie representeert in het optimalisatie probleem van het huidige element.

Een probleem matrix wordt gebruikt om te illustreren waar zich gekoppelde fysische responsies en koppelingen in het ontwerp probleem bevinden. Er zijn vier typisch verschillende patronen die in de probleem matrix kunnen worden geïdentificeerd:

1. een probleem matrix waar de koppeling aanwezig is tussen de ontwerpvariabelen;
2. een probleem matrix waar de koppeling aanwezig is tussen de ontwerp restricties;
3. een probleem matrix waar ontwerp variabelen en fysische responsies van verschillende elementen aanwezig zijn in enkele ontwerp restricties;
4. een probleem matrix waar geen hiërarchie kan worden geïdentificeerd. Alle ontwerp functies zijn afhankelijk van fysische responsies en ontwerp variabelen van alle hiërarchische elementen.

Aan de hand van de grafische illustratie van het multi-level optimalisatie probleem kunnen aanpassingen aan de individuele optimalisatie formuleringen op element niveau gemaakt worden. Deze aanpassingen zijn nodig om een specifieke multi-level optimalisatie toe te kunnen passen. Deze aanpassingen bestaan in het algemeen uit:

1. het herformuleren van ontwerp restricties naar het doel van het ontwerp. Dit is bijvoorbeeld mogelijk via zogenaamde omhulsel functies;
2. het introduceren van kopieën van ontwerp variabelen die gedeeld worden tussen elementen;
3. het verwaarlozen van koppelingen, zodat individuele elementen geïdentificeerd kunnen worden.

De resulterende probleem matrix bepaalt welke coördinatie strategie het meest geschikt is, om het ontkoppelde optimalisatie probleem op te kunnen lossen.

Via de probleem matrix worden de multi-level optimalisatie methoden, die in dit werk bestudeerd zijn, als volgt verdeeld:

- koppeling is aanwezig in de ontwerp restricties: “Optimization by Linear Decomposition” (OLD) en “Quasi-separable Subsystem Decomposition” (QSD) zijn methoden ontwikkeld voor dit type multi-level optimalisatie problemen;

- koppeling is aanwezig tussen de ontwerp restricties en de doel functie: "Collaborative Optimization" (CO) en "Analytical Target Cascading" (ATC) zijn methoden ontwikkeld om dit type multi-level optimalisatie problemen op te lossen;
- ontwerp variabelen en fysische responsies van meer dan één enkel element zijn aanwezig in een klein aantal ontwerp restricties: "Analytical Target Cascading" (ATC) en "Bi-Level Integrated System Synthesis" (BLISS) zijn methoden ontwikkeld om dit type multi-level optimalisatie problemen op te lossen;
- er kan geen hiërarchie geïdentificeerd worden: "Concurrent SubSystem Optimization" (CSSO).

De uniforme notatie die is geïntroduceerd in het huidige proefschrift en de toepassing van alle multi-level optimalisatie problemen die bestudeerd zijn in het huidige proefschrift en toegepast op eenzelfde testprobleem, suggereert dat, de multi-level optimalisatie methoden ook toegepast kunnen worden op problemen, die een ander patroon in de probleem matrix hebben, dan waarvoor ze zijn afgeleid. Echter, de efficiëntie van een methode kan nadelig uitpakken, wanneer het toegepast wordt op een ander type probleem, dan waarvoor de methode geïntroduceerd is.

Coördinatie van een optimalisatie probleem dat ontkoppeld is, in een hiërarchie van meerdere niveaus en/of meerdere elementen per niveau, is belangrijk om de individuele optimale ontwerpen te sturen, zodanig, dat een voor de gehele constructie optimaal ontwerp bereikt wordt.

Coördinatie hangt af van de gekozen ontkoppelingsmethode. Om die reden kunnen twee typen coördinatie onderscheiden worden:

1. overeenkomst of model gebaseerde coördinatie;
2. relaxatie of doel gebaseerde coördinatie.

Overeenkomst of model gebaseerde coördinatie houdt in dat, coördinatie plaats vindt via model data tussen de individuele elementen van de hiërarchie. De coördinator levert eigenschappen van naburige elementen aan het huidige element wanneer hierom gevraagd wordt. Bij relaxatie of doel gebaseerde coördinatie worden voor elk individueel element relaxatie parameters bepaald. De coördinatie vindt plaats via parameters die samenhangen met de relaxatie formulering van de consistentie restrictie.

Er worden drie typen beslissingen onderscheiden:

1. onafhankelijke beslissingen;
2. afhankelijke beslissingen;
3. afhankelijke en gekoppelde beslissingen.

Onafhankelijke beslissingen vormen geen probleem voor de coördinatie. Doordat deze beslissingen onafhankelijk van elkaar zijn, kan de beslissing om gekoppelde data te vernieuwen, of een element te evalueren, uitgevoerd worden zonder dat dit van invloed is op naburige elementen. Indien afhankelijke beslissingen aanwezig zijn, dan is de volgorde waarin gekoppelde data verzonden wordt, en/of een individueel element geëvalueerd wordt, wel belangrijk. Deze situaties doen zich voor wanneer de volgorde waarin elementen worden opgelost kan resulteren in naburige elementen die in conflict raken met individuele ontwerp restricties.

Om te voorkomen dat naburige gekoppelde elementen in conflict komen met ontwerp restricties, stelt de coördinator extra informatie over deze elementen beschikbaar aan het element dat geëvalueerd wordt. Elementen die eerder worden geëvalueerd in de hiërarchie ontvangen hierdoor extra informatie over de naburige gekoppelde elementen. De extra informatie stelt het element in staat om ontwerpen die een conflict veroorzaken in naburige elementen te mijden. De extra informatie over naburige elementen kan verkregen worden via:

- een gevoeligheidsanalyse van het optimum van een element;
- het in -en uitschakelen van ontwerp restricties die gedeeld worden tussen verschillende elementen;
- linearisering van ontwerp restrictie vergelijkingen van naburige elementen, ten opzichte van responsies van het huidige element;
- responsie oppervlakken, die het gedrag van naburige elementen modelleren ten opzichte van veranderingen in het huidige element;
- kopieën van de vergelijkingen die relaxatie van de consistentie restricties beschrijven in het geval van relaxatie gebaseerde consistentie ontkoppeling.

Afhankelijke en gekoppelde beslissingen zijn aangewezen op iteraties waarbij de gehele hiërarchie meerdere malen doorlopen wordt. Meestal zijn dit soort beslissingen ontstaan door het gebruik van relaxatie gebaseerde coördinatie technieken. De parameters, die samenhangen met de relaxatie formulering van de consistentie restricties, worden gebruikt om een balans te vinden tussen het optimaliseren van de individuele doelfunctie van ieder individueel element en het bereiken van consistentie met naburige elementen in de hiërarchie.

Coördinatie van individuele elementen is gebaseerd op het type ontkoppeling dat gebruikt wordt. De multi-level optimalisatie methoden die in dit proefschrift bestudeerd zijn, zijn onderverdeeld in:

- coördinatie gebaseerd op een overeenkomst formulering van de consistentie restricties, deze methoden zijn: "Optimization by Linear Decomposition" (OLD), Quasi-separable Subsystem Decomposition (QSD), "Concurrent SubSystem Optimization" (CSSO) en "Bi-Level Integrated Subsystem Decomposition" (BLISS);
- coördinatie gebaseerd op een relaxatie formulering van de consistentie restricties, deze methoden zijn: "Collaborative Optimization" (CO) en "Analytical Target Cascading" (ATC).

In het geval dat, coördinatie gebaseerd is op een overeenkomst formulering van de consistentie restricties, is het type data dat benodigd is belangrijk. Dit type data wordt verkregen via de coördinator. De data bestaat uit een benadering van het gedrag van naburige gekoppelde elementen. Er zijn verschillende manieren beschikbaar om benaderingen van de eigenschappen van naburige elementen te creëren. De belangrijkste zijn:

- het gebruik van functie waarden van doel en restrictie functies van naburige elementen, via extrapolatie;
- linearisering van restrictie functies van naburige elementen ten behoeve van gebruik in het huidige element;
- het gebruik van geprojecteerde fysische responsies van naburige elementen.

Numerieke tests met behulp van multi-level optimalisatie methoden, gebaseerd op een overeenkomst formulering van de consistentie restricties, lieten zien dat:

1. het bouwen van een responsie oppervlak van het optimum van individuele elementen via “Quasi-separable Subsystem Decomposition” (QSD), een grotere uitdaging bleek, dan het bouwen van een responsie oppervlak door geprojecteerde fysische responsies, die nodig zijn voor “Bi-Level Integrated System Synthesis” (BLISS). Het gebruik van optimale waarden bij verschillende geprojecteerde responsies en waarden voor gedeelde ontwerpvariabelen van doel en restrictie functies kan een uitdaging zijn, doordat deze data niet met behulp van een gladde functie benaderd kan worden. In overeenstemming met wat in de literatuur gemeld wordt, is deze data vaak discontinu. Echter, via het introduceren van bijvoorbeeld barrière functies in de doelfuncties van naburige elementen, kan het probleem van discontinuïteit van deze benaderingsfuncties verholpen worden;
2. linearisering van doel en/of restrictie functies via gevoeligheidsanalyse vraagt kleine bewegingslimieten, zodat accurate gevoeligheidsinformatie verkregen wordt. Het introduceren van wisselwerking en verantwoordelijkheidsvariabelen om restrictievergelijkingen aan of uit te zetten introduceert numerieke moeilijkheden.

Het type data dat tussen elementen wordt gestuurd bij het gebruik van coördinatie gebaseerd op een relaxatie formulering van de consistentie restricties, hangt af van de relaxatie formulering van de consistentie vergelijking. In het algemeen zijn er drie verschillende methoden beschikbaar. Deze methoden zijn:

1. een relaxatie formulering gebaseerd op een straf functie;
2. een relaxatie formulering gebaseerd op een “Lagrangian” functie;
3. een relaxatie formulering gebaseerd op een “Augmented Lagrangian” functie.

Strategieën voor het vernieuwen van relaxatie parameters hangen af van de relaxatie formulering waarvoor gekozen is. De relaxatie parameters worden iteratief vernieuwd, totdat zij de optimale relaxatie parameters benaderen. Deze kunnen berekend worden via analytische expressies of via linearisering welke afhangt van een gevoeligheidsanalyse van het optimum.

De eigenschap van een multi-level algoritme om een optimaal punt te vinden is moeilijk vast te stellen, omdat het optimale punt van te voren bepaald moet zijn. Deze eigenschap wordt echter vaak gebruikt als argument om voor een bepaald multi-level algoritme te kiezen. Met name wanneer een geldige wiskundige onderbouwing van de methode aanwezig is, wordt een wiskundige onderbouwing expliciet als geldige keuze genoemd, dit ten nadele van methoden waarvoor geen geldige wiskundige onderbouwing bekend is.

Een numerieke studie naar overeenkomst gebaseerde coördinatie methoden en relaxatie gebaseerde coördinatie methoden liet zien dat, voor het behandelde test probleem, overeenkomst gebaseerde coördinatie methoden de exacte optimale oplossing vinden. Relaxatie gebaseerde methoden convergeren naar ontwerpen dichtbij het optimale punt. "Collaborative Optimization" (CO) convergeerde echter naar een niet optimaal punt. Het slechte resultaat van CO is te wijten aan de gekozen straf functie. Het wijzigen van deze straf functie (aan de hand van voorstellen uit de literatuur) in een correcte variant, met bijbehorende verversingstechniek voor de relaxatie parameters, resulteert wel in het vinden van de juiste oplossing.

Numerieke tests die uitgevoerd werden op relaxatie gebaseerde coördinatie methoden lieten zien dat, de convergentie snelheid van dit type algoritme langzaam is, en dat extra inspanningen nodig zijn om deze eigenschappen te verbeteren, voordat deze algoritmen in een commerciële omgeving toepasbaar worden. Om de convergentie eigenschappen van coördinatie methoden te verbeteren zijn zogenaamde multi-level coördinatie methoden interessant. Deze methoden zijn afgeleid van methoden die grote systemen van vergelijkingen oplossen. Een nadeel van deze methoden is echter dat de wiskundige aannamen waarop deze methoden gebaseerd zijn, veel strikter zijn, dan de zogenaamde twee-niveau coördinatie methoden, die gebruikelijk zijn bij multi-level optimalisatie. Dit proefschrift toont aan dat problemen die geformuleerd zijn voor twee-niveau coördinatie, eenvoudig zijn om te schrijven naar problemen die opgelost kunnen worden via een multi-level coördinatie techniek.

Relaxatie gebaseerde coördinatie methoden geven weinig inzicht in het gedrag van individuele optimalisatie problemen en de invloed van koppelingen. Samengevat zijn alleen het initiële en het uiteindelijke ontwerp zinnige resultaten. De tussenliggende stappen van het optimalisatie proces zijn onbruikbaar, aangezien het ontwerpen betreft welke niet aan de ontwerp consistentie restricties voldoen.

De multi-level methoden die in dit werk bestudeerd zijn, zijn geformuleerd met bepaalde toepassingen in het achterhoofd. Om met verschillende multi-level optimalisatie methoden, op een efficiënte manier, te kunnen werken, is een gegeneraliseerd raamwerk geformuleerd. Dit raamwerk kan het overgrote deel aan multi-level optimalisatie problemen oplossen via verschillende multi-level optimalisatie technieken. Het voorgestelde raamwerk combineert generieke aspecten van de verschillende multi-level optimalisatie methoden in een enkel multi-level raamwerk.

Het multi-level raamwerk is getest op het twee staaf elementen probleem, dat in dit proefschrift gedefinieerd is, en op standaard problemen die overgenomen zijn uit de literatuur. Het testen is uitgevoerd met behulp van "Analytical Target Cascading" (ATC) waarbij de relaxatie van de consistentie restricties is uitgevoerd met behulp van een "Augmented Lagrangian" relaxatie implementatie. Flexibiliteit van het raamwerk is gedemonstreerd via twee verschillende ontkoppelingsformuleringen op elk van de multi-level optimalisatie problemen. Op elk van de test problemen is een top-down ontkoppeling en een niet hiërarchische ontkoppeling van de multi-level optimalisatie toegepast.

De numerieke kosten, die nodig waren om de top-down hiërarchische ontkoppeling en de niet hiërarchische ontkoppeling op te lossen, varieerden per test probleem. De kleinere test problemen convergeerden sneller, wanneer gebruik werd gemaakt van hiërarchische top-down ontkoppeling, en de test problemen met veel ontwerp restricties convergeerden sneller, wanneer een niet hiërarchische ontkoppeling werd toegepast.

Twee coördinatie strategieën zijn toegepast. Een sequentieel coördinatie proces en een parallel coördinatie proces. Het parallelle proces bleek duurder in termen van numerieke kosten in vergelijking tot het sequentiële proces. De verwachting is echter dat, wanneer de grootte van het probleem toeneemt, de kosten van het parallelle proces, die van het sequentiële proces benaderen. De tijd die nodig is om een gehele multi-level optimalisatie parallel uit te voeren zal korter zijn, dan wanneer hetzelfde proces sequentieel wordt doorlopen.

Verschillende convergentie criteria zijn toegepast om de prestatie van het geïmplementeerde algoritme te testen. De convergentie criteria zijn aangepast, samen met parameters die het vernieuwen van relaxatie parameters bepalen. Drie verschillende methoden zijn gebruikt: "Block Coordinate Descent" (BCD); "InExact" (InE); en "Alternating Descent" (AD). De keuze tussen één van de drie methoden is gebaseerd op het toewijzen van numerieke kosten aan de individuele elementen, of aan de coördinator en de uiteindelijke nauwkeurigheid van de oplossing. Voor alle behandelde test problemen was AD het goedkoopste in termen van numerieke kosten.

De resultaten van dit onderzoek laten zien dat het ontwerpen van complexe constructies die gedrag vertonen dat gekoppeld is over verschillende niveaus via methoden die voorgesteld zijn in de literatuur een uitdaging is.

- Overeenkomst gebaseerde methoden hebben het nadeel dat, voor elke parameter die van een individueel element naar een naburig element gestuurd wordt, gevoeligheidsinformatie nodig is, of, dat een responsie oppervlak gecreërd moet worden. Het vergaren van accurate gevoeligheidsinformatie is alleszins triviaal en kan op zichzelf al een kostbare procedure worden. Het passen van een responsie oppervlak, door de parameters die van een enkel individueel element naar een naburig element gestuurd worden, vormt een echte uitdaging, wanneer er tien (een klein aantal als het hier een gedetailleerd eindige elementen model betreft dat fysische eigenschappen en/of gedeelde ontwerpvariabelen stuurt naar een globaal eindige elementen model op hoger niveau) of meer parameters gepast moeten worden.

- Relaxatie gebaseerde methoden hebben het nadeel dat zij langzame convergentie eigenschappen bezitten. De lineaire convergentie snelheid van deze methoden, en het feit dat tussenliggende ontwerpen gedurende het multi-level optimalisatie proces onbruikbaar zijn, maakt deze methoden moeilijk te gebruiken in een commerciële omgeving.

Multi-level optimalisatie en multi-disciplinaire optimalisatie methoden zijn toepasbaar, wanneer de hoeveelheid data die tussen individuele elementen wordt gestuurd klein blijft en de elementen onderling zwak gekoppeld zijn. Met zwakke koppeling wordt hier bedoeld dat, er geen grote numerieke fouten ontstaan in naburige elementen, wanneer de data van één element naar het naburige element wordt vertaald. Het aantal parameters dat verstuurd kan worden, van het ene element naar het andere element, is beperkt. Hierdoor kan men zich afvragen of het werk dat verricht wordt, om een nauwkeurige oplossing voor elk individueel element afzonderlijk te verkrijgen, gerechtvaardigd is. Doordat er maar een klein aantal parameters geprojecteerd kan worden van het ene naar het andere element in de hiërarchie, kan de nauwkeurigheid van de oplossing verslechteren.

Albert Jan de Wit

Acknowledgements

By making something absolutely clear, somebody will be confused.
(Murphy's Law)

First of all, I thank my wife Julia, for all the love and support she gave me these years. I also thank my parents and brother for encouraging me to study at the university and being there whenever I needed help. I also thank my friends in Delft for a memorable studying time in Delft. Furthermore, I would like to thank my supervisor Fred van Keulen. He gave me the freedom to develop my own ideas and to participate in an international group of researchers. I thank my colleagues from the Structural Optimization and Computational Mechanics group in Delft for the nice working atmosphere and the endless coffee table discussions: Jan, Hans, Marianne, Jacqueline, Marten Jan, Gerard, Teun, Andriy, Sham, Peterjan, Chiara, Matthijs, Gih-Keong, Caspar, Daniel, Hammed, Nico, Qian-Li and Saputra. I also would like to thank the members of the STW-DFG project Multi-scale Methods in Computational Mechanics for the nice discussions during our half annual meetings.

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Albert Jan de Wit
Rotterdam, 2009

Curriculum Vitae

Albert Jan de Wit was born in Doetinchem on November 23, 1979. After one year his parents moved to Assendelft and there he attended primary school "'t Kofschip"(1984 - 1992). At the age of twelve his family moved to Medemblik where he attended secondary school (Atheneum) at "Wieringherlant"in Wieringerwerf (1992 - 1998).

After finishing secondary school in 1998 he moved to Delft to study Aerospace Engineering at Delft University of Technology. During his studies he participated in the IAESTE exchange program during the summer of 2002. He worked for a period of 3 months in Belfast at "the Queen's University Belfast", faculty of Aerospace Engineering. He graduated in 2004 in Delft specialized in aerospace materials and computational mechanics.

He continued his studies as a PhD student at the faculty of Mechanical, Maritime and Materials Engineering at Delft University of Technology as a member of the chair "Structural Optimization and Computational Mechanics". During his PhD he participated in the Multi-scale Methods in Computational Mechanics program, a collaboration between Dutch and German universities. As part of this program he worked for a period of 4 months at Stuttgart University, faculty of Civil Engineering in 2005.

Albert is currently employed as an R&D engineer at the Dutch National Aerospace Laboratory - NLR in Amsterdam.

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Stellingen

behorende bij het proefschrift

Een uniforme werkwijze voor ontkoppelen en coördineren ten behoeve van multi-level optimalisatie

Albert Jan de Wit

1. Huidige multi-level optimalisatietechnieken zijn dusdanig kostbaar in termen van rekentijd en modellering dat het multi-level optimaliseren van constructies nog niet tot een logische keuze voor het numerieke optimalisatieproces behoort.
   [dit proefschrift]

2. Bij het toepassen van multi-level optimalisatie met behulp van "Bi-Level Integrated System Synthesis" is niet de multi-level expertise van belang maar de "kunst" van het creëren van responsie-oppervlakken.
   [dit proefschrift]

3. Gecompliceerde ontwerpproblemen worden met behulp van multi-level optimalisatie niet makkelijker.
   [dit proefschrift]

4. Bij het ontwikkelen van multi-level optimalisatietechnieken voor constructies waarbij gebruik moet worden gemaakt van interactie tussen verschillende eindige-elementen modellen is het formuleren van het raakvlak tussen individuele deelproblemen onderbelicht.
   [dit proefschrift]

5. De huidige druk op onderzoekers om te publiceren, dan wel beloning naar rato van publicaties, zal uiteindelijk leiden tot een wetenschapscrisis.

6. De kniklast van een constructie is geen doel, maar een randvoorwaarde voor constructie optimalisatieproblemen.

7. Toegankelijkheid van onderwijs staat de kwaliteit van onderwijs in de weg.

8. Doelstellingen tot het reduceren van het verbruik van fossiele brandstoffen zijn overbodig.

9. Een hiërarchische kantoor structuur waarbij hoogleraren zo ver mogelijk van hun studenten zitten belemmert de kennisoverdracht en werpt een onzichtbare barrière tussen "kennis" generaties op.

10. Gezien het feit dat bij mooi weer heel werkend Nederland op het strand ligt, staat Nederland met de globale opwarming van de aarde nog een aardige crisis te wachten.

Deze stellingen worden verdedigbaar geacht en zijn als zodanig goedgekeurd door de promotor, Prof. dr. ir. A. van Keulen.

Propositions

accompanying the thesis

A unified approach towards decomposition and coordination for multi-level optimization

Albert Jan de Wit

1. Current multi-level optimization methods are so costly in terms of computation time and modeling effort that multi-level optimization of structures is not a logical choice for the process of numerical optimization.
   [this thesis]
2. Conducting multi-level optimization via Bi-Level Integrated System Synthesis does not require multi-level expertise but the expertise of creating response surfaces.
   [this thesis]
3. Complicated design problems do not become easier via multi-level optimization.
   [this thesis]
4. During the development of multi-level optimization techniques applied to structures where interaction between different finite element models has to be accounted for, not enough attention has been given to formulating the interface.
   [this thesis]
5. Current pressure on researchers to publish, or being rewarded according to the amount of publications, will ultimately result in a science crisis.
6. The buckling load of a structure is not an objective but a constraint of structural optimization problems.
7. Access to education stands in the way of quality of education.
8. Objectives to reduce the use of fossil fuels are superfluous.
9. A hierarchical office structure where professors are seated far from their students stands in the way of knowledge transfer and poses an invisible barrier between knowledge generations.
10. Given the fact that during nice weather Dutch workers spend their time lying on the beach, the Netherlands awaits a major crisis given earth's prospect of global warming.

These propositions are considered defendable and as such have been approved by the supervisor, Prof. dr. ir. A. van Keulen.