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Discrete-to-continuum limits of semilinear stochastic evolution equations in Banach spaces

Yves van Gennip^a, Jonas Latz^b, Joshua Willems^{a,*}

^a Delft Institute of Applied Mathematics, Delft University of Technology, the Netherlands

^b Department of Mathematics, University of Manchester, United Kingdom

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Abstract

We study the convergence of semilinear parabolic stochastic evolution equations, posed on a sequence of Banach spaces approximating a limiting space and driven by additive white noise projected onto the former spaces. Under appropriate uniformity and convergence conditions on the linear operators, nonlinear drifts and initial data, we establish convergence of the associated mild solution processes when lifted to a common state space. Our framework is applied to the case where the limiting problem is a stochastic partial differential equation whose linear part is a generalized Whittle–Matérn operator on a manifold $\mathcal{M}$, discretized by a sequence of graphs constructed from a (random) point cloud. In this setting we obtain various discrete-to-continuum convergence results for solutions lifted to $L^q(\mathcal{M})$ for $q \in [2, \infty]$, one of which recovers the L^∞ -convergence of a finite-difference discretization of certain (fractional) stochastic Allen–Cahn equations.

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* Corresponding author.

E-mail addresses: y.vangennip@tudelft.nl (Y. van Gennip), jonas.latz@manchester.ac.uk (J. Latz), j.willems@tudelft.nl (J. Willems).

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1. Introduction

1.1. Background and motivation

We establish discrete-to-continuum limits of stochastic evolution equations of the form (1.1), i.e., semilinear parabolic stochastic partial differential equations (SPDEs) driven by Gaussian white noise. Such SPDEs of evolution play an important role in the modeling of physical and other systems, such as fluid dynamics [5,27,32,63], quantum optics [13], phase separation [19], diffusion in random media [45,54], and population dynamics [88]. Given their significance, there has been a considerable interest in the analysis and numerical analysis of SPDEs; see the introductory textbooks [61] and [62], respectively.

We consider the convergence of a sequence of abstract continuous-time equations, each posed on a different Banach space in order to model the approximation of an evolution SPDE by equations that are continuous in time and discrete in space. This framework covers a typical setting where the spatial domains are finite graphs and the limiting differential operator in space is approximated by the corresponding graphical variants. If the finite graphs approximate an underlying manifold, then the graphical differential operators are related to a finite-difference approximation of the differential operator. As an example, we study a class of semilinear SPDEs whose linear part is given by a fractional-order elliptic differential operator on a manifold (known as a *Whittle–Matérn operator*), discretized by a sequence of graphs constructed from a (possibly randomly sampled) cloud of points. *Linear* equations of this type have previously been studied in the context of statistics and machine learning [66,72,73]. By verifying in detail that the hypotheses of our abstract framework are satisfied in this situation, we establish the discrete-to-continuum convergence of this scheme. Although the main advantage of our general results is their applicability to highly unstructured discretizations, we additionally show that they recover the L^∞ -convergence of finite-difference discretizations of the fractional stochastic Allen–Cahn equation on the one-dimensional flat torus. To further illustrate the significance of results of this form, we now discuss a few examples of semidiscrete SPDEs as well as their continuum limits.

Semidiscrete models appear frequently in the numerical analysis of (S)PDEs of evolution; since the challenges of spatial, temporal and spatiotemporal discretization are different, these settings are often analyzed separately. This leads to thorough studies of (S)PDEs that are discretized in space but not in time. In the context of SPDEs of evolution, we refer to [8,41,55] for examples.

Stochastic PDEs on graphs also appear naturally as models in the physical sciences, e.g., for interacting particle systems [18] or the representation of disordered media [45]. In the former case, the continuum limit represents the large particle limit in the interacting particle system.

In the data science literature, (S)PDEs on graphs have recently gained popularity as semi-supervised learning techniques. In a semi-supervised learning problem we are given a set of labeled features as well as a set of unlabeled features, and the goal is to use the former features to recover the labels of the latter. Features are, for example, images, text, or voice recordings; corresponding labels may be descriptors of the content of the images, the author of the text, or a transcript of the voice recording, respectively. Given an appropriate similarity measure on the space of features, an edge-weighted graph can be constructed in which nodes representing similar features are connected by highly weighted edges. The unknown labels can then be estimated by space-discretized PDEs on this graph, as in [6,10,80,90]. The PDEs often describe gradient flows that minimize a variational functional. Stochastic PDEs appear in this setting if, in addition to finding an estimate for the labels, the uncertainty in the labels is to be quantified as well [7,

72]. The SPDEs of evolution here either form the basis of Markov chain Monte Carlo sampling algorithms [16,43,44,70] or of a randomized global optimization scheme [14,15] for the solution of the variational problem in the deterministic setting. In this semi-supervised learning setting, discrete-to-continuum limits are of interest because they establish the consistency of the models in the large-data limit. For deterministic PDEs, the literature has grown to encompass pointwise limits of operators, as in [46], Γ -limits of the functionals that underlie the dynamics [34,59,77,79,86], and more recently, discrete-to-continuum limits for the dynamics themselves [28,31,37,48,60,64,85]. For a more in-depth overview of the literature of discrete-to-continuum limits, we refer to [81].

1.2. Main results

We will now summarize the abstract setting and main discrete-to-continuum convergence results from Sections 4–6, which will already be applied in Section 3 to the Whittle–Matérn and stochastic fractional Allen–Cahn equations described in Section 1.1.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $T \in (0, \infty)$ (called the *time horizon*), we consider a sequence of semilinear parabolic stochastic evolution equations

$$\begin{cases} dX_n(t) = -A_n X_n(t)dt + F_n(t, X_n(t))dt + dW_n(t), & t \in (0, T], \\ X_n(0) = \xi_n, \end{cases} \quad (1.1)$$

indexed by $n \in \overline{\mathbb{N}} := \{1, 2, \dots\} \cup \{\infty\}$. This problem will be rigorously formulated as a stochastic differential equation taking values in a real Banach space E_n (or a smaller embedded space $B_n \hookrightarrow E_n$) called the *state space*. For example, E_n can be the Lebesgue space $L^q(\mathcal{D}_n)$ (with $q \in [2, \infty)$) on some bounded spatial domain $\mathcal{D}_n$, and B_n can be $C(\mathcal{D}_n)$ the space of continuous functions. Since the Banach-space-theoretic properties of E_n and B_n are typically very different, each requires separate assumptions and arguments to handle the noise term.

In general, we assume that the terms appearing in (1.1) are as follows:

- $-A_n$ is the generator of a bounded analytic semigroup of bounded linear operators on E_n or B_n ;
- $u_n \mapsto F_n(\omega, t, u_n)$ is a possibly random and nonlinear drift operator on E_n or B_n for all $(\omega, t) \in \Omega \times [0, T]$;
- $(W_n(t))_{t \geq 0}$ is the projection, onto an appropriate subspace H_n , of an H -valued cylindrical Wiener process $(W(t))_{t \geq 0}$ for a real and separable Hilbert space H , e.g., $H_n = L^2(\mathcal{D}_n)$ and $H = L^2(\mathcal{D})$ (more details are specified in Theorem 1.1 below);
- ξ_n is a possibly random initial datum with values in E_n or B_n .

The precise assumptions (in particular, whether the operators and initial data are E_n -valued or B_n -valued) vary throughout Sections 3–6; an overview is provided in Table 1. Depending on the setting, the *mild solutions* to (1.1) are either well defined on the whole of $[0, T]$ almost surely, or cease to exist at a time $t < T$ with nonzero probability; such solutions are said to be *global* or *local*, respectively.

The aim of this work is to establish conditions on the data of (1.1) under which the corresponding solutions $(X_n)_{n \in \mathbb{N}}$ converge to X_∞ as $n \rightarrow \infty$. In order to compare processes which take their values in different Banach spaces, we need to assume that each of the sequences $(E_n)_{n \in \mathbb{N}}$,

$(H_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ embeds uniformly into a common space, which they approximate in some appropriate sense as $n \rightarrow \infty$. In particular, we shall assume that they share a common sequence of (linear) *lifting operators* $(\Lambda_n)_{n \in \mathbb{N}}$ mapping each E_n (resp. H_n, B_n) in a uniformly-bounded manner into the corresponding limiting spaces, and *projection operators* $(\Pi_n)_{n \in \mathbb{N}}$ which are left-inverses to the respective lifting operators. That is, each sequence satisfies Definition 2.1 below with common lifting and projection operators.

With these operators, we can compare the $(E_n$ - or B_n -valued) solution processes X_n by instead considering convergence of the lifted processes $\tilde{X}_n := \Lambda_n X_n$ to X_∞ as $n \rightarrow \infty$, which we call *discrete-to-continuum convergence*. Moreover, they allow us to formulate assumptions under which this occurs, by imposing conditions on the lifted resolvents $\tilde{R}_n := \Lambda_n (A_n + \text{Id}_n)^{-1} \Pi_n$ of the linear operators A_n , the lifted drift operators $\tilde{F}_n(\omega, t, u) := \Lambda_n F_n(t, \omega, \Pi_n u)$, and the lifted initial data $\tilde{\xi}_n := \Lambda_n \xi_n$. Roughly speaking, we assume that

- $\tilde{F}_n \rightarrow F_\infty$ ‘pointwise’ (see Assumption 5.5 or 6.8);
- $\tilde{R}_n \rightarrow R_\infty$ ‘pointwise’ and there exists a small enough $\beta \in [0, \frac{1}{2})$ such that the fractional powers $\tilde{R}_n^\beta$ converge to R_∞^β in an appropriate operator norm (see Assumption 4.5 or 6.3);
- $\tilde{\xi}_n \rightarrow \xi_\infty$ in $L^p(\Omega; E_\infty)$ or $L^p(\Omega; \tilde{B})$ for some $p \in [1, \infty)$ (see Assumption 5.6 or 6.6).

Again we refer to Table 1 for an overview of the different settings and types of solutions, with references to the precise formulations of the corresponding assumptions; the setting in the first row (i.e., Section 3.2) covers the (fractional) stochastic Allen–Cahn equations announced in Section 1.1, see Example 3.8 below. The following theorem is a summary of the discrete-to-continuum approximation results for solutions to the abstract equations (1.1) in these respective settings.

Theorem 1.1 (*Discrete-to-continuum convergence—summarized*). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T \in (0, \infty)$ be a terminal time. Consider equations (1.1), where the state spaces, linear operators, drift operators and initial data are as in one of the rows of Table 1. Let $p \in [1, \infty)$ be the stochastic integrability of the initial data and let $W_n := \Pi_n W$, where $(W(t))_{t \geq 0}$ is an H -valued cylindrical Wiener process. For all $n \in \mathbb{N}$, there exists a unique (local or global, see Table 1) mild solution X_n to (1.1), and the lifted solution processes $\tilde{X}_n := \Lambda_n X_n$ satisfy the following:*

- (i) *If the solutions are global and $p > 1$, then for all $p^- \in [1, p)$ we have*

$$\tilde{X}_n \rightarrow X_\infty \quad \text{as } n \rightarrow \infty$$

in $L^{p^-}(\Omega; C([0, T]; E_\infty))$ (resp. in $L^{p^-}(\Omega; C([0, T]; \tilde{B}))$). In the (semi)linear setting with globally Lipschitz drifts of linear growth, the above is also true on the endpoint $p^- := p$ for any $p \in [1, \infty)$.

- (ii) *If the solutions are local, with associated explosion times $\sigma_n: \Omega \rightarrow (0, T]$ (precisely defined in (5.9) below), then we have*

$$\tilde{X}_n \mathbf{1}_{[0, \sigma_n \wedge \sigma_n)} \rightarrow X_\infty \mathbf{1}_{[0, \sigma_\infty)} \quad \text{as } n \rightarrow \infty$$

in $L^0(\Omega \times [0, T]; E_\infty)$ (resp. in $L^0(\Omega \times [0, T]; \tilde{B})$), where L^0 indicates convergence in measure.

Table 1

Overview of the types of SPDEs considered in the different (sub)sections comprising this work. The row in which an assumption appears for the first time also indicates the (sub)section where its definition can be found.

Sec.	Description	Assumptions	Sol. type
§3.2	graph-based approximation of Whittle–Matérn operators on a manifold	• $A_n := [\mathcal{L}_n^{\tau,K}]^s$ (Whittle–Matérn operators) • $[F_n(t, u)](x) := f_n(t, u(x))$ (Nemytskii drift) • Assumption 3.7 (on the functions $(f_n)_{n \in \overline{\mathbb{N}}}$)	global
§4	E_n -valued linear	• Assumptions 4.1, 4.2 and 4.5 (linear operators) • $F_n := 0$ and $\xi_n := 0$	global
§5.1	E_n -valued semilinear; globally Lipschitz drifts of linear growth	• Assumptions 4.1, 4.2 and 4.5 (linear operators) • Assumptions 5.1 and 5.5 (drift operators) • Assumption 5.6 (initial data)	global
§5.2	E_n -valued semilinear; locally Lipschitz and locally bounded drifts	• Assumptions 4.1, 4.2 and 4.5 (linear operators) • Assumptions 5.5 and 5.10 (drift operators) • Assumption 5.6 (initial data)	local
§6.1	B_n -valued semilinear; globally Lipschitz drifts of linear growth	• Assumptions 6.1–6.4 with $\theta + 2\beta < 1$ (lin. ops.) • Assumptions 6.7 and 6.8 (drift operators) • Assumption 6.6 (initial data)	global
§6.2	B_n -valued semilinear; locally Lipschitz and locally bounded drifts	• Assumptions 6.1–6.4 with $\theta + 2\beta < 1$ (lin. ops.) • Assumptions 6.8 and 6.14 (drift operators) • Assumption 6.6 (initial data)	local
§6.3	B_n -valued semilinear; dissipative drifts	• Assumptions 6.1–6.4 with $\theta + 2\beta < 1$ (lin. ops.) • Assumptions 6.8 and 6.16 (drift operators) • Assumption 6.6 (initial data)	global

The full convergence statement for each setting is given in the corresponding part of Sections 4–6. To be precise, Theorem 1.1 is comprised of the following results, in order of appearance: Theorem 3.10, Proposition 4.8, Theorems 5.7 and 5.13, Proposition 6.5, Theorems 6.13 and 6.15 and Corollary 6.19.

1.3. Contributions

The abstract discrete-to-continuum approximation theorems for stochastic semilinear parabolic evolution equations driven by additive cylindrical Wiener noise—summarized in Theorem 1.1 and proved in Sections 4–6—complement the results from [56,57], which establish the *continuous dependence on the coefficients* of semilinear equations driven by multiplicative noise in a state space with unconditional martingale differences (UMD), i.e., convergence in the case where $E_n = E$ (and $B_n = B$) for all $n \in \overline{\mathbb{N}}$. Given the motivating applications and our aim to provide a self-contained exposition of the proofs, we make the simplifying assumptions that the UMD spaces $(E_n)_{n \in \overline{\mathbb{N}}}$ have Rademacher type 2 (which was also assumed for E in [57] but not in [56]) and the noise is additive.

Under these conditions, we provide a direct proof of convergence of the E_n -valued stochastic convolutions solving the linear parts of (1.1) using a Da Prato–Kwapień–Zabczyk factorization argument [20] and a discrete-to-continuum analog of the Trotter–Kato approximation theorem [52, Theorem 2.1], see Proposition 4.8. We extend it to the semilinear E_n -valued settings described in Table 1 by adapting the arguments from [56, Sections 3 and 4] and [57, Sub-

section 3.1] to incorporate the discrete-to-continuum projection and lifting operators, yielding Theorems 5.7 and 5.13, respectively. In order to state and prove the analogous Theorems 6.13 and 6.15 for the B_n -valued settings, we impose a uniform ultracontractivity condition on the semigroups which replaces the restriction in [57, Section 3] that the fractional domain spaces $\dot{E}_n^\alpha := D((\text{Id}_n + A_n)^{\alpha/2})$ also coincide for all $n \in \mathbb{N}$.

Theorem 3.10, regarding the graph discretization of equations whose linear operators are of generalized Whittle–Matérn type on a manifold $\mathcal{M}$, extends analogous convergence results for linear equations on a spatial domain (cf. [72, Theorem 4.2] and [72, Theorem 7] in $L^2(\mathcal{M})$ and $L^\infty(\mathcal{M})$, respectively) to spatiotemporal and semilinear equations. Like the cited theorems, its proof relies on recent spectral convergence results for graph Laplacians (see [11] and [12] for convergence of eigenfunctions in L^2 and L^∞ , respectively), which we use to verify that these SPDEs fit into the abstract framework from Sections 4–6.

1.4. Outline

The remainder of this article is structured as follows. In Section 2, we establish some notational conventions and collect preliminaries regarding the (deterministic) discrete-to-continuum Trotter–Kato approximation theorem. We demonstrate in Section 3 how the results summarized by Theorem 1.1 can be applied to graph discretizations of stochastic parabolic evolution equations whose linear part is a generalized Whittle–Matérn operator on a manifold. In Section 4, we consider the *linear* E_n -valued version of (1.1), whose solutions are also known as infinite-dimensional Ornstein–Uhlenbeck processes. These results are extended in Section 5 to allow for *semilinear* E_n -valued drift operators under (local or global) Lipschitz continuity and boundedness assumptions. In Section 6 we first treat the analogous results in the semilinear B_n -valued setting, and then establish global well-posedness and convergence for dissipative drifts. Finally, in Section 7 we discuss some potential directions for further research. This work is complemented by three appendices: Appendix A consists of postponed proofs of the results from Section 3. Appendices B and C are concerned with fractional parabolic integration and (uniformly) sectorial linear operators, respectively. See Fig. 1 for a schematic overview of the relations between Sections 3–6 and the appendices.

2. Preliminaries

2.1. Notation

Table 2 can be used as a reference for some of the most important notation used throughout this work.

The Cartesian product $\prod_{j \in \mathcal{I}} B_j$ of an indexed family of sets $(B_j)_{j \in \mathcal{I}}$ is comprised of all functions $f: \mathcal{I} \rightarrow \bigcup_{j \in \mathcal{I}} B_j$ satisfying $f(j) \in B_j$ for all $j \in \mathcal{I}$. Given two sets $\mathcal{P}, \mathcal{Q}$ and maps $\mathcal{F}, \mathcal{G}: \mathcal{P} \times \mathcal{Q} \rightarrow \mathbb{R}$, we write $\mathcal{F}(p, q) \lesssim_q \mathcal{G}(p, q)$ to indicate the existence of some $C: \mathcal{Q} \rightarrow [0, \infty)$ such that $\mathcal{F}(p, q) \leq C(q) \mathcal{G}(p, q)$ for all $(p, q) \in \mathcal{P} \times \mathcal{Q}$. We write $\mathcal{F}(p, q) \approx_q \mathcal{G}(p, q)$ if both $\mathcal{F}(p, q) \lesssim_q \mathcal{G}(p, q)$ and $\mathcal{G}(p, q) \lesssim_q \mathcal{F}(p, q)$ hold.

All normed spaces will be considered over the real or complex scalar field. Results concerning spectra and complex interpolation are naturally formulated for complex spaces; if the space is real, then we implicitly apply them to the complexification and subsequently restrict back to the original space.

Table 2

A selection of notation which is used throughout this work.

Elementary sets and operations		‘Discrete-to-continuum’ spaces	
$\mathbb{N}$	positive integers	$(E_n)_{n \in \mathbb{N}}, \tilde{E}$	UMD-type-2 Banach spaces from Assumption 4.1
$\mathbb{N}_0$	nonnegative integers	$(H_n)_{n \in \mathbb{N}}, \tilde{H}$	Hilbert spaces from Assumption 4.1
$\overline{\mathbb{N}}$	$\mathbb{N} \cup \{\infty\}$	$(B_n)_{n \in \mathbb{N}}, \tilde{B}$	Banach spaces from Assumption 6.1
Id_D	identity map on a set D		
$\mathbf{1}_{D_0}$	indicator map on $D_0 \subseteq D$		
$s \wedge t$	minimum of $s, t \in \mathbb{R}$		
Bounded linear operators		Projection and lifting	
H, K	separable Hilbert spaces	Π_n	projection operator from E_n (resp. H_n or B_n) to $\tilde{E}$ (resp. $\tilde{H}$ or $\tilde{B}$)
E, F	arbitrary Banach spaces	Λ_n	lifting operator from $\tilde{E}$ (resp. $\tilde{H}$ or $\tilde{B}$) to E_n (resp. H_n or B_n)
$\langle \cdot, \cdot \rangle_H$	inner product of H	$\tilde{T}_n$	lifted version $\Lambda_n T_n \Pi_n$ on $\tilde{E}$ (resp. $\tilde{H}$ or $\tilde{B}$) of operator T_n on E_n (resp. H_n or B_n)
$\ \cdot \ _E$	norm of E	$\tilde{Y}_n$	lifted version $\Lambda_n Y_n$ on $\tilde{E}$ (resp. $\tilde{H}$ or $\tilde{B}$) of process Y_n on E_n (resp. H_n or B_n)
E^*	dual space of E	M_Π	$\sup_{n \in \mathbb{N}} \ \Pi_n\ _{\mathcal{L}(\tilde{E}; E_n)}$
$\mathcal{L}(E; F)$	bounded linear operators (from E to F)	$\tilde{M}_\Pi$	$\sup_{n \in \mathbb{N}} \ \Pi_n\ _{\mathcal{L}(\tilde{B}; B_n)}$
$\mathcal{L}(E)$	abbreviation of $\mathcal{L}(E; E)$	M_Λ	$\sup_{n \in \mathbb{N}} \ \Lambda_n\ _{\mathcal{L}(E_n; \tilde{E})}$
$\gamma(H; E)$	γ -radonifying operators (see Section 4.1)	$\tilde{M}_\Lambda$	$\sup_{n \in \mathbb{N}} \ \Lambda_n\ _{\mathcal{L}(B_n; \tilde{B})}$
$\mathcal{L}_2(H; K)$	Hilbert–Schmidt operators		
Function spaces		Linear operators in evolution equations	
$C(K; E)$	continuous functions from a compact space K to E	A_n	linear operator on E_n with domain $D(A_n)$
$C(K)$	abbreviation of $C(K; \mathbb{R})$	S_n	semigroup generated by $-A_n$
$L^p(S; E)$	Bochner space of p -integrable functions from a measure space $(S, \mathcal{A}, \nu)$ to E	M_S	uniform-boundedness constant of $(S_n)_{n \in \mathbb{N}}$ in $(E_n)_{n \in \mathbb{N}}$, see (2.1)
$L^p(S)$	Lebesgue space $L^p(S; \mathbb{R})$	$\tilde{M}_S$	uniform-boundedness constant of $(S_n)_{n \in \mathbb{N}}$ in $(B_n)_{n \in \mathbb{N}}$, see (6.4)
		$\rho(A_n)$	resolvent set of A_n
		R_n^β	$(A_n + \text{Id}_n)^{-\beta}$
		$\mathcal{I}_{A_n}^s$	fractional parabolic integration operator, see Appendix B
Graph discretization		Stochastic evolution equations	
$\mathcal{M}$	manifold from Assumption 3.1	Q_n	covariance $\Pi_n \Pi_n^* \in \mathcal{L}(H_n)$
$\mathbb{T}^m$	m -dimensional flat torus	dW_n	H_n -valued Q_n -cylindrical Wiener noise on $(\Omega, \mathcal{F}, \mathbb{P})$
$d_{\mathcal{M}}$	geodesic metric on $\mathcal{M}$	W_{A_n}	stochastic convolution, see (4.5)
μ	volume measure on $\mathcal{M}$	ξ_n	initial datum
$\mathcal{M}_n$	point cloud $(x_n^{(j)})_{j=1}^n \subset \mathcal{M}$	F_n	drift operator
$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$	probability space of random point cloud from Example 3.3	L_F, C_F	Lipschitz and growth constants of E_n -valued drifts, see Assumption 5.1
μ_n	empirical measure on $\mathcal{M}_n$	$\tilde{L}_F, \tilde{C}_F$	Lipschitz and growth constants of B_n -valued drifts, see Assumption 6.7
T_n	transport map from $\mathcal{M}$ to $\mathcal{M}_n$	$L_F^{(r)}, C_{F,0}$	local Lipschitz and growth constants of E_n -valued drifts, see Assumption 5.10
ε_n	$\sup_{x \in \mathcal{M}} d_{\mathcal{M}}(x, T_n(x))$	f_n	real-valued drift coefficient function
h_n	graph connectivity length scale, see (3.7)	$\tilde{L}_f, \tilde{C}_f$	Lipschitz and growth constants of $(f_n)_{n \in \mathbb{N}}$, see Assumption 3.7(i)
$\mathcal{L}_n^{\tau, \kappa}$	(discretized) Whittle–Matérn operator with coefficient functions $\tau, \kappa: \mathcal{M} \rightarrow [0, \infty)$		
$(\lambda_n^{(j)})_{j=1}^n$	eigenvalues of $\mathcal{L}_n^{\tau, \kappa}$		
$(\psi_n^{(j)})_{j=1}^n$	$L^2(\mathcal{M}_n)$ -normalized eigenfunctions of $\mathcal{L}_n^{\tau, \kappa}$		
$M_{\psi, \infty}$	uniform L^∞ -bound of the eigenfunctions, see Assumption 3.9		
$M_{S, q}$	uniform-ultracontractivity constant, see (3.18)		

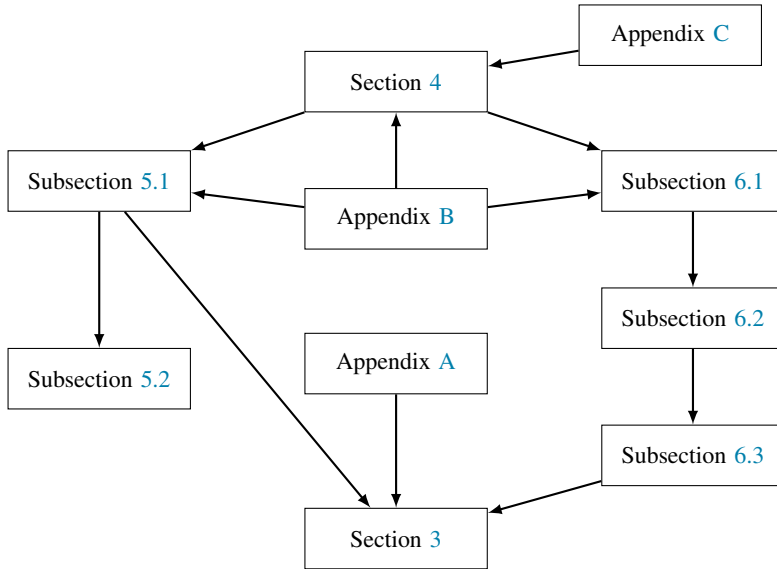


Fig. 1. Relations between the sections comprising the main part of this article and the appendix. Arrows indicate when the results of one section are applied in another.

The space $\mathcal{L}(E; F)$ of bounded linear operators from E to F shall be equipped with the norm $\|T\|_{\mathcal{L}(E; F)} := \sup_{\|x\|_E=1} \|Tx\|_F$. We call $T \in \mathcal{L}(E; F)$ a contraction if $\|T\|_{\mathcal{L}(E; F)} \leq 1$; in particular, the inequality need not be strict. An operator $T \in \mathcal{L}(H)$ is said to be positive definite if there exists $\theta \in (0, \infty)$ such that $\langle Tx, x \rangle_H \geq \theta \|x\|_H^2$ for all $x \in H$, and nonnegative definite if $\langle Tx, x \rangle_H \geq 0$. Given $T, S \in \mathcal{L}(H; K)$, we set $\langle T, S \rangle_{\mathcal{L}_2(H; K)} := \sum_{j=1}^{\infty} \langle Te_j, Se_j \rangle_K$, where $(e_j)_{j \in \mathbb{N}}$ is any orthonormal basis of H ; the space $\mathcal{L}_2(H; K)$ of Hilbert–Schmidt operators consists of those T for which $\|T\|_{\mathcal{L}_2(H; K)}^2 := \langle T, T \rangle_{\mathcal{L}_2(H; K)} < \infty$.

Given a measure space $(S, \mathcal{A}, \nu)$, a function $f: S \rightarrow E$ is said to be strongly measurable if it can be approximated ν -a.e. by simple functions. For $p \in [1, \infty]$, let $L^p(S; E) := L^p(S, \mathcal{A}, \nu; E)$ denote the Bochner space of (equivalence classes of) strongly measurable and p -integrable functions from S to E , with norm

$$\|f\|_{L^p(S; E)} := \begin{cases} (\int_S \|f(s)\|_E^p d\nu(s))^{1/p}, & \text{if } p \in [1, \infty); \\ \text{ess sup}_{s \in S} \|f(s)\|_E, & \text{if } p = \infty. \end{cases}$$

Sub-intervals $J \subseteq \mathbb{R}$ are equipped with the Lebesgue σ -algebra and measure. The Banach space of (bounded) continuous functions $u: J \rightarrow E$, endowed with the supremum norm, is denoted by $C(J; E)$.

The meaning of the tensor symbol $\otimes$ will depend on the context: Given a mapping $\Phi: (a, b) \rightarrow \mathcal{L}(E; F)$ and some $x \in E$, we define the function $\Phi \otimes x: (a, b) \rightarrow F$ by $[\Phi \otimes x](t) := \Phi(t)x$. If instead an $h \in H$ is given, we define $h \otimes x \in \mathcal{L}(H; E)$ to be the rank-one operator $[h \otimes x](u) := \langle h, u \rangle_H x$. The space of all (finite) linear combinations of such operators is denoted by $H \otimes E$. We define the convolution $\Psi * f: [0, T] \rightarrow F$ of the functions $\Psi: [0, T] \rightarrow \mathcal{L}(E; F)$ and $f: [0, T] \rightarrow E$ by $[\Psi * f](t) := \int_0^t \Psi(t-s)f(s)ds$.

2.2. Discrete-to-continuum Trotter–Kato approximation theorem

We encode the discrete-to-continuum setting in the following way:

Definition 2.1. Let $(E_n, \|\cdot\|_{E_n})_{n \in \mathbb{N}}$ and $(\tilde{E}, \|\cdot\|_{\tilde{E}})$ be (real or complex) Banach spaces and suppose that E_∞ is a closed linear subspace of $\tilde{E}$ (possibly $E_\infty = \tilde{E}$). We say that *the sequence $(E_n)_{n \in \mathbb{N}}$ is a discrete-to-continuum approximation of E_∞ (in $\tilde{E}$, if $E_\infty \neq \tilde{E}$)* if there exist operators $\Pi_n \in \mathcal{L}(\tilde{E}; E_n)$ and $\Lambda_n \in \mathcal{L}(E_n; \tilde{E})$ for all $n \in \mathbb{N}$ which satisfy

- (i) $M_\Pi := \sup_{n \in \mathbb{N}} \|\Pi_n\|_{\mathcal{L}(\tilde{E}; E_n)} < \infty$ and $M_\Lambda := \sup_{n \in \mathbb{N}} \|\Lambda_n\|_{\mathcal{L}(E_n; \tilde{E})} < \infty$;
- (ii) $\Lambda_n \Pi_n x \rightarrow x$ in $\tilde{E}$ as $n \rightarrow \infty$ for all $x \in E_\infty$;
- (iii) $\Pi_n \Lambda_n = \text{Id}_{E_n}$ for all $n \in \mathbb{N}$.

In this case, we additionally denote $\Pi_\infty = \Lambda_\infty := \text{Id}_{\tilde{E}}$ for convenience. The operators $(\Lambda_n)_{n \in \mathbb{N}}$ and $(\Pi_n)_{n \in \mathbb{N}}$ are called *lifting* and *projection operators*, respectively.

Note that parts (i) and (iii) of the above definition together imply that the lifting operators are continuous embeddings $\Lambda_n: E_n \hookrightarrow \tilde{E}$. In applications, they will typically be nested (in the sense that $E_n \hookrightarrow E_{n+1}$ for all $n \in \mathbb{N}$) and finite-dimensional, but neither of these assumptions is strictly necessary in the abstract theory. Moreover, we will often have $\tilde{E} = E_\infty$, but not always; see Section 6.

Now we consider the following sequence of linear operators on $(E_n)_{n \in \mathbb{N}}$:

Definition 2.2. For all $n \in \mathbb{N}$, let $-A_n: D(A_n) \subseteq E_n \rightarrow E_n$ be a linear operator generating a strongly continuous semigroup $(S_n(t))_{t \geq 0} \subseteq \mathcal{L}(E_n)$ on a Banach space $(E_n, \|\cdot\|_{E_n})$, meaning that $S_n(0) = \text{Id}_{E_n}$, $S_n(t+s) = S_n(t)S_n(s)$ for all $t, s \geq 0$, and $S_n(t)x \rightarrow x$ as $t \downarrow 0$ for all $x \in E_n$. If there exist $M_S \in [1, \infty)$, $w \in \mathbb{R}$ such that

$$\|S_n(t)\|_{\mathcal{L}(E_n)} \leq M_S e^{-wt} \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, \infty), \quad (2.1)$$

then we call $(S_n)_{n \in \mathbb{N}}$ a *uniformly bounded sequence of semigroups on $(E_n)_{n \in \mathbb{N}}$ with uniform growth exponent w* .

For any linear operator A on a Banach space E , we say that $\lambda \in \mathbb{C}$ belongs to the *resolvent set* $\rho(A)$ of A if the *resolvent (operator)* $R(\lambda, A) := (\lambda \text{Id}_E - A)^{-1}$ exists in $\mathcal{L}(E)$. Given sequences of operators on spaces as above, the Trotter–Kato approximation theorem (see, e.g., [30, Chapter III, Theorem 4.8]) establishes a link between the strong convergence of resolvents and uniform convergence of the semigroups on compact subintervals of $[0, \infty)$. The following discrete-to-continuum analog of this result was proved by Ito and Kappel [52, Theorem 2.1]:

Theorem 2.3 (*Discrete-to-continuum Trotter–Kato approximation*). *Let $(E_n)_{n \in \mathbb{N}}$ and $\tilde{E}$ be as in Definition 2.1, and let $(-A_n)_{n \in \mathbb{N}}$ generate a uniformly bounded sequence $(S_n)_{n \in \mathbb{N}}$ of semigroups on $(E_n)_{n \in \mathbb{N}}$ with growth exponent $w \in \mathbb{R}$ as in Definition 2.2. The following statements are equivalent:*

(a) *There exists a $\lambda \in \bigcap_{n \in \mathbb{N}} \rho(A_n)$ such that, for every $x \in E_\infty$,*

$$\Lambda_n R(\lambda, A_n) \Pi_n x \rightarrow R(\lambda, A_\infty) x \quad \text{in } \tilde{E} \quad \text{as } n \rightarrow \infty.$$

(b) *For all $x \in E_\infty$ and $T \in (0, \infty)$ it holds that*

$$\Lambda_n S_n \Pi_n \otimes x \rightarrow S_\infty \otimes x \quad \text{in } C([0, T]; \tilde{E}) \quad \text{as } n \rightarrow \infty.$$

If (a) holds for some $\lambda \in \bigcap_{n \in \mathbb{N}} \rho(A_n)$ (or, equivalently, if (b) holds), then (a) holds in fact for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda < w$.

3. Graph-based semilinear stochastic evolution equations with Whittle–Matérn linear operators

Before developing the general discrete-to-continuum convergence results summarized by Theorem 1.1 in the upcoming sections, in this section we demonstrate how they can be applied to the particular case of equations whose linear parts are graph discretizations of a certain linear operator on a manifold $\mathcal{M}$. We will consider the following semilinear SPDEs, indexed by $n \in \mathbb{N}$:

$$\begin{cases} du_n(t, x) + [\mathcal{L}_n^{\tau, \kappa}]^s u_n(t, x) dt = f_n(t, u_n(t, x)) dt + dW_n(t, x), & (t, x) \in (0, T] \times \mathcal{M}_n, \\ u_n(0, x) = \xi_n(x). \end{cases} \quad (3.1)$$

The components of these equations will be defined in more detail in the next subsections; we provide a brief overview here to put those upcoming definitions in context. The limiting problem ($n = \infty$) involves a *generalized Whittle–Matérn operator* on $\mathcal{M}$, which is a fractional power $[\mathcal{L}_\infty^{\tau, \kappa}]^s$ ($s \in (0, \infty)$) of an elliptic second-order differential operator $\mathcal{L}_\infty^{\tau, \kappa}$. Every $\mathcal{M}_n$ with $n \in \mathbb{N}$ represents a graph discretization (see Settings 3.2 and 3.3) of the manifold $\mathcal{M}$ (see Assumption 3.1), on which a corresponding discretization $\mathcal{L}_n^{\tau, \kappa}$ of $\mathcal{L}_\infty^{\tau, \kappa}$ can be defined (see equation (3.8)). These discretizations converge to the limiting operator in some appropriate sense. Each $(W_n(t))_{t \in [0, T]}$ is an $L^2(\mathcal{M}_n)$ -valued cylindrical Wiener process with respect to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, with finite time horizon $T \in (0, \infty)$. Its formal time derivative dW_n represents spatiotemporal Gaussian white noise on $[0, T] \times \mathcal{M}_n$. Finally, $f_n: \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are nonlinearities (also called *drift coefficients* in the stochastic setting), and $\xi_n: \Omega \times \mathcal{M}_n \rightarrow \mathbb{R}$ are the initial data.

We are interested in the convergence of the discrete solutions u_n on $\mathcal{M}_n$ to the limiting solution u_∞ on $\mathcal{M}$. In the spatial and linear case, such convergence results have been proven in [72, 73]. We also mention the work [66], in which the statistical properties of the spatiotemporal linear equation were investigated for fixed $n \in \mathbb{N}$.

3.1. Geometric graphs and generalized Whittle–Matérn operators

Assumption 3.1 (*Manifold assumption*). Given $m, d \in \mathbb{N}$, suppose that $\mathcal{M}$ is an m -dimensional smooth, connected, compact Riemannian manifold without boundary, embedded smoothly and isometrically into $\mathbb{R}^d$. Let μ and $d_{\mathcal{M}}$ denote the normalized volume measure and geodesic metric on $\mathcal{M}$, respectively. $\blacklozenge$

For each $n \in \mathbb{N}$, let a point cloud $\mathcal{M}_n := (x_n^{(j)})_{j=1}^n \subseteq \mathcal{M}$ be given. We suppose that $\mathcal{M}$ can be partitioned into n regions of mass $1/n$, which can be transported to the corresponding n points comprising $\mathcal{M}_n$, in such a way that the maximal geodetic displacement tends to zero as $n \rightarrow \infty$. More precisely, we assume that there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of transport maps $T_n: \mathcal{M} \rightarrow \mathcal{M}_n$ such that

$$\mu_n = T_{n\#} \mu \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad (3.2)$$

$$\varepsilon_n := \sup_{x \in \mathcal{M}} d_{\mathcal{M}}(x, T_n(x)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.3)$$

Here, $\mu_n := \frac{1}{n} \sum_{j=1}^n \delta_{x_n^{(j)}}$ is the empirical measure on $\mathcal{M}$ associated to $\mathcal{M}_n$, and $T_{n\#} \mu$ denotes the pushforward measure $T_{n\#} \mu(B) := \mu(\{T_n \in B\})$ on $\mathcal{M}_n$. Two different examples in which this assumption is satisfied are presented in Settings 3.2 and 3.3 below.

Given $u_n: \mathcal{M}_n \rightarrow \mathbb{R}$ for $n \in \mathbb{N}$, these transport maps enable us to define the functions $\Lambda_n u_n: \mathcal{M} \rightarrow \mathbb{R}$ and $\Pi_n u: \mathcal{M}_n \rightarrow \mathbb{R}$ by setting

$$\Lambda_n u_n(x) := u_n(T_n(x)) \quad \text{and} \quad \Pi_n u(x_n^{(j)}) := n \int_{V_n^{(j)}} u(x) d\mu(x), \quad (3.4)$$

respectively, for all $x \in \mathcal{M}$ and $j \in \{1, \dots, n\}$, where $V_n^{(j)} := \{T_n = x_n^{(j)}\} \subseteq \mathcal{M}$.

It turns out that the operations defined in (3.4) satisfy Definition 2.1 with respect to the following function spaces: Given $q \in [1, \infty]$ and $n \in \mathbb{N}$, we set $E_n := L^q(\mathcal{M}_n) := L^q(\mathcal{M}, \mu_n)$, as well as $\tilde{E} := L^q(\mathcal{M})$ and

$$E_\infty := \begin{cases} L^q(\mathcal{M}), & \text{if } q \in [1, \infty); \\ C(\mathcal{M}), & \text{if } q = \infty. \end{cases}$$

Later on, we will need $q \in [2, \infty)$, so that E_n is a UMD-type-2 space for use in stochastic integration (see Section 4.1), but the statements here hold for all $q \in [1, \infty]$.

For these spaces, Definition 2.1(i) is satisfied with $M_\Lambda = 1$ and $M_\Pi \leq 1$. Indeed, the fact that Λ_n is an isometry follows from (3.2) if $q \in [1, \infty)$, whereas for $q = \infty$ we see directly from the definition that

$$\|\Lambda_n u_n\|_{L^\infty(\mathcal{M})} = \sup_{x \in \mathcal{M}} |u_n(T_n(x))| = \max_{j=1}^n |u_n(x_n^{(j)})| = \|u_n\|_{L^\infty(\mathcal{M}_n)}.$$

To show that Π_n is a contraction, we first apply Hölder's inequality in (3.4) with $\frac{1}{q} + \frac{1}{q'} = 1$ to find $|\Pi_n u(x_n^{(j)})| \leq n \|u\|_{L^q(V_n^{(j)})} \mu(V_n^{(j)})^{\frac{1}{q'}} = n^{\frac{1}{q}} \|u\|_{L^q(V_n^{(j)})}$, so that

$$\|\Pi_n u\|_{L^q(\mathcal{M}_n)}^q = \frac{1}{n} \sum_{j=1}^n |\Pi_n u(x_n^{(j)})|^q \leq \sum_{j=1}^n \|u\|_{L^q(V_n^{(j)}, \mu)}^q = \|u\|_{L^q(\mathcal{M})}^q.$$

Definition 2.1(ii) is a consequence of (3.3), from which it follows that $\Lambda_n \Pi_n u \rightarrow u$ in $L^q(\mathcal{M})$ for any $u \in C(\mathcal{M})$. For $q = \infty$, this is what we wanted to show; if $q \in [1, \infty)$, then $C(\mathcal{M})$ is

dense in $L^q(\mathcal{M})$, and the fact that $(\Lambda_n \Pi_n)_{n \in \mathbb{N}}$ is uniformly bounded in $\mathcal{L}(L^q(\mathcal{M}))$ by Definition 2.1(i) which we have just proven to hold, yields $\Lambda_n \Pi_n u \rightarrow u$ for all $u \in L^q(\mathcal{M})$ as desired. Definition 2.1(iii) can be verified via direct computation using the definitions. Finally, we have

$$\begin{aligned} \int_{\mathcal{M}} \Lambda_n u_n(x) v(x) d\mu(x) &= \int_{\mathcal{M}} u_n(T_n(x)) v(x) d\mu(x) = \sum_{j=1}^n \int_{V_n^{(j)}} u_n(x_n^{(j)}) v(x) d\mu(x) \\ &= \frac{1}{n} \sum_{j=1}^n u_n(x_n^{(j)}) \Pi_n v(x_n^{(j)}) = \int_{\mathcal{M}_n} u_n(x) \Pi_n v(x) d\mu_n(x), \end{aligned} \quad (3.5)$$

which shows that the adjoint of $\Pi_n \in \mathcal{L}(L^q(\mathcal{M}); L^q(\mathcal{M}_n))$, where $q \in [1, \infty)$, is given by $\Pi_n^* = \Lambda_n \in \mathcal{L}(L^{q'}(\mathcal{M}_n); L^{q'}(\mathcal{M}))$.

The concrete choices of $\mathcal{M}$ and their discretizations which we will consider in this section are the following two:

Setting 3.2 (*Square grid on $\mathbb{T}^m$*). Let $\mathcal{M} := \mathbb{T}^m$ be the m -dimensional flat torus, which we view as the cube $[0, 1]^m$ endowed with periodic boundary conditions. For notational convenience, we will index our sequence of discretizations of $\mathbb{T}^m$ only by the natural numbers n such that $n^{1/m} \in \mathbb{N}$, for which we define the following square equidistant grid with mesh size $h_n := n^{-1/m}$:

$$\mathcal{M}_n := \{\tfrac{1}{2}n^{-1/m}, \tfrac{3}{2}n^{-1/m}, \dots, 1 - \tfrac{1}{2}n^{-1/m}\}^m.$$

Then the grid points of $\mathcal{M}_n$ can be written in the form $\mathbf{x}_n^{(\mathbf{j})} = n^{-1/m}(\mathbf{j} - \frac{1}{2}\mathbf{1})$ for some m -tuple $\mathbf{j} \in \{1, \dots, n^{1/m}\}^m$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^m$. To each of these points $\mathbf{x}_n^{(\mathbf{j})} \in \mathcal{M}_n$, we associate the half-open cube $U_n^{(\mathbf{j})} := \prod_{k=1}^m [n^{-1/m}(j_k - 1), n^{-1/m}j_k)$. Since these cubes form a partition of $\mathcal{M}$ (recalling that opposite sides are identified with each other), we can define the transport map $T_n: \mathcal{M} \rightarrow \mathcal{M}_n$ by $T_n(x) := \mathbf{x}_n^{(\mathbf{j})}$ whenever $x \in U_n^{(\mathbf{j})}$. It readily follows that (3.2) holds, as does (3.3), since $\varepsilon_n = \frac{1}{2}\sqrt{m}n^{-1/m}$ for all $n \in \mathbb{N}$.

Setting 3.3 (*Randomly sampled point cloud*). Let $\mathcal{M}$ be any manifold satisfying Assumption 3.1, and let $(x^{(n)})_{n \in \mathbb{N}} \subseteq \mathcal{M}$ be a sequence of points independently sampled from μ . We can view this sequence as a sample from the product probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) := \prod_{n \in \mathbb{N}} (\mathcal{M}, \mathcal{B}(\mathcal{M}), \mu)$, where $\mathcal{B}(\mathcal{M})$ denotes the Borel σ -algebra on $\mathcal{M}$. If we set $\mathcal{M}_n := (x^{(j)})_{j=1}^n$ for all $n \in \mathbb{N}$, then [72, Proposition 4.1] states that, $\tilde{\mathbb{P}}$ -a.s., there exists a sequence $(T_n)_{n \in \mathbb{N}}$ of transport maps $T_n: \mathcal{M} \rightarrow \mathcal{M}_n$ for which (3.2) holds and

$$\varepsilon_n \lesssim_{\mathcal{M}} (\log n)^{c_m} n^{-1/m} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $c_m = \frac{3}{4}$ if $m = 2$ and $c_m = \frac{1}{m}$ otherwise.

The transport maps $(T_n)_{n \in \mathbb{N}}$ enable us to compare discrete functions $u_n \in L^q(\mathcal{M}_n)$ with functions $u \in L^q(\mathcal{M})$ by considering the norms $\|\Lambda_n u_n - u\|_{L^q(\mathcal{M})}$, where $\Lambda_n u_n := u_n \circ T_n$ as in (3.4). In particular, we can say that $(u_n)_{n \in \mathbb{N}}$ converges to u in the discrete-to-continuum sense if $\Lambda_n u_n \rightarrow u$ in $L^q(\mathcal{M})$ as $n \rightarrow \infty$ (as we will do from Section 3.2 onward). We remark that this

implies the convergence of the pairs $(\mu_n, u_n)_{n \in \mathbb{N}}$ to (μ, u) in the metric space $TL^q(\mathcal{M})$ defined in [34]; this follows from (3.3) and the fact that every T_n induces a *coupling* between μ_n and μ .

Let us now introduce the linear operators which we consider on domains $\mathcal{M}$ and $(\mathcal{M}_n)_{n \in \mathbb{N}}$ as in Settings 3.2 and 3.3. Given the coefficient functions $\tau: \mathcal{M} \rightarrow [0, \infty)$ and $\kappa: \mathcal{M} \rightarrow [0, \infty)$, respectively assumed to be Lipschitz and continuously differentiable, we consider the nonnegative and symmetric second-order linear differential operator $\mathcal{L}_\infty^{\tau, \kappa}$ formally defined by

$$\mathcal{L}_\infty^{\tau, \kappa} u := \tau u - \nabla \cdot (\kappa \nabla u), \quad \mathcal{D}(\mathcal{L}_\infty^{\tau, \kappa}) \subseteq L^2(\mathcal{M}), \quad (3.6)$$

where the gradient and divergence operators on the manifold $\mathcal{M}$ are meant in the Riemannian sense (recall Assumption 3.1 on $\mathcal{M}$).

For each $n \in \mathbb{N}$, we endow $\mathcal{M}_n$ with a (weighted, undirected) graph structure by viewing its points as vertices and defining the weight matrix $\mathbf{W}_n \in \mathbb{R}^{n \times n}$ by

$$(\mathbf{W}_n)_{ij} := \frac{2(m+2)}{v_m} \frac{1}{nh_n^{m+2}} \mathbf{1}_{[0, h_n]}(\|x_n^{(i)} - x_n^{(j)}\|_{\mathbb{R}^d}), \quad (3.7)$$

where v_m denotes the volume of the unit sphere in $\mathbb{R}^m$ and $h_n \in (0, \infty)$ is a given graph connectivity length scale. With these weights, the resulting graph is an example of a *geometric graph* (or in fact a *random geometric graph* if the nodes are sampled randomly as in Setting 3.3). The results in this section are likely to remain valid if the indicator function $\mathbf{1}_{[0, h_n]}$ in (3.7) is replaced by a more general (e.g., Gaussian) cut-off kernel (such as in [12]), but we only consider $\eta = \mathbf{1}_{[0, h_n]}$ in order to also cite sources which are not formulated in this generality.

The graph-discretized counterpart $\mathcal{L}_n^{\tau, \kappa}$ of (3.6) is then the operator which acts on a given function $u: \mathcal{M}_n \rightarrow \mathbb{R}$ as

$$\mathcal{L}_n^{\tau, \kappa} u(x_n^{(i)}) := \tau(x_n^{(i)})u(x_n^{(i)}) + \sum_{j=1}^n (\mathbf{W}_n)_{ij} \sqrt{\kappa(x_n^{(i)})\kappa(x_n^{(j)})} (u(x_n^{(i)}) - u(x_n^{(j)})). \quad (3.8)$$

This can be seen as a generalized version of the (unnormalized) graph Laplacian $\Delta_{\mathcal{M}_n}$, and in fact reduces to it if $\tau \equiv 0$ and $\kappa \equiv 1$.

Assumption 3.4 (*Coefficients of $\mathcal{L}_n^{\tau, \kappa}$*). Let $\tau: \mathcal{M} \rightarrow [0, \infty)$ and $\kappa: \mathcal{M} \rightarrow [0, \infty)$ be the coefficient functions used to define the base operators $(\mathcal{L}_n^{\tau, \kappa})_{n \in \mathbb{N}}$ in (3.6) and (3.8). Suppose that

- (i) τ is Lipschitz, whereas κ is continuously differentiable and bounded below away from zero.

For some results, we specialize to the case that

- (ii) $\tau \equiv 0$ and $\kappa \equiv 1$, i.e., $\mathcal{L}_n^{\tau, \kappa} = \Delta_{\mathcal{M}_n}$ and $\mathcal{L}_\infty^{\tau, \kappa}$ reduces to the Laplace–Beltrami operator on $\mathcal{M}$. ◆

Assumption 3.5 (*Connectivity length scale of random graph*). Let the manifold $\mathcal{M}$ and the random point clouds $(\mathcal{M}_n)_{n \in \mathbb{N}}$ on the probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be as in Setting 3.3. Let $(h_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ determine the connectivity length scales of the graphs associated to $(\mathcal{M}_n)_{n \in \mathbb{N}}$ via the weights (3.7), and suppose that $s \in (0, \infty)$. We will assume one of the following:

- (i) There exists a $\beta > \frac{m}{4s}$ such that $(\log n)^{c_m} n^{-\frac{1}{m}} \ll h_n \ll n^{-\frac{1}{4s\beta}}$.
- (ii) There exists a $\delta > 0$ so small that $\frac{m}{1-\delta} < m + 4 + \delta$, as well as a $\beta > \frac{m+4+\delta}{2s}$ such that $n^{-\frac{1}{m+4+\delta}} \lesssim h_n \ll n^{-\frac{1}{2s\beta}}$.

Given two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ of positive real numbers, the notation $a_n \ll b_n$ means $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$. $\blacklozenge$

Since $\text{Id}_n + \mathcal{L}_n^{\tau, \kappa}$ is self-adjoint, positive definite and has a compact inverse for all $n \in \overline{\mathbb{N}}$ (cf. [76, Chapter XII] in the case $n = \infty$), there exists an orthonormal basis $(\psi_n^{(j)})_{j=1}^n$ of $L^2(\mathcal{M}_n)$ and a non-decreasing sequence $(\lambda_n^{(j)})_{j=1}^n \subseteq [0, \infty)$, accumulating only at infinity for $n = \infty$, such that $\mathcal{L}_n^{\tau, \kappa} \psi_n^{(j)} = \lambda_n^{(j)} \psi_n^{(j)}$ for all $j \in \{1, \dots, n\}$. We summarize this state of affairs by saying that $(\psi_n^{(j)}, \lambda_n^{(j)})_{j=1}^n$ is an orthonormal eigenbasis of $L^2(\mathcal{M}_n)$ associated to $\mathcal{L}_n^{\tau, \kappa}$. The asymptotic behavior of the eigenvalues $(\lambda_\infty^{(j)})_{j \in \mathbb{N}}$ is described by Weyl's law, cf. [76, Theorem XII.2.1]:

$$\lambda_\infty^{(j)} \asymp_{(\mathcal{M}, \tau, \kappa)} j^{2/m} \quad \text{for all } j \in \mathbb{N}. \quad (3.9)$$

Given any of the above settings and $n \in \overline{\mathbb{N}}$, we define the generalized Whittle–Matérn operator A_n on $L^2(\mathcal{M}_n)$ as a fractional power of the symmetric elliptic operator $\mathcal{L}_n^{\tau, \kappa}$ given by (3.6) and (3.8). That is, we set $A_n := [\mathcal{L}_n^{\tau, \kappa}]^s$ for some $s \in [0, \infty)$, where we use the spectral definition of fractional powers:

$$A_n u = [\mathcal{L}_n^{\tau, \kappa}]^s u := \sum_{j=1}^n [\lambda_n^{(j)}]^s \langle u, \psi_n^{(j)} \rangle_{L^2(\mathcal{M}_n)} \psi_n^{(j)}, \quad u \in \text{D}(A_n) \subseteq L^2(\mathcal{M}_n). \quad (3.10)$$

These will be used as the linear operators $(A_n)_{n \in \overline{\mathbb{N}}}$ in the stochastic partial differential equations in the next subsection.

Since A_n is a nonnegative definite and self-adjoint operator on $L^2(\mathcal{M}_n)$ for any $n \in \overline{\mathbb{N}}$, the Lumer–Phillips theorem [82, Theorem 13.35] implies that $-A_n$ generates a contractive analytic C_0 -semigroup $(S_n(z))_{z \in \Sigma_\eta} \subseteq \mathcal{L}(L^2(\mathcal{M}_n))$ on the sector

$$\Sigma_\eta := \{\lambda \in \mathbb{C} \setminus \{0\} : \arg \lambda \in (-\eta, \eta)\} \quad (3.11)$$

for every $\eta \in (0, \frac{1}{2}\pi)$. Thus, the operators $(A_n)_{n \in \overline{\mathbb{N}}}$ on $(L^2(\mathcal{M}_n))_{n \in \overline{\mathbb{N}}}$ are uniformly sectorial of angle 0, see Appendix C.

The following additional assumption(s) on the $L^\infty(\mathcal{M}_n)$ -boundedness of the semigroups will be needed for some of the results in Section 3.2:

Assumption 3.6 (Uniform L^∞ -boundedness of semigroups). Suppose that

- (i) there exists a constant $M_{S, \infty} \in [1, \infty)$ such that

$$\|S_n(t)\|_{\mathcal{L}(L^\infty(\mathcal{M}_n))} \leq M_{S, \infty} \quad \text{for all } n \in \overline{\mathbb{N}} \text{ and } t \geq 0.$$

We may sometimes additionally assume that

(ii) $(S_n(t))_{t \geq 0}$ is $L^\infty(\mathcal{M}_n)$ -contractive for all $n \in \overline{\mathbb{N}}$, i.e., $M_{S, \infty} = 1$ in (i). $\blacklozenge$

Under this assumption, it follows from [69, Proposition 3.12] that $(S_n(z))_{z \in \Sigma_{\eta_q}}$ is bounded analytic on $L^q(\mathcal{M}_n)$ with $\eta_q = \frac{2}{q}\eta$ for all $n \in \overline{\mathbb{N}}$ and $q \in (2, \infty)$, and its uniform norm bound on the sector Σ_{η_q} only depends on q and $M_{S, \infty}$. Therefore, the sequence of operators $(A_n)_{n \in \overline{\mathbb{N}}}$ on $(L^q(\mathcal{M}_n))_{n \in \overline{\mathbb{N}}}$ is uniformly sectorial of angle at most $(\frac{1}{2} - \frac{1}{q})\pi$.

3.2. Convergence of graph-discretized semilinear SPDEs

Now we turn back to the SPDEs (3.1) and the discrete-to-continuum convergence of their solutions. Solutions to (3.1)—and all the other (semi)linear SPDEs that we consider in this work—are always interpreted in the mild sense. This notion of solutions is defined using the semigroup $(S_n(t))_{t \geq 0}$ generated by $-\mathcal{L}_n^{\tau, \kappa}$. We say that u_n is a *global mild solution* to (3.1) if it satisfies the following relation for all $t \in [0, T]$:

$$u_n(t) = S_n(t)\xi_n + \int_0^t S_n(t-s)F_n(s, u_n(s))ds + \int_0^t S_n(t-s)dW_n(s), \quad \mathbb{P}\text{-a.s.}$$

Here, we interpret $u_n = (u_n(t))_{t \in [0, T]}$ as a process taking its values in an infinite-dimensional Banach space of functions on $\mathcal{M}_n$ (such as $L^q(\mathcal{M}_n)$ or $C(\mathcal{M}_n)$), and we define for every $(\omega, t) \in \Omega \times [0, T]$ the *Nemytskii operator* $u_n \mapsto F_n(\omega, t, u_n)$ on this function space by setting $[F_n(\omega, t, u_n)](\xi) := f_n(\omega, t, u_n(\xi))$ for all $\xi \in \mathcal{M}_n$.

This notion of solution is called “global” because it exists on the whole of $[0, T]$, in contrast with “local” solutions, which may blow up before time T . However, we note that global solutions generally grow unbounded as $T \rightarrow \infty$. We will not consider local solutions in this section, but we do work with them in Section 5.2.

In this section, the real-valued functions f_n are supposed to satisfy the following:

Assumption 3.7 (Nonlinearities). We will assume one of the following conditions:

- (i) The nonlinearities $(f_n)_{n \in \overline{\mathbb{N}}}$ are globally Lipschitz continuous and grow linearly, both uniformly in n . I.e., there exist $\tilde{L}_f, \tilde{C}_f \in [0, \infty)$ such that, for all $n \in \overline{\mathbb{N}}$ and $x, y \in \mathbb{R}$,

$$|f_n(\omega, t, x) - f_n(\omega, t, y)| \leq \tilde{L}_f |x - y| \quad \text{and} \quad |f_n(\omega, t, x)| \leq \tilde{C}_f (1 + |x|).$$

- (ii) The nonlinearities $(f_n)_{n \in \overline{\mathbb{N}}}$ are of the polynomial form

$$f_n(\omega, t, x) := -a_{2k+1, n}(\omega, t)x^{2k+1} + \sum_{j=0}^{2k} a_{j, n}(\omega, t)x^j, \quad (3.12)$$

where $k \in \mathbb{N}_0$ and $a_{j, n}: \Omega \times [0, T] \rightarrow \mathbb{R}$ for each $j \in \{0, \dots, 2k+1\}$, and there exist constants $c, C \in (0, \infty)$ such that

$$c \leq a_{2k+1, n}(\omega, t) \leq C \quad \text{and} \quad |a_{j, n}(\omega, t)| \leq C \quad (3.13)$$

for all $j \in \{0, \dots, 2k\}$, $n \in \overline{\mathbb{N}}$ and $(\omega, t) \in \Omega \times [0, T]$.

In either case, we suppose moreover that $f_n \rightarrow f$ uniformly on compact intervals; i.e., for all $r \in [0, \infty)$ and $(\omega, t) \in \Omega \times [0, T]$,

$$\sup_{x \in [-r, r]} |f_n(\omega, t, x) - f_\infty(\omega, t, x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \blacklozenge \quad (3.14)$$

Example 3.8. The cubic polynomial $f_n(\omega, t, x) := -x^3 + x$, which turns (3.1) into the (fractional) stochastic Allen–Cahn equation in case $\tau \equiv 0$, $\kappa \equiv 1$ and $s \leq 1$, is of the form asserted in Assumption 3.7(ii). Note that this is also an example of the important situation where (3.14) is trivially satisfied by taking the same function $f_n := f$ for all $n \in \overline{\mathbb{N}}$.

The final technical assumption that we record before moving on to the main theorem of this section is the following:

Assumption 3.9 (Uniform L^∞ -boundedness of eigenfunctions). Suppose that there exists a constant $M_{\psi, \infty} \in (0, \infty)$ such that

$$\|\psi_n^{(j)}\|_{L^\infty(\mathcal{M}_n)} \leq M_{\psi, \infty} \quad \text{for all } n \in \overline{\mathbb{N}} \text{ and } j \in \{1, \dots, n\}. \quad \blacklozenge$$

The interplay of the various choices of spatial domains $\mathcal{M}_n$, linear operators A_n and nonlinearity functions f_n determines the class of SPDEs to which (3.1) belongs. Rigorous definitions of the corresponding mild solution concepts, as well as well-posedness and discrete-to-continuum convergence results can be found in Sections 4–6, respectively. Applying these results in their respective regimes of applicability yields the following discrete-to-continuum convergence theorem for the solutions to (3.1); note that the setting of part (c) covers the stochastic (fractional) Allen–Cahn equation on the one-dimensional torus, see Example 3.8.

Theorem 3.10. Let $\mathcal{M}$ and $(\mathcal{M}_n)_{n \in \overline{\mathbb{N}}}$ be as in Setting 3.2 or 3.3.

- (a) Consider Setting 3.3. Let $s > \frac{1}{2}m$ and suppose that Assumption 3.5(i) is satisfied with $\beta \in (\frac{m}{4s}, \frac{1}{2})$. If Assumptions 3.4(i) and 3.7(i) are satisfied, and $p \in [1, \infty)$ is such that $\Lambda_n \xi_n \rightarrow \xi_\infty$ in $L^p(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(\mathcal{M}))$, then there exists a unique global mild solution u_n in $L^p(\Omega; C([0, T]; L^2(\mathcal{M}_n)))$ to (3.1) for every $n \in \overline{\mathbb{N}}$, and as $n \rightarrow \infty$ we have

$$\Lambda_n u_n \rightarrow u_\infty \quad \widetilde{\mathbb{P}}\text{-a.s. in } L^p(\Omega; C([0, T]; L^2(\mathcal{M}))).$$

- (b) In Setting 3.3, let $\delta > 0$ be such that $\frac{m}{1-\delta} < m + 4 + \delta$; suppose $s > m + 4 + \delta$ and Assumption 3.5(ii) holds with $\beta \in (\frac{m+4+\delta}{2s}, \frac{1}{2})$. Let Assumptions 3.4(ii), 3.6(i), 3.7(i) and 3.9 be satisfied. If $p \in [1, \infty)$ is such that $\Lambda_n \xi_n \rightarrow \xi_\infty$ in $L^p(\Omega, \mathcal{F}_0, \mathbb{P}; L^\infty(\mathcal{M}))$, then there exists a unique global mild solution u_n in $L^p(\Omega; C([0, T]; L^\infty(\mathcal{M}_n)))$ to (3.1) for every $n \in \mathbb{N}$, as well as u_∞ in $L^p(\Omega; C([0, T]; C(\mathcal{M})))$ for $n = \infty$, and as $n \rightarrow \infty$ we have

$$\Lambda_n u_n \rightarrow u_\infty \quad \text{in } L^0(\widetilde{\Omega}, L^p(\Omega; C([0, T]; L^\infty(\mathcal{M}))))).$$

(c) Consider Setting 3.2 with $\mathcal{M} := \mathbb{T}$. Let $s \in (\frac{1}{2}, 1]$ and suppose that Assumptions 3.4(ii) and 3.7(ii) are satisfied. If $p \in (1, \infty)$ is such that $\Lambda_n \xi_n \rightarrow \xi_\infty$ in $L^p(\Omega, \mathcal{F}_0, \mathbb{P}; L^\infty(\mathbb{T}))$, then there exists a unique global mild solution u_n in $L^p(\Omega; C([0, T]; L^\infty(\mathcal{M}_n)))$ to (3.1) for every $n \in \mathbb{N}$, as well as u_∞ in $L^p(\Omega; C([0, T]; C(\mathbb{T})))$ for $n = \infty$ and for all $p^- \in [1, p)$ we have, as $n \rightarrow \infty$,

$$\Lambda_n u_n \rightarrow u_\infty \quad \text{in } L^{p^-}(\Omega; C([0, T]; L^\infty(\mathbb{T}))).$$

The proof is deferred to Section A.2 in Appendix A. In the next section, we list the intermediate results on which it relies. The motivations behind the various assumptions listed above, as well as their role in Theorem 3.10, are discussed in Section 3.4.

Before, we noted that the stochastic (fractional) Allen–Cahn equation on the one-dimensional torus (i.e. in Setting 3.2) is covered by part (c) of the theorem above. A similar Allen–Cahn equation can be formulated on a point cloud randomly sampled from a manifold (i.e. in Setting 3.3), but this would not be covered by Theorem 3.10, since the cubic nonlinearity f_n from Example 3.8 does not satisfy Assumption 3.7(i) as it neither is globally Lipschitz continuous, nor has linear growth.

Indeed, such nonlinearities necessitate working with L^∞ as a spatial function space for well-posedness and discrete-to-continuum convergence results, i.e. with the framework from Section 6 (as explained at the start of that section) instead of that from Sections 4 or 5. In particular we need L^∞ -convergence of the linear operators $[\mathcal{L}_n^{\tau, \kappa}]^s$. Unfortunately, in the randomly sampled setting, the L^∞ -convergence of the eigenfunctions of the operators $\mathcal{L}_n^{\tau, \kappa}$ has been proven in the literature under more restrictive assumptions on the graph connectivities $(h_n)_{n \in \mathbb{N}}$ than the corresponding L^2 results (see Theorem 3.13). This difference in conditions propagates to the L^2 and L^∞ parts of Theorem 3.14 on the convergence of $[\mathcal{L}_n^{\tau, \kappa}]^s$: Part (a) uses Assumption 3.5(i) (which entails that $s > \frac{m}{4\beta}$) and (b) uses Assumption 3.5(ii) (implying that $s > \frac{m+4+\delta}{2\beta}$). It is conjectured that eigenvector convergence in L^∞ holds in the same (larger) graph-connectivity regime as for L^2 (see Section 3.4), but it remains unproven to the authors' best current knowledge and a proof lies beyond the scope of this work. In Setting 3.2 we explicitly prove the convergence of the eigenfunctions in L^∞ in Lemma 3.11 and thus we avoid difficulties like the one described above. The same can possibly be done for deterministic point clouds on more general one-dimensional manifolds, but we restricted ourselves to this simple domain.

Thus, if we were to include a part to Theorem 3.10 covering such polynomial nonlinearities in Setting 3.3, convergence in L^∞ of the linear operators would require most of the same conditions as those in Theorem 3.10(b): The existence of a $\delta > 0$ which satisfies $\frac{m}{1-\delta} < m + 4 + \delta$ and $s > m + 4 + \delta = 5 + \delta$, and that Assumptions 3.5(ii) (with $\beta \in (\frac{m+4+\delta}{2s}, \frac{1}{2})$), 3.4(ii), 3.6(i), and 3.9 are satisfied. However, since Assumption 3.7(i) does not hold in this case, this would lead only to the existence of local solutions and convergence along the lines of Section 6.2.

As a final note in this section, we explain why we do obtain global solutions and convergence in Theorem 3.10(c), even though there we also only have Assumption 3.7(ii) and not Assumption 3.7(i). This relies on Lemma 6.18 and Corollary 6.19 that hold under Assumption 6.16. In the proof of Theorem 3.10(c) we use the L^∞ -contractivity of the semigroups generated by the linear operators to verify Assumption 6.16, specifically to verify that Assumption 3.6(ii) is satisfied if $s \in (0, 1]$. This implies accretivity of $A_n|_{B_n}$ as in (6.6), which is an ingredient in the proof of Assumption 6.16, as is Example 6.17. Based on related results in the literature, we think it unlikely that Assumption 3.6(ii) can be satisfied when $s > 1$, or in fact, for any semigroup

generated by an elliptic differential operator of order larger than two. Whether Assumption 6.16 can be proven for $s > 1$ in a different way is not known to the authors. Since the condition $s \leq 1$ conflicts with the requirement that $s > 5 + \delta$, which appears in our discussion about the Allen–Cahn equation in Setting 3.3 (with $m = 1$), this route that will allow us to prove global results in Theorem 3.10(c) even without recourse to Assumption 3.7(i), would not be open to us for the stochastic (fractional) Allen–Cahn equation on a randomly sampled point cloud.

3.3. Intermediate results

In this subsection, we collect a number of intermediate results which imply that the conditions imposed in Theorem 3.10 are sufficient to fit into the setting of the various convergence theorems in Sections 4–6. More precisely, depending on the setting, we wish to verify a subset of the following: Assumptions 4.1, 4.2 and 4.5 on the linear operators, Assumptions 5.1, 5.5 and 5.6 on the nonlinearities and initial conditions, respectively, as well as their extended counterparts: Assumptions 6.1–6.4, 6.6–6.8 and 6.16. The proofs of the results in this section are deferred to Appendix A for ease of exposition.

The necessary convergence of the linear operators, given by Assumptions 4.5 and 6.3, will ultimately be derived from the spectral convergence of $(\mathcal{L}_n^{\tau, \kappa})_{n \in \mathbb{N}}$ to $\mathcal{L}_\infty^{\tau, \kappa}$, i.e., the convergence of the respective eigenvalues and (lifted) eigenfunctions. In the square-grid Setting 3.2, we can argue directly using closed-form expressions of all the eigenvalues and eigenfunctions involved, see Lemma 3.11 below. A subtlety arising in the random-graph Setting 3.3 is that, for any $n \in \mathbb{N}$, we cannot in general control the errors $|\lambda_n^{(j)} - \lambda_\infty^{(j)}|$ and $\|\psi_n^{(j)} \circ T_n - \psi_\infty^{(j)}\|_{L^q(\mathcal{M})}$ for all $j \in \{1, \dots, n\}$, but only for indices j up to a sufficiently small integer k_n . We present the precise statements below: Theorems 3.12 and 3.13(a), which cover eigenvalue convergence and $L^2(\mathcal{M})$ -convergence of eigenfunctions, are respectively taken from [72, Theorems 4.6 and 4.7]. Theorem 3.13(b), concerning the $L^\infty(\mathcal{M})$ -convergence of Laplacian eigenvalues, is a consequence of the main results from [12], as shown in (the discussion preceding) [73, Lemma 15].

Lemma 3.11 (Spectral convergence—square grid). *Let $\mathcal{M} = \mathbb{T}^m$ be discretized by the sequence of square grids from Setting 3.2. If $\tau \equiv 0$ and $\kappa \equiv 1$, then for all $n \in \mathbb{N}$ such that $n^{1/m} \in \mathbb{N}$, the eigenfunction–eigenvalue pairs $(\psi_n^{(j)}, \lambda_n^{(j)})_{j=1}^n$ and $(\psi_\infty^{(j)}, \lambda_\infty^{(j)})_{j \in \mathbb{N}}$ corresponding to the graph Laplacian $\mathcal{L}_n^{\tau, \kappa} = \Delta_n$ and the Laplace–Beltrami operator $\mathcal{L}_\infty^{\tau, \kappa} = -\Delta_{\mathcal{M}}$, respectively, satisfy*

$$0 \leq \lambda_\infty^{(j)} - \lambda_n^{(j)} \leq \frac{1}{12} j^4 \pi^4 n^{-\frac{2}{m}} \quad \text{for all } j \in \{1, \dots, n\}; \quad (3.15)$$

$$\|\psi_\infty^{(j)} - \psi_n^{(j)} \circ T_n\|_{L^\infty(\mathcal{M})} \leq \frac{1}{2} \sqrt{2} j \pi n^{-\frac{1}{m}} \quad \text{for all } j \in \{1, \dots, n-1\}. \quad (3.16)$$

Theorem 3.12 (Eigenvalue convergence—random graphs). *Let the manifold $\mathcal{M}$ and the random point clouds $(\mathcal{M}_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be as in Setting 3.3. Suppose that $\tau: \mathcal{M} \rightarrow [0, \infty)$ is Lipschitz, and that $\kappa: \mathcal{M} \rightarrow [0, \infty)$ is continuously differentiable and bounded below away from zero.*

If the graph connectivity length scales $(h_n)_{n \in \mathbb{N}}$ (see (3.7)) are chosen in such a way that there exist positive integers $(k_n)_{n \in \mathbb{N}}$ satisfying

$$\varepsilon_n \ll h_n \ll [\lambda_\infty^{(k_n)}]^{-\frac{1}{2}}, \quad (3.17)$$

then there exists a constant $C_{(\mathcal{M}, \tau, \kappa)} > 0$ such that the following holds:

$$\tilde{\mathbb{P}}\left(\frac{|\lambda_n^{(j)} - \lambda_\infty^{(j)}|}{\lambda_\infty^{(j)} + 1} \leq C_{(\mathcal{M}, \tau, \kappa)} \varepsilon_n h_n^{-1} + h_n [\lambda_\infty^{(j)}]^\frac{1}{2} \text{ for all } n \in \mathbb{N}, j \in \{1, \dots, k_n\}\right) = 1.$$

Theorem 3.13 (Eigenfunction convergence—random graphs). *Let the manifold $\mathcal{M}$ and the random point clouds $(\mathcal{M}_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be as in Setting 3.3. Let $(h_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ be the connectivity length scales of the graphs associated to $(\mathcal{M}_n)_{n \in \mathbb{N}}$ via the weights (3.7), and consider the (graph-discretized) differential operators $\mathcal{L}_n^{\tau, \kappa}$ with coefficients $\tau: \mathcal{M} \rightarrow [0, \infty)$ and $\kappa: \mathcal{M} \rightarrow [0, \infty)$.*

- (a) *If Assumption 3.4(i) holds, and there exist integers $(k_n)_{n \in \mathbb{N}}$ such that (3.17) is satisfied, then there exists a constant $C_{(\mathcal{M}, \tau, \kappa)} > 0$ such that, for all $n \in \mathbb{N}$,*

$$\tilde{\mathbb{P}}\left(\|\psi_n^{(j)} \circ T_n - \psi_\infty^{(j)}\|_{L^2(\mathcal{M})} \leq C_{(\mathcal{M}, \tau, \kappa)} j^{\frac{3}{2}} (\varepsilon_n h_n^{-1} + h_n [\lambda_\infty^{(j)}]^\frac{1}{2})^\frac{1}{2} \text{ for all } j \in \{1, \dots, k_n\}\right) = 1.$$

- (b) *Let Assumption 3.4(ii) be satisfied. If there exist $(k_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ and $\delta > 0$ such that*

$$n^{-\frac{1}{m+4+\delta}} \lesssim_{\mathcal{M}} h_n \lesssim_{\mathcal{M}} [\lambda_\infty^{(k_n)}]^{-1} \quad \text{and} \quad \lambda_\infty^{(k_n)} \lesssim_{\mathcal{M}} n^{\frac{1-\delta}{m}},$$

then there exists a constant $C_{\mathcal{M}} > 0$ such that, as $n \rightarrow \infty$,

$$\tilde{\mathbb{P}}\left(\|\psi_n^{(j)} \circ T_n - \psi_\infty^{(j)}\|_{L^\infty(\mathcal{M})} \leq C_{\mathcal{M}} [\lambda_\infty^{(j)}]^{m+1} j^{\frac{3}{2}} (\varepsilon_n h_n^{-1} + h_n [\lambda_\infty^{(j)}]^\frac{1}{2})^\frac{1}{2} \text{ for all } j \in \{1, \dots, k_n\}\right) \rightarrow 1.$$

From the above results, we can derive the following convergence of the sequence $(A_n)_{n \in \mathbb{N}}$. Its proof is analogous to that of [72, Theorem 4.2], see Appendix A.

Theorem 3.14. *Given $\tau: \mathcal{M} \rightarrow [0, \infty)$, $\kappa: \mathcal{M} \rightarrow [0, \infty)$ and $s \in [0, \infty)$, consider the generalized Whittle–Matérn operators $A_n := [\mathcal{L}_n^{\kappa, \tau}]^s$ defined by (3.10), and set $\tilde{R}_n^\alpha := \Lambda_n (\text{Id}_n + A_n)^{-\alpha} \Pi_n$, for all $\alpha \in [0, \infty)$ and $n \in \mathbb{N}$.*

- (a) *Suppose that $\mathcal{M}$ and its discretizations $(\mathcal{M}_n)_{n \in \mathbb{N}}$ are as in Setting 3.3, Assumption 3.5(i) holds with $\beta \in (\frac{m}{4s}, \infty)$ and Assumption 3.4(i) is satisfied. Then we have for all $\beta' \in [\beta, \infty)$:*

$$\tilde{R}_n^{\beta'} \rightarrow R_\infty^{\beta'} \quad \tilde{\mathbb{P}}\text{-a.s. in } \mathcal{L}_2(L^2(\mathcal{M})) \quad \text{as } n \rightarrow \infty.$$

- (b) *Suppose that $\mathcal{M}$ and its discretizations $(\mathcal{M}_n)_{n \in \mathbb{N}}$ are as in Setting 3.3, Assumption 3.5(ii) holds with $\beta \in (\frac{m+4+\delta}{2s}, \infty)$, and Assumptions 3.4(ii) and 3.9 are satisfied. Then we have for all $\beta' \in [\beta, \infty)$:*

$$\tilde{R}_n^{\beta'} \rightarrow R_\infty^{\beta'} \quad \text{in } L^0(\tilde{\Omega}; \mathcal{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M}))) \quad \text{as } n \rightarrow \infty.$$

Here, $L^0(\tilde{\Omega})$ denotes convergence in probability with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

- (c) Suppose that $\mathcal{M} := \mathbb{T}^m$ is discretized using the square grids $(\mathcal{M}_n)_{n \in \mathbb{N}}$ from Setting 3.2, and that Assumption 3.4(ii) holds. For all $\beta \in (\frac{m}{4s}, \infty)$,

$$\tilde{R}_n^\beta \rightarrow R_\infty^\beta \quad \text{in } \mathcal{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M})) \quad \text{as } n \rightarrow \infty.$$

The following property, which we call the *uniform ultracontractivity* of the semigroups $(S_n)_{n \in \mathbb{N}}$, will be needed in order to obtain the $L^\infty(\mathcal{M})$ -convergence in Theorem 3.10(b) and (c). We omit its proof, which relies on Riesz–Thorin interpolation, Assumption 3.6, and some arguments from Theorem 3.14.

Lemma 3.15 (*Uniform ultracontractivity*). *Let $s \in (0, \infty)$ and consider the generalized Whittle–Matérn operators $A_n := [\mathcal{L}_n^{\kappa, \tau}]^s$ defined by (3.10) for all $n \in \mathbb{N}$. Assume either of the following statements:*

- (a) *In Setting 3.3, Assumption 3.5(i) or (ii) holds with corresponding β , as well as Assumptions 3.4(i), 3.6(i) and 3.9.*
 (b) *In Setting 3.2, $\beta \in (\frac{m}{4s}, \infty)$ is arbitrary, and Assumptions 3.4(ii) and 3.6(i) hold.*

Then, for every $q \in [2, \infty]$, there exists $M_{S,q} \in [1, \infty)$ such that

$$\|S_n(t)\|_{\mathcal{L}(L^q(\mathcal{M}_n); L^\infty(\mathcal{M}_n))} \leq M_{S,q} t^{-\frac{2}{q}\beta} \quad \text{for all } n \in \mathbb{N} \text{ and } t > 0. \quad (3.18)$$

In case of (a), (3.18) holds $\tilde{\mathbb{P}}$ -a.s.

3.4. Discussion of the assumptions

In this subsection, we comment on the various assumptions made in Theorem 3.10, the extent to which they are necessary, and how one might check them in practice.

The distinction between parts (i) and (ii) of Assumption 3.4, i.e., whether to allow for spatially varying coefficient functions τ and κ in the second-order symmetric base operators $(\mathcal{L}_n^{\tau, \kappa})_{n \in \mathbb{N}}$ instead of merely considering Laplacians, is mainly due to the availability of spectral convergence theorems in the respective situations. Most of the literature on eigenfunction convergence of graph-discretized second-order operators is focused on the Laplacian case, see for instance [11, 33] for L^2 -convergence and [12, 26, 87] for L^∞ -convergence. However, the authors of [72] show how the L^2 -convergence results can be extended to coefficient functions satisfying Assumption 3.4(i). We expect that most spectral convergence results for graph Laplacians can be extended to allow for varying coefficients, but doing so requires significant effort, hence we sometimes make Assumption 3.4(ii) for the sake of convenience.

Similarly, the difference between the bounds on the graph connectivity length scales in the two parts of Assumption 3.5 is a result of the current availability of spectral convergence literature. Eigenfunction convergence of $(\mathcal{L}_n^{\tau, \kappa})_{n \in \mathbb{N}}$ in L^2 has for instance been proved in [72] under Assumption 3.5(i), but for graphs and manifolds as in our setting, the optimal available L^∞ -convergence results (for graph Laplacians) seem to be those of [12], which require Assumption 3.5(ii). However, according to [12, Remark 2.7], it is plausible that the $L^\infty(\mathcal{M})$ -convergence of Laplacian eigenfunctions can be proved under the same assumptions as the $L^2(\mathcal{M})$ -convergence, with the same rate. Some recent results in this direction can be found in [3],

where the authors show L^∞ -convergence of Laplacian eigenvectors with optimal rates and loose lower bounds on the connectivity lengths, using homogenization theory, for point clouds on less general spatial domains.

Assumption 3.6 is natural in the sense that the results regarding L^∞ -convergence in space (for instance Theorem 3.10(b) and (c)) rely on uniform L^∞ -convergence of semigroup orbits on compact time intervals. The latter necessitates that Assumption 3.6(i) is satisfied, at least for $t \in [0, T]$ with arbitrarily large $T \in (0, \infty)$.

Moreover, for typical choices of differential operators, one can often check that Assumption 3.6(ii) holds, meaning that the semigroups are in fact L^∞ -contractive. One such example is outlined in the proof of Theorem 3.10(c): Matrix exponentials $(e^{-tL_n})_{t \geq 0}$ are L^∞ -contractive if and only if $L_n \in \mathbb{R}^{n \times n}$ is diagonally dominant with nonnegative diagonal entries [65, Lemma 6.1]. Sufficient conditions for the L^∞ -contractivity of the semigroup $(S_\infty(t))_{t \geq 0}$ generated by the negative of a uniformly elliptic second-order differential operator on a Euclidean domain $\mathcal{D} \subsetneq \mathbb{R}^d$, subject to appropriate boundary conditions, can be found in [69, Section 4.3]. Likewise, the heat semigroup associated to the Laplace–Beltrami operator on a compact Riemannian manifold $\mathcal{M}$ is L^∞ -contractive, cf. [24, p. 148].

As mentioned in the proof of Theorem 3.10(c), all of the above L^∞ -contractivity results for second-order differential operators can be extended to fractional powers $s \in (0, 1)$ by using a subordination formula such as [51, Theorem 15.2.17], noting that the definition of fractional power operators in this reference coincides with ours (more details are given in the first half of the proof of Lemma C.1 below). The semigroups generated by higher-order differential operators, however, are in general not contractive on L^∞ (or any L^q for $q \neq 2$, see for instance [58]); this is closely related to their lack of positivity preservation. As an example, the fractional heat kernel associated to $(-\Delta)^s$ on $\mathbb{R}^d$ with $s \in (0, \infty)$ at time $t \in (0, \infty)$ is given by the inverse Fourier transform of $\xi \mapsto \exp(-t\|\xi\|_{\mathbb{R}^d}^{2s})$, which is positive for $s \leq 1$ but fails to be sign-definite if $s > 1$, see [29, p. 626 and pp. 632–633], respectively.

Thus, for operators $([L_n^{\tau, \kappa}]^s)_{n \in \mathbb{N}}$ with $s > 1$, we have to content ourselves with uniform L^∞ -boundedness of the semigroups $(S_n(t))_{t \geq 0}$ in n and t , i.e., Assumption 3.6(i). In the absence of positivity preservation, one route to verifying such uniformity is through Gaussian upper bounds on the integral kernels corresponding to the semigroups, cf. [69, Proposition 7.1]. Such bounds have been established for higher-order differential operators on Euclidean domains, as well as Laplacian operators on more general domains such as manifolds, graphs and fractals (see [69, pp. 194–196] and the references therein). While it may be possible to unify these results in the setting of graph-discretized higher-order differential operators on manifolds, and thus obtain the uniform L^∞ -bounds required by Assumption 3.6(i), this appears to be highly nontrivial and outside the scope of this work.

We also remark that certain higher-order operators have been shown to exhibit (local) *eventual positivity*, meaning that for every nonnegative initial datum $u_0 \geq 0$ and subset $\mathcal{D}^*$ of the spatial domain $\mathcal{D}$, there exists $t^* > 0$ such that $S(t)u_0 \geq 0$ on $\mathcal{D}^*$ for all $t \geq t^*$. For instance, in [35], this was shown for the bi-Laplacian Δ^2 on $\mathcal{D} = \mathbb{R}^d$. In [39], the authors apply the theory of [22, 23] to treat the squared graph Laplacian Δ_n^2 , deduce that $(e^{-\Delta_n^2 t})_{t \geq 0}$ is eventually L^∞ -contractive [39, Proposition 6.7], and note that this implies L^∞ -boundedness uniformly in $t \geq 0$ [39, Remark 6.8]. However, as $n \rightarrow \infty$, their upper bound $\|e^{-\Delta_n^2 t}\|_\infty \leq \exp(\|\Delta_n\|_\infty^2 t^*)$ blows up in our setting. Hence, these results do not appear to be directly useful for our purpose of verifying Assumption 3.6(i).

If we restrict ourselves to nonlinearities of the form $[F_n(\omega, t, u)](x) := f_n(\omega, t, u(x))$ (see (5.1) and (6.1)), then the conditions in Assumption 3.7 are the natural ones to ensure the

global (in time) convergence results formulated in Theorem 3.10. Convergence results for more general nonlinearities, possibly formulated only in terms of local-in-time convergence and in weaker norms, can be found in Sections 5 and 6.

Assumption 3.9 was used (explicitly or implicitly) to establish the L^∞ -convergence asserted in Theorem 3.14 and the uniform L^2 – L^∞ -ultracontractivity in Lemma 3.15. The L^∞ -norms of the L^2 -normalized eigenfunctions of the Laplace–Beltrami operator on a general compact Riemannian manifold $\mathcal{M}$ of dimension m satisfy the upper bound $\|\psi^{(j)}\|_{L^\infty(\mathcal{M})} \lesssim [\lambda^{(j)}]^{\frac{m-1}{4}}$ due to Hörmander [47]. This bound is sharp in the sense that it is attained by the symmetric spherical harmonics on the sphere. On the other hand, the L^∞ -norms are uniformly bounded (i.e., satisfy Assumption 3.9) if $m = 1$ or if $\mathcal{M} = \mathbb{T}^m$ is a flat torus. Some results relating the L^∞ -growth rate of these eigenfunctions to the geometry of the manifold can be found in [25,75,78].

These observations indicate that Assumption 3.9 poses strong restrictions on the curvature of the manifold, which raises the question whether this assumption could be removed. Its central role in the proofs of Theorem 3.14(b), (c) and Lemma 3.15 is due to the L^2 – L^∞ -norm bounds (A.2) of operators which are defined in terms of eigenvalue expansions, such as the fractional powers defined by (3.10). This suggests that disposing of Assumption 3.9 would involve techniques which are not based on spectral representations and spectral convergence of the operators involved. For Lemma 3.15 in particular, one indication that this might be possible is the fact that, like L^∞ -boundedness, the L^2 – L^∞ -ultracontractivity of $(S_\infty(t))_{t \geq 0}$ follows from certain upper bounds on its heat kernel [40, Theorem 3.2]. For the Laplace–Beltrami operator on a compact Riemannian manifold, we indeed have such bounds by [24, Proposition 5.5.1 and Theorem 5.5.2], which imply (3.18) with $\beta = \frac{m}{4}$ (for $n = \infty$ and $s = 1$).

4. Infinite-dimensional Ornstein–Uhlenbeck process

This section and the subsequent Sections 5 and 6 are devoted to proving the abstract discrete-to-continuum approximation results which are applied to the Whittle–Matérn graph discretization setting in Section 3. Thus, from this point onwards we no longer necessarily work with graphs or Whittle–Matérn operators. Instead, we have the following abstract setting.

Let the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be given, where $T \in (0, \infty)$ is a time horizon. For each $n \in \overline{\mathbb{N}}$, we consider a linear stochastic evolution equation whose state space is a UMD-type-2 Banach space $(E_n, \|\cdot\|_{E_n})$, and whose driving noise dW_n takes its values in a Hilbert space $(H_n, \langle \cdot, \cdot \rangle_{H_n})$:

$$\begin{cases} dX_n(t) = -A_n X_n(t) dt + dW_n(t), & t \in (0, T], \\ X_n(0) = 0. \end{cases} \quad (4.1)$$

In order to precisely describe the setting and assumptions, as well as defining the solution concept for equation (4.1), we need to introduce the concepts of UMD-type-2 Banach spaces and stochastic integrals with respect to them. This is done in Section 4.1. These notions are then used in Section 4.2 where we define the setting (Assumptions 4.1–4.5), the solution concept (Definition 4.4) and the main approximation result (Proposition 4.8).

4.1. Stochastic integration in UMD-type-2 Banach spaces

Given a real and separable Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$, let $(W(t))_{t \geq 0}$ be an H -valued cylindrical Wiener process with respect to $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$. A rigorous definition can be found in [61,

Section 2.5.1]. For our purpose of constructing the stochastic integral below, it suffices to define $\langle W(t), h \rangle_H := \sum_{j=1}^{\infty} \beta_j(t) \langle e_j, h \rangle_H$ for all $t \geq 0$ and $h \in H$, where $(e_j)_{j \in \mathbb{N}}$ is an orthonormal basis of H and $(\beta_j(\cdot))_{j \in \mathbb{N}}$ is a sequence of independent (real-valued) Brownian motions on $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$. Then $(\langle W(t), h \rangle_H)_{t \geq 0}$ is a well-defined Brownian motion for every $h \in H$, and intuitively one can think of $(W(t))_{t \geq 0}$ as being given by $W(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j$.

Let $(E, \|\cdot\|_E)$ be a real Banach space, and let $(\gamma_j)_{j \in \mathbb{N}}$ be a sequence of independent (real-valued) standard normal random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, independent of the probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ used in the rest of this work. We define the space $\gamma(H; E)$ of γ -radonifying operators from H to E as the completion of the finite-rank operators $H \otimes E$ with respect to the norm $\|\sum_{j=1}^n h_j \otimes x_j\|_{\gamma(H; E)} := \|\sum_{j=1}^n \gamma_j x_j\|_{L^2(\Omega'; E)}$, where we assume that the $(h_j)_{j=1}^n$ are H -orthonormal. This norm is well-defined, i.e., it can be checked that the right-hand side is independent of the choice of representation. An important feature of $\gamma(H; E)$ is its *ideal property* (in the algebraic sense) [50, Theorem 9.1.10], which states that for all $T \in \gamma(H; E)$, $U \in \mathcal{L}(E; F)$ and $S \in \mathcal{L}(K; H)$, we have

$$UTS \in \gamma(K; F) \quad \text{with} \quad \|UTS\|_{\gamma(K; F)} \leq \|U\|_{\mathcal{L}(E; F)} \|T\|_{\gamma(H; E)} \|S\|_{\mathcal{L}(K; H)}. \quad (4.2)$$

For any rank-one operator $h \otimes x \in \mathcal{L}(H; E)$, we have $h \otimes x \in \gamma(H; E)$ with

$$\|h \otimes x\|_{\gamma(H; E)} = \|h\|_H \|x\|_E. \quad (4.3)$$

The stochastic integral of an elementary integrand $\Phi: (0, \infty) \rightarrow H \otimes E$, i.e., a function of the form $\Phi(t) = \sum_{j=1}^n \mathbf{1}_{(a_j, b_j]}(t) h_j \otimes x_j$, is defined by

$$\int_0^\infty \Phi(t) dW(t) := \sum_{j=1}^n (\langle h_j, W(b_j) \rangle_H - \langle h_j, W(a_j) \rangle_H) x_j \in L^2(\Omega; E).$$

In order to extend the definition of the stochastic integral beyond elementary integrands, one needs to impose further geometric assumptions on the Banach space E . In this article we work in one of the standard settings, namely that of spaces with *unconditional martingale differences* and of *Rademacher type 2* (abbreviated to *UMD-type-2*). Definitions of these notions can be found in [49, Section 4.2] and [50, Section 7.1], respectively, but we will only use them to ensure existence of stochastic integrals. In this case, one can establish the *Itô inequality*

$$\left\| \int_0^\infty \Phi(t) dW(t) \right\|_{L^2(\Omega; E)} \lesssim_E \|\Phi\|_{L^2(0, \infty; \gamma(H; E))} \quad (4.4)$$

for elementary integrands [83, Proposition 4.2], and use it to extend the definition of the stochastic integral to all $\Phi \in L^2(0, \infty; \gamma(H; E))$. In fact, since we are only concerned with deterministic integrands in this work, we could suffice with the type-2 assumption. Despite this, we additionally impose the UMD assumption for the sake of compatibility with some of the literature, and because the concrete examples of Banach spaces in which we are interested (such as the Lebesgue L^q -spaces for $q \in [2, \infty)$) satisfy both properties. For more details on stochastic integration in Banach spaces, we refer to the survey article [83].

The exponent 2 in L^2 appearing on both sides of (4.4) can be replaced by any other $p \in [1, \infty)$ at the cost of a p -dependent constant, see for instance [83, Theorem 4.7]. If E is also a Hilbert space, then $\gamma(H; E)$ is isometrically isomorphic to the space $\mathcal{L}_2(H; E)$, see [50, Proposition 9.1.9], and instead of the inequality (4.4) we have the Itô *isometry* between $L^2(0, \infty; \mathcal{L}_2(H; E))$ and $L^2(\Omega; E)$.

4.2. Setting, well-posedness and approximation results

We can now formulate the precise setting of equation (4.1). We start with the spaces $(E_n)_{n \in \overline{\mathbb{N}}}$ and $(H_n)_{n \in \overline{\mathbb{N}}}$:

Assumption 4.1 (*Spaces are discrete-to-continuum approximations*). Let $(E_n)_{n \in \overline{\mathbb{N}}}$ be UMD-type-2 Banach spaces and $(H_n)_{n \in \overline{\mathbb{N}}}$ be Hilbert spaces; all spaces are assumed to be real and separable. Suppose the sequences $(E_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ are discrete-to-continuum approximations of E_∞ and H_∞ , respectively (recall Definition 2.1). Moreover, suppose that they share a common family $(\Lambda_n, \Pi_n)_{n \in \overline{\mathbb{N}}}$ of lifting and projection operators. $\blacklozenge$

The noise term in (4.1) should now be interpreted as $W_n := \Pi_n W_\infty$, where $(W_\infty(t))_{t \geq 0}$ denotes an H_∞ -valued cylindrical Wiener process. I.e., the formal time derivative dW_∞ of W_∞ represents space–time Gaussian white noise, and $(W_n)_{t \geq 0}$ is an H_n -valued Q_n -cylindrical Wiener process colored in space by the covariance operator $Q_n := \Pi_n \Pi_n^* \in \mathcal{L}(H_n)$.

We impose the following uniformity assumptions on the operators $(A_n)_{n \in \overline{\mathbb{N}}}$:

Assumption 4.2 (*Uniform sectoriality; γ -radonifying fractional resolvents*). Let the operators $(A_n)_{n \in \overline{\mathbb{N}}}$ on $(E_n)_{n \in \overline{\mathbb{N}}}$ are uniformly sectorial of angle less than $\frac{1}{2}\pi$ (see Appendix C for the definition of this concept). This implies that $(-A_n)_{n \in \overline{\mathbb{N}}}$ generates a uniformly bounded sequence of C_0 -semigroups $(S_n)_{n \in \overline{\mathbb{N}}}$ on $(E_n)_{n \in \overline{\mathbb{N}}}$ with uniform growth exponent $w = 0$ (see Definition 2.2).

Moreover, suppose that there exists a $\beta \in [0, \frac{1}{2})$ such that $R_n^\beta := (\text{Id}_n + A_n)^{-\beta} \in \gamma(H_n; E_n)$ for every $n \in \overline{\mathbb{N}}$. $\blacklozenge$

In general, the fractional powers of the sectorial operators A_n appearing in Assumption 4.2 can be defined using any of the equivalent definitions in [42, Chapter 3]. If, as in Section 3, the operator A_n is the restriction of an operator whose eigenvalues form an orthonormal basis on some Hilbert space containing E_n , then one can use the spectral definition (3.10) of fractional powers of A_n .

The solution concept which we consider for all of the equations in this work is that of a *mild solution*. For the linear equation (4.1), it is given by the following stochastic convolution:

Proposition 4.3. *Let $n \in \overline{\mathbb{N}}$ and $T \in (0, \infty)$. Under Assumptions 4.1–4.2, the stochastic convolution defined by*

$$W_{A_n}(t) := \int_0^t S_n(t-s) dW_n(s), \quad t \in [0, T], \quad (4.5)$$

is a well-defined process in $C([0, T]; L^p(\Omega; E_n))$ for every $p \in [1, \infty)$.

Proof. For every $p \in [1, \infty)$ and $t \in [0, T]$, we have by the Itô inequality (4.4):

$$\|W_{A_n}(t)\|_{L^p(\Omega; E_n)}^2 \lesssim_{(p, E_n)} \int_0^t \|S_n(t-s)\|_{\gamma(H_n; E_n)}^2 ds \leq \|S_n\|_{L^2(0, T; \gamma(H_n; E_n))}^2$$

To show that the right-hand side is bounded, we use the ideal property (4.2) of $\gamma(H_n; E_n)$ and the estimate (C.2) for analytic semigroups (in conjunction with Assumptions 4.1 and 4.2) to see

$$\begin{aligned} \|S_n\|_{L^2(0, T; \gamma(H_n; E_n))}^2 &= \int_0^T \|(\text{Id}_n + A_n)^\beta S_n(t) R_n^\beta\|_{\gamma(H_n; E_n)}^2 dt \\ &\leq \|R_n^\beta\|_{\gamma(H_n; E_n)}^2 \int_0^T \|(\text{Id}_n + A_n)^\beta S_n(t)\|_{\mathcal{L}(E_n)}^2 dt \\ &\lesssim_\beta \|R_n^\beta\|_{\gamma(H_n; E_n)}^2 \int_0^T t^{-2\beta} dt = \|R_n^\beta\|_{\gamma(H_n; E_n)}^2 \frac{T^{1-2\beta}}{1-2\beta} < \infty. \end{aligned}$$

Note that $\|R_n^\beta\|_{\gamma(H_n; E_n)}^2$ is finite by Assumption 4.2. Next, applying the Itô inequality (4.4) to the difference $W_{A_n}(t+h) - W_{A_n}(t)$ for small enough $h \in \mathbb{R}$ yields

$$\|W_{A_n}(t+h) - W_{A_n}(t)\|_{L^p(\Omega; E_n)} \lesssim_{(p, E_n)} \|S_n(\cdot + h) - S_n\|_{L^2(0, T; \gamma(H_n; E_n))} \rightarrow 0,$$

as $h \rightarrow 0$, by the strong continuity of translation operators on $L^2(0, T; \gamma(H_n; E_n))$. This shows $W_{A_n} \in L^p(\Omega; C([0, T]; E_n))$. $\square$

Two stochastic processes $(X(t))_{t \in [0, T]}$ and $(Y(t))_{t \in [0, T]}$ are said to be *modifications* of each other if $\mathbb{P}(X(t) = Y(t)) = 1$ for all $t \in [0, T]$.

Definition 4.4. An E_n -valued stochastic process $X_n = (X_n(t))_{t \in [0, T]}$ belonging to the space $C([0, T]; L^p(\Omega; E_n))$ for some $p \in [1, \infty)$ is said to be a *mild solution* to (4.1) if it is a modification of the process W_{A_n} defined in (4.5).

The existence and uniqueness (up to modification) of the mild solution to (4.1) in the space $C([0, T]; L^p(\Omega; E_n))$ is then immediate from Definition 4.4.

As remarked in Section 2.2, we mainly have applications in mind where the problem corresponding to $n = \infty$ is interpreted as a spatiotemporal stochastic partial differential equation. In the present linear setting, its solution is also known as an *infinite-dimensional Ornstein–Uhlenbeck process*, and solutions to (4.1) for $n \in \mathbb{N}$ are spatially discretized approximations. Therefore, it is natural to ask whether we can identify the right mode of convergence of the operators $(A_n)_{n \in \mathbb{N}}$ to A_∞ as $n \rightarrow \infty$ to ensure the convergence of the stochastic processes $(W_{A_n})_{n \in \mathbb{N}}$ to W_{A_∞} .

The answer is provided by Proposition 4.7 below, which is a stochastic counterpart of the discrete-to-continuum Trotter–Kato approximation theorem for strongly continuous semigroups

recalled in Theorem 2.3. In fact, with an eye towards the proof of Proposition 4.8 below, we consider a more general class of auxiliary processes, see equation (4.6).

Before stating any discrete-to-continuum results, let us introduce some convenient notation for this goal. Using the operators $(\Lambda_n)_{n \in \mathbb{N}}$ and $(\Pi_n)_{n \in \mathbb{N}}$, we can take a mapping which has E_n as its domain or state space, and turn it into an analogous mapping from or to E_∞ . For instance, we define the E_∞ -valued processes $\tilde{W}_{A_n} := \Lambda_n W_{A_n}$, as well as the operators $\tilde{R}_n^\alpha := \Lambda_n R_n^\alpha \Pi_n$ and $\tilde{S}_n(t) := \Lambda_n S_n(t) \Pi_n$ in $\mathcal{L}(E_\infty)$ for $\alpha, t \in [0, \infty)$. Now we can formulate our notion of the discrete-to-continuum convergence $A_n \rightarrow A_\infty$ as $n \rightarrow \infty$ as follows:

Assumption 4.5 (Convergence of (fractional) resolvents). Suppose that $\tilde{R}_n^1 x \rightarrow R_\infty^1 x$ in $\tilde{E}$ as $n \rightarrow \infty$ for every $x \in E_\infty$, and that $\beta \in [0, \frac{1}{2})$ from Assumption 4.2 satisfies $\tilde{R}_n^\beta \rightarrow R_\infty^\beta$ in $\gamma(H_\infty; E_\infty)$. $\blacklozenge$

Proposition 4.7 states that this type of convergence of the operators is sufficient to ensure convergence of the solutions to the linear stochastic evolution equation (4.1). Its proof is based on Lemma C.1 in Appendix C, as well as the following general approximation lemma for square-integrable functions with values in the space of γ -radonifying operators. It is a simpler analog to [56, Lemma 2.6], which was only necessary to allow for stochastic equations in UMD Banach spaces without type 2.

Lemma 4.6. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces, and let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space. Given $a, b \in \mathbb{R}$ with $a < b$, let $M_n: (a, b) \rightarrow \mathcal{L}(E; F)$ for all $n \in \overline{\mathbb{N}}$ and suppose that

- (i) $M_n \otimes x \rightarrow M \otimes x$ uniformly on compact subsets of (a, b) for all $x \in E$, and
- (ii) $\sup_{n \in \overline{\mathbb{N}}} \sup_{t \in (a, b)} \|M_n(t)\|_{\mathcal{L}(E; F)} < \infty$.

For all $R \in L^2(a, b; \gamma(H; E))$ and $n \in \overline{\mathbb{N}}$, we have $M_n \otimes R \in L^2(a, b; \gamma(H; F))$ and

$$M_n R \rightarrow M R \quad \text{in } L^2(a, b; \gamma(H; F)) \quad \text{as } n \rightarrow \infty.$$

Proof. Arguing as in the proof of Proposition B.3 and using (ii), it follows that we only need to prove the claim for all R belonging to some dense subset D of $L^2(a, b; \gamma(H; E))$. Note that every $R \in L^2(a, b; \gamma(H; E))$ can be approximated by a step function $\sum_{j=1}^N \mathbf{1}_{(a_j, b'_j)} \otimes T_j$ with $a < a'_j < b'_j < b$ and $T_j \in \gamma(H; E)$, and by definition of $\gamma(H; E)$ the latter can be approximated by finite-rank operators. By linearity, it thus suffices to prove the statement for R of the form

$$R(t) = \mathbf{1}_{(a', b')}(t) h \otimes x, \quad \text{where } a < a' < b' < b \text{ and } (h, x) \in H \times E.$$

Substituting this representation, using (4.3) and (i), we find as $n \rightarrow \infty$:

$$\begin{aligned} \|M_n R - M R\|_{L^2(a, b; \gamma(H; F))}^2 &= \int_{a'}^{b'} \|h \otimes [M_n(t)x - M(t)x]\|_{\gamma(H; F)}^2 dt \\ &= \|h\|_H^2 \int_{a'}^{b'} \|M_n(t)x - M(t)x\|_F^2 dt \leq \|h\|_H^2 (b' - a') \sup_{t \in (a', b')} \|M_n(t)x - M(t)x\|_F^2 \rightarrow 0. \quad \square \end{aligned}$$

Proposition 4.7. Suppose that Assumptions 4.1 and 4.2 hold. Let us define the auxiliary processes

$$W_{A_n}^\delta(t) := \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} S_n(t-s) dW_n(s), \quad \delta \in (1/2, \infty), \quad t \in [0, \infty), \quad (4.6)$$

where Γ denotes the Gamma function [68, Section 5.2]. Then, for every $\beta' \in (\beta, \infty)$, $T \in (0, \infty)$ and $p \in [1, \infty)$, we have $W_{A_n}^{\beta'+\frac{1}{2}} \in C([0, T]; L^p(\Omega; E_n))$ for all $n \in \overline{\mathbb{N}}$. If we suppose in addition that Assumption 4.5 is satisfied, then

$$\widetilde{W}_{A_n}^{\beta'+\frac{1}{2}} \rightarrow W_{A_\infty}^{\beta'+\frac{1}{2}} \quad \text{in } C([0, T]; L^p(\Omega; E_\infty)) \quad \text{as } n \rightarrow \infty.$$

Proof. The fact that $W_{A_n}^{\beta'+\frac{1}{2}} \in C([0, T]; L^p(\Omega; E_n))$ for all $n \in \overline{\mathbb{N}}$ can be established by arguing as in Proposition 4.3, thus using Assumptions 4.1 and 4.2. For all $t \in [0, T]$, the Itô inequality (4.4) yields

$$\begin{aligned} & \| \widetilde{W}_{A_n}^{\beta'+\frac{1}{2}}(t) - W_{A_\infty}^{\beta'+\frac{1}{2}}(t) \|_{L^p(\Omega; E_\infty)} \\ & \lesssim_{(p, E_\infty)} \frac{1}{\Gamma(\beta' + \frac{1}{2})} \left(\int_0^t (t-s)^{2\beta'-1} \| \widetilde{S}_n(t-s) - S_\infty(t-s) \|_{\gamma(H_\infty; E_\infty)}^2 ds \right)^{\frac{1}{2}} \\ & \leq \frac{1}{\Gamma(\beta' + \frac{1}{2})} \left(\int_0^T s^{2\beta'-1} \| \widetilde{S}_n(s) - S_\infty(s) \|_{\gamma(H_\infty; E_\infty)}^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Since semigroups commute with fractional powers of their infinitesimal generators, we can write the difference between the semigroups as follows:

$$\begin{aligned} \widetilde{S}_n(s) - S_\infty(s) &= \Lambda_n(\text{Id}_n + A_n)^\beta S_n(s) \Pi_n \widetilde{R}_n^\beta - (\text{Id}_\infty + A_\infty)^\beta S_\infty(s) R_\infty^\beta \\ &= \Lambda_n(\text{Id}_n + A_n)^\beta S_n(s) \Pi_n (\widetilde{R}_n^\beta - R_\infty^\beta) \\ &\quad + (\Lambda_n(\text{Id}_n + A_n)^\beta S_n(s) \Pi_n - (\text{Id}_\infty + A_\infty)^\beta S_\infty(s)) R_\infty^\beta. \end{aligned}$$

Thus, by the triangle inequality, it suffices to show that

$$\begin{aligned} \text{(I)} &:= \int_0^T s^{2\beta'-1} \| \Lambda_n(\text{Id}_n + A_n)^\beta S_n(s) \Pi_n (\widetilde{R}_n^\beta - R_\infty^\beta) \|_{\gamma(H_\infty; E_\infty)}^2 ds \quad \text{and} \\ \text{(II)} &:= \int_0^T s^{2\beta'-1} \| [\Lambda_n(\text{Id}_n + A_n)^\beta S_n(s) \Pi_n - (\text{Id}_\infty + A_\infty)^\beta S_\infty(s)] R_\infty^\beta \|_{\gamma(H_\infty; E_\infty)}^2 ds \end{aligned}$$

tend to zero as $n \rightarrow \infty$. Applying the ideal property (4.2) of $\gamma(H_\infty; E_\infty)$, followed by the analytic semigroup estimate (C.2) in conjunction with Assumptions 4.1 and 4.2, we find

$$\begin{aligned}
(\text{I}) &\leq \|\tilde{R}_n^\beta - R_\infty^\beta\|_{\gamma(H_\infty; E_\infty)}^2 \int_0^T s^{2\beta'-1} \|\Lambda_n(\text{Id}_n + A_n)^\beta S_n(s) \Pi_n\|_{\mathcal{L}(E_\infty)}^2 ds \\
&\lesssim_{(\beta, M_\Lambda, M_\Pi)} \|\tilde{R}_n^\beta - R_\infty^\beta\|_{\gamma(H_\infty; E_\infty)}^2 \int_0^T s^{2(\beta'-\beta)-1} ds \\
&= \frac{T^{2(\beta'-\beta)}}{2(\beta'-\beta)} \|\tilde{R}_n^\beta - R_\infty^\beta\|_{\gamma(H_\infty; E_\infty)}^2 \rightarrow 0,
\end{aligned} \tag{4.7}$$

where the convergence on the last line follows from the second part of Assumption 4.5. The convergence (II) $\rightarrow 0$ follows by applying Lemma 4.6 with

$$M_n(t) := t^\beta \Lambda_n(\text{Id}_n + A_n)^\beta S_n(t) \Pi_n \quad \text{and} \quad R(t) := t^{\beta'-\beta-\frac{1}{2}} R_\infty^\beta.$$

Indeed, this is justified since $R \in L^2(0, T; \gamma(H_\infty; E_\infty))$ with

$$\|R\|_{L^2(0, T; \gamma(H_\infty; E_\infty))}^2 = \frac{T^{2(\beta'-\beta)}}{2(\beta'-\beta)} \|R_\infty^\beta\|_{\gamma(H_\infty; E_\infty)}^2 < \infty,$$

condition (ii) is verified by applying (C.2) to $\|M_n(t)\|_{\mathcal{L}(E_\infty)}$ combined with Assumptions 4.1 and 4.2 as in (4.7), and Hypothesis (i) holds by Lemma C.1. $\square$

We will show that there exist modifications of W_{A_∞} and $(\tilde{W}_{A_n})_{n \in \mathbb{N}}$ which, for all $p \in [1, \infty)$ and $T \in (0, \infty)$, belong to $L^p(\Omega; C([0, T]; E_\infty))$ and converge in this norm. In particular, as $n \rightarrow \infty$, their trajectories converge uniformly on bounded time intervals, $\mathbb{P}$ -a.s.

The proof is based on the Da Prato–Kwapień–Zabczyk factorization method, first formulated in [20] for Hilbert spaces (see also [21, Section 5.3]), and later extended to UMD-type-2 Banach spaces in [9, Theorem 3.2]. The general idea is to express the process W_{A_n} as the ‘product’ $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}$ of a fractional parabolic integral operator $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'}$ as in Appendix B and auxiliary process $W_{A_n}^{\frac{1}{2}+\beta'}$ as in (4.6), and using the smoothing properties of the former.

Proposition 4.8. *Let $p \in [1, \infty)$ and $T \in (0, \infty)$. If Assumptions 4.1 and 4.2 hold, then for every $n \in \overline{\mathbb{N}}$ there exists a modification of W_{A_n} which belongs to $L^p(\Omega; C([0, T]; E_n))$, which we will identify with W_{A_n} itself.*

If, in addition, Assumption 4.5 holds, then the sequence $(\tilde{W}_{A_n})_{n \in \overline{\mathbb{N}}}$ satisfies

$$\tilde{W}_{A_n} \rightarrow W_{A_\infty} \quad \text{in } L^p(\Omega; C([0, T]; E_\infty)) \quad \text{as } n \rightarrow \infty.$$

Proof. Let $\beta' \in (\beta, \frac{1}{2})$, where $\beta \in [0, \frac{1}{2})$ is as in Assumption 4.2. Since $L^p(\Omega)$ -spaces with higher exponents are embedded in those with lower ones contractively (because $\mathbb{P}(\Omega) = 1$), we assume without loss of generality that $p \in ((\frac{1}{2} - \beta)^{-1}, \infty)$. By the first part of Proposition 4.7 (thus using Assumptions 4.1 and 4.2) and the Fubini theorem, we have for all $n \in \overline{\mathbb{N}}$:

$$W_{A_n}^{\frac{1}{2}+\beta'} \in C([0, T]; L^p(\Omega; E_n)) \hookrightarrow L^p(0, T; L^p(\Omega; E_n)) \cong L^p(\Omega; L^p(0, T; E_n)),$$

where the constants for the first embedding depend only on p and T . In particular, there exists an event $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that $W_{A_n}^{\frac{1}{2}+\beta'}(\omega, \cdot)$ belongs to $L^p(0, T; E_n)$ for all $\omega \in \Omega_1$. It then follows from Proposition B.1(b) that $\mathfrak{J}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}(\omega, \cdot)$ belongs to $C([0, T]; E_n)$, where $(\mathfrak{J}_{A_n}^s)_{s \in [0, \infty)}$ are the fractional parabolic integral operators defined by (B.2) in Appendix B. In this case, the process $\mathfrak{J}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}$ (set to zero outside of Ω_1) belongs to $L^p(\Omega; C([0, T]; E_n))$, and by the factorization theorem [9, Theorem 3.2] it is a modification of $\tilde{W}_{A_n}$.

For the lifted processes, the properties of the embeddings and projections from Assumption 4.1 imply that $\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} \tilde{W}_{A_n}^{\frac{1}{2}+\beta'} \in L^p(\Omega; C([0, T]; E_\infty))$ is a continuous modification of $\tilde{W}_{A_n}$, where $\tilde{\mathfrak{J}}_{A_n}^s := \Lambda_n \mathfrak{J}_{A_n}^s \Pi_n$. Identifying $\tilde{W}_{A_n}$ with its factorized continuous modification for every $n \in \overline{\mathbb{N}}$, we can estimate as follows:

$$\begin{aligned} \|\tilde{W}_{A_n} - W_{A_\infty}\|_{L^p(\Omega; C([0, T]; E_\infty))} &= \|\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} \tilde{W}_{A_n}^{\frac{1}{2}+\beta'} - \mathfrak{J}_{A_\infty}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega; C([0, T]; E_\infty))} \\ &\leq \|\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} (\tilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'})\|_{L^p(\Omega; C([0, T]; E_\infty))} \\ &\quad + \|(\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} - \mathfrak{J}_{A_\infty}^{\frac{1}{2}-\beta'}) W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega; C([0, T]; E_\infty))}. \end{aligned}$$

Since $\frac{1}{2} - \beta' > \frac{1}{p}$, we can apply Corollary B.2(b) to find that $\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'}$ is a bounded linear operator from $L^p(0, T; E_\infty)$ to $C([0, T]; E_\infty)$ whose norm can be bounded independently of n . Thus, by the above discussion and the second part of Proposition 4.7 (which uses Assumption 4.5), we find

$$\begin{aligned} &\|\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} (\tilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'})\|_{L^p(\Omega; C([0, T]; E_\infty))} \\ &\lesssim \|\tilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega \times (0, T); E_\infty)} \leq T^{\frac{1}{p}} \|\tilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{C([0, T]; L^p(\Omega; E_\infty))} \rightarrow 0. \end{aligned}$$

Now we note that, for all $\omega \in \Omega$, Proposition B.3(a) implies that

$$\|\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega, \cdot) - \mathfrak{J}_{A_\infty}^{\frac{1}{2}-\kappa} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega, \cdot)\|_{C([0, T]; E_\infty)} \rightarrow 0.$$

Again by Corollary B.2(b), we moreover have

$$\|\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega, \cdot) - \mathfrak{J}_{A_\infty}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega, \cdot)\|_{C([0, T]; E_\infty)} \lesssim 2 \|W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(0, T; E_\infty)}$$

with constant independent of $n \in \mathbb{N}$, and since $W_{A_\infty}^{\frac{1}{2}+\beta'} \in L^p((0, T) \times \Omega; E_\infty)$, the dominated convergence theorem yields: $\|(\tilde{\mathfrak{J}}_{A_n}^{\frac{1}{2}-\beta'} - \mathfrak{J}_{A_\infty}^{\frac{1}{2}-\beta'}) W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega; C([0, T]; E_\infty))} \rightarrow 0$. $\square$

5. Approximation of semilinear stochastic evolution equations with additive cylindrical Wiener noise

In this section, we shall extend the results from Section 4 regarding the *linear* E_n -valued equation (4.1) to the *semilinear* case:

$$\begin{cases} dX_n(t) = -A_n X_n(t)dt + F_n(t, X_n(t))dt + dW_n(t), & t \in (0, T], \\ X_n(0) = \xi_n. \end{cases} \quad (5.1)$$

That is, as before, we have the same spaces $(E_n)_{n \in \overline{\mathbb{N}}}$ and $(H_n)_{n \in \overline{\mathbb{N}}}$ (satisfying Assumption 4.1), linear operators $(A_n)_{n \in \overline{\mathbb{N}}}$ (satisfying Assumption 4.2) and noise H_n -valued Q_n -cylindrical Wiener processes $(W_n)_{n \in \overline{\mathbb{N}}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. But in this section, we suppose moreover that we are given a drift coefficient function $F_n: \Omega \times [0, T] \times E_n \rightarrow E_n$ and initial datum $\xi_n: \Omega \rightarrow E_n$. We will consider the following semilinear stochastic evolution equation.

Note that $F_n(\omega, t, \cdot)$ is a (nonlinear) operator on E_n for all $(\omega, t) \in \Omega \times [0, T]$; in Section 3, we considered the specific case $[F_n(\omega, t, u_n)](x) := f_n(\omega, t, u_n(x))$ for some real-valued nonlinearity f_n . We will revisit this case in Section 6.

In what follows, we shall impose more precise conditions on the F_n and ξ_n to ensure the well-posedness of (5.1) for every fixed $n \in \overline{\mathbb{N}}$ and to obtain discrete-to-continuum convergence of the respective solutions as $n \rightarrow \infty$.

In Section 5.1 we will assume in particular that the drifts $(F_n)_{n \in \overline{\mathbb{N}}}$ are uniformly globally Lipschitz and of linear growth to obtain the existence of unique global solutions $(X_n(t))_{t \in [0, T]}$, whose lifted counterparts $\tilde{X}_n$ converge to X_∞ in $L^p(\Omega; C([0, T]; E_\infty))$ as $n \rightarrow \infty$, where $p \in [1, \infty)$ is the stochastic integrability of the initial data. These assumptions are relaxed in Section 5.2, where we suppose that the drifts are uniformly locally Lipschitz and uniformly bounded near zero. In general, this comes at the cost of obtaining merely local solutions, converging in a weaker norm. However, if one can show independently that the solutions are global and the $L^p(\Omega; C([0, T]; E_n))$ -norms of X_n are uniformly bounded in $n \in \mathbb{N}$, then we recover the stronger sense of convergence.

5.1. Globally Lipschitz drifts of linear growth

In this section we suppose that the drift coefficients $F_n: \Omega \times [0, T] \times E_n \rightarrow E_n$ in (5.1) for $n \in \overline{\mathbb{N}}$ are uniformly globally Lipschitz continuous and of linear growth. More precisely:

Assumption 5.1. There exist $L_F, C_F \in (0, \infty)$ such that, for all $t \in [0, T]$, $\omega \in \Omega$, $n \in \overline{\mathbb{N}}$ and $x_n, y_n \in E_n$,

$$\begin{aligned} \|F_n(\omega, t, x_n) - F_n(\omega, t, y_n)\|_{E_n} &\leq L_F \|x_n - y_n\|_{E_n}; \\ \|F_n(\omega, t, x_n)\|_{E_n} &\leq C_F (1 + \|x_n\|_{E_n}). \end{aligned}$$

Moreover, the process $(\omega, t) \mapsto F_n(\omega, t, x_n)$ is strongly measurable and adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. $\blacklozenge$

Now let us comment on the existence and uniqueness of solutions to (5.1) for fixed $n \in \overline{\mathbb{N}}$. We will use the following concept of global mild solutions, see [84, pp. 969–970]. In Subsection 5.2, we also introduce the concept of local solutions, which may blow up in finite time. In particular, a local solution which exists $\mathbb{P}$ -a.s. on the whole of $[0, T]$ is in fact global.

Recall that $(S_n(t))_{t \geq 0}$ denotes the C_0 -semigroup on E_n generated by $-A_n$.

Definition 5.2. An E_n -valued stochastic process $X_n = (X_n(t))_{t \in [0, T]}$ is a global mild solution to (5.1) with coefficients (A_n, F_n, ξ_n) if

- (i) $X_n: \Omega \times [0, T] \rightarrow E_n$ is strongly measurable and $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted;
- (ii) $s \mapsto S_n(t-s)F_n(s, X_n(s)) \in L^0(\Omega; L^1(0, t; E_n))$ for every $t \in [0, T]$;
- (iii) $s \mapsto S_n(t-s) \in L^2(0, t; \gamma(H_n; E_n))$ for every $t \in [0, T]$;
- (iv) for all $t \in [0, T]$, we have

$$X_n(t) = S_n(t)\xi_n + \int_0^t S_n(t-s)F_n(s, X_n(s))ds + W_{A_n}(t), \quad \mathbb{P}\text{-a.s.}$$

In the present framework, existence and uniqueness can be proven by showing that the operator $\Phi_{n,T}$ given by

$$[\Phi_{n,T}(u_n)](t) := S_n(t)\xi_n + \int_0^t S_n(t-s)F_n(s, u_n(s))ds + W_{A_n}(t) \quad (5.2)$$

is a well-defined and Lipschitz-continuous mapping on $L^p(\Omega; C([0, T]; E_n))$, whose Lipschitz constant tends to zero as $T \downarrow 0$ (see [84, Proposition 6.1] or [56, Theorem 3.7] for more general results):

Proposition 5.3. *Suppose that Assumptions 4.1, 4.2 and 5.1 are satisfied. Let $n \in \overline{\mathbb{N}}$, $p \in [1, \infty)$, $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and $T \in (0, \infty)$. The operator $\Phi_{n,T}$ given by (5.2) is well defined and Lipschitz continuous on $L^p(\Omega; C([0, T]; E_n))$. Its Lipschitz constant is independent of ξ_n , depends on A_n and F_n only through M_S and L_F , and tends to zero as $T \downarrow 0$.*

Proof. The fact that $S_n \otimes \xi_n \in L^p(\Omega; C([0, T]; E_n))$ is immediate from Assumptions 4.1–4.2 and $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$. We also have $W_{A_n} \in L^p(\Omega; C([0, T]; E_n))$ by the first part of Proposition 4.8. Given $u_n \in L^p(\Omega; C([0, T]; E_n))$, it follows from Assumption 5.1 that $\|s \mapsto F_n(s, u_n(s))\|_{L^\infty(0, T; E_n)} \leq C_F(1 + \|u_n\|_{C([0, T]; E_n)})$, so that $S_n * F_n(\cdot, u_n)$ belongs to $L^p(\Omega; C([0, T]; E_n))$ with

$$\|S_n * F_n(\cdot, u_n)\|_{L^p(\Omega; C([0, T]; E_n))} \leq C_F(1 + \|u_n\|_{L^p(\Omega; C([0, T]; E_n))})$$

by Proposition B.1(b) with $E = F = E_n$, $\alpha = 0$ and $s = 1$ (noting that $S_n * f = \mathfrak{J}_{A_n}^1 f$, see Appendix B). This shows that $\Phi_{n,T}$ is well-defined.

Now let $u_n, v_n \in L^p(\Omega; C([0, T]; E_n))$ and observe that

$$\Phi_{n,T}(u_n) - \Phi_{n,T}(v_n) = \int_0^t S_n(t-s)[F_n(s, u_n(s)) - F_n(s, v_n(s))]ds.$$

A straightforward estimate involving Assumptions 4.1, 4.2 and 5.1 then yields

$$\|\Phi_{n,T}(u_n) - \Phi_{n,T}(v_n)\|_{L^p(\Omega; C([0, T]; E_n))} \leq M_S L_F T \|u_n - v_n\|_{L^p(\Omega; C([0, T]; E_n))}. \quad \square$$

Under the conditions of Proposition 5.3, it follows from the Banach fixed-point theorem that (5.1) has a unique solution on a small enough time interval $[0, T_0]$, which can be extended to a unique global mild solution on any $[0, T]$ by “patching together” solutions on small time intervals:

Proposition 5.4. *Suppose that Assumptions 4.1, 4.2 and 5.1 hold, and let $n \in \overline{\mathbb{N}}$, $p \in [1, \infty)$, $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and $T \in (0, \infty)$. Then (5.1) has a unique global mild solution X_n in $L^p(\Omega; C([0, T]; E_n))$.*

Proof. By Proposition 5.3, there exists $T_0 \in (0, \infty)$ such that Φ_{n, T_0} is a strict contraction on $L^p(\Omega; C([0, T_0]; E_n))$, and thus has a unique fixed point X_n . Since the bound on the Lipschitz constant of $\Phi_{n, T}$ only depended on M_S , L_F and T , we can repeat the previous argument to obtain a unique solution $Y \in L^p(\Omega; C([0, T_0]; E_n))$ for (5.1) with initial datum $\eta_n := X_n(\frac{1}{2}T_0)$, drift $G_n(\cdot, u_n) := F_n(\cdot + \frac{1}{2}T_0, u_n)$ and noise $\widehat{W}_n := W_n(\cdot + \frac{1}{2}T_0)$. It can then be argued directly using Definition 5.2 that the concatenation of the processes X_n and $Y_n(\cdot + \frac{1}{2}T_0)$ is the unique mild solution to (5.1) with the original data on $[0, \frac{3}{2}T_0]$. Proceeding inductively, we find the same conclusion for all intervals $[0, (k + \frac{1}{2})T_0]$ with $k \in \mathbb{N}$ and thus for $[0, T]$. $\square$

For every $n \in \mathbb{N}$, we analogously define the lifted initial datum $\widetilde{\xi}_n: \Omega \rightarrow E_\infty$ by $\widetilde{\xi}_n := \Lambda_n \xi_n$ and the lifted drift coefficient $\widetilde{F}_n: \Omega \times [0, T] \times E_\infty \rightarrow E_\infty$ by

$$\widetilde{F}_n(\omega, t, x) := \Lambda_n F_n(\omega, t, \Pi_n x), \quad (\omega, t, x) \in \Omega \times [0, T] \times E_\infty. \quad (5.3)$$

We will assume that the drift coefficients and initial data are approximated in the following way:

Assumption 5.5 (Pointwise approximation of drifts). For a.e. $(\omega, t) \in \Omega \times [0, T]$ and all $x \in E_\infty$, we have

$$\widetilde{F}_n(\omega, t, x) \rightarrow F_\infty(\omega, t, x) \quad \text{in } E_\infty \text{ as } n \rightarrow \infty. \quad \blacklozenge$$

Assumption 5.6 (Approximation of initial data). Suppose that there exists $p \in [1, \infty)$ such that $(\xi_n)_{n \in \overline{\mathbb{N}}} \in \prod_{n \in \overline{\mathbb{N}}} L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and

$$\widetilde{\xi}_n \rightarrow \xi_\infty \quad \text{in } L^p(\Omega; E_\infty) \text{ as } n \rightarrow \infty. \quad \blacklozenge$$

Under these assumptions, we obtain the main result of this section, namely the following discrete-to-continuum convergence theorem in the context of uniformly globally Lipschitz nonlinearities of linear growth. It is analogous to [56, Theorem 4.3].

Theorem 5.7. *Suppose that Assumptions 4.1, 4.2, 4.5, 5.1, 5.5 and 5.6 hold, with $p \in [1, \infty)$. For all $n \in \overline{\mathbb{N}}$ and $T \in (0, \infty)$, let $X_n = (X_n(t))_{t \in [0, T]}$ denote the unique global solution to (5.1), and let $\widetilde{X}_n := \Lambda_n X_n$. Then we have*

$$\widetilde{X}_n \rightarrow X_\infty \quad \text{in } L^p(\Omega; C([0, T]; E_\infty)) \quad \text{as } n \rightarrow \infty.$$

Its proof involves the lifted counterparts of $\Phi_{n,T}$, defined by

$$\tilde{\Phi}_{n,T} := \Lambda_n \Phi_{n,T} \circ \Pi_n : L^p(\Omega; C([0, T]; E_\infty)) \rightarrow L^p(\Omega; C([0, T]; E_\infty)),$$

i.e., for all $u \in L^p(\Omega; C([0, T]; E_\infty))$ and $t \in [0, T]$, we have

$$\begin{aligned} [\tilde{\Phi}_{n,T}(u)](t) &:= \Lambda_n S_n(t) \xi_n + \int_0^t \Lambda_n S_n(t-s) F_n(s, \Pi_n u(s)) ds + \Lambda_n W_{A_n}(t) \\ &= \tilde{S}_n(t) \tilde{\xi}_n + \int_0^t \tilde{S}_n(t-s) \tilde{F}_n(s, u(s)) ds + \tilde{W}_{A_n}(t), \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

where the second identity is due to Definition 2.1(iii). Using the tensor and convolution notations from Section 2.1, it can be expressed even more concisely as

$$\tilde{\Phi}_{n,T}(u) = \tilde{S}_n \otimes \tilde{\xi}_n + \tilde{S}_n * \tilde{F}_n(\cdot, u) + \tilde{W}_{A_n} \quad (5.4)$$

In particular, we will show that all three terms of (5.4) converge to their “continuum” counterparts; they are addressed by Lemmas 5.8–5.9 below (which are analogous to [56, Lemma 4.4, 4.5(1) and 4.5(3)]), as well as Proposition 4.8.

Lemma 5.8. *If Assumptions 4.1, 4.2, 4.5 and 5.6 hold with $p \in [1, \infty)$, then*

$$\tilde{S}_n \otimes \xi_n \rightarrow S_\infty \otimes \xi_\infty \quad \text{in } L^p(\Omega; C([0, T]; E_\infty)) \quad \text{as } n \rightarrow \infty.$$

Proof. As in the beginning of the proof of Proposition 5.3, it follows from Assumptions 4.1, 4.2 and 5.6 that $S_n \otimes \xi_n \in L^p(\Omega; C([0, T]; E_n))$ for all $n \in \mathbb{N}$. Applying the projection and lifting operators from 4.1, we thus find $\tilde{S}_n \otimes \tilde{\xi}_n \in L^p(\Omega; C([0, T]; E_\infty))$. The triangle inequality implies

$$\begin{aligned} &\|\tilde{S}_n \otimes \tilde{\xi}_n - S_\infty \otimes \xi_\infty\|_{L^p(\Omega; C([0, T]; E_\infty))} \\ &\leq \|\tilde{S}_n \otimes (\tilde{\xi}_n - \xi_\infty)\|_{L^p(\Omega; C([0, T]; E_\infty))} + \|(\tilde{S}_n - S_\infty) \otimes \xi_\infty\|_{L^p(\Omega; C([0, T]; E_\infty))}. \end{aligned}$$

By Assumptions 4.1, 4.2 and 5.6, for the first term we have, as $n \rightarrow \infty$:

$$\|\tilde{S}_n \otimes (\tilde{\xi}_n - \xi_\infty)\|_{L^p(\Omega; C([0, T]; E_\infty))} \leq M_\Lambda M_S M_\Pi \|\tilde{\xi}_n - \xi_\infty\|_{L^p(\Omega; E_\infty)} \rightarrow 0$$

For the second term, first note that $\tilde{S}_n \otimes \xi_\infty(\omega) \rightarrow S_\infty \otimes \xi_\infty(\omega)$ in $C([0, T]; E_\infty)$, $\mathbb{P}$ -a.s., by Theorem 2.3, where we now also use Assumption 4.5. Since we also have

$$\|\tilde{S}_n \otimes \xi_\infty(\omega) - S_\infty \otimes \xi_\infty(\omega)\|_{C([0, T]; E_\infty)} \leq M_S (M_\Lambda M_\Pi + 1) \|\xi_\infty(\omega)\|_{E_\infty},$$

and the right-hand side belongs to $L^p(\Omega)$ by assumption, we deduce that also

$$\|(\tilde{S}_n - S_\infty) \otimes \xi_\infty\|_{L^p(\Omega; C([0, T]; E_\infty))} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \square$$

Lemma 5.9. Suppose that Assumptions 4.1, 4.2, 5.1 and 5.5 are satisfied. Let $p \in [1, \infty)$ and $u \in L^p(\Omega; C([0, T]; E_\infty))$ be given. Then we have

$$\tilde{S}_n * \tilde{F}_n(\cdot, u) \rightarrow S_\infty * F_\infty(\cdot, u) \quad \text{in } L^p(\Omega; C([0, T]; E_\infty)) \text{ as } n \rightarrow \infty.$$

Proof. Similarly to the proof of Proposition 5.3, it follows from Assumptions 4.1, 4.2 and 5.1 that $\tilde{S}_n * \tilde{F}_n(\cdot, u) \in L^p(\Omega; C([0, T]; E_\infty))$ for all $n \in \mathbb{N}$. By the triangle inequality, we can split up the statement into the following two assertions:

- (i) $\tilde{S}_n * \tilde{F}_n(\cdot, u) - \tilde{S}_n * F_\infty(\cdot, u) \rightarrow 0$ in $L^p(\Omega; C([0, T]; E_\infty))$ as $n \rightarrow \infty$;
- (ii) $\tilde{S}_n * F_\infty(\cdot, u) \rightarrow S_\infty * F_\infty(\cdot, u)$ in $L^p(\Omega; C([0, T]; E_\infty))$ as $n \rightarrow \infty$.

For almost every $(\omega, t) \in \Omega \times [0, T]$, we have by Assumptions 5.1 and 4.1:

$$\begin{aligned} & \|\tilde{F}_n(\omega, t, u(\omega, t)) - F_\infty(\omega, t, u(\omega, t))\|_{E_\infty} \\ & \leq C_F(M_\Lambda + 1 + (M_\Pi M_\Lambda + 1)\|u(\omega, t)\|_{E_\infty}). \end{aligned} \quad (5.5)$$

It follows that

$$\begin{aligned} \|\tilde{F}_n(\cdot, u) - F_\infty(\cdot, u)\|_{L^p(\Omega \times (0, T); E_\infty)} & \lesssim_{(C_F, M_\Lambda, M_\Pi)} \|u\|_{L^p(\Omega \times (0, T); E_\infty)} \\ & \lesssim_{(p, T)} \|u\|_{L^p(\Omega; C([0, T]; E_\infty))} < \infty. \end{aligned} \quad (5.6)$$

Since $\tilde{S}_n * f = \tilde{\mathcal{J}}_{A_n}^1 f$ for all $f \in L^p(0, T; E_\infty)$, we can apply Proposition B.1(b) with the spaces $E = F = E_\infty$ and constants $\alpha = 0, s = 1$ to find that

$$\begin{aligned} & \|\tilde{S}_n * (\tilde{F}_n(\cdot, u) - F_\infty(\cdot, u))\|_{C([0, T]; E_\infty)} \\ & \lesssim_{(s, r, T, M_S)} \|\tilde{F}_n(\cdot, u) - F_\infty(\cdot, u)\|_{L^p(\Omega \times (0, T); E_\infty)}. \end{aligned}$$

The latter tends to zero as $n \rightarrow \infty$ by the dominated convergence theorem, which applies in view of Assumption 5.5 and (5.5)–(5.6). This shows (i).

For (ii), we derive in the same way that, for almost every $\omega \in \Omega$,

$$t \mapsto F_\infty(\omega, t, u(\omega, t)) \in L^p(0, T; E_\infty),$$

which implies, cf. Proposition B.3(a) with $\tilde{E} := E_\infty$, that

$$\tilde{S}_n * F_\infty(\omega, \cdot, u(\omega, \cdot)) \rightarrow S_\infty * F_\infty(\omega, \cdot, u(\omega, \cdot)) \quad \text{in } C([0, T]; E_\infty) \quad \text{as } n \rightarrow \infty.$$

The conclusion follows by using the uniform boundedness of the operators $(\tilde{\mathcal{J}}_{A_n}^1)_{n \in \mathbb{N}}$ in $\mathcal{L}(L^p(0, T; E_\infty); C([0, T]; E_\infty))$, asserted in Corollary B.2(b) (with $\tilde{E} := E_\infty$ once more), and finishing the dominated convergence argument as in part (i). $\square$

Proof of Theorem 5.7. By Proposition 5.3 and 4.1, for small enough $T_0 \in (0, \infty)$ there exists a constant $c \in [0, 1)$, depending only on L_F, M_S, M_Λ and M_Π , such that the following holds for all $u, v \in L^p(\Omega; C([0, T_0]; E_\infty))$:

$$\sup_{n \in \mathbb{N}} \|\tilde{\Phi}_{n,T_0}(u) - \tilde{\Phi}_{n,T_0}(v)\|_{L^p(\Omega; C([0, T_0]; E_\infty))} \leq c \|u - v\|_{L^p(\Omega; C([0, T_0]; E_\infty))}.$$

Moreover, by Proposition 5.4, for every $n \in \overline{\mathbb{N}}$, there exists a unique global solution X_n in $L^p(\Omega; C([0, T]; E_n))$ to (5.1), which in particular satisfies $X_n = \Phi_{n,T_0}(X_n)$ when restricted to $[0, T_0]$. By Assumption 4.1, this implies $\tilde{X}_n = \tilde{\Phi}_{n,T_0}(\tilde{X}_n)$. Hence,

$$\begin{aligned} \|X_\infty - \tilde{X}_n\|_{L^p(\Omega; C([0, T_0]; E_\infty))} &= \|\Phi_{\infty, T_0}(X_\infty) - \tilde{\Phi}_{n, T_0}(\tilde{X}_n)\|_{L^p(\Omega; C([0, T_0]; E_\infty))} \\ &\leq \|\Phi_{\infty, T_0}(X_\infty) - \tilde{\Phi}_{n, T_0}(X_\infty)\|_{L^p(\Omega; C([0, T_0]; E_\infty))} \\ &\quad + \|\tilde{\Phi}_{n, T_0}(X_\infty) - \tilde{\Phi}_{n, T_0}(\tilde{X}_n)\|_{L^p(\Omega; C([0, T_0]; E_\infty))} \\ &\leq \|\Phi_{\infty, T_0}(X_\infty) - \tilde{\Phi}_{n, T_0}(X_\infty)\|_{L^p(\Omega; C([0, T_0]; E_\infty))} \\ &\quad + c \|X_\infty - \tilde{X}_n\|_{L^p(\Omega; C([0, T_0]; E_\infty))}, \end{aligned}$$

so that Lemmas 5.8–5.9 and Proposition 4.8 yield (using all of Assumptions 4.1, 4.2, 4.5, 5.1, 5.5 and 5.6):

$$\begin{aligned} \|X_\infty - \tilde{X}_n\|_{L^p(\Omega; C([0, T_0]; E_\infty))} \\ \leq \frac{1}{1-c} \|\Phi_{\infty, T_0}(X_\infty) - \tilde{\Phi}_{n, T_0}(X_\infty)\|_{L^p(\Omega; C([0, T_0]; E_\infty))} \rightarrow 0 \end{aligned} \quad (5.7)$$

as $n \rightarrow \infty$. In order to extend the convergence to arbitrary time horizons, we write

$$\begin{aligned} \|X_\infty - \tilde{X}_n\|_{L^p(\Omega; C([0, \frac{3}{2}T_0]; E_\infty))} &\leq \|X_\infty - \tilde{X}_n\|_{L^p(\Omega; C([0, \frac{1}{2}T_0]; E_\infty))} \\ &\quad + \|X_\infty - \tilde{X}_n\|_{L^p(\Omega; C([\frac{1}{2}T_0, \frac{3}{2}T_0]; E_\infty))}. \end{aligned}$$

The first term tends to zero as $n \rightarrow \infty$ by (5.7). As for the second term, we note that $X_n|_{[\frac{1}{2}T_0, \frac{3}{2}T_0]}$ are the respective solutions to (5.1) with shifted drift functions $(F_n(\cdot + \frac{1}{2}T_0, \cdot))_{n \in \overline{\mathbb{N}}}$ and initial values $(X_n(\frac{1}{2}T_0))_{n \in \overline{\mathbb{N}}}$. Since the Lipschitz constants of the fixed point operators defined above did not depend on the initial datum and only depended on F through its (time-independent) Lipschitz constant L_F , we can repeat the same argument to find that the second term tends to zero. Proceeding by induction, we obtain $\tilde{X}_n \rightarrow X_\infty$ in $L^p(\Omega; C([0, (1 + \frac{k}{2})T_0]; E_\infty))$ for any $k \in \mathbb{N}$, and thus in $L^p(\Omega; C([0, T]; E_\infty))$ for any $T \in (0, \infty)$. $\square$

5.2. Locally Lipschitz nonlinearities

In this section, we work under the weaker assumption that the drift coefficients $(F_n)_{n \in \overline{\mathbb{N}}}$ are locally Lipschitz continuous, with local-Lipschitz-continuity constants uniformly bounded in $t \in [0, T]$, $\omega \in \Omega$ and $n \in \overline{\mathbb{N}}$. Moreover, we replace the uniform linear growth condition from Assumption 5.1 by the assumption that $F_n(t, \omega, 0)$ is bounded, again uniformly in t , ω and n ; we will also call this notion of boundedness local since it only involves $u = 0$. Thus, we assume that the $(F_n)_{n \in \overline{\mathbb{N}}}$ are locally uniformly Lipschitz and locally uniformly bounded:

Assumption 5.10. For every $r \in (0, \infty)$ there exists a constant $L_F^{(r)} \in (0, \infty)$ such that for almost every $(\omega, t) \in \Omega \times [0, T]$, all $n \in \overline{\mathbb{N}}$ and every $x_n, y_n \in E_n$ such that $\|x_n\|_{E_n}, \|y_n\|_{E_n} \leq r$, we have

$$\|F_n(\omega, t, x_n) - F_n(\omega, t, y_n)\|_{E_n} \leq L_F^{(r)} \|x_n - y_n\|_{E_n}.$$

Moreover, for every $x_n \in E_n$, $n \in \overline{\mathbb{N}}$ the process $(\omega, t) \mapsto F_n(\omega, t, x_n)$ is strongly measurable and adapted, and there exists a constant $C_{F,0}$ such that

$$\|F_n(\omega, t, 0)\|_{E_n} \leq C_{F,0} \quad \text{for all } n \in \overline{\mathbb{N}}. \quad \blacklozenge$$

Under these conditions, we can in general not expect to obtain global solutions of (5.1) in the sense of Definition 5.2. Instead, we need to work with locally defined E_n -valued stochastic processes, i.e., with mappings of the form

$$Y : \{(\omega, t) \in \Omega \times [0, T] : t \in [0, \tau(\omega))\} \rightarrow E_n \quad (5.8)$$

for some stopping time $\tau : \Omega \rightarrow [0, T]$. We denote such a process by $Y = (Y(t))_{t \in [0, \tau)}$. If the half-open interval $[0, \tau(\omega))$ in (5.8) is replaced by $[0, \tau(\omega)]$, then we write $Y = (Y(t))_{t \in [0, \tau]}$ instead. We say that $(Y(t))_{t \in [0, \tau)}$ is *admissible* if

- for all $t \in [0, T]$, the mapping $\{\omega \in \Omega : t < \tau(\omega)\} \ni \omega \mapsto Y(\omega, t) \in E_n$ is $\mathcal{F}_t$ -measurable;
- the mapping $[0, \tau(\omega)) \ni t \mapsto Y(\omega, t) \in E_n$ is continuous, $\mathbb{P}$ -a.s.

We denote by $V^{\text{loc}}([0, \tau) \times \Omega; E_n)$ the space of admissible E_n -valued processes $(Y(t))_{t \in [0, \tau)}$ for which there exists a sequence $(\tau_m)_{m \in \mathbb{N}}$ of stopping times such that, for $\mathbb{P}$ -a.e. $\omega \in \Omega$, we have $\tau_m(\omega) \uparrow \tau(\omega)$ as $m \rightarrow \infty$ and $\|Y\|_{C([0, \tau_m(\omega)]; E_n)} < \infty$ for all $m \in \mathbb{N}$. As in [84, Section 8], we define local solutions to (5.1) as follows:

Definition 5.11. An admissible E_n -valued stochastic process $X_n = (X_n(t))_{t \in [0, \tau)}$ is said to be a local solution to (5.1) with coefficients (A_n, F_n, ξ_n) if there exists a sequence $(\tau_m)_{m \in \mathbb{N}}$ of stopping times such that $\tau_m \uparrow \tau$ as $m \rightarrow \infty$, $\mathbb{P}$ -a.s., and for all $m \in \mathbb{N}$ we have

- for every $t \in [0, T]$, the process $(\omega, s) \mapsto S_n(t-s)F_n(\omega, s, X_n(\omega, s))\mathbf{1}_{[0, \tau_m]}(s)$ belongs to $L^0(\Omega; L^1(0, t; E_n))$;
- for every $t \in [0, T]$, $s \mapsto S_n(t-s)\mathbf{1}_{[0, \tau_m]}(s) \in L^2(0, t; \gamma(H_n; E_n))$;
- it holds $\mathbb{P}$ -a.s. that for all $t \in [0, \tau_m]$, we have

$$\begin{aligned} X_n(t) &= S_n(t)\xi_n + \int_0^t S_n(t-s)F_n(s, X_n(s))\mathbf{1}_{[0, \tau_m]}(s)ds \\ &\quad + \int_0^t S_n(t-s)\mathbf{1}_{[0, \tau_m]}(s)dW_n(s). \end{aligned}$$

We say that a local solution $(X_n(t))_{t \in [0, \tau)}$ to (5.1) is *maximal* if for any other local solution $(\bar{X}_n(t))_{t \in [0, \bar{\tau})}$ it holds $\mathbb{P}$ -a.s. that $\bar{\tau} \leq \tau$ and $X_n|_{[0, \bar{\tau})} \equiv \bar{X}_n$. It is called *global* if $\tau = T$ holds $\mathbb{P}$ -a.s. and there exists a solution $(\hat{X}_n(t))_{t \in [0, T]}$ to (5.1) in the sense of Definition 5.2 such that $\hat{X}_n|_{[0, \tau)} \equiv X_n$, $\mathbb{P}$ -a.s. The stopping time τ is called an *explosion time* if

$$\limsup_{t \uparrow \tau(\omega)} \|X_n(\omega, t)\|_{E_n} = \infty \quad \text{for a.e. } \omega \in \Omega \text{ such that } \tau(\omega) < T. \quad (5.9)$$

The following local well-posedness result then follows from [84, Theorem 8.1]:

Theorem 5.12 ([84, Theorem 8.1]). *Suppose that Assumptions 4.1, 4.2 and 5.10 are satisfied, and let $n \in \overline{\mathbb{N}}$, $p \in [1, \infty)$, $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and $T \in (0, \infty)$ be given. Then (5.1) has a unique maximal local mild solution $(X_n(t))_{t \in [0, \sigma_n]}$ in $V^{\text{loc}}([0, \sigma_n] \times \Omega; E_n)$, where $\sigma_n: \Omega \rightarrow [0, T]$ is an explosion time.*

Combined with the convergence Assumptions 5.6 and 5.5, we can argue analogously to [57, Theorem 3.3 and Corollary 3.4] to derive the following extension of Theorem 5.7 to the present setting.

Theorem 5.13. *Suppose that Assumptions 4.1, 4.2, 5.6, 5.10 and 5.5 are satisfied. For $n \in \overline{\mathbb{N}}$, let $(X_n(t))_{t \in [0, \sigma_n]}$ be the maximal local solution to (5.1) with explosion time $\sigma_n: \Omega \rightarrow [0, T]$, and set $\tilde{X}_n := \Lambda_n X_n$. Then the following is true:*

- (i) *We have $\tilde{X}_n \mathbf{1}_{[0, \sigma_n \wedge \sigma_\infty)} \rightarrow X_\infty \mathbf{1}_{[0, \sigma_\infty)}$ in $L^0(\Omega \times [0, T]; E_\infty)$ as $n \rightarrow \infty$.*

If, moreover, $\sigma_n = T$ holds $\mathbb{P}$ -a.s. for all $n \in \mathbb{N}$ and $p \in [1, \infty)$ is such that

$$\sup_{n \in \mathbb{N}} \|X_n\|_{L^p(\Omega; C([0, T]; E_n))} < \infty, \quad (5.10)$$

then the following assertions also hold:

- (ii) *We have $\sigma_\infty = T$, $\mathbb{P}$ -a.s.*
 (iii) *If $p \in (1, \infty)$, then for all $p^- \in [1, p)$ we have*

$$\tilde{X}_n \rightarrow X_\infty \quad \text{in } L^{p^-}(\Omega; C([0, T]; E_\infty)) \text{ as } n \rightarrow \infty.$$

Similarly to [57, Theorem 3.3 and Corollary 3.4], the proof of Theorem 5.13 relies on the following general approximation results for locally defined processes:

Theorem 5.14 ([57, Theorem 2.1 and Corollary 2.5]). *Let $(E, \|\cdot\|_E)$ be a real and separable Banach space and $T \in (0, \infty)$. For every $n \in \overline{\mathbb{N}}$, suppose that $(Y_n(t))_{t \in [0, \sigma_n]}$ is a continuous and adapted E -valued locally defined process with explosion time $\sigma_n: \Omega \rightarrow (0, T]$, and define the stopping times $\rho_n^{(r)}: \Omega \rightarrow [0, T]$ by*

$$\rho_n^{(r)} := \inf\{t \in (0, \sigma_n) : \|Y_n(t)\|_E > r\}, \quad r \in (0, \infty), \quad (5.11)$$

with the convention that $\inf \emptyset := T$. Moreover, suppose that for each $r \in (0, \infty)$ there exists a (globally defined) continuous and adapted E -valued process $(Y_n^{(r)}(t))_{t \in [0, T]}$ which satisfies the following two conditions:

- (a) *For all $n \in \overline{\mathbb{N}}$ and $r \in (0, \infty)$, it holds $\mathbb{P}$ -a.s. that*

$$Y_n^{(r)} \mathbf{1}_{[0, \rho_n^{(r)}]} \equiv Y_n \mathbf{1}_{[0, \rho_n^{(r)}]} \quad \text{on } [0, T];$$

(b) For all $r \in (0, \infty)$ we have

$$Y_n^{(r)} \rightarrow Y_\infty^{(r)} \quad \text{in } L^0(\Omega; C([0, T]; E)) \text{ as } n \rightarrow \infty.$$

Then the following assertions hold:

(i) For all $r \in (0, \infty)$ and $\varepsilon > 0$ it holds $\mathbb{P}$ -a.s. that

$$\liminf_{n \rightarrow \infty} \rho_n^{(r)} \leq \rho_\infty^{(r)} \leq \limsup_{n \rightarrow \infty} \rho_n^{(r+\varepsilon)}.$$

(ii) For all $r \in (0, \infty)$ and $\varepsilon > 0$, we have

$$Y_n \mathbf{1}_{[0, \rho_\infty^{(r)} \wedge \rho_n^{(r+\varepsilon)}]} \rightarrow Y_\infty \mathbf{1}_{[0, \rho_\infty^{(r)}]} \quad \text{in } L^0(\Omega; B_b([0, T]; E)) \text{ as } n \rightarrow \infty,$$

where $B_b([0, T]; E)$ denotes the space of bounded and strongly measurable functions from $[0, T]$ to E .

(iii) We have

$$Y_n \mathbf{1}_{[0, \sigma_\infty \wedge \sigma_n]} \rightarrow Y_\infty \mathbf{1}_{[0, \sigma_\infty]} \quad \text{in } L^0(\Omega \times [0, T]; E) \text{ as } n \rightarrow \infty.$$

If, in addition, we have $\mathbb{P}(\sigma_n = T) = 1$ for all $n \in \mathbb{N}$, and $p \in [1, \infty)$ is such that

$$\sup_{n \in \mathbb{N}} \|Y_n\|_{L^p(\Omega; C([0, T]; E))} < \infty,$$

then the following assertions also hold:

(iv) We have $\mathbb{P}(\sigma_\infty = T) = 1$ and $X_\infty \in L^q(\Omega; C([0, T]; E))$.

(v) If $p \in (1, \infty)$, then for all $p^- \in [1, p)$ we have

$$Y_n \rightarrow Y_\infty \quad \text{in } L^{p^-}(\Omega; C([0, T]; E)) \text{ as } n \rightarrow \infty.$$

Proof of Theorem 5.13. For all $n \in \overline{\mathbb{N}}$ and $r \in (0, \infty)$, define $F_n^{(r)}: \Omega \times [0, T] \times E_n \rightarrow E_n$ by

$$F_n^{(r)}(\omega, t, x_n) := \begin{cases} F_n(\omega, t, x_n), & \text{if } \|\Lambda_n x_n\|_{E_\infty} \leq r, \\ F_n\left(\omega, t, \frac{r x_n}{\|\Lambda_n x_n\|_{E_\infty}}\right), & \text{otherwise.} \end{cases} \quad (5.12)$$

For any fixed $r > 0$, the sequence $(F_n^{(r)})_{n \in \overline{\mathbb{N}}}$ satisfies Assumptions 5.1 and 5.5. Indeed, to establish the former, we first note that $F_n^{(r)}$ can be written as

$$F_n^{(r)}(\omega, t, x_n) = F_n(\omega, t, \Pi_n R_r(\Lambda_n x_n)), \quad (5.13)$$

where $R_r: E_\infty \rightarrow E_\infty$ denotes the canonical retraction of E_∞ onto the closed ball around 0 in E_∞ with radius r :

$$R_r(x) := \begin{cases} x, & \text{if } \|x\|_{E_\infty} \leq r; \\ \frac{rx}{\|x\|_{E_\infty}}, & \text{otherwise.} \end{cases}$$

An elementary estimate shows that R_r is Lipschitz with constant 2. It follows that $(F_n^{(r)})_{n \in \mathbb{N}}$ is uniformly globally Lipschitz with constant $L_{F^{(r)}} \leq L_F^{(rM_\Pi)} M_\Pi M_\Lambda$, and thus of linear growth with uniform constant $C_{F^{(r)}} \leq \max\{L_{F^{(r)}}, C_{F,0}\}$, so that it satisfies Assumption 5.1.

In order to show that $(F_n^{(r)})_{n \in \mathbb{N}}$ satisfies Assumption 5.5, first note that for every sequence $(y_n)_{n \in \mathbb{N}} \subseteq E_\infty$ converging to some y in E_∞ , we have $\tilde{F}_n(\omega, t, y_n) \rightarrow F_\infty(\omega, t, y)$. Indeed, by the triangle inequality it suffices to note that

$$\|\tilde{F}_n(\omega, t, y_n) - \tilde{F}_n(\omega, t, y)\|_{E_\infty} \rightarrow 0 \quad \text{and} \quad \|\tilde{F}_n(\omega, t, y) - F_\infty(\omega, t, y)\|_{E_\infty} \rightarrow 0$$

as $n \rightarrow \infty$, respectively because $(\tilde{F}_n)_{n \in \mathbb{N}}$ is uniformly locally Lipschitz (with local-Lipschitz-continuity constants $L_{\tilde{F}}^{(r)} \leq M_\Pi M_\Lambda L_F^{(r)}$) and since Assumption 5.5 holds for $(F_n)_{n \in \mathbb{N}}$. Writing

$$\tilde{F}_n^{(r)}(\omega, t, x) = \tilde{F}_n(\omega, t, R_r(\Lambda_n \Pi_n x)),$$

see (5.13), we can apply the above observation to the sequence $y_n := R_r(\Lambda_n \Pi_n x)$, which converges to $y := R_r(x)$ in E_∞ as $n \rightarrow \infty$ in view of Definition 2.1(ii) and the (Lipschitz) continuity of R_r . Therefore, we find $\tilde{F}_n^{(r)}(\omega, t, x) \rightarrow F_\infty^{(r)}(\omega, t, x)$, thus proving the claim that Assumption 5.5 holds for $(F_n^{(r)})_{n \in \mathbb{N}}$ as well.

For each $r > 0$, Assumption 5.1 for $(F_n^{(r)})_{n \in \mathbb{N}}$ yields the existence of a unique global solution $(X_n^{(r)}(t))_{t \in [0, T]}$ to (5.1) with coefficients $(A_n, F_n^{(r)}, \xi_n)$. In order to establish statement (i) of the present theorem, we will apply the corresponding parts (i)–(iii) of Theorem 5.14 to the processes $Y_n := \tilde{X}_n$; hence we need to verify its conditions (a) and (b) for $(\tilde{X}_n)_{n \in \mathbb{N}}$. First note that we have $\rho_n^{(r)} \leq \sigma_n$, where $\rho_n^{(r)}$ is defined by (5.11), and that the restrictions of $X_n^{(r)}$ and X_n to $[0, \rho_n^{(r)})$ are local solutions to (5.1) with coefficients $(A_n, F_n^{(r)}, \xi_n)$ and (A_n, F_n, ξ_n) , respectively. Since it holds $\mathbb{P}$ -a.s. that $\|\Lambda_n X_n(t)\|_{E_\infty} \leq r$ for $t \in [0, \rho_n^{(r)})$, we find

$$F_n(\cdot, X_n) \equiv F_n^{(r)}(\cdot, X_n) \quad \text{on } [0, \rho_n^{(r)}), \quad \mathbb{P}\text{-a.s.},$$

hence $(X_n(t))_{t \in [0, \rho_n^{(r)})}$ is in fact also a local solution of the equation with coefficients $(A_n, F_n^{(r)}, \xi_n)$. Therefore, the local uniqueness of (5.1) (cf. [84, Lemma 8.2]) implies that $X_n^{(r)} \equiv X_n$ on $[0, \rho_n^{(r)}]$ holds $\mathbb{P}$ -a.s., and applying Λ_n on both sides verifies (a). Condition (b) follows by applying Theorem 5.7 to $(F_n^{(r)})_{n \in \mathbb{N}}$, proving (i).

Finally, since $(\Lambda_n)_{n \in \mathbb{N}}$ is uniformly bounded in view of Definition 2.1(i), we see that (5.10) implies that the conditions of Theorem 5.14(iv) and (v) are satisfied, which directly yields the remaining assertions 5.10 and 5.10. $\square$

6. Reaction–diffusion type equations

In this section, we introduce another family of Banach spaces $(B_n)_{n \in \mathbb{N}}$ such that each B_n embeds into E_n and $\tilde{B} \supseteq B_\infty$ (along with other assumptions, given in Subsection 6.1), and consider

the B_n -valued counterparts of (5.1). The main purpose of this setting is to eventually specialize to the class of stochastic reaction–diffusion type equations which are formally given, for any $n \in \overline{\mathbb{N}}$, by

$$\begin{cases} dX_n(t, x) = -A_n X_n(t, x)dt + f_n(t, X_n(t, x))dt + dW_n(t, x), \\ X_n(0, x) = \xi_n(x), \end{cases} \quad (6.1)$$

where $t \in (0, T]$, $T \in (0, \infty)$, $x \in \mathcal{D}_n \subseteq \mathbb{R}^d$ and $f_n: \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz (real-valued) function. This problem amounts to letting the drift F_n in (5.1) be a Nemytskii operator (also called a *superposition operator*), i.e., defining it by

$$[F_n(\omega, t, u)](x) := f_n(\omega, t, u(x)), \quad (\omega, t, x) \in \Omega \times [0, T] \times \mathcal{D}_n, \quad (6.2)$$

for a given function $u: \mathcal{D}_n \rightarrow \mathbb{R}$. However, in order for a Nemytskii operator F_n to inherit the local Lipschitz continuity from f_n , we cannot view it as acting on the UMD-type-2 Banach space $E_n = L^q(\mathcal{D}_n)$ with $q \in [1, \infty)$. In fact, the thus-defined operator $F_n(\omega, t, \cdot)$ is locally Lipschitz if and only if f_n is globally Lipschitz; this follows by applying [1, Theorem 3.10] to $F_n(t, \omega, \cdot)$, which for fixed $(t, \omega) \in [0, T] \times \Omega$ is “the superposition operator generated by $\mathcal{D} \times \mathbb{R} \ni (x, \eta) \mapsto f_n(t, \omega, \eta)$ ” in the language of [1]. In particular, if $(F_n)_{n \in \overline{\mathbb{N}}}$ is a family of Nemytskii operators which is uniformly locally Lipschitz in the sense of Assumption 5.10 on the spaces $E_n = L^q(\mathcal{D}_n)$, then it is in fact globally Lipschitz and of linear growth in the sense of Assumption 5.1.

Thus, we will instead view F_n as an operator on $B_n := C(\overline{\mathcal{D}_n})$ (coinciding with $B_n = L^\infty(\mathcal{D}_n)$ if $\mathcal{D}_n$ is discrete). This, in turn, poses a difficulty for stochastic evolution equations, since there is no theory for the stochastic integration of integrands taking their values in a space of continuous functions. This is due to the poor geometric properties of $(C(\overline{\mathcal{D}_n}), \|\cdot\|_\infty)$ as a Banach space: The most general notion of stochastic integration in Banach spaces (see [83]) requires at least the UMD property; such spaces are, in particular, reflexive [49, Theorem 4.3.3], which $C(\overline{\mathcal{D}_n})$ fails to be.

One way to circumvent this issue is to proceed as in [57, Section 3.2]; namely, defining the fractional domain spaces

$$\dot{E}_n^\alpha := D(A_n^{\alpha/2}), \quad \|x_n\|_{\dot{E}_n^\alpha} := \|(\text{Id}_n + A_n)^{\alpha/2} x_n\|_{E_n}, \quad (6.3)$$

and supposing that $\dot{E}_n^\theta \hookrightarrow C(\overline{\mathcal{D}_n}) \hookrightarrow L^q(\mathcal{D}_n)$ continuously and densely for some $\theta \in [0, 1)$, one can carry out the stochastic integration in the space $\dot{E}_n^\theta$, while working with $C(\overline{\mathcal{D}_n})$ -valued processes for the fixed-point arguments.

In applications, we typically assume that A_n is an (unbounded) linear differential operator on $L^q(\mathcal{D}_n)$, where $q \in [2, \infty)$, augmented with some boundary conditions (“b.c.”) such that $\dot{E}_n^\alpha$ is the fractional Sobolev space $W_{\text{b.c.}}^{\alpha, q}(\mathcal{D}_n)$ of order α . We then suppose that θ is chosen large enough in relation to the dimension d that we have the continuous and dense Sobolev embedding $W_{\text{b.c.}}^{\alpha, q}(\mathcal{D}_n) \hookrightarrow C_{\text{b.c.}}(\overline{\mathcal{D}_n})$.

In Section 6.1 we will specify the abstract formulation of the setting outlined above, as well as some additional uniformity conditions with respect to $n \in \overline{\mathbb{N}}$. These will be used to, as a first step, derive B_n -valued counterparts to the E_n -valued discrete-to-continuum approximation results for globally Lipschitz drifts of linear growth from Subsection 5.1; in Subsection 6.2 we do the same

for the B_n -valued setting with locally Lipschitz and locally bounded drifts. In the latter case, the solution is local in general, but in Section 6.3 we state an extra dissipativity assumption on F_n under which the existence of global solutions to (6.1) has been proven in [57, Section 4]. These processes then also converge in an improved sense, and we can apply this to Section 3.

6.1. Setting and convergence for globally Lipschitz drifts

We start by specifying the abstract “ B_n -valued setting” for the treatment of reaction–diffusion type equations which was outlined at the beginning of this section. We use the following extensions of Assumptions 4.1, 4.2 and 4.5:

Assumption 6.1 (*Embedded discrete-to-continuum spaces*). In addition to the UMD-type-2 Banach spaces $(E_n)_{n \in \mathbb{N}}$ and the Hilbert spaces $(H_n)_{n \in \mathbb{N}}$ from Assumption 4.1, let $(B_n)_{n \in \mathbb{N}}$ and $\tilde{B}$ be real Banach spaces such that $B_n \hookrightarrow E_n$ continuously and densely for each $n \in \mathbb{N}$. Suppose that $(B_n)_{n \in \mathbb{N}}$ is a discrete-to-continuum approximation of B_∞ in $\tilde{B}$ (see Definition 2.1), sharing the same projection and lifting operators $(\Pi_n)_{n \in \mathbb{N}}$, $(\Lambda_n)_{n \in \mathbb{N}}$ with $(E_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$. We set $\tilde{M}_\Pi := \sup_{n \in \mathbb{N}} \|\Pi_n\|_{\mathcal{L}(\tilde{B}; B_n)}$ and $\tilde{M}_\Lambda := \sup_{n \in \mathbb{N}} \|\Lambda_n\|_{\mathcal{L}(B_n; \tilde{B})}$. ♦

Assumption 6.2 (*Uniform boundedness of restricted semigroups*). Let the semigroups $(S_n)_{n \in \mathbb{N}}$ on $(E_n)_{n \in \mathbb{N}}$ be as in Assumption 4.2. In addition, suppose that S_n leaves B_n invariant for every $n \in \mathbb{N}$, and that the restrictions $(S_n|_{B_n})_{n \in \mathbb{N}}$ form a uniformly bounded sequence of C_0 -semigroups on $(B_n)_{n \in \mathbb{N}}$ with uniform growth exponent $w = 0$. This means (recall Definition 2.2) that there exists $\tilde{M}_S \in [1, \infty)$ such that

$$\|S_n(t)\|_{\mathcal{L}(B_n)} \leq \tilde{M}_S \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, \infty). \quad \blacklozenge \quad (6.4)$$

Assumption 6.3 (*Resolvents converge in $\tilde{B}$*). In addition to the resolvent convergences with respect to $(E_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ from Assumption 4.5, suppose that $\tilde{R}_n x \rightarrow R_\infty x$ in $\tilde{B}$ as $n \rightarrow \infty$ for all $x \in B_\infty$. ♦

By [30, Chapter II, Proposition 2.3], Assumptions 6.1 and 6.2 imply that the generator of $(S_n(t)|_{B_n})_{t \geq 0}$ is the operator $-A_n|_{B_n} : D(A_n|_{B_n}) \subseteq B_n \rightarrow B_n$ defined by

$$-A_n|_{B_n} x_n := -A_n x_n \quad \text{on } D(A_n|_{B_n}) := \{x_n \in B_n \cap D(A_n) : A_n x_n \in B_n\},$$

which is known as *the part of $-A_n$ in B_n* . Therefore, by Theorem 2.3, Assumption 6.3 implies $\tilde{S}_n|_{B_n} \otimes x \rightarrow S|_{B_\infty} \otimes x$ in $C([0, T]; \tilde{B})$, as $n \rightarrow \infty$, for all $x \in B_\infty$ and $T \in (0, \infty)$.

The following assumption is necessary in order to circumvent the aforementioned problem regarding stochastic integration in arbitrary separable Banach spaces B_n . It replaces [57, Assumption (A3)], which forced all the spaces $(\dot{E}_n^\alpha)_{n \in \mathbb{N}}$ to essentially be the same, which is not satisfied in applications such as discrete-to-continuum approximation, where each $\dot{E}_n^\alpha$ consists of functions defined on a different domain. The following assumption does hold for instance in the graph-discretized setting of Section 3, see Lemma 3.15.

Assumption 6.4 (*Uniform ultracontractivity*). Let $\theta \in [0, 1)$, $M_\theta \in [0, \infty)$ be such that the embeddings $\dot{E}_n^\theta \hookrightarrow B_n \hookrightarrow E_n$ hold continuously and densely for all $n \in \mathbb{N}$, with

$$\|S_n(t)x_n\|_{B_n} \leq M_\theta t^{-\theta/2} \|x_n\|_{E_n} \quad \text{for all } x_n \in E_n \text{ and } t \in [0, \infty). \quad \blacklozenge$$

The uniformity in n of the constant M_θ enables us to prove the following discrete-to-continuum approximation result for the stochastic convolutions $(W_{A_n})_{n \in \mathbb{N}}$ as B_n -valued processes (i.e., a B_n -valued counterpart to Proposition 4.8):

Proposition 6.5. *Let $p \in [1, \infty)$ and $T \in (0, \infty)$ be given, and suppose that Assumptions 6.1, 6.2 and 6.4 hold, with $\theta + 2\beta < 1$. For every $n \in \mathbb{N}$, there exists a modification of W_{A_n} which belongs to $L^p(\Omega; C([0, T]; B_n))$, and we identify these modifications with the processes $(W_{A_n})_{n \in \mathbb{N}}$ themselves.*

Under the additional Assumption 6.3, we have

$$\tilde{W}_{A_n} \rightarrow W_{A_\infty} \quad \text{in } L^p(\Omega; C([0, T]; \tilde{B})) \text{ as } n \rightarrow \infty.$$

Proof. Fix a $\beta' \in (\beta, \frac{1}{2})$ such that $\theta + 2\beta' < 1$, where $\beta \in [0, \frac{1}{2})$ is as in 4.2. Without loss of generality, we may assume that $p \in (2, \infty)$ is so large that, in fact, $\theta + 2\beta' < 1 - \frac{2}{p}$. As in the proof of Proposition 4.8, we see that $W_{A_n}^{\frac{1}{2}+\beta'}(\omega, \cdot)$, where $W_{A_n}^{\frac{1}{2}+\beta'}$ is the auxiliary process defined by (4.6), belongs to $L^p(0, T; E_n)$ for $\mathbb{P}$ -a.e. $\omega \in \Omega$, hence $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}(\omega, \cdot) \in C([0, T]; B_n)$ by applying Proposition B.1(b) with $E := E_n$, $F := B_n$ and $\alpha := \theta$ from Assumption 6.4. Moreover, one finds that the process $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}$ is a continuous modification of W_{A_n} , belonging to $L^p(\Omega; C([0, T]; B_n))$. It now suffices to establish the following, as $n \rightarrow \infty$:

$$\begin{aligned} \|\mathfrak{I}_{A_n}^{\frac{1}{2}-\kappa} (\tilde{W}_{A_n}^{\frac{1}{2}+\kappa} - W_{A_\infty}^{\frac{1}{2}+\kappa})\|_{L^p(\Omega; C([0, T]; \tilde{B}))} &\rightarrow 0, \\ \|\mathfrak{I}_{A_n}^{\frac{1}{2}-\kappa} W_{A_\infty}^{\frac{1}{2}+\kappa} - \mathfrak{I}_{A_\infty}^{\frac{1}{2}-\kappa} W_{A_\infty}^{\frac{1}{2}+\kappa}\|_{L^p(\Omega; C([0, T]; \tilde{B}))} &\rightarrow 0. \end{aligned}$$

These convergences follow by arguing as in the proof of Proposition 4.8, where we now need Corollary B.2(c) and Proposition B.3(b) instead of Corollary B.2(b) and Proposition B.3(a), respectively. $\square$

For the initial data, we consider the following analog to Assumption 5.6:

Assumption 6.6 (Approximation of B_n -valued initial data). Suppose that there exists $p \in [1, \infty)$ such that $(\xi_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} L^p(\Omega; B_n)$ and

$$\tilde{\xi}_n \rightarrow \xi_\infty \quad \text{in } L^p(\Omega; \tilde{B}) \text{ as } n \rightarrow \infty. \quad \blacklozenge$$

Finally, regarding the drift coefficients $(F_n)_{n \in \mathbb{N}}$, we now suppose:

Assumption 6.7 (Globally Lipschitz B_n -valued drifts of linear growth). Suppose that Assumption 5.1 holds with $(B_n)_{n \in \mathbb{N}}$ in place of $(E_n)_{n \in \mathbb{N}}$, and B_n -valued Lipschitz and growth constants respectively denoted by $\tilde{L}_F$ and $\tilde{C}_F$. $\blacklozenge$

Assumption 6.8 (Pointwise approximation of B_n -valued drifts). Suppose that for almost every $(\omega, t) \in \Omega \times [0, T]$ and every $x \in B_\infty$ we have

$$\tilde{F}_n(\omega, t, x) \rightarrow F_\infty(\omega, t, x) \quad \text{in } \tilde{B} \text{ as } n \rightarrow \infty. \quad \blacklozenge$$

Note that in the approximation Assumptions 6.3 and 6.8, we only impose convergence for x belonging to $B_\infty \subseteq \tilde{B}$, and similarly we only consider $\xi_\infty \in L^p(\Omega; B_\infty)$ in Assumption 6.6. Recall that this is sufficient since Theorem 2.3, on which the approximation results ultimately rely, is formulated in this setting.

Under Assumptions 6.1, 6.2, 6.4 (with $\theta + 2\beta < 1$), 6.6 and 6.7, we can derive well-posedness of B_n -valued global solutions to (5.1); these are defined by simply replacing the space E_n by B_n in Definition 5.2. To this end, we again investigate the fixed-point operators $\Phi_{n,T}$ defined by (5.2), viewing them now as acting on $L^p(\Omega; C([0, T]; B_n))$. We have the following analog to Proposition 5.3:

Proposition 6.9. *Suppose that Assumptions 6.1, 6.2, 6.4 hold with $\theta + 2\beta < 1$, and 6.7 is satisfied. Let $n \in \mathbb{N}$, $p \in [1, \infty)$, $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; B_n)$ and $T \in (0, \infty)$ be given. The operator $\Phi_{n,T}$ given by (5.2) is well defined and Lipschitz continuous on $L^p(\Omega; C([0, T]; B_n))$. Its Lipschitz constant is independent of ξ_n , depends on A_n and F_n only through $\tilde{M}_S$ and $\tilde{L}_F$, and tends to zero as $T \downarrow 0$.*

Proof. The fact that $S_n \otimes \xi_n \in L^p(\Omega; C([0, T]; B_n))$ is immediate from Assumptions 6.1, 6.2 and $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$. By the first part of Proposition 6.5 (which also uses Assumption 6.4), we find $W_{A_n} \in L^p(\Omega; C([0, T]; B_n))$. By Assumption 6.7 and Proposition B.1(b) (with $E = F = B_n$, $\alpha = 0$ and $s = 1$), we have $S_n * F_n(\cdot, u_n) \in L^p(\Omega; C([0, T]; B_n))$ for all $u_n \in L^p(\Omega; C([0, T]; B_n))$. This shows that $\Phi_{n,T}$ is well-defined. A straightforward estimate involving Assumptions 6.1, 6.2 and 6.7 yields, for all $u_n, v_n \in L^p(\Omega; C([0, T]; B_n))$,

$$\|\Phi_{n,T}(u_n) - \Phi_{n,T}(v_n)\|_{L^p(\Omega; C([0, T]; B_n))} \leq \tilde{M}_S \tilde{L}_F T \|u_n - v_n\|_{L^p(\Omega; C([0, T]; B_n))}. \quad \square$$

Consequently, the proof of the following global well-posedness result is entirely analogous to that of Proposition 5.4:

Proposition 6.10. *Suppose that Assumptions 6.1, 6.2, 6.4 hold with $\theta + 2\beta < 1$, and 6.7 is satisfied. Let $n \in \mathbb{N}$, $p \in [1, \infty)$, $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and $T \in (0, \infty)$ be given. Then (5.1) has a unique global mild solution X_n in $L^p(\Omega; C([0, T]; B_n))$.*

Under the additional convergence Assumptions 6.3, 6.6 and 6.8, we can again set out to prove discrete-to-continuum convergence of B_n -valued global mild solutions to (5.1) by showing that all the expressions appearing in the fixed point maps $\tilde{\Phi}_{n,T}$ from (5.2) are continuous as mappings on $L^p(\Omega; C([0, T]; \tilde{B}))$. For the first term, the following can be proven exactly in the same way as Lemma 5.8:

Lemma 6.11. *If Assumptions 6.1–6.3 and 6.6 are satisfied, then we have $\tilde{S}_n \otimes \tilde{\xi}_n \rightarrow S_\infty \otimes \xi_\infty$ in $L^p(\Omega; C([0, T]; \tilde{B}))$ as $n \rightarrow \infty$.*

The following is an analog to Lemma 5.9:

Lemma 6.12. *Suppose that Assumptions 6.1–6.3, 6.7 and 6.8 are satisfied. Let $p \in [1, \infty)$ and $u \in L^p(\Omega; C([0, T]; B_\infty))$ be given. Then we have*

$$\tilde{S}_n * \tilde{F}_n(\cdot, u) \rightarrow S_\infty * F_\infty(\cdot, u) \quad \text{in } L^p(\Omega; C([0, T]; \tilde{B})) \text{ as } n \rightarrow \infty.$$

Proof. As in Lemma 5.9, we split up the statement into the following two assertions:

- (i) $\tilde{S}_n * \tilde{F}_n(\cdot, u) - \tilde{S}_n * F_\infty(\cdot, u) \rightarrow 0$ in $L^p(\Omega; C([0, T]; \tilde{B}))$ as $n \rightarrow \infty$;
- (ii) $\tilde{S}_n * F_\infty(\cdot, u) \rightarrow S_\infty * F_\infty(\cdot, u)$ in $L^p(\Omega; C([0, T]; \tilde{B}))$ as $n \rightarrow \infty$.

Part (i) is shown exactly as Lemma 5.9(i), up to replacing E_∞ by B_∞ (or $\tilde{B}$).

For (ii), we instead note that 6.7 implies, for almost every $\omega \in \Omega$,

$$t \mapsto F_\infty(\omega, t, u(\omega, t)) \in L^p(0, T; B_\infty).$$

Hence, in order to argue as in Lemma 5.9(ii), one needs to apply Proposition B.3(a) and Corollary B.2(b) with $E_n := B_n$ for all $n \in \mathbb{N}$ and $\tilde{E} := \tilde{B}$. $\square$

With these auxiliary results in place, we can prove the first main discrete-to-continuum approximation result for solutions to (6.1) with globally Lipschitz drift coefficients of linear growth, analogously to Theorem 5.7:

Theorem 6.13. *Let Assumptions 6.1–6.4, 6.7, 6.8 and 6.6 be satisfied, with $\theta + 2\beta < 1$ and $p \in [1, \infty)$. Denoting by X_n the unique B_n -valued global mild solution to (5.1), we have*

$$\tilde{X}_n \rightarrow X_\infty \quad \text{in } L^p(\Omega; C([0, T]; \tilde{B})) \quad \text{as } n \rightarrow \infty.$$

Proof. By Proposition 6.9 and 6.1, for small enough $T_0 \in (0, \infty)$ there exists a $c \in [0, 1)$, depending only on $\tilde{L}_F$, $\tilde{M}_S$, $\tilde{M}_\Lambda$ and $\tilde{M}_\Pi$, such that, for all $u, v \in L^p(\Omega; C([0, T_0]; B_\infty))$,

$$\sup_{n \in \mathbb{N}} \|\tilde{\Phi}_{n, T_0}(u) - \tilde{\Phi}_{n, T_0}(v)\|_{L^p(\Omega; C([0, T_0]; \tilde{B}))} \leq c \|u - v\|_{L^p(\Omega; C([0, T_0]; \tilde{B}))}.$$

By Proposition 6.10, there exists a unique global solution $X_n \in L^p(\Omega; C([0, T]; B_n))$ to (5.1) for every $n \in \mathbb{N}$. In particular, note that X_∞ takes its values in $B_\infty \subseteq \tilde{B}$. Thus, in order to finish the argument analogously to the proof of Theorem 5.7, it suffices to establish that $\tilde{\Phi}_{n, T}(\phi) \rightarrow \Phi_{\infty, T}(\phi)$ in $L^p(\Omega; C([0, T]; \tilde{B}))$ as $n \rightarrow \infty$ for all $\phi \in L^p(\Omega; C([0, T]; B_\infty))$. This is precisely the combined content of (the second part of) Proposition 6.5, along with Lemmas 6.11 and 6.12. $\square$

6.2. Locally Lipschitz drifts

As in Section 5.2, we can extend Theorem 6.13 to the locally Lipschitz setting. Namely, we assume that

Assumption 6.14 (Locally Lipschitz B_n -valued drifts). Assumption 5.10 holds with $(B_n)_{n \in \mathbb{N}}$ in place of $(E_n)_{n \in \mathbb{N}}$. For every $r > 0$, the B_n -valued local Lipschitz and local boundedness constants are respectively denoted by $\tilde{L}_F^{(r)}$ and $\tilde{C}_{F,0}^{(r)}$. $\blacklozenge$

Then, arguing in the same way as Theorem 5.13, we obtain the following result.

Theorem 6.15. Suppose that Assumptions 6.1–6.4, 6.6, 6.8 and 6.14 hold, with $\theta + 2\beta < 1$. For every $n \in \overline{\mathbb{N}}$, let $(X_n(t))_{t \in [0, \sigma_n]}$ be the maximal local solution to (6.1) with explosion time $\sigma_n : \Omega \rightarrow [0, T]$. Then we have

$$(i) \quad \tilde{X}_n \mathbf{1}_{[0, \sigma_n \wedge \sigma_\infty)} \rightarrow X_\infty \mathbf{1}_{[0, \sigma_\infty)} \text{ in } L^0(\Omega \times [0, T]; \tilde{B}) \text{ as } n \rightarrow \infty.$$

If, moreover, $\sigma_n = T$ holds $\mathbb{P}$ -a.s. for all $n \in \mathbb{N}$ and $p \in [1, \infty)$ is such that

$$\sup_{n \in \mathbb{N}} \|X_n\|_{L^p(\Omega; C([0, T]; B_n))} < \infty, \quad (6.5)$$

then the following assertions also hold:

- (ii) We have $\sigma_\infty = T$, $\mathbb{P}$ -a.s.
- (iii) If $p \in (1, \infty)$, then for all $p^- \in [1, p)$ we have

$$\tilde{X}_n \rightarrow X_\infty \quad \text{in } L^{p^-}(\Omega; C([0, T]; \tilde{B})) \text{ as } n \rightarrow \infty.$$

6.3. Global well-posedness and convergence for dissipative drifts

In this section, we consider a class of equations whose drift coefficients satisfy not only Assumption 6.14 (which would only guarantee *local* well-posedness), but additionally a type of dissipativity condition, also used in [57], which allows us to deduce global existence.

Let the subdifferential $\partial \|x_n\|_{B_n}$ of the norm $\|\cdot\|_{B_n}$ at $x_n \in B_n$ be given by

$$\partial \|x_n\|_{B_n} = \{x_n^* \in B_n^* : \|x_n^*\|_{B_n^*} \leq 1 \text{ and } B_n \langle x_n, x_n^* \rangle_{B_n^*} = \|x_n\|_{B_n}\}.$$

The assumptions on $(F_n)_{n \in \overline{\mathbb{N}}}$ under which we can derive global well-posedness, see Lemma 6.18 below (which is a simplified version of [57, Theorem 4.3] for equations driven by additive noise), are as follows:

Assumption 6.16. Let the conditions of Assumption 6.14 hold. Suppose that there exist constants $a', b' \in [0, \infty)$ and $N \in \mathbb{N}$ such that the following inequality holds for every $n \in \overline{\mathbb{N}}$, $\omega \in \Omega$, $t \in [0, T]$, $x_n \in D(A_n|_{B_n})$, $x_n^* \in \partial \|x_n\|_{B_n}$ and $y_n \in B_n$:

$$B_n \langle -A_n x_n + F_n(\omega, t, x_n + y_n), x_n^* \rangle_{B_n^*} \leq a'(1 + \|y_n\|_{B_n})^N + b' \|x_n\|_{B_n}. \quad \blacklozenge$$

If the semigroups $(S_n(t)|_{B_n})_{t \geq 0}$ are contractive on B_n , i.e., if $\tilde{M}_S = 1$ in Assumption 6.2, then we know that $A_n|_{B_n}$ is *accretive*, i.e.,

$$B_n \langle A_n x_n, x_n^* \rangle_{B_n^*} \geq 0 \quad \text{for all } x_n \in D(A_n|_{B_n}), x_n^* \in \partial \|x_n\|_{B_n}. \quad (6.6)$$

Thus, in this case, it suffices to check that

$$B_n \langle F_n(\omega, t, x_n + y_n), x_n^* \rangle_{B_n^*} \leq a'(1 + \|y_n\|_{B_n})^N, \quad (6.7)$$

in order to establish that Assumption 6.16 holds for $b' = 0$. The next example shows how the relation (6.7) can be verified in our situation of main interest. It is an elaborated version of [57, Example 4.2].

Example 6.17. Given $n \in \overline{\mathbb{N}}$, let $B_n := C(\mathcal{M}_n)$ be the space of continuous real-valued functions on a compact Hausdorff space $\mathcal{M}_n$ equipped with the norm $\|u_n\|_{B_n} := \sup_{\xi \in \mathcal{M}_n} |u_n(\xi)|$. In this case, for all $u_n \in B_n$, the subdifferential $\partial \|u_n\|_{B_n}$ is the weak*-closed convex hull of

$$\{r\delta_{\hat{\xi}} : r \in \operatorname{sgn} u_n(\hat{\xi}) \text{ for } \hat{\xi} \in \mathcal{M}_n \text{ such that } \|u_n\|_{B_n} = |u_n(\hat{\xi})|\}, \quad (6.8)$$

where $\delta_{\xi} \in C(\mathcal{M}_n)^*$ denotes the evaluation functional at $\xi \in \mathcal{M}_n$ and, for $y \in \mathbb{R}$,

$$\operatorname{sgn} y := \begin{cases} \{-1\}, & \text{if } y < 0; \\ \{-1, 1\}, & \text{if } y = 0; \\ \{1\}, & \text{if } y > 0. \end{cases}$$

Indeed, since subdifferential sets are convex by definition and $\partial \|u_n\|_{B_n}$, being contained in the closed unit ball in B_n^* , is weak* compact by the Banach–Alaoglu theorem, the Krein–Milman theorem implies that it suffices to argue that the extreme points of $\partial \|u_n\|_{B_n}$ are precisely given by (6.8). This, in turn, follows from a characterization of the extreme points of the closed unit ball in $C(\mathcal{M}_n)^*$ due to Arens and Kelley [2].

Moreover, suppose that $(F_n)_{n \in \overline{\mathbb{N}}}$ is a family of Nemytskii operators on $(B_n)_{n \in \overline{\mathbb{N}}}$ (see equation (6.2)), generated by a family $(f_n)_{n \in \overline{\mathbb{N}}}$ of functions satisfying the polynomial form introduced in (3.12)–(3.13). Fixing $n \in \overline{\mathbb{N}}$, $(\omega, t) \in \Omega \times [0, T]$ and $u_n, v_n \in B_n$, inequality (6.7) becomes

$$B_n \langle F_n(\omega, t, u_n + v_n), x_n^* \rangle_{B_n^*} \leq a'(1 + \|v_n\|_{B_n})^N \quad \text{for all } x_n^* \in \partial \|u_n\|_{B_n}. \quad (6.9)$$

Since the inequality is preserved under convex combinations and weak* limits of x_n^* , the above characterization of $\partial \|u_n\|_{B_n}$ shows that it only needs to be checked for $x_n^* = r\delta_{\hat{\xi}}$, where r belongs to $\operatorname{sgn} u_n(\hat{\xi})$ for $\hat{\xi} \in \mathcal{M}_n$ such that $\|u_n\|_{B_n} = |u_n(\hat{\xi})|$. That is, it suffices to verify that

$$rf_n(\omega, t, u_n(\hat{\xi}) + v_n(\hat{\xi})) \leq a'(1 + \|v_n\|_{B_n})^N.$$

Indeed, for $(f_n)_{n \in \overline{\mathbb{N}}}$ satisfying (3.12)–(3.13), we can establish the estimate

$$rf_n(\omega, t, y + z) \lesssim_{(c, C, k)} (1 + |z|)^{2k+1}$$

for all $(\omega, t) \in \Omega \times [0, T]$, $y, z \in \mathbb{R}$ and $r \in \operatorname{sgn} y$. This implies the existence of a constant $a' \in [0, \infty)$, depending only on $c, C \in (0, \infty)$ from (3.13) and $k \in \mathbb{N}_0$ from (3.12), such that (6.9) holds with $N = 2k + 1$.

Lemma 6.18. Let Assumptions 6.1–6.4 and 6.16 hold with $\tilde{M}_S = 1$, let $n \in \overline{\mathbb{N}}$ and suppose that $\theta + 2\beta < 1$. If $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; B_n)$ for some $p \in [1, \infty)$, then the maximal solution $(X_n(t))_{t \in [0, \sigma_n]}$ to (6.1) is global (i.e., it holds $\mathbb{P}$ -a.s. that $\sigma_n = T$), and we have

$$\|X_n\|_{L^p(\Omega; C([0, T]; B_n))} \lesssim_{(a', b', T, N)} 1 + \|\xi_n\|_{L^p(\Omega; B_n)} + \|W_{A_n}\|_{L^{Np}(\Omega; C([0, T]; B_n))}^N.$$

Proof. Fix $n \in \overline{\mathbb{N}}$. For each $m \in \mathbb{N}$, let $F_{n,m}$ denote the globally Lipschitz retraction of F_n onto the closed ball of radius m around $0 \in B_n$, cf. (5.12) (replacing E_n by B_n). Then $F_{n,m}$ satisfies the global Lipschitz and (global) linear growth estimates in Assumption 6.7, hence by Proposition 6.10 there exists a unique global B_n -valued mild solution $X_{n,m} \in L^p(\Omega; C([0, T]; B_n))$ to (5.1) with drift coefficient operator $F_{n,m}$. By the triangle inequality,

$$\begin{aligned} & \|X_{n,m}\|_{L^p(\Omega; C([0, T]; B_n))} \\ & \leq \|S_n \otimes \xi_n + S_n * F_{n,m}(\cdot, X_{n,m})\|_{L^p(\Omega; C([0, T]; B_n))} + \|W_{A_n}\|_{L^p(\Omega; C([0, T]; B_n))}. \end{aligned}$$

As shown in the proof of [57, Theorem 4.3], $F_{n,m}$ inherits the dissipativity estimate from Assumption 6.16, with the same constants a' , b' and N , from F_n . Thus, by [57, Lemma 4.4],

$$\begin{aligned} & \|S_n \otimes \xi_n + S_n * F_{n,m}(\cdot, X_m)\|_{C([0, T]; B_n)} \\ & \leq e^{b'T} \left(\|\xi\|_{B_n} + a' \int_0^T (1 + \|W_{A_n}(s)\|_{B_n})^N ds \right) \\ & \leq e^{b'T} \|\xi\|_{B_n} + a'T 2^{N-1} e^{b'T} (1 + \|W_{A_n}\|_{C([0, T]; B_n)}^N), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

It follows that

$$\begin{aligned} & \|S_n \otimes \xi_n + S_n * F_{n,m}(\cdot, X_m)\|_{L^p(\Omega; C([0, T]; B_n))} \\ & \lesssim_{(a', b', T, N)} 1 + \|\xi_n\|_{L^p(\Omega; B_n)} + \|W_{A_n}\|_{L^{Np}(\Omega; C([0, T]; B_n))}^N. \end{aligned}$$

Note that $W_{A_n} \in L^{Np}(\Omega; C([0, T]; B_n))$ by Proposition 6.5 (with Np taking the role of p). Combining these estimates, we find

$$\sup_{m \in \mathbb{N}} \|X_{n,m}\|_{L^p(\Omega; C([0, T]; B_n))} \lesssim_{(a', b', T, N)} 1 + \|\xi_n\|_{L^p(\Omega; B_n)} + \|W_{A_n}\|_{L^{Np}(\Omega; C([0, T]; B_n))}^N,$$

so the result follows by Theorem 5.14(iv)–(v) applied to $Y_m := X_{n,m}$. $\square$

Combined with Theorem 6.15(ii)–(iii), whose uniform-boundedness hypothesis (6.5) is verified by the combination of Lemma 6.18 and Proposition 6.5 under the assumption in the following corollary, we derive:

Corollary 6.19. *Suppose that Assumptions 6.1–6.4, 6.16, 6.8 and 6.6 are satisfied with $\tilde{M}_S = 1$, $p \in (1, \infty)$ and $\theta + 2\beta < 1$. Then for any $p^- \in [1, p)$, the sequence $((X_n)_{t \in [0, T]})_{n \in \overline{\mathbb{N}}}$ of B_n -valued global solutions to (6.1) satisfies*

$$\tilde{X}_n \rightarrow X_\infty \quad \text{in } L^{p^-}(\Omega; C([0, T]; \tilde{B})) \text{ as } n \rightarrow \infty.$$

7. Outlook

In this section we suggest some possible extensions of both the convergence of graph-discretized Whittle–Matérn SPDEs shown in Section 3 as well as the underlying abstract results from Sections 4–6.

7.1. Extensions of the Whittle–Matérn graph-approximation example

As discussed in Section 3.4, the approximation results from Theorem 3.10 for (3.1) might be extended to broader classes of domains $\mathcal{M}$, connectivity length regimes $(h_n)_{n \in \mathbb{N}}$, coefficient functions $\tau, \kappa: \mathcal{M} \rightarrow [0, \infty)$ and (fractional) powers $s \in (0, \infty)$. Possible advancements to this end include the establishment of more general L^∞ -convergence results for graph Laplacian eigenfunctions (or convergence of Whittle–Matérn operators without using spectral convergence), as well as uniform L^∞ -boundedness of the semigroups (for instance via heat kernel estimates).

7.2. Multiplicative noise and UMD Banach spaces of arbitrary type

Some of the continuous-dependence theorems from [56,57], which serve as the basis for our abstract discrete-to-continuum setting in Sections 4–6, were stated in more general settings than in the present work. The main difference is that the authors of [56,57] also treat semilinear SPDEs driven by *multiplicative* instead of additive noise, i.e., equations of the form

$$\begin{cases} dX_n(t) = -AX_n(t)dt + F_n(t, X_n(t))dt + G_n(t, X_n(t))dW(t), & t \in (0, T], \\ X_n(0) = \xi_n. \end{cases}$$

In addition to the *drift* coefficient operators F_n , these equations contain *diffusion* coefficient operators $G_n: \Omega \times [0, T] \times \dot{E}^\theta \rightarrow \gamma(H; \dot{E}^{-\kappa_G})$ (recall the definition (6.3) of fractional domain spaces), for certain $\theta, \kappa_G \in [0, \frac{1}{2})$. The authors establish continuous-approximation results in the case where the sequence $(G_n)_{n \in \mathbb{N}}$ satisfies certain “ γ -Lipschitz”, growth and pointwise-approximation conditions. That is, appropriate “ G_n -analogs” to “ F_n -conditions” such as Assumptions 5.1, 5.5 and/or 5.10 are used. The required analogs are significantly more technical than the assumptions on the drifts, so we will not state them here and instead refer to, e.g., [57, Assumptions (G1) and (G2)] for precise statements.

Using a procedure similar to the one we applied to the drift coefficient operators in the additive-noise case, it should equally be possible to treat the discrete-to-continuum multiplicative-noise setting by carefully incorporating our projection and lifting operators into these additional assumptions and arguments. However, as noted above, the resulting statements and proofs are likely to add significant technical and notational complexity when compared to the (already-technical) additive-noise case. In view of this fact, as well as our primary interest in additive-white-noise-driven equations, we have not pursued this in the current work.

In a similar vein, we have chosen not to extend the results from Sections 4 and 5 to the case where the UMD Banach spaces $(E_n)_{n \in \mathbb{N}}$ are of arbitrary Rademacher type (i.e., other than 2), despite our belief that this situation is also within reach, as the analogous setting was covered in [56].

7.3. Convergence rates

The most important next step for the practical applicability of the discretization methods described in this work would be to establish rates of convergence, i.e., to quantify how fast the approximations $(X_n)_{n \in \mathbb{N}}$ converge to the limiting solution X_∞ in terms of n . Here we give some indications of how our results might be extended in this direction.

When formulated in the setting of the present work, we can view (a particular case of) [17, Proposition 4.4] as an analog to the discrete-to-continuum Trotter–Kato semigroup-approximation theorem (recall Theorem 2.3), which yields bounds of the form

$$\|\tilde{S}_n - S_\infty\|_{C([0,T];\tilde{E})} \lesssim \|\tilde{R}(\lambda; A_n) - R(\lambda, A_\infty)\|_{\mathcal{L}(\tilde{E})},$$

where $\lambda \in \mathbb{C}$ has a sufficiently large real part and the implied constant is uniform in n . Since this result is proved using a Cauchy-integral argument (as was, for instance, Lemma C.1), it relies on the stronger assumption that the semigroups are uniformly sectorial instead of merely uniformly bounded and strongly continuous. Moreover, it is assumed throughout that the Banach spaces involved are UMD. Given that those two assumptions were also made in Sections 4 and 5, this result could be used as the basis for extending the theorems in these sections to include convergence rates, in situations where the resolvents of the linear operator admit such estimates. It is less clear which assumptions and arguments would be required to perform the same extension for the “ B_n -valued setting” of Section 6.

Assuming that the abstract convergence results of Sections 4–6 can indeed be enhanced along the lines sketched above, a natural next question would be whether convergence-rate estimates are available for the resolvents of the graph-discretized Whittle–Matérn operators from Section 3. Under more restrictive assumptions on the connectivity-parameter regime (i.e., more conditions than Assumption 3.5), rates of $L^2(\mathcal{M})$ -convergence for solutions of the purely spatial (i.e., stationary) random-graph-discretized SPDEs were established in [72, Theorem 4.2]. Under even stronger conditions, the same authors also obtain rates for $L^\infty(\mathcal{M})$ -convergence in [73, Theorem 7]. Since these theorems were proved by directly working with the solutions, they would first need to be converted to resolvent-convergence statements, similar to Theorem 3.14 (which is also a statement about resolvent convergence based on the arguments of [72,73]). In the deterministic and square-grid Setting 3.2, a more direct proof is likely possible, with which one might hope to recover some of the finite-difference convergence rates from [41].

7.4. Stochastic evolution inclusions

It would be interesting to consider the class of stochastic evolution *inclusions*, whose drift operators are allowed to be multi-valued. This occurs, for instance, in the Langevin setting where $F_n = \partial\varphi_n$ is the subdifferential of a convex and lower-semicontinuous but non-differentiable functional φ_n on the state space, taking values in $(-\infty, \infty]$. Continuous dependence results for stochastic inclusion problems have been established in two different settings in [36,74]; however, neither of these covers the class of *semilinear* inclusions driven by *cylindrical* (i.e., *white*) noise. Not much theory appears to be available for such problems, with even the question of well-posedness (for fixed $n \in \overline{\mathbb{N}}$) being highly nontrivial. In fact, to the best of our knowledge, the only results in this direction concern the (important) subclass of *stochastic reflection problems* (also known as *Skorokhod problems* in the scalar-valued case), given by

$$\begin{cases} dX(t) \in -AX(t)dt - \partial I_\Gamma(X(t))dt + dW(t), & t \in (0, T], \\ X(0) = \xi, \end{cases} \quad (7.1)$$

where Γ is a convex subset of a (Hilbertian) state space H and the *indicator functional* $I_\Gamma: H \rightarrow (-\infty, \infty]$ is defined to vanish on Γ and equal ∞ outside of it. In the first work on

this problem, Nualart and Pardoux [67] used a direct approach to show existence and uniqueness in a setting which corresponds to $H := L^2(0, 1)$, $A := -\frac{d^2}{dx^2}$ with homogeneous Dirichlet or Neumann boundary conditions, and $\Gamma := K_0$, where

$$K_\alpha := \{u \in L^2(0, 1) : u(x) \geq -\alpha \text{ for a.e. } x \in (0, 1)\}, \quad \alpha \in [0, \infty).$$

In [71], the authors first use the theory of Dirichlet forms to establish well-posedness of (7.1) in the case that Γ is a “regular” convex subset of a general Hilbert space H , a condition which includes $\Gamma := K_\alpha$ for $\alpha > 0$ but not K_0 , which is treated separately using different techniques. Lastly, the work [4] describes a variational approach to study (7.1) in a similar setting under the assumption that 0 belongs to the interior of Γ , which excludes the choice $\Gamma := K_0$. We point out that the argument used on [4, p. 362] to extract a weak*-convergent sequence from the set $(u_\varepsilon)_{\varepsilon>0}$ in the dual space of $L^\infty(0, T; H)$ appears to be flawed, as it seems to imply that the closed unit ball of the dual of this (non-separable) space is *sequentially* compact, which is not the case. Hence, the argument would need to be finished using a *generalized* subsequence (also known as a *subnet*) converging in the weak* sense to some u^* . For this reason, and since the theory for convergence of Dirichlet forms and their associated processes is well established—see for instance [53] for general results and [89] for an application to Markov chain Monte Carlo scaling—the setting of [71] is perhaps the most promising for an attempt at establishing discrete-to-continuum convergence results for (7.1).

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Appendix A. Proofs of results from Section 3

A.1. Proofs of intermediate results from Section 3.3

We start with the proof of Lemma 3.11 which establishes spectral convergence rates for the Laplace–Beltrami operator on the torus, discretized by a uniform grid.

Proof of Lemma 3.11. First suppose $m = 1$. In this case, the continuum Laplace–Beltrami operator reduces to the second derivative $-\frac{d^2}{dx^2}$ with periodic boundary conditions. Its eigenvalues and $L^2(\mathcal{M})$ -normalized eigenfunctions are respectively given, for all $j \in \mathbb{N}$ and $x \in [0, 1]$, by

$$\lambda_\infty^{(j)} = \begin{cases} j^2\pi^2 & \text{if } j \text{ is even,} \\ (j-1)^2\pi^2 & \text{if } j \text{ is odd;} \end{cases} \quad \psi_\infty^{(j)}(x) = \begin{cases} 1 & \text{if } j = 1, \\ \sqrt{2}\sin(j\pi x) & \text{if } j \text{ is even,} \\ \sqrt{2}\cos((j-1)\pi x) & \text{if } j \text{ is odd.} \end{cases}$$

That is, 0 is an eigenvalue corresponding to the constant 1 eigenfunction, and $(2k)^2\pi^2$ is an eigenvalue with eigenfunctions $x \mapsto \sin(2k\pi x)$, $\cos(2k\pi x)$ for all $k \in \mathbb{N}$.

The eigenvalues $(\lambda_n^{(j)})_{j=1}^n$ of the corresponding graph Laplacian, which in this case reduces to the finite difference approximation of the second derivative on the grid with $n \in \mathbb{N}$ points, are given by

$$\lambda_n^{(j)} = \begin{cases} 4n^2 \sin^2\left(\frac{\pi j}{2n}\right) & \text{if } j \text{ is even,} \\ 4n^2 \sin^2\left(\frac{\pi(j-1)}{2n}\right) & \text{if } j \text{ is odd.} \end{cases}$$

The corresponding $L^2(\mathcal{M}_n)$ -normalized eigenfunctions are

$$\psi_n^{(j)}(x_n^{(i)}) = \begin{cases} 1 & \text{if } j = 1, \\ (-1)^i & \text{if } j = n \text{ is even,} \\ \sqrt{2} \sin(j\pi x_n^{(i)}) & \text{if } j \neq n \text{ and } j \text{ is even,} \\ \sqrt{2} \cos((j-1)\pi x_n^{(i)}) & \text{if } j \text{ is odd.} \end{cases}$$

Let $j \in \{1, \dots, n\}$. Supposing that j is even (the odd case being analogous), we can write

$$\lambda_\infty^{(j)} - \lambda_n^{(j)} = j^2 \pi^2 \left(1 - \frac{4n^2}{j^2 \pi^2} \sin^2\left(\frac{\pi j}{2n}\right) \right) = j^2 \pi^2 \left(1 - \left[\frac{2n}{j\pi} \sin\left(\frac{\pi j}{2n}\right) \right]^2 \right),$$

so that the estimate in (3.15) for $m = 1$ follows from $0 \leq 1 - (\sin(x)/x)^2 \leq \frac{1}{3}x^2$, an elementary inequality which is valid for all $x \in \mathbb{R}$. Estimate (3.16) is a consequence of the fact that the sine and cosine functions are 1-Lipschitz; note that we only consider $j \in \{1, \dots, n-1\}$ to avoid the case where $j = n$ for even n .

The result for higher dimensions $m \in \mathbb{N}$ can be derived from the $m = 1$ case. Indeed, by separation of variables in the continuum case, or by writing the discretized operator as a Kronecker sum of m one-dimensional discretizations, one finds that the eigenvalues and eigenvectors of the m -dimensional operators are given by sums and products, respectively, of their 1-dimensional counterparts. From this, one can deduce the desired result. $\square$

Next we prove Theorem 3.14 regarding the convergence of the fractional resolvent operators $\widetilde{R}_n^{\beta'}$ in various settings and norms.

Proof of Theorem 3.14. Assertions (a)–(c) can all be shown using analogous arguments. Thus, we only provide a detailed proof for part (b), being the most involved case, and subsequently summarize the changes needed for (a) and (c).

(b) Step 1 (Setup and notation). The operator $\widetilde{R}_n^{\beta'}$ acts on functions $f \in L^2(\mathcal{M})$ in the following way:

$$\widetilde{R}_n^{\beta'} f = \sum_{j=1}^n (1 + [\lambda_n^{(j)}]^\beta)^{-\beta'} \langle \Pi_n f, \psi_n^{(j)} \rangle_{L^2(\mathcal{M}_n)} \Lambda_n \psi_n^{(j)}$$

$$= \sum_{j=1}^n (1 + [\lambda_n^{(j)}]^s)^{-\beta'} \langle f, \Lambda_n \psi_n^{(j)} \rangle_{L^2(\mathcal{M})} \Lambda_n \psi_n^{(j)}.$$

Here, we used the fact that $\Pi_n^* = \Lambda_n$ (see (3.5)) on the second line. Using the tensor notation from Section 2.1 and denoting $\tilde{\psi}_n^{(j)} := \Lambda_n \psi_n^{(j)}$, we can write these operators more concisely as follows:

$$U_n := \tilde{R}_n^{\beta'} = \sum_{j=1}^n (1 + [\lambda_n^{(j)}]^s)^{-\beta'} \tilde{\psi}_n^{(j)} \otimes \tilde{\psi}_n^{(j)}.$$

By Assumption 3.5(ii), there exists a sequence of natural numbers $(k_n)_{n \in \mathbb{N}}$ which satisfies the conditions of Theorem 3.13(b), as well as the relation

$$k_n \gg n^{\frac{m}{4s\beta}} \gtrsim n^{\frac{m}{4s\beta'}}. \quad (\text{A.1})$$

We use the sequence $(k_n)_{n \in \mathbb{N}}$ to define the following approximations for $n \in \mathbb{N}$:

$$\begin{aligned} U_n^1 &:= \sum_{j=1}^{k_n} (1 + [\lambda_\infty^{(j)}]^s)^{-\beta'} \psi_\infty^{(j)} \otimes \psi_\infty^{(j)}, \\ U_n^2 &:= \sum_{j=1}^{k_n} (1 + [\lambda_\infty^{(j)}]^s)^{-\beta'} \tilde{\psi}_n^{(j)} \otimes \tilde{\psi}_n^{(j)}, \\ U_n^3 &:= \sum_{j=1}^{k_n} (1 + [\lambda_n^{(j)}]^s)^{-\beta'} \tilde{\psi}_n^{(j)} \otimes \tilde{\psi}_n^{(j)}. \end{aligned}$$

For notational convenience, we will abbreviate the $\mathcal{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M}))$ -norm by $\|\cdot\|_{2 \rightarrow \infty}$ throughout this proof. We will make repeated use of the following estimate: Given an operator of the form $U = \sum_j \alpha_j e_j \otimes f_j$ for some scalars $(\alpha_j)_j \subseteq \mathbb{R}$, an orthonormal system $(e_j)_j \subseteq L^2(\mathcal{M})$ and some functions $(f_j)_j \subseteq L^\infty(\mathcal{M})$, we have by the Cauchy–Schwarz inequality:

$$\|U\|_{2 \rightarrow \infty} \leq \sup_j \|f_j\|_{L^\infty(\mathcal{M})} \left(\sum_j |\alpha_j|^2 \right)^{\frac{1}{2}}. \quad (\text{A.2})$$

Moreover, it is immediate from the definition of the $\|\cdot\|_{2 \rightarrow \infty}$ -norm that

$$\|h \otimes f\|_{2 \rightarrow \infty} = \|h\|_{L^2(\mathcal{M})} \|f\|_{L^\infty(\mathcal{M})} \quad \text{for all } h \in L^2(\mathcal{M}) \text{ and } f \in L^\infty(\mathcal{M}). \quad (\text{A.3})$$

Step 2 ($U_\infty - U_n^1$ and $U_n^3 - U_n$). For the difference between U_∞ and U_n^1 , we find using (A.2) and Assumption 3.9:

$$\|U_\infty - U_n^1\|_{2 \rightarrow \infty}^2 \leq M_{\psi, \infty}^2 \sum_{j=k_n+1}^{\infty} (1 + [\lambda_\infty^{(j)}]^s)^{-2\beta'}. \quad (\text{A.4})$$

Recalling Weyl's law (3.9), which implies that $(1 + [\lambda_\infty^{(j)}]^s)^{-2\beta'} \asymp_{\mathcal{M}} j^{-4s\beta'/m}$, we observe that the series on the right-hand side is convergent precisely when $\beta' > \frac{m}{4s}$. Since $k_n \rightarrow \infty$ as $n \rightarrow \infty$, this implies $U_\infty \rightarrow U_n^1$ in $\mathcal{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M}))$.

Similarly, for the difference between U_n^3 and U_n , we have

$$\|U_n^3 - U_n\|_{2 \rightarrow \infty}^2 \leq M_{\psi, \infty}^2 \sum_{j=k_n+1}^n (1 + [\lambda_n^{(j)}]^s)^{-2\beta'} \leq M_{\psi, \infty}^2 (n - k_n) (1 + [\lambda_n^{(k_n)}]^s)^{-2\beta'},$$

where the second inequality is due to the non-decreasing order of $(\lambda_n^{(j)})_{j=1}^n$. Moreover, as a consequence of Theorem 3.12, there exists a constant $C' > 0$ such that, $\tilde{\mathbb{P}}$ -a.s., for all $n \in \mathbb{N}$ and $j \in \{1, \dots, k_n\}$, we have $\lambda_n^{(j)} \geq C' \lambda_\infty^{(j)}$. In particular, $\lambda_n^{(k_n)} \gtrsim \lambda_\infty^{(k_n)}$. Together with Weyl's law, we find $1 + [\lambda_n^{(k_n)}]^s \gtrsim_{\mathcal{M}} k_n^{2s/m}$, so that the convergence of this difference is due to (A.1):

$$\|U_n^3 - U_n\|_{2 \rightarrow \infty}^2 \lesssim_{\mathcal{M}} M_{\psi, \infty}^2 n k_n^{-4s\beta'/m} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (\text{A.5})$$

Step 3 ($U_n^1 - U_n^2$). In order to show that

$$U_n^1 - U_n^2 \rightarrow 0 \quad \text{in } L^0(\tilde{\Omega}; \mathcal{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M}))) \quad \text{as } n \rightarrow \infty,$$

we first fix an arbitrary $\varepsilon > 0$. Then, for all $\ell, n \in \mathbb{N}$ such that $k_n > \ell$, we split off the first ℓ terms and use the triangle inequality to obtain

$$\begin{aligned} \|U_n^1 - U_n^2\|_{2 \rightarrow \infty} &\leq \sum_{j=1}^{\ell} (1 + [\lambda_\infty^{(j)}]^s)^{-\beta'} \|\psi_\infty^{(j)} \otimes \psi_\infty^{(j)} - \tilde{\psi}_n^{(j)} \otimes \tilde{\psi}_n^{(j)}\|_{2 \rightarrow \infty} \\ &+ \left\| \sum_{j=\ell+1}^{k_n} (1 + [\lambda_\infty^{(j)}]^s)^{-\beta'} \psi_\infty^{(j)} \otimes \psi_\infty^{(j)} \right\|_{2 \rightarrow \infty} + \left\| \sum_{j=\ell+1}^{k_n} (1 + [\lambda_\infty^{(j)}]^s)^{-\beta'} \tilde{\psi}_n^{(j)} \otimes \tilde{\psi}_n^{(j)} \right\|_{2 \rightarrow \infty}. \end{aligned}$$

Using the triangle inequality once more, followed by (A.3) and Assumption 3.9, the norms in the summation over $j \in \{1, \dots, \ell\}$ can be bounded by

$$\begin{aligned} &\|\psi_\infty^{(j)} \otimes \psi_\infty^{(j)} - \tilde{\psi}_n^{(j)} \otimes \tilde{\psi}_n^{(j)}\|_{2 \rightarrow \infty} \\ &\leq \|\psi_\infty^{(j)} \otimes (\psi_\infty^{(j)} - \tilde{\psi}_n^{(j)})\|_{2 \rightarrow \infty} + \|(\psi_\infty^{(j)} - \tilde{\psi}_n^{(j)}) \otimes \tilde{\psi}_n^{(j)}\|_{2 \rightarrow \infty} \\ &= \|\psi_\infty^{(j)} - \tilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})} + \|\psi_\infty^{(j)} - \tilde{\psi}_n^{(j)}\|_{L^2(\mathcal{M})} \|\tilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})} \\ &\leq (1 + M_{\psi, \infty}) \|\psi_\infty^{(j)} - \tilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})}, \end{aligned}$$

whereas the remaining two summations can be treated by arguing as for (A.4). Together, this yields

$$\begin{aligned} \|U_n^1 - U_n^2\|_{2 \rightarrow \infty} &\leq (1 + M_{\psi, \infty}) \sum_{j=1}^{\ell} (1 + [\lambda_{\infty}^{(j)}]^s)^{-\beta'} \|\psi_{\infty}^{(j)} - \tilde{\psi}_n^{(j)}\|_{L^{\infty}(\mathcal{M})} \\ &\quad + 2 \left(\sum_{j=\ell+1}^{\infty} (1 + [\lambda_{\infty}^{(j)}]^s)^{-2\beta'} \right)^{\frac{1}{2}}. \end{aligned}$$

Since we have already seen in Step 2 that the latter series converges, we can fix $\ell \in \mathbb{N}$ so large that the second sum on the right-hand side is less than $\frac{1}{2}\varepsilon$. Moreover, it follows from Theorem 3.13(b) that $\|\psi_{\infty}^{(j)} - \tilde{\psi}_n^{(j)}\|_{L^{\infty}(\mathcal{M})} \rightarrow 0$ in $L^0(\tilde{\Omega}, \tilde{\mathbb{P}})$ as $n \rightarrow \infty$ for every $j \in \{1, \dots, \ell\}$. In particular, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, the first sum on the right-hand side is less than $\frac{1}{2}\varepsilon$, and thus the whole right-hand side is less than ε , with probability $\tilde{\mathbb{P}} \geq 1 - \varepsilon$. This shows $\|U_n^1 - U_n^2\|_{2 \rightarrow \infty} \rightarrow 0$ in probability, as desired.

Step 4 ($U_n^2 - U_n^3$). Finally, the difference $U_n^2 - U_n^3$ can be treated in the same manner as Step 3, namely by writing, for all $\ell, n \in \mathbb{N}$ such that $k_n > \ell$,

$$\begin{aligned} \|U_n^2 - U_n^3\|_{2 \rightarrow \infty}^2 &\lesssim_{M_{\psi, \infty}} \sum_{j=1}^{k_n} |(1 + [\lambda_n^{(j)}]^s)^{-\beta'} - (1 + [\lambda_{\infty}^{(j)}]^s)^{-\beta'}|^2 \\ &\leq \sum_{j=1}^{\ell} |(1 + [\lambda_n^{(j)}]^s)^{-\beta'} - (1 + [\lambda_{\infty}^{(j)}]^s)^{-\beta'}|^2 \\ &\quad + 2 \sum_{j=\ell+1}^{k_n} (1 + [\lambda_n^{(j)}]^s)^{-2\beta'} + 2 \sum_{j=\ell+1}^{k_n} (1 + [\lambda_{\infty}^{(j)}]^s)^{-2\beta'}. \end{aligned}$$

Using the fact that, $\tilde{\mathbb{P}}$ -a.s., we have $\lambda_n^{(j)} \gtrsim \lambda_{\infty}^{(j)}$ for all $j \in \{1, \dots, k_n\}$ (see Step 2), the latter two summations can be bounded, up to a multiplicative constant, by the convergent series $\sum_{j=\ell+1}^{\infty} (1 + [\lambda_{\infty}^{(j)}]^s)^{-2\beta'}$. Combined with the eigenvalue convergence asserted by Theorem 3.12, which can be applied to the remaining summation, we obtain $\|U_n^2 - U_n^3\|_{2 \rightarrow \infty} \rightarrow 0$ as $n \rightarrow \infty$, $\tilde{\mathbb{P}}$ -a.s., by arguing as in Step 3. Thus, we have shown part (b).

(a) Replace (A.2) by the identity $\|U\|_{\mathcal{L}_2(L^2(\mathcal{M}))}^2 = \sum_j |\alpha_j|^2 \|f_j\|_{L^2(\mathcal{M})}^2$, which follows directly from the definition of the Hilbert–Schmidt norm. Assumption 3.9 is not required since all the eigenfunctions are L^2 -normalized. The sufficiency of Assumption 3.5(i) and the $\tilde{\mathbb{P}}$ -a.s. convergence in the conclusion are due to the use of Theorem 3.13(a) instead of Theorem 3.13(b).

(c) Recall from Setting 3.2 that $h_n := n^{-\frac{1}{m}}$ by definition. In view of Lemma 3.11, we can take $k_n := n - 1$, hence neither of the bounds on h_n from Assumption 3.5 is needed. Indeed, in the proof of (b), the lower bound (A.1) on k_n was only used in (A.5), which in the current situation becomes $\|U_n^3 - U_n\|_{2 \rightarrow \infty}^2 \lesssim_{\mathcal{M}} M_{\psi, \infty}^2 k_n^{-4s\beta'/m}$, and this tends to zero since, trivially, $k_n \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

Lastly, we prove Lemma 3.15 asserting the uniform ultracontractivity of the semigroups $(S_n(t))_{t \geq 0}$ associated to the (discretized) generalized Whittle–Matérn operators $-[\mathcal{L}_n^{\kappa, \tau}]^s$.

Proof of Lemma 3.15. (a) For $p = \infty$, the statement holds by Assumption 3.6(i). For $p = 2$, we note that, for all $n \in \mathbb{N}$, $t > 0$, and $f \in L^2(\mathcal{M}_n)$,

$$\|S_n(t)f\|_{L^\infty(\mathcal{M}_n)} = \|R_n^\beta(\text{Id}_n + A_n)^\beta S_n(t)f\|_{L^\infty(\mathcal{M}_n)} \leq t^{-\beta} \|R_n^\beta\|_{2 \rightarrow \infty} \|f\|_{L^2(\mathcal{M}_n)},$$

where we used (C.2), as $(S_n(t))_{t \geq 0}$ is a contractive analytic semigroup on $L^2(\mathcal{M}_n)$. In the proof of Theorem 3.14(b), we found $\|R_n^\beta\|_{2 \rightarrow \infty} \leq M_{\psi, \infty} \sum_{j=1}^n (1 + [\lambda_n^{(j)}]^s)^{-2\beta}$ under Assumption 3.9. Since Assumption 3.5(ii) implies 3.5(i) with the same β , and since the estimate of $\|R_n^\beta\|_{2 \rightarrow \infty}$ only involves the eigenvalues (and not the eigenfunctions), we can in both cases argue as in Theorem 3.14(a), under Assumption 3.4(i), to deduce that, $\mathbb{P}$ -a.s., the right-hand side can be bounded independently of n . This proves the statement for $p = 2$, hence by the Riesz–Thorin interpolation theorem [38, Theorem 1.3.4], the lemma holds for all $p \in [2, \infty]$, with the bound $M_{S,p} \leq M_{S,2}^{\frac{2}{p}} M_{S,\infty}^{1-\frac{2}{p}}$.

(b) The differences with part (a) are the use of Theorem 3.14(c) instead of 3.14(b), and the fact that Assumption 3.9 is automatically satisfied. $\square$

A.2. Proof of Theorem 3.10

Using the intermediate results stated in Section 3.3 and proved in Section A.1, we can now prove Theorem 3.10:

Proof of Theorem 3.10. To prove parts (a)–(c), we will apply Theorems 5.7, 6.13 and Corollary 6.19, respectively, which are the rigorous counterparts of the discrete-to-continuum Theorem 1.1 in the respective settings.

The argument preceding Setting 3.2 shows that Assumptions 4.1 and 6.1 hold in any of the given situations, with $H_n := L^2(\mathcal{M}_n)$ and $E_n := L^q(\mathcal{M}_n)$ for $n \in \overline{\mathbb{N}}$ and $q \in [2, \infty)$, as well as $B_n := L^\infty(\mathcal{M}_n)$ for all $n \in \mathbb{N}$, $B_\infty := C(\mathcal{M})$ and $\tilde{B} := L^\infty(\mathcal{M})$. Moreover, note that Assumption 5.6 (or 6.6) is explicitly assumed in each case.

(a) Here, we take $q = 2$, i.e., $E_n = H_n = L^2(\mathcal{M}_n)$ for all $n \in \overline{\mathbb{N}}$. As discussed at the end of Subsection 3.1, the operators $(A_n)_{n \in \overline{\mathbb{N}}} := ([\mathcal{L}_n^{\tau, \kappa}]^s)_{n \in \overline{\mathbb{N}}}$ are uniformly sectorial of angle 0 on $(L^2(\mathcal{M}_n))_{n \in \overline{\mathbb{N}}}$. Letting $\beta \in (\frac{m}{4s}, \frac{1}{2})$ be as in Assumption 3.5(i), it follows from Theorem 3.14(a) that $\tilde{R}_n^{\beta'} \rightarrow R_\infty^{\beta'}$, $\mathbb{P}$ -a.s., in $\mathcal{L}_2(L^2(\mathcal{M}))$ as $n \rightarrow \infty$, for all $\beta' \geq \beta$. Applying this with the values $\beta' := \beta \in (0, \frac{1}{2})$ and $\beta' := 1$ verifies Assumptions 4.2 and 4.5. Setting, for all $(\omega, t) \in \Omega \times [0, T]$, $u \in L^2(\mathcal{M}_n)$ and $x \in \mathcal{M}_n$,

$$[F_n(\omega, t, u)](x) := f_n(\omega, t, u(x)), \quad (\text{A.6})$$

it is immediate from Assumption 3.7(i) that Assumption 5.1 is satisfied. Moreover, combining the definition of $\tilde{F}_n$ from (5.3) with (A.6) yields

$$[\tilde{F}_n(\omega, t, u)](x) = [\Lambda_n F_n(\omega, t, \Pi_n u)](x) = f_n(t, \omega, \Lambda_n \Pi_n u(x)),$$

so that

$$\begin{aligned} \|\tilde{F}_n(\omega, t, u) - F_\infty(\omega, t, u)\|_{L^2(\mathcal{M})} &= \|f_n(\omega, t, \Lambda_n \Pi_n u(\cdot)) - f_\infty(\omega, t, u(\cdot))\|_{L^2(\mathcal{M})} \\ &\leq \tilde{L}_f \|\Lambda_n \Pi_n u - u\|_{L^2(\mathcal{M})} + \|f_n(\omega, t, u(\cdot)) - f_\infty(\omega, t, u(\cdot))\|_{L^2(\mathcal{M})}. \end{aligned}$$

As $n \rightarrow \infty$, the first term vanishes by Assumption 4.1, and the second term by dominated convergence using (3.14) and the uniform linear growth condition in Assumption 3.7(i). Therefore, Assumption 5.5 is also satisfied.

(b) Now we need Assumption 3.6(i) in order for $([\mathcal{L}_n^{\tau, \kappa}]^s)_{n \in \mathbb{N}}$ to be uniformly sectorial of angle less than $\frac{1}{2}\pi$ on $(L^q(\mathcal{M}_n))_{n \in \mathbb{N}}$ for all $q \in [2, \infty)$. Letting the constants $\delta > 0$ and $\beta \in (\frac{m+4+\delta}{2s}, \frac{1}{2})$ be as in Assumption 3.5(ii), it follows from Theorem 3.14(b) that $\tilde{R}_n^{\beta'} \rightarrow R_\infty^{\beta'}$ in $L^0(\tilde{\Omega}; \mathcal{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M})))$ as $n \rightarrow \infty$, for all $\beta' \geq \beta$, under Assumptions 3.4(ii) and 3.9. In particular, we have $\tilde{R}_n^\beta \rightarrow R_\infty^\beta$ in $L^0(\tilde{\Omega}; \gamma(L^2(\mathcal{M}); L^q(\mathcal{M})))$ for all $q \in [1, \infty)$ by [50, Corollary 9.3.3] (recall that γ -radonifying operators were defined in Section 4.1), and $\tilde{R}_n \rightarrow R_\infty$ in $L^0(\tilde{\Omega}; \mathcal{L}(L^\infty(\mathcal{M})))$. This verifies Assumptions 6.2 and 6.3. By Lemma 3.15(a), we get Assumption 6.4 with $\theta = \frac{4}{q}\beta$. Thus, choosing $q > \frac{4\beta}{1-2\beta}$ yields $\theta + 2\beta < 1$. Assumptions 6.7 and 6.8 follow similarly to part (a).

(c) As in part (b), we need to verify Assumptions 6.1–6.4, now with contractive semigroups $(S_n(t))_{t \geq 0}$, i.e., Assumption 3.6(ii). For $s = 1$, $(S_\infty(t))_{t \geq 0}$ is $L^\infty(\mathbb{T})$ -contractive since the $L^1(\mathbb{T})$ -norm of its heat kernel coincides with the $L^1(\mathbb{R})$ -norm of the Gauss–Weierstrass kernel, which is equal to 1. For finite n , the $L^\infty(\mathcal{M}_n)$ -contractivity of $S_n(t) = e^{-tA_n}$ is equivalent to A_n being diagonally dominant with positive diagonal by [65, Lemma 6.1], which holds for Laplacian matrices. Since these assertions can be extended to all $s \in (0, 1]$ by subordination, see for instance [51, Theorem 15.2.17], we indeed find that Assumption 3.6(ii) holds. Thus, we can proceed to argue as in (b), using Theorem 3.14(c) and Lemma 3.15(b) for an arbitrary $\beta \in (\frac{1}{4s}, \frac{1}{2})$, to obtain Assumptions 6.1–6.4 with $\tilde{M}_S = 1$ and $\theta + 2\beta < 1$ for $q \in [2, \infty)$ large enough.

It remains to establish that the nonlinearities from Assumption 3.7(ii) are such that Assumption 6.16 holds. This is done in Example 6.17, noting that the space $L^\infty(\mathcal{M}_n)$ coincides with $C(\mathcal{M}_n)$ if we equip $\mathcal{M}_n$ with the discrete topology. $\square$

Appendix B. Fractional parabolic integration

Let $(E, \|\cdot\|_E)$ be a Banach space. Suppose that $-A: D(A) \subseteq E \rightarrow E$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$. This implies the existence of constants $M \in [1, \infty)$ and $w \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathcal{L}(E)} \leq M e^{wt} \quad \text{for all } t \in [0, \infty). \quad (\text{B.1})$$

Fixing $T \in (0, \infty)$, we define $k_s: \mathbb{R} \rightarrow \mathcal{L}(E)$ by $k_s(\tau) := \frac{1}{\Gamma(s)} \tau^{s-1} S(s) \mathbf{1}_{[0, T]}(\tau)$ for $\tau \in \mathbb{R}$. For any function $f: (0, T) \rightarrow E$ such that the following Bochner integral converges in E for $s \in (0, \infty)$ and a.e. $t \in (0, T)$, we set

$$\mathfrak{I}_A^s f(t) := k_s * f(t) = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} S(t-\tau) f(\tau) d\tau. \quad (\text{B.2})$$

For $s = 0$ we set $\mathfrak{I}_A^0 := \text{Id}_E$. The following properties of the *fractional parabolic integration operator* $\mathfrak{I}_A^s$ (for a single operator A) are well known, see [21, Proposition 5.9], and used throughout the main text. We will state them here for the sake of self-containedness.

Proposition B.1. *Suppose that $-A: \mathcal{D}(A) \subseteq E \rightarrow E$ generates a strongly continuous semigroup $(S(t))_{t \geq 0} \subseteq \mathcal{L}(E)$ satisfying (B.1). Then, for every $s \in [0, \infty)$, $p \in [1, \infty]$ and $T \in (0, \infty)$, we have:*

- (a) $\mathfrak{I}_A^s$ is bounded from $L^p(0, T; E)$ to itself, with an operator norm depending only on s , p , T , w and M .

If $(F, \|\cdot\|_F)$ is a Banach space for which there exist $M' \in [1, \infty)$ and $\alpha \in [0, \infty)$ such that

$$S(t) \in \mathcal{L}(E; F) \quad \text{with} \quad \|S(t)\|_{\mathcal{L}(E; F)} \leq M' t^{-\alpha/2} \quad \text{for all } t \in [0, \infty),$$

and in addition we have either $p = 1$, $s \geq 1 + \frac{\alpha}{2}$ or $p > 1$, $s > \frac{1}{p} + \frac{\alpha}{2}$, then

- (b) $\mathfrak{I}_A^s$ is bounded from $L^p(0, T; E)$ to $C([0, T]; F)$, with an operator norm depending only on s , p , T and M' .

For a sequence of operators $(A_n)_{n \in \overline{\mathbb{N}}}$ on Banach spaces which satisfy the appropriate discrete-to-continuum assumptions from the main text, the proposition above implies the following corollary regarding uniform boundedness of the sequence $(\tilde{\mathfrak{I}}_{A_n}^s)_{n \in \overline{\mathbb{N}}}$, where $\tilde{\mathfrak{I}}_{A_n}^s := \Lambda_n \mathfrak{I}_{A_n}^s \Pi_n$ for all $n \in \overline{\mathbb{N}}$. From this, in turn, one can derive Proposition B.3 below asserting the strong convergence of these operators.

Corollary B.2. *Let the Banach spaces $(E_n, \|\cdot\|_{E_n})_{n \in \overline{\mathbb{N}}}$, $(\tilde{E}, \|\cdot\|_{\tilde{E}})$ and the linear operators $(A_n)_{n \in \overline{\mathbb{N}}}$ satisfy the assumptions of Theorem 2.3, and suppose that $p \in [1, \infty]$ and $s \in [0, \infty)$. The following assertions hold:*

- (a) *The sequence $(\tilde{\mathfrak{I}}_{A_n}^s)_{n \in \overline{\mathbb{N}}}$ is uniformly bounded in $\mathcal{L}(L^p(0, T; \tilde{E}))$.*
 (b) *The sequence $(\tilde{\mathfrak{I}}_{A_n}^s)_{n \in \overline{\mathbb{N}}}$ is uniformly bounded in $\mathcal{L}(L^p(0, T; \tilde{E}); C([0, T]; \tilde{E}))$ if, either, $p = 1$ and $s \geq 1$, or $p > 1$ and $s > \frac{1}{p}$.*

If the spaces $(E_n)_{n \in \overline{\mathbb{N}}}$, $(B_n)_{n \in \overline{\mathbb{N}}}$ and $\tilde{B}$ are as in Assumptions 6.1, 6.2 and 6.4, and we have $s > \frac{1}{p} + \frac{\theta}{2}$, then

- (c) *the sequence $(\tilde{\mathfrak{I}}_{A_n}^s)_{n \in \overline{\mathbb{N}}}$ is uniformly bounded in $\mathcal{L}(L^p(0, T; E_\infty); C([0, T]; \tilde{B}))$.*

Proposition B.3. *Let the Banach spaces $(E_n, \|\cdot\|_{E_n})_{n \in \overline{\mathbb{N}}}$, $(\tilde{E}, \|\cdot\|_{\tilde{E}})$ and the linear operators $(A_n)_{n \in \overline{\mathbb{N}}}$ satisfy the assumptions of Theorem 2.3. Let $p \in [1, \infty]$ and $s \in [0, \infty)$. The following assertions hold:*

- (a) *If either $p = 1$ and $s \geq 1$, or $p > 1$ and $s > \frac{1}{p}$, then we have $\tilde{\mathfrak{I}}_{A_n}^s f \rightarrow \mathfrak{I}_{A_\infty}^s f$ in $C([0, T]; \tilde{E})$, as $n \rightarrow \infty$, for every $f \in L^p(0, T; E_\infty)$.*

Moreover, let Assumptions 6.1–6.4 hold.

(b) If $s > \frac{1}{p} + \frac{\theta}{2}$, then $\tilde{\mathcal{I}}_{A_n}^s f \rightarrow \mathcal{I}_{A_\infty}^s f$ in $C([0, T]; \tilde{B})$ as $n \rightarrow \infty$ for every $f \in L^p(0, T; E_\infty)$.

Proof. We only present the details of the argument for part (b), the proof of (a) being similar.

Let $p \in [1, \infty)$, $s \in (\frac{1}{p} + \frac{\theta}{2}, \infty)$, $f \in L^p(0, T; E_\infty)$ and fix an arbitrary $\varepsilon > 0$. By the density of B_∞ in E_∞ (see 6.4), and that of B_∞ -valued simple functions in $L^p(0, T; B_\infty)$, there exists a function $g: [0, T] \rightarrow B_\infty$ of the form

$$g = \sum_{j=1}^K \mathbf{1}_{(a_j, b_j)} \otimes x_j, \quad K \in \mathbb{N}; \quad 0 \leq a_j < b_j \leq T, \quad x_j \in B_\infty \text{ for all } j \in \{1, \dots, K\}$$

such that

$$\|f - g\|_{L^p(0, T; E_\infty)} < \frac{\varepsilon}{4} \left(\sup_{n \in \mathbb{N}} \|\tilde{\mathcal{I}}_{A_n}^s\|_{\mathcal{L}(L^p(0, T; E_\infty); C([0, T]; \tilde{B}))} \right)^{-1}.$$

Note that the expression between the parentheses is finite by Corollary B.2(c) and can be assumed to be nonzero without loss of generality, as otherwise $\tilde{\mathcal{I}}_{A_n}^s = 0$ for all $n \in \mathbb{N}$ and the asserted convergence would be trivial. Thus, for every $n \in \mathbb{N}$,

$$\begin{aligned} & \|\tilde{\mathcal{I}}_{A_n}^s f - \mathcal{I}_A^s f\|_{C([0, T]; \tilde{B})} \\ & \leq \|\tilde{\mathcal{I}}_{A_n}^s (f - g)\|_{C([0, T]; \tilde{B})} + \|\tilde{\mathcal{I}}_{A_n}^s g - \mathcal{I}_A^s g\|_{C([0, T]; \tilde{B})} + \|\mathcal{I}_A^s (g - f)\|_{C([0, T]; \tilde{B})} \\ & < \frac{1}{2}\varepsilon + \|\tilde{\mathcal{I}}_{A_n}^s g - \mathcal{I}_A^s g\|_{C([0, T]; \tilde{B})}. \end{aligned}$$

For any $j \in \{1, \dots, K\}$, by Assumption 6.3 and the discrete-to-continuum Trotter–Kato approximation theorem, we can choose $N_j \in \mathbb{N}$ so large that

$$\|\tilde{\mathcal{S}}_n \otimes x_j - S \otimes x_j\|_{C([0, T]; \tilde{B})} < \frac{s\Gamma(s)}{2T^s K} \varepsilon \quad \text{for all } n \geq N_j.$$

Thus, setting $N := \max_{j=1}^K N_j$, we find for all $n \geq N$ and $t \in [0, T]$:

$$\begin{aligned} \|\tilde{\mathcal{I}}_{A_n}^s g(t) - \mathcal{I}_A^s g(t)\|_{\tilde{B}} & \leq \frac{1}{\Gamma(s)} \sum_{j=1}^K \int_{a_j}^{b_j} (t-r)^{s-1} \|\tilde{\mathcal{S}}_n(t-r)x_j - S(t-r)x_j\|_{\tilde{B}} dr \\ & \leq \frac{1}{\Gamma(s)} \sum_{j=1}^K \int_0^T r^{s-1} \|\tilde{\mathcal{S}}_n(r)x_j - S(r)x_j\|_{\tilde{B}} dr \\ & \leq \frac{T^s}{s\Gamma(s)} \sum_{j=1}^K \|\tilde{\mathcal{S}}_n \otimes x_j - S \otimes x_j\|_{C([0, T]; \tilde{B})} < \frac{\varepsilon}{2}. \end{aligned}$$

Since $t \in [0, T]$ was arbitrary, we conclude that $\|\tilde{\mathcal{I}}_{A_n}^s g - \mathcal{I}_A^s g\|_{C([0, T]; \tilde{B})} < \frac{1}{2}\varepsilon$, and therefore $\|\tilde{\mathcal{I}}_{A_n}^s f - \mathcal{I}_A^s f\|_{C([0, T]; \tilde{B})} < \varepsilon$ for all $n \geq N$. $\square$

Appendix C. Uniformly sectorial sequences of operators

We first recall the concept of sectorial operators. Given $\omega \in (0, \pi)$, we say that a linear operator $A: D(A) \subseteq E \rightarrow E$ on a (real or complex) Banach space E , with *spectrum* $\sigma(A) := \mathbb{C} \setminus \rho(A)$, is said to be ω -sectorial if

$$\sigma(A) \subseteq \overline{\Sigma_\omega} \quad \text{and} \quad M(\omega, A) := \sup\{\|\lambda R(\lambda, A)\|_{\mathcal{L}(E)} : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\omega}\} < \infty, \quad (\text{C.1})$$

where Σ_ω is as in (3.11) and $M(\omega, A)$ is called the ω -sectoriality constant. Its *angle of sectoriality* $\omega(A) \in [0, \pi)$ is defined as the infimum of all ω for which (C.1) holds.

If A is closed and densely defined, then by [82, Theorem 13.30], there exists $\omega \in (0, \frac{1}{2}\pi)$ such that A is ω -sectorial if and only if there exists $\eta \in (0, \frac{1}{2}\pi)$ such that $-A$ generates a bounded analytic semigroup $(S(t))_{t \geq 0}$ on Σ_η . The latter means that $[0, \infty) \ni t \mapsto S(t) \in \mathcal{L}(E)$ extends to a bounded holomorphic function $\Sigma_\eta \ni z \mapsto S(z) \in \mathcal{L}(E)$. Inspecting the proof of the cited theorem reveals that, whenever these equivalent conditions hold, we have

$$\sup_{z \in \Sigma_\eta} \|S(z)\|_{\mathcal{L}(E)} \sim_{(\omega, \eta)} M(\omega, A).$$

This theorem also asserts that the supremum of the set of $\eta \in (0, \frac{1}{2}\pi)$ for which $(S(t))_{t \geq 0}$ extends to a bounded analytic semigroup on Σ_η equals $\frac{1}{2}\pi - \omega(A)$.

Moreover, by [42, Propositions 3.4.1 and 3.4.3] we have

$$\|A^\alpha S(t)\|_{\mathcal{L}(E)} \lesssim_{(\omega, \alpha)} M(\omega, A) t^{-\alpha}, \quad (\text{C.2})$$

for all $\omega \in (\omega(A), \frac{1}{2}\pi)$ and $t \in (0, \infty)$, where the implicit constant is non-decreasing in α for any fixed ω .

We say that a sequence $(A_n)_{n \in \mathbb{N}}$ of linear operators $A_n: D(A_n) \subseteq E_n \rightarrow E_n$ is *uniformly sectorial of angle* $\omega \in [0, \pi)$ if A_n is sectorial of angle ω for all $n \in \mathbb{N}$ and

$$M_{\text{Unif}}(\omega', A) := \sup_{n \in \mathbb{N}} M(\omega', A_n) < \infty \quad \text{for all } \omega' \in (\omega, \pi). \quad (\text{C.3})$$

Lemma C.1 complements Theorem 2.3 in the situation where the semigroup generators are uniformly sectorial of angle less than $\frac{1}{2}\pi$, in which case we obtain the uniform convergence $\Lambda_n A_n^\alpha S(\cdot) \Pi_n x \rightarrow A_\infty^\alpha S_\infty(\cdot) x$ on compact subsets of $(0, \infty)$. It is an analog to [56, Lemma 4.1(2)] in the discrete-to-continuum setting and for general $\alpha \in (0, \infty)$ (instead of $\alpha = 1$).

Lemma C.1. *Let the sequence $(A_n)_{n \in \mathbb{N}}$ of linear operators $A_n: D(A_n) \subseteq E_n \rightarrow E_n$ on the Banach spaces $(E_n, \|\cdot\|_{E_n})$ be uniformly sectorial of angle $\omega \in [0, \frac{1}{2}\pi)$, and denote by $(S_n(t))_{t \geq 0}$ the bounded analytic C_0 -semigroups generated by $-A_n$. Let the conditions and the equivalent statements (a) and (b) of Theorem 2.3 hold. Then we have, for all $\alpha \in (0, \infty)$, $x \in E_\infty$ and $0 < a < b < \infty$,*

$$\sup_{t \in [a, b]} \|\Lambda_n A_n^\alpha S_n(t) \Pi_n x - A_\infty^\alpha S_\infty(t) x\|_{\tilde{E}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Fix $\omega' \in (\omega, \frac{1}{2}\pi)$, $n \in \overline{\mathbb{N}}$ and $t \in (0, \infty)$. We begin by sketching the functional calculus argument (see [51, Chapter 15] or [42] for a more comprehensive overview of this topic) which shows that we have the following Cauchy integral representation:

$$A_n^\alpha S_n(t) = \frac{1}{2\pi i} \int_{\partial \Sigma_{\omega'}} z^\alpha e^{-tz} R(z, A_n) dz. \quad (\text{C.4})$$

To see this, define the functions $f_\alpha, g_t: \Sigma_{\omega'} \rightarrow \mathbb{C}$ by $f_\alpha(z) := z^\alpha$ and $g_t(z) := e^{-tz}$ for $z \in \Sigma_{\omega'}$ and $t \in (0, \infty)$. Denote by $f_\alpha(A_n)$ and $g_t(A_n)$ the operators obtained via the extended Dunford calculus for sectorial operators as defined in [51, Definition 15.1.8]. Then $f_\alpha(A_n)$ is the fractional power A_n^α in the sense of [51, Definition 15.2.2], which satisfies $f_\alpha(A_n)x = f_\alpha(\lambda)x = \lambda^\alpha x$ if $A_n x = \lambda x$, hence this (more general) definition agrees with the spectrally defined fractional powers in the setting of (3.10). Moreover, $g_t(A_n) = S_n(t)$ by [51, Theorem 15.1.7]. Since $S_n(t)$ is bounded, we have $(f_\alpha g_t)(A_n) = f_\alpha(A_n)g_t(A_n)$ by [51, Proposition 15.1.12]. Finally, the function $(f_\alpha g_t)(z) = z^\alpha e^{-tz}$ is holomorphic and has (super)polynomial decay at 0 and ∞ , and thus belongs to the domain of the primary Dunford calculus [51, Definition 15.1.1], so that $(f_\alpha g_t)(A_n) = \frac{1}{2\pi i} \int_{\partial \Sigma_{\omega'}} z^\alpha e^{-tz} R(z, A_n) dz$. Putting all these observations together yields (C.4). Applying the projection and lifting operators and parametrizing the complex integral yields, for all $x \in E_\infty$,

$$\begin{aligned} \Lambda_n A_n^\alpha S_n(t) \Pi_n x &= -\frac{e^{i(\alpha+1)\omega'}}{2\pi i} \int_0^\infty r^\alpha \exp(-te^{i\omega'} r) \tilde{R}(e^{i\omega'} r, A_n) x dr \\ &\quad + \frac{e^{-i(\alpha+1)\omega'}}{2\pi i} \int_0^\infty r^\alpha \exp(-te^{-i\omega'} r) \tilde{R}(e^{-i\omega'} r, A_n) x dr, \end{aligned}$$

where we recall that $\Pi_\infty = \Lambda_\infty = \text{Id}_{\tilde{E}}$ for $n = \infty$. It follows that the above estimate implies the following uniform bound on the interval $[a, b]$:

$$\begin{aligned} &\sup_{t \in [a, b]} \|\Lambda_n A_n^\alpha S_n(t) \Pi_n x - A_\infty^\alpha S_\infty(t) x\|_{\tilde{E}} \\ &\leq \frac{1}{2\pi} \int_0^\infty r^\alpha e^{-a \cos(\omega') r} \left[\|\tilde{R}(re^{i\omega'}, A_n) x - R(re^{i\omega'}, A_\infty) x\|_{\tilde{E}} \right. \\ &\quad \left. + \|\tilde{R}(re^{-i\omega'}, A_n) x - R(re^{-i\omega'}, A_\infty) x\|_{\tilde{E}} \right] dr. \end{aligned} \quad (\text{C.5})$$

Setting $\eta := \frac{1}{2}\pi - \vartheta$ for some $\vartheta \in (\omega, \omega')$, the uniform sectoriality of $(A_n)_{n \in \overline{\mathbb{N}}}$ implies that the operators $(-A_n e^{\pm i\eta})_{n \in \overline{\mathbb{N}}}$ generate C_0 -semigroups $(S_n(te^{\pm i\eta}))_{t \geq 0}$ which are uniformly bounded in t and n . Therefore, we can apply Theorem 2.3 to these two sequences of semigroups to find that $\tilde{R}(\lambda, A_n)x \rightarrow R(\lambda, A_\infty)x$ for all $|\arg \lambda| > \vartheta$ if this convergence holds for one such λ . We have in fact $\tilde{R}(\lambda, A_n)x \rightarrow R(\lambda, A_\infty)x$ for all λ such that $|\arg \lambda| > \frac{1}{2}\pi > \vartheta$ by our hypothesis that the operators $(A_n)_{n \in \overline{\mathbb{N}}}$ satisfy statements (a) and (b) in Theorem 2.3, so we conclude

$\tilde{R}(re^{\pm i\omega'}, A_n)x \rightarrow R(re^{\pm i\omega'}, A_\infty)x$ for all $r \in (0, \infty)$. On the other hand, by (C.3) and Definition 2.1 we have $\|\tilde{R}(re^{\pm i\omega'}, A_n)x\|_{\tilde{E}} \leq M_\Pi M_\Lambda M_{\text{Unif}}(\vartheta, A)\|x\|_{\tilde{E}}/r$ for all $n \in \overline{\mathbb{N}}$. Hence, we can bound the integrand in (C.5), up to n -independent constants, by the integrable function $r \mapsto r^{\alpha-1} \exp(-a \cos(\omega')r)$, so that the integrals tend to zero by the dominated convergence theorem. $\square$

Data availability

No data was used for the research described in the article.

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