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GREEN'S FUNCTIONS AND THE NON-EQUILIBRIUM EQUATION WITH APPLICATIONS TO NON-EQUILIBRIUM FREE STREAMS

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SUMMARY

The propagation of small disturbances in a relaxing or reacting gas is governed by a third-order partial differential equation for the velocity potential (the non-equilibrium equation). A generalised Green's theorem which applies to this equation is established and Green's functions are found for supersonic and subsonic steady flows in two dimensions. These functions are used to find solutions for the flow past slender obstacles. For subsonic streams, the flow field is assumed to be of infinite extent; for a supersonic stream one can consider fields of finite extent. In particular, the method permits comparatively easy analysis of supersonic streams which are not necessarily in equilibrium or of uniform velocity ahead of the body. Three examples of such flows are worked out.

CONTENTS

1. Introduction

This examination of Green's function methods for solution of the non-equilibrium equation was motivated by the appearance of a recent paper on this topic by Ryhming[5]. Certain aspects of this latter work were felt to be unsatisfactory and we hope to clear the matter up in the present paper. In particular, in the example given by the aforementioned author for the flow velocity on a two-dimensional wedge in a supersonic non-equilibrium stream, the velocity is of the wrong sign: i.e. the pressure would be diminished on a wedge of positive turning angle according to this result, instead of increased as it should be. The reasons for this error in sign are quite fundamental, as we hope to point out below. In order to set about the task before us it is advisable to begin by establishing the appropriate generalisation of Green's theorem which applies to the non-equilibrium equation.

2. Green's Theorem

Restricting our attention to the two-dimensional steady flow case, the non-equilibrium equation satisfied by the perturbation potential $\varphi'(x', y')$ can be written in the form (Vincenti[6])

$$\Gamma \left\{ \beta_f^2 \varphi'_{x'x'x'} + \varphi'_{x'y'y'} \right\} + \beta_e^2 \varphi'_{x'x'} + \varphi'_{y'y'} = 0 \tag{1}$$

where Γ is the relaxation length and

$$\beta_f^2 = 1 - M_{f\infty}^2 \quad ; \quad \beta_e^2 = 1 - M_{e\infty}^2 \quad . \tag{2}$$

M_{f_∞} and M_{e_∞} are the frozen and equilibrium free stream Mach numbers, respectively. The free stream is assumed to be of velocity U directed along the x' - axis from left to right. It is convenient to define dimensionless co-ordinates x and y, such that

$$x = x'/\Gamma \quad ; \quad y = y'/\Gamma \quad , \tag{3}$$

whence, writing

$$\varphi'(x', y') = \varphi(x, y) , \tag{4}$$

equation 1 becomes

$$L[\varphi] \equiv \beta_f^2 \varphi_{xxx} + \varphi_{xyy} + \beta_e^2 \varphi_{xx} + \varphi_{yy} = 0 . \tag{5}$$

Equation 5 defines the operator L in terms of x, y co-ordinates. The operator which is adjoint to L is written as $\tilde{L}$ where

$$\tilde{L} \equiv -\beta_f^2 \frac{\partial^3}{\partial x^3} - \frac{\partial^3}{\partial x \partial y^2} + \beta_e^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad . \tag{6}$$

We shall write $\underset{\sim}{r}$ for the vector whose components are x, y and $\underset{\sim}{r}_o$ for the vector whose components are x_o, y_o, etc. where it is convenient to do so. Thus we can write

$$G = G(\underset{\sim}{r}/\underset{\sim}{r}_o) \tag{7}$$

for the Green's function, representing the influence at a field-point $\underset{\sim}{r}$ which results from a source located at a source-point $\underset{\sim}{r}_o$. G satisfies the equation

$$L[G] = -4\pi \; \delta(\underset{\sim}{r} - \underset{\sim}{r}_o) , \tag{8}$$

where δ is the impulse function (an even function of its argument). (The (-4π) term is not absolutely necessary here but is used in the book by Morse and Fesbach[3], from which the basic theory of this section is taken. To make reference to this work easier we therefore retain it).

We can also define the adjoint Green's function, written as $\tilde{G}$, where

$$\tilde{G} = \tilde{G}(\underset{\sim}{r}/\underset{\sim}{r}_o) . \tag{9}$$

$\tilde{G}$ satisfies the adjoint equation

$$\tilde{L}\,[\tilde{G}] = -4\pi\,\delta(\underset{\sim}{r} - \underset{\sim}{r}_o). \tag{10}$$

In the current notation the disturbance potential φ is a function $\varphi(\underset{\sim}{r})$. Let us therefore consider the quantity

$$\tilde{G}(\underset{\sim}{r}/\underset{\sim}{r}_o)\, L\left[\varphi(\underset{\sim}{r})\right] - \varphi(\underset{\sim}{r})\; \tilde{L}\left[\tilde{G}(\underset{\sim}{r}/\underset{\sim}{r}_o)\right] \equiv F . \tag{11}$$

Using the definitions of the operators in equations 5 and 6, writing out F in full and grouping the terms suitably, we see that

$$F = \frac{\partial}{\partial x}\left\{ \beta_f^2\left[\tilde{G}\varphi_{xx} + \varphi\tilde{G}_{xx} - \tilde{G}_x\varphi_x\right] + \beta_e^2\left[\tilde{G}\varphi_x - \varphi\tilde{G}_x\right] - \varphi_y\tilde{G}_y \right\}$$

$$+ \frac{\partial}{\partial y}\left\{ \left[\varphi_y + \varphi_{xy}\right]\tilde{G} - \left[\tilde{G}_y - \tilde{G}_{xy}\right]\varphi \right\} . \tag{12}$$

Clearly F can be re-written as the divergence of a vector $\underset{\sim}{P}$ whose components, P_x and P_y, are just the first and second bracket terms in equation 12, respectively. That is to say

$$F = \underset{\sim}{\nabla} \cdot \underset{\sim}{P} \tag{13}$$

where $\underset{\sim}{\nabla}$ is the gradient operator. Equations 11 and 13 constitute the necessary generalised Green's theorem, namely

$$\tilde{G}\,L\,[\varphi] - \varphi\,\tilde{L}\,[\tilde{G}] = \underset{\sim}{\nabla} \cdot \underset{\sim}{P} . \tag{14}$$

$\underset{\sim}{P}$ is of course a function of φ and $\tilde{G}$.

Equation 14 can now be used to find an expression for the potential φ within a closed surface in terms of the boundary values of φ, and its derivatives, and the adjoint Green's function. To do so we shall first interchange $\underset{\sim}{r}$ and $\underset{\sim}{r}_o$ in equation 14, so that it now reads

$$\tilde{G}_o L_o \left[\varphi_o\right] - \varphi_o \tilde{L}_o \left[\tilde{G}_o\right] = \underset{\sim}{\nabla}_o \cdot \underset{\sim}{P}_o , \tag{15}$$

where

$$\varphi_o \equiv \varphi(\underset{\sim}{r}_o) \;:\; \tilde{G}_o \equiv \tilde{G}(\underset{\sim}{r}_o/\underset{\sim}{r}) \;:\; \underset{\sim}{P}_o \equiv \underset{\sim}{P}(\varphi(\underset{\sim}{r}_o), \tilde{G}(\underset{\sim}{r}_o/\underset{\sim}{r})) \tag{16}$$

and L_o, $\tilde{L}_o$ and $\underset{\sim}{\nabla}_o$ are now operators involving x_o and y_o. We assume that the equations

$$L_o \left[\varphi_o\right] = 0 \;;\; \tilde{L}_o \left[\tilde{G}_o\right] = -4\pi\,\delta(\underset{\sim}{r} - \underset{\sim}{r}_o) \tag{17}$$

are satisfied within a region V_o bounded by a line S_o (see Fig. 1). The vector $\underset{\sim}{r}$ may or may not be inside S_o.

Multiplying the first of equations 17 by $\tilde{G}_o$ and the second by φ_o, subtracting the results and using equation 15, we find upon integrating throughout the region V_o that

$$\int_{V_o} \left\{ \tilde{G}_o L_o \left[\varphi_o\right] - \varphi_o \tilde{L}_o \left[\tilde{G}_o\right] \right\} dv_o = \int_{V_o} \underset{\sim}{\nabla}_o \cdot \underset{\sim}{P}_o \, dv_o \tag{18}$$

$$= 4\pi\,\varphi(\underset{\sim}{r}) \text{ if } \underset{\sim}{r} \text{ is inside } S_o$$

$$= 0 \text{ , if } \underset{\sim}{r} \text{ is outside } S_o .$$

(The last two results follow from the properties of the δ- function). But Gauss' theorem shows that

$$\int_{V_o} \underset{\sim}{\nabla}_o \cdot \underset{\sim}{P}_o \, dv_o = \int_{S_o} \underset{\sim}{n}_o \cdot \underset{\sim}{P}^s_o \, ds_o , \tag{19}$$

where n_o is the unit outwards normal vector to S_o and ds_o is an element of arc length on S_o ; $\underset{\sim}{P}^s_o$ is the value of vector $\underset{\sim}{P}_o$ on this line. Equations 18 and 19 show that

$$4\pi\,\varphi(\underset{\sim}{r}) = \int_{S_o} \underset{\sim}{n}_o \cdot \underset{\sim}{P}^s_o \, ds_o ,$$

or

$$4\pi\,\varphi(\underset{\sim}{r}) = \int_{S_o} \underset{\sim}{n}_o \cdot \underset{\sim}{P}\,(\varphi(\underset{\sim}{r}^s_o), \tilde{G}(\underset{\sim}{r}^s_o/\underset{\sim}{r}))\, ds_o , \tag{20}$$

if $\underset{\sim}{r}$ is within S_o. $\underset{\sim}{r}^s_o$ is the value of $\underset{\sim}{r}_o$ on S_o, and equation 20 is certainly one form of the desired result for $\varphi(\underset{\sim}{r})$.

However, it is more usual to express the value of $\varphi(\underset{\sim}{r})$ in terms of G rather than its adjoint $\tilde{G}$, so that a relation between these two latter functions is required. In order to derive such a relationship it is only necessary to relate the boundary conditions which must be satisfied by G and $\tilde{G}$, since we already know the differential equations satisfied by each function. For example we have

$$L\left[G(\underset{\sim}{r}/\underset{\sim}{r}_o)\right] = -4\pi\,\delta(\underset{\sim}{r} - \underset{\sim}{r}_o)\ ;\quad \tilde{L}\left[\tilde{G}(\underset{\sim}{r}/\underset{\sim}{r}_1)\right] = -4\pi\,\delta(\underset{\sim}{r} - \underset{\sim}{r}_1)\ . \tag{21}$$

Multiplying the first of equations 21 by $\tilde{G}(\underset{\sim}{r}/\underset{\sim}{r}_1)$, the second by $G(\underset{\sim}{r}/\underset{\sim}{r}_o)$, subtracting and integrating over the region V surrounded by the line S (which are just V_o and S_o in x, y rather than x_o, y_o co-ordinates; see Fig. 1) we find that

$$\int_V \left\{ \tilde{G}(\underset{\sim}{r}/\underset{\sim}{r}_1)\, L\left[G(\underset{\sim}{r}/\underset{\sim}{r}_o)\right] - G(\underset{\sim}{r}/\underset{\sim}{r}_o)\, \tilde{L}\left[\tilde{G}(\underset{\sim}{r}/\underset{\sim}{r}_1)\right] \right\} dv$$
$$= -4\pi\left\{ \tilde{G}(\underset{\sim}{r}_o/\underset{\sim}{r}_1) - G(\underset{\sim}{r}_1/\underset{\sim}{r}_o) \right\},$$

provided both $\underset{\sim}{r}_o$ and $\underset{\sim}{r}_1$ are within S. Using Green's theorem (equation 14 with $G(\underset{\sim}{r}/\underset{\sim}{r}_o)$ in place of $\varphi(\underset{\sim}{r})$) and, subsequently, Gauss' theorem, we see that

$$\int_S \underset{\sim}{n}\cdot\underset{\sim}{P}\left[G(\underset{\sim}{r}^s/\underset{\sim}{r}_o),\ \tilde{G}(\underset{\sim}{r}^s/\underset{\sim}{r}_1)\right] ds$$
$$= -4\pi\left\{ \tilde{G}(\underset{\sim}{r}_o/\underset{\sim}{r}_1) - G(\underset{\sim}{r}_1/\underset{\sim}{r}_o) \right\}.$$

The simple (reciprocity) relation

$$\tilde{G}(\underset{\sim}{r}_o/\underset{\sim}{r}_1) = G(\underset{\sim}{r}_1/\underset{\sim}{r}_o) \tag{22}$$

now exists between the Green's functions, provided that we choose the relationship between their boundary values so as to make

$$\underset{\sim}{n}\cdot\underset{\sim}{P}\left[G(\underset{\sim}{r}^s/\underset{\sim}{r}_o),\ \tilde{G}(\underset{\sim}{r}^s/\underset{\sim}{r}_1)\right] = 0 \tag{23}$$

everywhere on S. Thus the relation 23 limits the choice of boundary conditions for $\tilde{G}$ and G. For example, considering a rectangular boundary S made up of lines parallel to the x and y axes, equation 12, which defines the components P_x and P_y of $\underset{\sim}{P}$, enables us to deduce the following facts. On a line of constant y (so that $\underset{\sim}{n}\cdot\underset{\sim}{P}$ thereon is proportional to P_y) one can choose either homogenous Dirichlet or homogeneous Neumann conditions for G and $\tilde{G}$, thus making $\underset{\sim}{n}\cdot\underset{\sim}{P}$ vanish on such a line. On a line of constant x (involving only P_x, therefore), one must employ additional conditions in order to make $\underset{\sim}{n}\cdot\underset{\sim}{P}$ zero. It is not enough to make G and $\tilde{G}$ zero, for example. One must, in addition, choose either G_x or $\tilde{G}_x$ to be zero (G_y or $\tilde{G}_y$ will be zero automatically if G and $\tilde{G}$ are zero on a line of constant x). Alternatively, one could make $\underset{\sim}{n}\cdot\underset{\sim}{P}$ zero on such a line by choosing G, G_x and G_{xx} equal to zero, and simply ensuring that the appropriate terms in $\tilde{G}$ did not behave so badly as to make a product, like $G\,\tilde{G}_{xx}$ for example, other than zero on the line in question. The particular choice of conditions on lines of constant x will depend to some extent on the problem in hand, but the data certainly must be of the Cauchy type. We may also remark here that if the boundaries of S across the free stream should happen to be inclined to this direction, so that a fraction of P_y enters into $\underset{\sim}{n}\cdot\underset{\sim}{P}$ in addition to a fraction of P_x, Cauchy data is still required. One can also make $\underset{\sim}{n}\cdot\underset{\sim}{P}$ vanish on lines across the free stream by invoking a causality condition such as would apply in supersonic flow. That is to say, one requires that $G(x^S, y^S/x_o, y_o)$ should be zero for all points x^S, y^S ahead of the downstream pointing characteristic lines through the source point x_o, y_o. The adjoint condition on $\tilde{G}(x^S, y^S/x_o, y_o)$ makes this function zero for all x^S, y^S ahead of the upstream-facing characteristics through x_o y_o (since the direction of x is reversed in the adjoint problem). The appropriate parts of $\underset{\sim}{n} \cdot \underset{\sim}{P}$ then vanish because the source points are inside S. These particular causality conditions apply to downstream propagating waves of course; we shall use them in the section to follow. We remark that a causality condition is sufficient to find G, where it may apply ; one does not need additional data in such a case.

Using the reciprocity condition 22, we can now write equation 20 in the form

$$4\pi\ \varphi(\underset{\sim}{r}) = \int_{S_o} \underset{\sim}{n}_o \cdot \underset{\sim}{P}\ (\varphi(\underset{\sim}{r}_o^S)\ ,\ G(\underset{\sim}{r}/\underset{\sim}{r}_o^S))\ ds_o. \tag{24}$$

The Green's function $G(\underset{\sim}{r}/\underset{\sim}{r}_o^S)$ satisfies the inhomogeneous equation

$$L\left[G(\underset{\sim}{r}/\underset{\sim}{r}_o^S)\right] = -4\pi\ \delta(\underset{\sim}{r} - \underset{\sim}{r}_o^S)\ ; \tag{25}$$

i.e. as we would infer from elementary physical reasoning, the boundary value problem can be solved by distributing 'sources' of some kind along S_o. The type of 'source', or equivalently, the form of the Green's function will depend on the given data concerning $\varphi(r_o^S)$ and we shall say more about this later on. Meanwhile we note that the function $\underset{\sim}{P}$ in the integrand in equation 24 is $\underset{\sim}{P}_o^S$, so that it involves derivatives of φ and G with respect to x_o and y_o evaluated for $x_o = x_o^S$, $y_o = y_o^S$, (see the definitions of $\underset{\sim}{P}$ and $\underset{\sim}{P}_o$ in equations 12 and 16 for example). We can not find such derivatives of $G(\underset{\sim}{r}/\underset{\sim}{r}_o)$ from equation 25 as it stands: indeed we cannot solve 25 as it stands, because our boundary value data on $G(\underset{\sim}{r}/\underset{\sim}{r}_o)$ in equation 24 is given in terms of derivatives with respect to x_o, y_o and not x, y, so that we have no boundary value data for $G(\underset{\sim}{r}/\underset{\sim}{r}_o)$ applicable to the operator L, which is an operator in x, y co-ordinates. The proper evaluation of $G(\underset{\sim}{r}/\underset{\sim}{r}_o^S)$ with boundary value data given in r_o co-ordinates can be accomplished as follows. The second of equations 17 is

$$\tilde{L}_o\left[\tilde{G}(\underset{\sim}{r}_o/\underset{\sim}{r})\right] = -4\pi\ \delta\ (\underset{\sim}{r} - \underset{\sim}{r}_o)\ ,$$

which, using equation 22 with $\underset{\sim}{r}_1$ there written as $\underset{\sim}{r}$, is equivalent to

$$\tilde{L}_o\left[G(\underset{\sim}{r}/\underset{\sim}{r}_o)\right] = -4\pi\ \delta(\underset{\sim}{r} - \underset{\sim}{r}_o)\ . \tag{26}$$

We can now solve equation 26 for $G(\underset{\sim}{r}/\underset{\sim}{r}_o)$ in $\underset{\sim}{r}_o$ co-ordinates and, having satisfied the requisite conditions for $G(\underset{\sim}{r}/\underset{\sim}{r}_o)$ on S_o, then let $\underset{\sim}{r}_o \rightarrow \underset{\sim}{r}_o^S$ to find the appropriate value for use in equation 24. Of course $\underset{\sim}{r}_o$ in equation 26 must lie within S_o.

Ryhming[5] used a form of equation 24, which we can write as

$$\varphi(\underset{\sim}{r}) = \int_{S_o} A(r_o^s)\, G(\underset{\sim}{r}/\underset{\sim}{r}_o^s)\, ds_o, \tag{27}$$

the function A being undefined but eventually evaluated from the boundary conditions on $\varphi(\underset{\sim}{r})$. The result 27 was quoted by him without proof: he found the Green's function from equation 25. From what has gone before here, it is clear that a result like 27 above can only be true if $G(\underset{\sim}{r}/\underset{\sim}{r}_o^s)$ satisfies appropriate boundary conditions. How these conditions were ensured by solving equation 25 and taking only the particular solution is not entirely clear, although we do not imply that it is incorrect. We shall proceed with our analysis here, using the general results developed above, and find the solution for $\varphi(\underset{\sim}{r})$ in the half-plane $y > 0$, for supersonic flow in the first instance.

3. The Supersonic Problem

Consider the following problem; Find the potential $\varphi(\underset{\sim}{r})$ in the half-space $y > 0$, $-\infty < x < \infty$, for a supersonic flow, $\beta_e^2 \leqslant \beta_f^2 < 0$, when φ_y is given everywhere along the line $y = 0$; for example

$$\varphi_y(x, 0) = U\, h'(x), \quad -\infty < x < \infty \tag{28}$$

$y = h(x)$ represents the shape of a solid boundary adjacent to the flow in $y > 0$; equation 28 is then the linearised tangency condition. We shall assume that the supersonic flow is originally parallel, with velocity U along the x-axis direction, at some upstream location x.

Assume that the surface S_o is made up of the line $y_o = 0$ and straight lines parallel to the x_o and y_o axes (as shown in Fig. 2). Writing out equation 24 in full, and remembering that $\underset{\sim}{n}_o$ is an outwards unit normal vector, we find that

$$\left.\begin{aligned}
4\pi\varphi(x,y) = & -\int_{-H_2}^{H_1} \Big\{ \left[\varphi_{y_o}(x_o, 0) + \varphi_{x_o y_o}(x_o, 0)\right] G(x, y \mid x_o, 0) \\
& \qquad - \left[G_{y_o}(x, y/x_o, 0) - G_{x_o y_o}(x, y/x_o, 0)\right] \varphi(x_o, 0) \Big\}\, dx_o \\
& + \int_{-H_2}^{H_1} \Big\{ \left[\varphi_{y_o}(x_o, H_3) + \varphi_{x_o y_o}(x_o, H_3)\right] G(x, y/x_o, H_3) \\
& \qquad - \left[G_{y_o}(x, y/x_o\, H_3) - G_{x_o y_o}(x, y/x_o, H_3)\right] \varphi(x_o, H_3) \Big\}_{H_3 \to \infty} dx_o \\
& + \int_{0}^{H_3 \to \infty} \Big\{ G(x, y/H_1, y_o) \left[\varphi_{x_o x_o}(H_1, y_o)\beta_f^2 + \varphi_{x_o}(H_1, y_o)\beta_e^2\right] - G_{y_o}(x, y/H_1, y_o)\varphi_{y_o}(H_1, y_o) \\
& \qquad - G_{x_o}(x, y/H_1, y_o)\left[\varphi_{x_o}(H_1, y_o)\beta_f^2 - \varphi(H_1, y_o)\beta_e^2\right] - G_{x_o x_o}(x, y/H_1, y_o)\varphi(H_1, y_o)\beta_f^2 \Big\} dy_o \\
& - \int_{0}^{H_3 \to \infty} \Big\{ G(x, y/-H_2, y_o) \left[\varphi_{x_o x_o}(-H_2, y_o)\beta_f^2 + \varphi_{x_o}(-H_2, y_o)\beta_e^2\right] - G_{y_o}(x, y/-H_2, y_o)\varphi_{y_o}(-H_2 y_o) \\
& \qquad - G_{x_o}(x, y/-H_2, y_o)\left[\varphi_{x_o}(-H_2, y_o)\beta_f^2 - \varphi(-H_2, y_o)\beta_e^2\right] - G_{x_o x_o}(x, y/-H_2, y_o)\varphi(-H_2, y_o)\beta_f^2 \Big\} dy_o
\end{aligned}\right\} \tag{29}$$

The boundary value data given in equation 28 suggests that we should set $G_{y_o}(x, y/x_o, 0)$ equal to zero: we observe that specification of $\varphi_{y_o}(x_o, 0)$ means that $\varphi_{x_o y_o}(x_o, 0)$ is also known and similarly that, if $G_{y_o}(x, y/x_o, 0)$ is zero, $G_{x_o y_o}(x, y/x_o, 0)$ is zero too. We may anticipate that it should prove possible to find $G(x, y/x_o, H_3)$ and $G_{y_o}(x, y/x_o, H_3)$ equal to zero as $H_3 \to \infty$ for fixed $y < H_3$, thereby making the second integral in equation 29 equal to zero in the limit. In fact we can, strictly speaking, only choose one of either $G(x, y/x_o, H_3)$ or $G_{y_o}(x, y/x_o, H_3)$ equal to zero on $y_o = H_3$; we shall have to confirm our anticipation later on. In view of the nature of the problem one would certainly select G_{y_o} to be zero, since it is always possible to specify φ_{y_o} on $y_o = H_3$.

As far as the remaining integrals in equation 29 are concerned, we shall invoke the causality condition that $G(x, y/x_o, y_o)$ must be zero for all $x_o > x$ in a supersonic flow with down stream-propagating waves. Then the third integral will vanish because the point x, y is within S_o. With a suitable choice of Cauchy data for $\varphi(x_o, y_o)$ in the last integral we can make this vanish too: e.g. we can set φ, φ_{x_o}, $\varphi_{x_o x_o}$ equal to zero when $x_o = -H_2$ for all y_o in this integral. φ and φ_{x_o} are zero on $x_o = -H_2$ by hypothesis; the additional requirement on $\varphi_{x_o x_o}$ is interesting, and we shall comment upon it at a later stage.

On the assumption that it will be possible to verify all of these remarks about boundary value data, the final form for φ will read simply

$$4\pi\, \varphi(x, y) = -\int_{-H_2}^{H_1} \left[\varphi_{y_o}(x_o, 0) + \varphi_{x_o y_o}(x_o, 0)\right] G(x, y/x_o, 0)\, dx_o, \qquad (30)$$

for $\quad -H_2 < x < H_1, \quad 0 < y < +\infty.$

The task is now reduced to that of finding a Green's function which satisfies equation 26 and the boundary value data mentioned above. In conformity with the usual practice, we first find the Green's function for an unbounded domain and then find the function satisfying the required conditions on S_o by the method of images.

Noting the definition of the adjoint operator in equation 6, and writing out equation 26 in full, we solve

$$-\beta_f^2 G'_{x_o x_o x_o} - G'_{x_o y_o y_o} + \beta_e^2 G'_{x_o x_o} + G'_{y_o y_o} = -4\pi\, \delta(x - x_o)\, \delta(y - y_o) \qquad (31)$$

in the region $-\infty \leqslant y_o \leqslant \infty$, $-\infty \leqslant x_o \leqslant \infty$, where $G' = G'(x, y/x_o, y_o)$ is the unbounded domain Green's function. We use the Fourier transforms

$$g(x, y/y_o : \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G'(x, y/x_o, y_o) e^{i\zeta x_o}\, dx_o \; : \; G' = \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} g(x, y/y_o : \xi)\, e^{-ix_o\xi}\, d\xi \, . \qquad (32)$$

In the first of these ζ has a negative imaginary part, equal to $-i\epsilon'$ where $\epsilon' > \epsilon > 0$. Convergence of the integral for $g(\zeta)$ is thus secured as $x_o \to -\infty$; the behaviour of G' as $x_o \to +\infty$ is assumed to be suitable for convergence purposes*. We remark that the integration contour in the second of equations 32 runs below the real ξ-axis. Multiplying equation 31 by $\exp(i\zeta x_o)/\sqrt{2\pi}$ and integrating the result from x_o equals $-\infty$ to $+\infty$ gives the following equation for g (since $x_o = x$ lies within the range of integration):

$$-\beta_f^2(i\zeta^3)g - (-i\zeta)g_{y_o y_o} + \beta_e^2(-\zeta^2)g + g_{y_o y_o} = -4\pi\frac{e^{i\zeta x}}{\sqrt{2\pi}} \cdot \delta(y - y_o) . \tag{33}$$

Both β_f^2 and β_e^2 are negative in supersonic flow; accordingly equation 33 can be re-arranged so as to read

$$g_{y_o y_o} + |\beta_f^2|\zeta^2\left(\frac{i\zeta + a}{i\zeta + 1}\right) g = -4\pi f(\zeta)\,\delta(y - y_o) , \tag{34}$$

where

$$f(\zeta) \equiv -\frac{i}{\sqrt{2\pi}} \frac{e^{i\zeta x}}{(\zeta - i)} \tag{35}$$

and

$$a \equiv \beta_e^2/\beta_f^2 > 1 . \tag{36}$$

We shall also write

$$B^2 \equiv |\beta_f^2|\zeta^2\left(\frac{i\zeta + a}{i\zeta + 1}\right) = |\beta_f^2|\zeta^2\left(\frac{\zeta - ia}{\zeta - i}\right) \tag{37}$$

in what follows.

A general solution of equation 34 can be written in the form

$$g = Ce^{-iBy_o} + De^{iBy_o} + \frac{4\pi f(\zeta)}{i2B}\left\{e^{-iB(y_o - y)} - e^{iB(y_o - y)}\right\} H(y_o - y) , \tag{38}$$

where $H(y_o - y)$ is the Heaviside unit step function ($=0$, $y_o < y$; $= 1$, $y_o > y$). C and D are two quantities independent of y_o (they do depend on x, y, and ζ in general) which must be chosen for fit conditions on G'. We simply ask that g should remain bounded as $y_o \to \pm\infty$ for fixed x and y.

Before the values of C and D can be settled, we must decide which branch of the two-valued function B is to be used. Writing

$$B = |\beta_f|\zeta\sqrt{\frac{\zeta - ia}{\zeta - i}} \tag{39}$$

we see that the function $B(\xi)$, which will occur in the second integral of equations 32, has two branch points, at $\xi = ia$ and $\xi = i$. Fig. 3 shows the cut complex ξ-plane; $g(\xi)$ is regular for all Im $\xi < -\epsilon < 0$. If we let $\sqrt{\xi - ia}$ and $\sqrt{\xi - i}$ both behave like $\sqrt{|\xi|}$ as $\xi \to +\infty$ then $-iB$ (where B is now taken as a function

* This is in fact guaranteed by the causality condition, as we verify below.

of ξ rather than of ζ) will have a negative real part everywhere on the ξ integration contour. Converseley $+iB$ will have a positive real part and so, as $y_o(>y$, fixed) $\to +\infty$, g will behave like : -

$$g \sim De^{iBy_o} - \frac{4\pi f(\zeta)e^{-iBy}}{i2B} \cdot e^{iBy_o}.$$

To make g bounded in these circumstances we take

$$D = \frac{4\pi f(\zeta)\, e^{-iBy}}{i2B} \quad . \tag{40}$$

When $y_o \to -\infty$, g behaves as follows (note that now $y_o < y$):

$$g \sim C\, e^{-iBy_o} \quad .$$

Hence we must take

$$C = 0 . \tag{41}$$

The appropriate solution for g is therefore

$$g = \frac{4\pi f(\zeta)}{i2B} \left\{ e^{iB(y_o-y)} + \left[e^{-iB(y_o-y)} - e^{iB(y_o-y)} \right] H(y_o-y) \right\}. \tag{42}$$

The corresponding value of G' is found from equation 32 : -

$$G'(x,y/x_o,y_o) = -\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{i\xi(x-x_o)}}{(\xi - i)B} \left\{ e^{iB(y_o-y)} + \left[e^{-iB(y_o-y)} - e^{iB(y_o-y)} \right] H(y_o-y) \right\} d\xi \ . \tag{43}$$

We may close the contour $\pm\infty - i\epsilon$ in the region $\mathrm{Im}\,\xi < 0$ with an infinite semi-circle, on which $\xi = Re^{i\theta}$, $R \to \infty$, $-\pi < \theta < 0$. For $|\xi| = R \to \infty$ it is easy to show that

$$B = Re^{i\theta} \,|\beta_f| + 0(1).$$

Since $g(\xi)$ is regular in $\mathrm{Im}\,\xi < \epsilon$, we can replace the integration with respect to ξ from $-\infty - i\epsilon$ to $+\infty - i\epsilon$ by an integration with respect to θ from $\theta = -\pi +$ to $\theta = 0-$; $\sin\theta < 0$ in this interval, so that $iRe^{i\theta}$ has a positive real part. Each of the three exponential terms in equation 43 has a dominant part of the form

$$+ i\, Re^{i\theta} \left\{ x - x_o + |\beta_f|\, (y_o - y), \text{ or } x - x_o - |\beta_f|(y_o-y), \text{ or } x-x_o + |\beta_f|(y_o-y), \text{ respectively} \right\}.$$

When $y_o < y$ only the first term appears; when $y_o > y$ the first and third terms cancel, leaving only the second term. Thus, when $y_o < y$, $G'(x,y/x_o,y_o)$ will be zero if $x-x_o - |\beta_f|\,|y_o - y| < 0$; i.e. if $x_o > x - |\beta_f|\,|y_o-y|$. When $y_o > y$,

$G'(x,y|x_o,y_o)$ will be zero if $x_o > x - |\beta_f|(y_o - y)$. In general then G' is zero for all $x_o > x - |\beta_f|\,|y_o - y|$ and the same will be true of all of its derivatives with respect to x_o or y_o. Since the minimum value of $|y_o - y|$ is zero, the case is covered completely by demanding $x_o > x$. We have therefore verified that G' satisfies the causality condition which we had earlier required it to do. We can also write G' in the form

$$G'(x,y\ x_o,y_o) = -\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{i\xi(x-x_o) - iB|y_o-y|}}{(\xi - i)B}\, d\xi\ .$$

Now the reciprocity condition (equation 22) must be satisfied, so that

$$G'(x,y|x_o,y_o) = \tilde{G}'(x_o,y_o|x,y)\ .$$

Changing x_o, y_o for x, y and vice versa, this means that

$$\tilde{G}'(x,y|x_o,y_o) = -\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{i\xi(x_o-x) - iB|y_o-y|}}{(\xi - i)B}\, d\xi$$

so that $\tilde{G}'(x,y\ x_o,y_o)$ is zero if $x > x_o$. But the integral expression for G' above satisfies the equation

$$\tilde{\mathcal{L}}_o\left[G'(x,y|x_o,y_o)\right] = -4\pi\delta(x - x_o)\,\delta(y - y_o)$$

and it follows that the integral expression for $\tilde{G}'(x,y|x_o,y_o)$ must satisfy

$$\mathcal{L}\left[\tilde{G}'(x,y|x_o,y_o)\right] = -4\pi\,\delta(x - x_o)\,\delta(y - y_o),$$

since we merely write x,y for x_o, y_o and vice versa, and use the fact that the δ functions are even functions of their arguments. This latter equation is indeed the correct one for $\tilde{G}'(\underset{\sim}{r}/\underset{\sim}{r}_o)$ (note the last of equations 21), and so all the conditions of Section 2 are satisfied on account of the causality condition (i.e. in particular, condition 23 is confirmed on this account). Equation 43 is thus the proper choice for outgoing, or downstream propagating, waves.

We must now set about finding the proper value for $G(x,y|x_o,y_o)$ for use in equation 29. Suppose we place another source of the form given in equation 43 at the image of the source point x_o, y_o in the $y_o = 0$ plane, namely at x_o, $-y_o$. Writing G_i' for its potential we have (with $y > 0$),

$$G_i' = -\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{i\xi(x-x_o)}}{(\xi - i)B}\cdot e^{iB|-y_o-y|}\, d\xi\ . \tag{44}$$

Adding G_i' to G' gives

$$G(x,y|\,x_o,y_o) = -\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{i\xi(x-x_o)}}{(\xi-i)B}\left\{ e^{-iB(y_o+y)} + e^{iB(y_o-y)}\right.$$

$$\left. + \left[e^{-iB(y_o-y)} - e^{iB(y_o-y)}\right]H(y_o-y)\right\} d\xi \qquad (45)$$

Differentiating this function with respect to y_o (which is permissible because the resulting integral is still convergent) shows at once that $G_{y_o}(x,y\,|\,x_o,o)$ is zero for all x and x_o, and $y > o$. We also observe that both $G(x,y\,|\,x_o,y_o \to +\infty)$ and $G_{y_o}(x,y\,|\,x_o,y_o \to \infty)$ go to zero in the limit $y_o \to \infty$, $y_o > y$. G in equation (45) also satisfies the causality condition for $x_o > x$ and hence it is just the Green's function we require for equation (29). The third integral in equation (29) vanishes identically from causality, because $x_o = H_1 > x$, x being essentially <u>within</u> S_o; the second integral vanishes because G and G_{y_o} both vanish as $y_o = H_3 \to \infty$ for $y_o > y$ (i.e. point x,y within S_o again); also part of the first integral vanishes because $G_{y_o} = o$ when $y_o = o$, $y_o < y$. We can therefore write

$$4\pi\varphi\ (x,y) = -\int_{-H_2}^{H_1} \left[\varphi_{y_o}(x_o,o) + \varphi_{x_o y_o}(x_o,o)\right] G(x,y\,|\,x_o,o)\,dx_o$$

$$-\int_{o}^{\infty} P_x(x,y:-H_2,y_o)\,dy_o \qquad (46)$$

where P_x(etc.) is written for the long group of terms in { } brackets in the fourth integral of equation (29). Clearly P_x is zero if φ, φ_x and φ_{xx} are all zero for $x = -H_2$ and all y*: equation (46) is then identical with equation (30). For the

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* In this connection we note that setting φ, φ_x and φ_{xx} equal to zero everywhere on a line of constant x for $y > o$ we imply also that φ_y shall be zero. If we take $\varphi_y(x,o)$, $\varphi_{xy}(x,o)$ equal to zero for $x < o$ the first integral in equation (46) exerts no influence on φ for these values of x. Thus the station $x = -H_2$ must be upstream of $x = o$ (i.e. H_2 positive), or, more generally, $x = -H_2$ must be ahead of the nose of the body in order for the conditions on φ, φ_x and φ_y to hold that line.

moment we shall assume this to be the case; therefore we only need to evaluate $G(x,y|x_o,0)$ to complete the solution. From equation (45), keeping $y > y_o$ and letting $y_o \to 0$, we have

$$\frac{1}{4\pi} G(x,y|x_o,0) = -\frac{1}{2\pi} \int_{-\infty-i\epsilon}^{\infty-i\xi} \frac{e^{i\xi(x-x_o) - i\xi\sqrt{\frac{\xi-ia}{\xi-i}}|\beta_f|y}}{|\beta_f|\,\xi\sqrt{\frac{\xi-ia}{\xi-i}}\,(\xi - i)}\, d\xi \qquad (47)$$

having written in the full value for B as a function of ξ. We can use the arguments following equation (43) to show that $G(x,y|x_o,0)$ is zero for all $x_o > x - |\beta_f|y$. Going further, and writing

$$x_o = x - |\beta_f|y - \delta,$$

and then letting $\delta \to o$ from above (i.e. δ is essentially positive) we can show from the integral in equation (47) that

$$G(x,y|x_o,0) = 0 \;;\; x_o \geq x - |\beta_f|\, y, \qquad (48)$$

so that the Green's function is continuous across the downstream - facing frozen Mach line through the source point,

Let us now suppose that the solid boundary, whose shape has been given by

$$y = h(x)$$

(see the tangency condition 28), really has the form

$$y = h(x)\, H(x),$$

where H(x) is the unit step function and h(x) is a smooth continuous function of x which is zero when x = o. Then equation (28) is modified so as to read

$$\varphi_y(x,o) = Uh'(x)\, H(x)\,, \quad -\infty \leq x \leq \infty \qquad (49)$$

Consequently,

$$\varphi_{xy}(x,o) = Uh''(x)\, H(x) + Uh'(x)\delta(x) \qquad (50)$$

If h'(0) should happen to be zero, the last term in equation (50) vanishes.

Equation (30) for the potential φ will now become

$$\varphi(x,y) = -\int_{-H_2}^{x-|\beta_f|y-} U[h'(x_o)H(x_o)+h'(x_o)\delta(x_o)+h''(x_o)H(x_o)]\frac{1}{4\pi}G(x,y|x_o,o)dx_o$$

or

$$\varphi(x,y) = -Uh'(o)\frac{1}{4\pi}G(x,y|o,o) - \int_{o+}^{x-|\beta_f|y-} U[h'(x_o)+h''(x_o)]\frac{1}{4\pi}G(x,y|x_o,0)dx_o, \tag{51}$$

if $x-|\beta_f|y > 0$.

We can now find, for example, the streamwise disturbance velocity on the solid surface $y \to o+$. The general expression for this quantity, from equation (51), is

$$\varphi_x(x,o) = -Uh'(o)\frac{1}{4\pi}G_x(x,o|o,o) - \int_{o+}^{x-} U[h'(x_o)+h''(x_o)]\frac{1}{4\pi}G_x(x,o|x_o,o)dx_o \tag{52*}$$

and G_x can be found directly by differentiating equation (47) (since the resulting integral is convergent), i.e.

$$\frac{1}{4\pi}G_x(x,o|x_o,o) = -\frac{i}{2\pi}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{i\xi(x-x_o)}}{|\beta_f|(\xi-i)\sqrt{\frac{\xi-ia}{\xi-i}}}d\xi \tag{53}$$

For $x-x_o > o$ the integral in equation (53) can be reduced to an integration around the dumbell contour (depicted as ABCD in Figure 3). The contributions from the circles around the two branch points clearly vanish in the limit of vanishing radius; on the straight parts DA and BC we set $\xi = +iv$ and take care to see that the phases of $\sqrt{\xi - ia}$ and $\sqrt{\xi - i}$ are correct. In transferring to the dumbell contour from the $\pm\infty - i\epsilon$ contour, the latter is clearly achieved if we set:

On BC, $$\left.\begin{aligned}\sqrt{\xi-ia} &= \sqrt{a-v}\,e^{-i\pi/4}\\ \sqrt{\xi-i} &= \sqrt{v-1}\,e^{i\pi/4}\end{aligned}\right\} \therefore \sqrt{\frac{\xi-ia}{\xi-i}} = -i\sqrt{\frac{a-v}{v-1}}$$

On DA, $$\left.\begin{aligned}\sqrt{\xi-ia} &= \sqrt{a-v}\,e^{-i\pi/4}\\ \sqrt{\xi-i} &= \sqrt{v-1}\,e^{-i3\pi/4}\end{aligned}\right\} \therefore \sqrt{\frac{\xi-ia}{\xi-i}} = +i\sqrt{\frac{a-v}{v-1}}$$

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* There is no contribution from differentiation of the upper limit in equation (52) since $G(x,o|x-,o) = o$, see equation (48). This result should be compared with Ryhming's whose Green's function has an imaginary part in these circumstances. This part is then discarded for no other reason than that it is imaginary!

With these results we see that

$$|\beta_f| \frac{1}{4\pi} G_x(x, o | x_o, o) = -\frac{i}{2\pi}\left\{ \int_1^a \frac{e^{-(x-x_o)v}}{(iv-i)(-i)\sqrt{\frac{a-v}{v-1}}} idv + \int_a^1 \frac{e^{-(x-x_o)v}}{(iv-i)(+i)\sqrt{\frac{a-v}{v-1}}} idv \right\}$$

$$= \frac{1}{\pi} \int_1^a \frac{e^{-(x-x_o)v}}{\sqrt{a-v}\,\sqrt{v-1}} dv,$$

or, in other words

$$\frac{1}{4\pi} G_x(x, o | x_o, o) = \frac{e^{-(x-x_o)(\frac{a+1}{2})}}{|\beta_f|} I_o\left[(x-x_o)(\frac{a-1}{2})\right], \tag{54}$$

where I_o is the modified Bessel function of the first kind and zero order (Watson,[8]). We have gone through the analysis leading up to equation (54) in some moderate degree of detail in order to emphasise that G_x here is a positive function, a result which should be contrasted with Ryhming's solution which, whilst agreeing with the present analysis in respect of the sign and form of the source strength (in equation (51) for example), makes G_x negative. Furthermore, and of equal importance, there is no question of G or G_x (or indeed any other derivative of G) being anything other than purely real: Ryhming is forced to discard sundry embarrassing imaginary parts in his analysis, which seem to have arisen as a result of an incorrect choice of ξ-integration contour and phases for the radicals appearing in the quantity B when attempting to solve for G.

The value of $\varphi_x(x, o)$ for a wedge, which has h(x) equal to $\theta . x$, is found at once from equations (52) and (54): it is

$$\varphi_x(x, o) = -\frac{U\theta}{|\beta_f|}\left\{ e^{-x(\frac{a+1}{2})} I_o(x(\frac{a-1}{2})) + \int_o^x e^{-w(\frac{a+1}{2})} I_o(w(\frac{a-1}{2}))\, dw \right\} \tag{55}$$

following an obvious change of variable in the integral. Equation (55) agrees in all respects with the earlier results of Clarke[1] and Der[2] obtained by using different techniques.

To summarise the results arrived at so far for the supersonic non-equilibrium problem, we have shown, by using a generalised Green's theorem, that it is possible to find the disturbance velocity potential in a half-plane by employing homogeneous boundary conditions along one edge of the half-plane (in the present case homogeneous Neumann conditions) and Cauchy-type data along a boundary extending across the on-coming stream. The boundary value data is therefore adequately specified on an open surface; since the supersonic non-equilibrium equation for φ is of the hyperbolic type this conclusion comes as no surprise. It is interesting to note that the third-order character of the potential equation requires that we should specify not only φ and φ_x in the Cauchy conditions, but φ_{xx} too. (Just φ and φ_x are adequate for second-order hyperbolic equations). We can also take note of the fact that, for a solid surface which only begins to deviate from a free stream direction in the region $x > o$, the Cauchy data can be specified anywhere ahead of or on the line $x-|\beta_f|y = o$, and not just on $x \to -H_2$. We shall exploit this fact in the following section (where we shall also note the physical significance of specifying φ_{xx} in non-equilibrium problems). Finally we remark that the homogenous Neumann data specified along the solid boundary here is the same as that required for the more familiar second order hyperbolic equation; that is to say, no extra data than that required for a second order problem is necessary. However, we do remark that whilst φ will be a smooth continuous function of x and y if only the boundary slope is smooth in the second order case, the present third-order non-equilibrium equation requires that boundary curvature ($h''(x)$) should be smooth too for the same result in φ. (N.B. Although we have stated that $h(x)$ is a smooth continuous function of x, see equation (49), we may still find $h''(x)$ discontinuous. Equation (51) shows that $\varphi(x, y)$ will not be a smooth function in the event of a sudden jump in this quantity). In physical terms, a sudden change in boundary curvature can influence the disturbance field in the linear approximation.

4. Supersonic Free Streams Which Are Not in Equilibrium or Are Not Uniform

The solution obtained in the previous section was for the case φ, φ_x and $\varphi_{xx} = 0$ on the line $x = -H_2$ (constant). It follows that all y-derivatives of these quantities are zero on the same line and therefore that the basic equation for φ, namely $L[\varphi] = 0$, is satisfied there. In deriving the equation for φ a relation which arises during the analysis can be written for supersonic flow in the form

$$|\beta_f^2|\,\varphi'_{x'x'} - \varphi'_{y'y'} + \frac{h_{q_\infty}}{\rho_\infty h_{\rho_\infty}} \frac{(\bar{q} - q)}{\tau_o} = 0, \tag{56*}$$

(see, for example, Vincenti[6]). Here h is the specific enthalpy, ρ the density, q the non-equilibrium variable (e.g. degree of dissociation or internal mode temperature) and $\bar{q}$ its local equilibrium value. τ_o is the relaxation time, and suffix ∞ refers to the basic "undisturbed" state from which φ represents the degree of perturbation. h_q and h_ρ are the appropriate partial derivatives of h. If $\varphi_{yy} = o$, as is the case if $\varphi = o$ everywhere on $x = -H_2$ for example, then setting $\varphi_{xx} = o$ means that the initial stream, at the location $x = -H_2$, must be in equilibrium since q will equal $\bar{q}$ (τ_o being assumed finite and non-zero).

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* The variables x', y' are the dimensional variables, see Section 2, equations (1 to 5).

Now we shall attempt to relax the conditions φ, φ_x, φ_{xx} all zero. If, as we shall find, this can be done, then equation (56) indicates that we shall be studying free streams which may not be in equilibrium (the "free stream" here being taken as the flow crossing the boundary of our region S at $x = -H_2$). It is of course necessary to ensure that any conditions that we do impose on φ, φ_x, φ_y, φ_{xx} etc. are consistent with the equation $L[\varphi] = 0$, since this must be satisfied throughout the region bounded by S.

Let us first write equation (56) in the dimensionless variables x and y (see equation 3);

$$\beta_f^2 \varphi_{xx} + \varphi_{yy} = \frac{\Gamma^2 h_{q_\infty}}{\rho_\infty h_{p\infty} \tau_0} (\bar{q} - q) \equiv Q, \text{ say} \tag{57}$$

and reiterate equation (5) for convenience;

$$\frac{\partial}{\partial x} (\beta_f^2 \varphi_{xx} + \varphi_{yy}) + \beta_e^2 \varphi_{xx} + \varphi_{yy} = 0 \tag{58}$$

With equation (57), an alternative form for equation (58) is

$$\frac{\partial Q}{\partial x} + Q + (\beta_e^2 - \beta_f^2) \varphi_{xx} = 0 \tag{59}$$

We shall now suppose that φ, φ_x and φ_{xx} are all specified on the line $x = -H_2$ = constant. Let us write

$$\varphi(-H_2, y) = V(y) \; ; \; \varphi_x(-H_2, y) = W(y) \; ; \; \varphi_{xx}(-H_2, y) = X(y) \tag{60}$$

Then we shall also know φ_{yy} and φ_y:-

$$\varphi_{yy} = V''(y) \; ; \; \varphi_y = V'(y) \tag{61}$$

where each prime denotes a differentiation with respect to y. We have assumed that φ_y is zero on $y = 0$ ahead of the nose of the body in the previous section and we shall do so again here; thus

$$V'(o) = 0 \tag{62}$$

We must also note that, whilst φ_{yy} is given by V'', φ_{xyy} is given by W'', i.e.

$$\varphi_{xyy}(-H_2, y) = W''(y) \tag{63}$$

In general then, we are saying (from equation (57) et. seq) that

$$Q = \beta_f^2 X(y) + V''(y) \tag{64}$$

Equation (58) is satisfied with this value for $Q(-H_2, y)$ if

$$\beta_f^2 \varphi_{xxx}(-H_2, y) = - W''(y) - \beta_e^2 X(y) - V''(y), \tag{65}$$

and there is no reason why this should not be so. Using equations (64) and (60) in (59), we see that

$$\frac{\partial Q}{\partial x}(-H_2, y) = -\beta_e^2 X(y) - V''(y) \tag{66}$$

Equations (64) and (66) show that in general both Q and $\frac{\partial Q}{\partial x}$ differ from zero if one of them does; if $V''(y)$ should happen to be zero (meaning that $V'(y)$, and hence, reasonably enough, $V(y)$ is equal to zero too because of condition 62), then Q , $\partial Q/\partial x$ depend only on $X(y)$ and are only both zero if $X(y) = 0$. The latter case is the equilibrium stream, of course. However, we can make either Q or $\partial Q/\partial x$ zero independently of the other if $V''(y) \neq 0$. That is to say

$$Q = 0 \text{ if } \beta_f^2 X(y) = - V''(y) \;:\; \frac{\partial Q}{\partial x} = (a-1)V''(y) \tag{67}$$

$$\frac{\partial Q}{\partial x} = 0 \quad \text{if } \beta_e^2 X(y) = - V''(y) \;:\; Q = (1 - \frac{1}{a})V''(y) \tag{68}$$

where a is defined in equation (36).

We may certainly select a variety of values for Q therefore, and it is important to note that nowhere does the value of $\varphi_x = W(y)$ interfere with this selection; it merely serves to determine φ_{xxx} from equation (65) once the other quantities are specified.

Referring to the last integral in equation (29) (which is also the second one in equation (46)), we see that selection of φ, φ_y, φ_x, φ_{xx} on $x = -H_2$ for all y is all that is necessary to find $\varphi(x, y)$ in $y > 0$, $-H_2 < x < +\infty$, once φ_y on $y = 0$ has been chosen. In addition we see from the form of equation (46) that the non-equilibrium, non-uniform, free stream conditions at $x = -H_2$ simply add a part to $\varphi(x, y)$ over and above that due to the boundary shape (namely the first term in equation (46)). The latter we may write as $4\pi\varphi_{eq.}$, since it is all that remains of $4\pi\varphi$ when the free stream is a uniform, equilibrium one.

We may now write a general result for a non-uniform, non-equilibrium free stream in the form

$$\varphi(x,y) = \varphi_{eq}(x,y) - \frac{1}{4\pi}\int_0^\infty \Big\{ G(x,y\,|\,-H_2, y_o)[\beta_f^2 X(y_o) + \beta_e^2 W(y_o)]$$

$$- G_{y_o}(x,y\,|\,-H_2, y_o)V'(y_o) - G_{x_o}(x,y\,|\,-H_2, y_o)[\beta_f^2 W(y_o) - \beta_e^2 V(y_o)] \tag{69}$$

$$- G_{x_o x_o}(x,y\,|\,-H_2, y_o)\,\beta_f^2 V(y_o)\Big\}\,dy_o,$$

the Green's function being given by equation (45) with $x_o = -H_2$. It is now time to consider some specific examples.

(i) A Parallel Non-Equilibrium Stream of Constant Velocity.

For a parallel, constant velocity stream at $x = -H_2$ we can set $V(y_o) = 0 = W(y_o)$. The stream speed is then equal to U on this line, and equation (69) reduces to

$$\varphi(x,y) = \varphi_{eq}(x,y) - \frac{1}{4\pi}\int_0^\infty \beta_f^2 X(y_o) G(x,y\,|\,-H_2, y_o)\,dy_o \tag{70}$$

An especially simple case occurs if $X(y_o)$ is a constant (and hence Q = constant, see equation 64). Integrating equation (45) we readily show that

$$\varphi(x,y) = \varphi_{eq}(x,y) + \frac{2Q_c}{4\pi}\int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i(x+H_2)\xi}}{i|\beta_f^2|\,\xi^2(\xi - ia)}\,d\xi\,;$$

$$\therefore\quad \varphi(x,y) = \varphi_{eq}(x,y) + \frac{Q_c}{a|\beta_e^2|}\Big\{1 - e^{-(x+H_2)a} - a(x+H_2)\Big\}, \tag{71}$$

where Q_c is the constant value of Q (defined in equation (57)). The simple non-equilibrium free stream therefore has the effect of adding on an x-wise velocity component equal to

$$\frac{Q_c}{|\beta_e^2|}\Big\{e^{-a(x+H_2)} - 1\Big\} \tag{72}$$

over the entire flow field. Since $e^{-a(x+H_2)} \leqslant 1$ for $x \geqslant -H_2$, the velocity is negative for positive Q_c. Referring to equation (57), Q_c is positive if $\bar{q} > q$ (since the other quantities are positive). The gradual excitation of the internal energy mode up towards its equilibrium value drains kinetic energy from the gas stream, and tends to slow it down. Any body (whose disturbance field is summarised by φ_{eq}) which is immersed in such a flow, will therefore lie in a region of gradually increasing pressure. Taking (for example) a wedge, for which $\varphi_{eq\,x}$ on the surface is given by equation (55), we could reach an asymptotic state in which the streamwise velocity on the wedge surface was exactly U (namely $\varphi_x = 0$) if we made $\theta = -Q_c/U|\beta_e|$. (The right-hand side of equation (55) $\rightarrow -(U\theta/|\beta_f|)(1/\sqrt{a}) = -U\theta/|\beta_e|$ as $x \rightarrow +\infty$). Figure 4 in fact shows $|\beta_f|\varphi_x/U\theta$ plotted against x, φ_x being evaluated on y = 0, for this special relation between θ, Q_c etc. Two values of H_2 (namely 0 and 1) are used, and the curve for $Q_c = 0$ is drawn for comparison purposes (this solution is precisely the one previously obtained by one of us; Clarke [1]). Figure 4 is enough to show that the possible effects of a lack of equilibrium in the oncoming stream may be quite profound, completely altering pressure variations on a wedge, for example.

It is also of interest to see how φ_x behaves on the characteristic $x = |\beta_f|\, y$ through the nose of the body. $\varphi_{eq\,x}$ on this line follows at once from equation (51); it is

$$\varphi_{eq\,x} = -U\,h'(o)\,\frac{1}{4\pi}\,G_x(|\beta_f|\,y, y \mid o, o)$$

$$= -\frac{U\,h'(o)}{|\beta_f|}\,e^{-\frac{1}{2}(a-1)|\beta_f|\,y}, \qquad (73)$$

as one may readily show from the expression 47, for example, for G. In the particular non-equilibrium stream considered above we therefore find at once that

$$\varphi_x(|\beta_f|\,y, y) = -\frac{U\,h'(o)}{|\beta_f|}\,e^{-\frac{1}{2}(a-1)|\beta_f|\,y} - \frac{Q_c}{|\beta_e^2|}\left\{1 - e^{-a(|\beta_f|\,y + H_2)}\right\} \qquad (74)$$

(For the wedge $h'(o) = \theta$).

These few remarks about the behaviour of φ_x on the first characteristic through the corner lead us naturally on to a consideration of a different type of non-equilibrium free stream.

(ii) Vincenti's Non-Equilibrium Free Stream Problem.

Recently Vincenti [7] dealt with a particular type of non-equilibrium free stream which has some of the attributes of a real flow about a wedge of positive opening angle. Briefly, the idea is that the free supersonic stream approaching a wedge-like obstacle may be in a non-equilibrium state and additionally may well be <u>frozen</u> in this condition. The compressive disturbance across the leading edge shock wave is then assumed to "trigger off" the non-equilibrium processes (i.e. one assumes that the relaxation length Γ, see equation (1), jumps from infinity to some finite value across the shock front), and that subsequently the disturbances propagate according to equation (1), or its equivalent for finite Γ, equation (5). This situation is an idealisation of what may really occur in practice with a frozen non-equilibrium oncoming flow; clearly the objection on practical grounds is that such a dramatic change of Γ would hardly be expected for an obstacle of sufficient slenderness to render equation (5) applicable, but the gas model is at the very least of pedagogical interest and certainly helps one to assess the possible effects in a real situation. It will therefore be of some interest to attempt to repeat Vincenti's results here using the Green's function approach. (The original results were obtained by standard Laplace transform techniques).

A basic assumption in Vincenti's treatment is that the shock wave lies along the frozen characteristic $x - |\beta_f|\, y = 0$ passing through the nose of the obstacle to a sufficient order of accuracy. The non-equilibrium free stream is connected with the region downstream of $x - |\beta_f|\, y = 0$ by linearised Rankine-Hugoniot conditions (assuming no change of value of the non-equilibrium variable q, see equation (56) et seq). We shall not repeat this analysis here, but simply note that the following conditions on φ emerge;

$$\varphi(0+, \eta) = 0, \tag{75}$$

$$\left[2\frac{\partial}{\partial\eta} + a - 1\right] \varphi_\alpha(0+, \eta) = -\frac{(a-1)}{|\beta_f|} H_\infty q'_\infty U \tag{76}$$

We have written φ as a function of the semi-characteristic co-ordinates α and η, where

$$\alpha = x - |\beta_f|\, y \; ; \quad \eta = |\beta_f|\, y \tag{77}$$

H_∞ is defined (Vincenti, loc. cit.) as

$$H_\infty = -\frac{|\beta_f|}{U}\,\frac{h_{q_\infty}}{\rho_\infty h_{\rho_\infty}}\,\frac{\Gamma}{\tau_0(|\beta_e|^2 - |\beta_f|^2)}, \tag{78}$$

the subscript ∞ implying evaluation in the free stream, and we write

$$q'_\infty = q_\infty - \bar{q}_\infty \tag{79}$$

Thus q'_∞ is the extent to which the oncoming stream departs from an equilibrium state.

To use the Green's function technique we must now make up the boundary S_o from the lines $y_o = 0$, $0+ \leq x_o < \infty$ and $x_o - |\beta_f|\, y_o = 0+$, $0 \leq y_o \leq \infty$; or what is equivalent in the latter case, the whole of the line $\alpha_o = 0+$, with an obvious choice of nomenclature. The unit outwards vector normal to $\alpha_o = 0+$ is given by

$$\underset{\sim}{n}_o = -\frac{1}{M_f}\,\underset{\sim}{i} + \frac{|\beta_f|}{M_f}\,\underset{\sim}{j},$$

(where $\underset{\sim}{i}$, $\underset{\sim}{j}$ are the unit vectors along $0x_o$, $0y_o$ respectively) and the element of S_o on this line is clearly $dy_o . M_f$. Accordingly, equation (20) shows that

$$4\pi\varphi\ (x,y) = - \int_{o+}^{\infty} [\varphi y_o(x_o, 0) + \varphi x_o y_o(x_o, 0)] G(x, y\,|x_o, o) dx_o$$

$$- \int_{o}^{\infty} \left\{ P_{x_o}(\varphi(0+, \eta_o), G(x,y|\, 0+, \eta_o)) - |\beta_f|\, P_{y_o}(\varphi\,(0+, \eta_o), G(x,y|\, 0+, \eta_o)) \right\} dy_o, \tag{80}$$

where P_{x_o}, P_{y_o} are the components of $\underset{\sim}{P}$ along $0x_o$, $0y_o$, respectively, and we have replaced the x_o, y_o functional dependence of φ_o and G here by α_o (which must equal 0+) and η_o. The Green's function is given in equation (45).

It is convenient to express $P_{x_o} - |\beta_f|\, P_{y_o}$ in terms of quantities involving derivatives with respect to α_o and η_o; noting that

$$\frac{\partial}{\partial x_o} = \frac{\partial}{\partial \alpha_o} \quad ; \quad \frac{\partial}{\partial y_o} = |\beta_f| \left(\frac{\partial}{\partial \eta_o} - \frac{\partial}{\partial \alpha_o} \right)$$

this is readily carried out and we find that

$$\begin{aligned} P_{x_o} - |\beta_f|\, P_{y_o} = & - |\beta_f|^2 \left[G\varphi_{\alpha_o\alpha_o} + \varphi G_{\alpha_o\alpha_o} - \varphi_{\alpha_o} G_{\alpha_o} \right] \\ & - |\beta_e|^2 \left[G\varphi_{\alpha_o} - \varphi G_{\alpha_o} \right] - |\beta_f|^2 \left[\varphi_{\eta_o} - \varphi_{\alpha_o} \right] \left[G_{\eta_o} - G_{\alpha_o} \right] \\ & - |\beta_f|^2 \left[\varphi_{\eta_o} - \varphi_{\alpha_o} + \varphi_{\alpha_o\eta_o} - \varphi_{\alpha_o\alpha_o} \right] G \\ & + |\beta_f|^2 \left[G_{\eta_o} - G_{\alpha_o} - G_{\alpha_o\eta_o} + G_{\alpha_o\alpha_o} \right] \varphi . \end{aligned} \tag{81}$$

Certain terms in equation (81) cancel in the general case; we note in particular however that all terms in φ and φ_{η_o} will vanish on account of condition 75 when equation (81) is put into the integrand in equation (80) and hence specialised to its value on the line $\alpha_o = 0+$. For inclusion in equation (80) then, we can rewrite equation (81) in the form

$$\begin{aligned} P_{x_o} - |\beta_f|\, P_{y_o} = & - |\beta_e|^2\, G\varphi_{\alpha_o} + |\beta_f|^2\, \varphi_{\alpha_o} G_{\eta_o} \\ & + |\beta_f|^2\, \varphi_{\alpha_o} G - |\beta_f|^2 \varphi_{\alpha_o\eta_o} G \end{aligned}$$

or

$$P_{x_o} - |\beta_f|\, P_{y_o} = -|\beta_f|^2 \left\{ 2\frac{\partial}{\partial\eta_o} + (a-1) \right\} \varphi_{\alpha_o} G + |\beta_f|^2 \frac{\partial}{\partial\eta_o} (\varphi_{\alpha_o} G) \tag{81a}$$

Thus equation (80) becomes

$$4\pi\, \varphi(x,y) = -\int_{0+}^{\infty} [\varphi_{y_o}(x_o,0) + \varphi_{x_o y_o}(x_o,0)]\, G(x,y|\, x_o,0)dx_o$$

$$+ |\beta_f| \int_0^{\infty} \left\{ 2\frac{\partial}{\partial\eta_o} + (a-1) \right\} \varphi_{\alpha_o}(0+,\eta_o) G(x,y\,|\,0+,\eta_o)d\eta_o$$

$$- |\beta_f| \Big[[\varphi_{\alpha_o}(0+,\eta_o) G(x,y|\,0+,\eta_o) \Big]_{\eta_o=0.}^{\infty} \tag{82}$$

(In evaluating equation (80) to give equation (82) we have set $dy_o = d\eta_o/|\beta_f|$ and integrated the last term in equation (81a) directly). Using condition 76 the second integral in equation (82) becomes

$$-(a-1)H_\infty q'_\infty U \int_0^{\infty} G(x,y\,|\,0+,\eta_o)d\eta_o.$$

One must remember to split the integration from 0 to ∞ into two parts, from 0 to $|\beta_f|\, y$ and from $|\beta_f|\, y$ to ∞ ; after a little tedious manipulation, using equation (45) for G, we find that the integral is equal to

$$+ 2H_\infty q'_\infty U \left\{ \int_{-\infty - i\epsilon}^{\infty - i\epsilon} e^{i\xi(x-|\beta_f|\, y)} \frac{d\xi}{\xi^2 |\beta_f|} - \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i\xi x - iBy}}{(\xi - i)B} \left(1 - \frac{i}{\xi}\right) d\xi \right\} \tag{83}$$

The first integral in expression 83 gives simply $-2\pi(x-|\beta_f|\, y)/|\beta_f|$; the next integral we identify from equation (47) as $\frac{1}{2}G(x,y\,|\,0,0)$ and the third and last integral is easily shown to be equal to

$$\frac{1}{2} \int_{0+}^{x-|\beta_f|\, y} G(x,y|\, x_o,0)dx_o.$$

We note that the value of $\varphi_{\alpha_o}(0+,\eta_o)$ is needed in order to evaluate the last term in equation (82); it can be found by integrating condition 76 directly and putting in the value

$$\varphi_{\alpha_o}(0+, 0) = -\frac{Uh'(o)}{|\beta_f|}$$

This latter result comes directly from the shock relations and surface boundary condition (see Vincenti[7]), and clearly is consistent with the physical notion that immediately behind the shock at the corner, the flow is completely frozen, even though Γ there is finite. The value of $\varphi_{\alpha_o}(0+, \infty)$ is zero, as indeed is the value of $G(x, y \mid 0+, \infty)$. All that remains of the last term in equation (82) is therefore, the quantity $- Uh'(o)G(x, y \mid o, o)$.

With the first term in equation (82) it is important to note that, because the surface S_o lies just downstream of the leading characteristic, the lower integration limit is equal to 0+. Accordingly, the first term in equation (82) is equal to 4π times the right-hand side of equation (51), without the term in $h'(o)$. The term in $\delta(x_o)$ in the first version of equation (51) does not contribute, indeed its effect has been included in the boundary conditions applied along the line $x_o - |\beta_f| y_o = 0+$, and we have just seen that it is recovered from the last term of equation (82).

Collecting together all of these results, we find that

$$\begin{aligned}\varphi(x, y) = &- U[h'(o) - H_\infty q'_\infty] \frac{1}{4\pi} G(x, y \mid o, o) \\ &- \int_o^{x - |\beta_f| y} U[h'(x_o) + h''(x_o) - H_\infty q'_\infty] \frac{1}{4\pi} G(x, y \mid x_o, o) dx_o \\ &- (x - |\beta_f| y) \frac{UH_\infty q'_\infty}{|\beta_f|},\end{aligned} \tag{84}$$

for the appropriate non-equilibrium flow over a body whose shape is given by $y = h(x) H(x)$.

Insofar as the results given by Vincenti for the small disturbance conditions behind the leading edge shock wave apply to a body of any shape (following on behind the initial positive opening angle), equation (84) represents a mild generalisation of his solution, which was for a wedge of constant angle δ (equivalent to our θ in the previous section) only. Putting $h'(x_o)$ equal to δ and $h''(x_o)$ equal to zero, equation (84) can be recognised at once as exactly equivalent to Vincenti's transform relation in equation (56) of his paper; that is to say, the equilibrium free stream result (which has $q'_\infty = 0$) is simply multiplied by the factor $1 - (H_\infty q'_\infty / \delta)$ and the last term in equation (84) added on. There is no need to proceed further with the development of this kind of flow, since all aspects of it have been covered in the paper cited several times above. The agreement between the results of that work and those of the present analysis obtained by very different techniques is, at the least, reassuring.

(iii) A Non-Uniform, Non-Equilibrium Free Stream.

Finally, in this brief treatment of supersonic non-equilibrium oncoming streams, we shall consider a case which combines some of the aspects of the previous cases, (i) and (ii), together with a certain amount of non-uniformity. Let us suppose that on the line $x = o-$, immediately ahead of the nose of the obstacle, the non-equilibrium parameter Q (see Equation (57)), has the constant value Q_c (as in case (i)). In addition, let us suppose that $\partial Q/\partial x = o$ everywhere on this line. This last supposition makes our flow bear a remote relationship to Vincenti's case, for which $\partial Q/\partial x = o$ everywhere ahead of the line $x - |\beta_f|\, y = o$. The two cases are dissimilar in the sense that our specification of zero reaction rate is a statement of a purely local phenomenon; immediately downstream of the line $x = o-$ we may expect $\partial Q/\partial x$ to change. In order to have $Q = Q_c$, $\partial Q/\partial x = o$ on $x = o-$, eq. (68) shows that $|\beta_e^2|\, X(y)$ must equal $+V''(y)$ and

$$V''(y) = \left(\frac{a}{a-1}\right), \quad Q_c = |\beta_e|^2 X(y) \tag{85}$$

Thus

$$V'(y) = \left(\frac{a}{a-1}\right) Q_c\, y, \tag{86}$$

since $V'(o)$ is zero, and we can take

$$V(y) = \left(\frac{a}{a-1}\right) Q_c\, \frac{y^2}{2}. \tag{87}$$

Reference to equations (60) and (61) shows that, in particular, the oncoming stream at $x = o-$ has a divergent character, since φ_y increases linearly with y. We shall take $\varphi_x(o, y) = o$ for simplicity, as we are entitled to do. Clearly we should not allow φ_y to go on increasing indefinitely with y, since the basic small disturbance hypothesis would eventually be violated. However, there is no reason why we should not stop the oncoming flow at $y = L$, say, and deal only with the segment of the flow lying between $x = o-$, $y = o$ and $x = |\beta_f|\,(L-y)$. The latter line is the "rightwards-running" frozen characteristic through $x = o$, $y = L$; we may state from a knowledge of hyperbolic equation behaviour (or indeed infer directly from the nature of the Green's function in equation (45)) that any variations of φ, φ_y, etc. above $y = L$ will not interfere with the flow in the segment mentioned.

From a practical point of view, the oncoming flow that we are dealing with here may be thought of as a crude estimate of the flow from a straight-sided divergent nozzle.

We have only to deal with the addition to φ_{eq}, as in case (i), and can therefore write, from equations (69) and (85) to (87);

$$4\pi \; [\varphi(x,y) - \varphi_{eq}] = -\int_{o}^{L} \Big\{ -\frac{1}{a} G.V''(y_o) - G_{y_o} V'(y_o) - |\beta_e^2| \, G_{x_o} V(y_o)$$

$$+ |\beta_f^2| \; G_{x_o x_o} V(y_o) \Big\} dy_o$$

$$= [G.V]_{y_o=o}^{L} - \left(\frac{a-1}{a}\right) \int_{o}^{L} GV'' \, dy_o + |\beta_f|^2 \left\{ a \frac{\partial}{\partial x_o} - \frac{\partial^2}{\partial x_o^2} \right\} \int_{o}^{L} G.V.dy_o.$$

(88)

Dealing only with the case $y < L$, one must remember to split the integrals from O to L into the two parts, o to y and y to L. The product GV is zero when $y_o = o$ and is continuous at $y_o = y$ (see eq. (45)): the upper limit $y_o = L$ gives terms in $G(x, y \,|\, o, L)$, which do not contribute for $x < |\beta_f|(L-y)$: hence the first term in (88) is zero for present purposes. The remarks about contributions from terms evaluated at $y_o = L$ apply also to the other integrals in eq. (88); with this in mind one may show, after a little analysis, that for $x < |\beta_f|(L-y)$ we have

$$4\pi \; [\varphi(x,y) - \varphi_{eq}] = 2Q_c \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i\xi x} \, d\xi}{(\xi - i) i B^2}$$

$$+ |\beta_f^2| \left(\frac{a}{a-1}\right) \frac{Q_c}{2} \left\{ a \frac{\partial}{\partial x_o} - \frac{\partial^2}{\partial x_o^2} \right\} \Big\{ 4 \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i\xi(x-x_o)}}{(\xi - i) i B^4} d\xi$$

$$- 2y^2 \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i\xi(x-x_o)}}{(\xi - i) i B^2} \, d\xi \Big\}_{x_o = o} \qquad (89)$$

(We have used equations (45) and (85) to (87) in deriving this result). The x_o - derivatives can be taken inside the integrals in equation (89) and, after some manipulation, we find

$$4\pi \; [\varphi(x,o) - \varphi_{eq}] = 2Q_c \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i\xi x}}{i\,|\beta_f|^2} \left\{ -\frac{1}{ia\xi^2} + \frac{1}{a^2\xi} - a\frac{1}{a^2(\xi - ia)} \right\} d\xi$$

$$+ 2Q_c \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i\xi x}}{i\,|\beta_f^2|} \left\{ -\frac{1}{ia\xi^2} + \frac{1}{a^2\xi} - \frac{1}{a^2(\xi - ia)} + \frac{1}{i(a-1)\xi^3} \right\} d\xi,$$

whence

$$\varphi(x,o) - \varphi_{eq} = \frac{2Q_c}{a|\beta_e^2|}\left\{1 - e^{-ax} - ax\right\} - \frac{Q_c}{2(a-1)|\beta_f^2|} x^2, \tag{90}$$

having set y = o.

The similarity between this result and equation (71) in case (i) is apparent; the first term here is precisely twice the value of $\varphi - \varphi_{eq}$ in the earlier case. Since the first integral in equation (89) is exactly the same as the integral in equation (71), we infer that the doubling of the non-equilibrium effect here arises *from the divergence of the free stream; i.e. since this stream is expanding the* rate at which energy must be supplied to the relaxing mode is reinforced. In addition of course, one has the primary effect of the stream's divergence to add on; this is represented by the last term in equation (90). (N.B. Q_c is essentially positive here). We note from equation (68) that $Q = Q_c = 0$ if $a = 1$, leaving $V''(y)$ etc. finite and non-zero; i.e. $aQ_c/(a-1)$ remains finite as $a \to 1$. When $a = 1$ there are no relaxation effects, the first term on the right-hand side of equation (90) vanishes, but the last term remains and expresses the effect of the diverging stream.

The previous three examples show what can be done with some simple non-uniform, non-equilibrium streams. Problems of considerable generality could be worked out, but it may not always be possible to evaluate analytically the integrals involved in more complex cases.

5. Subsonic Flow

The subsonic flow regime is defined by requiring that both $M_{f\infty}$ and $M_{e\infty}$ shall be less than unity. Since $M_{e\infty} \geqslant M_{f\infty}$ this means that $\beta_f^2 \geqslant \beta_e > o$ and the quantity a, defined in equation (36), is always positive, but less than or equal to unity, i.e.

$$0 < (a = \beta_e^2/\beta_f^2) \leqslant 1. \tag{91}$$

As before, we set out to find the Green's function in the infinite domain $-\infty \leqslant y_o \leqslant \infty$, $-\infty \leqslant x_o \leqslant \infty$ for some fixed x, y lying within this region. *That is to say, we wish to solve equation* (31) *with* β_e^2 *and* β_f^2 *both positive*, in contrast to the supersonic case, for which these parameters are both negative. We shall use Fourier transforms for this purpose, defining $g(\zeta)$ as the appropriate transform of G', as in the first of equation (32), but with reservations here about the quantity ζ. In particular we shall say nothing for the moment about the imaginary part of this variable; the range of variation which is permissible for $\mathrm{Im}\,\zeta$ is clearly associated with the behaviour of G' for $|x_o| \to \infty$ and we hope to pronounce on this question shortly. Meanwhile, let us formally apply the first of equations (32) to equation (31) and assume that G', G'_{x_o}, $G'_{x_o x_o}$, etc., times $\exp(i\zeta x_o)$ all vanish for $|x_o| \to \infty$. The result is similar to equation (34), except that, in view of the behaviour of β_f^2, we should now write this in the form

$$g_{y_o y_o} - \beta_f^2 \zeta^2 \left(\frac{i\zeta + a}{i\zeta + 1}\right) g = -4\pi f(\zeta)\,\delta\,(y - y_o). \tag{92}$$

$f(\zeta)$ is exactly as written in equation (35), since x lies within the range of integration over x_o.

Clearly the general solution of equation (92) will be identical with equation (38) except that the quantity iB in this equation will be replaced by B', where

$$B'^2 = \beta_f^2 \;\zeta^2 \left(\frac{\zeta - ia}{\zeta - i}\right) \tag{93}$$

With this form of solution it would appear that $g(\zeta)$ is regular from $\mathrm{Im}\zeta < 0$ (there is, apparently, a pole at $\zeta = o$), for $o < \mathrm{Im}\zeta < a$, for $a < \mathrm{Im}\zeta < 1$ and for $\mathrm{Im}\zeta > 1$, since there are branch points at $\zeta = ia$ and i. Not all of these latter regions of regularity will occur together, since the ζ-plane must be cut in some fashion from $\zeta = ia$ and i in order to render $g(\zeta)$ single-valued. However, we must take note of the theorem regarding Fourier transforms which states that, if $g(\zeta)$ is a transform, regular in a strip $\tau_- < \mathrm{Im}\zeta < \tau_+$ of the ζ-plane, then G' its inverse, behaves in such a way that $|G'| < \exp(\tau_- + \delta)x_o$ as $x_o \to +\infty$ and $|G'| < \exp(\tau_+ - \delta)x_o$ as $x_o \to -\infty$ where δ is an arbitrarily small positive number, (see, e.g., Noble,[(4)] p. 24). If both τ_- and τ_+ are negative, $|G'|$ may increase without limit as $x_o \to -\infty$, whilst it must vanish exponentially as $x_o \to +\infty$. The converse must be true if both τ_- and τ_+ are positive. Neither type of behaviour is what we should expect for G', since the vanishing of the function at one extreme and not at the other is at variance with the physical idea that a source in a subsonic stream should exert both upstream and downstream influences. These remarks are certainly true for the "non-relaxing" case, $a = 1$, since it is then easy to prove that G' will vanish for all $x_o > x$ if $\tau_- < \tau_+ < 0$ and all $x_o < x$ if $\tau_+ > \tau_- > 0$. Behaviour of this kind is very much more "supersonic" in character than subsonic. The situation in a relaxing gas can hardly alter too radically from that just described for $a = 1$, if any of the previously mentioned strips of regularity are used for the inversion of $g(\zeta)$, and we must of course ensure that the relaxing gas solution goes over properly into the non-relaxing case as $a \to 1$ anyway. Since we have eliminated all those regions of the ζ-plane for which τ_- and τ_+ are both of one sign, we can only conclude that $g(\zeta)$ must somehow be inverted in a strip for which $\tau_- < 0$ and $\tau_+ > 0$. The form of $g(\zeta)$ found by solving equation (92) is apparently not regular in such a strip, however, on account of the pole at $\zeta = 0$, but we can find our way out of this dilemma as follows.

Consider the equation

$$g'_{y_o y_o} - \beta_f^2(\zeta^2 + \epsilon^2)\left(\frac{\zeta - ia}{\zeta - i}\right) g' = -4\pi\, f(\zeta)\,\delta\,(y - y_o) \tag{94}$$

whose general solution is

$$g' = Ce^{-B_\epsilon y_o} + De^{B_\epsilon y_o} + \frac{4\pi f}{2B_\epsilon}\left\{e^{-B_\epsilon (y_o - y)} - e^{B_\epsilon (y_o - y)}\right\} H(y_o - y), \tag{95}$$

where

$$B_\epsilon^2 = \beta_f^2 (\zeta^2 + \epsilon^2)\left(\frac{\zeta - ia}{\zeta - i}\right). \tag{96}$$

We can take ϵ real, positive and less than a: in the limit $\epsilon \to 0$, equation (94) reduces to equation (92) and equation (95) reduces to the general solution of the latter equation. However, g' has branch points at $\zeta = \pm i\epsilon$, ia and i, and is therefore regular in the strip $-\epsilon < \mathrm{Im}\zeta < \epsilon$. From the theorem quoted above this implies that the function of x_o (of which g' is the transform) is less than $\exp(-\epsilon + \delta)x_o$ as $x_o \to +\infty$ and less than $\exp(\epsilon - \delta)x_o$ as $x_o \to -\infty$, (since we identify τ_- with $-\epsilon$ and τ_+ with $+\epsilon$). In the limit as $\epsilon \to 0$, the function represented by $g' \to g$ has the proper behaviour for a Green's function in subsonic flow. That is to say, it need not vanish at either extreme location of the source point x_o, y_o in relation to the field point x, y; just how it does behave at these extremes we must investigate later on.

With the ζ-plane cut between the branch points at ia and i, from $i\epsilon$ to $i\infty$ in the upper half-plane and from $-i\epsilon$ to $-i\infty$ in the lower half-plane, we choose the phase of the radicals in B_ϵ, where

$$B_\epsilon = \beta_f \sqrt{\zeta + i\epsilon}\,\sqrt{\zeta - i\epsilon}\,\sqrt{\frac{\zeta - ia}{\zeta - i}}, \tag{97}$$

to be such that they all behave like $\sqrt{|\zeta|}$ as $\zeta \to +\infty$. Then it is easy to see that the real part of B_ϵ remains positive everywhere on the real $-\zeta$ axis. Using this line for the inversion contour, we can now find G' from the integral

$$\mathcal{G} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g'(\xi) e^{-i\xi x_o}\, d\xi \tag{98}$$

in the limit as $\epsilon \to 0$.

It is of course necessary to find the constants C and D in equation (95). We require that g' should not increase without limit as $|y_o| \to \infty$; with $y > y_o$ it is clear that C must be zero since $\exp(-By_o)$ does not fulfil this condition. Letting $y_o \to +\infty$ with $y < y_o$, the proper behaviour of g' is assured by setting

$$D = \frac{4\pi f}{2B_\epsilon}\, e^{-B_\epsilon y}; \tag{99}$$

thus

$$g' = \frac{4\pi f}{2B_\epsilon}\left\{ e^{B_\epsilon (y_o - y)} + \left[e^{-B_\epsilon (y_o-y)} - e^{B_\epsilon (y_o-y)} \right] H(y_o - y) \right\}$$

(100)

Alternatively, g' can be written as

$$g' = \frac{4\pi f}{2B_\epsilon} e^{-B_\epsilon | y_o - y|} \tag{101}$$

In any case we can now write, from equations (98) and (101),

$$\mathcal{g} = -i \int_{-\infty}^{\infty} e^{i\xi(x-x_o) - B_\epsilon | y_o - y|} \frac{d\xi}{(\xi - i)B_\epsilon} . \tag{102}$$

This integral is not in a particularly suitable form for further investigation and we must now attempt to re-arrange it somewhat in order to make further progress. Figure 5 shows the cut ξ-plane, with the strip within which $g'(\xi)$ is regular. When $x - x_o < 0$, integration in the strip $-\epsilon < \mathrm{Im}\,\xi < \epsilon$ is equivalent to integration round the loop contour marked "$x - x_o < o$" in Figure 5 and likewise, for the case $x - x_o > 0$, it is equivalent to integration along the contour marked "$x - x_o > 0$".

First consider the case $x - x_o < 0$: on the contour shown the phases of $\sqrt{\xi - ia}$ and $\sqrt{\xi - i}$ are both equal to $-\pi/4$ and the phase of $\sqrt{\xi - i\epsilon}$ is equal to $-\pi/4$. The phase of $\sqrt{\xi + i\epsilon}$ changes from $3\pi/4$ to $-\pi/4$ on rounding the branch point at $\xi = -i\epsilon$ and integration around the circle about this point vanishes in the limit as the circle's radius approaches zero. With these results we can write

$$\mathcal{g}(x - x_o < 0) = -i\left\{ \int_{\infty}^{\epsilon} \frac{e^{-v(x_o - x) - \beta_f i \sqrt{v^2 - \epsilon^2}\sqrt{\frac{v+a}{v+1}}\,|y_o - y|}}{(-i)(v+1)\beta_f i \sqrt{v^2 - \epsilon^2}\sqrt{\frac{v+a}{v+1}}} (-i)dv \right.$$

$$+ \int_{\epsilon}^{\infty} \frac{e^{-v(x_o - x) + \beta_f i \sqrt{v^2 - \epsilon^2}\sqrt{\frac{v+a}{v+1}}\,|y_o - y|}}{(-i)(v+1)\beta_f(-i)\sqrt{v^2 - \epsilon^2}\sqrt{\frac{v+a}{v+1}}} (-i)dv$$

$$= \frac{2}{\beta_f}\int_{\epsilon}^{\infty} \frac{e^{-v(x_o - x)} \cos\left[\beta_f \sqrt{v^2 - \epsilon^2}\sqrt{\frac{v+a}{v+1}}\;|y_o - y|\right]}{(v+1)\sqrt{v^2 - \epsilon^2}\sqrt{\frac{v+a}{v+1}}} dv. \tag{103}$$

The integral in equation (103) cannot be evaluated simply in terms of known functions, indeed it is not apparent as it stands how one may pass to the limit $\epsilon \to 0$. However, the integral is absolutely and uniformly convergent and so it can be differentiated with respect to either $x_o - x$ or $|y_o - y|$ under the integral *sign.* Let us choose the latter course, we have

$$\frac{\partial \phi}{\partial |y_o - y|} = -2\int_{\epsilon}^{\infty} e^{-v(x_o - x)} \sin\left[\beta_f \sqrt{v^2 - \epsilon^2}\sqrt{\frac{v+a}{v+1}}\,|y_o - y|\right]\frac{dv}{v+1}$$

(104)

Letting $\epsilon \to 0$ in this integral will give the corresponding derivative of the Green's function for $x - x_o < 0$, i.e.

$$\frac{\partial G'}{\partial |y_o - y|} = -2\int_{0}^{\infty} e^{-v(x_o - x)} \sin\left[\beta_f v\sqrt{\frac{v+a}{v+1}}\,|y_o - y|\right]\frac{dv}{v+1}\,. \tag{105}$$

In this form we can find how G' behaves for large values of $x_o - x$. Before doing so, we can make one slight modification which will help later on, that is to say we shall look at $(-\partial G'/\partial x_o + G')$ rather than at G' directly. Clearly

$$\frac{\partial}{\partial |y_o - y|}\left\{G' - \frac{\partial G}{\partial x_o}\right\} = -2\int_{0}^{\infty} e^{-v(x_o - x)} \sin\left[\beta_f v\sqrt{\frac{v+a}{v+1}}\,|y_o - y|\right] dv.$$

(106)

Now, writing the sine term in complex exponential form, we can expand the quantity $\sqrt{(v+a)/(v+1)}$ for small values of v and develop the integral to give an asymptotic representation of the left-hand side for $x_o - x \gg 0$. The dominant term gives

$$\frac{\partial}{\partial |y_o - y|}\left\{G' - \frac{\partial G'}{\partial x_o}\right\} \sim \frac{-2\,\beta_e |y_o - y|}{(x_o - x)^2 + \beta_e^2 (y_o - y)^2}\,, \tag{107}$$

whence it follows that

$$G' - \frac{\partial G'}{\partial x_o} \sim -\frac{1}{\beta_e}\log\left\{(x_o - x)^2 + \beta_e^2 (y_o - y)^2\right\}. \tag{108}$$

Before commenting on this result, let us examine the case a = 1. Referring to equation (31) in this event, we see that $G' - \partial G'/\partial x_o$ is really the Green's function for the subsonic, non-relaxing, small-disturbance equation, which is Laplacian in form. Equation (106) can be evaluated exactly in this case and gives results like (107) or (108) with the "asymptotically equals" sign replaced by an "equals" sign. (With a = 1 we can write either β_e or β_f, since they are equal). Equation (108) with = written in place of ~ is the well known result for Green's function in the subsonic flow mentioned above and so we infer that at a field point x, y a long way ahead of a source at x_o, y_o in a relaxing flow, the effect is similar to that produced by a source operating under fully equilibrium conditions (since it is essentially β_e which occurs in this case). We are, of course, entitled to say that, asymptotically, G' is given by the logarithm in equation (108), since $\partial G'/\partial x_o$ would behave like $1/(x_o-x)$ in this case and would be negligible for $x_o-x \gg 0$. It is the fact that G' behaves in this way which makes it necessary for us to use the Fourier transforms described earlier and which, in the limit $\epsilon \to 0$, are regular in an infinitesimally thin strip which just embraces the real ξ-axis.

Let us now turn to the case $x-x_o > 0$. The contour marked "$x-x_o > 0$" in Figure 5 is not particularly suitable for the case $|y_o-y| \neq 0$, since the term $(\xi - i)^{-\frac{1}{2}}$ which appears in the exponential in equation (102) behaves rather badly on the small circle surrounding $\xi = i$. However, the contour is quite adequate for the case $|y_o-y| = 0$, and we shall consider this situation in more detail below. Before doing so, it is interesting to note that G' for $x_o < x$ will be of radically different form from G' for $x_o > x$ in a relaxing flow*, as is evidenced by the direct intrusion of the branch points at ia and i in the former case, but not in the latter. One may contrast this with the non-relaxing case, $a = 1$, for which we may readily show that

$$G' - \frac{\partial G'}{\partial x_o} = - \frac{1}{\beta_e} \log \left\{ (x_o-x)^2 + \beta_e^2 (y_o-y)^2 \right\}$$

for $x-x_o > 0$, which is precisely the same as its value for $x-x_o < 0$. This symmetrical behaviour of the Green's function in a non-relaxing subsonic flow is symptomatic of the simple reciprocity relation $G(\mathbf{x}/\mathbf{x}_o) = G(\mathbf{x}_o/\mathbf{x})$ which exists in such a case, and which demonstrates a lack of directionality in both the basic equation and the boundary conditions. The introduction of the relaxation equation into the basic set describing the flow does introduce a specific direction (the streamwise one) into the equation for G', and we are therefore not surprised to observe the differences in G' mentioned above.

In order to help with the re-writing of the function $\mathcal{G}$ in terms of integrals on the contour "$x-x_o > 0$", we tabulate the phases of the various square root terms which appear in equation (102) (the contour is lettered suitably in Figure 5).

phase of / on the section	$\sqrt{\xi + i\epsilon}$	$\sqrt{\xi - i\epsilon}$	$\sqrt{\xi - ia}$	$\sqrt{\xi - i}$
AB	$\pi/4$	$-3\pi/4$	$-3\pi/4$	$-3\pi/4$
CD	$\pi/4$	$-3\pi/4$	$-3\pi/4$	$-\pi/4$
EF	$\pi/4$	$-3\pi/4$	$-\pi/4$	$-\pi/4$
GH	$\pi/4$	$\pi/4$	$-\pi/4$	$-\pi/4$
JK	$\pi/4$	$\pi/4$	$\pi/4$	$-\pi/4$
LM	$\pi/4$	$\pi/4$	$\pi/4$	$\pi/4$

* But see below for the case when $|x-x_o| \gg 0$.

Writing $+iv$ for ξ it is now easy to show that

$$\phi = -i\left\{\int_{\infty}^{1}\frac{e^{-v(x-x_o)}}{i(v-1)\beta_f(-i)\sqrt{v^2-\epsilon^2}\sqrt{\frac{v-a}{v-1}}}\,idv+\int_{1}^{\infty}\frac{e^{-v(x-x_o)}}{i(v-1)\beta_f(i)\sqrt{v^2-\epsilon^2}\sqrt{\frac{v-a}{v-1}}}\,idv\right.$$

$$+\int_{1}^{a}\frac{e^{-v(x-x_o)}}{i(v-1)\beta_f(-i)\sqrt{v^2-\epsilon^2}\,(-i)\sqrt{\frac{v-a}{1-v}}}\,idv+\int_{a}^{1}\frac{e^{-v(x-x_o)}}{i(v-1)\beta_f(i)\sqrt{v^2-\epsilon^2}\,(i)\sqrt{\frac{v-a}{1-v}}}\,idv$$

$$+\int_{a}^{\epsilon}\frac{e^{-v(x-x_o)}}{i(v-1)\beta_f(-i)\sqrt{v^2-\epsilon^2}\sqrt{\frac{a-v}{1-v}}}\,idv+\int_{\epsilon}^{a}\frac{e^{-v(x-x_o)}}{i(v-1)\beta_f(i)\sqrt{v^2-\epsilon^2}\sqrt{\frac{a-v}{1-v}}}\,idv\ .$$

Clearly the third and fourth integrals cancel and the remainder combine to give

$$\phi = -\frac{2}{\beta_f}\int_{\epsilon}^{a}\frac{e^{-v(x-x_o)}}{\sqrt{v^2-\epsilon^2}\,(v-1)\sqrt{\frac{a-v}{1-v}}}\,dv-\frac{2}{\beta_f}\int_{1}^{\infty}\frac{e^{-v(x-x_o)}}{\sqrt{v^2-\epsilon^2}\,(v-1)\sqrt{\frac{v-a}{v-1}}}\,dv. \qquad (109)$$

Differentiating this expression with respect to x_o and letting $\epsilon \to 0$, we find

$$G'_{x_o}(x,y|x_o,y) = +\frac{2}{\beta_f}\int_{o}^{a}\frac{e^{-v(x-x_o)}}{\sqrt{(a-v)(1-v)}}\,dv-\frac{2}{\beta_f}\int_{1}^{\infty}\frac{e^{-v(x-x_o)}}{\sqrt{(v-a)(v-1)}}\,dv. \qquad (110)$$

An alternative form of this result which may prove useful for computational purposes can be found by changing the variable from v to $\frac{1}{2}(a+1)-\frac{1}{2}(1-a)t$ in the first integral here, and from v to $\frac{1}{2}(a+1)+\frac{1}{2}(1-a)\cosh\theta$ in the second. We find that

$$G'_{x_o}(x,y|x_o,y) = +\frac{2}{\beta_f}\,e^{-\frac{1}{2}(a+1)(x-x_o)}\left\{\int_{1}^{(a+1)/(1-a)}\frac{e^{\frac{1}{2}(1-a)(x-x_o)t}}{\sqrt{t^2-1}}\,dt\right.$$

$$\left.-K_o\left[\tfrac{1}{2}(1-a)(x-x_o)\right]\right\} \qquad (111)$$

where K_o is the modified Bessel function of the second kind and zero order (Watson).

We may compare the results just obtained for $x-x_o > 0$ with the corresponding value for $G'_{x_o}(x,y|x_o,y)$ when $x-x_o < 0$. Referring to equation (103), the latter is found to be given by

$$G'_{x_o}(x,y\,|\,x_o,y) = -\frac{2}{\beta_f}\int_o^{\infty}\frac{e^{-v(x_o-x)}}{\sqrt{(v+a)(v+1)}}\,dv\,,$$

$$= -\frac{2}{\beta_f}\,e^{\frac{1}{2}(a+1)(x_o-x)}\int_{\frac{a+1}{1-a}}^{\infty}\frac{e^{-\frac{1}{2}(1-a)(x_o-x)t}}{\sqrt{t^2-1}}\,dt \qquad (112)$$

The very different forms of upstream (eq. 112) and downstream (eq. 111) influences are at once apparent. One should compare these results (particularly equation (111) and the second equation (112) with Ryhming's results for G_x. First of all, Ryhming's Green's function G should, for his subsonic problem, be such as to make $G_{y_o}(x,o\,|\,x_o,o) = o$. As we shall see below, this latter condition introduces a factor of 2, by which G' should be multiplied to give G. In addition, noting that equations (111) and (112) are expressions for G'_{x_o} and not G'_x, it appears that Rhyming's expression for G_x is additionally in error in respect of its sign for the case $x-x_o > o$.

We have shown (in equations (105) to (108)) how G' behaves for $x_o-x \gg 0$ and have found that it takes on the form of an equilibrium Green's function in this region. Since it will be of interest in some developments to follow, we should also attempt to find out how G' behaves as $|y_o-y| \to +\infty$ and any value of x_o-x, positive or negative, and also for $x-x_o \gg 0$ for any values of y_o and y. Since the integral in equation (102) is absolutely and uniformly convergent for all $y_o \neq y$ (and hence for $|y_o-y| > o$), we can form the function

$$\left(1-\frac{\partial}{\partial x_o}\right)\frac{\partial g}{\partial|y_o-y|} = \int_{-\infty}^{\infty} e^{i\xi(x-x_o) - B_\epsilon|y_o-y|}\,d\xi \qquad (113)$$

Now on either side of the point $\xi = 0$, the exponential $\exp(-B_\epsilon\,|y_o-y|)$ decreases very rapidly if $|y_o-y| \gg 0$. Accordingly the left-hand side of equation (113) will differ negligibly from

$$\int_o^{\infty} e^{-i\xi(x-x_o)-\xi\beta_e|y_o-y|}\,d\xi + \int_o^{\infty} e^{i\xi(x-x_o)-\xi\beta_e|y_o-y|}\,d\xi$$

in these circumstances (having let $\epsilon \to o$ here). Accordingly, we find that

$$\left\{1-\frac{\partial}{\partial x_o}\right\}\frac{\partial G'}{\partial|y_o-y|} \sim \frac{-2\beta_e\,|y_o-y|}{(x-x_o)^2+\beta_e^2(y_o-y)^2} \qquad (114)$$

in these circumstances. Not surprisingly, the result is the same as that found for $x_o-x \gg o$, and we can also show that the same behaviour is found for $x-x_o \gg o$ too. In other words, G' looks like an equilibrium flow source at large radial distances from the source point. The particular significance of this is that, whilst all derivatives of G' vanish in these regions, G' itself becomes logarithmically large. These conclusions do not conflict with the remarks at the end of the previous paragraph, since there we are comparing exact values of $G'(x,y|x_o,y)$ in the two regions of x_o for <u>any</u> values of $|x-x_o|$; indeed, examining the integrals in equation (110) and the first version of equation (112) shows how G'_{x_o} will be roughly equal as $|x-x_o| \to \infty$ in either case. Of course the precise manner in which they tend to their asymptotic values will be quite different in each case.

We can now attempt to solve a half-plane ("thickness") problem like that considered in Section 3 for supersonic flow. Thus, let us find $\varphi_x(x,y)$ in a region $y > o$, $-\infty \leqslant x \leqslant \infty$ with Neumann data (namely $\varphi_y(x,o)$) specified on the line $y = o$. (We choose to find φ_x here rather than φ because it is slightly easier to do so). Reference to equation (29), which applies equally well here if we let H_1 and $H_2 \to \infty$, shows that we should ask for a G such that $G_{y_o}(x,y|x_o,o) = o$. Using image methods, let us place another source like $G'(x,y|x_o,y_o)$ at the reflection of the point x_o, y_o in the plane $y_o = o$, i.e. at $x_o, -y_o$. With $y > o$, this means (see equation (102)) that we should consider the function

$$\mathcal{G}' = -i\int_{-\infty}^{\infty} \frac{e^{i\xi(x-x_o)}}{(\xi-i)B_\epsilon}\left\{ e^{-B_\epsilon(y_o+y)} + e^{-B_\epsilon|y_o-y|}\right\}d\xi \tag{115}$$

When $y_o < y$ it is clear that $\partial\mathcal{G}'/\partial y_o = o$ when $y_o = o$ and, accordingly, the appropriate value of G should follow on taking the limit $\epsilon \to o$ in $\mathcal{G}'$. Since we are solving for φ_x rather than φ, we are more interested in G_x (see equation (29) once again). Then we can write directly from equation (115),

$$G_x(x,y|x_o,y_o) = \int_{-\infty}^{\infty} \frac{e^{i\xi(x-x_o)}}{(\xi-i)(B'/\xi)}\left\{e^{-B'(y_o+y)} + e^{-B'|y_o-y|}\right\}d\xi \tag{116}$$

where B' (see equation (93)) is the proper limit of B_ϵ (equation (97)) as $\epsilon \to o$. (The ξ - plane must remain cut as in Figure 5 as $\epsilon \to o$). From the results of the previous paragraph we note that all the terms in G_x, G_{xx_o} etc. etc., which appear in the last three integrals in equation (29) will vanish as $H_1, H_2, H_3 \to \infty$ Since G_{y_o} (and hence $G_{x_oy_o}$) vanishes on $y = o$, we are left with the result;

$$4\pi\ \varphi_x(x,y) = -\int_{-\infty}^{\infty} [\varphi y_o(x_o,o) + \varphi_{x_oy_o}(x_o,o)]\ G_x(x,y|x_o,o)dx_o, \tag{117}$$

where

$$G_x(x,y|x_o,o) = 2\int_{-\infty}^{\infty} \frac{e^{i\xi(x-x_o)-B'y}}{(\xi-i)(B'/\xi)}\, d\xi \tag{118}$$

Let us assume that the body shape is given by the equation

$$y = h(x)H(x)\{1-H(x-L)\}, \tag{119}$$

where h(x) is a function which is zero at x = o and x = L. Then the tangency condition is

$$\varphi_y(x,o) = Uh'(x)H(x)\{1-H(x-L)\}, \tag{120}$$

and we easily find that

$$\varphi_{xy}(x,o) = Uh''(x)H(x)\{1-H(x-L)\} + Uh'(x)\,\delta(x)\,\{1-H(x-L)\}$$

$$- Uh'(x)H(x)\,\delta(x-L). \tag{121}$$

Then, putting equations (120) and (121) into equation (117) we have

$$4\pi\ \varphi_x(x,y) = -Uh'(o)G_x(x,y|\,o,o) + Uh'(L)G_x(x,y|\,L,O)$$

$$-\int_{o+}^{L-} U[h'(x_o) + h''(x_o)]\,G_x(x,y|\,x_o,o)dx_o. \tag{122}$$

We must note that G_x is different in the two regions $x-x_o \gtrless o$, so that integration from O to L may have to be split at $x_o = x$ if we want to know φ_x for $O < x < L$. In fact let us examine $\varphi_x(x,o)$ for $x < o$. In equations (110) to (112) we have found G'_{x_o} for $y_o = y$ (in particular y = o then) and, noting from equations (102) and (115) that $\mathscr{I}'$ is just twice the value of $\mathscr{I}$ when $y_o = y = o$, we can find the required values of G_x by multiplying equations (110) to (112) by -2, the minus sign arising since $\partial/\partial x = -\partial/\partial x_o$ for G or G'. For $x < o$ we require equation (112), and find that

$$4\pi\ \varphi_x(x,o) = -Uh'(o)\frac{4}{\beta_f}\int_o^\infty \frac{e^{-v|x|}}{\sqrt{(v+a)(v+1)}}dv + Uh'(L)\frac{4}{\beta_f}\int_o^\infty \frac{e^{-v(L+|x|)}}{\sqrt{(v+a)(v+1)}}dv$$

$$-\int_o^L U[h'(x_o) + h''(x_o)]\frac{4}{\beta_f}\int_o^\infty \frac{e^{-v(x_o+|x|)}}{\sqrt{(v+a)(v+1)}}dv.dx_o. \tag{123}$$

This result is not particularly tractable and it would appear to be necessary to evaluate φ_x numerically in the final analysis. (The integrals from o to ∞ in equation (123) can be related to the K_o-type Bessel functions). When a = 1, equation (123) can be simplified by integrating the last term (involving $h''(x_o)$) by parts. The terms in h(o) and h(L) cancel and one is left with the result that

$$4\pi \underset{x<o;a=1}{\varphi_x(x,o)} = -U \int_o^L \frac{h'(x_o)}{x_o+|x|}\, dx_o,$$

since the infinite integrals are easily evaluated in this case. It is easy to see from this that $|\varphi_x| \to \infty$ like $1/|x|$ as $|x| \to o$ and also that φ_x is essentially negative in this region. The velocity disturbance given by equation (123) appears to be of the proper sign and provides a partial check on the foregoing analysis.

We conclude this short look at subsonic relaxing flow Green's functions by remarking that, in order to find $\varphi(x,y)$, a closed boundary is necessary, in line with the elliptic nature of the problem. It is also necessary to prescribe data for φ everywhere on this closed boundary; more strictly, we need to specify φ_y on lines y = constant and φ_x on lines x = constant. In the particular case of the semi-infinite domain, one must set $\varphi_x = o$ for $|x| \to \infty$.

References

1. Clarke, J.F. — J. Fluid Mech., Vol. 7, 1960, pp. 577-595.
2. Der, J. — N.A.S.A. TR R-119, 1961.
3. Morse, P.M. and Fesbach, H. — Methods of theoretical physics. McGraw-Hill, 1953.
4. Noble, B. — The Wiener-Hopf technique. Pergamon Press, 1958.
5. Ryhming, I.L. — J. Aero/Space Sci., Vol. 29, 1962, pp. 1076-1080.
6. Vincenti, W.G. — J. Fluid Mech., Vol. 6, 1959, p. 481.
7. Vincenti, W.G. — J. de Mecanique, Vol. 1, 1962, pp. 193-212.
8. Watson, G.N. — Theory of Bessel functions. 2nd ed. C.U.P., 1944.

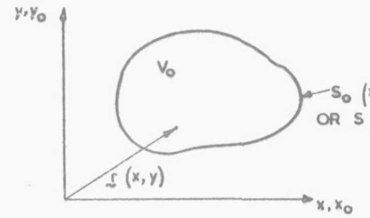

FIG. 1. THE SURFACE S_o OR S. VECTOR $\underset{\sim}{r}$ IS DRAWN WITHIN S_o.

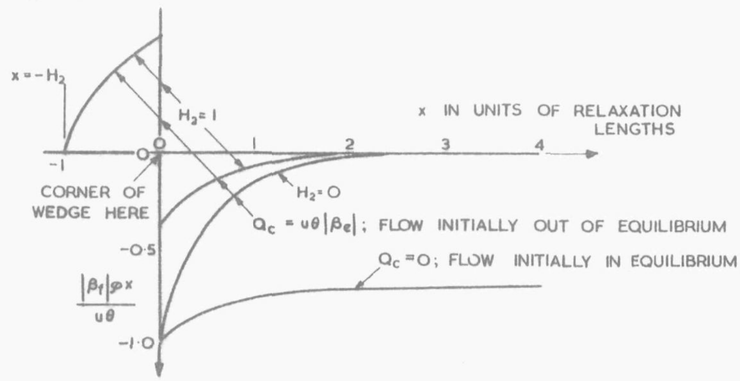

FIG. 4. DISTURBANCE VELOCITY ON THE SURFACE FOR A WEDGE IN A SIMPLE NON-EQUILIBRIUM ONCOMING STREAM, COMPARED WITH THE EQUILIBRIUM CASE. $a = 2$.

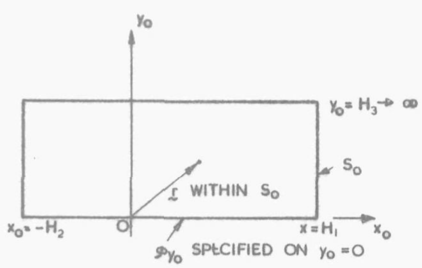

FIG. 2. THE SURFACE S_o USED IN SECTION 3.

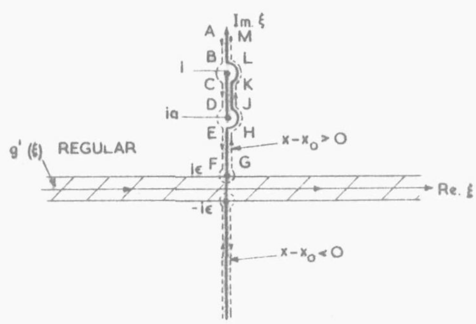

FIG. 5. THE ξ-PLANE AND CONTOURS FOR THE INVERSION OF g'. THE BRANCH CUTS ARE SHOWN AS HEAVY LINES; DOTTED LINES ARE CONTOURS EQUIVALENT TO Im. $\xi = O$ FOR THE $x - x_o$ VALUES SHOWN. (SUBSONIC CASE, $a < 1$).

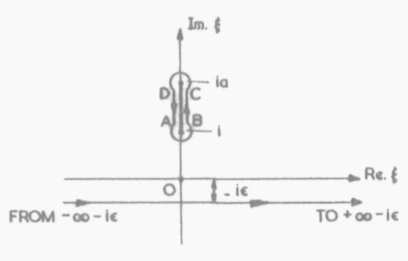

FIG. 3. INTEGRATION CONTOURS FOR THE GREEN'S FUNCTION IN THE COMPLEX ξ-PLANE; SUPERSONIC CASE $a > 1$.