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ON THE INTERACTION OF A LAMINAR HYPERSONIC BOUNDARY LAYER AND A CORNER EXPANSION WAVE

by

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SUMMARY

A simple method of calculating the extent of the interaction of a corner expansion wave with a hypersonic boundary layer is presented. Arguments are presented to suggest that for small turning angles the principal feature of the flow is the interaction of the boundary layer and the expansion wave downstream of the corner. The net result is a large increase in the thickness of the boundary layer, and a greatly extended region of pressure decay. Numerical results based on the "cold wall" or local similarity approximation are given.

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SYMBOLS

C_{f_∞}	skin friction coefficient
C_∞	Chapman Rubesin factor, $C_\infty = \dfrac{\mu_w T_\infty}{\mu_\infty T_w}$
f	$\int u/u_e \, d\eta$, defined by Eq. 3.6
g	H/H_e
G	$\int_o^\infty (g-f'^2)\, d\eta$ see Eq. 3.14
h	static enthalpy $h = \dfrac{\gamma}{\gamma-1}\dfrac{p}{\rho}$ for a perfect gas
H	total enthalpy, $H = h + 1/2\,(u^2 + v^2) \approx h + \frac{1}{2}u^2$
k	thermal conductivity
K	$M_\infty \, d\delta^*/dx$
M	Mach number
N	$\rho\mu/\rho_w\mu_w$
p	static pressure
P	p_e/p_∞
R	$\int P \, dZ$
Re	Reynolds number; $Re_{x,\infty}$ is based on running length x and free stream properties.
St_∞	Stanton number; $St_\infty = \dfrac{q_w}{\rho_\infty U C_p (T_r - T_w)}$
T	static temperature
u,v	components of velocity in the x & y directions respectively
U	free stream velocity
x, y	co-ordinates along the wall and normal to it respectively
Z	$\dfrac{\rho_\infty U M_\infty^{-6} x}{\mu_\infty C_\infty} = \bar{\chi}_\infty^{-2}$
α_C	amount by which the flow is turned in the region where the boundary layer equations do not apply

α_T — $\alpha_w + d\delta^*/dx|_U$

α_w — corner turning angle of body

γ — C_p/C_v ; specific heat ratio

δ — boundary layer disturbance thickness

δ^* — boundary layer displacement thickness

$\delta^* = \int_o^\delta (1 - \rho u/\rho_e\, u_e))dy$

η, ξ — defined by Eq. 3.5

μ — viscosity

ρ — density

$\bar{\chi}_\infty$ — viscous interaction parameter $\bar{\chi}_\infty = \dfrac{M_\infty^3 \sqrt{C}_\infty}{\sqrt{Re_{x,\infty}}}$

Subscripts

e — denotes conditions at the edge of the boundary layer

U — denotes conditions immediately upstream of the corner calculated as if the corner was not present

w — conditions at the wall

∞ — conditions in the free stream

d — denotes conditions immediately downstream of the corner

1. INTRODUCTION

The interaction of a laminar supersonic or hypersonic boundary layer with a steady corner expansion wave is a problem of considerable current theoretical and practical interest. When the boundary layer remains attached downstream of the corner, the turning process involves at least three mechanisms. Firstly, the presence of the corner is signalled upstream through the subsonic portion of the boundary layer, and the surface pressure commences to fall ahead of the corner. Secondly in the immediate neighbourhood of the corner the boundary layer created shear flow is turned by predominantly inviscid forces. Pressure gradients normal to the streamlines are expected to be significant and viscous shear stresses are believed to be important only in a very small layer near the wall, where the zero slip condition must be satisfied. Finally continuity of pressure along the wall and the pressure gradient required to sustain centrifugal acceleration requires that immediately downstream of the corner, the turning process in the external inviscid flow, which is assumed to be accomplished through a non-centered simple wave, is incomplete. Furthermore the expansion of the boundary layer through the corner implies that just downstream of the corner it is growing rapidly. Therefore in the region downstream of the corner an interaction occurs between the inviscid flow and boundary layer. The net result is a gradual decay in surface pressure to the Prandtl Meyer value for the wall turning angle α_w.

For moderate and high Mach numbers the downstream interaction process can extend for many boundary layer thicknesses beyond the corner. The relative importance of the three mechanisms depends on the turning angle α_w and the Mach number M_U of the inviscid flow just upstream of the corner. In addition to the effects just mentioned separation of the boundary layer can occur at or near the corner for certain downstream geometries. The resultant flow is then very complex. The present work is concerned only with the attached flow case.

Attempts have been made to treat the problem within the framework of boundary layer theory (Refs. 1, 2, 3, and 4). Zakkay et al (Ref. 1) calculated heat transfer for axisymmetric shapes by treating the upstream flow as if the upstream effect did not occur. They allowed the boundary-layer-created shear flow to undergo frictionless expansion around the corner and then evaluated the subsequent development of the boundary layer by representing it as a new boundary layer which starts at the corner on top of which the upstream developed boundary layer turned by the corner was matched as a viscous shear layer. The pressure gradient downstream of the corner was assumed to be zero for the numerical results presented for laminar flow. No upstream influence effects were allowed for in this formulation.

Hunt and Sibulkin (Ref. 2) examined the change in momentum thickness and shape factor δ^*/ϑ of the boundary layer through the corner region by means of a momentum integral technique modified to account for radial pressure gradients. They predicted large changes in the momentum thickness of the boundary layer as it passed through the corner region. More recently, Oosthuizen (Ref. 3) undertook an extensive analysis of the problem. His work, which was a generalization of an earlier, much more approximate treatment by Curle (Ref. 4) used the momentum integral form of the boundary layer equations and the equations of a non-centred Prandtl-Meyer simple wave to describe the flow. He calculated the flow field both upstream and downstream of the corner region by allowing the boundary layer and expansion wave to interact, and matched the upstream and downstream solutions he so obtained by specifying continuity of inviscid flow properties and by matching the shape factor δ^*/ϑ of the boundary layer at the corner. He ignored centrifugal effects in the neighbourhood of the corner and modelled upstream influence effects using the Prandtl boundary layer equations.

Weinbaum (Ref. 5) examined the flow field in the immediate neighbourhood of the corner by using the method of characteristics in the supersonic portion of the flow and assuming that the sonic line remained parallel to the wall downstream of the corner. His results suggested that for locally hypersonic flows and small turning angles the flow was highly underexpanded just downstream of the corner.

It must be noted that the use of the boundary layer equations to model upstream influence effects in supersonic flow involves conceptual difficulties. It has been pointed out (see, for example Ref. 6) that since the boundary layer equations are parabolic and the inviscid supersonic flow equations are hyperbolic, nowhere in the flow field in this model is there a mechanism for upstream influence effects. That is to say an elliptic behaviour is required to correctly model upstream influence. There are two possible mechanisms; the supersonic diffusion of vorticity and the propagation of signals in the subsonic portion of the boundary layer. The former mechanism is almost certainly negligible (Ref. 7) so that the equations for the subsonic portion of the boundary layer should be elliptic. This type of approach has been used by several authors, perhaps the most complete of which is the paper by Lighthill (Ref. 8). Lighthill treats the boundary layer as a parallel shear flow, on top of which a perturbation was superimposed which was inviscid in the subsonic portion of the boundary layer with the exception of the region immediately adjacent to the wall. His analysis is limited to small disturbances and moderate external Mach numbers, M_u.

It follows that the use of the boundary layer equations to model the upstream influence problem in the manner described in Refs. 3 and 4 implies the assumption that, although the upstream influence effect is not correctly represented, if a solution can be obtained it is probably reasonable since, except in the immediate neighbourhood of the corner, the boundary layer equations are a realistic representation of the conservation laws. In the case of hypersonic flow a solution upstream using the boundary layer equations is not possible, since in contrast to supersonic flow, the hypersonic boundary layer equations do not permit thinning to occur under a falling pressure gradient. This point is developed in Sections 2 and 4.

It is evident that a complete solution to this problem will probably require direct solution of the Navier Stokes equations, especially if the details of the flow in the neighbourhood of the corner are required. However, for locally hypersonic flows simplifications are possible which render the calculation of the major features relatively straightforward. In Section 2 it is argued that the dominant feature of the flow for large Mach number the interaction of the expansion wave and boundary layer downstream of the corner. Calculations based on this simplified model are presented here.

2. BEHAVIOUR IN HYPERSONIC FLOW

A characteristic feature of inviscid hypersonic flow is that small deflections produced by slender bodies generate large changes in pressure and density, but very small changes in fluid speed. Typically, an expansion of a perfect gas with a specific heat ratio $\gamma = 1.4$ through a 10^o turning angle by a simple wave from a Mach number $M_U = 10$ causes the pressure to drop by a factor of 21 and the density by a factor of 9, whereas the speed increases by less than 2%. This affects the growth of a locally hypersonic boundary layer. A decrease in pressure at the edge of the boundary layer is associated with a negligible velocity change so that the principal effect on the boundary layer is a decrease in the density and an increase in the displacement thickness. This behaviour is demonstrated formally using an approximate model in Sect.4. It is in direct contrast with the behaviour of the incompressible boundary layer, where a decrease in pressure tends to thin the boundary layer because the only effect is an increase in the velocity at the edge of the boundary layer. For supersonic boundary layers if the Mach number M_e at the edge is low enough it is possible for the boundary layer to thin, or at least grow at a slower rate than it does under constant pressure.

This behaviour was first reported in the literature by Crocco and Lees (Ref 19) in connection with their study of the supersonic base flow problem. They introduced the concept of a critical Mach number $M_{ecr} > 1$ for a given isentropic external flow. For $M_e < M_{ecr}$, $d\delta/dp_e > 0$ whereas for $M_e > M_{ecr}$, $d\delta^*/dp_e < 0$. The two modes of behaviour have been called subcritical and supercritical respectively. A locally hypersonic boundary layer, which may be defined as one in which $u_e^2 \simeq 2H_e$, is always supercritical.

The significance of the critical point in the present problem is that, within the framework of the boundary layer theory, for $M_e < M_{ecr}$ the boundary layer will thin under the action of an isentropic simple wave expansion generated by the displacement effects of the boundary layer itself. In the locally hypersonic flow case the boundary layer will thicken and a contradiction arises since this will tend to generate a compression wave. Consequently, the type of analysis used by Oosthuizen (Ref 3) in which upstream of the corner, the boundary layer equations were matched to an expansion wave is only possible if $M_u < M_{ecr}$. The corresponding behaviour in problems involving separated flow (see for example, Holden Ref.10) is that if the boundary is originally supercritical it cannot generate its own adverse pressure gradient in the external inviscid flow. It is therefore required to undergo a supercritical-subcritical "jump" before separation by means of a shock wave at the edge of the boundary layer. Of course, such a jump cannot occur in the corner expansion problem.

An alternative way of viewing the effect just described is to remember that the supersonic boundary layer contains both subsonic and supersonic streamtubes. Any acceleration associated with a decrease in pressure in the external flow will cause a decrease in the thickness of the subsonic streamtubes and an increase in the thickness of the supersonic streamtubes. For sufficiently large M_e, the expansion of the supersonic streamtubes will eventually dominate so that the boundary layer reacts by thickening at a greater rate than it would at constant pressure. The magnitude of M_{ecr} should increase with increase in the wall temperature T_w, but it can be readily demonstrated that even for adiabatic walls the boundary layer should become supercritical for sufficiently large M_e. This is done in Sect. 4 by reference to the locally hypersonic similar solutions.

The implication of the above remarks is that no mechanism exists by which large pressure decay can occur upstream of the corner. Since in the hypersonic flow even small values of α_w can be associated with very large changes in pressure through a homentropic simple wave, continuity of pressure along the wall requires that almost all of the pressure decay occur downstream of the corner. There will of course be some decay of pressure upstream of the corner; the presence of the subsonic portion of the boundary layer ensures this. However, it has been noted that it is not possible to use the boundary layer equations to describe the upstream influence. The significant pressure decay should be confined to a region fairly close to the corner, because it should scale with the thickness of the subsonic portion of the boundary layer rather than δ itself. To apply the boundary layer type analysis upstream in the manner used by Oosthuizen (Ref. 3) the decay should extend many boundary layer thicknesses upstream of the corner. For locally hypersonic boundary layers the subsonic layer thickness may well be a small fraction of the total boundary layer thickness so that significant pressure decay should be confined to a region which is relatively small when compared with δ, and in which centrifugal effects are important. Hence, the boundary layer equations are not appropriate to the analysis of the upstream effects in the locally hypersonic flow. It is necessary to use a more realistic analysis such as that given by Lighthill (Ref. 8) or more recently by Olson and Messiter (Ref. 18) and Weiss and Nelson (Ref. 10) for the base flow problem.

For the present analysis it is assumed that the pressure decay downstream of the corner is sufficiently spread out so that the majority occurs away from the immediate neighbourhood of the corner and in a region where the boundary layer equations can be used. Then a relatively simple interaction analysis can be used for this part of the flow. If the external inviscid flow is turned through an angle α_c in the corner region where the boundary layer equations do not apply, then it is assumed that $\alpha_c/\alpha_w \ll 1$. The assumed behaviour is illustrated in Fig. 1.

The validity of the model just suggested will have to be checked by experiment. At the time of writing there does not appear to be suitable experiments for this purpose available in the literature. However, some recent experiments by Holden Ref. 11 on the interaction of boundary layers with compression corners suggested that the present model may be very good. His experiments, which included measurements of heat transfer and pressure for completely attached boundary layers showed that for this case, if the wall was "cold", that is $T_W/T_O \ll 1$, upstream influence effects were negligible. Presumably, the compression corner would be a more severe test of the upstream influence effects than would the expansion corner.

3. COLD WALL SIMILARITY IN HYPERSONIC BOUNDARY LAYERS

A number of methods for the calculation of laminar hypersonic boundary layers for arbitrary external pressure distributions are described in the literature (Ref. 12). Integral methods (Ref. 13) for example, have been developed to apply to this type of interaction problem. However in view of the approximate nature of the present model, the "cold wall similarity" method suggested by Lees (Ref. 14) was applied since it enabled a very simple formulation to be developed. In spite of its simplicity, for favorable pressure gradients and cold walls ($T_w/T_o \ll 1$) this method is known to yield reasonably accurate results. In this section "cold wall" similarity theory is developed in a form suitable for the present calculations.

For two dimensional bodies the boundary layer equations are

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{3.1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{3.2}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu\left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right) \tag{3.3}$$

where

$$H = h(p,\rho) + \frac{1}{2}u^2 \tag{3.4}$$

The usual transformations of compressible boundary layer theory are applied:

$$\xi = \int_0 \rho_w \mu_w u_e dx \; ; \; \eta = \frac{u_e}{\sqrt{2\xi}} \int \rho \, dy \tag{3.5}$$

$$\frac{u}{u_e} = \frac{\partial f}{\partial \eta} \qquad g = \frac{H}{H_e} \tag{3.6}$$

Continuity is automatically satisfied so that if the solutions are assumed to be a function of η only the equations reduce to

$$(Nf'')' + ff'' + \frac{2\xi}{u_e}\frac{du_e}{d\xi}\left[\frac{\rho_e}{\rho} - f'^2\right] = 0 \tag{3.7}$$

$$\left(\frac{N}{Pr}g'\right)' + fg' + \frac{u_e^2(\xi)}{H_e}\left[N\left(1 - \frac{1}{Pr}\right)f' f''\right]' = 0 \tag{3.8}$$

where

$$\frac{\partial}{\partial \eta} = (\;)' \quad \text{and} \quad N = \frac{\rho\mu}{\rho_w \mu_w}$$

The boundary conditions are, with $u_e(\xi)$ or $p_e(\xi)$ given

i) at y or $\eta = 0$, $\quad f = f' = 0, \quad g = g_w(\xi)$

ii) as $\eta \to \infty$, $\quad f' \to 1, \quad g \to 1$ (3.9)

In general, ξ independence, or self similarity exists only under very restricted circumstances. If it is assumed that

(i) the flow is locally hypersonic so that $u_e^2 - 2H_e$, and

(ii) the gas is calorically perfect so that

$$h = \frac{\gamma}{\gamma-1} \frac{p}{\rho} \quad ; \quad \frac{p}{\rho} = RT$$

(iii) Pr = constant

(iv) $\mu \propto T$ so that N = 1,

(v) g_w = constant

the equations (3.7) and (3.8) reduce to

$$f''' + ff'' + \beta(x)\,(g - f'^2) = 0$$
$$g'' + Prfg' + 2(Pr-1)\,(f'\,f'')' = 0 \qquad (3.10)$$

where

$$\beta(x) = \frac{\gamma-1}{\gamma} \frac{[\int p_e\,dx]}{p_e^2} \frac{dp_e}{dx} \qquad (3.11)$$

Self similarity exists only if β = constant or $p_e \propto x^n$. If, in place of assumption (v) it is assumed that $g \ll 1$, that is, the wall is "cold" then the boundary conditions suggest that the term $\beta(x)\,(g-f'^2) \ll 1$ over the range of integration. The boundary equations are approximately self similar for arbitrary $p_e = p_e(x)$ since the momentum becomes approximately

$$f''' + ff'' = 0 \qquad (3.12)$$

The usefulness of the "cold wall" assumption was first pointed out by Lees Ref (14).

The "cold wall" similarity approach is distinct from the local similarity method (Ref. 12, p 312) which treats β as a parameter which varies slowly with x along the boundary layer. The local value of dp_e/dx is used to determine β and the boundary layer profiles are then determined from the similar solutions for the same value of β.

For self similar profiles the displacement thickness δ^* is given by

$$\delta^* = \int_0^\infty \left[1 - \frac{\rho u}{\rho_e u_e}\right] dy = \frac{\sqrt{2\xi}}{\rho_e u_e} \int_0^\infty \left[\frac{\rho_e}{\rho} - \frac{u}{u_e}\right] d\eta \qquad (3.13)$$

Application of the assumptions (i) to (v) above leads to

$$\delta^* = \frac{\gamma-1}{2\gamma} U_\infty^{3/2} \frac{\mu_w}{RT_w} G \frac{[\int_0 p_e dx]^{\frac{1}{2}}}{p_e(x)} \qquad (3.14)$$

where

$$G = \int_0^\infty \left(g - f'^{\,2} \right) d\eta \qquad (3.15)$$

It is convenient to write this relation in terms of the appropriate free stream variables M_∞, p_∞ & μ_∞. After some algebra it turns out that (3.14) can be cast into the form

$$\frac{M_\infty \delta^*}{x} = \frac{\gamma - 1}{\sqrt{2}} \quad \frac{G\ (Pr)}{Z} \quad \frac{[\int PdZ]^{\frac{1}{2}}}{P} \qquad (3.16)$$

where

$$P = \frac{p_e}{p_\infty} \quad Z = \frac{R_{e_{x,\infty}}}{M_\infty{}^6 C_\infty}, \quad R_{e_{x,\infty}} = \frac{xU_\infty \rho_\infty}{\mu_\infty} \qquad (3.17)$$

and C_∞ is the Chapman Rubesin factor, given by

$$C = \frac{\mu_w}{\mu_\infty} \quad \frac{T_\infty}{T_w} \qquad (3.18)$$

The quantity Z is proportional to x and is related to the hypersonic viscous interaction parameter $\overline{\chi}_\infty = Z^{-2}$.

Simple expressions for heat transfer and skin friction can be similarly derived:

$$St_\infty \ M_\infty^3 = \sqrt{2} \ \frac{g_w'}{Pr\ [gr - g_w]} \frac{P}{(\int PdZ)^{1/2}} \qquad (3.19)$$

$$C_{f\infty} \ M_\infty^3 = \sqrt{2} \quad f''\ (o) \quad \frac{P}{(\int PdZ)^{1/2}} \qquad (3.20)$$

where $g_r = H_r/H_e$, and H_r is the recovery enthalpy, and

$$St_\infty = \frac{q_w}{\rho_\infty UC_p\ (T_r - T_w)} \qquad (3.21)$$

$$Cf_\infty = \frac{2}{\rho_\infty U^2} \left(\mu \ \frac{\partial u}{\partial y} \right) \Bigg|_{y=0} \qquad (3.22)$$

It has been pointed out (Ref. 15) that even if the cold wall assumption does not apply and the pressure gradient term in equation (3.7) is retained its contribution can be small for favorable pressure gradients. From equation 3.12 for β = constant

$$p_e \propto x^n ; \ \beta = - \frac{\gamma - 1}{\gamma} \quad \frac{n}{n+1},$$

so that if $p_e \propto x^{-\frac{1}{2}}$ as in strong shock boundary layer interaction, $\beta = 0.286$. Hence the cold wall similarity concept should be applicable over a wider range of g_w than suggested solely by consideration of the term $(g - (f')^2)$.

Since in the present analysis the momentum equation is reduced to Blasius' equation, then $f''_w = 0.470$ (Ref. 12). Some values of G and g'_w were computed by integrating equation (3.11) for N=1, and the other parameters in the range $0.6 \leq P_r \leq 1.2$ and $0 \leq g_w \leq 0.4$. They are given in Table 1.

4. BEHAVIOUR OF THE HYPERSONIC BOUNDARY LAYER IN A FALLING PRESSURE GRADIENT

The expression for δ^*, equation (3.14), can be used to demonstrate that for hypersonic flows, a decrease in pressure p_e causes an increase in δ^*. For given free stream conditions

$$\delta^* = \frac{W\left[\int p_e \, dx\right]^{\frac{1}{2}}}{p_e} \qquad (4.1)$$

so that

$$\frac{d\delta^*}{dx} = W\left\{\frac{1}{2}\left(\int p_e \, dx\right)^{-\frac{1}{2}} - \frac{\left[\int p_e dx\right]^{\frac{1}{2}}}{p_e^{2}}\frac{dp_e}{dx}\right\} \qquad (4.2)$$

Now $W > 0$, so that for $d\,\delta^*/dx < 0$, it is required that

$$2\int p_e dx \; \frac{dp_e}{dx} > p_e^{2} \qquad (4.3)$$

Since $\int p_e dx > 0$ equation (4.3) requires $dp_e/dx > 0$. Alternatively it can be shown that when $dp_e/dx < 0$, $d\delta^*/dx > 0$ always. In the present problem $dp_e/dx < 0$ upstream of the corner, so that a contradiction arises, and the boundary layer equations are unable to provide a mechanism for significant pressure decrease upstream.

A very important point to note is that equation (4.1) applies to all self-similar flows, and in particular to those self similar solutions for which $p_e \propto x^n$. Since these solutions require g_w constant, but not necessarily small, then they demonstrate that a locally hypersonic boundary layer is supercritical even if the wall is adiabatic or heated.

5. APPLICATION TO SHOCK BOUNDARY LAYER INTERACTION THEORY

The growth of the boundary layer on a flat plate is now calculated by the present theory. This is necessary to provide the initial condition, that is P and $R = \int PdZ$ for the expansion wave boundary layer interaction process downstream of the corner. It also serves as a convenient check on the accuracy of the present theory since a direct comparison with more accurate calculations of this problem can be made.

For shock boundary layer interaction calculations the tangent wedge rule is normally used to estimate the pressure at the edge of the boundary layer (Ref. 12). In this problem the effective body is usually slender, so that the pressure at the edge of the boundary layer is accurately given by the hypersonic small disturbance solution for oblique shocks. The tangent wedge relation is [Ref.12,p 279]

$$P - 1 = \gamma K^2 \left[\left\{ \left(\frac{\gamma + 1}{4} \right)^2 + \frac{1}{K^2} \right\}^{\frac{1}{2}} + \frac{\gamma + 1}{4} \right] \tag{5.1}$$

where in the present problem

$$P = p_e/p_\infty, \quad K = M_\infty \frac{d\delta^*}{dx} \tag{5.2}$$

Differentiating Eq. (3.16) and inserting into (5.1) leads to the following expression for the shock-boundary layer interaction problem:

$$\frac{dR}{dZ} = P$$

$$\frac{dP}{dZ} = \frac{P^2}{2R} \left[1 - \frac{2^{3/2} (P-1) R^{\frac{1}{2}}}{\gamma (\gamma-1) G} \left\{ \frac{P(\gamma + 1) + (\gamma-1)}{2\gamma} \right\}^{-\frac{1}{2}} \right] \tag{5.3}$$

Solution of these two ordinary differential equations by standard Runge-Kutta techniques leads to a pressure distribution $P = P(Z)$ or $p_e = p_e(x)$. It is known in the strong interaction limit $\bar{\chi} \to \infty$ or $Z \to 0$ that $P \to \infty$. Hence to start the integration the appropriate expression corresponding to the strong interaction limit must be provided. This is done by simplifying (5.3) with the approximation $P \gg 1$ to obtain

$$\frac{2R}{P^2} \frac{dP}{dZ} = \left[1 - \frac{4 P^{\frac{1}{2}} R^{\frac{1}{2}}}{(\gamma-1)\sqrt{\gamma(\gamma+1)}\, G} \right] \tag{5.4}$$

Note that as $P \to \infty$, $R \to 0$ so that both terms in the square brackets of equation (5.3) have to be retained. By assuming a solution of the form $P = AZ^n$ it is found that

$$P = \frac{3}{\sqrt{2}} (\gamma-1) \sqrt{\gamma (\gamma + 1)}\; G\; Z^{-\frac{1}{2}} \tag{5.5}$$

This solution agrees with that given in Ref. 12, p. 358-9. Integration of equation 5.3 then proceeds by obtaining starting values of P and R at a value of $\bar{\chi}_\infty = Z^{-\frac{1}{2}}$ which is chosen such that the strong interaction solution (5.5) and the complete solution give the same value of dP/dZ to within acceptable error.

The heat transfer and pressure distribution on a cold flat plate in hypersonic flow were computed by the cold wall similarity method and the results are given in Figures 2 and 3. The validity of the present approximation was verified by comparison with other theoretical methods and some experimental results. The theoretical methods are the " local similarity " approach described in detail by Dewey (Ref. 16) and a momentum integral method described by Chan (Ref. 13) and the experimental results were obtained by Hall and Golian Ref. 17. The strong interaction solution as given by Ref. 12 is included in these comparisons. Excellent agreement is obtained between the theories and experiment in the case of the pressure distribution. Agreement in the case of the heat transfer distribution is somewhat less satisfactory; but the "flat plate" similarity method used here does not appear to be significantly worse than the other methods. Its use in the present problem appears to be justified.

6. FORMULATION OF THE DOWNSTREAM FLOW FIELD

The rate of growth of the boundary layer displacement thickness immediately downstream of the corner region $d\delta^*/dx|_d$ is given by (see Figure 1)

$$\left.\frac{d\delta^*}{dx}\right|_d = \left.\frac{d\delta^*}{dx}\right|_u + \alpha_w - \alpha_c \tag{6.1}$$

where $d\delta^*/dx|_u$ is the rate of growth of the boundary layer displacement thickness immediately upstream of the corner. For the present analysis $d\delta^*/dx|_u$ is assumed to be that value which would occur if no corner were present. Also with $\alpha_c \ll \alpha_w$ equation (6.1) can be written

$$\left.\frac{d\delta^*}{dx}\right|_d \approx \left.\frac{d\delta^*}{dx}\right|_U + \alpha_w \equiv \alpha_T \tag{6.2}$$

That is to say the turning process is in the first approximation assumed to be carried out entirely in the region where the boundary layer equations apply and therefore continuity of pressure requires $p_u = p_d$.

Since the inviscid flow is assumed to be a simple wave the pressure downstream of the corner at the edge of the boundary layer is given by (Ref 20, p 36).

$$\frac{P}{P_U} = \left\{1 - \frac{\gamma - 1}{2}\left\{\alpha_T - \frac{d\delta^*}{dx}\right\} M_U\right\}^{\frac{2\gamma}{\gamma-1}} \tag{6.3}$$

where M_u is the Mach number of the external flow just upstream of the corner, and the hypersonic small disturbance approximations have been used. Equation (6.3) can be combined with equation (3.16) to give the governing equation for the present problem:

$$\frac{dP}{dZ} = \frac{P^2}{2R}\left\{1 - \frac{2\sqrt{2}\,R^{\frac{1}{2}}}{(\gamma-1)G}\left[M_\infty\alpha_T + \left(\frac{M_\infty}{M_U}\right)\frac{2}{\gamma-1}\left\{\left(\frac{P}{P_U}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right\}\right]\right\}$$

$$\frac{dR}{dZ} = P \tag{6.4}$$

The initial conditions are obtained from the shock-boundary layer interaction solution upstream of the corner:

$$\text{at } Z = Z_u,\quad P = P_u,\quad R = R_u \tag{6.5}$$

which in turn are obtained from the solution of Equation 5.3.

The two quantities $M_\infty\alpha_T$ and M_u/M_∞ have to be specified to complete the formulation. Since the tangent wedge formula was used to relate the flow deflection to the pressure, $d\delta^*/dx|_u$ can be written down in terms of P_u:

$$M_\infty \left.\frac{d\delta^*}{dx}\right|_u = \frac{P_u - 1}{\gamma}\left[\frac{P_u\,(\gamma+1) + (\gamma-1)}{2\gamma}\right]^{-\frac{1}{2}} \tag{6.6}$$

Hence

$$M_\infty \alpha_T = M_\infty \alpha_w + \frac{P_u - 1}{\gamma}\left[\frac{P_u\ (\gamma + 1) + (\gamma - 1)}{2\gamma}\right]^{-\frac{1}{2}} \tag{6.7}$$

The choice of a suitable value of M_u is not quite as straightforward. The use of the tangent wedge concept to compute M_u is in general inadequate, since the fluid streamlines at the edge of the boundary layer at a given value of x or Z cross the shock wave at point where the entropy increase can be much higher than would be computed by the tangent wedge formula. Consequently the sound speed by the tangent wedge formula should be low. However, it can be argued that it is not reasonable to apply a simple wave description to the inviscid flow if there is a large difference between the Mach numbers at the edge of the boundary layer and at a point just behind the shock wave. Hence it is necessary to restrict the present model to values of x or Z sufficiently large that x_u is not in the strong interaction regime where $\bar{\chi}_\infty \gg 1$. Consequently it is then consistent to use the oblique shock wave relations to estimate M_u. The required relation is obtained from hypersonic small disturbance theory and is

$$\left(\frac{M_u}{M_\infty}\right) = \frac{(\gamma+1)\ P_u + (\gamma - 1)}{(\gamma - 1)\ P_u + (\gamma + 1)}\ \frac{1}{P_u} \tag{6.8}$$

An additional reason for the constraint that $\bar{\chi}_\infty$ should not be too large at the corner is that in the strong interaction regime the shock wave is relatively close to the body. Then reflections of the corner expansion wave from the shock could intersect the boundary layer relatively close to the corner, and cause a significant deviation of the pressure distribution from the simple wave law used here.

The present theory is readily generalized to the case when the upstream body is a wedge. For a wedge angle ϑ and corner angle α the flow field upstream of the corner is given by

$$\frac{dP}{dZ} = \frac{P^2}{2R}\left[1 + \frac{2\sqrt{2}\ R^{\frac{1}{2}}}{G(\gamma - 1)}\left\{K_\vartheta - \frac{(P-1)}{\gamma}\left[\frac{P(\gamma+1) + (\gamma-1)}{2\gamma}\right]^{-\frac{1}{2}}\right\}\right]$$

$$dR/dZ = P \tag{6.9}$$

where $K_\vartheta = M_\infty \vartheta$. The equation for the flow downstream of the corner is the same as for the flat plate model. However the expression for the effective total turning angle at the corner α_T is different:

$$M_\infty \alpha_T = M_\infty \frac{d\delta^*}{dx} + \alpha = K_T + M_\infty \alpha - M_\infty \vartheta \tag{6.10}$$

where K_T instead of $M_\infty d\delta^*/dx$ is known in terms of P_u through the tangent wedge rule. Hence

$$M_\infty \alpha_T = \frac{P_u - 1}{\gamma}\left[\frac{P_u\ (\gamma + 1) - (\gamma - 1)}{2\gamma}\right]^{-\frac{1}{2}} - M_U \vartheta - M_\infty \alpha \tag{6.11}$$

The expression for M_u remains the same as for the flat plate case.

7. RESULTS AND CONCLUSIONS

Some calculated values of surface pressure distribution, heat transfer and displacement thickness downstream of a corner are given in Figures 4 to 6 respectively. In each case the forebody is a flat plate. The most striking feature is the greatly extended region of pressure decay downstream of the corner and the very large growth in the thickness of the boundary layer. Significant pressure decay occurs over a region, the length of which can be many times longer than the original plate length. Typically, at a value of χ_∞ such that $P_U = 1.97$ the pressure decays to 49% of the corner value in a distance equal to the original plate length x_u whereas the asympototic value is 11% of the corner pressure. At a distance of $10x_u$ the pressure is still a factor of 1.8 larger than the asymptotic value. Almost all of the pressure decay occurs in a region well away from the corner. For the above conditions, at a distance of $3\delta^*_U$ downstream of the corner the surface pressure is approximately 80% of the upstream value. Similar comments can be made for displacement thickness effects. The heat transfer is found to be very greatly reduced by the corner expansion.

It can be concluded that the present model is self consistent since the calculations confirm the basic assumption that the great majority of the pressure decay occurs well away from the corner and in a region where the boundary layer equations can be expected to hold. The present calculations can be regarded as a first approximation to a complete solution to the problem which would include the details of the flow in the neighbourhood of the corner. Such a solution would supply a value of α_c which would in turn enable an appropriate correction to be made to the calculations described here. Ultimately of course suitable experiments will be needed to verify the use of this model. There does not appear to be available in the literature any experiments which can be applied to test its validity. The crucial experimental test will be pressure measurements on a two dimensional body, since in this case the pressure decay downstream of the corner is related purely to the interaction between the expansion wave and the boundary layer. The use of axially symmetric shapes or heat transfer measurements create difficulties of interpretation since in other cases additional mechanisms exist which cause a decay in the observed quantities. Nevertheless, heat transfer measurements will be required to obtain the magnitude of the expected peak in heat transfer rate at the corner; a quantity which is of considerable practical importance. yet which is not predicted by any of the present theories, including the one presented here.

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TABLE 1.

Tabulated values of $G = \int_0^\infty (g - f'^2) d\eta$ and $g'(0)$ for $0.6 \leq Pr \leq 1.2$ and $0 \leq g_w \leq 0.4$ obtained by solution of equation (3.11) for $\delta = 1.4$ and $u_e^2 \simeq 2He$ ($M_e >> 1.$).

Pr	g_w	G	g' (0)	G. test	Error %
0.6	0.0	0.1801	0.3805		
	0.1	0.3322	0.3425		
	0.2	0.4842	0.3045		
	0.4	0.7884	0.2284		
	0.6	1.092	0.1523		
0.8	0.0	0.3488	0.4220		
	0.1	0.4828	0.3798		
	0.2	0.6168	0.3376		
	0.4	0.8849	0.2532		
	0.6	1.1520	0.1688		
1.0	0.0	0.4624	0.4550	0.4698	1.6%
	0.1	0.5849	0.4095	0.5915	1.1%
	0.2	0.7074	0.3640	0.7133	0.8%
	0.4	0.9523	0.2730	0.9567	0.5%
	0.6	1.1970	0.1820	1.2000	0.3%
1.2	0.0	0.5440	0.4827		
	0.1	0.6582	0.4344		
	0.2	0.7725	0.3861		
	0.4	1.0010	0.2896		
	0.6	1.229	0.1930		

Note: For Pr = 1 the solution of equation (3.11) is available in closed form and is $g(\eta) = g_w + f'(\eta)(1-g_w)$. This expression was used to compute G_{test}. The comparison of G_{test} with G is a measure of the computational error in the present calculations.

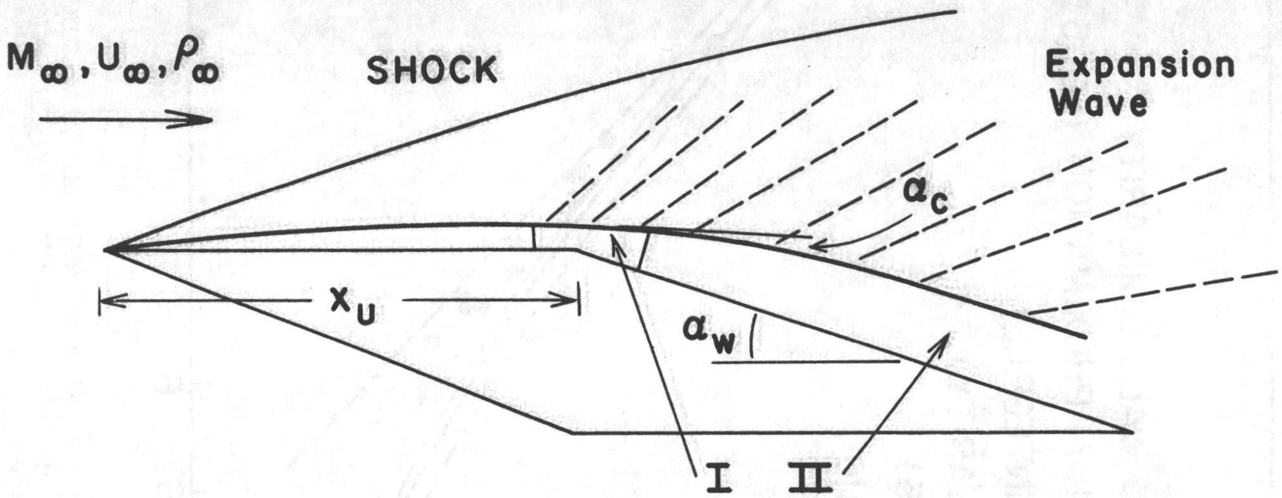

FIGURE 1: PROPOSED MODEL FOR HYPERSONIC FLOW AND SMALL TURNING ANGLES α_w. REGION I IS THE AREA WHERE THE BOUNDARY LAYER EQUATIONS ARE PRESUMED TO BE INAPPLICABLE, AND REGION II IS THE DOWN-STREAM INTERACTION REGION.

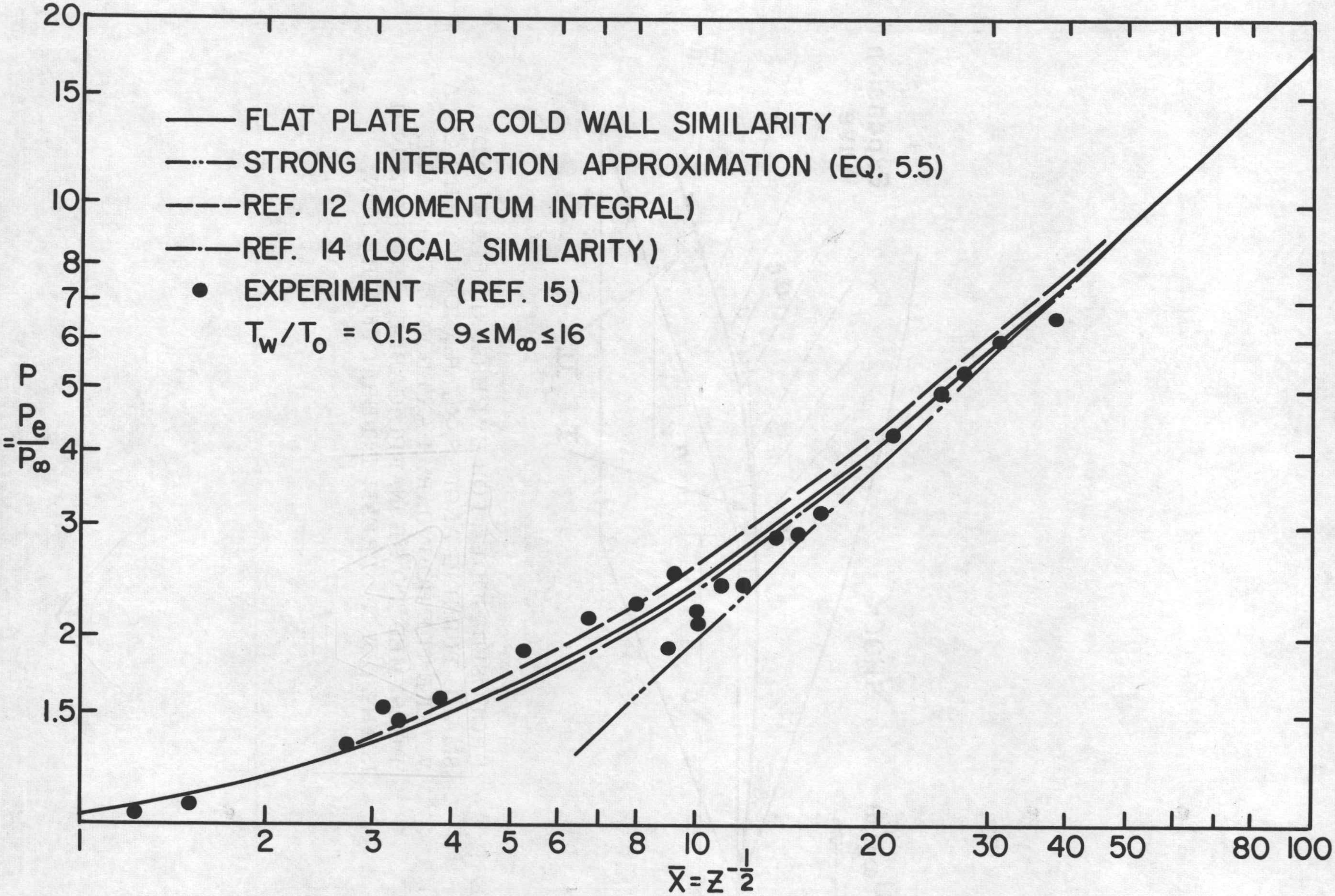

FIGURE 2: COMPARISON OF VARIOUS APPROXIMATE METHODS FOR PREDICTING THE PRESSURE DISTRIBUTION ON A HYPERSONIC FLOW WITH EXPERIMENTAL RESULTS.

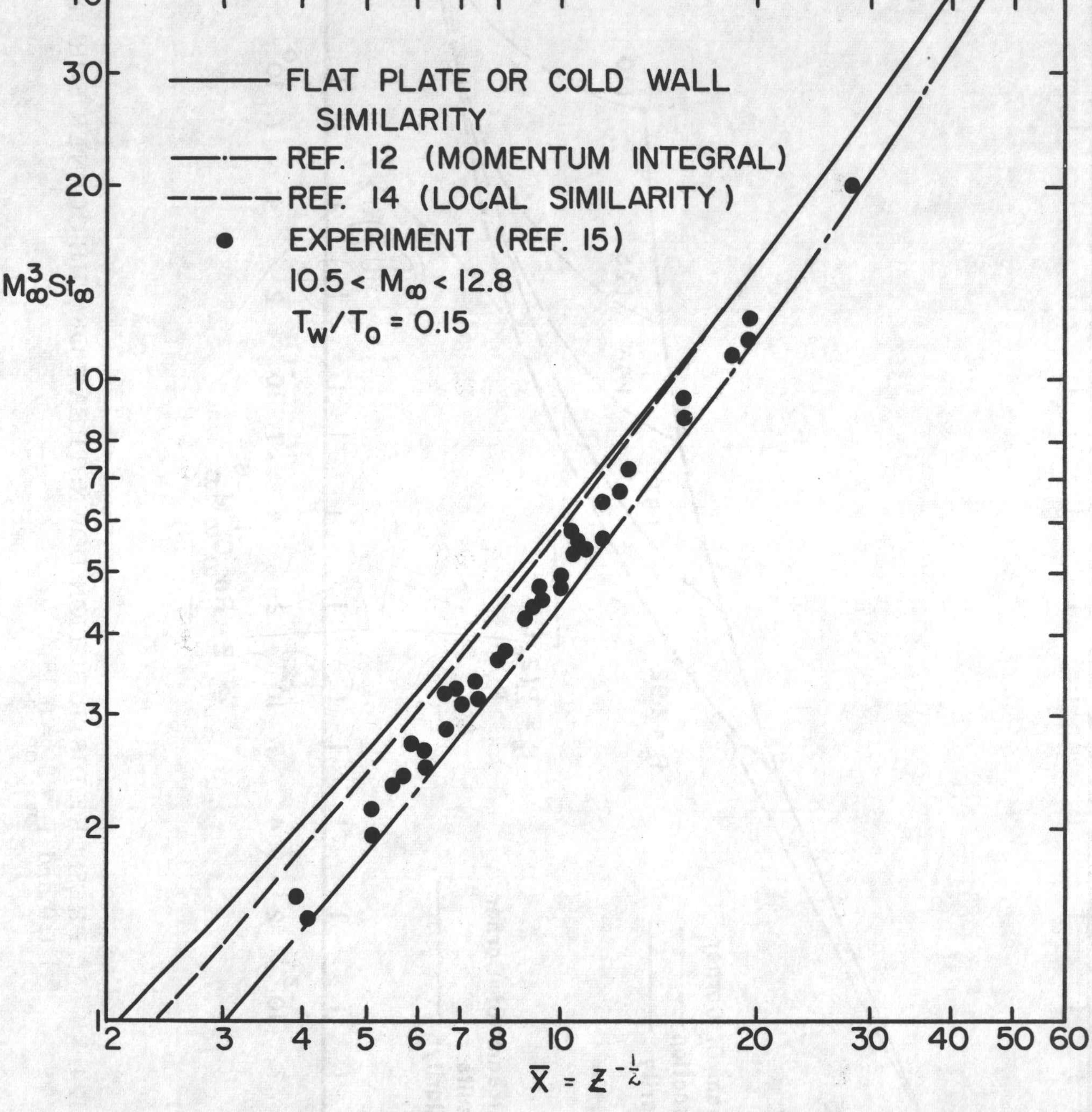

FIGURE 3: COMPARISON OF VARIOUS APPROXIMATE THEORIES FOR HEAT TRANSFER TO A FLAT PLATE IN HYPERSONIC FLOW WITH EXPERIMENT.

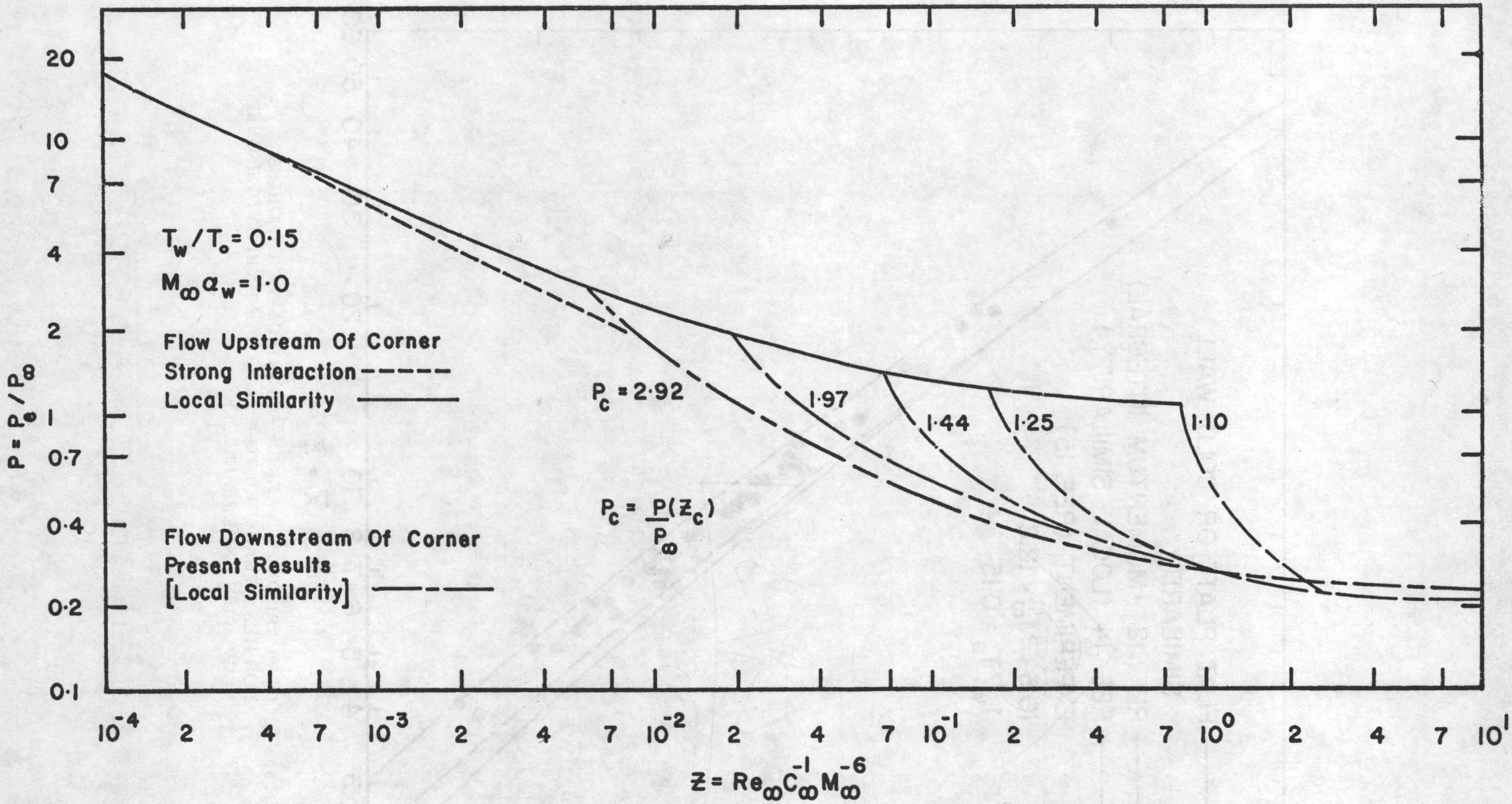

FIGURE 4: PRESSURE DISTRIBUTION DOWNSTREAM OF THE CORNER FOR $M_\infty \alpha_w$ = 1.0 and T_w/T_o = 0.15.

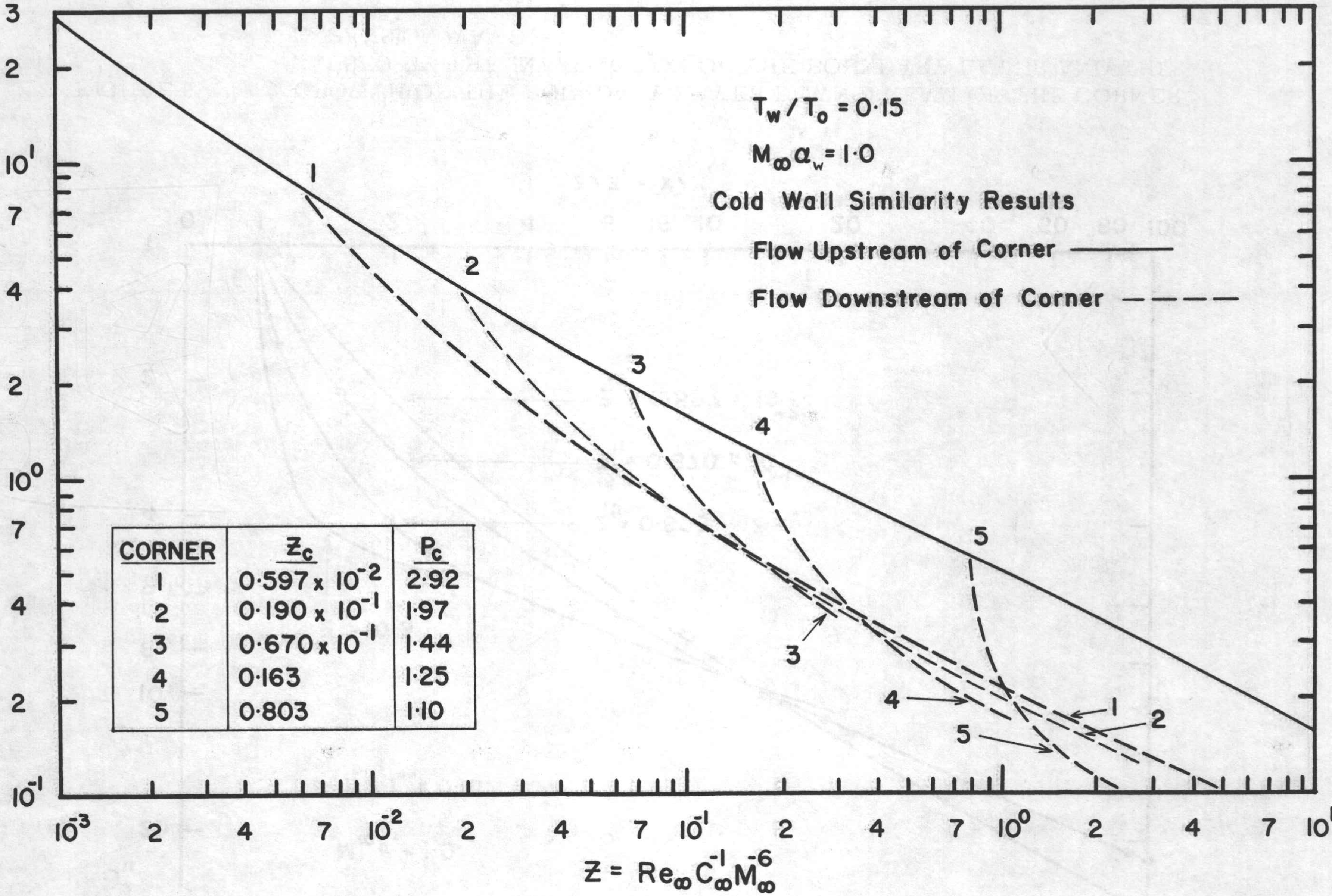

FIGURE 5: HEAT TRANSFER DISTRIBUTION DOWNSTREAM OF AN EXPANSION CORNER IN HYPERSONIC FLOW FOR $M_\infty \alpha_w = 1.0$ and $T_w/T_o = 0.15$

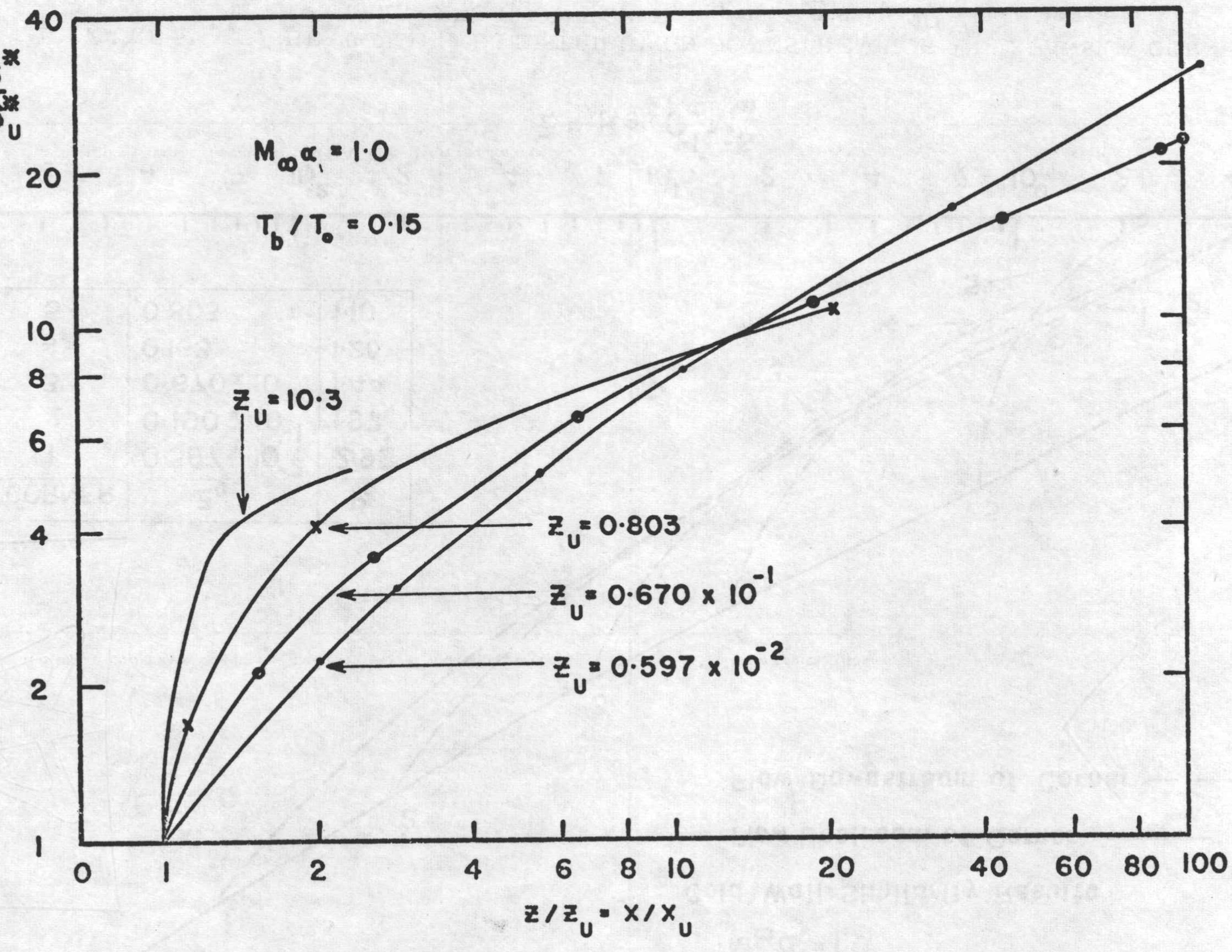

FIGURE 6: GROWTH OF THE BOUNDARY LAYER DOWNSTREAM OF THE CORNER CAUSED BY THE INTERACTION OF THE BOUNDARY LAYER AND THE EXPANSION WAVE