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Large deviations for stochastic evolution equations in the critical variational setting

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ABSTRACT

Using the weak convergence approach, we prove the large deviation principle (LDP) for solutions to quasilinear stochastic evolution equations with small Gaussian noise in the critical variational setting, a recently developed general variational framework. No additional assumptions are made apart from those required for well-posedness. In particular, no monotonicity is required, nor a compact embedding in the Gelfand triple. Moreover, we allow for flexible growth of the diffusion coefficient, including gradient noise. This leads to numerous applications for which the LDP was not established yet, in particular equations on unbounded domains with gradient noise. Since our framework includes the 2D Navier–Stokes and Boussinesq equations with gradient noise and unbounded domains, our results resolve an open problem that has remained unsolved for over 15 years.

1. Introduction

In this paper we study large deviations for solutions to small-noise stochastic evolution equations of the form

$$\mathrm{d}Y^\varepsilon(t) = -A(t, Y^\varepsilon(t))\,\mathrm{d}t + \sqrt{\varepsilon}B(t, Y^\varepsilon(t))\,\mathrm{d}W(t) \tag{1.1}$$

in the new variational framework of [3] by Agresti and the second author. This framework, the *critical variational setting*, has been developed to extend the classical variational approach to stochastic evolution equations originating from [8,26,34]. In the classical variational approach, the drift and diffusion coefficients A and B need to satisfy several conditions to ensure well-posedness of (1.1). The usual weak monotonicity condition is especially restrictive. It is therefore no surprise that efforts have been made to weaken the monotonicity condition, e.g. in [29, §5.2] and very recently [43] with a much weaker local monotonicity condition. One of the advantages of the critical variational setting of [3] used in this paper, is that no form of monotonicity is assumed. In return, A and B are of a slightly less (but still very) general form:

$$A(t, v) = A_0(t, v)v - F(t, v) + f(t), \quad B(t, v) = B_0(t, v)v + G(t, v) + g(t)$$

for $t \in \mathbb{R}_+$ and $v \in V$, where (V, H, V^*) is a Gelfand triple belonging to the stochastic evolution equation. That is, (A, B) contains a quasilinear part (A_0, B_0) and a semilinear part (F, G) and it is assumed that both parts satisfy certain critical local Lipschitz conditions,

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where the Lipschitz constant may depend arbitrarily on $\|v\|_H$ and, allowing even more flexibility, polynomially on interpolation norms $\|v\|_{V_\beta}$, where $V_\beta = [V^*, V]_\beta$ denotes the complex interpolation space. Besides the absence of any monotonicity assumption, another major improvement of the critical variational setting is the weakening of the usual growth conditions on the diffusion coefficient B, allowing e.g. for gradient noise. Lastly, a special feature is that critical nonlinearities are allowed (see (2.6)), which is not the case in other settings. The critical variational setting covers many semilinear and some quasilinear equations that were not covered by more classical variational settings. In particular, this holds for many equations that require an (analytically) strong setting, in which monotonicity often fails, for example the Cahn–Hilliard equation, the tamed Navier–Stokes equations and the Allen–Cahn equation. See [3, §5] for details. The exact assumptions in the critical variational setting can be found in Section 2. Finally, it should be stressed that unlike the settings in [43], the critical variational setting does not require a compact (Sobolev) embedding $V \hookrightarrow H$ and is thus suited to treat equations on unbounded spatial domains.

The goal of this paper is to establish the LDP for solutions to small-noise stochastic evolution equations in the critical variational setting. Large deviations have been studied for SPDEs in many different frameworks. The first results for SPDEs were inspired by the pioneering paper for SDEs by Freidlin and Wentzell [18] (see also [14, §5.6]), relying on discretizations and the contraction principle. These techniques were extended to several SPDE settings with Gaussian noise, notably in [10,13,35] (stochastic reaction-diffusion equations), [11] (semilinear parabolic equations) and [40] (stochastic porous media equations). However, for less regular A and B such techniques are difficult to use in general settings. In 2001, Budhiraja and Dupuis proved a substantially generalized contraction principle, the so-called weak convergence approach to large deviations [9]. This approach turned out to be extremely powerful for SPDEs and subsequently, it was applied to many SPDEs with less regular coefficients, e.g. in [12,45] (2D Navier–Stokes and hydrodynamical models), [15] (Boussinesq equations), as well as [39] and [28] (general classical variational settings). A more detailed discussion on applications to fluid dynamics can be found below.

Also for the recent variational settings with even weaker conditions on the coefficients A and B, the weak convergence approach has led to new LDP proofs. In [20] the LDP is obtained for McKean-Vlasov quasilinear stochastic evolution equations, in [32] for a setting from [43], in [27] for the same setting extended to Lévy noise, and most recently, [33] obtained the LDP for the strongest setting of [43]. The latter allows flexible growth bounds on B, including gradient noise. Still, the combination of flexible growth of B and unbounded spatial domains (that is, no compact embedding $V \hookrightarrow H$) has not been covered in any of the papers so far. The main improvement of our work is that we allow for both. In fact, no additional bounds on A and B are assumed for the LDP apart from those in [3] required for well-posedness, nor do we assume a compact embedding in the Gelfand triple. New techniques are used to replace the usual compactness arguments. The paper contains new approaches for

- well-posedness of the skeleton equation and compact sublevel sets of the rate function in the LDP, by means of maximal regularity theory and a strong approximation argument,
- the stochastic continuity criterion from the weak convergence approach, using critical estimates for the nonlinearities and an effective combination of deterministic and stochastic Gronwall inequalities.

The LDP result in this paper opens up many new applications. In particular, the following examples are included on bounded and unbounded domains in $\mathbb{R}^d$ and with gradient noise:

- Navier–Stokes equations for $d = 2$ [4, App. A],
- tamed Navier–Stokes equations for $d = 3$ [3, §5.2],
- Cahn–Hilliard equation for $d = 1, 2$ [3, §5.1],
- Swift–Hohenberg equations for $d = 1, 2, 3$ [3, §5.6],
- many reaction-diffusion equations, e.g. for $d \leq 4$:
    - Allen–Cahn equations [3, §5.4],
    - symbiotic Lotka–Volterra equations [5, Th. 3.11],
    - coagulation equations [5, Th. 3.9].

This list is far from extensive.

To make our results concrete for some of the models discussed above, we present an application to a general fluid dynamics model in Section 5. Specifying further, in Section 5.2, we derive the LDP for the 2D Navier–Stokes equations with gradient noise and do not assume that the underlying domain is bounded.

The LDP for the 2D Navier–Stokes equations with gradient noise and unbounded domains was already considered in the pioneering work [45]. However, the proof of [45, Lem. 4.5] is incomplete – specifically, the argument in the final line. The gap was already indicated in [15, p. 2054] and concerns the compactness of the sublevel sets of the rate function. In [12,15], the gap is resolved for the 2D Navier–Stokes and Boussinesq equations under the assumption that the noise is gradient-free (see [12, Th. 3.2], [15, Ass. A Bis, p. 2072]), but the gradient noise case has remained open since then. The possibly unbounded spatial domains rule out alternative arguments based on compact Sobolev embeddings. We have now covered the gradient noise case in Theorems 5.2 and 5.4 – extending the results of [12,15] and completing the proof of [45].

Another application that we would like to highlight are the 3D tamed Navier–Stokes equations, for which a large deviation principle was established in [42]. However, gradient noise was not considered in that work, and it is far from straightforward to extend their approach to settings where such noise is present. Our main result, Theorem 2.6, now includes the gradient noise case and also applies to a broad class of other models (see the list above).

Closing the above indicated gap requires several intricate approximation techniques, which are detailed in Section 4.2. Furthermore, in the full abstract setting, the stochastic continuity criterion also necessitates new ideas, which we develop in Section 4.3.

Notation

We let $\mathbb{R}_+ := [0, \infty)$. For $T > 0$ and a normed space X we let $C([0, T]; X)$ denote the space of continuous functions from $[0, T]$ to X equipped with supremum norm $\|f\|_{C([0,T];X)} := \sup_{t \in [0,T]} \|f(t)\|_X$. For $(S, \mathcal{A}, \mu)$ a measure space, we denote by $L^0(S; X)$ the space of strongly measurable functions $f : S \to X$, with identification of a.e. equal functions. For $p \in (0, \infty]$, we let $L^p(S; X)$ denote the subset consisting of all $f \in L^0(S; X)$ for which $\|f\|_{L^p(S;X)} < \infty$, where

$$\|f\|_{L^p(S;X)} := \begin{cases} \left(\int_S \|f(s)\|_X^p \, d\mu(s) \right)^{\frac{1}{p}}, & p < \infty, \\ \operatorname{ess\,sup}_{s \in S} \|f(s)\|_X, & p = \infty. \end{cases}$$

We write $L^p(S) := L^p(S; \mathbb{R})$ and if $S = [0, T] \subset \mathbb{R}$, we write $L^p(0, T; X) := L^p(S; X)$. Moreover, we let $L^p_{\text{loc}}(\mathbb{R}_+; X) := \{u : \mathbb{R}_+ \to X : u|_{[0,T]} \in L^2(0, T; X) \text{ for all } T \in \mathbb{R}_+\}$.

For Hilbert spaces U and H we let $\mathcal{L}(U, H)$ and $\mathcal{L}_2(U, H)$ denote the continuous linear operators and Hilbert-Schmidt operators from U to H, respectively. For brevity, we write

$$\|\cdot\|_H := \|\cdot\|_{\mathcal{L}_2(U,H)}.$$

Furthermore, we denote the dual of a Hilbert space V by V^* and for $\beta \in (0, 1)$, we denote the complex interpolation space at β by

$$V_\beta := [V^*, V]_\beta, \quad \|\cdot\|_\beta := \|\cdot\|_{V_\beta}.$$

For a metric space M we denote its Borel σ-algebra by $\mathcal{B}(M)$. The unique product measure space of two σ-finite measure spaces $(S_1, \mathcal{A}_1, \mu_1)$ and $(S_2, \mathcal{A}_2, \mu_2)$ is denoted by $(S_1 \times S_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$. Let $I = [0, T]$ or $I = \mathbb{R}_+$ and let X be a Banach space. A process $(\Phi(t))_{t \in I}$ is a strongly measurable function $\Phi : I \times \Omega \to X$. It is called strongly progressively measurable if for every $t \in I$, $\Phi|_{[0,t] \times \Omega}$ is strongly $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable. For $I = \mathbb{R}_+$, we denote the σ-algebra generated by the strongly progressively measurable processes by $\mathcal{P}$.

We write $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$ for $a, b \in \mathbb{R}$.

2. Main result

We specify our setting for stochastic evolution equations and recall the definition of the large deviation principle before we state our main result, Theorem 2.6.

2.1. The critical variational setting

We let (V, H, V^*) be a Gelfand triple of real Hilbert spaces. That is, $(V, (\cdot, \cdot)_V)$ and $(H, (\cdot, \cdot)_H)$ are real Hilbert spaces such that there exists a continuous and dense embedding $\iota : V \hookrightarrow H$. Then, $j : H \hookrightarrow V^* : x \mapsto (x, \iota(\cdot))_H$ is a continuous embedding and $j(H)$ is dense in V^* by reflexivity of V ($j = \iota^*$ under Riesz' identification $H \cong H^*$). From now on we identify $x \in V$ with $\iota(x) \in H$ and $x \in H$ with $j(x) \in V^*$. Then, if $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the abstract dual V^* and V, one has

$$\langle x, v \rangle = (v, x)_H \quad \text{for all } x \in H, v \in V.$$

For convenience of the reader, we recall that in applications, one does not work with the abstract dual V^* but with a space V' which, under some assumptions and with the correct duality pairing, is isomorphic to V^*, see also [26, p. 1244]. One starts with reflexive Banach (or Hilbert) (sub)spaces $V \subset H \subset V'$, where each inclusion is dense and continuous and one defines $j : H \hookrightarrow V^* : x \mapsto (x, \cdot)_H$. Then, provided that

$$|(x, v)_H| \le \|x\|_{V'} \|v\|_V \quad \text{for all } x \in H, v \in V, \tag{2.1}$$

there exists a unique continuous extension to a map $j_1 : V' \to V^*$. Furthermore, if j_1 is bijective, then it follows that $j_1 : V' \cong V^*$ as normed spaces, although not necessarily isometrically. The duality pairing is then given by $\langle v', v \rangle := j_1(v')(v)$ and for $x \in H, v \in V$ we have $\langle x, v \rangle = (x, v)_H$ since j_1 is the extension of j. The triple (V, H, V') is also called a Gelfand triple and simply denoted by (V, H, V^*), where as explained, the correct duality pairing $\langle \cdot, \cdot \rangle : V' \times V \to \mathbb{R}$ is given by $\langle v', v \rangle := j_1(v')(v)$.

In fact, bijectivity of j_1 holds if and only if there exists $\alpha > 0$ such that

$$\alpha \|x\|_{V'} \le \sup_{v \in V, \|v\|_V \le 1} |(x, v)_H| =: \|j_1(x)\|_{V^*} \quad \text{for all } x \in H. \tag{2.2}$$

The equivalence follows from [44, Th. 4.48], density of $H \subset V'$ and continuity of j_1, and density of $\operatorname{Im}(j_1) \subset V^*$. The latter holds since $j_1(V') \supset j_1(H) = j(H)$ and one can verify that $j(H)$ is dense in V^* using reflexivity of V. In conclusion, provided that $V \subset H \subset V'$ continuously and densely, one only has to verify (2.1) and (2.2) to have $j_1 : V' \cong V^*$.

Popular choices for the Gelfand triple are the weak and strong setting for a given differential operator. For example, if $A(t, u) := \Delta u$ on $\mathbb{R}^d$, then one can use

$$V = H^1(\mathbb{R}^d), \ H = L^2(\mathbb{R}^d), \ V' = H^{-1}(\mathbb{R}^d) = V^* \qquad \text{(weak setting)},$$

$$V = H^2(\mathbb{R}^d), \ H = H^1(\mathbb{R}^d), \ V' = L^2(\mathbb{R}^d) \cong V^* \qquad \text{(strong setting)}.$$

See also [3, Ex. 2.1, Ex. 2.2].

Recall that $H = [V^*, V]_{\frac{1}{2}}$ [7, §5.5.2] and the following interpolation estimate holds for $\beta \in (\frac{1}{2}, 1)$:

$$\|v\|_\beta \leq K \|v\|_H^{2-2\beta} \|v\|_V^{2\beta-1}, \qquad v \in V. \tag{2.3}$$

Since strong solutions are required to be strongly measurable, see Definition 2.1 below, one can assume without loss of generality that V and H are separable, see also [26, p. 1244]. Thus, from now on we assume that V and H are separable.

As mentioned in the introduction, we work with the critical variational setting from [3]. We consider stochastic evolution equations of the form

$$\begin{cases} du(t) = -A(t, u(t)) \, dt + B(t, u(t)) \, dW(t), & t \in [0, T], \\ u(0) = x, \end{cases} \tag{2.4}$$

where $x \in H$, $T > 0$ and W is a U-cylindrical Brownian motion (see Definition 4.1).

If $\Phi : [0, T] \times \Omega \to \mathcal{L}_2(U, H)$ is strongly progressively measurable and $\Phi \in L^2(0, T; \mathcal{L}_2(U, H))$ a.s., then one can define the stochastic integral $\int_0^t \Phi(s) \, dW(s)$ for $t \in [0, T]$, see [31, §5.4 ($p = 0$)].

We now specify what we mean by a strong solution to (2.4). In our definition we also allow for $L^1(0, T; H)$-valued integrands, which is only needed to treat the skeleton equation associated to (2.4), see Definition 2.5 below.

Definition 2.1. For $T > 0$, we define the *maximal regularity space* by

$$\mathrm{MR}(0, T) := C([0, T]; H) \cap L^2(0, T; V), \quad \| \cdot \|_{\mathrm{MR}(0,T)} := \| \cdot \|_{C([0,T];H)} + \| \cdot \|_{L^2(0,T;V)}.$$

Let $A : \mathbb{R}_+ \times V \to V^*$, $B : \mathbb{R}_+ \times V \to \mathcal{L}_2(U, H)$ and let $x \in H$. Let W be a U-cylindrical Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and let $T > 0$. We say that a strongly progressively measurable process $u : [0, T] \times \Omega \to V$ is a *strong solution* to (2.4) if a.s.

$$u \in \mathrm{MR}(0, T), \ A(\cdot, u(\cdot)) \in L^2(0, T; V^*) + L^1(0, T; H), \ B(\cdot, u(\cdot)) \in L^2(0, T; \mathcal{L}_2(U, H))$$

and a.s.

$$u(t) = x - \int_0^t A(s, u(s)) \, ds + \int_0^t B(s, u(s)) \, dW(s) \ \text{ in } V^* \text{ for all } t \in [0, T]. \tag{2.5}$$

A strong solution u is *unique* if for any other strong solution v we have a.s. $u = v$ in $\mathrm{MR}(0, T)$.

If $B = 0$, we write $u'(t) = -A(t, u(t))$ instead of $du(t) = -A(t, u(t)) \, dt$ in (2.4) and we call $u \in \mathrm{MR}(0, T)$ a strong solution if $A(\cdot, u(\cdot)) \in L^2(0, T; V^*) + L^1(0, T; H)$ and (2.5) holds.

For the weak convergence approach to large deviations it is necessary to let A and B be defined on $\mathbb{R}_+ \times V$ rather than $\mathbb{R}_+ \times \Omega \times V$, meaning that stochasticity enters A and B through the solution u in (2.4) and not separately. Also, the initial value x in (1.1) has to be deterministic. Other than that, we make exactly the same assumptions as those required for global well-posedness [3, Th. 3.5]. Let us introduce these assumptions.

Assumption 2.2. We assume that:

(1) $A(t, v) = A_0(t, v)v - F(t, v) - f$ and $B(t, v) = B_0(t, v)v + G(t, v) + g$, where

$$A_0 : \mathbb{R}_+ \times H \to \mathcal{L}(V, V^*) \text{ and } B_0 : \mathbb{R}_+ \times H \to \mathcal{L}(V, \mathcal{L}_2(U, H))),$$

are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(H)$-measurable, and

$$F : \mathbb{R}_+ \times V \to V^* \text{ and } G : \mathbb{R}_+ \times V \to \mathcal{L}_2(U, H)$$

are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(V)$-measurable, and $f : \mathbb{R}_+ \to V^*$ and $g : \mathbb{R}_+ \to \mathcal{L}_2(U, H)$ are $\mathcal{B}(\mathbb{R}_+)$-measurable maps with

$$f \in L^2_{\mathrm{loc}}(\mathbb{R}_+; V^*) \text{ and } g \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{L}_2(U, H)).$$

(2) For all $T > 0$ and $n \in \mathbb{R}_+$, there exist $\theta_{n,T}, M_{n,T} > 0$ such that for any $t \in [0, T]$, $u \in H$, $v \in V$ with $\|u\|_H \leq n$, we have

$$\langle A_0(t, u)v, v \rangle - \frac{1}{2} \|\!| B_0(t, u)v \|\!|_H^2 \geq \theta_{n,T} \|v\|_V^2 - M_{n,T} \|v\|_H^2.$$

(3) There exist $\rho_j \geq 0$ and $\beta_j \in (\frac{1}{2}, 1)$ such that

$$2\beta_j \leq 1 + \frac{1}{1 + \rho_j}, \quad j \in \{1, \dots, m_F + m_G\}, \qquad \text{((sub)criticality)} \tag{2.6}$$

for some $m_F, m_G \in \mathbb{N}$ and for all $T > 0$, $n \in \mathbb{R}_+$ there exists a constant $C_{n,T}$ such that for all $t \in [0, T]$ and $u, v, w \in V$ with $\|u\|_H, \|v\|_H \leq n$, we have

$$\|A_0(t, u)w\|_{V^*} \leq C_{n,T} \|w\|_V,$$

$$\|A_0(t, u)w - A_0(t, v)w\|_{V^*} \leq C_{n,T} \|u - v\|_H \|w\|_V,$$

$$\|\!| B_0(t, u)w \|\!|_H \leq C_{n,T} \|w\|_V,$$

$$\||B_0(t,u)w - B_0(t,v)w\||_H \leq C_{n,T} \|u-v\|_H \|w\|_V,$$

$$\|F(t,u)\|_{V^*} \leq C_{n,T} \sum_{j=1}^{m_F} (1 + \|u\|_{\beta_j}^{\rho_j+1}),$$

$$\|F(t,u) - F(t,v)\|_{V^*} \leq C_{n,T} \sum_{j=1}^{m_F} (1 + \|u\|_{\beta_j}^{\rho_j} + \|v\|_{\beta_j}^{\rho_j}) \|u-v\|_{\beta_j},$$

$$\||G(t,u)\||_H \leq C_{n,T} \sum_{j=m_F+1}^{m_F+m_G} (1 + \|u\|_{\beta_j}^{\rho_j+1}),$$

$$\||G(t,u) - G(t,v)\||_H \leq C_{n,T} \sum_{j=m_F+1}^{m_F+m_G} (1 + \|u\|_{\beta_j}^{\rho_j} + \|v\|_{\beta_j}^{\rho_j}) \|u-v\|_{\beta_j}.$$

Without loss of generality, we assume that the constants $C_{n,T}$ are non-decreasing in n and T.

Because the coefficients are defined on $\mathbb{R}_+ \times V$ instead of $\mathbb{R}_+ \times \Omega \times V$, the measurability in Assumption 2.2(1) is different than in [3, Ass. 3.1]. However, (A, B) satisfies our assumption if and only if $(\bar{A}, \bar{B})$ satisfies [3, Ass. 3.1], where $\bar{A}(t, \omega, v) := A(t, v)$ and $\bar{B}(t, \omega, v) := B(t, v)$ are trivial extensions.

A_0 and B_0 determine the leading differential order of the drift and the noise, respectively, and constitute the principal part of the quasilinear equation. In the semilinear case, $A_0(t, u)$ and $B_0(t, u)$ do not depend on u, rendering the principal part linear.

Condition (2.6) describes a balance between the growth rate $\rho_j + 1$ of the nonlinearities F and G and the regularity coefficient β_j (whose value is usually determined by Sobolev embeddings). In case of equality in (2.6) for some j, the nonlinearity is called *critical*.

From (2.3) and Assumption 2(3), it is clear that $\|F(t,v)\|_{V^*} + \||G(t,v)\||_H \leq \tilde{C}_{\|v\|_H,T}(1 + \|v\|_V)$ for all $t \in [0,T]$ if $\|v\|_H \leq n$, where $\tilde{C}_{\|v\|_H,T}$ is a constant. Thus we have integrability of $F(\cdot, u(\cdot))$ and $G(\cdot, u(\cdot))$ if $u \in \mathrm{MR}(0, T)$.

In [3, Th. 3.3] it is shown that under Assumption 2.2, there exists a unique local solution to (2.4). In [3, Th. 3.5], this is extended to a global well-posedness result under a coercivity condition on (A, B). The next result follows from [3, Th. 3.5].

Theorem 2.3. *Let (A, B) satisfy Assumption 2.2 and suppose that (A, B) is coercive in the following sense: for all $T > 0$, there exist $\theta, M > 0$ and $\phi \in L^2(0, T)$ such that for all $v \in V$ and $t \in [0, T]$,*

$$\langle A(t,v), v \rangle - \frac{1}{2}\||B(t,v)\||_H^2 \geq \theta\|v\|_V^2 - M\|v\|_H^2 - |\phi(t)|^2. \tag{2.7}$$

Then, for any $x \in H$ and $T > 0$, there exists a unique strong solution u to (2.4) on $[0, T]$.

Energy estimates can also be found in [3, Th. 3.5], but these will not be used. More general theory in an L^p-setting was developed in [1,2].

2.2. Statement of the main result

Definition 2.4. Let $\mathcal{E}$ be a Polish space, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(Y^\epsilon)_{\epsilon>0}$ be a collection of $\mathcal{E}$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $I : \mathcal{E} \to [0, \infty]$ be a function. Then (Y^ϵ) *satisfies the large deviation principle (LDP) on $\mathcal{E}$ with rate function* $I : S \to [0, \infty]$ if

(i) I has compact sublevel sets,
(ii) for all open $E \subset \mathcal{E}$: $\liminf_{\epsilon \downarrow 0} \epsilon \log \mathbb{P}(Y^\epsilon \in E) \geq -\inf_{z \in E} I(z)$,
(iii) for all closed $E \subset \mathcal{E}$: $\limsup_{\epsilon \downarrow 0} \epsilon \log \mathbb{P}(Y^\epsilon \in E) \leq -\inf_{z \in E} I(z)$.

Before we formulate our LDP result, we define the skeleton equation, which appears in the rate function of our LDP.

Definition 2.5. Let $x \in H$ be fixed. For $\psi \in L^2(0, T; U)$, the *skeleton equation* associated to the stochastic evolution Eq. (2.4) is given by

$$\begin{cases} (u^\psi)'(t) = -A(t, u^\psi(t)) + B(t, u^\psi(t))\psi(t), & t \in [0, T], \\ u^\psi(0) = x. \end{cases} \tag{2.8}$$

The main theorem of this paper is as follows.

Theorem 2.6. *Suppose that (A, B) satisfies Assumption 2.2 and coercivity (2.7). Let $x \in H$. For $\epsilon \in (0, 1]$, let Y^ϵ be the strong solution to*

$$\begin{cases} dY^\epsilon(t) = -A(t, Y^\epsilon(t))\,dt + \sqrt{\epsilon}B(t, Y^\epsilon(t))\,dW(t), & t \in [0, T], \\ u(0) = x, \end{cases}$$

Then (Y^ϵ) satisfies the LDP on $\mathrm{MR}(0, T)$ with rate function $I : \mathrm{MR}(0, T) \to [0, +\infty]$ given by

$$I(z) = \frac{1}{2}\inf\left\{\int_0^T \|\psi(s)\|_U^2\,ds : \psi \in L^2(0, T; U), z = u^\psi\right\}, \tag{2.9}$$

where $\inf \varnothing := +\infty$ and u^ψ is the strong solution to (2.8).

We have taken $\epsilon \in (0,1]$ to ensure that $(A, \sqrt{\epsilon} B)$ satisfies coercivity (2.7), so that the equation for Y^{ϵ} is well-posed by Theorem 2.3.

To have u^{ψ} appearing in (2.9) well-defined, (2.8) needs to be (globally) well-posed. In Section 3, we prove that this is the case. Finally, we recall that the LDP is equivalent to the *Laplace principle* [16, Def. 1.2.2, Th. 1.2.1, Th. 1.2.3]. The weak convergence approach from [9] offers sufficient conditions for the latter, hence for the LDP. The approach is stated in Section 4.1, after which we apply it to prove Theorem 2.6 in the remainder of Section 4.

3. Well-posedness of the skeleton equation

Before we turn to large deviations, we prove global well-posedness of the skeleton Eq. (2.8) under Assumption 2.2 and coercivity (2.7). This is needed, since the solution to (2.8) appears in the rate function (2.9) of the LDP.

Unfortunately, well-posedness cannot be proved at once. Instead, we first achieve well-posedness of an appropriate linearized version of the skeleton equation in Corollary 3.5, together with a maximal regularity estimate. Then, we can borrow the strategies from [22, Chap. 18] and [36–38]. That is, we use the maximal regularity estimate of Corollary 3.5 for the linearized equation in a fixed point argument, yielding existence of a local solution to the skeleton equation in Theorem 3.7. Finally, we extend to a global solution in Theorem 3.16, making use of a blow-up criterion. Uniqueness will be obtained along the way.

3.1. Linearized skeleton equation

We consider the following linearization of (2.8). We discard the non-linearities F and G and for fixed $w \in L^{\infty}(0, T; H)$, we consider

$$\begin{cases} u'(t) + A_0(t, w(t))u(t) - B_0(t, w(t))u(t)\psi(t) = \bar{f}(t) + \bar{g}(t)\psi(t), \\ u(0) = x, \end{cases} \tag{3.1}$$

where A_0 and B_0 are as in Assumption 2.2 and $\bar{f} \in L^2(0, T; V^*)$, $\bar{g} \in L^2(0, T; \mathcal{L}_2(U, H))$. In this subsection we prove well-posedness of (3.1) using the method of continuity [22, Lem. 16.2.2], together with a suitable maximal regularity estimate. We prove it for more general equations as this does not require any more effort and makes the exposition more transparent. Let us introduce spaces S and E that will be used in the method of continuity.

Definition 3.1. For $T > 0$, we let

$$S := L^2(0, T; V^*) + L^1(0, T; H)$$

be the sum space of the interpolation couple $(L^2(0, T; V^*), L^1(0, T; H))$, where we note that both components embed continuously into the Hausdorff topological vector space $L^1(0, T; V^*)$. The norm on S is given by

$$\|h\|_S := \inf \left\{ \|f\|_{L^2(0,T;V^*)} + \|g\|_{L^1(0,T;H)} : h = f + g, f \in L^2(0, T; V^*), g \in L^1(0, T; H) \right\}.$$

Note that S is a Banach space [21, Prop. C.1.3] and $S \hookrightarrow L^1(0, T; V^*)$. Moreover, we define

$$E := \{u \in \mathrm{MR}(0, T) : u \text{ is weakly differentiable}, u' \in S\}, \quad \|u\|_E := \|u\|_{\mathrm{MR}(0,T)} + \|u'\|_S.$$

Note that trivially, $E \hookrightarrow \mathrm{MR}(0, T)$.

Dealing with the sum space S is not standard in part of the literature. However, it is covered excellently in Pardoux' thesis [34].

The following proposition is a direct consequence of [34, Th. 2.1].

Proposition 3.2. *Let $\bar{A} : [0, T] \to \mathcal{L}(V, V^*)$ be such that $[0, T] \to V^* : t \mapsto \bar{A}(t)v$ is strongly Borel measurable for all $v \in V$ and suppose that $a_T := \sup_{t \in [0,T]} \|\bar{A}(t)\|_{\mathcal{L}(V,V^*)} < \infty$. Suppose that there exists $\theta > 0$ such that for all $t \in [0, T]$ and $v \in V$:*

$$\langle \bar{A}(t)v, v \rangle \geq \theta \|v\|_V^2.$$

Then, for any $h \in S$ and $x \in H$, there exists a unique $u \in E$ satisfying

$$\begin{cases} u'(t) + \bar{A}(t)u(t) = h(t), \quad t \in [0, T], \\ u(0) = x. \end{cases} \tag{3.2}$$

We will need an extension of Proposition 3.2 with the coercivity condition replaced by the weaker condition (3.3) below. As a preparation, we first prove a maximal regularity estimate.

Lemma 3.3. *Let $\bar{A} : [0, T] \to \mathcal{L}(V, V^*)$ be such that $\bar{A}(\cdot)u(\cdot) \in S$ for any $u \in \mathrm{MR}(0, T)$. Suppose that there exist $\theta > 0$ and $M \in L^1(0, T)$, $M \geq 0$ such that for all $t \in [0, T]$ and $v \in V$:*

$$\langle \bar{A}(t)v, v \rangle \geq \theta \|v\|_V^2 - M(t)\|v\|_H^2. \tag{3.3}$$

Let $h \in S$ and $x \in H$ and suppose that $u \in \mathrm{MR}(0, T)$ is a strong solution to (3.2). Then

$$\|u\|_{\mathrm{MR}(0,T)} \leq C_{\theta} \exp(2\|M\|_{L^1(0,T)})(\|h\|_S + \|x\|_H), \tag{3.4}$$

for a constant $C_{\theta} > 0$ depending only on θ.

Proof. Write $h = f + g$ with $f \in L^2(0, T; V^*)$, $g \in L^1(0, T; H)$. We apply [34, Th. 2.2]. Since $u \in \mathrm{MR}(0, T)$ is a strong solution, we have $u(t) = x + \int_0^t v(s)\,ds$ with $v := h(\cdot) - \bar{A}(\cdot)u(\cdot) \in S = L^2(0, T; V^*) + L^1(0, T; H)$. Hence, the chain rule (A.2) and (3.3) yield for all $t \in [0, T]$:

$$\|u(t)\|_H^2 = \|x\|_H^2 + 2\int_0^t \langle h(s), u(s)\rangle\,ds - 2\int_0^t \langle \bar{A}(s)u(s), u(s)\rangle\,ds$$

$$\leq \|x\|_H^2 + 2\int_0^t \langle h(s), u(s)\rangle\,ds - 2\theta\|u\|_{L^2(0,t;V)}^2 + 2\int_0^t M(s)\|u(s)\|_H^2\,ds. \tag{3.5}$$

Note that by Young's inequality, we have for all $s \in [0, t]$:

$$\langle h(s), u(s)\rangle = \langle f(s), u(s)\rangle + \langle g(s), u(s)\rangle \leq \|u(s)\|_V \|f(s)\|_{V^*} + \|u(s)\|_H \|g(s)\|_H \leq \frac{\theta}{2}\|u(s)\|_V^2 + \frac{1}{2\theta}\|f(s)\|_{V^*}^2 + \sup_{r \in [0,t]} \|u(r)\|_H \|g(s)\|_H.$$

Entering this into (3.5) we obtain for all $0 \leq t \leq t_1 \leq T$:

$$\|u(t)\|_H^2 + \theta\|u\|_{L^2(0,t;V)}^2 \leq \|x\|_H^2 + \frac{1}{\theta}\|f\|_{L^2(0,T;V^*)}^2 + 2\sup_{r \in [0,t_1]} \|u(r)\|_H \|g\|_{L^1(0,T;H)} + \int_0^{t_1} 2M(s)\|u(s)\|_H^2\,ds$$

$$\leq \|x\|_H^2 + \frac{1}{\theta}\|f\|_{L^2(0,T;V^*)}^2 + \frac{1}{2}\sup_{r \in [0,t_1]} \|u(r)\|_H^2 + 2\|g\|_{L^1(0,T;H)}^2 + \int_0^t 2M(s)\|u(s)\|_H^2\,ds.$$

Hence, taking $\sup_{t \in [0,t_1]}$ in the above and writing $F(t_1) := \frac{1}{2}\sup_{t \in [0,t_1]}\left(\|u(t)\|_H^2 + \theta\|u\|_{L^2(0,t;V)}\right)$ gives for all $0 \leq t_1 \leq T$:

$$F(t_1) \leq 2F(t_1) - \frac{1}{2}\sup_{r \in [0,t_1]} \|u(r)\|_H^2 \leq \|x\|_H^2 + \frac{1}{\theta}\|f\|_{L^2(0,T;V^*)}^2 + 2\|g\|_{L^1(0,T;H)}^2 + \int_0^{t_1} 2M(s)\|u(s)\|_H^2\,ds$$

$$\leq \|x\|_H^2 + \frac{1}{\theta}\|f\|_{L^2(0,T;V^*)}^2 + 2\|g\|_{L^1(0,T;H)}^2 + \int_0^{t_1} 4M(s)F(s)\,ds,$$

so by Gronwall's inequality, we obtain

$$\|u\|_{C([0,T];H)}^2 + \theta\|u\|_{L^2(0,T;V)}^2 \leq 4F(T) \leq 4\left(\|x\|_H^2 + \frac{1}{\theta}\|f\|_{L^2(0,T;V^*)}^2 + 2\|g\|_{L^1(0,T;H)}^2\right)\exp(4\|M\|_{L^1(0,T)}).$$

Thus

$$\|u\|_{\mathrm{MR}(0,T)}^2 \leq (1 \vee \theta^{-1})4\left(\|x\|_H^2 + (\theta^{-1} \vee 2)(\|f\|_{L^2(0,T;V^*)} + \|g\|_{L^1(0,T;H)})^2\right)\exp(4\|M\|_{L^1(0,T)}).$$

Since f and g with $h = f + g$ were arbitrary, taking the infimum over $\{(f, g) \in L^2(0, T; V^*) \times L^1(0, T; H) : h = f + g\}$ gives

$$\|u\|_{\mathrm{MR}(0,T)}^2 \leq C_\theta^2\left(\|x\|_H^2 + \|h\|_S^2\right)\exp(4\|M\|_{L^1(0,T)}).$$

where $C_\theta := \left(4(1 \vee \theta^{-1})(\theta^{-1} \vee 2)\right)^{\frac{1}{2}}$. Taking square roots on both sides yields (3.4). $\quad\square$

We now prove Proposition 3.2 under the weaker coercivity (3.3).

Theorem 3.4. *Let $\bar{A} : [0, T] \to \mathcal{L}(V, V^*)$ and suppose that for all $u \in \mathrm{MR}(0, T)$:*

$$\bar{A}(\cdot)u(\cdot) \in S, \qquad \|\bar{A}(\cdot)u(\cdot)\|_S \leq \alpha\|u\|_{\mathrm{MR}(0,T)}, \tag{3.6}$$

for some constant $\alpha > 0$ independent of u. Suppose that coercivity (3.3) is satisfied for some $\theta > 0$ and $M \in L^1(0, T)$. Then for any $h \in S$, there exists a unique strong solution $u \in \mathrm{MR}(0, T)$ to (3.2). Moreover, the estimate (3.4) holds.

Proof. We use the method of continuity [22, Lem. 16.2.2]. Define $A_0 \in \mathcal{L}(V, V^*)$ by $A_0 v := \theta\langle\cdot, v\rangle_V$. For $\lambda \in [0, 1]$, put

$$A_\lambda : [0, T] \to \mathcal{L}(V, V^*) : t \mapsto (1 - \lambda)A_0 + \lambda\bar{A}(t),$$

$$L_\lambda : E \to S \times H : (L_\lambda u) := \left(u'(\cdot) + A_\lambda(\cdot)(u(\cdot)), u(0)\right).$$

Clearly, L_λ is linear. We show that $L_\lambda \in \mathcal{L}(E, S \times H)$ and that $[0, 1] \to \mathcal{L}(E, S \times H) : \lambda \mapsto L_\lambda$ is continuous.

Let $u \in E$ be arbitrary. For all $t \in [0, T]$ we have $A_0 u(t) = \theta\langle\cdot, u(t)\rangle \in V^*$, so by the Riesz isomorphism, $\|A_0 u(t)\|_{V^*} = \theta\|u(t)\|_V$. Since $u \in L^2(0, T; V)$, it follows that $A_0 u(\cdot) \in L^2(0, T; V^*) \subset S$ and

$$\|A_0 u(\cdot)\|_S \leq \|A_0 u(\cdot)\|_{L^2(0,T;V^*)} = \theta\|u\|_{L^2(0,T;V)} \leq \theta\|u\|_{\mathrm{MR}(0,T)}$$

Combining with (3.6) gives $A_\lambda(\cdot)u(\cdot) \in S$ and

$$\|A_\lambda(\cdot)u(\cdot)\|_S \leq (1 - \lambda)\|A_0 u(\cdot)\|_S + \lambda\|\bar{A}(\cdot)u(\cdot)\|_S \leq (\theta + \alpha)\|u\|_{\mathrm{MR}(0,T)}. \tag{3.7}$$

Note that $u' \in S$ and $\|u'\|_S \leq \|u\|_E$ by definition of E. Moreover, $E \hookrightarrow \mathrm{MR}(0, T)$, thus

$$\|L_\lambda u\|_{S \times H} \leq \|u'\|_S + \|A_\lambda(\cdot)u(\cdot)\|_S + \|u(0)\|_H \leq \|u\|_E + (\theta + \alpha)\|u\|_{\mathrm{MR}(0,T)} + \|u\|_{C([0,T];H)} \leq (2 + \theta + \alpha)\|u\|_E,$$

proving $L_\lambda \in \mathcal{L}(E, S \times H)$. Moreover, we have for any $\lambda, \mu \in [0, 1]$ and $u \in E$:

$$\|(L_\lambda - L_\mu)u\|_{S \times H} = \|\left((\mu - \lambda)A_0 u(\cdot) + (\lambda - \mu)\bar{A}(\cdot)u(\cdot), 0\right)\|_{S \times H} \leq |\mu - \lambda|\|A_0 u(\cdot)\|_S + |\lambda - \mu|\|\bar{A}(\cdot)u(\cdot)\|_S$$

$$\leq |\mu - \lambda|(\theta + \alpha)\|u\|_{\mathrm{MR}(0,T)}$$
$$\leq |\mu - \lambda|(\theta + \alpha)\|u\|_E,$$

i.e. $\|L_\lambda - L_\mu\|_{\mathcal{L}(E,S\times H)} \leq |\mu - \lambda|(\theta + \alpha)$. Thus $\lambda \mapsto L_\lambda$ is (Lipschitz) continuous.

Next, we verify that $\|u\|_E \leq K\|L_\lambda u\|_{S\times H}$ for some $K > 0$ independent of λ. Note that A_λ satisfies all conditions of Lemma 3.3. Coercivity (3.3) holds since

$$\langle v, A_\lambda(t)v\rangle = (1-\lambda)\langle v, A_0 v\rangle + \lambda\langle v, \bar{A}(t)v\rangle \geq (1-\lambda)\theta\|v\|_V^2 + \lambda(\theta\|v\|_V^2 - |M(t)|\|v\|_H^2) \geq \theta\|v\|_V^2 - |M(t)|\|v\|_H^2.$$

Thus, by (3.4) applied to $h := u' + A_\lambda u \in S$ and $x := u(0)$:

$$\|u\|_{\mathrm{MR}(0,T)} \leq C_\theta \exp(2\|M\|_{L^1(0,T)})\big(\|h\|_S + \|u(0)\|_H\big) = C\|L_\lambda u\|_{S\times H},$$

with $C := C_\theta \exp(2\|M\|_{L^1(0,T)})$. Together with (3.7) this gives for all $u \in E$:

$$\|u\|_E = \|u\|_{\mathrm{MR}(0,T)} + \|u'\|_S = \|u\|_{\mathrm{MR}(0,T)} + \|h(\cdot) - A_\lambda(\cdot)u(\cdot)\|_S \leq \|u\|_{\mathrm{MR}(0,T)} + \|L_\lambda u\|_{S\times H} + \|A_\lambda(\cdot)u(\cdot)\|_S$$
$$\leq (1 + \theta + \alpha)\|u\|_{\mathrm{MR}(0,T)} + \|L_\lambda u\|_{S\times H}$$
$$\leq (1 + C(1 + \theta + \alpha))\|L_\lambda u\|_{S\times H}.$$

Finally, note that $L_0 : E \to S \times H : (L_0 u) = (u'(\cdot) + A_0 u(\cdot), u(0))$ is surjective. This follows from Proposition 3.2 applied to $\bar{A}_0 : [0,T] \to \mathcal{L}(V,V^*)$ given by $\bar{A}_0(t)v := A_0 v = \theta(\cdot, v)_V$.

All requirements for the method of continuity are fulfilled and we conclude that L_1 is surjective, giving existence of strong solutions. The a priori estimate (3.4) now follows from Lemma 3.3 and proves uniqueness of strong solutions at once, since $\bar{A}(t)$ is linear. □

As promised, a mere application of Theorem 3.4 now gives us the desired well-posedness and maximal regularity estimate for (3.1).

Corollary 3.5. *Let A_0 and B_0 satisfy the conditions concerning A_0, B_0 in Assumption 2.2 and let $\psi \in L^2(0,T;U)$. Let $T > 0$ and $w \in L^\infty(0,T;H)$. Define $\bar{A} : [0,T] \to \mathcal{L}(V,V^*)$ by $\bar{A}(t)v := A_0(t,w(t))v - B_0(t,w(t))v\psi(t)$. Then $\bar{A}$ satisfies all conditions of Theorem 3.4. Consequently, for any $\bar{f} \in L^2(0,T;V^*)$ and $\bar{g} \in L^2(0,T;\mathcal{L}_2(U,H))$, there exists a unique strong solution $u \in \mathrm{MR}(0,T)$ to*

$$\begin{cases} u'(t) + A_0(t,w(t))u(t) - B_0(t,w(t))u(t)\psi(t) = \bar{f}(t) + \bar{g}(t)\psi(t), \\ u(0) = x, \end{cases}$$

Moreover, for any $\tilde{T} \in [0,T]$ there exists a constant $K_{\tilde{T}} > 0$ such that

$$\|u\|_{\mathrm{MR}(0,\tilde{T})} \leq K_{\tilde{T}}\Big(\|x\|_H + \|\bar{f}\|_{L^2(0,\tilde{T};V^*)} + \|\bar{g}\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))}\Big), \tag{3.8}$$

and $K_{\tilde{T}}$ is non-decreasing in $\tilde{T}$ and depends further only on T, $\|w\|_{L^\infty(0,T;H)}$ and $\|\psi\|_{L^2(0,\tilde{T};U)}$.

Proof. Put $n := \|w\|_{L^\infty(0,T;H)}$. Since strong solutions only depend on $\bar{A}$ through an integral, we can fix a strongly measurable, pointwise defined measurable version of w which satisfies $\|w(t)\|_H \leq n$ for all $t \in [0,T]$. Strong measurability of $\bar{A}(\cdot)u(\cdot)$ is then satisfied if $u \in \mathrm{MR}(0,T)$, see Remark 3.6. Moreover, Assumption 2.2(3) gives for all $u \in \mathrm{MR}(0,T)$:

$$\|A_0(\cdot,w(\cdot))u(\cdot)\|_{L^2(0,T;V^*)} \leq C_{n,T}\|u\|_{L^2(0,T;V)} < \infty$$

and by the Cauchy–Schwarz inequality,

$$\|B_0(\cdot,w(\cdot))u(\cdot)\psi(\cdot)\|_{L^1(0,T;H)} \leq \|B_0(\cdot,w(\cdot))u(\cdot)\|_{L^2(0,T;\mathcal{L}_2(U,H))}\|\psi\|_{L^2(0,T;U)} \leq C_{n,T}\|u\|_{L^2(0,T;V)}\|\psi\|_{L^2(0,T;U)} < \infty.$$

Hence, $\bar{A}(\cdot)u(\cdot) \in S$ and we have

$$\|\bar{A}(\cdot)u(\cdot)\|_S \leq \|A_0(\cdot,w(\cdot))u(\cdot)\|_{L^2(0,T;V^*)} + \|B_0(\cdot,w(\cdot))u(\cdot)\psi(\cdot)\|_{L^1(0,T;H)} \leq C_{n,T}(1 + \|\psi\|_{L^2(0,T;U)})\|u\|_{L^2(0,T;V)} \leq \alpha\|u\|_{\mathrm{MR}(0,T)},$$

where $\alpha := C_{n,T}(1 + \|\psi\|_{L^2(0,T;U)})$.

Furthermore, by Assumption 2.2(2), we have for all $v \in V$ and $t \in [0,T]$:

$$\langle \bar{A}(t)v, v\rangle = \langle A_0(t,w(t))v, v\rangle - \langle B_0(t,w(t))\psi(t)v, v\rangle$$
$$\geq \langle A_0(t,w(t))v, v\rangle - \frac{1}{2}\|B_0(t,w(t))v\|_H^2 - \frac{1}{2}\|\psi(t)\|_U^2\|v\|_H^2$$
$$\geq \theta_{n,T}\|v\|_V^2 - (M_{n,T} + \frac{1}{2}\|\psi(t)\|_U^2)\|v\|_H^2,$$

so coercivity (3.3) is satisfied with $\theta := \theta_{n,T}$ and $M(\cdot) := M_{n,T} + \frac{1}{2}\|\psi(\cdot)\|_U^2 \in L^1(0,T)$.

As before by the Cauchy–Schwarz inequality, $h := \bar{f} + \bar{g}\psi \in S$. Now Theorem 3.4 yields existence of a unique strong solution $u \in \mathrm{MR}(0,T)$ to (3.1). Finally, let $\tilde{T} \in (0,T]$ be arbitrary and put $\tilde{S} := L^2(0,\tilde{T};V^*) + L^1(0,\tilde{T};H)$. We have

$$\|h\|_{\tilde{S}} \leq \|\bar{f}\|_{L^2(0,\tilde{T};V^*)} + \|\bar{g}\psi\|_{L^1(0,\tilde{T};H)} \leq \|\bar{f}\|_{L^2(0,\tilde{T};V^*)} + \|\bar{g}\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))}\|\psi\|_{L^2(0,\tilde{T};U)}. \tag{3.9}$$

As $u|_{[0,\tilde{T}]}$ is a strong solution to (3.1) on $[0,\tilde{T}]$, (3.4) and (3.9) yield

$$\|u\|_{\mathrm{MR}(0,\tilde{T})} \leq C_\theta \exp(2\|M\|_{L^1(0,\tilde{T})})\big(\|h\|_{\tilde{S}} + \|x\|_H\big) \leq K_{\tilde{T}}\big(\|x\|_H + \|\bar{f}\|_{L^2(0,\tilde{T};V^*)} + \|\bar{g}\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))}\big),$$

where $K_{\tilde{T}} := C_\theta \exp(2\|M\|_{L^1(0,\tilde{T})})(1 \vee \|\psi\|_{L^2(0,\tilde{T};U)})$ is non-decreasing in $\tilde{T}$. Note that apart from $\tilde{T}$, $K_{\tilde{T}}$ only depends on T, n and $\|\psi\|_{L^2(0,\tilde{T};U)}$, since these determine θ and $\|M\|_{L^1(0,\tilde{T})}$. □

Remark 3.6. In Corollary 3.5, the map $t \mapsto \bar{A}(t)v$ is strongly Borel measurable for any $v \in V$, and even more is true. Assumption 2.2 assures that we have strong Borel measurability of $A_0(\cdot, w(\cdot))u(\cdot), F(\cdot, u(\cdot)) : [0, T] \to V^*$ and $B_0(\cdot, w(\cdot))u(\cdot), G(\cdot, u(\cdot)) : [0, T] \to \mathcal{L}_2(U, H)$, for any $u \in L^0(0, T; V)$ and $w \in L^0(0, T; H)$. This follows from strong measurability of $u : [0, T] \to V$ and $w : [0, T] \to H$ and the fact that by Assumption 2.2(3), $F(t, \cdot), G(t, \cdot)$ are continuous on V and $A_0(t, \cdot)\cdot, B_0(t, \cdot)\cdot$ are continuous on $H \times V$. Moreover, one uses the measurability of Assumption 2.2(1), separability of $V, H, V^*, \mathcal{L}_2(U, H)$ and continuity of $V \hookrightarrow H \hookrightarrow V^*$.

3.2. Local well-posedness

From now on, we let $\psi \in L^2(0, T; U)$ be arbitrary but fixed. Using Corollary 3.5, we will prove local well-posedness of the actual skeleton Eq. (2.8). Local well-posedness is established in Theorem 3.7. Its proof and preparatory lemma's are analogous to [22, §18.2], which was inspired by [37] and [38].

The skeleton equation does not fit in the setting of [22,37] or [38], for the reason that we only have L^1-(instead of L^2-)integrability of the term $B(\cdot, u^\psi(\cdot))\psi(\cdot)$ in (2.8). Besides that, our maximal regularity space $\mathrm{MR}(0, T) = C([0, T]; H) \cap L^2(0, T; V)$ is different.

When no confusion can arise, we omit the time input in our notations for brevity. For example, for $u, v \in \mathrm{MR}(0, T)$ we denote by $A_0(u)v$ and $B_0(u)v\psi$ the maps $t \mapsto A_0(t, u(t))v(t)$ and $t \mapsto B_0(t, u_0)v(t)\psi(t)$ respectively and similarly for $F(u)$ and $G(u)\psi$. We define the following V^*-valued mappings:

$$\tilde{A}(u)v := A_0(u)v - B_0(u)v\psi, \qquad \tilde{F}(u) = F(u) + f + (G(u) + g)\psi.$$

Theorem 3.7 (Local well-posedness of the skeleton equation). *Suppose that (A, B) satisfies Assumption 2.2. Let $u_0 \in H$ be fixed. Then there exist $\tilde{T}, \epsilon > 0$ such that for each $v_0 \in B_H(u_0, \epsilon)$, there exists a unique strong solution $u_{v_0} \in \mathrm{MR}(0, \tilde{T})$ to*

$$\begin{cases} u' + \tilde{A}(u)u = \tilde{F}(u) & \text{on } [0, \tilde{T}] \\ u(0) = v_0. \end{cases} \tag{3.10}$$

Moreover, there exists a constant $C > 0$ such that for all $v_0, w_0 \in B_H(u_0, \epsilon)$:

$$\|u_{v_0} - u_{w_0}\|_{\mathrm{MR}(0, \tilde{T})} \leq C\|v_0 - w_0\|_H. \tag{3.11}$$

Theorem 3.7 will be proved using the Banach fixed point theorem, applied to the map $\Psi_{v_0} : \mathrm{MR}(0, \tilde{T}) \to \mathrm{MR}(0, \tilde{T})$ defined by $\Psi_{v_0}(v) := u$, where u is the unique strong solution to

$$\begin{cases} u' + \tilde{A}(u_0)u = (\tilde{A}(u_0) - \tilde{A}(v))v + \tilde{F}(v) & \text{on } [0, \tilde{T}], \\ u(0) = v_0. \end{cases} \tag{3.12}$$

Note that $u \in \mathrm{MR}(0, \tilde{T})$ is a strong solution to (3.10) if and only if $\Psi_{v_0}(u) = u$.

Our first task is to prove that Ψ_{v_0} is well-defined, i.e. (3.12) is well-posed. By Corollary 3.5 ($w(t) := u_0$) it suffices to show that

$$\tilde{f} := (A_0(u_0) - A_0(v))v + F(v) + f, \quad \tilde{g} := (B_0(u_0) - B_0(v))v + G(v) + g \tag{3.13}$$

satisfy $\tilde{f} \in L^2(0, \tilde{T}; V^*)$ and $\tilde{g} \in L^2(0, \tilde{T}; \mathcal{L}_2(U, H))$. The latter will be ascertained by the following lemma, which will also be used later on in Section 4.

Lemma 3.8. *Let $\rho_j \geq 0, \beta_j \in (\frac{1}{2}, 1)$ be such that $(2\beta_j - 1)(\rho_j + 1) \leq 1$. Let $V_{\beta_j} = [V^*, V]_{\beta_j}$ be the complex interpolation space with norm $\|\cdot\|_{\beta_j} := \|\cdot\|_{V_{\beta_j}}$. Then, for any $T > 0$:*

(i) $\iota_{j,T} : \mathrm{MR}(0, T) \hookrightarrow L^{2(\rho_j+1)}(0, T; V_{\beta_j})$. The embedding satisfies $\|\iota_{j,T}\| \leq M_T^j$ with $M_T^j \in \mathbb{R}_+$ non-decreasing in T. Suppose that (A, B) satisfies Assumption 2.2. Let $n \in \mathbb{R}_+$ and $T > 0$. For $C_{n,T}$ the constant from Assumption 2.2(3) (non-decreasing in n and T), it holds that

(ii) for all $u \in C([0, T]; H), w \in L^2(0, T; V)$ with $\|u\|_{C([0,T];H)} \leq n$:

$$\|A_0(u)w\|_{L^2(0,T;V^*)} \vee \|B_0(u)w\|_{L^2(0,T;\mathcal{L}_2(U,H))} \leq C_{n,T}\|w\|_{L^2(0,T;V)},$$

(iii) for all $u, v \in C([0, T]; H), w \in L^2(0, T; V)$ with $\|u\|_{C([0,T];H)}, \|v\|_{C([0,T];H)} \leq n$:

$$\|(A_0(u) - A_0(v))w\|_{L^2(0,T;V^*)} \vee \|(B_0(u) - B_0(v))w\|_{L^2(0,T;\mathcal{L}_2(U,H))} \leq C_{n,T}\left(\int_0^T \|u(s) - v(s)\|_H^2 \|w(s)\|_V^2 \, ds\right)^{\frac{1}{2}}$$

$$\leq C_{n,T}\|u - v\|_{C([0,T];H)}\|w\|_{L^2(0,T;V)}.$$

Moreover, there exists a constant $\tilde{C}_{n,T}$ non-decreasing in T such that

(iv) for all $u \in \mathrm{MR}(0, T)$ with $\|u\|_{C([0,T];H)} \leq n$:

$$\|F(u)\|_{L^2(0,T;V^*)} \vee \|G(u)\|_{L^2(0,T;\mathcal{L}_2(U,H))} \leq \tilde{C}_{n,T}(1 + \|u\|_{L^2(0,T;V)}).$$

Lastly, for each $\sigma > 0$ there exists a constant $C_{n,T,\sigma}$ non-decreasing in T such that

(v) for all $u, v \in \mathrm{MR}(0, T)$ with $\|u\|_{C([0,T];H)}, \|v\|_{C([0,T];H)} \leq n$:

$$\|F(u) - F(v)\|_{L^2(0,T;V^*)}^2 \vee \|G(u) - G(v)\|_{L^2(0,T;\mathcal{L}_2(U,H))}^2$$

$$\leq C_{n,T,\sigma}\int_0^t (1 + \|u(s)\|_V^2 + \|v(s)\|_V^2)\|u(s) - v(s)\|_H^2 \, ds + \sigma C_{n,T}^2\|u - v\|_{L^2(0,T;V)}^2.$$

Proof. (i): By the interpolation estimate (2.3), we have for any $u \in \mathrm{MR}(0, T)$:

$$\int_0^T \|u(t)\|_{\beta_j}^{2(\rho_j + 1)} \, dt \leq K \int_0^T \|u(t)\|_H^{2(\rho_j+1)(2-2\beta_j)} \|u(t)\|_V^{2(\rho_j+1)(2\beta_j-1)} \, dt$$

$$\leq K \|u\|_{C([0,T];H)}^{2(\rho_j+1)(2-2\beta_j)} \int_0^T \|u(t)\|_V^{2(\rho_j+1)(2\beta_j-1)} \, dt$$

$$\leq K \|u\|_{C([0,T];H)}^{2(\rho_j+1)(2-2\beta_j)} \|1\|_{L^{p_j'}(0,T)} \|\|u\|_V^{2(\rho_j+1)(2\beta_j-1)}\|_{L^{p_j}(0,T)}$$

$$\leq K \|u\|_{C([0,T];H)}^{2(\rho_j+1)(2-2\beta_j)} (1 \vee T)^{\frac{p_j-1}{p_j}} \|u\|_{L^2(0,T;V)}^{2(\rho_j+1)(2\beta_j-1)},$$

where we applied Hölder's inequality for each j with $p_j := \frac{1}{(\rho_j+1)(2\beta_j-1)} \in [1, \infty)$, $p_j' := \frac{p_j}{p_j-1} \in [1, \infty]$ and included the maximum with 1 to cover the case $p_j' = \infty$. We conclude that

$$\|u\|_{L^{2(\rho_j+1)}(0,T;V_{\beta_j})} \leq M_T^j \|u\|_{C([0,T];H)}^{(2-2\beta_j)} \|u\|_{L^2(0,T;V)}^{(2\beta_j-1)} \leq M_T^j \big((2-2\beta_j)\|u\|_{C([0,T];H)} + (2\beta_j-1)\|u\|_{L^2(0,T;V)}\big) \leq M_T^j \|u\|_{\mathrm{MR}(0,T)},$$

where $M_T^j \in \mathbb{R}_+$ is non-decreasing in T. We used Young's inequality and the fact that $\beta_j \in (\frac{1}{2}, 1)$.

In (ii)–(v), note that strong measurability of $A_0(\cdot, u(\cdot))w(\cdot)$, $F(\cdot, u(\cdot))$, $B_0(\cdot, u(\cdot))w(\cdot)$ and $G(\cdot, u(\cdot))$ holds, as was mentioned in Remark 3.6. Moreover, by symmetry in Assumption 2.2, B_0 and G can be estimated in the same way as A_0 and F. We provide the estimates for the latter.

Assumption 2.2(3) immediately yields (ii) and (iii).

For (iv): by Assumption 2.2(3) and (2.3), we have pointwise in $t \in [0, T]$:

$$\|F(u)\|_{V^*} \leq C_{n,T} \sum_{j=1}^{m_F} (1 + \|u\|_{\beta_j}^{\rho_j+1}) \leq C_{n,T} \sum_{j=1}^{m_F} (1 + (Kn^{2-2\beta_j})^{\rho_j+1} \|u\|_V^{(2\beta_j-1)(\rho_j+1)}) \leq C_{n,T} \sum_{j=1}^{m_F} (1 + C_n(1 + \|u\|_V)) \leq \bar{C}_{n,T}(1 + \|u\|_V),$$

where we used that $(2\beta_j - 1)(\rho_j + 1) \leq 1$ and put $C_n := \max_{j=1,\dots,m_F} (Kn^{2-2\beta_j})^{\rho_j+1} < \infty$ and $\bar{C}_{n,T} := m_F C_{n,T}(1 + C_n)$. Thus

$$\|F(u)\|_{L^2(0,T;V^*)} \leq \bar{C}_{n,T}(T^{\frac{1}{2}} + \|u\|_{L^2(0,T;V)}) \leq \tilde{C}_{n,T}(1 + \|u\|_{L^2(0,T;V)}),$$

with $\tilde{C}_{n,T} = \bar{C}_{n,T}(T^{\frac{1}{2}} \vee 1)$. Since $C_{n,T}$ is non-decreasing in T, the same holds for $\bar{C}_{n,T}$ and $\tilde{C}_{n,T}$.

For (v): the following estimates can be found in the proof of [3, Prop. 4.5]. By Assumption 2.2(3) we have pointwise in $t \in [0, T]$:

$$\|F(u) - F(v)\|_{V^*} \leq C_{n,T} \sum_{j=1}^{m_F} \left(1 + \|u\|_{\beta_j}^{\rho_j} + \|v\|_{\beta_j}^{\rho_j}\right) \|u - v\|_{\beta_j}. \tag{3.14}$$

By the interpolation estimate (2.3) and Young's inequality (with powers $\frac{1}{2-2\beta}$ and $\frac{1}{2\beta-1}$), we have for all $y, z \in V$, $\beta \in (\frac{1}{2}, 1)$, $\rho \geq 0$ with $(2\beta-1)(\rho+1) \leq 1$ and for all $\sigma > 0$:

$$\|y\|_\beta^\rho \|z\|_\beta \leq \left(K^{\rho+1} \|y\|_H^{(2-2\beta)\rho} \|y\|_V^{(2\beta-1)\rho} \|z\|_H^{2-2\beta}\right) \|z\|_V^{2\beta-1} \leq \sigma^{-\frac{2\beta-1}{2-2\beta}} (2-2\beta) K^{\frac{\rho+1}{2-2\beta}} \|y\|_H^\rho \|y\|_V^{\frac{(2\beta-1)\rho}{2-2\beta}} \|z\|_H + \sigma(2\beta-1)\|z\|_V$$

$$\leq \sigma^{-\frac{2\beta-1}{2-2\beta}} K^{\frac{\rho+1}{2-2\beta}} \|y\|_H^\rho (1 + \|y\|_V) \|z\|_H + \sigma\|z\|_V$$

$$\leq M_{\sigma,\beta,\rho} \|y\|_H^\rho (1 + \|y\|_V) \|z\|_H + \sigma\|z\|_V, \tag{3.15}$$

where $M_{\sigma,\beta,\rho} > 0$ is a constant depending only on σ, β and ρ and we let $0^0 = 1$. In the above we used that $a := \frac{(2\beta-1)\rho}{2-2\beta} \in [0, 1]$, hence $x^a \leq 1 + x$ for $x \geq 0$. For $j \in \{1, \dots, m_F\}$, application of (3.15) gives pointwise in $t \in [0, T]$:

$$\left(1 + \|u\|_{\beta_j}^{\rho_j} + \|v\|_{\beta_j}^{\rho_j}\right) \|u - v\|_{\beta_j} \leq \left(M_{\sigma,\beta_j,0} + M_{\sigma,\beta_j,\rho_j} \|u\|_H^{\rho_j}(1 + \|u\|_V) + M_{\sigma,\beta_j,\rho_j} \|v\|_H^{\rho_j}(1 + \|v\|_V)\right) \|u - v\|_H + 3\sigma\|u - v\|_V$$

$$\leq M_\sigma(1 + \|u\|_V + \|v\|_V) \|u - v\|_H + 3\sigma\|u - v\|_V, \tag{3.16}$$

with $M_\sigma := \max_{j=1,\dots,m_F}(M_{\sigma,\beta_j,0} + 2M_{\sigma,\beta_j,\rho_j} N^{\rho_j}) < \infty$. Now, (3.14) and (3.16) imply

$$\|F(u) - F(v)\|_{V^*} \leq C_{n,T} m_F M_\sigma(1 + \|u\|_V + \|v\|_V) \|u - v\|_H + 3\sigma C_{n,T} m_F \|u - v\|_V$$

and hence, applying $(x_1 + \dots + x_d)^2 \leq d(x_1^2 + \dots + x_d^2)$ with $d = 2, 3$:

$$\|F(u) - F(v)\|_{L^2(0,T;V^*)}^2 \leq \bar{C}_{n,T,\sigma} \int_0^T (1 + \|u(t)\|_V^2 + \|v(t)\|_V^2) \|u(t) - v(t)\|_H^2 \, dt + 2(3\sigma C_{n,T} m_F)^2 \int_0^T \|u(t) - v(t)\|_V^2 \, dt,$$

with $\bar{C}_{N,T,\sigma} = 6(C_{n,T} m_F M_\sigma)^2$. Since $C_{n,T}$ is non-decreasing in T, the same holds for $\bar{C}_{n,T,\sigma}$. Substituting $\sigma = 18\bar{\sigma}^2 m_F^2$, $C_{n,T,\sigma} := \bar{C}_{n,T,\bar{\sigma}}$ now yields (v). $\square$

Remark 3.9. Lemma 3.8 yields $A(\cdot, u(\cdot)) \in L^2(0, T; V^*)$ and $B(\cdot, u(\cdot)) \in L^2(0, T; \mathcal{L}_2(U, H))$ a.s. if $u \in \mathrm{MR}(0, T)$ a.s. Hence, under Assumption 2.2, this condition is redundant in the definition of a strong solution (Definition 2.1).

From Lemma 3.8, we see that $\tilde{f}$ and $\tilde{g}$ defined by (3.13) lie in $L^2(0, \tilde{T}; V^*)$ and $L^2(0, \tilde{T}; \mathcal{L}_2(U, H))$ respectively, for any $\tilde{T} > 0$ and $v \in \text{MR}(0, \tilde{T})$ (put $n = \|u_0\|_H \vee \|v\|_{C([0,\tilde{T}];H)}$ and apply (iii) and (v)). Thus Corollary 3.5 gives that (3.12) is well-posed, i.e. Ψ_{v_0} is well-defined.

Our next concern is to prove that Ψ_{v_0} is contractive on a suitable smaller subspace of $\text{MR}(0, T)$. To define this subspace, let us introduce some notations. For what follows, we fix an arbitrary $u_0 \in H$ and $T > 0$. For $v_0 \in H$, we let $z_{v_0} \in \text{MR}(0, T)$ be the *reference solution*, defined as the unique strong solution to the linear problem

$$\begin{cases} z' + \tilde{A}(u_0)z = 0 & \text{on } [0, T], \\ z(0) = v_0. \end{cases} \tag{3.17}$$

Well-posedness holds by Corollary 3.5. Note that $z_{u_0}(0) = u_0$ and $z_{u_0} \in \text{MR}(0, T)$, so there exists a $T_1 \in (0, T]$ such that

$$\|z_{u_0} - u_0\|_{C([0,T_1];H)} \leq \frac{1}{3}. \tag{3.18}$$

We fix such a T_1. Finally, for $v_0 \in H$, $r > 0$ and $\tilde{T} \in [0, T]$, we define

$$Z_{r,\tilde{T}}(v_0) := \{v \in \text{MR}(0, \tilde{T}) : v(0) = v_0, \|v - z_{u_0}\|_{\text{MR}(0,\tilde{T})} \leq r\}. \tag{3.19}$$

Note that $Z_{r,\tilde{T}}(v_0)$ is closed in $\text{MR}(0, \tilde{T})$, hence complete. Eventually, we will find that Ψ_{v_0} is contractive on some $Z_{r,\tilde{T}}(v_0)$. Several crucial estimates will be gathered in the next lemma's.

Lemma 3.10. *There exist $\varepsilon_1, r_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$, $r \in (0, r_1]$, $\tilde{T} \in (0, T_1]$, $v_0 \in B_H(u_0, \varepsilon)$ and $v \in Z_{r,\tilde{T}}(v_0)$ it holds that $\|v - u_0\|_{C([0,\tilde{T}];H)} \leq 1$.*

Proof. Let $\varepsilon, r > 0$, $\tilde{T} \in (0, T_1]$ and let $v \in Z_{r,\tilde{T}}(v_0)$. We have

$$\|v - z_{v_0}\|_{\text{MR}(0,\tilde{T})} \leq \|v - z_{u_0}\|_{\text{MR}(0,\tilde{T})} + \|z_{u_0} - z_{v_0}\|_{\text{MR}(0,T_1)} = \|v - z_{u_0}\|_{\text{MR}(0,\tilde{T})} + \|z_{u_0 - v_0}\|_{\text{MR}(0,T_1)} \leq r + K_{T_1}\|u_0 - v_0\|_H,$$

where the last inequality follows from the definition of $Z_{r,\tilde{T}}(v_0)$ and (3.8). Therefore,

$$\|v - u_0\|_{C([0,\tilde{T}];H)} \leq \|v - z_{v_0}\|_{\text{MR}(0,\tilde{T})} + \|z_{v_0} - z_{u_0}\|_{\text{MR}(0,T_1)} + \|z_{u_0} - u_0\|_{C([0,T_1];H)} \leq \left(r + K_{T_1}\|u_0 - v_0\|_H\right) + K_{T_1}\|u_0 - v_0\|_H + \frac{1}{3}$$

$$\leq r + 2K_{T_1}\varepsilon + \frac{1}{3},$$

whenever $v_0 \in B_H(u_0, \varepsilon)$. Taking $r_1 = \frac{1}{3}$ and $\varepsilon_1 = (6K_{T_1})^{-1}$, the claim is proved. $\square$

The next lemma is analogous to [22, Lem. 18.2.10].

Lemma 3.11. *Let $u_0 \in H$ and suppose that (A, B) satisfies Assumption 2.2. Let $\tilde{f} \in L^2(0, \tilde{T}; V^*)$ and $\tilde{g} \in L^2(0, \tilde{T}; \mathcal{L}_2(U, H))$ be defined by (3.13). For ε_1 and r_1 from Lemma 3.10, the following estimates hold for any $\tilde{T} \in (0, T_1]$, $\varepsilon \in (0, \varepsilon_1]$, $r \in (0, r_1]$, $v_0 \in B_H(u_0, \varepsilon)$, $v \in Z_{r,\tilde{T}}(v_0)$ and $\sigma > 0$:*

$$\|\tilde{f}\|_{L^2(0,\tilde{T};V^*)} \vee \|\tilde{g}\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))} \leq \alpha_{T_1}(\tilde{T}) + \beta_{T_1,\sigma}(\tilde{T}, r)r + \sigma r, \tag{3.20}$$

where $\alpha_{T_1}(\tilde{T}), \beta_{T_1,\sigma}(\tilde{T}, r) \downarrow 0$ as $\tilde{T}, r \downarrow 0$ and $\alpha_{T_1}(\tilde{T})$ and $\beta_{T_1,\sigma}(\tilde{T}, r)$ are independent of v_0 and v.

Proof. Let $v_0 \in B_H(u_0, \varepsilon)$, $\tilde{T} \in (0, T_1]$ and $v \in Z_{r,\tilde{T}}(v_0)$ be arbitrary. We estimate each term appearing in the definition of $\tilde{f}$. By Lemma 3.10,

$$\|v\|_{C([0,\tilde{T}];H)} \leq \|v - u_0\|_{C([0,T_1];H)} + \|u_0\|_H \leq \|u_0\|_H + 1. \tag{3.21}$$

Putting $C_{T_1} := C_{\|u_0\|_H+1,T_1}$, Lemma 3.8(iii) gives

$$\|A_0(u_0)v - A_0(v)v\|_{L^2(0,\tilde{T};V^*)} \leq C_{T_1}\|u_0 - v\|_{C([0,\tilde{T}];H)}\|v\|_{L^2(0,\tilde{T};V)}$$

$$\leq C_{T_1}\left(\|u_0 - z_{u_0}\|_{C([0,\tilde{T}];H)} + \|z_{u_0} - v\|_{C([0,\tilde{T}];H)}\right)\left(\|v - z_{u_0}\|_{L^2(0,\tilde{T};V)} + \|z_{u_0}\|_{L^2(0,\tilde{T};V)}\right)$$

$$\leq C_{T_1}(\alpha(\tilde{T}) + r)^2 \leq 2C_{T_1}(\alpha(\tilde{T})^2 + r^2) \tag{3.22}$$

with

$$\alpha(\tilde{T}) := \|u_0 - z_{u_0}\|_{C([0,\tilde{T}];H)} \vee \|z_{u_0}\|_{L^2(0,\tilde{T};V)}.$$

Note that $\alpha(\tilde{T}) \downarrow 0$ as $\tilde{T} \downarrow 0$, since $z_{u_0} \in C([0, T_1]; H) \cap L^2(0, T_1; V)$ and $z_{u_0}(0) = u_0$.

We turn to the term $F(v)$ appearing in $\tilde{f}$. By (3.18),

$$\|z_{u_0}\|_{C([0,\tilde{T}];H)} \leq \|z_{u_0} - u_0\|_{C([0,T_1];H)} + \|u_0\|_H < 1 + \|u_0\|_H. \tag{3.23}$$

Now we apply Lemma 3.8(v) with $\tilde{\sigma} := \sigma^2 C_{\|u_0\|_H+1,T_1}^{-2}$ and let $\tilde{C}_{T_1,\sigma} := C_{\|u_0\|_H+1,T_1,\tilde{\sigma}}$ denote the constant of Lemma 3.8(v) corresponding to $\tilde{\sigma}$. Recalling (3.23) and (3.21), we obtain

$$\|F(v)\|_{L^2(0,\tilde{T};V^*)} \leq \|F(v) - F(z_{u_0})\|_{L^2(0,\tilde{T};V^*)} + \|F(z_{u_0})\|_{L^2(0,\tilde{T};V^*)}$$

$$\leq \left(\tilde{C}_{T_1,\sigma} \int_0^{\tilde{T}} (1 + \|v\|_V^2 + \|z_{u_0}\|_V^2) \|v - z_{u_0}\|_H^2 \, \mathrm{d}s \right)^{\frac{1}{2}} + \sigma \|v - z_{u_0}\|_{L^2(0,\tilde{T};V)} + \|F(z_{u_0})\|_{L^2(0,\tilde{T},V^*)}$$

$$\leq \left(\tilde{C}_{T_1,\sigma} \int_0^{\tilde{T}} (1 + \|v\|_V^2 + \|z_{u_0}\|_V^2) r^2 \, \mathrm{d}s \right)^{\frac{1}{2}} + \sigma r + \|F(z_{u_0})\|_{L^2(0,\tilde{T},V^*)}$$

$$\leq r \tilde{C}_{T_1,\sigma}^{\frac{1}{2}} (\tilde{T}^{\frac{1}{2}} + \|v\|_{L^2(0,\tilde{T};V)} + \|z_{u_0}\|_{L^2(0,\tilde{T};V)}) + \sigma r + \|F(z_{u_0})\|_{L^2(0,\tilde{T},V^*)}$$

$$\leq r \tilde{C}_{T_1,\sigma}^{\frac{1}{2}} (\tilde{T}^{\frac{1}{2}} + \|v - z_{u_0}\|_{L^2(0,\tilde{T};V)} + 2\|z_{u_0}\|_{L^2(0,\tilde{T};V)}) + \sigma r + \|F(z_{u_0})\|_{L^2(0,\tilde{T},V^*)}$$

$$\leq r \tilde{C}_{T_1,\sigma}^{\frac{1}{2}} (\tilde{T}^{\frac{1}{2}} + r + 2\|z_{u_0}\|_{L^2(0,\tilde{T};V)}) + \sigma r + \|F(z_{u_0})\|_{L^2(0,\tilde{T},V^*)}.$$

It follows that

$$\|F(v)\|_{L^2(0,\tilde{T};V^*)} + \|f\|_{L^2(0,\tilde{T};V^*)} \leq \tilde{\beta}_{T_1,\sigma}(\tilde{T},r)r + \sigma r + \gamma(\tilde{T}), \tag{3.24}$$

with

$$\tilde{\beta}_{T_1,\sigma}(\tilde{T},r) := \tilde{C}_{T_1,\sigma}^{\frac{1}{2}} (\tilde{T}^{\frac{1}{2}} + r + 2\|z_{u_0}\|_{L^2(0,\tilde{T};V)}), \quad \gamma(\tilde{T}) := (\|f\|_{L^2(0,\tilde{T};V^*)} \vee \|g\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))}) + \|F(z_{u_0})\|_{L^2(0,\tilde{T},V^*)}.$$

Recall that $z_{u_0} \in \mathrm{MR}(0,T_1) \subset L^2(0,T_1;V)$ and by Lemma 3.8(iv), $F(z_{u_0}) \in L^2(0,T_1;V^*)$. So $\tilde{\beta}_{T_1,\sigma}(\tilde{T},r) \downarrow 0$ as $\tilde{T}, r \downarrow 0$ and $\gamma(\tilde{T}) \downarrow 0$ as $\tilde{T} \downarrow 0$ by the Dominated Convergence Theorem. Combining (3.24) and (3.22) and putting

$$\beta_{T_1,\sigma}(\tilde{T},r) := \tilde{\beta}_{T_1,\sigma}(\tilde{T},r) + 2C_{T_1}r, \quad \alpha_{T_1}(\tilde{T}) := 2C_{T_1}\alpha(\tilde{T})^2 + \gamma(\tilde{T}),$$

proves (3.20) for $\tilde{f}$. By symmetry in Lemma 3.8, the estimate for $\tilde{g}$ follows similarly. $\quad\square$

Before we prove Theorem 3.7, we need one more lemma, a modification of [22, Lemma 18.2.12].

Lemma 3.12. *Let $u_0 \in H$ and suppose that (A, B) satisfies Assumption 2.2. For ϵ_1 and r_1 from Lemma 3.10, the following estimates hold for any $\tilde{T} \in (0, T_1]$, $\epsilon \in (0, \epsilon_1]$, $r \in (0, r_1]$, $v_0, w_0 \in B_H(u_0, \epsilon)$, $v \in Z_{r,\tilde{T}}(v_0)$, $w \in Z_{r,\tilde{T}}(w_0)$, $u \in \mathrm{MR}(0, \tilde{T})$ and $\sigma > 0$:*

$$\|(A_0(v) - A_0(w))v\|_{L^2(0,\tilde{T};V^*)} \vee \|(B_0(v) - B_0(w))v\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))} \leq c_{T_1}(r + \alpha(\tilde{T}))\|v - w\|_{\mathrm{MR}(0,\tilde{T})},$$

$$\|(A_0(u_0) - A_0(w))u\|_{L^2(0,\tilde{T};V^*)} \vee \|(B_0(u_0) - B_0(w))u\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))} \leq c_{T_1}(r + \beta(\tilde{T}))\|u\|_{\mathrm{MR}(0,\tilde{T})},$$

$$\|F(v) - F(w)\|_{L^2(0,\tilde{T};V^*)} \vee \|G(v) - G(w)\|_{L^2(0,\tilde{T};\mathcal{L}_2(U,H))} \leq (\gamma_{T_1,\sigma}(\tilde{T},r) + \sigma)\|v - w\|_{\mathrm{MR}(0,\tilde{T})},$$

where c_{T_1} is a constant and $\alpha(\tilde{T}), \beta(\tilde{T}), \gamma_{T_1,\sigma}(\tilde{T},r) \downarrow 0$ as $\tilde{T}, r \downarrow 0$. Moreover, $c_{T_1}, \alpha(\tilde{T}), \beta(\tilde{T})$ and $\gamma_{T_1,\sigma}(\tilde{T},r)$ are independent of v_0, w_0, v and w.

Proof. Fix $n := 2\|u_0\|_H + 2$, $c_{T_1} := C_{n,T_1}$ and note that $\|v\|_{C([0,\tilde{T}];H)} + \|w\|_{C([0,\tilde{T}];H)} \leq n$ by Lemma 3.10. By Lemma 3.8(iii), we have

$$\|(A_0(v) - A_0(w))v\|_{L^2(0,\tilde{T};V^*)} \leq c_{T_1}\|v - w\|_{C([0,\tilde{T}];H)}\|v\|_{L^2(0,\tilde{T};V)} \leq c_{T_1}\|v - w\|_{\mathrm{MR}(0,\tilde{T})}(\|v - z_{u_0}\|_{\mathrm{MR}(0,\tilde{T})} + \|z_{u_0}\|_{L^2(0,\tilde{T};V)})$$

$$\leq c_{T_1}\|v - w\|_{\mathrm{MR}(0,\tilde{T})}(r + \alpha(\tilde{T})),$$

where $\alpha(\tilde{T}) := \|z_{u_0}\|_{L^2(0,\tilde{T};V)} \downarrow 0$ as $\tilde{T} \downarrow 0$. Similarly,

$$\|(A_0(u_0) - A_0(w))u\|_{L^2(0,\tilde{T};V^*)} \leq c_{T_1}\|u_0 - w\|_{C([0,\tilde{T}];H)}\|u\|_{L^2(0,\tilde{T};V)} \leq c_{T_1}(\|u_0 - z_{u_0}\|_{C([0,\tilde{T}];H)} + \|z_{u_0} - w\|_{\mathrm{MR}(0,\tilde{T})})\|u\|_{\mathrm{MR}(0,\tilde{T})}$$

$$\leq c_{T_1}(\beta(\tilde{T}) + r)\|u\|_{\mathrm{MR}(0,\tilde{T})},$$

where $\beta(\tilde{T}) := \|u_0 - z_{u_0}\|_{C([0,\tilde{T}];H)} \downarrow 0$ as $\tilde{T} \downarrow 0$ since $z_{u_0}(0) = u_0$.

Now we turn to F. By Lemma 3.8(v), we have for any $\tilde{\sigma} > 0$:

$$\|F(v) - F(w)\|_{L^2(0,\tilde{T};V^*)}^2 \leq C_{n,T_1,\tilde{\sigma}}\|v - w\|_{\mathrm{MR}(0,\tilde{T})}^2 \int_0^{\tilde{T}} 1 + \|v\|_V^2 + \|w\|_V^2 \, \mathrm{d}t + \tilde{\sigma}C_{n,T_1}^2\|v - w\|_{\mathrm{MR}(0,\tilde{T})}^2.$$

Moreover,

$$\|v\|_{L^2(0,\tilde{T};V^*)} \leq \|v - z_{u_0}\|_{L^2(0,\tilde{T};V^*)} + \|z_{u_0}\|_{L^2(0,\tilde{T};V^*)} \leq r + \|z_{u_0}\|_{L^2(0,\tilde{T};V^*)}$$

and similarly for w. Applying the above with $\tilde{\sigma} := \sigma^2 C_{n,T_1}^{-2}$, putting $\tilde{C}_{T_1,\sigma} := C_{n,T_1,\tilde{\sigma}}$ and taking square roots, we find

$$\|F(v) - F(w)\|_{L^2(0,\tilde{T};V^*)} \leq \tilde{C}_{T_1,\sigma}^{\frac{1}{2}}\|v - w\|_{\mathrm{MR}(0,\tilde{T})}(\tilde{T}^{\frac{1}{2}} + 2r + 2\|z_{u_0}\|_{L^2(0,\tilde{T};V^*)}) + \sigma\|v - w\|_{\mathrm{MR}(0,\tilde{T})}.$$

The desired estimate thus holds with $\gamma_{T_1,\sigma}(\tilde{T},r) := \tilde{C}_{T_1,\sigma}^{\frac{1}{2}}(\tilde{T}^{\frac{1}{2}} + 2r + 2\|z_{u_0}\|_{L^2(0,\tilde{T};V^*)})$.

By symmetry in Assumption 2.2(3), B_0 and G can be estimated similarly. $\quad\square$

We are now ready to prove Theorem 3.7. The proof is adapted from [22, Th. 18.2.6].

Proof of Theorem 3.7. Let $\epsilon_1, r_1 > 0$ be as in Lemma 3.10 and let $\tilde{T} \in (0, T_1]$, $\epsilon \in (0, \epsilon_1]$, $r \in (0, r_1]$. As above, define $\Psi_{v_0} : \mathrm{MR}(0, \tilde{T}) \to \mathrm{MR}(0, \tilde{T})$ by $\Psi_{v_0}(v) := u$, where u is the unique strong solution to (3.12). Recall that u solves (3.10) if and only if $\Psi_{v_0}(u) = u$ and recall

that $Z_{r,\bar{T}}(v_0)$ defined by (3.19) is closed in MR$(0,\bar{T})$, hence complete. We show that for $\bar{T}, \varepsilon, r$ small enough, the mapping Ψ_{v_0} maps $Z_{r,\bar{T}}(v_0)$ to itself and is contractive. The Banach fixed point theorem then gives existence of a unique fixed point in $Z_{r,\bar{T}}(v_0)$, hence existence of a solution to (3.10). We will extend the uniqueness within $Z_{r,\bar{T}}(v_0)$ to uniqueness in MR$(0,\bar{T})$.

Let $v \in Z_{r,\bar{T}}(v_0)$ and let $u := \Psi_{v_0}(v)$. Let z_{u_0} be defined as in (3.17) and define $\tilde{f}, \tilde{g}$ by (3.13). Note that $u - z_{u_0} = \Psi_{v_0 - u_0}(v)$, so by (3.8) and (3.20), we have for any $\sigma > 0$:

$$\|u - z_{u_0}\|_{\mathrm{MR}(0,\bar{T})} \leq \|u - z_{u_0}\|_{\mathrm{MR}(0,T_1)} \leq K_{T_1}\left(\|v_0 - u_0\|_H + \|\tilde{f}\|_{L^2(0,T_1;V^*)} + \|\tilde{g}\|_{L^2(0,T_1;\mathcal{L}_2(U,H))}\right)$$
$$\leq K_{T_1}\left(\varepsilon + 2\alpha_{T_1}(\bar{T}) + 2\beta_{T_1,\sigma}(\bar{T},r)r + 2\sigma r\right),$$

with $\alpha_{T_1}(\bar{T}), \beta_{T_1,\sigma}(\bar{T},r) \downarrow 0$ as $\bar{T}, r \downarrow 0$. Recall that K_{T_1} from (3.8) only depends on $T_1, T, \|u_0\|_H$ and ψ, not on v_0 or v. Fixing first $\sigma := (4K_{T_1})^{-1}$, we find

$$\|u - z_{u_0}\|_{\mathrm{MR}(0,\bar{T})} \leq \frac{r}{2} + K_{T_1}\left(\varepsilon + 2\alpha_{T_1}(\bar{T}) + 2\beta_{T_1,\sigma}(\bar{T},r)r\right).$$

For all small enough r and all small enough $\bar{T}, \varepsilon$ (dependent on r), one thus has $\|u - z_{u_0}\|_{\mathrm{MR}(0,\bar{T})} \leq r$, i.e. $\Psi_{v_0}(v) = u \in Z_{r,\bar{T}}(v_0)$. In particular, for all such $r, \bar{T}, \varepsilon$ and for all $v_0 \in B_H(u_0, \varepsilon)$, Ψ_{v_0} maps $Z_{r,\bar{T}}(v_0)$ to itself.

Now we show that for some (even smaller) $r, \bar{T}, \varepsilon > 0$, the map $\Psi_{v_0} : Z_{r,\bar{T}}(v_0) \to Z_{r,\bar{T}}(v_0)$ is contractive for all $v_0 \in B_H(u_0, \varepsilon)$ and we prove continuous dependence on the initial value v_0. Let $v_0, w_0 \in B_H(u_0, \varepsilon)$, $v \in Z_{r,\bar{T}}(v_0)$, $w \in Z_{r,\bar{T}}(w_0)$ and note that $u := \Psi_{v_0}(v) - \Psi_{w_0}(w)$ is a strong solution to

$$\begin{cases} u' + \tilde{A}(u_0)u = (\tilde{A}(u_0) - \tilde{A}(v))v - (\tilde{A}(u_0) - \tilde{A}(w))w + \tilde{F}(v) - \tilde{F}(w) & \text{on } [0,\bar{T}], \\ u(0) = v_0 - w_0. \end{cases}$$

Hence, by (3.8):

$$\|u\|_{\mathrm{MR}(0,\bar{T})} \leq K_{T_1}(\|v_0 - w_0\|_H + \|\bar{f}\|_{L^2(0,\bar{T};V^*)} + \|\bar{g}\|_{L^2(0,\bar{T};\mathcal{L}_2(U,H))}),$$

with $\bar{f} := (A_0(u_0) - A_0(v))v - (A_0(u_0) - A_0(w))w + F(v) - F(w)$ and $\bar{g} := (B_0(u_0) - B_0(v))v - (B_0(u_0) - B_0(w))w + G(v) - G(w)$. We have by Lemma 3.12, for any $\sigma > 0$:

$$\|\bar{f}\|_{L^2(0,\bar{T};V^*)} \leq \|(A_0(v) - A_0(w))v\|_{L^2(0,\bar{T};V^*)} + \|(A_0(u_0) - A_0(w))(v - w)\|_{L^2(0,\bar{T};V^*)} + \|F(v) - F(w)\|_{L^2(0,\bar{T};V^*)}$$
$$\leq \left(c_{T_1}(2r + \alpha(\bar{T}) + \beta(\bar{T})) + \gamma_{T_1,\sigma}(\bar{T},r) + \sigma\right)\|v - w\|_{\mathrm{MR}(0,\bar{T})},$$

with $\alpha(\bar{T}), \beta(\bar{T}), \gamma_{T_1,\sigma}(\bar{T},r) \downarrow 0$ as $\bar{T}, r \downarrow 0$. The same estimate applies to $\|\bar{g}\|_{L^2(0,\bar{T};\mathcal{L}_2(U,H))}$ by symmetry. Fixing $\sigma := (8K_{T_1})^{-1}$ and putting $C(\bar{T}, r) := 2\left(c_{T_1}(2r + \alpha(\bar{T}) + \beta(\bar{T})) + \gamma_{T_1,\sigma}(\bar{T}, r)\right)$, we conclude that

$$\|u\|_{\mathrm{MR}(0,\bar{T})} \leq K_{T_1}\|v_0 - w_0\|_H + \left(K_{T_1}C(\bar{T},r) + \frac{1}{4}\right)\|v - w\|_{\mathrm{MR}(0,\bar{T})},$$

with $C(\bar{T}, r) \downarrow 0$ as $\bar{T}, r \downarrow 0$. For all small enough $r, \bar{T}, \varepsilon$ we thus have $K_{T_1}C(\bar{T}, r) \leq \frac{1}{4}$ and

$$\|\Psi_{v_0}(v) - \Psi_{w_0}(w)\|_{\mathrm{MR}(0,\bar{T})} = \|u\|_{\mathrm{MR}(0,\bar{T})} \leq K_{T_1}\|v_0 - w_0\|_H + \frac{1}{2}\|v - w\|_{\mathrm{MR}(0,\bar{T})}. \tag{3.25}$$

Application to $w_0 = v_0$ shows that $\Psi_{v_0} : Z_{r,\bar{T}}(v_0) \to Z_{r,\bar{T}}(v_0)$ is a strict contraction. A unique fixed point is thus guaranteed by the Banach fixed point theorem. Now let $u_{v_0} \in Z_{r,\bar{T}}(v_0)$ and $u_{w_0} \in Z_{r,\bar{T}}(w_0)$ be fixed points of Ψ_{v_0} and Ψ_{w_0}, respectively. Then (3.25) yields

$$\|u_{v_0} - u_{w_0}\|_{\mathrm{MR}(0,\bar{T})} = \|\Psi_{v_0}(u_{v_0}) - \Psi_{w_0}(u_{w_0})\|_{\mathrm{MR}(0,\bar{T})} \leq K_{T_1}\|v_0 - w_0\|_H + \frac{1}{2}\|u_{v_0} - u_{w_0}\|_{\mathrm{MR}(0,\bar{T})}.$$

Consequently, (3.11) holds with $C := 2K_{T_1} > 0$.

It remains to show that uniqueness not only holds within $Z_{r,\bar{T}}(v_0)$ but also within the larger space MR$(0,\bar{T})$. Let $v, \tilde{v} \in \mathrm{MR}(0,\bar{T})$ be strong solutions to (3.10) and suppose that $v \neq \tilde{v}$. Then we have $s := \inf\{t \in [0,\bar{T}] : v(t) \neq \tilde{v}(t) \text{ in } H\} \in [0,\bar{T})$ since MR$(0,\bar{T}) = C([0,\bar{T}];H) \cap L^2(0,\bar{T};V)$ and $V \hookrightarrow H$ is injective. Moreover, $v(s) = \tilde{v}(s) =: w_0$ as $v, \tilde{v} \in C([0,\bar{T}];H)$ and $v(\cdot + s)$ and $\tilde{v}(\cdot + s)$ are strong solutions to

$$\begin{cases} u' + \tilde{A}(u)u = \tilde{F}(u) & \text{on } [0,\bar{T} - s], \\ u(0) = w_0. \end{cases} \tag{3.26}$$

Now, by the first part of the proof, there exist $r_0, T_0 > 0$ such that (3.26) has a unique solution in $Z_{r,\delta}(w_0)$ for all $r \in (0, r_0]$ and $\delta \in (0, T_0]$ (take $u_0 = v_0 = w_0$). Fix

$$\delta := \sup\{t \in [0, \min\{T_0, \bar{T} - s\}) : \|v(\cdot + s) - z_{w_0}\|_{\mathrm{MR}(0,t)} \vee \|\tilde{v}(\cdot + s) - z_{w_0}\|_{\mathrm{MR}(0,t)} < r_0\}$$

and note that $\delta \in (0, \min\{T_0, \bar{T} - s\}]$ since $v(0 + s) = \tilde{v}(0 + s) = w_0 = z_{w_0}(0)$. In particular, $\delta \in (0, T_0]$ and $v(\cdot + s), \tilde{v}(\cdot + s) \in Z_{r_0,T}(w_0)$ by definition of δ. Uniqueness of solutions in $Z_{r_0,\delta}(w_0)$ implies that $v(\cdot + s) = \tilde{v}(\cdot + s)$ on $[0, \delta]$. Therefore $v = \tilde{v}$ on $[0, s + \delta]$, contradicting the definition of s. We conclude that $v = \tilde{v}$. $\square$

Remark 3.13. Observe that the local well-posedness could also have been proved under mere coercivity of A_0 instead of co-ercivity of (A_0, B_0) (Assumption 2.2(2)). Indeed, in the current section, we have only used Corollary 3.5 and the estimates from Assumption 2.2(3). Now, the proof of Corollary 3.5 continues when we only assume $\langle A_0(t, u)v, v \rangle \geq \theta_{n,T} \|v\|_V^2 - M_{n,T} \|v\|_H^2$, since then, combined with Assumption 2.2(3) and Young's inequality:

$$\langle \bar{A}(t)v, v \rangle \geq \langle A_0(t, w(t))v, v \rangle - \sigma \|\|B_0(t, w(t))v\|\|_H^2 - C_\sigma \|\psi(t)\|_U^2 \|v\|_H^2 \geq \theta_{n,T} \|v\|_V^2 - (M_{n,T} + C_\sigma \|\psi(t)\|_U^2) \|v\|_H^2 - \sigma C_{n,T}^2 \|v\|_V^2.$$

Putting $\sigma := \theta_{n,T}(2 C_{n,T}^2)^{-1}$, the required coercivity (3.3) for $\bar{A}$ follows.

3.3. Global well-posedness

Similar to [36, Chap. 5] and [22, §18.2], we will extend Theorem 3.7 to a global well-posedness result by means of maximal solutions and a blow-up criterion.

Definition 3.14. For $T \in (0, \infty]$, we define

$$\text{MR}_{\text{loc}}(0, T) := \{u : [0, T) \to H : u|_{[0, \tilde{T}]} \in \text{MR}(0, \tilde{T}) \text{ for all } \tilde{T} \in [0, T)\}.$$

A *maximal solution* to (2.8) is a pair $(u_*, T_*) \in \text{MR}_{\text{loc}}(0, T_*) \times (0, \infty]$ such that

(i) for all $T \in (0, T_*)$, $u_*|_{[0,T]}$ is a strong solution to (2.8),
(ii) for any $T > 0$ and for any strong solution $u \in \text{MR}(0, T)$ to (2.8) it holds that $T \leq T_*$ and $u = u_*$ on $[0, T]$.

Note that maximal solutions are unique by definition. The proof of the next proposition is adapted from [22, Th. 18.2.14, Th. 18.2.15] and [36].

Proposition 3.15 (Blow-up criterion). *Let $x \in H$ and $\psi \in L^2_{\text{loc}}(\mathbb{R}_+; U)$. Suppose that (A, B) satisfies Assumption 2.2. Then Eq. (2.8) has a maximal solution (u_*, T_*). Moreover, if $T_* < \infty$ and $\sup_{T \in [0, T_*)} \|u_*\|_{L^2(0, T; V)} < \infty$, then $\lim_{t \uparrow T_*} u_*(t)$ does not exist in H.*

Proof. The proof of Theorem 3.7 (with $u_0 = v_0 = x$) shows that there exists a local solution and that any strong solution on any finite time interval is unique. Hence, there exists a maximal solution (u_*, T_*) for some $T_* \in (0, \infty]$ and $u_* \in \text{MR}_{\text{loc}}(0, T_*)$.

Suppose that $T_* < \infty$, $\sup_{T \in [0, T_*)} \|u_*\|_{L^2(0, T; V)} < \infty$ and $u^* := \lim_{t \uparrow T_*} u_*(t)$ does exist in H. We will derive a contradiction. Note that the second assumption implies $u_* \in L^2(0, T_*; V)$.

By Theorem 3.7, there exists $\delta > 0$ and a strong solution $u \in \text{MR}(T_*, T_* + \delta)$ to

$$\begin{cases} u' + \tilde{A}(u)u = \tilde{F}(u) & \text{on } [T_*, T_* + \delta], \\ u(T_*) = u^*, \end{cases} \tag{3.27}$$

where we use that the translated pair $(A(T_* + \cdot, \cdot), B(T_* + \cdot, \cdot))$ also satisfies Assumption 2.2. Then

$$\bar{u}(t) := \begin{cases} u_*(t), & t \in [0, T_*), \\ u(t), & t \in [T_*, T_* + \delta] \end{cases}$$

satisfies $\bar{u} \in \text{MR}(0, T_* + \delta)$ and $\bar{u}$ is a strong solution to (3.10) on $[0, T_* + \delta]$, contradicting maximality of (u_*, T_*). $\square$

Using the blow-up criterion, we finally prove global well-posedness for the skeleton equation. Besides Assumption 2.2, we now also assume the coercivity condition (2.7) for the pair (A, B). This condition has not been used so far, but it is also needed for the global well-posedness result for the stochastic evolution equation [3, Th. 3.5], see Theorem 2.3.

Theorem 3.16 (Global well-posedness skeleton equation). *Suppose that (A, B) satisfies Assumption 2.2 and coercivity (2.7). Then for any $\psi \in L^2_{\text{loc}}(\mathbb{R}_+; U)$, $x \in H$ and $T > 0$, there exists a unique strong solution $u \in \text{MR}(0, T)$ to (2.8). Moreover,*

$$\|u\|_{\text{MR}(0,T)} \leq (2 + \tfrac{1}{\theta})^{\frac{1}{2}} \left(\|x\|_H + \sqrt{2} \|\phi\|_{L^2(0,T)} \right) \exp[MT + \tfrac{1}{2} \|\psi\|_{L^2(0,T;U)}^2], \tag{3.28}$$

where $\theta, M > 0$ and $\phi \in L^2(0, T)$ are such that (2.7) holds for $t \in [0, T]$.

Proof. By Proposition 3.15 we have a maximal solution (u_*, T_*) to (2.8). If $T_* = \infty$, then well-posedness for every $T > 0$ follows. Suppose that $T_* < \infty$. We will derive a contradiction. Let $\theta, M > 0$ and $\phi \in L^2(0, T_*)$ be such that the coercivity condition (2.7) holds with $T = T^*$. By definition of the maximal solution, $u_*|_{[0,T]}$ is a strong solution to (2.8) on $[0, T]$ for all $T \in [0, T_*)$. The chain rule (A.2) thus gives for all $t \in [0, T_*)$:

$$\|u_*(t)\|_H^2 = \|x\|_H^2 + 2 \int_0^t \langle -A(s, u_*(s)), u_*(s) \rangle + \langle B(s, u_*(s))\psi(s), u_*(s) \rangle \, ds$$

$$\leq \|x\|_H^2 + 2 \int_0^t -\tfrac{1}{2} \|\|B(s, u_*(s))\|\|_H^2 - \theta \|u_*(s)\|_V^2 + M \|u_*(s)\|_H^2 + |\phi(s)|^2 + \|\|B(s, u_*(s))\|\|_H \|\psi(s)\|_U \|u_*(s)\|_H \, ds$$

$$\leq \|x\|_H^2 + 2 \int_0^t -\tfrac{1}{2} \|\|B(s, u_*(s))\|\|_H^2 - \theta \|u_*(s)\|_V^2 + M \|u_*(s)\|_H^2 + |\phi(s)|^2 + \tfrac{1}{2} \|\|B(s, u_*(s))\|\|_H^2 + \tfrac{1}{2} \|\psi(s)\|_U^2 \|u_*(s)\|_H^2 \, ds$$

$$= -2\theta \|u_*\|_{L^2(0,t;V)}^2 + \|x\|_H^2 + 2 \|\phi\|_{L^2(0,t)}^2 + \int_0^t (2M + \|\psi(s)\|_U^2) \|u_*(s)\|_H^2 \, ds.$$

By Lemma A.1(Gronwall), we obtain for all $T \in (0, T_*)$:

$$\|u_*\|^2_{C([0,T];H)} + \|u_*\|^2_{L^2(0,T;V)} \leq (1 + \frac{1}{2\theta})\Big(\|x\|^2_H + 2\|\phi\|^2_{L^2(0,T)}\Big) \exp[2MT + \|\psi\|^2_{L^2(0,T;U)}],$$

hence

$$\|u_*\|_{\mathrm{MR}(0,T)} \leq (2 + \frac{1}{\theta})^{\frac{1}{2}} \Big(\|x\|_H + \sqrt{2}\|\phi\|_{L^2(0,T)}\Big) \exp[MT + \frac{1}{2}\|\psi\|^2_{L^2(0,T;U)}] =: K(T), \tag{3.29}$$

where $K : [0, T_*] \to \mathbb{R}_+$ is increasing. Applying Lemma 3.8(iv) with $n = K(T_*) < \infty$ we find that $F(u_*) \in L^2(0, t; V^*)$ for all $t \in (0, T_*)$ and $L := \sup_{t \in [0, T_*)} \|F(u_*)\|_{L^2(0,t;V^*)} < \infty$. Thus, by the Monotone Convergence Theorem, $\|F(u_*)\|_{L^2(0,T_*;V^*)} \leq L < \infty$. Similarly, $G(u_*) \in L^2(0, T_*; \mathcal{L}_2(U, H))$. Now we apply Corollary 3.5 with $T := T^*$, $w := u_* \in C([0, T_*]; H) \subset L^\infty(0, T; H)$ (extend by $w(T) := u_*(0)$ on the Lebesgue null set $\{T\}$), $n := K(T_*)$ and $\bar{f} := F(u_*) + f \in L^2(0, T_*; V^*)$, $\bar{g} := G(u_*) + g \in L^2(0, T_*; \mathcal{L}_2(U, H))$. Corollary 3.5 gives existence of a strong solution $\bar{u} \in \mathrm{MR}(0, T_*)$ to (2.8) on $[0, T_*]$. By uniqueness of the maximal solution, it follows that $u_*|_{[0,T]} = \bar{u}|_{[0,T]}$ for all $T \in [0, T_*)$. Hence $\lim_{t \uparrow T_*} u_*(t) = \lim_{t \uparrow T_*} \bar{u}(t) = \bar{u}(T_*) \in H$, contradicting Proposition 3.15.

We conclude that the assumption $T_* < \infty$ was false, i.e. $T_* = \infty$ and for any $T > 0$, $u := u_*|_{[0,T]} \in \mathrm{MR}(0, T)$ is the desired strong solution on $[0, T]$. Finally, note that the estimates leading to (3.29) can be repeated with θ, M, ϕ of the coercivity condition belonging to T instead of T_*, proving (3.28). $\quad\square$

4. Proof of the large deviation principle

4.1. Weak convergence approach

We return to our original setting of Section 2 and start with the proof of the LDP of Theorem 2.6. From now on, assume that U is a real separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ is a filtered probability space. For $\varepsilon > 0$, we let Y^ε be the unique strong solution to

$$\begin{cases} \mathrm{d}Y^\varepsilon(t) = -A(t, Y^\varepsilon(t)) + \sqrt{\varepsilon} B(t, Y^\varepsilon(t)) \, \mathrm{d}W(t), & t \in [0, T], \\ Y^\varepsilon(0) = x. \end{cases} \tag{4.1}$$

Here, W is a U-cylindrical Brownian motion, which is defined as follows.

Definition 4.1. Let $W \in \mathcal{L}(L^2(\mathbb{R}_+; U), L^2(\Omega))$. Then W is called a *U-cylindrical Brownian motion* with respect to $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ if for all $f, g \in L^2(\mathbb{R}_+; U)$ and $t \in \mathbb{R}_+$:

(i) Wf is normally distributed with mean zero and $\mathbb{E}[Wf Wg] = \langle f, g \rangle_{L^2(\mathbb{R}_+;U)}$,
(ii) if $\mathrm{supp}(f) \subset [0, t]$, then Wf is $\mathcal{F}_t$-measurable,
(iii) if $\mathrm{supp}(f) \subset [t, \infty)$, then Wf is independent of $\mathcal{F}_t$.

There exist several different definitions of a cylindrical Brownian motion or cylindrical Wiener process in the literature. Some references in our proof of the LDP use (an equivalent of) an $\mathbb{R}^\infty$-*Brownian motion*, defined below.

Definition 4.2. An $\mathbb{R}^\infty$-Brownian motion (in U) is a pair $\tilde{W} := ((\beta_k)_{k \in \mathbb{N}}, (e_k)_{k \in \mathbb{N}})$, with $(\beta_k)_{k \in \mathbb{N}}$ a sequence of independent standard real-valued $(\mathcal{F}_t)$-Brownian motions and $(e_k)_{k \in \mathbb{N}}$ an orthonormal basis for U.

In Proposition A.4 of Appendix A, the connection between the U-cylindrical Brownian motion and the $\mathbb{R}^\infty$-Brownian motion is summarized, as well as their equivalent, but differently constructed stochastic integrals. The $\mathbb{R}^\infty$-Brownian motion of Definition 4.2 is e.g. used in [29], where it is called a cylindrical Q-Wiener process (with $Q := I \in \mathcal{L}(U; U)$ the identity operator). Often, the notation $\tilde{W}(t) = \sum_{k \in \mathbb{N}} \beta_k(t) e_k$ is also used, which is only formal as the series does not converge in $L^2(\Omega; U)$. However, we will write $\tilde{W} = ((\beta_k)_{k \in \mathbb{N}}, (e_k)_{k \in \mathbb{N}})$.

Remark 4.3. For the proof of the LDP for (Y^ε), without loss of generality, we can assume that the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous and complete. Indeed, one can fix any orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of U and put $\mathcal{H}_t^k := \sigma(W(\mathbb{1}_{(0,s]} \otimes e_k) : s \in [0, t])$ for $k \in \mathbb{N}$ and

$$\mathcal{F}_t^0 := \sigma(\bigcup_{k \in \mathbb{N}} \mathcal{H}_t^k), \quad \mathcal{H}_t^0 := \sigma(\bigcup_{k \in \mathbb{N}} \mathcal{H}_t^k \cup \mathcal{N}), \quad \mathcal{H}_t := \mathcal{H}_{t+}^0 := \bigcap_{h > 0} \mathcal{H}_{t+h}^0,$$

where $\mathcal{N}$ is the collection of all $(\Omega, \mathcal{F}, \mathbb{P})$-null sets. Then $(\mathcal{H}_t)_{t \geq 0}$ is a complete, right-continuous filtration on $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. Moreover, one can show that W is a U-cylindrical Brownian motion with respect to $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\mathcal{H}_t)_{t \geq 0})$ and with respect to $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t^0)_{t \geq 0})$. Let Y_0^ε and $\bar{Y}^\varepsilon$ be the unique strong solution to (4.1) on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t^0)_{t \geq 0})$ and $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\mathcal{H}_t)_{t \geq 0})$, respectively. Since $\mathcal{F}_t^0 \subset \mathcal{F}_t \cap \mathcal{H}_t$, Y_0^ε is also a strong solution to (4.1) on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ and on $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\mathcal{H}_t)_{t \geq 0})$. Pathwise uniqueness gives $Y^\varepsilon = Y_0^\varepsilon = \bar{Y}^\varepsilon$ $\mathbb{P}$-a.s. Now trivially from Definition 2.4, if we prove the LDP for $(\bar{Y}^\varepsilon)$, then the LDP carries over to (Y^ε).

In view of the above remark, we assume that the filtration $(\mathcal{F}_t)_{t \geq 0}$ *is right-continuous and complete* from now on, and we assume that W is a U-cylindrical Brownian motion with respect to $(\mathcal{F}_t)_{t \geq 0}$. Moreover, we fix any orthonormal basis $(e_k)_{k \in \mathbb{N}}$ for U. We let $\tilde{W} = (\beta_k)_{k \in \mathbb{N}}, (e_k)_{k \in \mathbb{N}})$ denote the unique $\mathbb{R}^\infty$-Brownian motion associated to W from Proposition A.4, i.e. satisfying (A.6). In the upcoming proofs $\tilde{W}$ will be useful, since we will be applying the Yamada-Watanabe theorem and Girsanov's theorem for $\mathbb{R}^\infty$-Brownian

motions. Finally, from now on we fix a separable Hilbert space U_1 and a Hilbert-Schmidt *inclusion* $J : U \hookrightarrow U_1$. This is always possible: let $\langle u, v \rangle_1 := \sum_{k=1}^{\infty} \frac{1}{k} \langle u, e_k \rangle_U \langle e_k, v \rangle_U$ for $u, v \in U$ and let $U_1 :=$ completion$(U, \langle \cdot, \cdot \rangle_1)$. We associate to $\bar{W}$ the following U_1-valued process:

$$\bar{W}_1(t) := \sum_{k=1}^{\infty} \beta_k(t) J e_k, \qquad t \in [0, T]. \tag{4.2}$$

By [29, Prop. 2.5.2], $\bar{W}_1$ is a Q_1-Wiener process on U_1, with $Q_1 := JJ^*$. In what follows, $\bar{W}_1$ denotes this Q_1-Wiener process defined by (4.2). We note that the paths of $\bar{W}_1$ are in $C([0, T]; U_1)$.

Definition 4.4. We define

$$\mathcal{A} := \{\Psi : [0, T] \times \Omega \to U : \Psi \text{ is an } (\mathcal{F}_t)\text{-predictable process, } \|\Psi\|_{L^2(0,T;U)} < \infty \ \mathbb{P}\text{-a.s.}\}$$

and for $K > 0$,

$$S_K := \{\psi \in L^2(0, T; U) : \|\psi\|_{L^2(0,T;U)} \le K\}, \qquad \mathcal{A}_K := \{\Psi \in \mathcal{A} : \Psi \in S_K \ \mathbb{P}\text{-a.s.}\}.$$

We write (S_K, weak) for the topological space consisting of S_K, equipped with the weak topology inherited from $L^2(0, T; U)$.

The next theorem gives sufficient conditions for the LDP and is known as the *weak convergence approach*, which originates from [9, Th. 4.4]. In [30], a useful adaptation was proved. The following version is immediately derived from [30, Th. 3.2]. We will use it to prove Theorem 2.6.

Theorem 4.5. *Let $\mathcal{E}$ be a Polish space and let $(Y^\varepsilon)_{\varepsilon > 0}$ be a collection of $\mathcal{E}$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\bar{W} = ((\beta_k)_{k \in \mathbb{N}}, (e_k)_{k \in \mathbb{N}})$ be an $\mathbb{R}^\infty$-Brownian motion. Let $\bar{W}_1 : \Omega \to C([0, T]; U_1)$ be the associated Q_1-Wiener process on U_1 defined by (4.2). Suppose that for $\varepsilon \ge 0$, there exist measurable maps $\mathcal{G}^\varepsilon : C([0, T]; U_1) \to \mathcal{E}$ such that*

(i) $Y^\varepsilon = \mathcal{G}^\varepsilon(\bar{W}_1(\cdot))$ a.s. for all $\varepsilon > 0$,

(ii) for any $K < \infty$, $(\psi_n) \subset S_K$ and $\psi \in S_K$ with $\psi_n \to \psi$ weakly in $L^2(0, T; U)$, it holds that

$$\mathcal{G}^0 \left(\int_0^\cdot \psi_n(s) \, ds \right) \to \mathcal{G}^0 \left(\int_0^\cdot \psi(s) \, ds \right) \text{ in } \mathcal{E},$$

(iii) for any $K < \infty$ and $(\Psi^\varepsilon) \subset \mathcal{A}_K$, it holds that

$$\mathcal{G}^\varepsilon \left(\bar{W}_1(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \Psi^\varepsilon(s) \, ds \right) - \mathcal{G}^0 \left(\int_0^\cdot \Psi^\varepsilon(s) \, ds \right) \to 0 \text{ in probability}$$

as $\mathcal{E}$-valued random variables.

Then $(Y^\varepsilon)_{\varepsilon > 0}$ satisfies the LDP on $\mathcal{E}$ with good rate function

$$I(z) := \frac{1}{2} \inf \left\{ \int_0^T \|\psi(s)\|_U^2 \, ds : \psi \in L^2(0, T; U), \ z = \mathcal{G}^0 \left(\int_0^\cdot \psi(s) \, ds \right) \right\}. \tag{4.3}$$

Conditions (ii) and (iii) imply the conditions of the original weak convergence approach of [9]. For the latter, instead of (ii) and (iii), one would require

(II) for any $K < \infty$, $\{\mathcal{G}^0(\int_0^\cdot \psi(s) \, ds) : \psi \in S_K\}$ is a compact subset of $\mathcal{E}$,

(III) for any $K < \infty$, if $(\Psi^\varepsilon) \subset \mathcal{A}_K$ with $\Psi^\varepsilon \to \Psi$ in distribution with respect to the weak topology on $L^2(0, T; U)$, then $\mathcal{G}^\varepsilon \left(\bar{W}_1(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \Psi^\varepsilon(s) \, ds \right) \to \mathcal{G}^0(\int_0^\cdot \Psi(s) \, ds)$ in distribution.

Here, [9, Th. 4.4] is applied with Q_1-Wiener process $\bar{W}_1$, $H := U_1$, $H_0 := Q_1^{\frac{1}{2}}(U_1)$ and one uses that $Q_1^{\frac{1}{2}}(U_1) = J(U) = U$ as a subspace of U_1, see [29, Prop. 2.5.2] (with $Q := I$, $U_0 := I^{\frac{1}{2}}(U) = U$).

Note that (II) means that the sublevel sets of the rate function I defined by (4.3) are compact, as is also required in Definition 2.4. On the other hand, (ii) means that the map $\tau : (S_K, \text{weak}) \to \mathcal{E} : \psi \mapsto \mathcal{G}^0(\int_0^\cdot \psi \, ds) = u^\psi$ is continuous (S_K is weakly metrizable as opposed to $L^2(0, T; U)$, thus sequential continuity suffices). In particular, this implies (II). Indeed, $S_K \subset L^2(0, T; U)$ is weakly compact by the Banach-Alaoglu theorem and reflexivity of $L^2(0, T; U)$, so $\{\mathcal{G}^0(\int_0^\cdot \psi(s) \, ds) : \psi \in S_K\} = \tau(S_K)$ is the continuous image of a compact set, hence it is compact.

We will apply Theorem 4.5 with the map $\mathcal{G}^0 : C([0, T]; U_1) \to \text{MR}(0, T)$ given by

$$\mathcal{G}^0(\gamma) := \begin{cases} u^\psi, & \text{if } \gamma = \int_0^\cdot \psi(s) \, ds, \ \psi \in L^2(0, T; U), \\ 0, & \text{otherwise,} \end{cases} \tag{4.4}$$

where u^ψ is the strong solution to (2.8). Note that the rate function I defined by (2.9) is then precisely equal to the rate function given by (4.3).

We will verify that all conditions in Theorem 4.5 are satisfied for Y^ε defined as the strong solution to (4.1). Condition (i) follows from the Yamada-Watanabe theorem in [41]. The details are given in Lemma 4.6, as well as a preparation for the proof of condition (iii).

Lemma 4.6. *Suppose that Assumption 2.2 holds and suppose that (A, B) satisfies (2.7). Let $x \in H$. Then for each $\varepsilon > 0$, there exists a measurable map $\mathcal{G}^{\varepsilon} : C([0, T]; U_1) \to \mathrm{MR}(0, T)$ such that the unique strong solution Y^{ε} to (4.1) satisfies $Y^{\varepsilon} = \mathcal{G}^{\varepsilon}(\bar{W}_1)$ a.s., where $\bar{W}_1$ is given by (4.2). Moreover, for any $\Psi^{\varepsilon} \in \mathcal{A}_K$, $X^{\varepsilon} := \mathcal{G}^{\varepsilon}(\bar{W}_1(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \Psi^{\varepsilon}(s) \, ds)$ is a strong solution to*

$$
\begin{cases}
dX^{\varepsilon}(t) = -A(t, X^{\varepsilon}(t)) + B(t, X^{\varepsilon}(t))\Psi^{\varepsilon}(t) + \sqrt{\varepsilon} B(t, X^{\varepsilon}(t)) \, dW(t), \quad t \in [0, T], \\
X^{\varepsilon}(0) = x.
\end{cases}
\tag{4.5}
$$

Proof. To prove the first statement, we use the Yamada-Watanabe theorem from [41, Th. 2.1] on $[0, T]$ with $L^1(0, T; V)$ replaced by $L^2(0, T; V)$. Let $\varepsilon > 0$. For any Y^{ε} with $Y^{\varepsilon} \in \mathrm{MR}(0, T)$ a.s. and for any $\xi \in L^0((\Omega, \mathcal{F}_0); H)$, we have that $(Y^{\varepsilon}, \bar{W})$ is a weak solution in the sense of [41, Def. 1.4] to

$$
\begin{cases}
d\tilde{Y}^{\varepsilon}(t) = -A(t, \tilde{Y}^{\varepsilon}(t)) + \sqrt{\varepsilon} B(t, \tilde{Y}^{\varepsilon}(t)) \, d\tilde{W}(t), \\
\tilde{Y}^{\varepsilon}(0) = \xi,
\end{cases}
$$

if and only if Y^{ε} is a strong solution in the sense of [3, Def. 3.2] to (4.1) with x replaced by ξ. This is a mere consequence of (A.7) and the fact that $B(\cdot, Y(\cdot)) \in L^2(([0, T] \times \Omega, \mathcal{P}, \lambda \times \mathbb{P}); \mathcal{L}_2(U, H)) \subset \mathcal{N}(0, T)$ for any $Y \in \mathrm{MR}(0, T)$. By [3, Th. 3.5], (4.1) has a unique strong solution Y^{ε}, also when x is replaced by random initial data ξ. Thus we have pathwise uniqueness in the sense of [41, Def. 1.7] and we have existence of a.s. $\mathrm{MR}(0, T)$-valued weak solutions. Now fix $x \in H$ and $\varepsilon > 0$ and let Y^{ε} be the unique strong solution to (4.1). By [41, Th. 2.1, Def. 1.9(2), Def. 1.8] there exists a measurable map $\mathcal{G}^{\varepsilon} : C([0, T]; U_1) \to \mathrm{MR}(0, T)$ such that a.s. $Y^{\varepsilon} = \mathcal{G}^{\varepsilon}(\bar{W}_1(\cdot))$.

Next, let $X^{\varepsilon} := \mathcal{G}^{\varepsilon}(\bar{W}_1(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \Psi^{\varepsilon}(s) \, ds)$. We prove that X^{ε} solves (4.5) Define

$$
\hat{W} := \tilde{W} + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \Psi^{\varepsilon}(s) \, ds := ((\hat{\beta}_k)_{k \in \mathbb{N}}, (e_k)_{k \in \mathbb{N}}), \qquad \hat{\beta}_k := \beta_k + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \langle \Psi^{\varepsilon}(s), e_k \rangle_U \, ds.
$$

We have $\mathbb{E}[\exp(\frac{1}{2} \| - \frac{1}{\sqrt{\varepsilon}} \Psi^{\varepsilon} \|_{L^2(0,T;U)}^2)] \le \exp(\frac{K^2}{2\varepsilon}) < \infty$, so by Novikov's condition [24, Prop. 5.12],

$$
\mathbb{E}\left[\exp\left(\int_0^T \langle -\frac{1}{\sqrt{\varepsilon}} \Psi^{\varepsilon}(s), d\tilde{W}(s) \rangle_U - \frac{1}{2} \| \frac{1}{\sqrt{\varepsilon}} \Psi^{\varepsilon} \|_{L^2(0,T;U)}^2 \right) \right] = 1.
$$

Now Girsanov's theorem [29, Proposition I.0.6] and [17, Th. 2.3] yields that $\hat{W}$ is an $\mathbb{R}^{\infty}$-Brownian motion on $(\Omega, \mathcal{F}, \hat{\mathbb{P}}, (\mathcal{F}_t)_{t \ge 0})$, where

$$
\hat{\mathbb{P}} := \exp\left(-\frac{1}{\sqrt{\varepsilon}} \int_0^T \langle \Psi^{\varepsilon}(s), d\tilde{W}(s) \rangle_U - \frac{1}{2\varepsilon} \| \Psi^{\varepsilon} \|_{L^2(0,T;U)}^2 \right) d\mathbb{P}.
$$

Moreover, $\hat{W}$ induces a U_1-valued Q_1-Wiener process $\hat{W}_1$ on $(\Omega, \mathcal{F}, \hat{\mathbb{P}}, (\mathcal{F}_t)_{t \ge 0})$ using the same Hilbert-Schmidt inclusion $J : U \hookrightarrow U_1$ as we used for $\bar{W}_1$ in (4.2), resulting in:

$$
\begin{aligned}
\hat{W}_1(t) &:= \sum_{k \in \mathbb{N}} \hat{\beta}_k(t) J e_k = \sum_{k \in \mathbb{N}} \beta_k(t) J e_k + \frac{1}{\sqrt{\varepsilon}} \sum_{k \in \mathbb{N}} \left(\int_0^t \langle \Psi^{\varepsilon}(s), e_k \rangle_U \, ds \right) J e_k = \bar{W}_1(t) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \sum_{k \in \mathbb{N}} \langle \Psi^{\varepsilon}(s), e_k \rangle_U e_k \, ds \\
&= \bar{W}_1(t) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \Psi^{\varepsilon}(s) \, ds
\end{aligned}
$$

$\mathbb{P}$-a.s. in U_1, where we used that $\Psi^{\varepsilon} \in \mathcal{A}_K$ to apply Fubini's theorem in the second line. Thus, recalling the definition of X^{ε} and noting that $\hat{\mathbb{P}} \ll \mathbb{P} \ll \hat{\mathbb{P}}$, we have $\hat{\mathbb{P}}$-a.s. $X^{\varepsilon} = \mathcal{G}^{\varepsilon}(\hat{W}_1(\cdot))$. By the Yamada-Watanabe theorem [41, Th. 2.1, Def. 1.9], for $X^{\varepsilon} = \mathcal{G}^{\varepsilon}(\hat{W}_1(\cdot))$ we have that $(X^{\varepsilon}, \hat{W})$ is a weak solution to (4.1). That is, X^{ε} satisfies $\hat{\mathbb{P}}$-a.s. in V^*:

$$
X^{\varepsilon}(t) = x + \int_0^t -A(s, X^{\varepsilon}(s)) \, ds + \int_0^t \sqrt{\varepsilon} B(s, X^{\varepsilon}(s)) \, d\hat{W}(s).
\tag{4.6}
$$

By Proposition A.4, there exists a unique U-cylindrical Brownian motion $\hat{\mathcal{W}} \in \mathcal{L}(L^2(\mathbb{R}_+; U); L^2(\Omega))$ with respect to $(\Omega, \mathcal{F}, \hat{\mathbb{P}}, (\mathcal{F}_t)_{t \ge 0})$, satisfying for all $u \in U$ and $t \in [0, T]$:

$$
\hat{\mathcal{W}}(\mathbb{1}_{(0,t]} \otimes u) = \sum_{k=1}^{\infty} \hat{\beta}_k(t) \langle u, e_k \rangle_U = \mathcal{W}(\mathbb{1}_{(0,t]} \otimes u) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \langle \Psi^{\varepsilon}(s), u \rangle_U \, ds,
\tag{4.7}
$$

where the last equality follows from the definition of $\hat{\beta}_k$ and (A.6). Let $\hat{\mathcal{N}}(0, T)$ denote the stochastically integrable processes with respect to $\hat{\mathcal{W}}$ and $\hat{W}$ on $(\Omega, \mathcal{F}, \hat{\mathbb{P}}, (\mathcal{F}_t)_{t \ge 0})$, i.e. (A.3) with $\mathbb{P}$ replaced by $\hat{\mathbb{P}}$. Note that $\hat{\mathcal{N}}(0, T) = \mathcal{N}(0, T)$, since $\mathbb{P} \ll \hat{\mathbb{P}} \ll \mathbb{P}$. Thus, Proposition A.4 gives $\int_0^t \Phi(s) \, d\hat{\mathcal{W}}(s) = \int_0^t \Phi(s) \, d\hat{W}(s)$ $\hat{\mathbb{P}}$-a.s. for all $\Phi \in \mathcal{N}(0, T)$ and $t \in [0, T]$. Therefore, combined with (4.6), X^{ε} satisfies $\hat{\mathbb{P}}$-a.s. (hence $\mathbb{P}$-a.s.) in V^*:

$$
\begin{aligned}
X^{\varepsilon}(t) &= x + \int_0^t -A(s, X^{\varepsilon}(s)) \, ds + \int_0^t \sqrt{\varepsilon} B(s, X^{\varepsilon}(s)) \, d\hat{\mathcal{W}}(s) \\
&= x + \int_0^t -A(s, X^{\varepsilon}(s)) \, ds + \int_0^t \sqrt{\varepsilon} B(s, X^{\varepsilon}(s)) \, dW(s) + \int_0^t B(s, X^{\varepsilon}(s))\Psi^{\varepsilon}(s) \, ds.
\end{aligned}
$$

In the last line we used that $\int_0^t \Phi(s)\,d\hat{W}(s) = \int_0^t \Phi(s)\,dW(s) + \frac{1}{\sqrt{\varepsilon}}\int_0^t \Phi(s)\Psi^\varepsilon(s)\,ds$ for $\Phi \in \mathcal{N}(0,T)$ and $t \in [0,T]$. For $\Phi = \mathbb{1}_{A\times(t_1,t_2]} \otimes (u \otimes x)$ with $0 \leq t_1 < t_2 \leq T$, $A \in \mathcal{F}_{t_1}$, $u \in U$, $x \in H$, the identity follows from (4.7) and the definition of the stochastic integral for elementary processes [31, p. 305]. By linearity and continuity of the integrals and by a density argument and localization, the identity extends for $\Phi \in \mathcal{N}(0,T)$. This finishes the proof of the last claim of the lemma. $\square$

Remark 4.7. The above proof also yields existence and uniqueness of strong solutions to (4.5), since it was actually shown that X^ε is a strong solution to (4.5) if and only if it is a strong solution to (4.1) with W replaced by the U-cylindrical Brownian motion $\hat{\mathcal{W}}$. The latter was already considered in Theorem 2.3.

4.2. Weakly continuous dependence in the skeleton equation

In this subsection we prove that condition (ii) of Theorem 4.5 is satisfied. This will be achieved in the upcoming Proposition 4.9. Its proof was inspired by [20, Th. 3.2]. Using an additional approximation by Bochner-simple functions, we can omit the time(-Hölder) regularity assumptions on B of [20, (H5)].

Lemma 4.8. Let $(w_n) \subset C([0,T];H)$, $(\alpha_n) \subset L^1(0,T;V^*)$ and $(\psi_n) \subset L^2(0,T;U)$ be such that

$$w_n(t) = \int_0^t \alpha_n(s)\,ds$$

and such that $C_\alpha := \sup_{n\in\mathbb{N}}\|\alpha_n\|_{L^1(0,T;V^*)} < \infty$, $C_w := \sup_{n\in\mathbb{N}}\|w_n\|_{C([0,T];H)} < \infty$ and $\psi_n \to \psi$ weakly in $L^2(0,T;U)$. Let $b \in L^2(0,T;\mathcal{L}_2(U,H))$. Then,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\Big|\int_0^t \langle b(s)(\psi_n(s)-\psi(s)), w_n(s)\rangle\,ds\Big| = 0. \tag{4.8}$$

Proof. Without loss of generality, we can assume $\psi = 0$, i.e. $\psi_n \to 0$ weakly in $L^2(0,T;U)$ (apply to $\psi_n - \psi$). Since (ψ_n) is weakly convergent, it is bounded. Throughout the proof we let

$$C_\psi := \sup_{n\in\mathbb{N}}\|\psi_n\|_{L^2(0,T;U)} < \infty.$$

First, let us observe that it suffices to prove (4.8) for all b in the collection

$$S := \{\mathbb{1}_D \otimes u \otimes v : D \in \mathcal{B}([0,T]), u \in U, v \in V\} \subset L^2(0,T;\mathcal{L}_2(U,H)),$$

where $(\mathbb{1}_D \otimes u \otimes v)(t)x := \mathbb{1}_D(t)(u,x)_U v \in H$ for $t \in [0,T]$ and $x \in U$. Note that $\mathrm{span}(S)$ is dense in $L^2(0,T;\mathcal{L}_2(U,H))$, using consecutively density of Bochner-simple functions, density of finite rank operators in $\mathcal{L}_2(U,H)$ and density of V in H. Define for $n \in \mathbb{N}$:

$$I_n : L^2(0,T;\mathcal{L}_2(U,H)) \to C([0,T];\mathbb{R}), \quad I_n(b) := \int_0^{\cdot} \langle b(s)\psi_n(s), w_n(s)\rangle\,ds.$$

Each I_n is linear and continuous with $\|I_n\| \leq C_\psi C_w$, independent of n:

$$\|I_n(b)\|_{C([0,T];\mathbb{R})} \leq \|b\|_{L^2(0,T;\mathcal{L}_2(U,H))}\|\psi_n\|_{L^2(0,T;U)}\|w_n\|_{C(0,T;H)} \leq \|b\|_{L^2(0,T;\mathcal{L}_2(U,H))}C_\psi C_w. \tag{4.9}$$

If (4.8) holds for all $b \in S$, i.e. $\lim_{n\to\infty}\|I_n(b)\|_{C([0,T];\mathbb{R})} = 0$, then it also holds for all $b \in \mathrm{span}(S)$, by the triangle inequality in $C([0,T];\mathbb{R})$. Moreover, for $b \in L^2(0,T;\mathcal{L}_2(U,H))$, we find $(b_k) \subset \mathrm{span}(S)$ with $b_k \to b$ in $L^2(0,T;\mathcal{L}_2(U,H))$ by density. Now (4.9) and a standard 2ε-argument yield (4.8) for b.

It remains to prove (4.8) for $b = \mathbb{1}_D \otimes u \otimes v$ with $D \in \mathcal{B}([0,T])$, $u \in U$ and $v \in V$. Note that in this case,

$$I_n(b) = \int_0^{\cdot} \mathbb{1}_D(s)(u,\psi_n(s))_U \langle v, w_n(s)\rangle\,ds \tag{4.10}$$

and we have

$$(v, w_n(s))_H = \langle v, w_n(s)\rangle = \int_0^t \langle \alpha_n(s), v\rangle\,ds.$$

Since u and v are fixed, we have $(v,w_n(\cdot))_H \in C([0,T];\mathbb{R})$ and $\langle \alpha_n(\cdot), v\rangle \in L^1(0,T)$ with norms uniformly bounded in n. Moreover, $(u,\psi_n(\cdot))_U \in L^2(0,T)$ and $\psi_n \to 0$ weakly in $L^2(0,T;U)$ implies $(u,\psi_n(\cdot))_U \to 0$ weakly in $L^2(0,T)$. Combined with (4.10), we conclude that it suffices to prove the lemma for $U = V = H = V^* = \mathbb{R}$ and $b = \mathbb{1}_D \in L^2(0,T) = L^2(0,T;\mathcal{L}_2(\mathbb{R};\mathbb{R}))$.

Let (w_n), (α_n), (ψ_n) be as in the statement, now real-valued, and with $\psi = 0$. Define $I^n(t) := \int_0^t \mathbb{1}_D(s)\psi_n(s)w_n(s)\,ds$. We have to show that $\lim_{n\to\infty}\sup_{t\in[0,T]}|I^n(t)| = 0$.

We use an equidistant time discretization to approximate w_n. For $\delta > 0$ and $t \in [0,T]$, put $t_\delta := \lfloor\frac{t}{\delta}\rfloor\delta$. We have for all $n \in \mathbb{N}$ and $\delta > 0$:

$$|I^n(t)| \leq \Big|\int_0^t \mathbb{1}_D(s)\psi_n(s)(w_n(s)-w_n(s_\delta))\,ds\Big| + \Big|\int_0^t \mathbb{1}_D(s)\psi_n(s)(w_n(s_\delta))\,ds\Big|$$

$$\leq \Big|\int_0^t \mathbb{1}_D(s)\psi_n(s)(w_n(s)-w_n(s_\delta))\,ds\Big| + \sum_{l=0}^{\lfloor\frac{t}{\delta}\rfloor-1}\Big|\int_{l\delta}^{(l+1)\delta} \mathbb{1}_D(s)\psi_n(s)w_n(s_\delta)\,ds\Big| + \Big|\int_{t_\delta}^t \mathbb{1}_D(s)\psi_n(s)w_n(s_\delta)\,ds\Big|$$

$$=: J_1^{n,\delta}(t) + \sum_{l=0}^{\lfloor \frac{T}{\delta} \rfloor - 1} J_2^{n,\delta,l} + J_3^{n,\delta}(t). \tag{4.11}$$

We estimate each term. Since $\psi_n \to 0$ weakly in $L^2(0,T)$, we have for all $\delta > 0$ and $l \in \mathbb{N}$:

$$J_2^{n,\delta,l} = |w_n(l\delta)| \left| \int_{l\delta}^{(l+1)\delta} \mathbb{1}_D(s)\psi_n(s)\,\mathrm{d}s \right| \leq C_w \left| \int_{l\delta}^{(l+1)\delta} \mathbb{1}_D(s)\psi_n(s)\,\mathrm{d}s \right| \to 0 \quad \text{as } n \to \infty. \tag{4.12}$$

Furthermore, we have for all $n \in \mathbb{N}$:

$$\sup_{t \in [0,T]} J_3^{n,\delta}(t) \leq C_w \sup_{t \in [0,T]} \int_{t_\delta}^t |\psi_n(s)|\,\mathrm{d}s \leq C_w C_\psi \delta^{\frac{1}{2}} \to 0 \quad \text{as } \delta \downarrow 0, \tag{4.13}$$

where we used that $|t - t_\delta| < \delta$ for all $t \in [0,T]$. Note that the convergence is uniform in n.

Finally, we estimate $J_1^{n,\delta}(t)$ uniformly in n and t. By the Cauchy–Schwarz inequality, we have for all $n \in \mathbb{N}$ and $\delta > 0$:

$$\sup_{t \in [0,T]} J_1^{n,\delta}(t) \leq \int_0^T |\psi_n(s)|\,|w_n(s) - w_n(s_\delta)|\,\mathrm{d}s \leq C_\psi \|w_n(\cdot) - w_n(\cdot_\delta)\|_{L^2(0,T)} \tag{4.14}$$

To estimate further, we use an argument inspired by [20, Lem. 3.3]. Note that $w_n(0) = 0$ and

$$\int_0^T |w_n(t) - w_n(t_\delta)|^2\,\mathrm{d}t = \int_0^\delta |w_n(t)|^2\,\mathrm{d}t + \int_\delta^T |w_n(t) - w_n(t_\delta)|^2\,\mathrm{d}t \leq \delta C_w^2 + \int_\delta^T |w_n(t) - w_n(t_\delta)|^2\,\mathrm{d}t. \tag{4.15}$$

For any $t \in [\delta, T]$, we can apply the chain rule (A.2) to $v_n^{t,\delta}(\cdot) := w_n(\cdot) - w_n(t_\delta) = \int_{t_\delta}^\cdot \alpha_n(s)\,\mathrm{d}s$ on $[t_\delta, T]$ and obtain for all $\tilde{t} \in [t_\delta, T]$:

$$|w_n(\tilde{t}) - w_n(t_\delta)|^2 = 2 \int_{t_\delta}^{\tilde{t}} \alpha_n(s)(w_n(s) - w_n(t_\delta))\,\mathrm{d}s.$$

Applying the above expression with $\tilde{t} = t$ we estimate the second term from (4.15):

$$\begin{aligned}
\int_\delta^T |w_n(t) - w_n(t_\delta)|^2\,\mathrm{d}t &= 2 \int_\delta^T \int_{t_\delta}^t \alpha_n(s)(w_n(s) - w_n(t_\delta))\,\mathrm{d}s\,\mathrm{d}t \\
&\leq 4C_w \int_\delta^T \int_{t-\delta}^t |\alpha_n(s)|\,\mathrm{d}s\,\mathrm{d}t \\
&\leq 4C_w \int_0^T \int_\delta^T \mathbb{1}_{[s,(s+\delta)\wedge T]}(t)\,\mathrm{d}t\,|\alpha_n(s)|\,\mathrm{d}s \\
&\leq 4\delta C_w C_\alpha, \tag{4.16}
\end{aligned}$$

where we used that $\mathbb{1}_{[t-\delta,t]}(s) \leq \mathbb{1}_{[s,(s+\delta)\wedge T]}(t)$ for all $(s,t) \in [0,T] \times [\delta,T]$. Combining (4.14)–(4.16) we conclude that for all $\delta > 0$:

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} J_1^{n,\delta}(t) \leq C_\psi \left(\delta C_w^2 + 4\delta C_w C_\alpha \right)^{\frac{1}{2}}. \tag{4.17}$$

Now let $\varepsilon > 0$. According to (4.17) and (4.13), fix $\delta > 0$ sufficiently small such that we have $\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} J_1^{n,\delta}(t) < \frac{\varepsilon}{3}$ and $\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} J_3^{n,\delta}(t) < \frac{\varepsilon}{3}$. Then, according to (4.12), pick $N \in \mathbb{N}$ such that for all $n \geq N$: $J_2^{n,\delta,l} < \frac{\varepsilon}{3\lfloor \frac{T}{\delta} \rfloor}$. By (4.11), we obtain $\sup_{t \in [0,T]} |I^n(t)| < \varepsilon$ for all $n \geq N$. Thus $\lim_{n \to \infty} \sup_{t \in [0,T]} |I^n(t)| = 0$. $\quad \square$

Equipped with the lemma above, we now prove that condition (ii) of Theorem 4.5 is satisfied. Note that the growth bounds on B in Assumption 2.2 contain V-norms (instead of merely H-norms), making it more difficult to apply Gronwall inequalities. To deal with this, the estimates from Lemma 3.8 will be used.

Proposition 4.9. *Suppose that Assumption 2.2 holds and suppose that (A, B) satisfies (2.7). For $\psi \in L^2(0,T;U)$ let u^ψ be the unique strong solution to (2.8). Then for any $K \geq 0$, the map $(S_K, \text{weak}) \to \mathrm{MR}(0,T) : \psi \mapsto u^\psi$ is continuous.*

Proof. Note that S_K is weakly metrizable (as opposed to $L^2(0,T;U)$), so we may verify sequential continuity. Suppose that $\psi_n \to \psi$ weakly in $L^2(0,T;U)$ and write $w_n := u^{\psi_n} - u^\psi$. We show that $w_n \to 0$ in $\mathrm{MR}(0,T)$. For each $n \in \mathbb{N}$, w_n is a strong solution to

$$\begin{cases} w_n' + \bar{A}_0 w_n = f_n + \left(\bar{B}_0 w_n + g_n \right)\psi_n + b(\psi_n - \psi), \\ w_n(0) = 0, \end{cases}$$

where $\bar{A}_0 := A_0(u^\psi)$, $\bar{B}_0 := B_0(u^\psi)$ and

$$\begin{aligned}
f_n &:= (A_0(u^\psi) - A_0(u^{\psi_n}))u^{\psi_n} + F(u^{\psi_n}) - F(u^\psi) \in L^2(0,T;V^*), \\
g_n &:= -(B_0(u^\psi) - B_0(u^{\psi_n}))u^{\psi_n} + G(u^{\psi_n}) - G(u^\psi) \in L^2(0,T;\mathcal{L}_2(U,H)), \\
b &:= B(u^\psi) = B_0(u^\psi)u^\psi + G(u^\psi) + g \in L^2(0,T;\mathcal{L}_2(U,H)).
\end{aligned}$$

By the chain rule (A.2), we have for all $t \in [0, T]$:

$$\frac{1}{2}\|w_n(t)\|_H^2 = \int_0^t -\langle \bar{A}_0 w_n(s), w_n(s) \rangle + \langle \bar{B}_0 w_n(s)\psi_n(s), w_n(s) \rangle \, \mathrm{d}s$$

$$+ \int_0^t \langle f_n(s), w_n(s) \rangle + \langle g_n(s)\psi_n(s), w_n(s) \rangle \, \mathrm{d}s$$

$$+ \int_0^t \langle b(s)(\psi_n(s) - \psi(s)), w_n(s) \rangle \, \mathrm{d}s$$

$$=: I_1^n(t) + I_2^n(t) + I_3^n(t). \tag{4.18}$$

The strategy is now to use Lemma A.1 (Gronwall) for deriving an estimate of the form

$$\|w_n\|_{\mathrm{MR}(0,T)}^2 \leq C \sup_{t \in [0,T]} |I_3^n(t)|,$$

after which we will apply Lemma 4.8 to I_3^n and obtain $w_n \to 0$ in $\mathrm{MR}(0, T)$. Using the maximal regularity estimate (3.28) and boundedness of (ψ_n) in $L^2(0, T; U)$, we put

$$N := \|u^\psi\|_{\mathrm{MR}(0,T)} + \sup_{n \in \mathbb{N}} \|u^{\psi_n}\|_{\mathrm{MR}(0,T)} < \infty. \tag{4.19}$$

Let $\theta_{N,T}$, $M_{N,T}$ and $C_{N,T}$ be as in Assumption 2.2. We estimate I_1^n and I_2^n appearing in (4.18). The coercivity of (A_0, B_0) in Assumption 2.2(2) gives

$$I_1^n(t) \leq \int_0^t -\langle \bar{A}_0 w_n(s), w_n(s) \rangle + \|\bar{B}_0 w_n(s)\|_H \|\psi_n(s)\|_U \|w_n(s)\|_H \, \mathrm{d}s$$

$$\leq \int_0^t -\langle \bar{A}_0 w_n(s), w_n(s) \rangle + \frac{1}{2}\|\bar{B}_0 w_n(s)\|_H^2 + \frac{1}{2}\|\psi_n(s)\|_U^2 \|w_n(s)\|_H^2 \, \mathrm{d}s$$

$$\leq \int_0^t -\theta_{N,T}\|w_n(s)\|_V^2 + (M_{N,T} + \frac{1}{2}\|\psi_n(s)\|_U^2)\|w_n(s)\|_H^2 \, \mathrm{d}s. \tag{4.20}$$

Moreover,

$$I_2^n(t) \leq \int_0^t \|f_n(s)\|_{V^*}\|w_n(s)\|_V + \|g_n(s)\|_H \|\psi_n(s)\|_U \|w_n(s)\|_H \, \mathrm{d}s$$

$$\leq \int_0^t \frac{1}{\theta_{N,T}}\|f_n(s)\|_{V^*}^2 + \frac{\theta_{N,T}}{4}\|w_n(s)\|_V^2 + \frac{1}{2}\|g_n(s)\|_H^2 + \frac{1}{2}\|\psi_n(s)\|_U^2 \|w_n(s)\|_H^2 \, \mathrm{d}s. \tag{4.21}$$

For f_n, Lemma 3.8(iii)(v) gives for any $\sigma > 0$:

$$\|f_n\|_{L^2(0,t;V^*)}^2 \leq 2\|(A_0(u^\psi) - A_0(u^{\psi_n}))u^{\psi_n}\|_{L^2(0,t;V^*)}^2 + 2\|F(u^{\psi_n}) - F(u^\psi)\|_{L^2(0,t;V^*)}^2$$

$$\leq 2C_{N,T}^2 \int_0^t \|u^{\psi_n}\|_V^2 \|w_n\|_H^2 \, \mathrm{d}s + 2C_{N,T,\sigma} \int_0^t (1 + \|u^\psi\|_V^2 + \|u^{\psi_n}\|_V^2)\|w_n\|_H^2 \, \mathrm{d}s + 2\sigma C_{N,T}^2 \|w_n\|_{L^2(0,t;V)}^2. \tag{4.22}$$

Similarly, $\|g_n\|_{L^2(0,t;\mathcal{L}_2(U,H))}^2$ is bounded by the right-hand side of (4.22), by Lemma 3.8(iii)(v). Fix $\bar{\sigma} := \theta_{N,T}^2(4(2 + \theta_{N,T})C_{N,T}^2)^{-1} > 0$. Combining (4.21) and (4.22) yields

$$I_2^n(t) \leq (\frac{2}{\theta_{N,T}} + 1)\left(C_{N,T}^2 \int_0^t \|u^{\psi_n}\|_V^2 \|w_n\|_H^2 \, \mathrm{d}s + C_{N,T,\bar{\sigma}} \int_0^t (1 + \|u^\psi\|_V^2 + \|u^{\psi_n}\|_V^2)\|w_n\|_H^2 \, \mathrm{d}s\right)$$

$$+ (\frac{2}{\theta_{N,T}} + 1)\bar{\sigma} C_{N,T}^2 \|w_n\|_{L^2(0,t;V)}^2 + \frac{\theta_{N,T}}{4}\|w_n\|_{L^2(0,t;V)}^2 + \int_0^t \frac{1}{2}\|\psi_n\|_U^2 \|w_n\|_H^2 \, \mathrm{d}s$$

$$= \int_0^t h_n(s)\|w_n(s)\|_H^2 \, \mathrm{d}s + \frac{\theta_{N,T}}{2}\|w_n\|_{L^2(0,t;V)}^2, \tag{4.23}$$

where

$$h_n(s) := (\frac{2}{\theta_{N,T}} + 1)\left(C_{N,T}^2 \|u^{\psi_n}(s)\|_V^2 + C_{N,T,\bar{\sigma}}(1 + \|u^\psi(s)\|_V^2 + \|u^{\psi_n}(s)\|_V^2)\right) + \frac{1}{2}\|\psi_n(s)\|_U^2.$$

Note that $\sup_{n \in \mathbb{N}} \|h_n\|_{L^1(0,T)} < \infty$, by (4.19) and since $(\psi_n) \subset S_K$. Now (4.20) and (4.23) give

$$I_1^n(t) + I_2^n(t) \leq -\frac{\theta_{N,T}}{2}\|w_n\|_{L^2(0,t;V)}^2 + \int_0^t \left(h_n(s) + M_{N,T} + \frac{1}{2}\|\psi_n(s)\|_U^2\right)\|w_n(s)\|_H^2 \, \mathrm{d}s.$$

Hence, combined with (4.18):

$$\|w_n(t)\|_H^2 \leq -\theta_{N,T}\|w_n\|_{L^2(0,t;V)}^2 + 2\int_0^t \left(h_n(s) + M_{N,T} + \frac{1}{2}\|\psi_n(s)\|_U^2\right)\|w_n(s)\|_H^2 \, \mathrm{d}s + 2\sup_{s \in [0,t]}|I_3^n(s)|.$$

Lemma A.1 (Gronwall) gives for all $n \in \mathbb{N}$:

$$\frac{1}{2}\|w_n\|^2_{\text{MR}(0,T)} \leq \sup_{t\in[0,T]}\|w_n(t)\|^2_H + \|w_n\|^2_{L^2(0,T;V)} \leq 2(1 + \frac{1}{\theta_{N,T}})\sup_{s\in[0,t]}|I^n_3(s)|\exp(2\kappa), \tag{4.24}$$

with constant $\kappa := \sup_{n\in\mathbb{N}}(\|h_n\|_{L^1(0,T)} + \frac{1}{2}\|\psi_n\|^2_{L^2(0,T;U)}) + M_{N,T} < \infty$.

By (4.24), it remains to show that $\lim_{n\to\infty}\sup_{t\in[0,T]}|I^n_3(t)| = 0$. We use Lemma 4.8. Note that $\sup_{n\in\mathbb{N}}\|w_n\|_{\text{MR}(0,T)} \leq N$ by (4.19), so we only have to verify boundedness of $(\alpha_n) \subset L^1(0,T;V^*)$, where $\alpha_n := -\bar{A}_0 w_n + f_n + (\bar{B}_0 w_n + g_n)\psi_n + b(\psi_n - \psi) \in L^2(0,T;V^*) + L^1(0,T;H) \subset L^1(0,T;V^*)$. The last inclusion is continuous, so it suffices to prove boundedness of $(-\bar{A}_0 w_n + f_n) \subset L^2(0,T;V^*)$ and $(\beta_n) := ((\bar{B}_0 w_n + g_n)\psi_n + b(\psi_n - \psi)) \subset L^1(0,T;H)$. Note that $\|(\bar{B}_0 w_n + g_n)\psi_n\|_H \leq \|\bar{B}_0 w_n + g_n\|_H \|\psi_n\|_U$ with (ψ_n) bounded in $L^2(0,T;U)$ and similar for $b(\psi_n - \psi)$. Thus by the Cauchy–Schwarz inequality, if we show that $(\bar{B}_0 w_n)$ and (g_n) are bounded in $L^2(0,T;\mathcal{L}_2(U,H))$, then boundedness of $(\beta_n) \subset L^1(0,T;H)$ follows ($b \in L^2(0,T;\mathcal{L}_2(U,H))$ does not depend on n). By symmetry in Assumption 2.2(3), $\bar{B}_0 w_n$ and g_n can be estimated in the same way as $\bar{A}_0 w_n := A_0(u^\psi)w_n$ and $f_n := (A_0(u^\psi) - A_0(u^{\psi_n}))u^{\psi_n} + F(u^{\psi_n}) - F(u^\psi)$, respectively. We provide the estimates for the latter here. By Lemma 3.8(ii)(iii), $\|A_0(u^\psi)w_n\|_{L^2(0,T;V^*)} \leq C_{N,T}N < \infty$ and $\|(A_0(u^\psi) - A_0(u^{\psi_n}))u^{\psi_n}\|_{L^2(0,T;V^*)} \leq C_{N,T}N^2 < \infty$. Furthermore, Lemma 3.8(iv) gives $\|F(u^\psi)\|_{L^2(0,T;V^*)} \leq \tilde{C}_{N,T}(1 + N) < \infty$. Finally, $F(u^\psi) \in L^2(0,T;V^*)$ does not depend on n. We conclude that $(-\bar{A}_0 w_n + f_n)$ is bounded in $L^2(0,T;V^*)$ and by the considerations above, (β_n) is bounded in $L^1(0,T;H)$. Lemma 4.8 thus yields $\lim_{n\to\infty}\sup_{t\in[0,T]}|I^n_3(t)| = 0$ and (4.24) gives $w_n \to 0$ in MR(0,T). $\square$

Remark 4.10. Proposition 4.9 also ensures measurability of the map $\mathcal{G}^0 : C([0,T];U_1) \to \text{MR}(0,T)$ defined by (4.4), as required in Theorem 4.5. Note that $\{\int_0^\cdot \psi(s)\,ds : \psi \in L^2(0,T;U)\} = \{v \in W^{1,2}(0,T;U) : v(0) = 0\} =: W_0^{1,2}$. By Sobolev embedding [22, Corollary L.4.6], $W_0^{1,2}$ embeds continuously into $C([0,T];U)$ ($W_0^{1,2}$ is a closed subspace of $W^{1,2}(0,T;U)$). Hence Kuratowski's theorem [25, Th. 15.1] gives $\mathcal{B}(W_0^{1,2}) \subset \mathcal{B}(C([0,T];U))$. Moreover, $\gamma : W_0^{1,2} \to \text{MR}(0,T) : \int_0^\cdot \psi(s)\,ds \mapsto u^\psi$ is continuous, since $\int_0^\cdot \psi_n(s)\,ds \to \int_0^\cdot \psi(s)\,ds$ in $W_0^{1,2}$ implies $\psi_n \to \psi$ in $L^2(0,T;U)$, and $L^2(0,T;U) \to \text{MR}(0,T) : \psi \mapsto u^\psi$ is norm-continuous since it is weakly sequentially continuous by Proposition 4.9. It follows that for $E \in \mathcal{B}(\text{MR}(0,T))$, we have

$$\left(\mathcal{G}^0\right)^{-1}(E) = \begin{cases} \gamma^{-1}(E) \in \mathcal{B}(W_0^{1,2}) \subset \mathcal{B}(C([0,T];U)), & 0 \notin E, \\ \gamma^{-1}(E) \cup (C([0,T];U) \setminus W^{1,2}(0)) \in \mathcal{B}(C([0,T];U)), & 0 \in E. \end{cases}$$

Since $U \hookrightarrow U_1$, Kuratowski's theorem yields $\mathcal{B}(C([0,T];U)) \subset \mathcal{B}(C([0,T];U_1))$. Thus $\mathcal{G}^0$ is measurable.

4.3. Stochastic continuity criterion

It remains to verify the stochastic continuity criterion (iii) of Theorem 4.5. Before we prove that (iii) is satisfied, we first derive some stochastic bounds which we will later apply to $X^\varepsilon := \mathcal{G}^\varepsilon\left(\tilde{W}_1(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot \Psi^\varepsilon(s)\,ds\right)$. In the next lemma we use a stochastic Gronwall lemma as in [3] to avoid further growth bound assumptions on B.

Lemma 4.11. *Suppose that Assumption 2.2 holds and suppose that (A,B) satisfies (2.7). Let $K > 0$, $(\Psi^\varepsilon)_{0<\varepsilon<\frac{1}{2}} \subset \mathcal{A}_K$ and let $x \in H$. For $\varepsilon \in (0,\frac{1}{2})$, let X^ε be a strong solution to*

$$\begin{cases} dX^\varepsilon(t) = (-A(t,X^\varepsilon(t)) + B(t,X^\varepsilon(t))\Psi^\varepsilon(t))\,dt + \sqrt{\varepsilon}B(t,X^\varepsilon(t))\,dW(t), & t \in [0,T], \\ X^\varepsilon(0) = x, \end{cases} \tag{4.25}$$

Then there exists $C > 0$ such that for all $\gamma > 0$,

$$\begin{cases} \sup_{\varepsilon\in(0,\frac{1}{2})}\mathbb{P}(\|X^\varepsilon\|_{\text{MR}(0,T)} > \gamma) \leq \frac{C}{\gamma^2}, \\ \sup_{\varepsilon\in(0,\frac{1}{2})}\mathbb{P}(\|B(\cdot,X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))} > \gamma) \leq \frac{C}{\gamma^2}. \end{cases}$$

The constant C depends only on x,K,T and ϕ,M,θ from (2.7).

Proof. By the Itô formula (A.4), by (2.7) and since $\varepsilon < \frac{1}{2}$, we have a.s. for all $t \in [0,T]$:

$$\|X^\varepsilon(t)\|^2_H - \|x\|^2_H = 2\int_0^t \langle -A(s,X^\varepsilon(s)), X^\varepsilon(s)\rangle + \langle B(s,X^\varepsilon(s))\Psi^\varepsilon(s), X^\varepsilon(s)\rangle\,ds$$

$$+ 2\sqrt{\varepsilon}\int_0^t \langle X^\varepsilon(s), B(s,X^\varepsilon(s))\,dW(s)\rangle + \varepsilon\int_0^t \|B(s,X^\varepsilon(s))\|^2_H\,ds$$

$$\leq 2\int_0^t -\frac{1}{2}\|B(s,X^\varepsilon(s))\|^2_H - \theta\|X^\varepsilon(s)\|^2_V + M\|X^\varepsilon(s)\|^2_H + |\phi(s)|^2\,ds$$

$$+ 2\int_0^t \langle B(s,X^\varepsilon(s))\Psi^\varepsilon(s), X^\varepsilon(s)\rangle\,ds$$

$$+ \varepsilon\int_0^t \|B(s,X^\varepsilon(s))\|^2_H\,ds + 2\sqrt{\varepsilon}\int_0^r \langle X^\varepsilon(s), B(s,X^\varepsilon(s)(\cdot))\,dW(s)\rangle$$

$$\leq -\int_0^t \|\!|B(s, X^\varepsilon(s))|\!\|_H^2 \, ds - 2\theta \|X^\varepsilon\|_{L^2(0,t;V)}^2 + \int_0^t 2M \|X^\varepsilon(s)\|_H^2 \, ds + 2\|\phi\|_{L^2(0,t)}^2$$

$$+ 2\int_0^t \frac{1}{8} \|\!|B(s, X^\varepsilon(s))|\!\|_H^2 + 2\|\Psi^\varepsilon(s)\|_U^2 \|X^\varepsilon(s)\|_H^2 \, ds$$

$$+ \frac{1}{2}\int_0^t \|\!|B(s, X^\varepsilon(s))|\!\|_H^2 \, ds + 2\sqrt{\varepsilon}\int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s)(\cdot)) \rangle \, dW(s)$$

$$= -\frac{1}{4}\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,t;\mathcal{L}_2(U,H))}^2 - 2\theta \|X^\varepsilon\|_{L^2(0,t;V)}^2 + 2\|\phi\|_{L^2(0,t)}^2$$

$$+ \int_0^t 2(M + 2\|\Psi^\varepsilon(s)\|_U^2)\|X^\varepsilon(s)\|_H^2 \, ds + 2\sqrt{\varepsilon}\int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s)(\cdot)) \rangle \, dW(s). \tag{4.26}$$

We conclude that $y_\varepsilon(t) \leq h(t) + \int_0^t y_\varepsilon(s)a_\varepsilon(s) \, ds + 2\sqrt{\varepsilon}\int_0^t \langle X^\varepsilon(s), B(s, X^\varepsilon(s)(\cdot)) \rangle \, dW(s)$, where

$$y_\varepsilon(t) := \|X^\varepsilon(t)\|_H^2 + 2\theta \|X^\varepsilon\|_{L^2(0,t;V)}^2 + \frac{1}{4}\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,t;\mathcal{L}_2(U,H))}^2,$$

$$h(t) := \|x\|_H^2 + 2\|\phi\|_{L^2(0,T)}^2, \qquad a_\varepsilon(t) := 2(M + 2\|\Psi^\varepsilon(t)\|_U^2).$$

Now the stochastic Gronwall inequality [19, Cor. 5.4b), (50)] (with $R := 2MT + 4K^2$) gives

$$\mathbb{P}\Big(\sup_{t\in[0,T]} y_\varepsilon(t) > \gamma\Big) \leq \frac{\exp(2MT + 4K^2)}{\gamma}\mathbb{E}[h(T)] \leq \frac{\exp(2MT + 4K^2)}{\gamma}\big(\|x\|_H^2 + 2\|\phi\|_{L^2(0,T)}^2\big)$$

for all $\gamma > 0$, where we used that $\|\Psi^\varepsilon\|_{L^2(0,T;U)} \leq K$ a.s. since $(\Psi^\varepsilon) \subset \mathcal{A}_K$. Using

$$\Big\{\sup_{t\in[0,T]} y_\varepsilon(t) > \gamma\Big\}$$

$$\supset \Big\{\|X^\varepsilon\|_{C([0,T];H)}^2 + 2\theta\|X^\varepsilon\|_{L^2(0,T;V)}^2 > 2\gamma\Big\} \cup \Big\{\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))}^2 > 4\gamma\Big\}$$

$$\supset \Big\{\|X^\varepsilon\|_{\mathrm{MR}(0,T)}^2 > \frac{4\gamma}{1 \wedge 2\theta}\Big\} \cup \Big\{\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))}^2 > 4\gamma\Big\}$$

and putting $C := \frac{4}{1\wedge 2\theta}\exp(2MT + 4K^2)(\|x\|_H^2 + 2\|\phi\|_{L^2(0,T)}^2)$, yields for all $\varepsilon \in (0, \frac{1}{2})$:

$$\mathbb{P}(\|X^\varepsilon\|_{\mathrm{MR}(0,T)}^2 > \gamma) \leq \frac{C}{\gamma}, \quad \mathbb{P}(\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))}^2 > \gamma) \leq \frac{C}{\gamma}.$$

Consequently, we have $\mathbb{P}(\|X^\varepsilon\|_{\mathrm{MR}(0,T)} > \gamma) = \mathbb{P}(\|X^\varepsilon\|_{\mathrm{MR}(0,T)}^2 > \gamma^2) \leq \frac{C}{\gamma^2}$ and in the same way, $\mathbb{P}(\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))} > \gamma) \leq \frac{C}{\gamma^2}$, uniformly in $\varepsilon \in (0, \frac{1}{2})$. $\quad\square$

We now prove that condition (iii) of Theorem 4.5 is satisfied.

Proposition 4.12. *Suppose that Assumption 2.2 holds and suppose that (A, B) satisfies (2.7). Let $(\Psi^\varepsilon)_{0<\varepsilon<\frac{1}{2}} \subset \mathcal{A}_K$ for some $K > 0$ and let $x \in H$. For $\varepsilon \in (0, \frac{1}{2})$, let X^ε and u^ε be defined by*

$$X^\varepsilon := \mathcal{G}^\varepsilon\Big(\tilde{W}_1(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^{\cdot} \Psi^\varepsilon(s) \, ds\Big), \quad u^\varepsilon := \mathcal{G}^0\Big(\int_0^{\cdot} \Psi^\varepsilon(s) \, ds\Big),$$

where $\mathcal{G}^\varepsilon : C([0,T];U_1) \to \mathrm{MR}(0,T)$ is the measurable map from Lemma 4.6 for $\varepsilon > 0$, $\mathcal{G}^0$ is defined by (4.4) and $\tilde{W}_1$ by (4.2). Then $X^\varepsilon - u^\varepsilon \to 0$ in probability in $\mathrm{MR}(0,T)$ as $\varepsilon \downarrow 0$.

Proof. We will apply Itô's formula and Assumption 2.2. However, because the estimates in Assumption 2.2 are n-dependent, below we use a cut-off argument to reduce to processes that are bounded by n in H-norm.

By Definition 4.4, we have a.s. $\|\Psi^\varepsilon\|_{L^2(0,T;U)} \leq K < \infty$. Thus, recalling (4.4), we have for a.e. $\omega \in \Omega$: $u^\varepsilon(\omega) = u^{\Psi^\varepsilon(\omega)}$, where the latter is the unique strong solution (Theorem 3.16) to (2.8) with $\psi = \Psi^\varepsilon(\omega) \in S_K$. Furthermore, the maximal regularity estimate (3.28) gives

$$N := \mathrm{ess\,sup}_{\omega\in\Omega} \sup_{\varepsilon\in(0,\frac{1}{2})} \|u^\varepsilon(\omega)\|_{\mathrm{MR}(0,T)} < \infty. \tag{4.27}$$

On the other hand, for X^ε we do not have a.s. $\sup_{\varepsilon\in(0,\frac{1}{2})} \|X^\varepsilon\|_{C([0,T];H)} < \infty$, but we do have the boundedness in probability from Lemma 4.11. For $\varepsilon \in (0, \frac{1}{2})$ and $n \in \mathbb{N}$, define

$$E_{n,\varepsilon} := \{\|X^\varepsilon\|_{\mathrm{MR}(0,T)} \leq n\} \cap \{\|u^\varepsilon\|_{\mathrm{MR}(0,T)} \leq N\}.$$

By Lemma 4.6, X^ε is a strong solution to (4.25), so thanks to Lemma 4.11 and (4.27),

$$\mathbb{P}(E_{n,\varepsilon}^c) = \mathbb{P}(\|X^\varepsilon\|_{\mathrm{MR}(0,T)} > n) \leq \frac{C}{n^2},$$

where C is a constant independent of ε. Hence, for all $\varepsilon \in (0, \frac{1}{2})$ and $n \in \mathbb{N}$:

$$\mathbb{P}(\|X^\varepsilon - u^\varepsilon\|_{\text{MR}(0,T)} > \gamma) \leq \mathbb{P}(\{\|X^\varepsilon - u^\varepsilon\|_{\text{MR}(0,T)} > \gamma\} \cap E_{n,\varepsilon}) + \mathbb{P}(E_{n,\varepsilon}^c)$$

$$\leq \mathbb{P}(\{\|X^\varepsilon - u^\varepsilon\|_{\text{MR}(0,T)} > \gamma\} \cap E_{n,\varepsilon}) + \frac{C}{n^2}.$$

Therefore, to have the stated convergence in probability, it suffices to prove that for any $\delta > 0$ and any large enough $n \in \mathbb{N}$:

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\{\|X^\varepsilon - u^\varepsilon\|_{\text{MR}(0,T)} > \delta\} \cap E_{n,\varepsilon}) = 0. \tag{4.28}$$

Let $n \geq N$ be arbitrary, where N is given by (4.27). We prove (4.28). By the Itô formula (A.4), we have for all $t \in [0, T]$:

$$\|X^\varepsilon(t) - u^\varepsilon(t)\|_H^2 = 2 \int_0^t \langle -A(s, X^\varepsilon(s)) + A(s, u^\varepsilon(s)), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ 2 \int_0^t \langle \left(B(s, X^\varepsilon(s)) - B(s, u^\varepsilon(s)) \right) \Psi^\varepsilon(s), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ \varepsilon \int_0^t \|\!|\!| B(s, X^\varepsilon(s)) \|\!|\!|_H^2 \, ds$$

$$+ 2\sqrt{\varepsilon} \int_0^t \langle X^\varepsilon(s) - u^\varepsilon(s), B(s, X^\varepsilon(s)) \, dW(s) \rangle$$

$$=: I_1^\varepsilon(t) + I_2^\varepsilon(t) + I_3^\varepsilon(t) + I_4^\varepsilon(t).$$

Below we derive an estimate of the form

$$I_1^\varepsilon(t) + I_2^\varepsilon(t) \leq -\theta_{n,T} \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2 + \int_0^t |h_{n,\varepsilon}(s)| \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \, ds \tag{4.29}$$

that holds a.s. on the set $E_{n,\varepsilon}$, for every $t \in [0, T]$ and $\varepsilon \in (0, \frac{1}{2})$. Here, $\theta_{n,T}$ is a constant and a.s. $h_{n,\varepsilon} \in L^1(0,T)$, with $\alpha_n := \sup_{\varepsilon \in (0, \frac{1}{2})} \text{ess sup}_\Omega \|h_{n,\varepsilon} \mathbb{1}_{E_{n,\varepsilon}}\|_{L^1(0,T)} < \infty$. Then, a.s. on $E_{n,\varepsilon}$,

$$\|X^\varepsilon(t) - u^\varepsilon(t)\|_H^2 \leq -\theta_{n,T} \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2 + I_3^\varepsilon(t) + \sup_{r \in [0,t]} I_4^\varepsilon(r) + \int_0^t |h_{n,\varepsilon}(s)| \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \, ds,$$

so Lemma A.1 (Gronwall) gives pointwise in a.e. $\omega \in E_{n,\varepsilon}$:

$$\|X^\varepsilon - u^\varepsilon\|_{C([0,T];H)}^2 + \|X^\varepsilon - u^\varepsilon\|_{L^2(0,T;V)}^2 \leq (1 + \theta_{n,T}^{-1}) \exp(\alpha_n) \left(I_3^\varepsilon(T) + \sup_{t \in [0,T]} |I_4^\varepsilon(t)| \right).$$

Putting $c_n := 2(1 + \theta_{n,T}^{-1}) \exp(\alpha_n)$, we thus have $\|X^\varepsilon - u^\varepsilon\|_{\text{MR}(0,T)}^2 \leq c_n \left(I_3^\varepsilon(T) + \sup_{t \in [0,T]} |I_4^\varepsilon(t)| \right)$ a.s. on $E_{n,\varepsilon}$, and therefore,

$$\mathbb{P}(\{\|X^\varepsilon - u^\varepsilon\|_{\text{MR}(0,T)}^2 > \delta\} \cap E_{n,\varepsilon}) \leq \sum_{i=3}^4 \mathbb{P}(\{\sup_{t \in [0,T]} |I_i^\varepsilon(t)| > \frac{\delta}{2c_n}\} \cap E_{n,\varepsilon}).$$

Hence, after we have proved (4.29), for (4.28), it suffices to prove two convergences in probability:

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(I_3^\varepsilon(T) > \delta) = 0 \text{ for any } \delta > 0, \tag{4.30}$$

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\sup_{t \in [0,T]} |I_4^\varepsilon(t)| > \delta) = 0 \text{ for any } \delta > 0. \tag{4.31}$$

All in all, recalling that we reduced the original problem to proving (4.28), by the reasoning above it remains to establish (4.29), (4.30) and (4.31).

Let us prove (4.29). Recall that $A(t, v) = A_0(t, v)v - F(t, v) - f$ and $B(t, v) = B_0(t, v)v + G(t, v) + g$, see Assumption 2.2(1). We have pointwise on $E_{n,\varepsilon}$, for all $\varepsilon \in (0, \frac{1}{2})$:

$$\frac{1}{2}(I_1^\varepsilon(t) + I_2^\varepsilon(t)) = \int_0^t \langle -A_0(s, u^\varepsilon(s))(X^\varepsilon(s) - u^\varepsilon(s)), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ \int_0^t \langle \left(A_0(s, u^\varepsilon(s)) - A_0(s, X^\varepsilon(s)) \right) X^\varepsilon(s), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ \int_0^t \langle F(X^\varepsilon(s)) - F(u^\varepsilon(s)), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ \int_0^t \langle B_0(s, u^\varepsilon(s))(X^\varepsilon(s) - u^\varepsilon(s))\Psi^\varepsilon(s), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ \int_0^t \langle \left(B_0(s, X^\varepsilon(s)) - B_0(s, u^\varepsilon(s)) \right) X^\varepsilon(s)\Psi^\varepsilon(s), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ \int_0^t \langle \big(G(X^\varepsilon(s)) - G(u^\varepsilon(s)) \big) \Psi^\varepsilon(s), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$\leq \int_0^t \langle -A_0(s, u^\varepsilon(s))(X^\varepsilon(s) - u^\varepsilon(s)), X^\varepsilon(s) - u^\varepsilon(s) \rangle \, ds$$

$$+ \int_0^t \frac{1}{2} \|B_0(s, u^\varepsilon(s))(X^\varepsilon(s) - u^\varepsilon(s))\|_H^2 + \frac{1}{2} \|\Psi^\varepsilon(s)\|_U^2 \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \, ds$$

$$+ \int_0^t \| \big(A_0(s, u^\varepsilon(s)) - A_0(s, X^\varepsilon(s)) \big) X^\varepsilon(s) \|_{V*} \|X^\varepsilon(s) - u^\varepsilon(s)\|_V \, ds$$

$$+ \int_0^t \| F(X^\varepsilon(s)) - F(u^\varepsilon(s)) \|_{V*} \|X^\varepsilon(s) - u^\varepsilon(s)\|_V \, ds$$

$$+ \int_0^t \| \big(B_0(s, X^\varepsilon(s)) - B_0(s, u^\varepsilon(s)) \big) X^\varepsilon(s) \|_H \|\Psi^\varepsilon(s)\|_U \|X^\varepsilon(s) - u^\varepsilon(s)\|_H \, ds$$

$$+ \int_0^t \| G(X^\varepsilon(s)) - G(u^\varepsilon(s)) \|_H \|\Psi^\varepsilon(s)\|_U \|X^\varepsilon(s) - u^\varepsilon(s)\|_H \, ds$$

$$\leq \int_0^t -\theta_{n,T} \|X^\varepsilon(s) - u^\varepsilon(s)\|_V^2 \, ds$$

$$+ \int_0^t \Big(M_{n,T} + \frac{1}{2} \|\Psi^\varepsilon(s)\|_U^2 \Big) \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \, ds$$

$$+ \int_0^t C_\sigma \| \big(A_0(s, u^\varepsilon(s)) - A_0(s, X^\varepsilon(s)) \big) X^\varepsilon(s) \|_{V*}^2 + \sigma \|X^\varepsilon(s) - u^\varepsilon(s)\|_V^2 \, ds$$

$$+ \int_0^t C_\sigma \| F(X^\varepsilon(s)) - F(u^\varepsilon(s)) \|_{V*}^2 + \sigma \|X^\varepsilon(s) - u^\varepsilon(s)\|_V^2 \, ds$$

$$+ \int_0^t \frac{1}{2} \| \big(B_0(s, X^\varepsilon(s)) - B_0(s, u^\varepsilon(s)) \big) X^\varepsilon(s) \|_H^2 + \frac{1}{2} \|\Psi^\varepsilon(s)\|_U^2 \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \, ds$$

$$+ \frac{1}{2} \int_0^t \| G(X^\varepsilon(s)) - G(u^\varepsilon(s)) \|_H^2 + \frac{1}{2} \|\Psi^\varepsilon(s)\|_U^2 \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \, ds$$

$$=: -\theta_{n,T} \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2 + J_1^\varepsilon(t) + J_2^{\varepsilon,\sigma}(t) + J_3^{\varepsilon,\sigma}(t) + J_4^\varepsilon(t) + J_5^\varepsilon(t) \tag{4.32}$$

for any $\sigma > 0$, where $C_\sigma := \frac{1}{4\sigma}$ from Young's inequality and $\theta_{n,T}$ and $M_{n,T}$ are the constants from the local coercivity of (A_0, B_0) in Assumption 2.2(2).

Next, we estimate the terms of (4.32). J_1^ε is already in the desired form for application of Gronwall's inequality. Moreover, Lemma 3.8(iii) yields

$$J_2^{\varepsilon,\sigma}(t) \leq C_\sigma C_{n,T}^2 \int_0^t \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \|X^\varepsilon(s)\|_V^2 \, ds + \sigma \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2, \tag{4.33}$$

$$J_4^\varepsilon(t) \leq \frac{1}{2} \int_0^t \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \big(C_{n,T}^2 \|X^\varepsilon(s)\|_V^2 + \|\Psi^\varepsilon(s)\|_U^2 \big) \, ds. \tag{4.34}$$

Similarly, Lemma 3.8(v) gives for any $\tilde{\sigma} > 0$:

$$J_3^{\varepsilon,\sigma}(t) \leq C_\sigma C_{n,T,\tilde{\sigma}} \int_0^t \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \big(1 + \|X^\varepsilon(s)\|_V^2 + \|u^\varepsilon(s)\|_V^2 \big) \, ds + C_\sigma \tilde{\sigma} C_{n,T}^2 \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2 + \sigma \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2 \tag{4.35}$$

for some constant $C_{n,T,\tilde{\sigma}} > 0$, and

$$J_5^\varepsilon(t) \leq \frac{1}{2} \int_0^t \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \big(C_{n,T,\tilde{\sigma}}(1 + \|X^\varepsilon(s)\|_V^2 + \|u^\varepsilon(s)\|_V^2) + \|\Psi^\varepsilon(s)\|_U^2 \big) \, ds + \frac{1}{2} \tilde{\sigma} C_{n,T}^2 \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2. \tag{4.36}$$

Now we fix $\sigma := \frac{\theta_{n,T}}{8}$. Then we fix $\tilde{\sigma} := \frac{\theta_{n,T}}{8 C_{n,T}^2 (C_\sigma \vee \frac{1}{2})}$. Combining estimates (4.33)–(4.36) with (4.32), we obtain

$$I_1^\varepsilon(t) + I_2^\varepsilon(t) \leq -\theta_{n,T} \|X^\varepsilon - u^\varepsilon\|_{L^2(0,t;V)}^2 + \int_0^t h_{n,\varepsilon}(s) \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \, ds,$$

where $h_{n,\varepsilon}$ is of the form

$$h_{n,\varepsilon}(s) = C_{n,T,\sigma,\tilde{\sigma}} \big(1 + \|\Psi^\varepsilon(s)\|_U^2 + \|X^\varepsilon(s)\|_V^2 + \|u^\varepsilon(s)\|_V^2 \big), \tag{4.37}$$

for a constant $C_{n,T,\sigma,\tilde{\sigma}} > 0$. Now, a.s. $\Psi^\varepsilon \in L^2(0,T;U)$ and a.s. $X^\varepsilon, u^\varepsilon \in L^2(0,T;V)$, thus a.s. $h_{n,\varepsilon} \in L^1(0,T)$. By definition of $E_{n,\varepsilon}$, by (4.37) and since $(\Psi^\varepsilon) \subset \mathcal{A}_K$, we have a.s. $\|h_{n,\varepsilon} \mathbb{1}_{E_{n,\varepsilon}}\|_{L^1(0,T)} \leq C_{n,T,\sigma,\tilde{\sigma}}(T + K^2 + 2n^2)$ for every $\varepsilon \in (0, \frac{1}{2})$. Thus, $h_{n,\varepsilon}$ has all required properties and (4.29) is satisfied a.s. on the set $E_{n,\varepsilon}$, for every $\varepsilon \in (0, \frac{1}{2})$, as desired.

Regarding (4.30), by Lemma 4.11 we have for any (fixed) $\delta > 0$:

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(I_3^\varepsilon(T) > \delta) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))}^2 > \delta \varepsilon^{-1}) \leq \lim_{\varepsilon \downarrow 0} C \varepsilon \delta^{-1} = 0.$$

It remains to prove (4.31). Note that I_4^ε is a continuous local martingale (using Lemma 4.11) starting at zero, with $[I_4^\varepsilon](T) = \int_0^T \varepsilon \|\langle X^\varepsilon(s) - u^\varepsilon(s), B(s, X^\varepsilon(s))(\cdot)\rangle\|_{\mathcal{L}_2(U,\mathbb{R})}^2 \, ds$, where $[I_4^\varepsilon]$ denotes the quadratic variation. Thus, by [23, Prop. 18.6], (4.31) is equivalent to

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}\Big(\varepsilon \int_0^T \|\langle X^\varepsilon(s) - u^\varepsilon(s), B(s, X^\varepsilon(s))(\cdot)\rangle\|_{\mathcal{L}_2(U,\mathbb{R})}^2 \, ds > \delta \Big) = 0 \text{ for all } \delta > 0. \tag{4.38}$$

We prove the latter. We have for all $\delta > 0$ and $\varepsilon \in (0, \frac{1}{2})$:

$$\mathbb{P}\Big(\varepsilon \int_0^T \|\langle X^\varepsilon(s) - u^\varepsilon(s), B(s, X^\varepsilon(s))(\cdot)\rangle\|_{\mathcal{L}_2(U,\mathbb{R})}^2 \, ds > \delta \Big)$$

$$\leq \mathbb{P}\Big(\int_0^T \|X^\varepsilon(s) - u^\varepsilon(s)\|_H^2 \|B(s, X^\varepsilon(s))\|_H^2 \, ds > \delta \varepsilon^{-1} \Big)$$

$$\leq \mathbb{P}\Big(\|X^\varepsilon - u^\varepsilon\|_{C([0,T];H)}^2 \int_0^T \|B(s, X^\varepsilon(s))\|_H^2 \, ds > \delta \varepsilon^{-1} \Big)$$

$$\leq \mathbb{P}\Big(\|X^\varepsilon - u^\varepsilon\|_{C([0,T];H)} > (\delta \varepsilon^{-1})^{\frac{1}{4}} \Big) + \mathbb{P}\Big(\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))} > (\delta \varepsilon^{-1})^{\frac{1}{4}} \Big). \tag{4.39}$$

Due to Lemma 4.11, we have

$$\mathbb{P}\Big(\|X^\varepsilon - u^\varepsilon\|_{C([0,T];H)} > (\delta \varepsilon^{-1})^{\frac{1}{4}} \Big) \leq \mathbb{P}\Big(\|X^\varepsilon\|_{C([0,T];H)} > \tfrac{1}{2}(\delta \varepsilon^{-1})^{\frac{1}{4}} \Big) + \mathbb{P}\Big(\|u^\varepsilon\|_{C([0,T];H)} > \tfrac{1}{2}(\delta \varepsilon^{-1})^{\frac{1}{4}} \Big)$$

$$\leq 4C(\varepsilon \delta^{-1})^{\frac{1}{2}} + \mathbb{P}\Big(\|u^\varepsilon\|_{C([0,T];H)} > \tfrac{1}{2}(\delta \varepsilon^{-1})^{\frac{1}{4}} \Big), \tag{4.40}$$

$$\mathbb{P}\Big(\|B(\cdot, X^\varepsilon(\cdot))\|_{L^2(0,T;\mathcal{L}_2(U,H))} > (\delta \varepsilon^{-1})^{\frac{1}{4}} \Big) \leq C(\varepsilon \delta^{-1})^{\frac{1}{2}}. \tag{4.41}$$

Note that $\mathbb{P}\big(\|u^\varepsilon\|_{C([0,T];H)} > \frac{1}{2}(\delta \varepsilon^{-1})^{\frac{1}{4}} \big) = 0$ for all $\varepsilon \in (0, \frac{\delta}{16N^4} \wedge \frac{1}{2})$ by (4.27). Thus, combining (4.40), (4.41) and continuing from (4.39), we see that for all $\varepsilon \in (0, \frac{\delta}{16N^4} \wedge \frac{1}{2})$:

$$\mathbb{P}\Big(\varepsilon \int_0^T \|\langle X^\varepsilon(s) - u^\varepsilon(s), B(s, X^\varepsilon(s))(\cdot)\rangle\|_{\mathcal{L}_2(U,\mathbb{R})}^2 \, ds > \delta \Big) \leq 5C(\varepsilon \delta^{-1})^{\frac{1}{2}}.$$

Letting $\varepsilon \downarrow 0$ we arrive at (4.38). □

4.4. Proof of Theorem 2.6

Proving Theorem 2.6 is now only a matter of combining.

Proof. We verify the criteria of Theorem 4.5. Note that $\mathcal{E} := \mathrm{MR}(0,T)$ is Polish. Define $\mathcal{G}^0$ by (4.4) and for $\varepsilon > 0$, let $\mathcal{G}^\varepsilon$ be the measurable map from Lemma 4.6. Now, (i) holds by Lemma 4.6, (ii) holds by Proposition 4.9 and (iii) holds by Proposition 4.12. The proof is complete. □

Lastly, a small remark.

Remark 4.13. The LDP of Theorem 2.6 implies the following Strong Law of Large Numbers: we have $Y^\varepsilon \to Y^0$ a.s. as $\varepsilon \downarrow 0$, where Y^0 solves (4.1) with $\varepsilon = 0$, i.e. with only the drift term. This follows from the Borel-Cantelli lemma and the fact that the rate function has a unique zero at Y^0. Indeed, $I(Y^0) = I(u^0) = 0$ and if $I(z) = 0$, one finds $(\psi_n) \subset L^2(0,T;U)$ with $z = u^{\psi_n}$ and $\|\psi_n\|_{L^2(0,T;U)} \to 0$. Then, $u^{\psi_n} \to z$ in $\mathrm{MR}(0,T)$ and by Proposition 4.9, $u^{\psi_n} \to u^0 = Y^0$ in $\mathrm{MR}(0,T)$, thus $\{z \in \mathrm{MR}(0,T) : I(z) = 0\} = \{Y^0\}$.

5. Application to fluid dynamics

In this subsection, we apply our results to an abstract fluid dynamics model considered in several earlier works. We closely follow the presentation of [6,12] and focus on what the large deviation principle of Theorem 2.6 becomes in this setting. Afterwards, we specialize to the Navier–Stokes equations with gradient noise to make our results even more concrete.

5.1. Abstract model

The abstract form of the problem we consider is as follows

$$\begin{cases} dY^\varepsilon(t) + A_0(t)Y^\varepsilon \, dt = \Phi(Y^\varepsilon(t), Y^\varepsilon(t)) \, dt + \sqrt{\varepsilon}\big(B_0(t)Y^\varepsilon(t) + G(t, Y^\varepsilon(t)) \big) \, dW(t), \\ u(0) = x. \end{cases} \tag{5.1}$$

Here, Φ is supposed to take care of the typical bilinear term appearing in equations in fluid dynamics. In particular, all of the following models can be included in the abstract framework below: 2D Navier–Stokes, 2D Boussinesq equations, quasigeostrophic equations, 2D magneto-hydrodynamic equations, 2D magnetic Bénard problem, 3D Leray α-model for Navier–Stokes equations and shell models of turbulence.

To put this problem in the setting of (2.4) and Assumption 2.2, we assume the following.

Assumption 5.1.

(1) $A_0 : \mathbb{R}_+ \to \mathcal{L}(V, V^*)$ and $B_0 : \mathbb{R}_+ \to \mathcal{L}(V, \mathcal{L}_2(U, H))$ are measurable and for all $T > 0$, $\sup_{t \in [0,T]} \|A_0(t)\|_{\mathcal{L}(V,V^*)} < \infty$ and $\sup_{t \in [0,T]} \|B_0(t)\|_{\mathcal{L}(V, \mathcal{L}_2(U,H))} < \infty$. Moreover, for all $T > 0$, there exist $\theta > 0$ and $M \geq 0$ such that for all $v \in V$ and $t \in [0, T]$,

$$\langle v, A_0(t)v \rangle - \tfrac{1}{2}\|B_0(t)v\|^2_{\mathcal{L}_2(U,H)} \geq \theta \|v\|^2_V - M\|v\|^2_H.$$

(2) For some $\beta_1 \in (\tfrac{1}{2}, \tfrac{3}{4}]$, $\Phi : V_{\beta_1} \times V_{\beta_1} \to V^*$ is bilinear and satisfies

$$\|\Phi(u, v)\|_{V^*} \leq C \|u\|_{\beta_1} \|v\|_{\beta_1}, \quad \langle u, \Phi(u, u) \rangle = 0, \qquad u, v \in V.$$

(3) For some $\beta_2 \in (\tfrac{1}{2}, 1)$, $G : \mathbb{R}_+ \times V_{\beta_2} \to \mathcal{L}_2(U, H)$ is measurable and satisfies the following Lipschitz conditions: for all $T > 0$, there exists a constant C such that for all $u, v \in V_{\beta_2}$ and $t \in [0, T]$,

$$\|G(t, u) - G(t, v)\|_{\mathcal{L}_2(U,H)} \leq C \|u - v\|_{V_{\beta_2}} \quad \text{and} \quad \|G(t, u)\|_{\mathcal{L}_2(U,H)} \leq C(1 + \|u\|_{V_{\beta_2}}).$$

The associated skeleton equation is given by

$$\begin{cases} (u^\psi)'(t) + A_0(t)u^\psi(t) = \Phi(u^\psi(t), u^\psi(t)) + \big(B_0(t)u^\psi(t) + G(t, u^\psi(t))\big)\psi(t), & t \in [0, T], \\ u^\psi(0) = x. \end{cases} \tag{5.2}$$

Theorem 5.2. *Suppose that Assumption 5.1 holds, Then for every $x \in H$ and $\varepsilon \in (0, 1]$, the problem (5.1) has a unique global solution*

$$Y^\varepsilon \in L^2_{\mathrm{loc}}([0, \infty); V) \cap C([0, \infty); H) \ \text{a.s.}$$

Moreover, for every $T > 0$, (Y^ε) satisfies the LDP on $L^2(0, T; V) \cap C([0, T]; H)$ with rate function $I : L^2_{\mathrm{loc}}(0, T; V) \cap C([0, T]; H) \to [0, +\infty]$ given by

$$I(z) = \frac{1}{2} \inf \left\{ \int_0^T \|\psi(s)\|^2_U \, \mathrm{d}s : \psi \in L^2(0, T; U), z = u^\psi \right\},$$

where $\inf \varnothing := +\infty$ and u^ψ is the strong solution to (5.2).

Proof. In [6, Th. 7.10] it is shown that Assumption 5.1 is satisfied, noting that the arguments also work for the time-dependent setting. Thus well-posedness follows from Theorem 2.3 and the large deviation principle follows from Theorem 2.6. $\quad\square$

5.2. LDP for Navier–Stokes equations with gradient noise

Next we specialize the result to the 2D Navier–Stokes equations on an arbitrary open set $\mathcal{O} \subseteq \mathbb{R}^2$ (possibly unbounded), and we let the noise term contain a transport/gradient term. The large deviation principle is new even for the case $\mathcal{O} = \mathbb{R}^2$. Indeed, as explained in the introduction, previous results in the literature either contain a gap, or do not have gradient noise, or assume boundedness of the domain $\mathcal{O}$.

For simplicity we only consider the case of Itô noise. For details on Stratonovich noise, see [4, App. A]. We follow the presentation of [6, §7.3.4].

Consider the following Navier–Stokes system with no-slip condition on domain $\mathcal{O}$:

$$\begin{cases} \mathrm{d}Y^\varepsilon = \big[\nu \Delta Y^\varepsilon - (Y^\varepsilon \cdot \nabla)Y^\varepsilon - \nabla P^\varepsilon\big] \, \mathrm{d}t + \sqrt{\varepsilon} \sum_{n \geq 1} \big[(b_n \cdot \nabla)Y^\varepsilon + g_n(\cdot, Y^\varepsilon) - \nabla \widetilde{P}_n^\varepsilon\big] \, \mathrm{d}W_t^n, \\ \operatorname{div} Y^\varepsilon = 0, \\ Y^\varepsilon = 0 \quad \text{on } \partial\mathcal{O}, \\ Y^\varepsilon(0, \cdot) = u_0. \end{cases} \tag{5.3}$$

Here, $Y^\varepsilon := (Y^{\varepsilon,1}, Y^{\varepsilon,2}) : [0, \infty) \times \Omega \times \mathcal{O} \to \mathbb{R}^2$ denotes the unknown velocity field, $P^\varepsilon, \widetilde{P}_n^\varepsilon : [0, \infty) \times \Omega \times \mathcal{O} \to \mathbb{R}$ the unknown pressures, $(W_t^n : t \geq 0)_{n \geq 1}$ a given sequence of independent standard Brownian motions and

$$(b_n \cdot \nabla)u := \left(\sum_{j \in \{1,2\}} b_n^j \partial_j u^k \right)_{k=1,2}, \qquad (u \cdot \nabla)u := \left(\sum_{j \in \{1,2\}} u^j \partial_j u^k \right)_{k=1,2}.$$

Assumption 5.3. Let $d = 2$. Let $b^j = (b_n^j)_{n \geq 1} : \mathbb{R}_+ \times \mathcal{O} \to \ell^2$ be measurable and bounded and suppose that for every $T > 0$ there exists a $\mu \in (0, \nu)$ such that for all $x \in \mathcal{O}$ and $t \in [0, T]$,

$$\frac{1}{2} \sum_{n \geq 1} \sum_{i,j \in \{1,2\}} b_n^i(x) b_n^j(x) \xi_i \xi_j \leq \mu |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d.$$

Moreover, $g^1, g^2 : \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}^2 \to \ell^2$ and for every $T > 0$ there exists a constant L_g such that

$$\|g^j(t, x, y) - g^j(t, x, y')\|_{\ell^2} \leq L_g |y - y'|$$

$$\|g^j(t, x, y)\|_{\ell^2} \leq L_g(1 + |y|), \quad x \in \mathcal{O}, y, y' \in \mathbb{R}^2, \ t \in [0, T], \ j \in \{1, 2\}.$$

As in [6, §7.3.4], we can use the Helmholtz projection $\mathbb{P}$ to rewrite (5.3) as (5.1). To this end, let $\mathcal{U} = \ell^2$ with standard basis $(e_n)_{n \geq 1}$ and let

$$H = \mathbb{L}^2(\mathcal{O}), \quad V = \mathbb{H}_0^1(\mathcal{O}) = H_0^1(\mathcal{O}; \mathbb{R}^2) \cap \mathbb{L}^2(\mathcal{O}), \quad V^* := \mathbb{H}^{-1}(\mathcal{O}) = (\mathbb{H}_0^1(\mathcal{O}))^*,$$

where $\mathbb{L}^2(\mathcal{O})$ denotes the range of the Helmholtz projection in $L^2(\mathcal{O}; \mathbb{R}^2)$. By the divergence free condition, $(u \cdot \nabla)u = \mathrm{div}(u \otimes u)$, where $u \otimes u$ is the matrix with components $u_j u_k$. Assuming $x \in \mathbb{L}^2(\mathcal{O})$, after applying the Helmholtz projection $\mathbb{P}$ to (5.3), we can write (5.3) in the form (5.1) with

$$A_0 = -\nu \mathbb{P} \Delta, \quad \Phi(u, v) = -\mathbb{P}\mathrm{div}[u \otimes v], \quad (B_0 u)e_n = \mathbb{P}[(b_n \cdot \nabla)u], \quad G(u)e_n = \mathbb{P}g_n(\cdot, u).$$

For $\psi \in L^2(0, T; \ell^2)$, consider the following skeleton equation on $\mathcal{O}$:

$$\begin{cases} \mathrm{d}u^\psi = \left[\nu \Delta u^\psi - \mathbb{P}\mathrm{div}(u^\psi \otimes u^\psi)\right] \mathrm{d}t + \sum_{n \geq 1} \left(\mathbb{P}[(b_n \cdot \nabla)u^\psi] + \mathbb{P}g_n(\cdot, u^\psi)\right)\psi_n, \\ u^\psi = 0 \quad \text{on } \partial\mathcal{O}, \\ u^\psi(0, \cdot) = u_0. \end{cases} \tag{5.4}$$

In [6, §7.3.4] it is verified that Assumption 5.1 is fulfilled for the above setting. Thus we obtain the next result immediately from Theorem 5.2.

Theorem 5.4 (LDP for the 2D Navier–Stokes equations with transport noise). *Let $d = 2$. Suppose that Assumption 5.3 holds, Then for every $x \in \mathbb{L}^2(\mathcal{O})$ and $\varepsilon \in (0, 1]$, there exists a unique global solution $Y^\varepsilon \in L^2_{\mathrm{loc}}([0, \infty); \mathbb{H}_0^1(\mathcal{O})) \cap C([0, \infty); \mathbb{L}^2(\mathcal{O}))$ to (5.3). Moreover, for every $T > 0$, (Y^ε) satisfies the LDP on $\mathrm{MR}(0, T) := L^2(0, T; \mathbb{H}_0^1(\mathcal{O})) \cap C([0, T]; \mathbb{L}^2(\mathcal{O}))$ with rate function $I : \mathrm{MR}(0, T) \to [0, +\infty]$ given by*

$$I(z) = \frac{1}{2} \inf \left\{ \int_0^T \|\psi(s)\|_{\ell^2}^2 \, \mathrm{d}s \, : \, \psi \in L^2(0, T; \ell^2), z = u^\psi \right\},$$

where $\inf \varnothing := +\infty$ and u^ψ is the strong solution to (5.4).

Acknowledgement

Appendix A.

For convenience we state some tools that are used repeatedly. To begin, let us state a direct consequence of Gronwall's inequality.

Lemma A.1 (Gronwall). *Let $T > 0$ and let $F, G, H, K : [0, T] \to \mathbb{R}_+$ with F and G continuous, K non-decreasing and $H \in L^1(0, T)$. Suppose that $F(t) \leq -G(t) + K(t) + \int_0^t F(s)H(s) \, \mathrm{d}s$ for all $t \in [0, T]$. Then*

$$\sup_{t \in [0, T]} F(t) \vee \sup_{t \in [0, T]} G(t) \leq K(T) \exp[\|H\|_{L^1(0, T)}].$$

The following special case of a chain rule from [34] is useful, since it applies to $L^2(0, T; V^*) + L^1(0, T; H)$-valued integrands.

Lemma A.2. *[34, Lem. 2.2 p. 30] Let (V, H, V^*) be a Gelfand triple of Hilbert spaces. Let $x \in H$, $u \in C([0, T]; H) \cap L^2(0, T; V)$ and $v \in L^2(0, T; V^*) + L^1(0, T; H)$ be such that*

$$u(t) = x + \int_0^t v(s) \, \mathrm{d}s \text{ in } V^*, \quad \text{for all } t \in [0, T]. \tag{A.1}$$

Then for all $t \in [0, T]$:

$$\|u(t)\|_H^2 = \|x\|_H^2 + 2 \int_0^t \langle v(s), u(s) \rangle \, \mathrm{d}s. \tag{A.2}$$

Proof. Note that $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ and $v \in L^1(0, T; V^*)$. Thus by (A.1), u is weakly differentiable with $u' = v$ a.e. on $[0, T]$, see [21, Lem. 2.5.8]. Hence, $u' = v \in L^2(0, T; V^*) + L^1(0, T; H)$. Also, (A.1) implies absolute continuity of $u : [0, T] \to V^*$. Now [34, Lem. 2.2 p. 30, $p = 2$] gives $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 = 2\langle u'(t), u(t) \rangle = 2\langle v(t), u(t) \rangle$ a.e., proving (A.2). $\quad\square$

Stochastic versions of the chain rule, or Itô formula, are also given in [34]. The following special case is suited for random, $L^2(0, T; V^*) + L^1(0, T; H)$-valued integrands. We recall that the class of integrable processes for a U-cylindrical Brownian motion (Definition 4.1) is given by

$$\mathcal{N}(0, T) := \left\{ \Phi : [0, T] \times \Omega \to \mathcal{L}_2(U, H) : \Phi \text{ strongly progressively measurable}, \mathbb{P}(\|\Phi\|_{L^2(0, T; \mathcal{L}_2(U, H))} < \infty) = 1 \right\}. \tag{A.3}$$

Lemma A.3. *[34, Th. 3.1 p. 57, Th. 3.3 p. 59] Let (V, H, V^*) be a Gelfand triple of Hilbert spaces and let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in \mathbb{R}_+})$ be a filtered probability space. Suppose that*

(i) $u \in L^0(\Omega; L^2(0,T;V))$, $u_0 \in L^0(\Omega, \mathcal{F}_0, \mathbb{P}; H)$,

(ii) $v \in L^0(\Omega; L^1(0,T;H)) + L^0(\Omega; L^2(0,T;V^*))$, v is adapted,

(iii) $\Phi \in \mathcal{N}(0,T)$, W is a U-cylindrical Brownian motion,

(iv) a.s. for all $t \in [0,T]$: $u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}W(s)$.

Then, $u \in L^0(\Omega; C([0,T];H))$ and a.s. for all $t \in [0,T]$:

$$\|u(t)\|_H^2 = \|u_0\|_H^2 + 2\int_0^t \langle v(s), u(s)\rangle\,\mathrm{d}s + 2\int_0^t \langle u(s), \Phi(s)\,\mathrm{d}W(s)\rangle + \int_0^t \|\|\Phi(s)\|\|_H^2\,\mathrm{d}s. \tag{A.4}$$

Finally, we relate the U-cylindrical Brownian motion W of Definition 4.1 to the $\mathbb{R}^\infty$-Brownian motion $\tilde{W}$ of Definition 4.2, as well as their stochastic integrals constructed in [31] and [29], respectively.

An $\mathbb{R}^\infty$-Brownian motion $\tilde{W} = ((\beta_k)_{k\in\mathbb{N}}, (e_k)_{k\in\mathbb{N}})$ in U corresponds to a Wiener process $\tilde{W}_1$ in a larger space U_1, with trace class covariance. That is, for any Hilbert-Schmidt embedding $J : U \hookrightarrow U_1$, the U_1-valued process given by

$$\tilde{W}_1(t) := \sum_{k=1}^\infty \beta_k(t) J e_k, \qquad t \in [0,T], \tag{A.5}$$

defines a Q_1-Wiener process on U_1, with $Q_1 := JJ^* \in \mathcal{L}(U_1, U_1)$ nonnegative definite, symmetric and of trace class [29, Prop. 2.5.2].

It is well-known that $\mathcal{N}(0,T)$ from (A.3) is the class of integrable processes for both $\tilde{W}$ and W, see [29, p. 52, p. 53] and [1, Prop. 2.13] and the proof in [31, p. 306, §5.4 ($p = 0$)]. The next proposition relates the stochastic integrals corresponding to $\tilde{W}$ and W.

Proposition A.4. *For any U-cylindrical Brownian motion $W \in \mathcal{L}(L^2(\mathbb{R}_+; U), L^2(\Omega))$ and any orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of U, there exists an $\mathbb{R}^\infty$-Brownian motion $\tilde{W} = ((\beta_k)_{k\in\mathbb{N}}, (e_k)_{k\in\mathbb{N}})$ with*

$$W(\mathbb{1}_{(0,t]} \otimes e_k) = \beta_k(t) \quad \text{in } L^2(\Omega), \quad \text{for all } k \in \mathbb{N} \text{ and } t \in \mathbb{R}_+. \tag{A.6}$$

The sequence $(\beta_k)_{k\in\mathbb{N}}$ in $\tilde{W}$ is unique up to indistinguishability.

Reversely, given an $\mathbb{R}^\infty$-Brownian motion $\tilde{W} = ((\beta_k)_{k\in\mathbb{N}}, (e_k)_{k\in\mathbb{N}})$, there exists a unique U-cylindrical Brownian motion $W \in \mathcal{L}(L^2(\mathbb{R}_+; U), L^2(\Omega))$ that satisfies (A.6).

If (A.6) holds, then for any $\Phi \in \mathcal{N}(0,T)$ and $t \in [0,T]$, we have $\mathbb{P}$-a.s. in $C([0,T];H)$:

$$\int_0^t \Phi(s)\,\mathrm{d}W(s) = \int_0^t \Phi(s) \circ J^{-1}\,\mathrm{d}\tilde{W}_1(s) =: \int_0^t \Phi(s)\,\mathrm{d}\tilde{W}(s), \qquad t \in [0,T], \tag{A.7}$$

with $\tilde{W}_1$ as in (A.5). Here, the integral on the left-hand side is the one constructed in [31] and the middle and right integral are those constructed in [29, §2.3, §2.5].

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