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Gentile, C.; Morales-España, G.; Ramos, A

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ORIGINAL PAPER

A tight MIP formulation of the unit commitment problem with start-up and shut-down constraints

C. Gentile[1] · G. Morales-España[2] · A. Ramos[3]

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Abstract This paper provides the convex hull description of the single thermal Unit Commitment (UC) problem with the following basic operating constraints: (1) generation limits, (2) start-up and shut-down capabilities, and (3) minimum up and down times. The proposed constraints can be used as the core of any unit commitment formulation to strengthen the lower bound in enumerative approaches. We provide evidence that dramatic improvements in computational time are obtained by solving the self-UC problem and the network-constrained UC problem with the new inequalities for different case studies.

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✉ C. Gentile
gentile@iasi.cnr.it

G. Morales-España
g.a.moralesespana@tudelft.nl

A. Ramos
andres.ramos@upcomillas.es

1 Istituto di Analisi dei Sistemi ed Informatica "A. Ruberti", Consiglio Nazionale delle Ricerche, Via dei Taurini 19, 00185 Roma, Italy

2 Department of Electrical Sustainable Energy, Delft University of Technology, 2628 CD Delft, The Netherlands

3 Institute for Research in Technology (IIT) of the School of Engineering (ICAI), Universidad Pontificia Comillas, Madrid, Spain

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1 Introduction

The unit commitment (UC) problem requires to optimally operate a set of power generation units over a time horizon ranging from one day to one week. Despite the breakthrough in mixed-integer programming (MIP) solvers, the UC problem remains restricted in size and scope due to the time requested for its solution. Variants of the UC problem can be solved significantly faster by improving their MIP formulations by providing the convex hull description of some set of constraints. Even though other constraints in the problem might add some fractional vertices, this solution should be nearer to the optimal solution than the solution of the original model would be (Wolsey 1998; Williams 2013). Some efforts in tightening specific set of constraints have been done, such as the convex hull of the minimum up and down times (Lee et al. 2004; Malkin 2003; Rajan and Takriti 2005), cuts to tighten ramping limits (Ostrowski et al. 2012; Damci et al. 2015), tighter approximation for quadratic generation costs (Frangioni et al. 2009), new formulations for the time-depending start-up costs (Silbernagl et al. 2014), and simultaneously tight and compact description of thermal units operation (Morales-España et al. 2013a, b, 2015a, b).

The main contribution of this paper is a slight modification of the constraints presented in Morales-España et al. (2013a) plus the proof that the new model provides the convex hull description of the solutions satisfying the following set of constraints: (1) generation limits, (2) start-up and shut-down capabilities, and (3) minimum up and down times. This result is a basic step towards the definition of a formulation describing the convex hull of the set of solutions satisfying also general ramp constraints with a linear number of variables. Recently, a formulation with $O(T^3)$ variables (where T is the length of the time horizon) describing the convex hull of the feasible solutions has been obtained independently in Frangioni and Gentile (2015) and Knueven et al. (2015), but using formulations based on the dynamic programming algorithm in Frangioni and Gentile (2006). Moreover, the techniques used in this paper could be possibly used also to achieve this more general result. These results are in some sense orthogonal to those in Damci et al. (2015). In this paper, we consider both start-up and shut-down capabilities together but we do not consider ramp constraints; in Damci et al. (2015) two separate polytopes are defined: the ramp-up polytope considering solutions satisfying ramp-up and start-up limits and the ramp-down polytope considering solutions satisfying ramp-down and shut-down limits. In Damci et al. (2015) the convex hull descriptions for ramp-up and ramp-down polytopes are provided for the case of only two periods and some facet defining inequalities are presented for the same polytopes with arbitrary time horizon.

On the application side tighter formulations are usually solved in less time by MIP solvers; however, this must be tested by computational experiments. We compare the new formulation with two other MIP formulations obtaining results significantly faster for three different case studies. The first one consists in solving a self-UC problem only taking into account the constraints proposed in this paper. The self-UC problem requires to optimize the net profit of a price-taker generation company, that is a relatively small company that is not able to influence the market price. If we restrict to the above mentioned constraints, we have a convex hull description also for the self-UC problem. The second and third case studies solve the network-constrained UC problem for two IEEE power systems, where other common constraints are taken into account, such as demand balance, reserves, ramping, and transmission limits.

The remainder of this paper is organized as follows: Sect. 2 introduces the main notation used to describe the proposed formulation. Section 3 details the basic operating constraints of a single generating unit. Section 4 contains the facet inducing and convex hull proofs for the proposed linear description of the self-UC subproblem. Section 5 provides and discusses results from several case studies, where a comparison with other three UC formulations is made. Finally, some relevant conclusions are drawn in Sect. 6.

2 Notation

Here, we introduce the main notation used in this paper. The length of the time horizon is denoted by T and the time is indexed by t. The set of generating units is denoted by $\mathcal{G}$ and indexed with g running from 1 to G.

2.1 Unit's technical parameters

$\overline{P}_g/\underline{P}_g$	Maximum/minimum power output [MW] for unit g.
$\mathrm{SD}_g/\mathrm{SU}_g$	Shut-down/start-up capability [MW] for unit g.
$\mathrm{TD}_g/\mathrm{TU}_g$	Minimum down/up time [h] for unit g.

2.2 Decision variables

- u_{gt} Binary variable for the commitment status of unit g for period t, which is equal to 1 if the unit is online and 0 otherwise.
- v_{gt} Binary variable for the start-up status of unit g, which is equal to 1 if the unit starts up in period t and 0 otherwise.
- w_{gt} Binary variable for the shut-down status of unit g, which is equal to 1 if the unit shuts down in period t and 0 otherwise.
- p_{gt} Power production above the unit's minimum output $\underline{P}_g$ [MW] for unit g in period t. The total generation output is equal to $u_{gt}\underline{P}_g + p_{gt}$.

3 Modeling the unit's operation

This section describes the mathematical formulations of the basic operation of a single generating unit in UC problems. To simplify the notation, here we do not report the unit index. In Sect. 5 we consider two multi-units UC problems where the single generating unit formulations must be replicated for each unit.

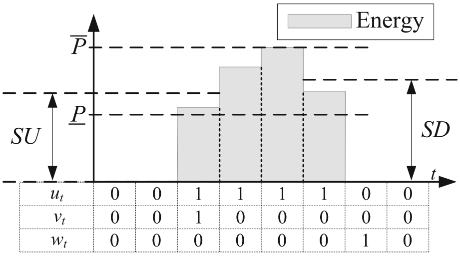

Fig. 1 Unit's operation including its start-up and shut-down capabilities

Two main formulations can be found in the literature: *1bin* formulation, so called because it uses only one vector of binary variables u_t denoting the status ON/OFF of the unit for each time period t; *3bin* formulation, so called because it uses three vectors of binary variables by adding to the state variables also the start-up v_t and shut-down w_t variables. The basic constraints of the *1bin* and *3bin* formulations are presented in Appendix 1.

In this paper, the following set of constraints is modeled: generation limits, minimum up and down times, and start-up and shut-down capabilities. As shown in Fig. 1, the start-up capability SU is the maximum power that a generating unit could produce when it starts up. Similarly, the unit should be producing below its shut-down capability SD when it shuts down.

First, we use the following constraints, which were proposed in (Rajan and Takriti 2005) to describe the convex hull formulation of the minimum-up and -down time constraints:

$$u_t - u_{t-1} = v_t - w_t \quad t = 2, \ldots, T \tag{1}$$

$$\sum_{j=t-\mathrm{TU}+1}^{t} v_j \le u_t \quad t = 2, \ldots, T \tag{2}$$

$$\sum_{j=t-\mathrm{TD}+1}^{t} w_j \le 1 - u_t \quad t = 2, \ldots, T \tag{3}$$

where inequalities in (2) state that in an interval of TU consecutive time periods a unit can be started-up at most once; inequalities (3) works similarly for the shut-down case.

Here, we present the formulation that we now denote as *TC* obtained by adding to constraints (1)–(3) the following constraints with start-up and shut-down capabilities:

$$p_1 \le \left(\overline{P} - \underline{P}\right) u_1 - \left(\overline{P} - \mathrm{SD}\right) w_2 \tag{4}$$

$$p_t \le \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - \mathrm{SU}\right) v_t - \left(\overline{P} - \mathrm{SD}\right) w_{t+1} \quad t = 2, \ldots, T-1 \tag{5}$$

$$p_T \le \left(\overline{P} - \underline{P}\right) u_T - \left(\overline{P} - \mathrm{SU}\right) v_T \tag{6}$$

Constraint (5) states that the maximum power above the minimum output in period t when the unit is started-up (e.g., $u_t = v_t = 1$ and $w_{t+1} = 0$) is equal to $\mathrm{SU} - \underline{P}$, when the unit is shut-down at time $t+1$ (e.g., $u_t = w_{t+1} = 1$ and $v_t = 0$) is equal to $\mathrm{SD} - \underline{P}$, and when the unit is continuously online (e.g, $u_t = 1$ and $v_t = w_{t+1} = 0$) is equal to $\overline{P} - \underline{P}$. Constraints (4) and (6) describe the first and the last period cases.

Be aware that (5) may be infeasible in the event that the unit is online for just one period. Indeed, when $v_t = w_{t+1} = 1$ the right side of (5) can be negative. Consequently, (5) is only valid for units with uptime $\mathrm{TU} \geq 2$. The correct formulation for units with $\mathrm{TU} = 1$ is given by substituting (5) with the following pair of constraints:

$$p_t \leq \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - \mathrm{SD}\right) w_{t+1} - \max\left(\mathrm{SD}{-}\mathrm{SU}, 0\right) v_t \quad t = 2, \ldots, T-1 \tag{7}$$

$$p_t \leq \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - \mathrm{SU}\right) v_t - \max\left(\mathrm{SU}{-}\mathrm{SD}, 0\right) w_{t+1} \quad t = 2, \ldots, T-1. \tag{8}$$

Finally, the variable bounds are given by

$$0 \leq u_t \leq 1 \quad t = 1, \ldots, T \tag{9}$$

$$v_t \geq 0, \quad w_t \geq 0 \quad t = 2, \ldots, T \tag{10}$$

$$p_t \geq 0 \quad t = 1, \ldots, T. \tag{11}$$

In summary, inequalities (4)–(6) together with inequalities (1)–(3) and (9)–(11) describe the operations for units with $\mathrm{TU} \geq 2$, and inequalities (4), (6), (7), (8) together with inequalities (1)–(3) and (9)–(11) for units with $\mathrm{TU} = 1$. The main contribution of this paper is that the polytopes described with these formulations always have integral vertices with respect to the binary variables.

In Morales-España et al. (2013a), a slightly different formulation was presented, where instead of constraints (7) and (8) the following ones were used:

$$p_t \leq \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - \mathrm{SD}\right) w_{t+1} \quad t = 2, \ldots, T-1 \tag{12}$$

$$p_t \leq \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - \mathrm{SU}\right) v_t \quad t = 2, \ldots, T-1. \tag{13}$$

Note that if $\mathrm{SU} = \mathrm{SD}$ then (7) and (8) and (12) and (13) would be equivalent. We denote the old formulation (Morales-España et al. 2013a) with the latter constraints as *TC0*.

4 Strength of the proposed inequalities

In this section, we prove that inequalities (1)–(11) describe the convex hull of the feasible solutions. Note that constraints (1) uniquely define the value of the variables w as a function of u and v. Unless differently specified, in the following, we will consider only the space defined by the variables u, v, and p. Moreover, we suppose that all constraints (3)–(5), (7), (8), and (10) are rewritten by substituting the w variables accordingly.

Definition 1 Let $\overline{C}_T\left(\mathrm{TU}, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\right)$ be the convex hull of the feasible integer solution for the problem. That is, for $\mathrm{TU} \geq 2$, we write

$$\overline{C}_T(\mathrm{TU} \geq 2, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}) = \\ \mathrm{conv}\{(u, v, p) \in \{0,1\}^{2T-1} \times \mathbb{R}^T_+ | \, (u, v, p) \;\; \text{satisfy} \;\; (1){-}(6) \text{ and } (9){-}(11)\};$$

for $\mathrm{TU} = 1$, we write

$$\overline{C}_T\left(\mathrm{TU} = 1, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\right) = \mathrm{conv}\left\{(u, v, p) \in \{0,1\}^{2T-1} \times \mathbb{R}^T_+ | \, (u, v, p) \right. \\ \left. \text{satisfy} \;\; (1){-}(4), (6){-}(8), \;\text{and}\; (9){-}(11)\right\}.$$

In short we write $\overline{C}_T$ for $\overline{C}_T\left(\mathrm{TU}, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\right)$, $\overline{C}_T(\mathrm{TU}{\geq}2)$ for $\overline{C}_T(\mathrm{TU}{\geq}2$, $\mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD})$, and $\overline{C}_T(\mathrm{TU} = 1)$ for $\overline{C}_T(\mathrm{TU} = 1, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD})$.

Proposition 2 *$dim(\overline{C}_T) = 3T - 1$ and thus $\overline{C}_T$ is full-dimensional.*

Proposition 3 *The inequalities (4), (6) and (11) describe facets of the polytope $\overline{C}_T$. Moreover, inequalities (5) describe facets of the polytope $\overline{C}_T\left(\mathrm{TU} \geq 2\right)$, and inequalities (7) and (8) describe facets of the polytope $\overline{C}_T\left(\mathrm{TU} = 1\right)$.*

The proofs of Propositions 2 and 3 can be performed by exhibiting the right number of affinely independent points (details of the proofs can be requested to the authors).

For the convex hull proof, we need a preliminary lemma that is very easy to prove from well-known results (for completeness we report a proof suggested by a referee):

Lemma 4 *Suppose that $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ is an integer polyhedron. Suppose that $y \in \mathbb{R}^K$ are new variables and that $Q = \{(x, y) : d^k x \leq y_k \leq c^k x, k = 1, \ldots, K\}$, with at most one lower bound $d^k x$ and one upper bound $c^k x$ for each variable y_k. If $d^k x \leq c^k x$ for each $x \in P$, then $P \cap Q$ has extreme points with x integer.*

Proof Consider the linear program LP(P,Q): $\min\{qx + \sum_{k=1}^K h_k y_k : (x, y) \in P \cap Q\}$. We prove that for each objective function this LP has an integer solution with respect to x. Set $y_k = d^k x$ if $h_k \geq 0$ and $y_k = c^k x$ otherwise. Solve the resulting LP in the x-space. Then x is integer and the corresponding (x, y) is optimal for the linear program LP(P,Q). □

Theorem 5 *Let $\overline{D}_T\left(\mathrm{TU}, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\right)$ be a polyhedron defined as follows:*

- *for* $\mathrm{TU} \geq 2$

$$\overline{D}_T\left(\mathrm{TU} \geq 2, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\right) = \\ \left\{(u, v, p) \in [0,1]^{2T-1} \times \mathbb{R}^T_+ | \, (u, v, p) \;\text{satisfy}\; (1){-}(6) \text{ and } (9){-}(11)\right\};$$

- *for* $\mathrm{TU} = 1$

$$\overline{D}_T\left(\mathrm{TU} = 1, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\right) = \left\{(u, v, p) \in [0,1]^{2T-1} \times \mathbb{R}^T_+ | \, (u, v, p) \right. \\ \left. \text{satisfy}\; (1){-}(4), (6){-}(8), \;\text{and}\; (9){-}(11)\right\}.$$

Then $\overline{C}_T\big(\mathrm{TU}, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\big) = \overline{D}_T\big(\mathrm{TU}, \mathrm{TD}, \overline{P}, \underline{P}, \mathrm{SU}, \mathrm{SD}\big)$.

Proof As for $\overline{C}_T$, we use short notations $\overline{D}_T$, $\overline{D}_T\big(\mathrm{TU} \geq 2\big)$, and $\overline{D}_T\big(\mathrm{TU} = 1\big)$. The proof for $\mathrm{TU} \geq 2$ easily follows from Lemma 4. Indeed, $\overline{D}_T\big(\mathrm{TU} \geq 2\big)$ is described by the inequalities (1)–(3) and (9) and (10) that describe an integral polyhedron in u and v as proved in (Rajan and Takriti 2005), together with inequalities (4)–(6) and (11) satisfying the hypothesis of Lemma 4.

For $\mathrm{TU} = 1$ let us suppose that $\mathrm{SU} \geq \mathrm{SD}$. We follow Approach 8 in (Wolsey 1998) (see Section 9.2.3, Problem 2, Approach 8). We first introduce an extended formulation of the problem, then we prove that the extended formulation is integral, and finally we prove that the projection of the new polyhedron correspond to $\overline{D}_T\big(\mathrm{TU} = 1\big)$. We divide the proof into a series of claims. We define the following new binary variables for $t = 2, \ldots, T-1$: $x_t = 1$ if and only if $v_t = 1$ and $w_{t+1} = 1$, $\tilde{v}_t = 1$ if and only if and $w_{t+1} = 0$, $\tilde{w}_{t+1} = 1$ if and only if $v_t = 0$ and $w_{t+1} = 1$, $\tilde{u}_t = 1$, if and only if $u_t = 1$, $v_t = 0$, and $w_{t+1} = 0$. Moreover, $\tilde{u}_T = 1$ if and only if $u_T = 1$ and $v_T = 0$.

Claim 1. The polyhedron P defined by the points $(u, v, w, \tilde{u}, \tilde{v}, \tilde{w}, x)$ satisfying the following inequalities is integral:

$$v_t \leq u_t \quad t = 2, \ldots, T \tag{14}$$

$$\sum_{i=t-\mathrm{TD}+1}^{t} w_i \leq 1 - u_t \quad t \in [\mathrm{TD}+1, T] \tag{15}$$

$$u_t - u_{t-1} = v_t - w_t \quad t \in [2, T] \tag{16}$$

$$w_{t+1} = \tilde{w}_{t+1} + x_t \quad t \in [2, T-1] \tag{17}$$

$$v_t = \tilde{v}_t + x_t \quad t \in [2, T-1] \tag{18}$$

$$u_t = \tilde{v}_t + \tilde{w}_{t+1} + x_t + \tilde{u}_t \quad t \in [2, T-1] \tag{19}$$

$$u_T = v_T + \tilde{u}_T \tag{20}$$

$$0 \leq u_t \leq 1 \quad t \in [1, T] \tag{21}$$

$$v_t, w_t, \tilde{u}_t \geq 0 \quad t \in [2, T] \tag{22}$$

$$\tilde{v}_t, x_t \geq 0 \quad t \in [2, T-1] \tag{23}$$

$$\tilde{w}_t \geq 0 \quad t \in [3, T] \tag{24}$$

Proof of Claim 1 The proof is carried on by showing that the coefficient matrix associated with the above linear system is totally unimodular.

We exploit this well-known property (proved by Ghouila-Houri, see (Nemhauser and Wolsey 1999), Chapter III.1, Theorem 2.7): let A be a $\{0, 1, -1\}$-matrix, if each subset J of columns of A can be partitioned into J_1 and J_2 such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1 \tag{25}$$

for each row i, then A is totally unimodular. This part of the proof has been inspired by the proof of Malkin (2003) for the polyhedron defined by (1)–(3).

First we assign the variables $w_i \in J$ alternatively to J_1 and to J_2 in lexicographic order. Then the variables $u_t \in J$ are assigned either to J_1 if $w_k \in J_2$, where $k = \max\{i | 1 \leq i \leq t, w_i \in J\}$, or to J_2 if $w_k \in J_1$, or to the same set with respect to u_{t-1} if $\{i | 1 \leq i \leq t, w_i \in J\}$ is empty. Thus condition (25) is satisfied for constraints (15).

Variables $v_t \in J$ are assigned either to J_1 if $u_t \in J_1$, or to J_2 if $u_t \in J_2$, or to the opposite set with respect to u_{t-1} if $u_t \notin J$, or to the same set as w_t if both $u_{t-1}, u_t \notin J$. This ensures that condition (25) is satisfied for constraints (14) and (16).

If $v_t, w_{t+1} \in J$, then assign $\tilde{v}_t \in J$ to the same subset as v_t, $x_t \in J$ to the opposite set with respect to $\tilde{v}_t$, and $\tilde{w}_t \in J$ to the same subset as w_t. These assignments guarantee that condition (25) is satisfied for constraints (17) and (18) both in the case that v_t and w_{t+1} are in the same set or in different sets. Moreover, the assignment for $\tilde{u}_t$ can be chosen to satisfy condition (25) for constraints (19). If one between v_t and w_{t+1} does not belong to J then proceed as follows: suppose w.l.o.g. that $v_t \notin J$, then assign w_{t+1}, $\tilde{w}_{t+1}$, and $\tilde{v}_t$ to the same set and x_t to the other set, then $\tilde{u}_t$ can be chosen to satisfy condition (25) for constraints (19). Similar choices can be done if some of the variables $\tilde{v}_t, \tilde{w}_{t+1}, x_t, \tilde{u}_t$ do not belong to J and the claim follows. **End of Claim 1.**

Then we define the polyhedron $\tilde{Q}$ by adding to (14)–(24)

$$p_t^v \leq (\mathrm{SU} - \underline{P})\tilde{v}_t \quad t \in [2, T-1] \tag{26}$$
$$p_t^x \leq (\mathrm{SD} - \underline{P})x_t \quad t \in [2, T-1] \tag{27}$$
$$p_t^w \leq (\mathrm{SD} - \underline{P})\tilde{w}_{t+1} \quad t \in [2, T-1] \tag{28}$$
$$p_t^u \leq (\overline{P} - \underline{P})\tilde{u}_t \quad t \in [2, T] \tag{29}$$
$$p_T^v \leq (\mathrm{SU} - \underline{P})v_T \tag{30}$$
$$p_1 \leq (\overline{P} - \underline{P})u_1 - (\overline{P} - \mathrm{SD})w_2, \tag{31}$$

where p^v, p^x, p^w, p^u and p_1 are new non-negative variables.

Claim 2. The polyhedron $\tilde{Q}$ is integral with respect to variables $u, v, w, x, \tilde{u}, \tilde{v}, \tilde{w}$. **End of Claim 2.**

Claim 2 follows by applying Lemma 4 to the polyhedron P of Claim 1. Then we define the polyhedron Q by adding to (14)–(24), (26)–(31)

$$p_t = p_t^v + p_t^x + p_t^w + p_t^u \qquad t \in [2, \ldots, T-1] \tag{32}$$
$$p_T = p_T^v + p_T^u, \tag{33}$$

where p_t for $t \in [2 \ldots T]$ are non-negative variables.

Claim 3. The polyhedron Q is integral with respect to variables $u, v, w, x, \tilde{u}, \tilde{v}, \tilde{w}$. **End of Claim 3.**

Claim 3 follows from Claim 2 and by the straightforward extension of Lemma 4, where the role of P is played by the integral polyhedron $\tilde{Q}$. Finally we prove that

Claim 4. The projection of Q onto the space of variables u, v, p is equivalent to $\overline{D}_T$.

Proof of Claim 4 We start by eliminating the variables p_t^v, p_t^x, p_t^w, and p_t^u by simply substituting constraints (32) and (33) with the following:

$$p_t \leq (\mathrm{SU} - \underline{P})\tilde{v}_t + (\mathrm{SD} - \underline{P})x_t + (\mathrm{SD} - \underline{P})\tilde{w}_{t+1} + (\overline{P} - \underline{P})\tilde{u}_t \quad t \in [2, T-1] \tag{34}$$

$$p_T \leq (\mathrm{SU} - \underline{P})v_T + (\overline{P} - \underline{P})\tilde{u}_T, \tag{35}$$

which are obtained by using constraints (26)–(30). □

Now, we replace $\tilde{u}_T$ from (20) in (35) to obtain $p_T \leq \left(\overline{P} - \underline{P}\right) u_T - \left(\overline{P} - \mathrm{SU}\right) v_T$ that coincides with (6). Then we eliminate variables in (34) according to the following order: $\tilde{u}_t$ by using the Eq. (19); $\tilde{w}_{t+1}$ by using the Eq. (17); $\tilde{v}_t$ by using the Eq. (18). It is not difficult to see that for $t \in [2, T-1]$ we obtain the following constraints:

$$p_t \leq (\overline{P} - \underline{P})u_t - (\overline{P} - \mathrm{SU})v_t - (\overline{P} - \mathrm{SD})w_{t+1} + (\overline{P} - \mathrm{SU})x_t \tag{36}$$

$$x_t \geq 0 \tag{37}$$

$$x_t \geq v_t + w_{t+1} - u_t \tag{38}$$

$$x_t \leq v_t \tag{39}$$

$$x_t \leq w_{t+1}. \tag{40}$$

Now we can apply the Fourier-Motzkin elimination procedure to variables x_t as follows: (i) from constraints (39) and (36) we obtain constraint (7); (ii) from constraints (39) and (37) we obtain $v_t \geq 0$; (iii) from constraints (39) and (38) we obtain $w_{t+1} \leq u_t$ that is dominated by constraints (2); (iv) from constraints (40) and (36) we obtain constraint (8); (v) from constraints (40) and (37) we obtain $w_{t+1} \geq 0$; and from constraints (40) and (38) we obtain $u_t \geq v_t$. Finally, the claim follows by observing that (31) coincides with (4). **End of Claim 4.**

From Claim 4 it follows that $\overline{D}_T$ is integral with respect to the variables u and v. The proof for $\mathrm{SD} \geq \mathrm{SU}$ can be performed in a symmetric way. □

5 Numerical results

To illustrate the computational performance of the formulation presented in this paper, three sets of case studies are carried out: one for a self-UC problem and two others for a network-constrained UC problem. This section compares the computational performance of the proposed *TC* formulation with two other formulations, (Carrion and Arroyo 2006) and (Ostrowski et al. 2012), which have been recognized as computationally efficient in the literature (Morales-España et al. 2013a, 2014; Tahanan et al. 2015).

The following three formulations are then implemented:

- *TC* This is the complete formulation presented in this paper. For the network-constrained UC, we include other common constraints such as demand-balance, reserves, ramping, and transmission limits. The complete network-constrained UC is presented in Appendix 2.

- *1bin* This formulation is presented in (Carrion and Arroyo 2006) and requires a single set of binary variables (per unit and per period), i.e., the start-up and shut-down decisions are expressed as a function of the commitment decision variables.
- *3bin* The convex hull of the minimum up/down time constraints proposed in (Rajan and Takriti 2005) (see (1)–(3) and (9)–(10)) is implemented with the three-binary formulation. This formulation is presented in (Ostrowski et al. 2012)

Notice that different set of constraints is used for the self-UC and for the network-constrained UC problems. For the self-UC problems, *1bin* and *3bin* are modeled only considering (1) generation limits, (2) minimum up and down times, and (3) start-up and shut-down capabilities. For the network-constrained UC problems, *1bin* and *3bin* are modeled taking into account the full set of constraints presented in (Carrion and Arroyo 2006) and its *3bin* equivalent (Ostrowski et al. 2012), respectively; in addition, these formulations are further extended by introducing downwards reserve (which is modeled in the same fashion as the upward reserve, see Appendix 2), transmission limits [see (48) in Appendix 2), and wind generation [which is taken into account in the demand-balance (45) and transmission-limit constraints (48)].

All tests were carried out using CPLEX 12.5 with its default parameters on an Intel-i7 3.4-GHz personal computer with 8 GB of RAM memory. The problems are solved until they hit the time limit of 10,000 s or until they reach optimality (more precisely to 10^{-4} % of relative optimality gap).

5.1 Self-UC

We illustrate the computational performance of the formulation proposed in this paper by solving the self-UC problem for a price-taker producer for different time spans. The goal of a price-taker producer is to maximize his profit [which is the difference between the revenue and the total operating cost (Morales-España et al. 2013b)] during the planning horizon:

$$\max \sum_{t=1}^{T} \sum_{g=1}^{G} \Big[\pi_t \left[u_{gt} \underline{P}_g + p_{gt} \right] - \Big(C_g^{\mathrm{NL}} u_{gt} + C_g^{\mathrm{LV}} \left[u_{gt} \underline{P}_g + p_{gt} \right] + C_g^{\mathrm{SU}} v_{gt} + C_g^{\mathrm{SD}} w_{gt} \Big) \Big] \tag{41}$$

where subscript g stands for generating units and G is the number of units; π_t refers to the energy prices; C_g^{NL}, C_g^{LV}, C_g^{SU} and C_g^{SD} are the no-load, linear-variable, start-up, and shut-down costs of unit g, respectively (for this case study $C_g^{\mathrm{SD}} = 0$ for all units). The objective function (41) is optimized over the solution set described by generation limits, start-up, and shut-down capabilities, and minimum up and down times constraints. The self-UC also arises when solving UC with decomposition methods such as Lagrangian Relaxation (Frangioni et al. 2008; Frangioni and Gentile 2006) (where the prices are the Lagrangian multipliers).

Table 1 Generator data

Gen	Technical information							Cost coefficients[c]		
	$\overline{P}$ [MW]	$\underline{P}$ [MW]	TU/TD [h]	SU [MW]	SD [MW]	p_0 [MW/h][a]	Ste_0 [h][b]	C^{NL} [$/h]	C^{LV} [$/MWh]	C^{SU} [$]
1	455	150	8	252	303	150	8	1000	16.19	9000
2	455	150	8	252	303	150	8	970	17.26	10,000
3	130	20	5	57	75	20	5	700	16.60	1100
4	130	20	5	57	75	20	5	680	16.50	1120
5	162	25	6	71	94	25	6	450	19.70	1800
6	80	20	3	40	50	20	3	370	22.26	340
7	85	25	3	45	55	25	3	480	27.74	520
8	55	10	1	25	33	10	1	660	25.92	60
9	55	10	1	25	33	10	1	665	27.74	60
10	55	10	1	25	33	10	1	670	27.79	60

[a] p_0 is the unit's initial production prior to the first period of the time span
[b] Ste_0 is the number of hours that the unit has been online prior to the first period of the time span
[c] C^{NL}, C^{LV} and C^{SU} stand for no-load, linear-variable, and startup costs, respectively

Table 2 Energy price ($/MWh)

$t = 1 \ldots 12 \rightarrow$	13.0	7.2	4.6	3.3	3.9	5.9	9.8	15.0	22.1	31.3	33.2	24.8
$t = 13 \ldots 24 \rightarrow$	19.5	16.3	14.3	13.7	15.0	17.6	20.2	29.3	49.5	53.4	30.0	20.2

The 10-unit system data are presented in Table 1 and the energy prices are shown in Table 2. The power system data are based on information presented in (Carrion and Arroyo 2006; Morales-España et al. 2013a).

Here, apart from *TC*, *1bin* and *3bin*, the tight and compact formulation presented in (Morales-España et al. 2013a), labeled as *TC0,* is also implemented. It is important to note that the formulation *TC0* uses constraints (12) and (13) instead of (7) and (8) for units with TU $= 1$. Apart from those constraints, *TC* and *TC0* are identical. Note, however, that (7) and (8) are needed to describe the convex hull, as proved in Sect. 4.

Table 3 shows the computational performances for four cases with different time spans. All formulations achieve the same MIP optimum since all of them model the same MIP problem. However, they present different LP optima, the relative distance between their MIP and LP optima is measured with the Integrality Gap (Williams 2013; Morales-España et al. 2013a). Note that the MIP optima of *TC* were achieved by just solving the LP over (1)–(11), IntGap$=$0, hence solving the problems in LP time. On the other hand, as usual, the Branch-and-Cut (B&C) method was needed to solve the MIP for *TC0*, *3bin* and *1bin*. Table 3 also shows the MIP time and B&C nodes explored that were required by the different formulations to reach optimality. It is interesting to note that although *TC0* reached optimality exploring zero B&C nodes, *TC0* needed to make use of the solver's cutting planes strategy because the relaxed LP solution did not achieve the integer one, IntGap$\neq$0 (the solver used 227 and 1224 cuts for the smallest and largest case, respectively). This tightening process took more

Table 3 Self-UC: computational performance comparison

Case (days)	Optimum (M$)	IntGap (%)				LP time (s)				MIP time (s)[a]				B&C Nodes			
		TC	$TC0$	$3bin$	$1bin$	TC	$TC0$	$3bin$	$1bin$	TC	$TC0$	$3bin$	$1bin$	TC	$TC0$	$3bin$	$1bin$
64	7.259361	0	0.09	0.88	2.57	0.57	0.47	0.80	0.95	0.57	1.92	12.01	13.79	0	0	496	487
128	14.517096	0	0.09	0.87	2.57	1.17	1.20	2.06	2.60	1.17	4.81	45.54	(0.033)	0	0	528	603,915
256	29.032567	0	0.09	0.87	2.57	3.16	3.29	5.38	6.88	3.16	7.75	199.18	(0.052)	0	0	533	229,035
512	58.063509	0	0.09	0.87	2.57	8.08	8.39	14.29	18.83	8.08	17.29	734.03	(0.054)	0	0	488	136,128

[a] If the time limit is reached then the final % of optimality tolerance is shown between parentheses

time than the time required to solve the initial LP relaxation, which is why the MIP time for *TC0* is more than twice its LP relaxation time.

Table 4 shows the dimensions of all the formulations for four selected instances. Note that *TC* and *TC0* are more compact, in terms of quantity of constraints and nonzero elements, than *3bin* and *1bin*. The formulation *1bin* presents a third of binary variables in comparison with the other formulations, but three times more continuous variables. This is because the work in (Carrion and Arroyo 2006) reformulated the units' operation model to avoid the start-up and shut-down binary variables, claiming that this would reduce the node enumeration in the B&C process. Note, however, that this reformulation considerably damaged the strength of *1bin*; hence it presented the worst computational performance; similar results are obtained in (Ostrowski et al. 2012; Morales-España et al. 2013a). The formulation *1bin* presents more continuous variables than the other formulations because it requires the introduction of new continuous variables to model the start-up and shut-down costs of the generating units.

In conclusion, *TC* presents a dramatic improvement in computation in comparison with *3bin* and *1bin* due to its tightness (speedups above 90x and 8500x, respectively); and it also presents a lower LP burden due to its compactness, see Table 4. Compared with *TC0*, the formulation *TC* is tighter; consequently, *TC* requires less time to solve the MIP problem (speedup above 4.1x).

5.2 Network-constrained UC

Here, two IEEE systems are used for different time spans, from 24 to 96 hours, the IEEE 118-bus system and the IEEE 73-bus reliability test system. All data for these two systems can be found in (Morales-España 2014) and (Wong et al. 1999; Hedman et al. 2010), respectively. The IEEE-118 bus system has 118 buses; 186 transmission lines; 54 thermal units; 91 loads, with average and maximum levels of 3991 and 5592 MW, respectively; and three wind generation units, with aggregated average and maximum production of 867 and 1333 MW, respectively. For this system, the upward and downward reserve requirement are set as the 5 % of the total expected wind production for each hour. The IEEE 73-bus reliability test system has 73 buses; 120 transmission lines; 99 thermal units; 51 loads, with average and maximum levels of 7094 and 8547 MW, respectively; and no wind generation. For this system, the upward and downward reserve requirement are set as the 1 % of the total expected demand for each hour.

Bear in mind that the network-constrained UC problem is considerably more complex than the self-UC problem, described in Sect. 5.1, due to the new complicating constraints that are now included (into all the formulations), such as demand-balance, reserves, ramping and transmission limits (see Appendix 2).

Table 5 shows the problem size for all formulations for the two IEEE systems. This table shows the problem size for a time span of 24 h; larger problem sizes are proportional (approximately) to the quantity of hours. On the other hand, there is no direct size relation between the two systems because they have different proportions in thermal and wind units as well as transmission lines. For example, the IEEE 73-bus system has 45 (83 %) more units than the IEEE 118-bus system, but 66 (35 %) less

Table 4 Self-UC: problem size comparison

Case (days)	# Constraints			# Nonzero elements				# Real var		# Binary var	
	TC[a]	*3bin*	*1bin*	TC	$TC0$	*3bin*	*1bin*	TC[b]	*1bin*	$TC^{\dagger}$	*1bin*
64	65,997	107,459	138,225	338,994	334,389	417,313	469,719	15,360	46,080	46,080	15,360
128	132,045	214,979	276,465	678,450	669,237	835,105	939,735	30,720	92,160	92,160	30,720
256	264,141	430,019	552,945	1,357,362	1,338,933	1,670,689	1,879,767	61,440	184,320	184,320	61,440
512	528,333	860,099	1,105,905	2,715,186	2,678,325	3,341,857	3,759,831	12,288	368,640	368,640	122,880

[a] *TC* is equal to *TC0* for these cases
[b] *TC*, *TC0* and *3bin* are equal for these cases

Table 5 IEEE 118-bus & 73-bus systems: problem size comparison of the network constrained UC formulations for a time span of 24 h

System	# Constraints			# Nonzero elements			# Real var			# Binary var	
	TC	*3bin*	*1bin*	*TC*	*3bin*	*1bin*	*TC*	*3bin*	*1bin*	*TC*[a]	*1bin*
IEEE 118-bus	15,903	37,803	38,141	536,815	473,791	472,969	8424	9720	11,016	3888	1296
IEEE 73-bus	23,425	82,846	83,524	581,704	786,268	786,310	11,862	12,384	14,760	7110	2358

[a] *TC* is equal to *3bin* for these cases

transmission lines. Similarly to the self-UC case study (Sect. 5.1), *TC* is more compact than the others, in terms of quantity of constraints. For the IEEE 118-bus system, having a larger number of transmission lines, *TC* presents more nonzeros than the others because *TC* uses $\underline{P}_g u_{gt} + p_{gt}$, which appear in each of the line constraints, to represent the total unit's production, unlike other formulations that use one variable to represent the total production. Beware, however, that a new variable could be introduced representing the total unit's production, thus decreasing the number of nonzeros, but this will increase the number of variables and constraints. Despite this increase in nonzeros, the LP complexity of *TC* for the IEEE 118-bus system is significantly lower than that of both *3bin* and *1bin*, which took in average 15.1 and 17.9 times longer than *TC* to solve the LP problem, respectively (see Table 6). Similarly, for the IEEE 73-bus system, *TC* could solve the LP problem in average 15.6 and 14.2 times faster than *3bin* and *1bin*, respectively (see Table 7). In short, *TC* presents a lower LP burden than the others due to its compactness, as also concluded in the self-UC case in Sect. 5.1.

Tables 6 and 7 show the computational performance of the network-constrained UC problem for both IEEE test systems and for all formulations and different time spans (up to 96 h). For these experiments, *TC* is the tightest formulation since its IntGap is always lower than that of *1bin* and *3bin*. On the other hand, although *1bin* has a third of binary variables in comparison with the others, it has the largest quantity of constraints and it is the least tight (see IntGap Table 6), consequently presenting the worst computational performance, as also discussed in Sect. 5.1.

Interestingly, for the IEEE 118-bus system, all three formulations achieved the same optimum integer solution (all of them model the same integer problem), although *TC* was the only formulation that could prove optimality within the time limit. *3bin* could prove optimality for only one case, the smallest case; and *1bin* could not prove optimality for any of the cases. Notice that due to the tightness, *TC* could prove optimality exploring considerably fewer B&C nodes less than (an order of magnitude) *3bin* and *1bin*, which could not even converge to optimality.

For the IEEE 118-bus system, *TC* always found better integer solutions (reported in Table 6) than the other formulations. *3bin* and *1bin* could not prove optimality for any of the cases. *TC* could prove optimality for the two smallest cases, where *TC* explored fewer nodes than the others, which could not even reach optimality. For the two largest cases, none of the formulations could reach optimality, but *TC* was an order of magnitude nearer to optimality. Also notice that for these two large cases, *TC* could explore more nodes within the time limit due to its compactness, which lower the LP complexity solved during the iterations.

Tables 6 and 7 show the computational performance of the UC formulations trying to reach optimality (more precisely to 10^{-4} % of relative optimality gap) within a 10,000 s time limit. Notice that *1bin* could only reach optimality gaps above 0.13 % for 7 out of 8 cases, and in the best case the optimality gap was above 0.09 %. Similarly, *3bin* presented optimality gaps above 0.09 % for 7 of the cases. In short, only *3bin* could reach an optimality gap below 0.09 % in just one case. To observe the performance of *TC* around these orders of magnitude of optimality gaps, Table 8 shows the performance of *TC* for a required optimality gap of 0.05 % for the two IEEE test systems. Notice that 4 cases could even be solved before branching (0 B&C nodes), 5 cases were solved in less than 15 s, and all the cases could be solved in less than 170 s,

Table 6 IEEE 118-bus system results: computational performance of the network constrained UC formulations for different time spans

Hours	Optimum	IntGap (%)			LP time (s)			MIP time (s)[a]			B&C Nodes		
	M$	*TC*	*3bin*	*1bin*	*TC*	*3bin*	*1bin*	*TC*	*3bin*	*1bin*	*TC*	*3bin*	*1bin*
24	0.826814	0.53	1.13	1.75	0.33	2.48	2.9	4.13	585.22	(0.094)	77	93,285	889,610
48	1.649732	0.49	0.70	1.37	1.17	17.88	19.19	26.15	(0.095)	(0.269)	546	260,545	40,115
72	2.472651	0.46	0.56	1.24	2.57	40.59	57.21	474.85	(0.136)	(0.336)	2411	50,593	20,657
96	3.295570	0.44	0.48	1.18	4.29	93.85	102.79	1193.92	(0.180)	(0.317)	4295	40,601	14,605

[a] If the time limit (10,000 s) is reached then the final % of optimality tolerance is shown between parentheses

Table 7 IEEE 73-bus system results: computational performance of the network constrained UC formulations for different time spans

Hours	Optimum	IntGap (%)			LP time (s)			MIP time (s)[a]			B&C Nodes		
	M$	*TC*	*3bin*	*1bin*	*TC*	*3bin*	*1bin*	*TC*	*3bin*	*1bin*	*TC*	*3bin*	*1bin*
24	1.695434	0.02	0.23	1.18	0.36	1.42	1.31	22.60	(0.107)	(0.134)	24,510	1,009,500	897,063
48	3.327422	0.02	0.24	1.13	0.80	15.3	9.66	123.17	(0.151)	(0.180)	22,378	245,587	264,653
72	4.959410	0.02	0.24	1.11	1.33	22.14	21.87	(0.010)	(0.187)	(0.239)	1,245,643	100,358	27,756
96	6.591398	0.02	0.24	1.10	1.97	45.07	48.66	(0.012)	(0.175)	(0.374)	655,694	12,768	2363

[a] If the time limit (10,000 s) is reached then the final % of optimality gap is shown between parentheses

Table 8 IEEE 118-bus and 73-bus system results: computational performance of *TC* for 0.05 % of optimality gap and different time spans

Hours	MIP time (s)		Optimality gap (%)		B&C nodes	
	118-bus	73-bus	118-bus	73-bus	118-bus	73-bus
24	3.45	3.54	0.030	0.045	0	5
48	9.25	7.94	0.036	0.032	0	0
72	68.2	13.09	0.034	0.049	625	0
96	167.44	45.76	0.041	0.049	560	490

Table 9 IEEE 73-bus system: initial vs. final lower bounds of UC formulations for different time spans

Hours	LP relaxations (M$)			Final best lower bound (M$)		
	TC	*3bin*	*1bin*	*TC*	*3bin*	*1bin*
24	1.695161	1.691586	1.675454	1.695434	1.693621	1.693167
48	3.326716	3.319535	3.289971	3.327422	3.322417	3.32144
72	4.958264	4.947482	4.904489	4.958887	4.951332	4.947532
96	6.589812	6.575429	6.519006	6.590607	6.57985	6.569458

unlike *3bin* and *1bin* which could not reach that low optimality gaps within 10,000 seconds. Due to the simultaneous tightness and compactness, *TC* could reach 0.05 % optimality tolerance for four cases (one for the IEEE 118-bus system and three for the IEEE 73-bus system) in less time than that required by *1bin* and *3bin* to solve their LP problem.

Furthermore, for the IEEE 73-bus system, *TC* presented better (higher) lower bounds in the initial LP relaxation than the final lower bounds found by *3bin* and *1bin* within the time limit, as shown in Table 9 (this was not the case for the IEEE 118-bus system). Thanks to the convex hull provided in this paper, for the IEEE 73-test system, *TC* could provide initial lower bounds, in less than 2 s (see LP time in Table 7), which were better than the final lower bounds obtained by *3bin* and *1bin* within 10,000 s.

6 Conclusion

This paper presented the convex hull description of the single thermal Unit Commitment problem with the following basic constraints: generation limits, start-up and shut-down capabilities, and minimum up and down times. The model does not include some crucial constraints, such as ramping, but the proposed constraints can be used as the core of any UC formulation and they can help to tighten the final UC model.

Computational experiments have been carried out among the new proposed formulation and two previous formulations called *1bin* and *3bin* considering two UC variants: the self-UC and the network-constrained UC problems. For both problems, the new proposed formulation presents a dramatic improvement in computation in comparison with *3bin* and *1bin* due to its tightness; and it also presents a lower LP burden due to its compactness (see Tables 4 and 5).

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Appendix 1: 1bin and 3bin UC formulations

This section presents the basic constraints for the *1bin* and *3bin* UC formulations. The nomenclature used here is the same one presented in Sect. 2; the new nomenclature is defined once it is introduced. It is important to highlight that *1bin* and *3bin* formulations consider the total energy production variable $\widehat{p}_t$ from 0 to $\overline{P}$, unlike the formulation presented in this paper where p_t represents the energy production above $\underline{P}$.

1bin formulation

The *1bin* formulation is the following (see Carrion and Arroyo (2006)):

$$
\begin{aligned}
& \underline{P}u_t \leq \widehat{p}_t \leq \overline{P}u_t && t=1,\ldots,T \\
& \widehat{p}_t \leq \widehat{p}_{t-1} + RUu_{t-1} + \mathrm{SU}(u_t - u_{t-1}) + \overline{P}(1-u_t) && t=2,\ldots,T \\
& \widehat{p}_{t-1} \leq \widehat{p}_t + RDu_t + \mathrm{SD}(u_{t-1} - u_t) + \overline{P}(1-u_{t-1}) && t=2,\ldots,T \\
& \sum_{j=1}^{G}(1-u_j) = 0 && \\
& \sum_{j=t}^{t+\mathrm{TU}-1} u_j \geq \mathrm{TU}(u_t - u_{t-1}) && t=H+1,\ldots,T-\mathrm{TU}+1 \\
& \sum_{j=t}^{T}\left[u_j - (u_t - u_{t-1})\right] \geq 0 && t=T-\mathrm{TU}+2,\ldots,T \\
& \sum_{j=1}^{L} u_j = 0 && \\
& \sum_{j=t}^{t+\mathrm{TD}-1} (1-u_j) \geq \mathrm{TD}(u_{t-1} - u_t) && t=L+1,\ldots,T-\mathrm{TD}+1 \\
& \sum_{j=t}^{T}\left[1-u_j - (u_{t-1} - u_t)\right] \geq 0 && t=T-\mathrm{TD}+2,\ldots,T \\
& suc_t \geq C^{\mathrm{SU}}(u_t - u_{t-1}) && t=2,\ldots,T \\
& sdc_t \geq C^{\mathrm{SD}}(u_{t-1} - u_t) && t=2,\ldots,T \\
& 0 \leq u_t \leq 1 && t=1,\ldots,T
\end{aligned}
\tag{42}
$$

where $H = min\{T, (\mathrm{TU} - \tau_0)u_0\}$ and $L = \min\{T, (\mathrm{TD} + \tau_0)(1 - u_0)\}$ are the minimum number of time instants the unit must be initially on or off, respectively (τ_0 indicates the number of time instants the unit has been on prior to time 0 if $\tau_0 > 0$, while $-\tau_0$ indicates the number of time instants the unit has been off prior to time 0 if $\tau_0 < 0$).

Note that *1bin* models the unit's start-up and shut-down capabilities inside the ramping constraints. For the set of experiments presented in 5.1, where no ramping constraints are considered, the ramping constraints of *1bin* were adapted to only model the start-up and shut-down capabilities. Therefore, the constraints for the unit's start-up and shut-down capability become $\widehat{p}_t \leq \mathrm{SU}(u_t - u_{t-1}) + \overline{P}(1 + u_{t-1} - u_t)$ and $\widehat{p}_{t-1} \leq \mathrm{SD}(u_{t-1} - u_t) + \overline{P}(1 + u_t - u_{t-1})$, respectively.

3bin formulation

The *3bin* formulation is the following Ostrowski et al. (2012):

$$\begin{aligned}
&\underline{P}u_t \leq \widehat{p}_t \leq \overline{P}u_t && t = 1, \ldots, T \\
&\widehat{p}_t \leq \widehat{p}_{t-1} + RUu_{t-1} + SUv_t && t = 2, \ldots, T \\
&\widehat{p}_{t-1} \leq \widehat{p}_t + RDu_t + SDw_t && t = 2, \ldots, T
\end{aligned} \tag{43}$$

where the minimum up and down constraints are guaranteed using (1)–(3), and the initial conditions of those constraints are ensured in the same way as *1bin* (see Appendix 1.1).

Similarly to *1bin*, *3bin* also models the unit's start-up and shut-down capabilities inside the ramping constraints. Then, for the set of experiments presented in Sect. 5.1, the ramping constraints of *3bin* were adapted to only model the start-up and shut-down capabilities. Therefore, the constraints for the unit's start-up and shut-down capability become $\widehat{p}_t \leq \overline{P}u_{t-1} + SUv_t$ and $\widehat{p}_{t-1} \leq \overline{P}u_t + SDv_t$, respectively.

Note that, unlike *1bin*, *3bin* and *TC* do not need extra variables suc_t and std_t for the start-up and shut-down costs since these costs can be directly expressed with variables v_t and w_t and included in the objective function, see (41).

Appendix 2: Network-constrained UC formulation

Here, we present the network-constrained UC formulation, whose core is based on the tight and compact model presented in Sect. 3.

Nomenclature

In the following we present the additional needed notations beyond the ones presented in Sect. 2.

Indexes and Sets

$b \in \mathcal{B}$	Buses, running from 1 to B.
$\mathcal{B}^{\mathrm{W}}$	Set of buses in $\mathcal{B}$ with wind power injection.
$l \in \mathcal{L}$	Transmission lines, running from 1 to L.
$g \in \mathcal{G}$	Generating units, running from 1 to G.

$t \in \mathcal{T}$ Hourly periods, running from 1 to T hours.

System parameters

D_{bt} Energy demand on bus b at the end of hour t [MWh].
$\underline{D}_t^- / D_t^+$ System requirements for downward/upward reserve for hour t [MW].
$\overline{F}_l$ Power flow limit on transmission line l [MW].
$\Gamma_{lb}/\Gamma_{lg}^{\mathrm{G}}$ Shift factor for line l associated with bus b/unit g [p.u.].
P_{bt}^{W} Nominal forecasted wind energy for hour t [MWh].

Unit's parameters

$C_g^{\mathrm{SD}}/C_g^{\mathrm{SU}}$ Shut-down/start-up cost [\$].
$\mathrm{RD}_g/\mathrm{RU}_g$ Ramp-down/ramp-up capability [MW/h].

Decision variables

p_{bt}^{W} Wind energy output for hour t [MWh].
r_{gt}^-/r_{gt}^+ Downward/upward power reserve [MW].

Objective function

The UC seeks to minimize all production costs:

$$\min \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left[C_g^{\mathrm{LV}} \left(\underline{P}_g u_{gt} + p_{gt} \right) + C_g^{\mathrm{NL}} u_{gt} + C_g^{\mathrm{SU}} v_{gt} + C_g^{\mathrm{SD}} w_{gt} \right] \tag{44}$$

The proposed formulation also takes into account variable start-up costs, which depend on how long the unit has been offline. The reader is referred to (Morales-España et al. 2013a, b) for further details.

System-wide constraints

Energy demand balance and upward/downward reserves requirements are guaranteed as follows:

$$\sum_{g\in\mathcal{G}} \left(\underline{P}_g u_{gt} + p_{gt} \right) = \sum_{b\in\mathcal{B}} D_{bt} - \sum_{b\in\mathcal{B}^{\mathrm{W}}} p_{bt}^{\mathrm{W}} \quad \forall\, t \tag{45}$$

$$\sum_{g\in\mathcal{G}} r_{gt}^+ \geq D_t^+ \quad \forall\, t \tag{46}$$

$$\sum_{g\in\mathcal{G}} r_{gt}^- \geq D_t^- \quad \forall\, t \tag{47}$$

Transmission limits are ensured with

$$-\overline{F}_l \leq \sum_{g\in\mathcal{G}} \Gamma^{\mathrm{G}}_{lg} \left(\underline{P}_g u_{gt} + p_{gt}\right) + \sum_{b\in\mathcal{B}^{\mathrm{W}}} \Gamma_{lb} p^{\mathrm{W}}_{bt} - \sum_{\forall\, b\in\mathcal{B}} \Gamma_{lb} D_{bt} \leq \overline{F}_l \quad \forall\, l, t \tag{48}$$

Individual unit constraints

The commitment, start-up/shut-down logic, and the minimum up/down times are guaranteed by constraints (1)–(3) and (9)–(10) replicated for each generation unit g and where the initial conditions for the minimum up/down constraints are detailed in (Morales-España et al. 2013a). Basically, u_{gt} is fixed (become constant) to 0 or 1 for the initial periods where the unit must remain offline or online, respectively.

The energy production and reserves must be within the power capacity limits:

$$\begin{aligned} p_{gt} + r^{+}_{gt} \leq \left(\overline{P}_g - \underline{P}_g\right) u_{gt} - \left(\overline{P}_g - \mathrm{SD}_g\right) w_{g,t+1} \\ - \max\left(\mathrm{SD}_g - \mathrm{SU}_g, 0\right) v_{g,t} \quad \forall\, g \in \mathcal{G}^1, t \end{aligned} \tag{49}$$

$$\begin{aligned} p_{gt} + r^{+}_{gt} \leq \left(\overline{P}_g - \underline{P}_g\right) u_{gt} - \left(\overline{P}_g - \mathrm{SU}_g\right) v_{gt} \\ - \max\left(\mathrm{SU}_g - \mathrm{SD}_g, 0\right) w_{g,t+1} \quad \forall\, g \in \mathcal{G}^1, t \end{aligned} \tag{50}$$

$$\begin{aligned} p_{gt} + r^{+}_{gt} \leq \left(\overline{P}_g - \underline{P}_g\right) u_{gt} - \left(\overline{P}_g - \mathrm{SU}_g\right) v_{gt} \\ - \left(\overline{P}_g - \mathrm{SD}_g\right) w_{g,t+1} \quad \forall\, g \notin \mathcal{G}^1, t \end{aligned} \tag{51}$$

$$p_{gt} - r^{-}_{gt} \geq 0 \quad \forall\, g, t \tag{52}$$

where $\mathcal{G}^1$ is defined as the units in $\mathcal{G}$ with $\mathrm{TU}_g = 1$.

Ramping capability limits are ensured with

$$\left(p_{gt} + r^{+}_{gt}\right) - p_{g,t-1} \leq \mathrm{RU}_g \quad \forall\, g, t \tag{53}$$

$$-\left(p_{gt} - r^{-}_{gt}\right) + p_{g,t-1} \leq \mathrm{RD}_g \quad \forall\, g, t \tag{54}$$

notice that by modeling the generation output p_{gt} above $\underline{P}_g$, the proposed formulation avoids introducing binary variables into the ramping constraints (53) and (54), unlike *1bin* and *3bin*, see Appendix 1 respectively. In other words, when the generation output variable is defined between 0 and $\overline{P}_g$, then the ramping constraints should consider the case when a generator's output level should not be limited by the ramp rate, when it is starting up or shutting down; such complicating situations are usually tackled by introducing big-M parameters together with binary variables into the ramping constraints.

Wind production limits are represented by

$$p_{bt}^{\mathrm{W}} \leq P_{bt}^{\mathrm{W}} \quad \forall\, b \in \mathcal{B}^{\mathrm{W}}, t \tag{55}$$

Finally, non-negative constraints for all decision variables:

$$p_{gt}, r_{gt}^{+}, r_{gt}^{-} \geq 0 \quad \forall\, g, t \tag{56}$$

$$p_{bt}^{\mathrm{W}} \geq 0 \quad \forall\, b \in \mathcal{B}^{\mathrm{W}}, t \tag{57}$$

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