Stability for Discrete Event Max-Min-Plus (MMP) and Max-Min-Plus-Scaling (MMPS) Systems

Max-Plus Lyapunov Functions for Stability Analysis & Control

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Master of Science Thesis

Delft University of Technology

Delft Center for Systems and Control

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Abstract

This research presents a framework for analysing the stability and control of discrete-event systems, specifically emphasising max-min-plus (MMP) and max-min-min-plus-scaling (MMPS) systems. These systems are valuable modelling tools for various applications, including production systems and urban railway traffic management, respectively. However, a critical challenge in discrete-event systems is the lack of a generalised approach to assessing the stability of time signals, particularly in the context of MMPS systems. To address this challenge, this research will use max-plus Lyapunov functions already used to study the buffer stability in discrete-event switching-max-plus-linear (SMPL) systems.

This thesis provides a framework to use max-plus Lyapunov functions to determine buffer stability of MMP and MMPS systems, focusing on their time signals. The max-plus Lyapunov function uses a buffer for each pair of states. The system is considered stable if the difference converges to the buffer levels for every pair of states. Given the structure of MMP and MMPS systems, the difference between the states after one state update will often be bounded. To determine this boundedness of the buffer levels, a novel concept of "fully correlated" MMP and MMPS systems is introduced. Using the properties of fully correlated systems, an algorithm is proposed to determine the buffer levels for both MMP and MMPS systems. We also derive analytical methods using Markov properties to assess the additive eigenvalue of fully correlated time-invariant monotonic MMPS systems. Using the property of fully correlatedness, it is also derived that fully correlated time-invariant non-monotonic MMPS systems will always have a bounded buffer and growth rate and can have multiple additive eigenvalues. The findings show that fully correlated time-invariant systems consistently exhibit bounded growth rates.

In addition to providing theoretical insights, this study demonstrates the practical use of max-plus Lyapunov functions as a control Lyapunov function (CLF) in model predicitive control (MPC). A novel control technique is proposed to stabilise naturally unstable discrete-event systems. This approach has been effectively applied to stabilise inherently unstable discrete-event max-plus-linear (MPL) and MMP systems, indicating the practical significance of the proposed framework.

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Preface & Acknowlagements

Dear reader,

After a year, I am finally finished with my thesis work. Many of my friends found completing their final thesis agonizing, but I must say that I quite enjoyed it. Spending a whole year on one subject makes you invested. In one of my first meetings after the literature research, I was assured that if I found no working results, it would still be okay, 'Just write down what you did'. My research subject was described to me as a quest, and I experienced it as such. I often thought about the subject outside the TU, trying to figure out why something worked or did not. I believe that most of the ideas for this thesis came to me while biking or walking.

I want to thank Ton and Sreeshma for their excellent guidance and fun meetings this year. Ton, your enthusiasm around the subject is infectious. In the beginning stage of my thesis, every meeting had an enormous flow of information. This helped me understand the max-plus world and made me enthusiastic about the topic as well.

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Furthermore, I would like to express my heartfelt gratitude to my parents. Although confidence in my capabilities was not always present in the first year of my bachelor's, I hope I made up for it in later years. Although the path was not always without its challenges, your support and genuine interest in my life were constant sources of encouragement.

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“Simplicity is the highest goal, achievable when you have overcome all difficulties. After one has played a vast quantity of notes and more notes, it is simplicity that emerges as the crowning reward of art.’

— *Frédéric Chopin*

Chapter 1

Introduction

1-1 Background

Systems and control is a diverse field with applications in all engineering domains. Besides, engineering systems and control can also be found in biological and economic systems. Examples are how blood sugar and blood pressure are regulated in the human body or how unemployment and inflation are controlled by governmental fiscal decisions[1]. While it may not always be physically present, systems and control are all around us. One of the main principles of systems and control is studying the behaviour of dynamical systems. To control dynamical systems often, feedback is used to keep the states close to the desired value.

Stability is a crucial concept in systems and control. It can be characterized as the property of a system that describes its ability to maintain a desired state or trajectory. There are different notions of stability for dynamical systems. One of the most essential notions of stability is Lyapunov stability.

There is significant interest in better understanding and controlling manufacturing processes typically studied in disciplines such as Operations Research, and this has led to interdisciplinary research to study the control of discrete-event systems (DES) that cannot be described by traditional differential or difference equations[1]. Events are considered to be sudden changes in a process[2]. Examples of events are a train that departs from a station, a message sent, and a product finished in a production line.

Often, models that describe the behaviour of a discrete event system are nonlinear in conventional algebra. Max-plus algebra is a mathematical framework that uses the operations maximum and addition. The maximum operator replaces the conventional plus, and the plus operation replaces the conventional times operation. Using max-plus algebra, a class of discrete-event systems (DES) can be described: max-plus linear discrete-event systems (DES). These systems will be linear in the max-plus algebra[3]. Examples of MPL discrete-event systems are flow-shop scheduling problems and printers[4].

The modelling of discrete-event systems can be extended to model a larger class of systems. An extension of max-plus-linear (MPL) systems are switching-max-plus-linear (SMPL) systems, and such models extend the max-plus linear modelling framework by allowing changes in synchronization and ordering constraints in the system evolution[5]. Another extension is max-min-plus (MMP) systems, which are max-plus systems, described by the max and plus operation, to which the min operation, being minimization, is added. Using MMP systems, it is possible to model systems that experience synchronization and competition. Thus, a larger class of problems can be modeled[2]. Finally, MMP systems can be extended with the operation scaling, MMPS systems, which results in a hybrid discrete-event system.

The scaling operator may be included if external parameters rely on prior state or input values. MMPS systems can also arise when examining the closed-loop setup of an MPL system with a resudation or model predicitive control (MPC) controller[6]. Lastly, MMPS systems are very suitable to approximate nonlinear (in max-plus algebra) discrete event functions. The theory on the stability of MMPS systems is still in its early stages. In [7], two different methods are presented to determine the additive eigenvalue (growth rate) of the time signals of MMPS systems.

In Max-plus-algebraic hybrid automata: beyond synchronization and linearity [8], an approach is proposed that uses max-plus Lyapunov functions. It is possible to prove the stability of the buffer levels of SMPL systems using the max-plus Lyapunov function. The buffer level in discrete-event systems described in max-plus algebra is the time delay between the occurrences of different events in either the same event cycle or consecutive ones [8]. This research aims to give a methodology to prove the buffer stability of MMP and MMPS systems using max-plus Lyapunov functions.

1-2 Problem description

For time-driven systems, stability can be characterized as the property of a system that describes its ability to maintain a desired state or trajectory. The states of discrete-event systems (DES) often represent time. As time is monotonically increasing, the states will never converge to a certain point. Therefore, stability theory for conventional time-driven systems is unsuitable for accessing the stability of DES.

The buffer level in discrete-event systems described in max-plus algebra is the time delay between the occurrences of different events in either the same event cycle or consecutive ones [8]. For SMPL systems, a method is derived to validate the stability of the buffer levels using max-plus Lyapunov functions. As the introduction of minimization in MMP and scaling in MMPS systems will be considered nonlinear in the max-plus algebra sense, the approach presented in [8] has to be modified.

There is no established theory regarding the buffer stability of MMPS systems. Given that these systems can simulate complicated DES, it would be advantageous to develop a means of determining their buffer stability. Moreover, the existence of a stabilizing controller can be proven through control Lyapunov function (CLF). Consequently, if we can devise a way to utilize the max-plus Lyapunov function, it may also be feasible to use it for control purposes.

1-2-1 Research questions

- How can buffer stability be proven using Lyapunov or Lyapunov-like functions for discrete event max-min-plus (MMP) systems?
  - What kind of Lyapunov function can be used to determine the buffer stability of a max-min-plus (MMP) system?
  - Can we provide a generalized methodology to determine the buffer stability of max-min-plus (MMP) systems?

- How can buffer stability be proven using Lyapunov or Lyapunov-like functions for the time signals of discrete event max-min-min-plus-scaling (MMPS) systems?
  - What kind of Lyapunov function can be used to determine the buffer stability of a max-min-min-plus-scaling (MMPS) system?
  - Can we provide a generalized methodology to determine the buffer stability of max-min-min-plus-scaling (MMPS) systems?
  - Is it possible to determine the additive eigenvalues (growth rates) for discrete-event max-min-min-plus-scaling (MMPS) systems?

- Can max-plus Lyapunov Functions serve as Control Lyapunov Functions for Model Predictive Control in Discrete-Event Systems?
  - Can a generalized approach be developed to stabilize the buffers of unstable discrete-event systems?
  - Is it possible to utilize max-plus Lyapunov functions as control Lyapunov function (CLF) to stabilize the buffer of unstable max-plus-linear (MPL) systems?
  - Is it possible to utilize max-plus Lyapunov functions as control Lyapunov function (CLF) to stabilize the buffer of unstable max-min-plus (MMP) systems?

1-2-2 Approach

The first chapter of the thesis gives background information about systems and control, stability and discrete-event systems (DES) and the importance of these subjects. The problem description comprises the research questions and the approach applied in this thesis.

To properly grasp the concepts discussed in the background section, it is crucial to have a solid understanding of the mathematics involved. As such, the second chapter begins by defining important mathematical terms related to discrete-event systems (DES). The chapter covers definitions of max-plus algebra and max-plus convex geometry. Furthermore, it delves into the mathematical representation of max-plus-linear (MPL), switching-max-plus-linear (SMPL), max-min-plus (MMP), and max-min-min-plus-scaling (MMPS) systems, including the corresponding canonical forms and time invariance of MMPS systems.

The third chapter discusses the stability analysis of conventional time-driven systems and discrete-event SMPL systems. We will use the notions of stability for discrete-event SMPL as a basis to prove the stability of MMP and MMPS systems in the fourth chapter. This concludes the literature section.

The fourth and fifth chapters present the contributions of this thesis. The fourth chapter provides a framework for proving the stability of MMP and MMPS systems. We define fully correlated MMP and MMPS to demonstrate buffer stability and prove their boundedness. We will present an algorithm to find the buffer of fully correlated MMPS systems for the max-plus Lyapunov function. Additionally, the chapter provides insights concerning the behaviour of MMPS systems. This will focus on the growth rate (additive eigenvalue) of MMPS systems. For both MMP and MMPS systems, we provide examples using Matlab simulations. All the optimizations from the examples are performed using Yalmip and Gurobi.

The fifth chapter explores using max-plus Lyapunov functions as cost functions for model predicitive control (MPC). The chapter provides two examples which show promising results in stabilizing inherent unstable discrete-event systems. We give a general approach that can be applied to discrete-event systems. The approach is validated by stabilizing an unstable MPL and MMP system. This is done by simulating the system in Matlab, with and without disturbance. The final two chapters present the main conclusions, contributions, and recommendations for future research.

1-3 Outline

The first chapter presents the introduction with a background, problem description and outline. The second chapter provides the mathematical background of discrete-event systems from the literature. The third chapter will give the stability theory of conventional time-driven systems and discrete-event switching-max-plus-linear (SMPL) systems from the literature. The fourth and fifth chapters present the contributions of this thesis. Chapter four introduces a framework for proving the stability of max-min-plus (MMP) and max-min-min-plus-scaling (MMPS) systems, which is accompanied by analytical solutions for the additive eigenvalue of MMPS systems. Chapter five presents a model predicitive control (MPC) method that uses max-plus Lyapunov functions as a control Lyapunov function (CLF). Finally, the last two chapters provide the main conclusions, contributions, and recommendations for future research.

Chapter 2

Discrete-Event Systems

This chapter explores discrete event systems' characteristics and mathematical representations. We discuss the importance of max-plus algebra and max-plus convex geometry. Additionally, we define several systems, including max-plus-linear (MPL), switching-max-plus-linear (SMPL), max-min-plus (MMP), and max-min-min-plus-scaling (MMPS) systems. We explore canonical forms and the property of time invariance for MMPS systems. This chapter lays the foundation for understanding discrete event systems' principles, representations, and characteristics and prepares for deeper explorations for stability analysis and control.

2-1 Discrete event systems

Discrete event systems are dynamic systems where state updates occur at events. An example of such events is in production lines, where changes in product status are registered as events. These systems describe man-made systems like production lines, railway systems, etc. They consist of a finite number of resources like processors, memories, communication channels, and machines, which are processed by a certain amount of users, such as jobs, packets, and manufactured objects. The system can be expressed using the important terms synchronization and concurrency. Synchronization requires the availability of resources or users simultaneously, while competition occurs when some users must choose among several resources[9].

Suppose the processing times in a system depend on external parameters or previous values of the state and input. In that case, such a system can be written as a max-plus-linear parameter-varying (MP-LPV) system, which is equivalent to an MMPS system [4].

2-2 Max-plus algebra

Max-plus algebra consists of two operations: max-plus addition ($\oplus$) and multiplication ($\otimes$). The zero element is defined as $\epsilon = -\infty$ where $\mathbb{R}_\epsilon = \mathbb{R} \cup \{\epsilon\}$. The unit element is $\mathbb{1} = 0$. The two operations are equivalent to:

$$a \oplus b = \max(a, b) \tag{2-1}$$
$$a \otimes b = a + b \tag{2-2}$$

For $a, b \in \mathbb{R}_\epsilon$. When describing n-dimensional vectors or $m \times n$-matrices this is defined respectively as $\mathbb{R}_\epsilon^n$ and $\mathbb{R}_\epsilon^{m\times n}$. A matrix element is denoted with i and j, the same as the classical algebraic definition. Matrix operations in max-plus algebra are equivalent to:

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max\left(a_{ij}, b_{ij}\right) \tag{2-3}$$
$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_k \left(a_{ik} + c_{kj}\right) \tag{2-4}$$

The max-plus identity matrix $\mathscr{I}_n^{\otimes}$ has zeros on the diagonal and all non-diagonal elements are equal to ϵ[8].

Definition 2.1 (Min-plus algebra, [8])**.** *The min-plus algebra, $\mathbb{R}_{min} = (\mathbb{R}_\top, \oplus', \otimes')$, defined as a dual of the max-plus algebra acting on the set $\mathbb{R}_\top = \mathbb{R} \cup \{+\infty\}$, is also a tropical semiring. The zero element is $\top = +\infty$ and the unit element is $\mathbb{1}$. The vector and matrix operations are defined analogously as in the max-plus algebra.*

Definition 2.2 (Regularity, [2])**.** *A matrix $A \in \mathbb{R}_\epsilon^{n\times m}$ is called regular if A contains at least one finite element in each row.*

Definition 2.3 (Irreducibility, [2])**.** *The term irreducibility for max-plus algebra is a graph-theoretical concept. Let a graph $\mathcal{G} = (\mathcal{N}, \mathcal{D})$ denote a graph with node set $\mathcal{N}$ and arc set $\mathcal{D}$. For $i, j \in \mathcal{N}$, node j is said to be reachable from node i, denoted as $i\mathcal{R}j$, if there exists a path from i to j. A graph $\mathcal{G}$ is called strongly connected if for any two nodes $i, j \in \mathcal{N}$, node j is reachable from node i. A matrix $A \in \mathbb{R}_\epsilon^{n\times n}$ is called irreducible if its communication graph $\mathcal{G}(a)$ is strongly connected. If a matrix is not irreducible, it is called reducible.*

2-2-1 Max-plus convex geometry

The max-plus convex geometry is a basis for proving the stability of discrete event SMPL in [8], which is described in section 3-2. This section gives max-plus analogues to linear spaces, convex sets, and cones.

Definition 2.4 (Supremum norm, [8])**.** *Let $x \in \mathbb{R}_\epsilon^n$. The max-plus algebra is equipped with the conventional l^∞ norm defined as:*

$$||x||_\infty = \max_{i\in\underline{n}} |x_i| = \max(\max_{i\in\underline{n}}(x_i), \max_{j\in\underline{n}}(-x_j)) \tag{2-5}$$

The metric induces by l^∞ norm is denoted as $d(x, y) = ||x - y||_\infty$.

Definition 2.5 (Max-plus Hilbert projective metric, [8]). *The max-plus Hilbert projective (semi) norm in max-plus algebra is defined as:*

$$||x||_{\mathbb{P}} = \max_{i \in \underline{n}}(x_i) - \min_{j \in \underline{n}}(x_j), \quad x \in \mathbb{R}^n \tag{2-6}$$

$$\|A\|_{\mathbb{P}} = \max\left\{\|[A]_{\cdot i}\|_p \mid i \in \underline{m}\right\}, \quad A \in \mathbb{R}^{n \times m}. \tag{2-7}$$

The max-plus Hilbert projective (semi-) norm induces the max-plus Hilbert projective (pseudo-) metric as $d_{\mathrm{H}}(x, y) = \|x - y\|_{\mathbb{P}}$ for $x, y \in \mathbb{R}^n$. The projective norm satisfies the triangle inequality, definiteness and absolute homogeneity.

Definition 2.6 (Open ball (projective norm), [8]). *An open ball with radius $\delta > 0$ and with center $\{\lambda + x\}$ with $\lambda \in \mathbb{R}$, with respect to the max-plus Hilbert projective norm is defined as:*

$$\mathscr{B}_\delta(x) := \{y \in \mathbb{R}^n | \; ||y - x||_{\mathbb{P}} < \delta\} \tag{2-8}$$

Definition 2.7 (Open ball (infinity norm), [4]). *An open ball of radius $\delta > 0$ centered at x with respect to the infinity norm is defined as:*

$$\mathscr{B}_\delta(x) := \{y \in \mathbb{R} | \; ||y - x||_\infty < \delta\} \tag{2-9}$$

Definition 2.8 (Max-plus cones, [8]). *A subset $\mathscr{W} \subseteq \mathbb{R}^n_\epsilon$ is said to be a max-plus cone if it is closed under addition ($\oplus$) of its elements and under multiplication ($\otimes$) with scalars in $\mathbb{R}_\epsilon$:*

$$\lambda \otimes u \oplus \mu \otimes v \in \mathscr{W} \tag{2-10}$$

$\forall u, v \in \mathscr{W}$ and $\lambda, \mu \in \mathbb{R}_\epsilon$. The subset $\mathscr{W}$ is said to be a convex max-plus cone if (2-10) *holds $\forall u, v \in \mathscr{W}$ and $\lambda, \mu \in \mathbb{R}_\epsilon$ such that $\lambda \oplus \mu = \mathbb{1}$.*

Definition 2.9 (Finitely generated max-plus cones, [8]). *A max-plus span, span$_\oplus$, is defined analogously to conventional algebra. A max plus cone $\mathscr{W} \subseteq \mathbb{R}^n_\epsilon$ is a finitely generated max-plus cone if there exists a set of vectors $W = \{w_1, w_2, ..., w_m\}$ such that:*

$$\mathscr{W} = span_\oplus(W) = \left\{\bigoplus_{i=1}^{m} \alpha_i \otimes w_i | \alpha_i \in \mathbb{R}_\epsilon\right\} \tag{2-11}$$

Definition 2.10 (Kleene cones, [8]). *A Kleene cone is a max-plus cone generated as a max-plus column span of a Kleene star matrix.*

A Kleene cone has the property that it is also convex in the Euclidean sense. A max-plus cone bounded in the max plus Hilbert projective norm is Euclidean convex if and only if it is generated as the max-plus column span of a Kleene star matrix. It can be noted that $\mathscr{B}_\delta = \text{span}_\oplus(K^{(\delta)})$ where

$$K^{(\delta)} = \begin{bmatrix} 0 & -\delta & \cdots & -\delta \\ -\delta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\delta \\ -\delta & \dots & -\delta & 0 \end{bmatrix} \tag{2-12}$$

is a Kleene-star matrix with finite columns[8].

Definition 2.11 (Max-plus C-set, [8]). *A max-plus C-set is a subset $\mathscr{K} \subseteq \mathbb{R}^n$ such that it is a finitely generated max-plus cone and convex cone and bounded in the max-plus Hilbert projective norm.*

2-2-2 Max-plus-linear (MPL) systems

Definition 2.12 (Max-plus linear systems, [10])**.** *Discrete-event systems with only synchronization and no concurrency can be modelled by a max-plus-algebraic model of the following form[9]:*

$$\begin{aligned} x(k+1) &= A \otimes x(k) \oplus B \otimes u(k), \\ y(k) &= C \otimes x(k) \end{aligned} \tag{2-13}$$

with $A \in \mathbb{R}_\varepsilon^{n\times n}, B \in \mathbb{R}_\varepsilon^{n\times m}$ *and* $C \in \mathbb{R}_\varepsilon^{l\times n}$ *where* m *is the number of inputs and* l *the number of outputs. A discrete-event system that can be modelled by* (2-13) *will be called a max-plus-linear time-invariant discrete-event system or MPL system for short.*

2-2-3 Switching-max-plus-linear (SMPL) systems

Definition 2.13 (Switching-max-plus-linear (SMPL) systems, [8])**.** *Switching max-plus-linear (SMPL) systems can be described with the following dynamics:*

$$\begin{aligned} x(k) &= f(l(k), x(k-1), u(k), r(k)) \\ l(k) &= \phi(l(k-1), x(k-1), u(k), v(k), w(k)) \\ y(k) &= h(l(k), x(k), u(k), r(k)) \end{aligned} \tag{2-14}$$

The functions $f(\cdot)$ *and* $h(\cdot)$ *are max-plus linear and describe the state update and the output of the system. The function* $\phi(\cdot)$ *encodes the switching mechanism of the state dynamics.*

2-3 Max-min-plus (MMP) systems

We have already discussed the zero elements of max-plus (ϵ) and min-plus algebra ($\top$). The complete set is defined as $\mathbb{R}_c = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ (complete). The following conventions are introduced $0 \cdot \epsilon = 0$, $0 \cdot \top = 0$ and $\top + \epsilon = 0$. To refer to either $\mathbb{R}_\epsilon$, $\mathbb{R}_\top$, $\mathbb{R}$ or $\mathbb{R}_c$ often $\mathcal{R}$ is used[4].

Definition 2.14 (Max-min-plus (MMP) systems, [2])**.** *MMP systems are described by expressions in which the three operations, minimization, maximization, and addition, appear. They can be viewed as an extension of max-plus expressions in that minimization has been added as a possible operation.*

An MMP function can be defined by (the symbol | stands for 'or' and is recursive):

$$f := x_i | \alpha | f_k \oplus f_l | f_k \oplus' f_l | f_k \otimes f_l \tag{2-15}$$

with $i \in \{1, ..., n\}$, $\alpha \in \mathcal{R}$ and where f_k and f_l are MMP functions as well.

Definition 2.15 (Conjunctive normal form (MMP), [2])**.** *The conjunctive normal form can be represented as:*

$$f = f_1 \oplus' f_2 \oplus' \cdots \oplus' f_p \tag{2-16}$$

for some finite $p \in \mathbb{N}$ *and where each* f_i *is a max-plus expression.*

Using identities:

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus b \oplus c, \\ a \oplus' (b \oplus' c) &= a \oplus' b \oplus' c, \\ c \oplus (a \oplus' b) &= (c \oplus a) \oplus' (c \oplus b), \\ c \oplus' (a \oplus b) &= (c \oplus' a) \oplus (c \oplus' b)\,, \end{aligned} \tag{2-17}$$

each max-min-plus expression f can be transformed in conunctive normal form.

2-3-1 Bipartite systems

Definition 2.16 (Bipartite systems, [2]). *A bipartite system is an MMP system characterized by two matrices $B \in \mathbb{R}_\epsilon^{n\times m}$ and $C \in \mathbb{R}_\top^{m\times n}$, such that*

$$x(k+1) = B \otimes y(k), \quad y(k+1) = C \otimes' x(k) \tag{2-18}$$

The system matrices are assumed to be regular.

Definition 2.17 (Irreducible pair, [2]). *The matrix pair (B, C) is irreducible if no permutations σ of $\underline{n}$ and τ of $\underline{m}$ exist such that*

$$B(\sigma,\tau) = \begin{pmatrix} B_{11} & B_{12} \\ \epsilon & B_{22} \end{pmatrix}, \quad C(\tau,\sigma) = \begin{pmatrix} C_{11} & \top \\ C_{21} & C_{22} \end{pmatrix},$$

where

- *the sizes of B_{ij} and $C_{ji}^\top, i, j \in \underline{2}$, are identical (the submatrices B_{ii} and C_{jj} are not necessarily square), and*
- *B_{11} and C_{22} are regular. Otherwise, the pair (B, C) is reducible.*

Theorem 2.1 (Existence eigenvalue and eigenvector of bipartite system, [2]). *Consider the regular bipartite system (2-18). If the matrix pair (B, C) is irreducible, then an (additive[1]) eigenvalue (with corresponding additive eigenvector) exists.*

2-4 Max-min-plus-scaling (MMPS) systems

Definition 2.18 (Max-min-plus-scaling (MMPS) function, [11]). *Max-min-plus-scaling functions consist of four operations: max, min, plus and scaling. It can be defined by (the symbol | stands for 'or' and is recursive):*

$$f := x_i|\alpha|f_k \oplus f_l|f_k \oplus' f_l|f_k \otimes f_l|\beta \cdot f_k, \tag{2-19}$$

with $i \in \{1, ..., n\}$, $\alpha \in \mathcal{R}$, $\beta \in \mathbb{R}$ and where f_k and f_l are MMPS functions as well.

[1] See definition 2.19.

An MMPS system can be described as:

$$\begin{aligned} x(k+1) =& \mathcal{M}_x(x(k), u(k), d(k)) \\ y(k) =& \mathcal{M}_y(x(k), u(k), d(k)) \\ & \mathcal{M}_c(x(k), u(k), d(k)) \leq c \end{aligned} \tag{2-20}$$

with states $x(k)$, inputs $u(k)$ and disturbances $d(k)$[12]. The term (2-20) is for constrained MMPS systems. When absent, the system will be an unconstrained MMPS system. $\mathcal{M}_x$, $\mathcal{M}_y$ and $\mathcal{M}_c$ are MMPS expressions. The classes of Max-plus linear parameter varying (Max-Plus LPV) systems and MMPS systems coincide[4].

Definition 2.19 (Additive eigenvalue, additive eigenvector, [7])**.** *The system $x(k) = f(x(k-1))$, $k \in \mathbb{Z}^+$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to have an additive eigenvalue if there exists a real number $\lambda \in \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ such that*

$$f(v) = v + \lambda. \tag{2-21}$$

The scalar λ is called an eigenvalue, and the vector v is called a corresponding eigenvector.

As we use conventional algebraic eigenvalues in this thesis, we will refer to additive eigenvalues and additive eigenvectors for MMPS systems as λ_g and v_g.

2-4-1 Canonical forms of MMPS systems

The canonical form of an MMPS system is rewriting it in a standard way while preserving the behaviour of the system. The following canonical forms coincide with MMPS systems.

Definition 2.20 (Conjunctive MMPS systems (cMMPS), [4])**.** *Conjunctive MMPS systems describe a state-space model of the form:*

$$x(k) = \min_{i=1,\ldots,K} \max_{j=1,\ldots,n_i} (\alpha_{i,j}^T p(k) + \beta_{i,j}) \tag{2-22}$$

for some integers $K, n_1, ..., n_K$ vectors $\alpha_{i,j}$ and real numbers $\beta_{i,j}$.

Definition 2.21 (Disjunctive MMPS systems (dMMPS), [4])**.** *Disjunctive MMPS systems describe a state-space model of the form:*

$$x(k) = \max_{i=1,\ldots,L} \min_{j=1,\ldots,m_i} (\sigma_{i,j}^T p(k) + \rho_{i,j}) \tag{2-23}$$

for some integers $L, m_1, ..., m_L$, vectors $\sigma_{i,j}$, and real numbers $\rho_{i,j}$.

Definition 2.22 (Kripfganz MMPS systems (kMMPS), [4])**.** *Kripfganz MMPS systems describe a state-space model of the form:*

$$x(k) = \max_{i=1,\ldots,M} (\mu_{1,i}^T p(k) + \nu_{1,i}) - \max_{j=1,\ldots,K} (\mu_{2,j}^T p(k) + \nu_{2,j}) \tag{2-24}$$

for some integers $L, m_1, ..., m_L$ vectors $\mu_{i,j}$, and real numbers $\nu_{i,j}$.

Definition 2.23 (ABC canonical form, [7]). *Consider the following system:*

$$x(k+1) = C \otimes' (B \otimes (A \cdot x(k))) \tag{2-25}$$

The system is an MMPS system in the conjunctive ABC canonical form [2].

2-4-2 Time invariance for MMPS systems

Definition 2.24 (Additive homogeneous system, [6]). *Consider an MMPS system $x(k) = f(p(k))$; the system is said to be additive homogeneous if the function $f : \mathcal{P} \rightarrow \mathcal{R}^n$ has the following property :*

$$f(p+\lambda) = f(p) + \lambda,\ \forall \lambda \in \mathbb{R} \tag{2-26}$$

Definition 2.25 (Partly additive homogeneous system, [6]). *Now we consider an MMPS system that is split $p \in \{p_1, p_2\}$ where $p_1 \in \mathbb{R}^{n_1}$ and $p_2 \in \mathbb{R}^{n_2}$ with the functions $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_1}$ and $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_2}$. The system is said to be a partly additive homogeneous system if the following holds:*

$$\begin{bmatrix} f_1(p_1+\lambda, p_2) \\ f_2(p_1+\lambda, p_2) \end{bmatrix} = \begin{bmatrix} f_1(p_1, p_2)+\lambda \\ f_2(p_1, p_2) \end{bmatrix},\ \forall \lambda \in \mathbb{R} \tag{2-27}$$

2-4-3 Time invariance for time and quantity signals

Consider an MMPS system that only has time signals $x_t = f_t(p_t(k))$. The system will be time-invariant if a shift in the signal p_t will result in a shift in the value of the state.

$$x_t(k) = f_t(p_t(k)) \tag{2-28}$$

$$x_t(k) + \tau = f_t(p_t(k) + \tau) \tag{2-29}$$

Thus, such an MMPS system will be time-invariant if the system is additive homogeneous. Consider an MMPS system with the split $p \in \{p_t, p_q\}$ dividing the time and quantity signals.

$$x(k) = \begin{cases} x_t(k) = f_t(p_t(k), p_q(k)) \\ x_q(k) = f_q(p_t(k), p_q(k)) \end{cases} \tag{2-30}$$

The MMPS system is said to be time-invariant if the system is partly additive homogeneous. The term of the system that should be additive homogeneous is the term with the time signals [4].

$$\begin{bmatrix} f_t(p_t+\lambda, p_q) \\ f_q(p_t+\lambda, p_q) \end{bmatrix} = \begin{bmatrix} f_t(p_t, p_q)+\lambda \\ f_q(p_t, p_q) \end{bmatrix},\ \forall \lambda \in \mathbb{R} \tag{2-31}$$

Lemma 2.2 (Time-invariance for MMPS systems in ABC conjunctive form, [7]). *An MMPS in ABC-canonical form is homogeneous/time-invariant iff $\sum_j a_{i,j} = 1,\ \forall i$ where $a_{i,j}$ are the entries of A.*

Lemma 2.3 (Monotonic MMPS system, [7]). *An MMPS in ABC-canonical form is monotonic when $a_{i,j} \geq 0,\ \forall i,j$ where $a_{i,j}$ are the components of A.*

[2] The notation for the ABC canonical form in [7] differs from this one due to a change in notation during the thesis.

2-5 Conclusion

This chapter explored discrete-event systems, including their characteristics and algebraic representations. Discrete-event systems are dynamic systems where state updates occur at events. Examples of how they manifest in real-world scenarios are production lines and railway systems. These systems consist of a finite number of resources and users, and their behaviour can be described through the concepts of synchronization and concurrency.

We studied the algebraic framework of max-plus algebra, with operations of max-plus addition and multiplication. We also discussed different important concepts of max-plus algebra. The chapter explored max-plus convex geometry, a foundation for establishing the stability of discrete-event switching-max-plus-linear (SMPL) systems, which will be discussed in section 3-2. Various geometric concepts were discussed, such as the supremum norm, max-plus Hilbert projective metric, and max-plus cones.

Furthermore, we discussed the representation of max-plus-linear (MPL), switching-max-plus-linear (SMPL), max-min-plus (MMP) and max-min-min-plus-scaling (MMPS) systems. For MMP systems, we discussed bipartite systems for which the buffer stability will be proven in chapter four. For MMPS systems, we examined different canonical forms —conjunctive, disjunctive, Kripfganz, and ABC—that help standardize their representation while preserving their behaviour. Lastly, we discussed the notions of time invariance and monotonicity for MMPS systems. We examined how time invariance can be interpreted in the context of time and quantity signals within MMPS systems.

In conclusion, this chapter has comprehensively overviews discrete-event systems, max-plus algebra, max-plus convex geometry and MPL, SMPL, MMP, and MMPS systems. These concepts lay the foundation for understanding the behaviour and analysis of discrete-event systems.

Chapter 3

Stability Analysis

This chapter presents stability analysis for time-driven and discrete-event SMPL systems. It covers Lyapunov stability, positive invariant sets, local stability and ultimate boundedness. The second part presents a stability analysis for SMPL systems. It discusses redefined notions of stability based on conventional stability theory to determine stability for SMPL systems.

3-1 Stability analysis of time-driven systems

3-1-1 Lyapunov's Method

Consider the autonomous system $\dot{x} = f(x)$ and let this be a locally Lipschitz function defined over a domain $D \subset R^n$, which contains the origin, and $f(0) = 0$. Let $V(x)$ be a continuously differentiable function defined over D such that the conditions in equations (3-1) and (3-2) hold.

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \; \forall \; x \in D \quad \text{with} \quad x \neq 0 \tag{3-1}$$

$$\dot{V}(x) \leq 0 \; \forall \; x \in D \tag{3-2}$$

Then, the origin is a stable equilibrium point of $\dot{x} = f(x)$. Moreover, if

$$\dot{V}(x) < 0 \; \forall x \in D \text{ with } x \neq 0, \tag{3-3}$$

then the origin is asymptotically stable. Furthermore, if $D = R^n$, equations (3-1) and (3-2) hold $\forall \; x \neq 0$, and $V(x)$ is radially unbounded:

$$||x|| \to \infty \Rightarrow V(x) \to \infty, \tag{3-4}$$

then the origin is globally asymptotically stable[13].

Definition 3.1 (Exponential stability, [13]). *Let $f(x)$ be a locally Lipschitz function defined over the domain $D \subset \mathbb{R}^n$, which contains the origin and let $f(0) = 0$. Let $V(x)$ be a continuously differentiable function defined over D such that equations* (3-5) *and* (3-6) *hold.*

$$k_1||x||^a \leq V(x) \leq k_2||x||^a \tag{3-5}$$
$$\dot{V}(x) \leq -k_3||x||^a \tag{3-6}$$

For all $x \in D$, where k_1, k_2, k_3 and a are positive constants. Then the origin is an exponentially stable equilibrium point of $\dot{x} = f(x)$. If the assumptions hold globally, the origin will be globally exponentially stable.

3-1-2 Lyapunov Function, discrete-time

Consider the autonomous discrete-time system $x(k+1) = f(x(k))$. Now consider a Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, which is defined and continuous on the state space. The discrete-time derivative is equal to the difference between the discrete-time Lyapunov function.

$$\Delta V(x(k)) = V(k+1) - V(k) \tag{3-7}$$

If the condition $f(0) = 0$ holds, the global uniform asymptotic stability for the discrete-time model has the same definition as in section 3-1-1, where the Lyapunov function is replaced with the Lyapunovs' difference [14].

Definition 3.2 (Exponential Stability, discrete-time, [14]). *The discrete-time system $x(k+1) = f(x(k))$ is said to be globally exponentially stable if there exists a $0 < \lambda < 1$ and a positive μ such that $\forall$ $||x(0)||$,*

$$||x(k)|| \leq \mu||x(0)||\lambda^k, \quad \forall t \geq 0 \tag{3-8}$$

3-1-3 Positive invariant sets

Definition 3.3 (Positive invariant sets, [15]). *A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for the autonomous system $x^+ = f(x,u)$ subject to the constraints in $x \in \mathcal{X}, u \in \mathcal{U}$ if $x(0) \in \mathcal{O} \Rightarrow x(k) \in \mathcal{O}$ $\forall k \in \mathbb{Z}^+$.*

Positive invariant sets determine if an autonomous system will not violate constraints. Rewriting a non-autonomous to an autonomous system using a feedback controller $u = f(x)$ can validate if the controlled system will violate the constraints. A set $\mathcal{O} \subseteq \mathcal{X}$ is a positive invariant set for the autonomous system $x^+ = f(x,u)$ subjected to the constraint in $x \in \mathcal{X}$, $u \in \mathcal{U}$, if $x(0) \in \mathcal{O} \rightarrow x(k) \in \mathcal{O}$ $\forall k \in \mathbb{Z}^+$[4]. This means that for any initial state $x(0)$ that lies in the positive invariant set, the state of the autonomous system will stay in that set for all $k \geq 0$. Consider the algorithm 1, this algorithm will calculate the positive invariant subset $\mathcal{O}_\infty$ for the autonomous system $x^+ = f_a(x)$ subject to state constraints $x(k) \in \mathcal{X}$, $\forall k \geq 0$.

Algorithm 1 generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k$, $\forall k \in \mathbb{Z}^+$ and it terminates when $\Omega_{k+1} = \Omega_k$. If it terminates, then Ω_k is the maximal positive invariant set $\mathcal{O}_\infty$ for the

Algorithm 1 Computation of $\mathcal{O}_\infty$[15]

Input: $f_a, \mathcal{X}$;
Output: $\mathcal{O}_\infty$
$\Omega_0 \leftarrow \mathcal{X}, \quad k \leftarrow -1$
repeat
    $k \leftarrow k + 1$
    $\Omega_{k+1} \leftarrow \mathrm{Pre}\,(\Omega_k) \cap \Omega_k$
until $\Omega_{k+1} = \Omega_k$
$\mathcal{O}_\infty \leftarrow \Omega_k$

autonomous system $x^+ = f_a(x)$ subject to state constraints $x(k) \in \mathcal{X}$ and $\forall k \geq 0$. If $\Omega_k = \emptyset$ for some integer k, then the simple conclusion is that $\mathcal{O}_\infty = \emptyset$. In general, algorithm 1 may never terminate. If the algorithm does not terminate in a finite number of iterations, it can be proven that $\mathcal{O}_\infty = \lim_{k\to+\infty} \Omega_k$[15].

3-1-4 Local stability and ultimate boundedness

Global stability can be a strict notion of stability. Control problems often can be solved using less strict notions of local stability because the convergence with arbitrary initial conditions can be too restrictive. Another reason to introduce local stability is that disturbances can prevent the system from asymptotically approaching the origin. In such a case, convergence to a set could be considered.

Definition 3.4 (Uniform local asymptotic stability, [14]). *Consider the set $\mathcal{S}$ and let it be a neighbourhood of the origin. The system is said to be uniformly locally asymptotically stable with the domain of attraction $\mathcal{S}$ if the following two conditions hold:*

1. *Local stability holds if $\forall \mu > 0$ there exists $\delta > 0$ such that $||x(0)|| \leq \delta$ implies $||x(t)|| \leq \mu$, $\forall t \geq 0$.*
2. *Local uniform convergence holds if $\forall\ \gamma > 0$ there exists $T(\gamma) > 0$ such that if $x(0) \in \mathcal{S}$, then $||x(t)|| \leq \gamma$, $\forall t \geq T(\gamma)$.*

Definition 3.5 (Uniform ultimate boundedness, [14]). *Consider the set $\mathcal{S}$ and let it be a neighbourhood of the origin. The system is said to be uniformly ultimately bounded in $\mathcal{S}$, if $\forall \mu > 0$ there exists $T(\mu) > 0$, such that for every $||x(0)|| \leq \mu$, $x(t) \in \mathcal{S}$, $\forall t \geq T(\mu)$.*

Definition 3.6 (Lyapunov function inside a set, [14]). *Consider the positively invariant set:*

$$\mathscr{N}[\psi, \nu] = \{x : \psi(x) \leq \nu\} \tag{3-9}$$

Now consider the set $\mathcal{S}$ and let it be a neighbourhood of the origin. The locally Lipschitz positive definite function V is said to be a Lyapunov function inside set $\mathcal{S}$ if there exists a $\nu > 0$ such that $\mathcal{S} \subseteq \mathscr{N}[\psi, \nu]$ and the inequality holds:

$$\dot{V}(x) \leq -\phi(||x(t)||) \tag{3-10}$$

for some κ-function ψ.

Definition 3.7 (Lyapunov function outside a set, [14]). *Now consider the set $\mathcal{S}$ and let it be a neighbourhood of the origin. The locally Lipschitz positive definite function V is said to be a Lyapunov function outside set $\mathcal{S}$ if there exists a $\nu > 0$ such that $\mathcal{S} \subseteq \mathscr{N}[\psi, \nu]$ and $\forall$ $x \notin \mathscr{N}[\psi, \nu]$ the inequality holds:*

$$\dot{V}(x) \leq -\phi(||x(t)||) \tag{3-11}$$

3-2 Stability analysis of switching-max-plus-linear (SMPL) systems

The stability analysis of SMPL systems in [8] has modelling assumptions. Consider the dynamics for the continuous state $x \in \mathbb{R}^n_\epsilon$ at event step $k \in \mathbb{N}$ is written in the form:

$$x(k) = f(l, x(k-1)), \quad l \in \underline{n_L} \tag{3-12}$$

The switching sequence (possible infinite) $\sigma_k = (l_k)_{k \in \mathbb{N}}$ along with an initial state $x(0)$ for system dynamics (3-12) completely describes the trajectory of the discrete-event system. For the rest of this section, it is assumed that the discrete-event system is structurally finite and the function $f(l, x)$ is continuous and additively homogeneous with the state $x \in \mathbb{R}^n$ for every $l \in \underline{n_L}$.

The notions of stability will often use the term "buffer". The buffer level of discrete-event systems described in max-plus algebra is the time delay between the occurrences of different events in either the same event cycle or consecutive ones[8]. The notion of stability is associated with the boundedness of these buffer levels.

3-2-1 Autonomous notions of stability

Definition 3.8 (Max-plus bounded-buffer stability, [8]). *A discrete event-system* (3-12) *is said to be:*

1. *uniformly max-plus bounded-buffer stable if there exists a constant $\delta > 0$ and $\forall \mu > 0$ there exists a constant $T = T(\mu, \delta) > 0$ such that if $x(0) \in \mathscr{B}_\mu$, then $x(k) \in \mathscr{B}_\delta$ $\forall k \geq T(\mu, \delta)$.*
2. *uniformly locally asymptotically max-plus bounded-buffer stable with respect to a closed set $\mathscr{K} \subseteq \mathscr{B}_\tau$ for some $\tau > 0$ if:*
   - *(a) For every $\delta > 0$ there is $\mu = \mu(\delta) > 0$ such that if $x(0) \in \mathscr{B}_\mu(\mathscr{K})$, then $x(k) \in \mathscr{B}_\delta(\mathscr{K})$ $\forall k \geq 0$.*
   - *(b) There exists a constant $\mu > 0$ and for every $\eta > 0$, there exists a scalar $T = T(\eta) > 0$ such that if $x(0) \in \mathscr{B}_\mu(\mathscr{K})$, there is a $x(k) \in \mathscr{B}_\eta(\mathscr{K})$ $\forall k \geq T(\eta)$.*

Definition 3.9 (Max-plus Lipschitz stability, [8]). *A discrete event-system* (3-12) *is said to be:*

1. *uniformly max-plus Lipschitz stable if for every $\mu > 0$ there exists a scalar $\delta > 0$ and a constant $T(\mu, \delta) \in \mathbb{N}$ such that if $x(0) \in \mathscr{B}_\mu$, then $x(k) \in \mathscr{B}_\delta(x(k-1))$ $\forall k \geq T(\mu, \delta)$.*

2. *uniformly locally asymptotically max-plus Lipschitz stable with a basin of attraction* $\mathscr{B}_\eta$*, if for a value of* $\eta > 0$ *and the same stability (as in 1) and there exists a scalar* $\rho \in \mathbb{R}$ *and a constant* $T(\eta) > 0$ *such that if* $x(0) \in \mathscr{B}_\eta$*, then* $||x(k) - \rho \cdot k||_\infty$ *is bounded* $\forall k \geq T(\eta)$*.*

To prove the uniform asymptotic max-plus bounded-buffer stability later in this chapter, it is required to have a discrete event system that is uniform max-plus Lipschitz stable. The asymptotic max-plus Lipschitz stability implies that the asymptotic growth rate is a constant; the definition requires that growth rate $\rho \in \mathbb{R}$ exists.

Definition 3.10 (Max-plus incremental stability, [8])**.** *A discrete event system* (3-12) *evolving on a positively invariant set* $\mathscr{K} \subseteq \mathbb{R}^n$ *for a given switching sequence* $\sigma_k = \{l(k)\}_{k\in\mathbb{N}}$ *is said to be*

1. *Uniformly max-plus incrementally stable in* $\mathscr{K} \subseteq \mathbb{R}^n$ *if for every* $\delta > 0$*, there is a* $\mu = \mu(\delta) > 0$ *such that for any* $x_\sigma^{(1)}(0)$*,* $x_\sigma^{(2)}(0) \in \mathscr{K}$*, if* $x_\sigma^{(2)}(0) \in \mathscr{B}_\mu(x_\sigma^{(1)}(0))$ *there is* $x_\sigma^{(2)}(k) \in \mathscr{B}_\delta(x_\sigma^{(1)}(k))\ \forall k \geq 0$*.*

2. *Uniformly asymptotically max-plus incrementally stable in* $\mathscr{K} \subseteq \mathbb{R}^n$ *if the system is uniformly max-plus incrementally stable, and for each* $\eta > 0$*, there exists a scalar* $T = T(\eta) > 0$ *such that for any* $x_\sigma^{(1)}(0)$*,* $x_\sigma^{(2)}(0) \in \mathscr{K}$*, there is* $x_\sigma^{(2)}(k) \in \mathscr{B}_\eta(x_\sigma^{(1)}(k))\ \forall k \geq 0$*.*

Definition 3.11 (Max-plus convergent dynamics, [8])**.** *A discrete event system* (3-12) *is said to be uniformly max-plus convergent in a positively invariant set* $\mathscr{K} \subseteq \mathbb{R}^n$ *for a given switching sequence* $\sigma_k = \{l(k)\}_{k\in\mathbb{N}}$ *if the following conditions hold:*

1. *there exists a unique solution* $\tilde{x}_\sigma(k)$ *of a discrete event system defined in K and bounded in the max-plus Hilbert projective norm* $\forall k \in \mathbb{N}$*.*

2. *the system is uniformly asymptotically max-plus bounded-buffer stable with respect to solution* $\tilde{x}_\sigma(k)$ *if:*

   (a) *for every* $\delta > 0$*, there is a* $\mu = \mu(\delta) > 0$ *such that* $x(0) \in \mathscr{B}_\mu(\tilde{x}_\sigma)(0)$*, there is* $x(k) \in \mathscr{B}_\delta(\tilde{x}_\sigma(k))\ \forall k \geq 0$*.*

   (b) *there exists a scalar* $T = T(\eta) > 0$ *such that if* $x(0) \in \mathscr{B}_\mu(\tilde{x}_\sigma(0))$*, there is* $x(k) \in \mathscr{B}_\eta(\tilde{x}_\sigma(k))\ \forall k \geq T(\eta)$*.*

3-2-2 Ultimate boundedness

Definition 3.12 (Lyapunov function outside a set, [8])**.** *A positive definite continuous function* $\Psi : \mathbb{R}^n \to \mathbb{R}$ *is a Lyapunov function outside* $\mathscr{B}_\delta, \delta > 0$*, for the system* (3-12) *if:*

1. *there exists* $v > 0$ *such that* $\mathscr{N}(\Psi, v) \subseteq \mathscr{B}_\delta$ *and for all* $x \notin \mathscr{N}(\Psi, v)$ *we have*

$$\Delta\Psi(x) = \Psi(f(l, x)) - \Psi(x) \leq -\alpha\left(\|x\|_{\mathbb{P}} + \delta'\right)$$

   for some $\delta' \geq 0$*, a function* α *of class* $\mathscr{K}$ *and for all* $l \in \underline{n_{\mathrm{L}}}$*.*

2. *The set* $\mathscr{N}(\Psi, v)$ *is positively invariant, so for all* $x \in \mathscr{N}(\Psi, v)$ *and for all* $l \in \underline{n_{\mathrm{L}}}$*, we have*
$$\Psi(f(l,x)) \leq v$$

Theorem 3.1 (Ultimate buffer boundedness, [8]). *A discrete-event system* (3-12) *is uniformly max-plus bounded buffer stable if it admits a Lyapunov function outside* $\mathscr{B}_\delta$*, for a finite* $\delta > 0$ *as in definition 3.12.*

Definition 3.13 (Max-plus gauge function, [8]). *Given a max-plus C-set* $\mathcal{K} \subseteq \mathbb{R}^n = \operatorname{span}_\oplus(K)$*, its max-plus gauge function is defined as:*

$$\Psi_{\mathcal{K}}(x) = \min_{\mu \geq 0} \left\{ \mu \in \mathbb{R} \mid x \in \operatorname{eig}\left(\widetilde{K}_\mu, 0\right) \right\}, \quad \forall x \in \mathbb{R}^n. \tag{3-13}$$

Here, the minimum is attained since the max-plus eigenspace of a Kleene star matrix is finitely generated.

The following properties of a max-plus gauge function can be verified.

Definition 3.14 (Max-plus gauge function properties, [8]). *Given a max-plus C-set* $\mathscr{K} \subseteq \mathbb{R}^n$*, the associated max-plus gauge function* $\Psi_{\mathscr{K}} : \mathbb{R}^n \to \mathbb{R}$ *satisfies the following properties:*

a) Max-plus sub-linearity: $\Psi_{\mathcal{K}}(x \oplus y) \leq \Psi_{\mathscr{K}}(x) \oplus \psi(y)$*, for all* $x, y \in \mathbb{R}^n$*;*

b) Scale freeness: $\Psi_{\mathscr{K}}(\mu \otimes x) = \Psi_{\mathscr{K}}(x)$*, for all* $x \in \mathbb{R}^n$ *and* $\mu \in \mathbb{R}$*;*

c) Positive definiteness: $\Psi_{\mathscr{K}}(x) \geq 0, \Psi_{\mathscr{K}}(x) = 0 \Leftrightarrow x \in \mathscr{K}$*;*

d) Continuity.

The max-plus convexity of the sublevel-set of a max-plus gauge function, $\mathscr{N}(\Psi_{\mathcal{K}}, \delta)$ for a given $\delta > 0$, follows from [65, Theorem 18-9]. As the max-plus C-set $\mathscr{K}$ is closed, the function $\Psi_{\mathcal{K}}(\cdot)$ is max-plus convex.

Definition 3.15 (Closed-form expression of a max-plus gauge function, [8]). *For a given max-plus C-set* $\mathscr{K} \subseteq \mathbb{R}^n$ *and a vector* $x \in \mathbb{R}^n$*, max-plus gauge function can be described as:*

$$\Psi_{\mathscr{K}}(x) = x^* \otimes K \otimes x = (-x)^T \otimes K \otimes x, \tag{3-14}$$

where $K \in \mathbb{R}$ *is the generating Kleene star matrix. The max-plus gauge function provides the minimum max plus Hilbert projective distance between* x *and* $\mathscr{K}$*.*

The smallest max-plus Hilbert ball $\mathscr{B}_\delta$ enclosing the set $\operatorname{span}_\oplus(K)$, for a given Kleene star matrix $K \in \mathbb{R}_\epsilon^{n \times n}$, can be obtained by existing algorithms[16]. This provides a global ultimate upper boundary on the buffer level of the discrete event system [8, theorem 4.3.1].

Theorem 3.2 (Asymptotic Lipschitz stability, [8]). *Consider again the discrete-event system in* (3-12)*. Let* $\rho \in \mathbb{R}$ *be given and let the normalised state be denoted as* $x_\rho(k) = x(k) - (\rho \cdot k) \otimes \mathbb{1}_n$*. The following statements are equivalent:*

1. *The system is uniformly max-plus bounded-buffer stable.*

   In addition, given any monotone and additively homogeneous function $\Phi : \mathbb{R}^n \to \mathbb{R}$*, for every* $\mu > 0$ *there exists a scalar* $T = T(\mu) \in \mathbb{N}$ *such that if* $x(0) \in \mathscr{B}_\mu$*, then* $|\Phi(x_\rho(k)) - \Phi(x_\rho(k-1))|$ *is uniformly bounded for all* $k \geq T$*.*

2. *The system is uniformly asymptotically max-plus Lipschitz stable.*

3-2-3 Lyapunov stability

Definition 3.16 (Max-plus Lyapunov function, [8]). *Let a discrete event time system have a closed positive invariant set $\mathscr{K} \subseteq \mathbb{R}^n$.*

Let there be an open domain $\mathscr{D} \subseteq \mathbb{R}^n$ and let $\mathscr{K} \subset \mathscr{D}$ be a closed set. With the continuous function $V : \mathscr{D} \to \mathbb{R}_+$ is a common max-plus Lyapunov function with respect to $\mathscr{K}$ defined over the domain $\mathscr{D}$ for the system dynamics when the three following conditions hold:

1. *The function V is scale-free such that:*

$$V(\mu \otimes x) = V(x), \quad \forall x \in \mathscr{D} \text{ and } \mu \in \mathbb{R}$$

2. *There exist two functions of class $\mathscr{K}_\infty$, α_1 and α_2 such that:*

$$\alpha_1(||x||_{\mathscr{K},\mathbb{P}}) \leq V(x) \leq \alpha_2(||x||_{\mathscr{K},\mathbb{P}})$$

for all $x \in \mathscr{D}$;

3. *There exists a continuous, positive definite function α_3 such that:*

$$V(f(l,x)) - V(x) \leq -\alpha_3(||x||_{\mathscr{K},\mathbb{P}})$$

for all $x \in \mathscr{D}$ and for all $l \in n_L$

3-2-4 LaSalle-Like relaxations

Definition 3.17 (Weak max-plus Lyapunov function, [8]). *Let $\mathscr{K} \subseteq \mathbb{R}^n$ be any set. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a weak Lyapunov function in $\mathscr{K}$ for the* (3-12) *dynamics if the following conditions hold:*

1. *The function V is scale-free such that:*

$$V(\mu \otimes x) = V(x), \quad \forall x \in \mathscr{D} \text{ and } \mu \in \mathbb{R}$$

2. *$V(\cdot)$ is continuous on $\mathscr{K}$.*
3. *$V(f(l,x) - V(x)) \leq 0, \quad \forall x \in \mathscr{K}$ and $\forall l \in n_L$*

The function V is positive definite with respect to a set $\mathscr{K}_c \subseteq \mathbb{R}^n$ if:

1. *$V(x) = 0, \quad \forall x \in \mathscr{K}_c$*
2. *There exists an $\eta > 0$ such that $V(x) > 0$ wherever $x \in \mathscr{B}_\eta(\mathscr{K}_c)$ and $x \notin \mathscr{K}_c$*

3-3 Stability analysis of max-min-min-plus-scaling (MMPS) systems

An extension of the framework provided in the last section is worked out for MMPS systems in [4]. In this section, the definitions are presented such that notions of autonomous stability can be made for both time and quantity signals. Consider an MMPS discrete event with time signals x_t and quantity signals x_q:

$$\begin{aligned} x_t(k) &= f_t(x_t(k-1), x_q(k-1)), \\ x_q(k) &= f_c(x_t(k-1), x_q(k-1)). \end{aligned} \tag{3-15}$$

We assume that the discrete-event system (3-15) is structurally finite. In addition, the function is assumed to be continuous and partly additively homogenous in the states $x_t, x_q \in \mathbb{R}^n$. We will define an open ball produced using the projective norm (2.6) as $\overline{\mathscr{B}_\delta}$, and the open ball produced using the infinity norm (2.7) as $\mathscr{B}_\delta$.

Definition 3.18 (Uniformly bounded stable, [4])**.** *An MMPS system with both a time state x_t and a quantity state x_q is said to be uniformly bounded stable for every $\mu_t, \mu_q > 0$ there exists constants $\delta_t, \delta_q > 0$ and a constant $T = T(\mu_t, \mu_q, \delta_t, \delta_q) > 0$ such that if $x_t(0) \in \overline{\mathscr{B}}_{\mu_t}$ and $x_q(0) \in \mathscr{B}_{\mu_q}$, we have $x_t(k) \in \overline{\mathscr{B}}_{\delta_t}, x(k) \in \mathscr{B}_{\delta_q}$ for all $k \geq T$;*

Definition 3.19 (Uniform Lipschitz stability, [4])**.** *An MMPS system with both a time state x_t and a quantity state x_q is said to be uniformly Lipschitz stable if there exists constants $\bar{\mu}_t, \bar{\mu}_q > 0$ such that for every $\mu_t \in (0; \bar{\mu}_t)$ and $\mu_q \in (0; \bar{\mu}_q)$ there exists a $\delta_t = \delta_t(\mu_t, \mu_q) > 0$ and $\delta_q = \delta_q(\mu_t, \mu_q) > 0$ such that if $x_t(0) \in \overline{\mathscr{B}}_{\mu_t}$ and $x_q(0) \in \mathscr{B}_{\mu_q}$, we have $x_t(k) \in \overline{\mathscr{B}}_{\bar{\delta}_t}(x(k-1))$ and $x_q(k) \in \mathscr{B}_{\bar{\delta}_q}$ for all $k \geq 0$;*

Definition 3.20 (Uniform locally asymptotical Lipschitz stability, [4])**.** *An MMPS system is said to be uniformly locally asymptotically Lipschitz stable with a basin of attraction $\left(\overline{\mathscr{B}}_{\mu_t}(0), \mathscr{B}_{\mu_q}(0)\right)$ if the system is stable (as in 3.19) and there exist scalars $c \in \mathbb{N}$ and $T = T(\mu_t, \mu_q) > 0$ such that if $x_t(0) \in \overline{\mathscr{B}}_{\mu_t}(0)$ and $x_q(0) \in \mathscr{B}_{\mu_q}(0)$, we have $x_t(k) \in \overline{\mathscr{B}}_0(x(k-c))$ and $x_q(k) \in \mathscr{B}_0(x(k-c))$ for all $k \geq T(\mu_t, \mu_q)$.*

Definition 3.21. *The sub-level sets generated by a continuous function $\Psi : \mathbb{R}^{n_t} \times \mathbb{R}^{n_q} \to \mathbb{R}$ for $\delta \geq 0$ are denoted as:*

$$\mathcal{N}(\Psi, \nu) = \{(x_\mathrm{t} \in \mathbb{R}^{n_t}, x_\mathrm{t} \in \mathbb{R}^{n_q}) \mid \Psi(x_\mathrm{t}, x_\mathrm{q}) \leq \delta\}.$$

Definition 3.22. *A positive definite continuous function $\Psi : \mathbb{R}^{n_e} \times \mathbb{R}^{n_q} \to \mathbb{R}$ is a Lyapunov function outside $\overline{\mathscr{B}}_{\delta_t} \times \mathscr{B}_{\delta_q}, \delta_t, \delta_q > 0$, for the system (3-15) if:*

- *there exists $\nu > 0$ such that $\mathcal{N}(\Psi, \nu) \subseteq \mathcal{B}_\delta$ and for all $x \notin \mathcal{N}(\Psi, \nu)$ we have*

  $$\Delta\Psi = \Psi(f(x_\mathrm{t}, x_\mathrm{q})) - \Psi(x_\mathrm{t}, x_\mathrm{q}) \leq -\alpha\left(\|x_\mathrm{t}\|_\mathbb{P} + \|x_\mathrm{q}\|_\infty + \delta'\right)$$

  for some $\delta' \geq 0, \kappa$-function α and for all $k \in \mathbb{N}$.

- *The set $\mathcal{N}(\Psi, \nu)$ is positively invariant, so we have for all $(x_\mathrm{t}, x_\mathrm{q}) \in \mathcal{N}(\Psi, \nu)$*

  $$\Psi(f(x_\mathrm{t}, x_\mathrm{q})) \leq \nu$$

  for all $k \in \mathbb{N}$.

Theorem 3.3. *A discrete-event dynamical system* (3-15) *is uniformly bounded stable in* $\overline{\mathscr{B}}_{\delta_t} \times \mathscr{B}_{\delta_q}$ *if it admits a Lyapunov function outside* $\overline{\mathscr{B}}_{\delta_t}, \mathscr{B}_{\bar{\delta}_q}$*, for a finite* $\delta > 0$*, as in definition 3.18[4].*

3-4 Conclusion

This chapter explored various aspects of stability analysis for different types of systems. We explored stability analysis for time-driven systems, focusing on Lyapunov's method. We discussed conditions for stable, asymptotically stable, and globally asymptotically stable equilibrium points. We also studied the concept of exponential stability for continuous-time systems.

Moving on, we discussed stability analysis for discrete-time systems using Lyapunov functions. We highlighted the difference between Lyapunov functions for continuous and discrete-time systems and examined exponential stability in discrete time. Next, we discussed positive invariant sets, explaining how they are used to ensure that systems do not violate constraints. We examined an algorithm for computing positive invariant sets and discussed their significance in ensuring system stability under constraints.

We discussed definitions for assessing the stability of SMPL systems, including max-plus bounded-buffer stability, Lipschitz stability, incremental stability, ultimate boundedness and asymptotic Lipschitz stability. We studied the concept of max-plus Lyapunov functions and weak max-plus Lyapunov functions for discrete event systems. These functions can provide insight into the buffer stability of SMPL systems. Finally, we discussed the autonomous stability notions for MMPS systems with both time and quantity signals.

Overall, this chapter covered a wide range of stability analysis techniques for different types of systems, providing insights into understanding and ensuring the stability of complex dynamic systems.

Chapter 4

Max-plus Lyapunov Functions for MMP and MMPS Systems

This chapter presents a framework for proving Lyapunov stability for MMP and MMPS systems using the max-plus gauge function (3-13) as the max-plus Lyapunov function. This research focuses on the time signals of these systems. Stability is proven in a similar manner for both types of systems. The first section provides a method to determine the boundedness of the buffers of MMP and MMPS systems. The second section presents a method to determine the allowable buffer for the max-plus Lyapunov function. This is done by providing an algorithm for constructing an attractive positive invariant set for MMP and MMPS. The third section presents analytical methods to determine the maximal buffer for bipartite MMP systems. The fourth section gives extra conditions for a bounded growth rate for MMPS systems and presents analytical methods to determine the buffer for two-dimensional MMPS systems. Furthermore, it yields additional insights into the additive eigenvalues and additive eigenvectors of MMPS in relation to their properties of full correlation, time-invariance, and monotonicity. The chapter ends with a conclusion and technical proofs.

4-1 Fully correlated systems

The term fully correlated system is introduced in this thesis. It can be used to determine if the buffer between the states of MMP and MMPS systems will be bounded. This section defines and provides examples of such systems.

4-1-1 Fully correlated MMP system

Consider an MMP system that is rewritten in conjunctive normal form 2.15. Within this system, we introduce a set of variables ω_i selected from $\{\omega_1, \cdots, \omega_p\}$, which corresponds to the power set[1] of the set $\{x_1, \cdots, x_n\}$. We can rewrite the state update of $x_i(k+1)$ as:

$$x_i(k+1) = f_1(\omega_1) \oplus' f_2(\omega_2) \oplus' \cdots \oplus' f_q(\omega_q), \tag{4-1}$$

Here, the functions $f_i \in \{f_1, \cdots, f_q\}$ are max-plus functions.

Definition 4.1 (Fully correlated MMP system)**.** *An MMP system is classified as fully correlated if, for every state, the update as in* (4-1) *is composed of max-plus functions that are dependent on the same set of ω_i.*

Theorem 4.1. *A fully correlated MMP system will have a bounded absolute difference between all the states.*

Proof. see section 4-6.

Example 4.1. *Consider an MMP system which is of the form:*

$$x(k+1) = C \otimes' (B \otimes x(k)) \tag{4-2}$$

Now consider the system matrices:

$$B = \begin{bmatrix} 2 & 3 & \epsilon \\ 1 & 2 & \epsilon \\ \epsilon & \epsilon & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \top & 3 \\ \top & 2 & 4 \\ 2 & 9 & 2 \end{bmatrix} \tag{4-3}$$

To check that the system is fully correlated, we write out the state updates:

$$\begin{aligned} x_1(k+1) &= ((x_1(k) \otimes 3) \oplus (x_2(k) \otimes 4)) \otimes' (x_3(k) \otimes 9) \\ x_2(k+1) &= ((x_1(k) \otimes 3) \oplus (x_2(k) \otimes 4)) \otimes' (x_3(k) \otimes 10) \\ x_3(k+1) &= ((x_1(k) \otimes 4) \oplus (x_2(k) \otimes 5)) \otimes' ((x_1(k) \otimes 10) \oplus (x_2(k) \otimes 11)) \otimes' (x_3(k) \otimes 8) \end{aligned} \tag{4-4}$$

We define the set $\omega_1 = \{x_1, x_2\}$ and $\omega_2 = \{x_3\}$. All state updates are composed of max-plus functions dependent on ω_1 and ω_2. Therefore, we can conclude that the system is fully correlated.

Example 4.2. *Consider the following MMP system:*

$$\begin{aligned} x_1(k+1) &= ((x_1(k) \otimes 3) \oplus (x_2(k) \otimes 9)) \oplus' (x_2(k) \otimes 4) \\ x_2(k+1) &= ((x_1(k) \otimes 7) \oplus' (x_2(k) \otimes 4)) \oplus (x_1(k) \otimes 6) \end{aligned} \tag{4-5}$$

The MMP is rewritten in conjunctive normal form to check that the system is fully correlated. As the first state update is already in this form, this is only necessary for $x_2(k+1)$:

$$\begin{aligned} x_2(k+1) &= ((x_1(k) \otimes 7) \oplus (x_1(k) \otimes 6)) \oplus' ((x_1 \otimes 6) \oplus (x_2(k) \otimes 4)) \\ &= (x_1(k) \otimes 7) \oplus' ((x_1(k) \otimes 6) \oplus (x_2(k) \otimes 4)) \end{aligned} \tag{4-6}$$

We define the set $\omega_1 = \{x_1, x_2\}$, $\omega_2 = \{x_1\}$ and $\omega_3 = \{x_2\}$. The first state update is composed of max-plus functions dependent ω_1 and ω_3, and the second state update is composed of max-plus functions dependent ω_1 and ω_2. Therefore, we can conclude that the system is not fully correlated.

[1]Let A be a set; by the power set of A we mean the class of all the subsets of A[17].

4-1-2 Fully correlated MMPS system

Consider an MMPS system that has been transformed into the ABC normal form. Let the vector $A \cdot x$ be represented as $\begin{bmatrix}\alpha_1 & \cdots & \alpha_n\end{bmatrix}^T$. Within this system, we introduce a set of variables ω_i selected from $\{\omega_1, \cdots, \omega_p\}$, which corresponds to the power set of the set $\{\alpha_1, \cdots, \alpha_n\}$. We can rewrite the state update of $x_i(k+1)$ as:

$$x_i(k+1) = f_1(\omega_1) \oplus' f_2(\omega_2) \oplus' \cdots \oplus' f_q(\omega_q), \tag{4-7}$$

Here, the functions $f_i \in \{f_1, \cdots, f_q\}$ are max-plus functions.

Definition 4.2 (Fully correlated MMPS system). *An MMPS system is classified as fully correlated if, for every state, the update as in* (4-7) *is composed of max-plus functions that are dependent on the same set of* ω_i.

Theorem 4.2. *A fully correlated MMPS system will have a bounded absolute difference between all the states.*

Proof. See section 4-6.

Example 4.3. *Consider an MMPS system in ABC canonical form, with system matrices:*

$$A = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & \epsilon \\ \epsilon & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 3 \\ 2 & 4 \end{bmatrix} \tag{4-8}$$

We substitute $A \cdot x = \alpha$, with $\alpha = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}^T$ for the states. Substituting α and writing out the state updates gives:

$$\begin{aligned} x_1(k+1) &= (9 \otimes \alpha_1) \oplus' (8 \otimes \alpha_2) \\ x_2(k+1) &= (5 \otimes \alpha_1) \oplus' (9 \otimes \alpha_2) \end{aligned} \tag{4-9}$$

We define the set $\omega_1 = \{\alpha_1\}$ and $\omega_2 = \{\alpha_2\}$. Both state updates are composed of max-plus functions dependent on ω_1 and ω_2. Therefore, we can conclude that the system is fully correlated.

Example 4.4. *Now consider a nearly identical system, except for the values of $B_{2,1}$ and $C_{2,1}$:*

$$A = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & \epsilon \\ 7 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 3 \\ \top & 4 \end{bmatrix} \tag{4-10}$$

Again let $A \times x$ be α:

$$\begin{aligned} x_1(k+1) =& (\alpha_1 \otimes 9) \otimes' ((\alpha_1 \otimes 10) \oplus (\alpha_2 \oplus 8)) \\ x_2(k+1) =& ((\alpha_1 \otimes 11) \oplus (\alpha_2 \otimes 9)) \end{aligned} \tag{4-11}$$

We define the set $\omega_1 = \{\alpha_1, \alpha_2\}$ and $\omega_2 = \{\alpha_2\}$. The first state update is composed of max-plus functions dependent on ω_1 and ω_2; the second state update is composed of one max-plus function dependent on ω_1. Therefore, we can conclude that the system is not fully correlated.

4-2 Redefining max-plus Lyapunov functions

Proof of the stability for switching-max-plus-linear (SMPL) systems uses the invariant max-plus C-set $\mathscr{K}$. In [8], a general setup for constructing the smallest non-empty positively invariant set for an open-loop SMPL system is provided. Here, the radius of the set is measured as the (negative) second-largest max-plus eigenvalue of the associated Kleene star matrix[18].

In this thesis, the same max-plus gauge function is used. The set will represent the allowable buffer between the states. This buffer will ensure that the max-plus Lyapunov function and its derivative are semi-positive and semi-negative definite, respectively. This method is easily interpreted when the max-plus Lyapunov function is written out. Consider a matrix K that has zeros on its diagonal and finite values $\delta_{i,j}$ with $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, n\}$:

$$K = \begin{bmatrix} 0 & -\delta_{1,2} & -\delta_{1,3} & \cdots & -\delta_{1,n} \\ -\delta_{1,2} & 0 & -\delta_{2,3} & \cdots & -\delta_{2,n} \\ -\delta_{1,3} & -\delta_{2,3} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\delta_{1,n} & -\delta_{2,n} & \dots & \dots & 0 \end{bmatrix} \tag{4-12}$$

$$\begin{aligned} V(x) =& (-x)^T \otimes K \otimes x \\ =& \max(\max(-x_1, -x_2 - \delta_{1,2}, -x_3 - \delta_{1,3}) + x_1, \max(-x_1 - \delta_{1,2}, -x_2, -x_3 - \delta_{1,3}) + x_2, \cdots) \\ =& \max(0, \pm(x_1 - x_2) - \delta_{1,2}, \pm(x_1 - x_3) - \delta_{1,3}, \pm(x_2 - x_3) - \delta_{2,3}, \cdots) \end{aligned} \tag{4-13}$$

The lower bound of the max-plus Lyapunov function will be equal to zero. The function will give the maximal difference between the states minus the corresponding value of $\delta_{i,j}$. The value of $\delta_{i,j}$ will be the allowable buffer between the states. This is convenient as when all the absolute differences are smaller than the buffer, the value of the max-plus Lyapunov function will be zero. This will be equivalent to that the states are in set $\mathscr{K} = \text{span}_{\oplus}(K)$.

MMP or MMPS systems can have oscillating behaviour. This may lead to a divergence between the states. The system is considered stable if the divergence is within the bounds of the buffer level $\delta_{i,j}$. This method has a significant advantage over traditional Lyapunov functions such as the two-norm. By modifying the original system to compare all states to a single reference state, we could investigate the Lyapunov stability of the buffers using the two-norm. However, if we used the conventional two-norm, the oscillation would result in a positive Lyapunov function.

Using a visual example helps to show why fully correlated systems can be used to determine the attractivity and positive invariance of the set. We will first show an example of an SMPL system [8, example 4.3.3.].

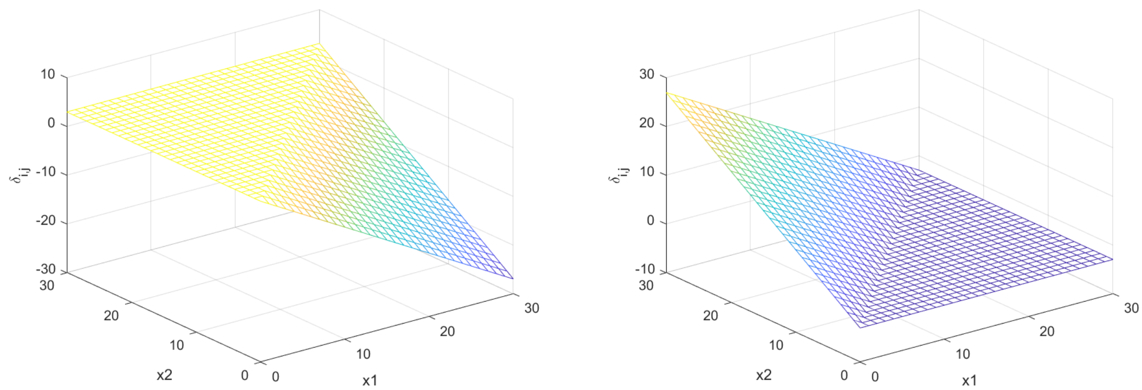
Example 4.5. *Consider a bimodal open-loop switching max-plus-linear (MPL) system defined by the following matrices:*

$$A^{(1)} = \begin{bmatrix} 4 & \epsilon \\ 1 & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 3 & 3 \\ \epsilon & 6 \end{bmatrix}. \tag{4-14}$$

In example [8, example 4.3.3.], it is given that the buffer will be $\delta_{1,2} = 3$. The plots of figure 4-1 display the absolute difference between the states after a single state update along the z-axis. The x and y-axes show the initial system conditions for the system. Matrix $A^{(1)}$ corresponds to figure 4-1a and matrix $A^{(2)}$ corresponds to figure 4-1b. In both plots, there is a plane that creates the upper or lower bound where the difference is constant. Using the matrices, we can easily find the constant buffer and the conditions necessary for this constant buffer. For system matrix $A^{(1)}$ the buffer will stay constant if $x_1(k) \geq x_2(k)$:

$$x_1(k+1) - x_2(k+1) = 4 + x_1(k) - \max(1 + x_1(k), 1 + x_2(k)) = 3 \tag{4-15}$$

For system matrix $A^{(2)}$, the buffer is constant if $x_2 \geq x_1$ with a buffer of minus three. If both subsystems always converge to the set where the absolute difference is smaller than three, then we consider the system stable. Using [8, theorem 4.4.1], it is possible to prove that the states will converge to the set $|x_1(k) - x_2(k)| \leq 3$. This set will be positively invariant with respect to both system dynamics. Thus, we can conclude that the states will converge to the set, and the buffer of the SMPL system will stay smaller or equal to three.



(a) MPL system with system matrix $A^{(1)}$ **(b)** MPL system with system matrix $A^{(2)}$

Figure 4-1: Difference between states after one state update $(x_1(k+1) - x_2(k+1)$ of a SMPL system

Figure 4-2 shows the difference between the states for a fully correlated MMP system. The plot shows that the difference between the states will be upper and lower-bounded. This means that after one state update, the difference between the states will always be equal or smaller than the maximal constant buffer. We can use the buffer values to create a set $\mathscr{K} = \text{span}_{\oplus}(K)$. The set created by the maximal constant buffer will, therefore, be an attractive set and positively invariant with respect to the system dynamics. Using an updated domain, we can use this property to calculate the updated maximal buffer. The domain will be within the previously calculated maximal buffer. We will iterate until we find the smallest allowable buffer for MMP and MMPS systems. The following section will present the algorithm to calculate the buffers for all states.

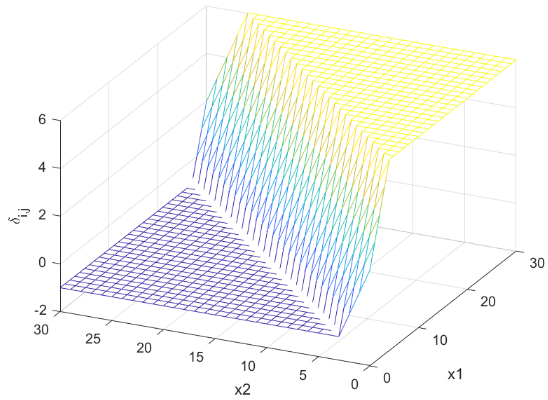


Figure 4-2: Difference between states after one state update $(x_1(k+1) - x_2(k+1))$ of a fully correlated MMP system

4-2-1 Construction

Consider a discrete event fully correlated MMP or MMPS system. Using optimisation, the absolute maximal buffer is easily calculated. This can be done for all combinations of states separately. The optimisation will be over a cost function that is the absolute value of the difference between each pair of states. This optimisation is performed using a dummy variable. The problem can be recast in a mixed-integer linear program (MILP) problem. Using a binary variable b and introducing a sufficient big term N, the initial optimisation will be equal to [2]:

$$\begin{aligned} &\max_{x(k)} \quad \delta_{i,j} \\ \text{s.t.} \quad &\delta_{i,j} \geq f_i(x(k)) - f_j(x(k)) \\ &\delta_{i,j} \geq -f_i(x(k)) + f_j(x(k)) \\ &\delta_{i,j} \leq f_i(x(k)) - f_j(x(k)) + Nb \\ &\delta_{i,j} \leq -(f_i(x(k)) - f_j(x(k))) + N(b-1) \end{aligned} \tag{4-16}$$

In the first iteration, the algorithm will find the maximal buffer for the whole state space. After calculating this maximal difference for all combinations of states, the constraints are updated. Because the difference between the states will be equal to or smaller than the maximal buffer, the following pair of constraints will be included for all combinations of states

$$\begin{aligned} x_j &\leq x_i + \delta_{i,j} \\ x_j &\geq x_i - \delta_{i,j} \end{aligned} \tag{4-17}$$

[2]Note that the max-plus functions need to be rewritten to make it a MILP problem.

The algorithm 2 will repeat this procedure with updated constraints until all $\delta_{i,j}$ equals the value of the previous iteration. The code for the algorithm used in all Matlab simulations can be found in appendix B-0-1.

Algorithm 2 Calculating matrix K

$z_{i,j} = \infty, \quad \forall i,j$ ▷ Initial domain for optimisation
repeat $\forall i,j$
  Converged ← true ▷ Assume convergence unless condition is not met

$$
\begin{aligned}
\max_{x(k)} \quad & \delta_{i,j} \\
\text{s.t.} \quad & \delta_{i,j} \geq f_i(x(k)) - f_j(x(k)) \\
& \delta_{i,j} \geq -f_i(x(k)) + f_j(x(k)) \\
& \delta_{i,j} \leq f_i(x(k)) - f_j(x(k)) + Nb \\
& \delta_{i,j} \leq -(f_i(x(k)) - f_j(x(k))) + N(b-1) \\
& x_2 \leq x_1 + z_{1,2} \\
& x_2 \geq x_1 - z_{1,2} \\
& \quad \vdots \\
& x_n \leq x_{n-1} + z_{n-1,n} \\
& x_n \geq x_{n-1} - z_{n-1,n}
\end{aligned}
$$

  $\delta_{i,j} = \delta_{j,i}$ ▷ The absolute difference is equal for the pair i,j
  if $z_{i,j} \neq \delta_{i,j}$ **then**
    Converged ← false ▷ Update convergence status
  end if
  $z_{i,j} = \delta_{i,j}$
until Converged

4-3 MMP systems

This section presents analytical methods to calculate the maximal buffer for bipartite systems. We propose a method for bipartite systems that have specific initial conditions, including eigenvectors and those dependent on the C matrix. This section focuses on the buffer stability of bipartite systems. In the next section about MMPS systems, we draw conclusions about the additive eigenvalues of MMPS systems. Due to the similarities, we can also apply the method to bipartite systems with partially arbitrary initial conditions. This will be elaborated on in the section about MMPS systems.

4-3-1 Bipartite system with initial conditions equal to eigenvector

Consider a bipartite system from (2-18). If a valid additive eigenvector exists and the initial conditions are equal to an additive eigenvector, there will be no transient period [19]. This means that all states will have a linear drift with no oscillations.

Corollary 4.2.1. *The buffer between the states of a bipartite system with initial conditions equal to an additive eigenvector will equal the absolute difference between the additive eigenvector entries.*

Proof of corollary 4.2.1. All the states will have a constant growth rate equal to the additive eigenvalue. As a result, the difference or buffer between the states will be equal to the difference between the initial values of each state. The initial values of the states are equal to the eigenvector. This completes the proof. □

Our interest is in the buffer stability of states x. The states of y are only used for modelling. The maximal difference between the states of x equals the eigenvector if there is no transient behaviour. Consider the augmented state z with the corresponding state update:

$$z(k) = \begin{pmatrix} x(k) \\ y(k) \end{pmatrix}, \quad \mathcal{M}\left(\begin{pmatrix} x(k) \\ y(k) \end{pmatrix}\right) = \begin{pmatrix} B \otimes y(k) \\ C \otimes' x(k) \end{pmatrix} \tag{4-18}$$

Algorithm 3 Power algorithm for bipartite systems[20]

Take arbitrary initial state vector $z(0)$
Iterate $z(k+1) = \mathscr{M}(z(k))$ until there are integers p, q with $p > q \geq 0$ and a real number c such that $z(p) = z(q) \otimes c$
Define as eigenvalue: $\lambda = \frac{c}{p-q}$
Define as eigenvector: $v = \bigoplus_{j=1}^{p-q} \left(\lambda^{\otimes(p-q-j)} \otimes z(q+j-1)\right)$
if $\mathscr{M}(v) = \lambda \otimes v$ **then**
    v is correct eigenvector of the system for eigenvalue λ
else if $\mathscr{M}(v) \neq \lambda \otimes v$ **then**
    Take $z(0) = v$ as a new initial state and iterate $z(k+1) = \mathscr{M}(z(k))$, until for some r $z(r+1) = \lambda \otimes z(r)$, then $z(r)$ is an eigenvector of the system for eigenvalue λ
end if

Consider the additive eigenvector v_g:

$$v_g = \begin{bmatrix} v_{b,1} & v_{b,2} & \cdots & v_{b,n} & v_{c,1} & v_{c,2} & \cdots & v_{c,n} \end{bmatrix}^T \tag{4-19}$$

The buffer will equal the absolute difference between each eigenvalue corresponding to the state. Thus, matrix K can be defined as:

$$\begin{aligned} K^{(\delta)} &= \begin{bmatrix} 0 & -\delta_{1,2} & \cdots & -\delta_{1,n} \\ -\delta_{1,2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\delta_{(n-1),(n-1)} \\ -\delta_{1,n} & \cdots & -\delta_{(n-1),(n-1)} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -|v_{b,1} - v_{b,2}| & \cdots & -|v_{b,1} - v_{b,n}| \\ -|v_{b,1} - v_{b,2}| & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -|v_{b,n-1} - v_{b,n}| \\ -|v_{b,1} - v_{b,n}| & \cdots & -|v_{b,n-1} - v_{b,n}| & 0 \end{bmatrix} \end{aligned} \tag{4-20}$$

Note that it is possible to include the additive eigenvector of the states y in this method. However, it was not utilized as bipartite systems where a setup for determining buffer levels for MMPS systems. The ABC canonical form shares a similar form with bipartite systems, except for the division of states, which is why the states y were not included.

4-3-2 Bipartite systems with partly arbitrary initial conditions

Consider the bipartite system of (2-18). Let the initial conditions of x_0 be arbitrary finite values and let $y_0 = C \otimes x_0$; we will denote this as partly arbitrary initial conditions.

Theorem 4.3. *A bipartite system with partly arbitrary initial conditions and only finite entries for matrix B and a regular matrix C will have a maximal buffer between states i, j equal to:*

$$\delta_{i,j} = \max(|b_{i,1} - b_{j,1}|, |b_{i,2} - b_{j,2}|, \cdots, |b_{i,m} - b_{j,m}|) \tag{4-21}$$

Proof. see section 4-6.

Example 4.6. *Consider a bipartite system with system matrix B with only finite values and a regular matrix C:*

$$B = \begin{bmatrix} 6 & 7 & 0 \\ 3 & 5 & 2 \\ 7 & 9 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 1 & 8 \\ 1 & 6 & 4 \\ 8 & 5 & 5 \end{bmatrix} \tag{4-22}$$

According to theorem 4.3, the maximal buffer between each state will be equal to:

$$\delta_{i,j} = \max(|b_{i,1} - b_{j,1}|, |b_{i,2} - b_{j,2}|, \cdots, |b_{i,m} - b_{j,m}|) \tag{4-23}$$

Filling in the entries of the B matrix gives:

$$\delta_{1,2} = \max(3, 2, 2) = 3, \quad \delta_{1,3} = \max(1, 2, 3) = 3, \quad \delta_{2,3} = \max(4, 4, 1) = 4 \tag{4-24}$$

Now the K matrix will be equal to:

$$K = \begin{bmatrix} 0 & -3 & -3 \\ -3 & 0 & -4 \\ -3 & -4 & 0 \end{bmatrix} \tag{4-25}$$

When we use the algorithm 2, the result for matrix K will be equal. The system is simulated using partly arbitrary initial conditions in figure 4-3. It converges to the set $\mathscr{K}$ after one event and converges to the edge of the set where it remains.

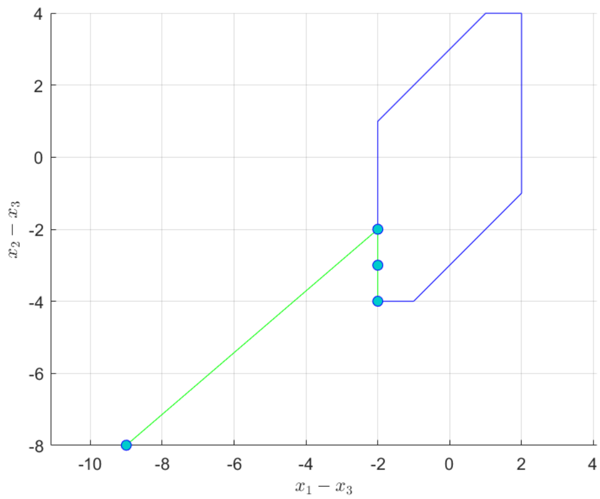


Figure 4-3: Set $\mathscr{K}$ using matrix K for the analytical solution (in blue –) plotted on the hyperplane$\{x \in \mathbb{R}^3 | x_3 = 0\}$. The system dynamics of the bipartite system (green line with sky blue markers –).

Example 4.7. *Consider a bipartite system with system matrix B with only finite values and a regular matrix C:*

$$B = \begin{bmatrix} 56 & 16 & 71 \\ 6 & 95 & 97 \\ 6 & 81 & 100 \end{bmatrix}, \quad C = \begin{bmatrix} 99 & 53 & 90 \\ 15 & 7 & 84 \\ 96 & 31 & 0 \end{bmatrix} \tag{4-26}$$

According to theorem 4.3, the maximal buffer between each state will be equal to:

$$\delta_{1,2} = \max(50, 79, 26) = 79, \quad \delta_{1,3} = \max(50, 65, 29) = 65, \quad \delta_{2,3} = \max(0, 14, 3) = 14 \tag{4-27}$$

We will compare the analytical solution with the matrix K calculated by the algorithm 2:

$$K_{an} = \begin{bmatrix} 0 & -79 & -65 \\ -79 & 0 & -14 \\ -65 & -14 & 0 \end{bmatrix}, \quad K_{alg} = \begin{bmatrix} 0 & -18 & -23 \\ -18 & 0 & -14 \\ -23 & -14 & 0 \end{bmatrix} \tag{4-28}$$

In figure 4-4, the system is simulated using arbitrary initial conditions. It is evident that the final set generated by matrix K_{alg} is significantly smaller. The difference between the states converges to the set $\mathscr{K}_{alg}$ in two events, and the system dynamics converge to the border of the set and remain constant.

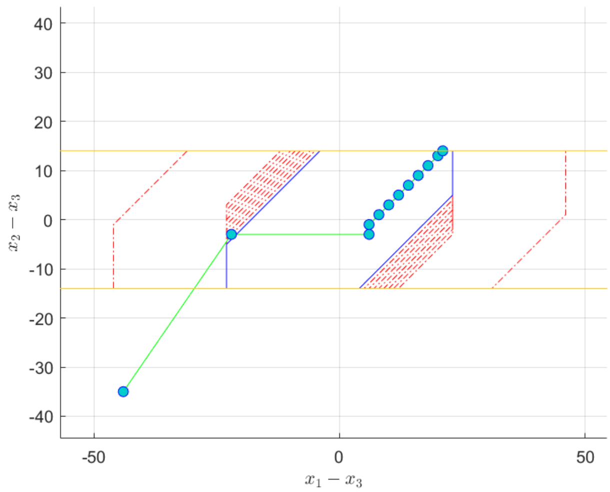


Figure 4-4: Set $\mathscr{K}_{an}$ using matrix K_{an} for the analytical solution (in mustard yellow –), the final set $\mathscr{K}_{alg}$ using matrix K_{alg} (in blue –), the intermediate results of set $\mathscr{K}_{alg}$ (red dotted line – –) all plotted on the hyperplane$\{x \in \mathbb{R}^3 | x_3 = 0\}$. The system dynamics of the bipartite system (green line with sky blue markers –).

4-4 MMPS systems

4-4-1 MMPS functions in conjunctive ABC canonical form

Consider an MMPS system that is the conjunctive ABC canonical form (2-25). Let the system with state $x \in \mathbb{R}^n$ and system matrices $A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}_{\epsilon}^{m \times n}$ and $C \in \mathbb{R}_{\top}^{n \times m}$. We define the vector $A \cdot x = \alpha$. We denote the linear combination of each row as α_i with $i \in \{1, \cdots, n\}$. For every state, the update will be a linear combination α_i together with a constant addition. Due to the permutation, the whole system will have many possible combinations of α_i. We denote a unique vector with a combination of α_i as $\alpha^{(i)}$ and the specific addition from the B and C matrix as $\gamma^{(i)}$. The combination of both will be considered the mode of the MMPS system. If an MMPS system always converges to the same mode, we denote this as the dominant mode. Assuming that for a state i the state update will be:

$$x_i(k+1) = \min_j(\max_j(b_{ij} + \alpha_j) + c_{ij}) = \alpha_{j_0} + b_{ij_0} + c_{ij_0}, \tag{4-29}$$

we can write the state update of all states of mode (i) as:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} \alpha_{j0} \\ \alpha_{j1} \\ \vdots \\ \alpha_{jn} \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \\ x(k+1) &= \alpha^{(i)} + \gamma^{(i)} = \Phi^{(i)} \cdot x + \gamma^{(i)} \end{aligned} \tag{4-30}$$

In conventional algebra, this would be an affine system. We denote the combinations of α_i for each mode as $\{\alpha^{(1)}, \cdots, \alpha^{(p)}\}$ with corresponding system matrix $\{\Phi^{(1)}, \cdots, \Phi^{(p)}\}$, and the corresponding addition $\{\gamma^{(1)}, \cdots, \gamma^{(p)}\}$.

Lemma 4.4. *If a time-invariant monotonic MMPS system stays in the same mode with system matrix $\Phi^{(i)}$ and vector $\gamma^{(i)}$, the growth rate of each state will converge to a steady state that will be equal to the steady state value of the Markov chain $(\Phi^{(i)})^{(n-1)} \cdot \gamma^{(i)}$. If the growth rates are equal for each state, it will be an additive eigenvalue λ_g of the system.*

Proof. See section 4-6.

Lemma 4.5. *If a time-invariant non-monotonic MMPS system stays in a mode which is dependent on the same α_i with system matrix $\Phi^{(i)}$ and vector $\gamma^{(i)}$, the additive eigenvalue will be equal to $\lambda_g = \Phi^{(i)}_{1.} \cdot \gamma^{(i)}$.*

Proof. See section 4-6. Note that any row of the matrix can be used to calculate the additive eigenvalue, as all rows are equal.

Corollary 4.5.1. *The additive eigenvectors v_g of an MMPS system can be constructed using the buffer of the corresponding dominant mode $\alpha^{(i)}$.*

To construct an additive eigenvector, use initial states with differences equal to buffers of the dominant mode.

Theorem 4.6. *A fully correlated MMPS system will have a bounded growth rate if it is time-invariant.*

Proof. See section 4-6.

Corollary 4.6.1. *Consider an MMPS system in ABC canonical form. The states will have a bounded buffer and growth rate if the system is fully correlated and a time-invariant MMPS system.*

These results provide valuable insights into the behaviour of MMPS systems. One crucial observation is that in a fully correlated time-invariant monotonic MMPS system, the buffer may reach a steady state value that depends on different α_i. This happens because the growth rate of each mode will eventually reach a steady state, given the Markov properties. The buffer will remain constant if the growth rates converge to an equal rate for every state. However, when growth rates differ, the states can diverge until the system reaches another mode.

For non-monotonic time-invariant MMPS systems, fully correlatedness will always ensure that the system will have a bounded growth rate. Many modes will experience exponential growth due to absolute eigenvalues bigger than one. The unstable nature of these modes will let the states diverge until the system reaches another mode. If the system reaches a mode dependent on the same α_i, it will experience linear growth (lemma 4.5). The maximal buffer will be from a mode dependent on the same α_i, this can be derived from the proof of theorem 4.1. Because the maximal divergence between the states will equal the maximal buffer, the system will always experience linear growth at the maximal buffer. Another possibility is that the system keeps switching between modes, resulting in small oscillations and, again, bounded growth. Additionally, using the same logic, non-monotonic time-invariant MMPS systems that are not fully correlated can become unstable because this bound will not exist.

Another finding is that fully correlated time-invariant monotonic MMPS systems will always have one dominant mode and only one additive eigenvalue. This property still needs to be proved, but from many (10^6) simulations, it is clear that the growth rate will always converge to the same value. Additionally, simulation made it clear that fully correlated time-invariant non-monotonic MMPS systems can have multiple additive eigenvalues and, thus, not always have a dominant mode.

Conjecture 4.1. *If a fully correlated max-min-min-plus-scaling (MMPS) system is time-invariant and monotonic, it will have one dominant mode and, thus, one additive eigenvalue.*

Lemma 4.7. *If a fully correlated MMPS system is time-invariant and non-monotonic, it can have multiple additive eigenvalues.*

Proof. See example 4.9.

The last thing to note is that a bipartite system can be rewritten as an MMPS system that is in ABC canonical form. We first rewrite the system such that it is only dependent on $x(k)$ and choose the initial conditions to be partly arbitrary. The scaling matrix A will equal an identity matrix in conventional algebra. Therefore the A matrix will be time-invariant and monotonic. If the system is fully correlated and time-invariant, we can calculate the additive eigenvalue of each mode. According to conjecture 4.1, we can conclude that a fully correlated bipartite system with partly arbitrary initial conditions will also have one eigenvalue as it is time-invariant and monotonic.

Example 4.8. *Consider an MMPS system in the ABC canonical form 2-25, with system matrices:*

$$A = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 8 \\ 5 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 10 & 5 \\ 3 & 2 \end{bmatrix} \tag{4-31}$$

Using lemma 4.4, we can derive that if the buffer goes to a steady state, the additive eigenvalue rate will be equal to $\lambda_g = \Phi^{(n-1)} \cdot \gamma$. The number of possible modes is sixteen. Now, using Matlab, we calculate each mode's additive eigenvalues. The mode that gives the eigenvalue is equal to:

$$\lambda_g = \Phi^n \cdot \gamma = \begin{bmatrix} 0.8 & 0.2 \\ 0.8 & 0.2 \end{bmatrix}^n \cdot \begin{bmatrix} 5+5 \\ 2+5 \end{bmatrix} = \begin{bmatrix} 9.4 \\ 9.4 \end{bmatrix} \tag{4-32}$$

The growth rate in figure 4-5 converges to $\lambda_g = 9.4$.

We have computed all potential additive eigenvalues and can match the mode with the real additive eigenvalue. Before simulation, we know the highest and lowest additive eigenvalue but cannot ascertain which mode it will converge to. It would, therefore, be beneficial to find a way to determine the dominant mode beforehand to calculate the additive eigenvalue.

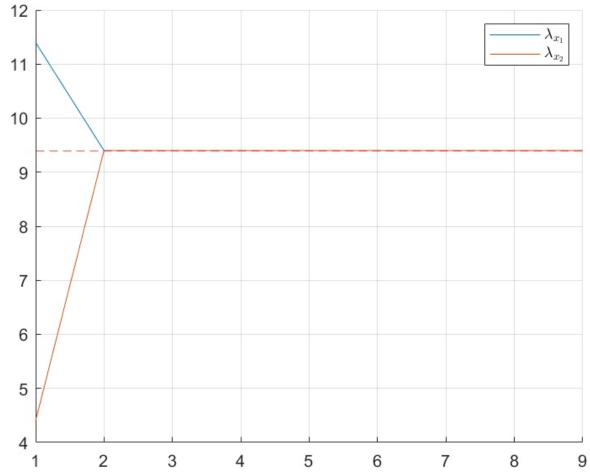


Figure 4-5: Additive eigenvalue of the MMPS system

Example 4.9. *Consider a fully correlated, non-monotonic time-invariant MMPS system with the following system matrices:*

$$A = \begin{bmatrix} -93.7939 & 29.8611 & 64.9328 \\ -35.5214 & 80.7297 & -44.2082 \\ -13.3511 & 20.2736 & -5.9225 \end{bmatrix}, \quad B = \begin{bmatrix} 89 & 56 & 1 \\ 29 & 85 & 36 \\ 87 & 47 & 23 \end{bmatrix}, \quad C = \begin{bmatrix} 37 & 60 & 5 \\ 1 & 54 & 45 \\ 41 & 99 & 4 \end{bmatrix} \tag{4-33}$$

We simulate the system with different arbitrary initial conditions. In figure 4-6, the first state is simulated fifty times. It is clear from figure 4-6 that this system has three eigenvalues. The corresponding eigenvalues are equal to: $\lambda_{g,1} = 499.9$, $\lambda_{g,2} = 39.7$ *and* $\lambda_{g,3} = -492.2$. *Something to note is that the growth rate will be equal for all the states, but only the first state is plotted.*

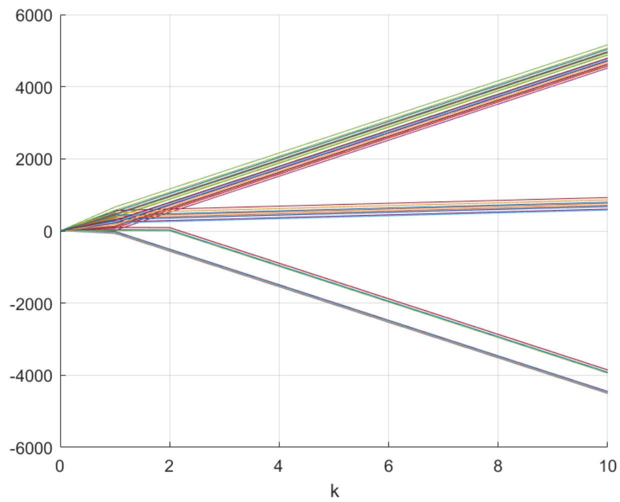


Figure 4-6: Simulation with different initial conditions, state x_1

4-4-2 Analytical solution

This section will present a method to analytically find the maximal buffer for a fully correlated MMPS system. It includes examples and discusses the limitations of the analytical solution.

Corollary 4.7.1. *If an MMPS system is in ABC normal form* (2-25) *and is fully correlated, it is possible to find the maximal buffer analytically.*

By writing out the equation from the ABC canonical form, the maximal difference between each state can be found dependent on the elements of the system matrices. Consider the system matrices for a two-dimensional system:

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \tag{4-34}$$

The state updates will be equal to:

$$\begin{aligned} x_1(k) = \min(&\max(b_1 + a_1x_1 + a_2x_2, b_2 + a_3x_1 + a_4x_2) + c_1, \\ &\max(b_3 + a_1x_1 + a_2x_2, b_4 + a_3x_1 + a_4x_2) + c_2) \\ x_2(k) = \min(&\max(b_1 + a_1x_1 + a_2x_2, b_2 + a_3x_1 + a_4x_2) + c_3, \\ &\max(b_3 + a_1x_1 + a_2x_2, b_4 + a_3x_1 + a_4x_2) + c_4) \end{aligned} \tag{4-35}$$

The bounded buffer between the states results from the states cancelling each other out. The linear combination of the states and the elements of the A matrix will cancel out at some point, and this will be between the upper and lower bounds of the buffer (see proof of theorem 4.2). As a result, we only have to look at which combination of the elements of the B and C matrices will be smaller. Using a set of fourteen inequalities, we can derive the maximal buffer for the whole state space. In appendix A, all the combinations of inequalities are presented with the corresponding minimal and maximal buffer. After one state update, the buffer will either have a lower and upper bound or a constant difference.

Example 4.10. *Consider the system matrices:*

$$A = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & \epsilon \\ \epsilon & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 3 \\ 2 & 4 \end{bmatrix} \tag{4-36}$$

Using the inequalities of appendix A, we can find that:

$$\begin{aligned} b_1 + c_1 &\geq b_3 + c_2 \\ b_4 + c_2 &\geq b_2 + c_1 \\ b_1 + c_3 &\geq b_3 + c_4 \\ b_4 + c_4 &\geq b_2 + c_3 \end{aligned}$$

Thus the maximal buffer will be $\delta_{1,2} = 4$ *and the minimal buffer will be* $\delta_{1,2} = 1$.

When dealing with a two-dimensional system containing square B and C matrices, there are already fourteen possible combinations of inequalities to determine the maximum buffer. However, when attempting to identify the difference between two states of a three-dimensional system with square B and C matrices, the number of combinations of inequalities increases significantly to a total of 117. This means that to find the matrix K, 351 inequalities will be considered. Finding the maximal buffer using analytical methods becomes exceedingly difficult for higher-order systems. Fortunately, optimisation provides a viable solution to this problem and can be used to overcome the challenges posed by higher-order systems.

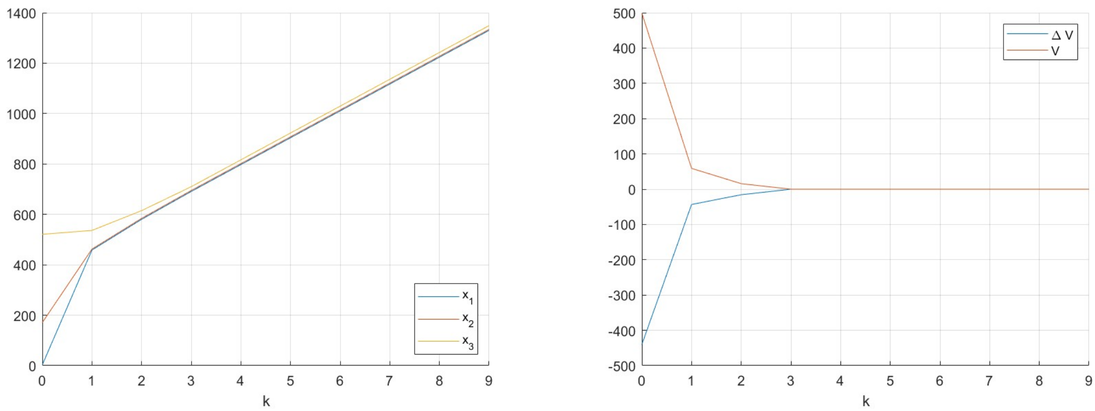
Example 4.11. *Consider a fully correlated, monotonic and time-invariant MMPS system in ABC canonical form with system matrices A, B and C:*

$$A = \begin{bmatrix} 0.341 & 0.399 & 0.260 \\ 0.567 & 0.298 & 0.135 \\ 0.087 & 0.090 & 0.823 \end{bmatrix}, \quad B = \begin{bmatrix} 94 & 68 & 8 \\ 3 & 56 & 12 \\ 88 & 52 & 63 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 74 & 58 \\ 10 & 81 & 78 \\ 96 & 80 & 31 \end{bmatrix} \tag{4-37}$$

Using the method presented in algorithm 2, we can derive that the matrix K is equal to:

$$K = \begin{bmatrix} 0 & -4 & -19 \\ -4 & 0 & -15 \\ -19 & -15 & 0 \end{bmatrix} \tag{4-38}$$

Using arbitrary initial conditions, the MMPS system is simulated, and figure 4-8 shows the states and the max-plus Lyapunov function. It can be observed that after three events, the states get a linear drift with no oscillations. The difference between the states is bounded and equal to the corresponding values of matrix K. As a result, the max-plus Lyapunov function is positive semi-definite, and the derivative is negative semi-definite.



(a) States **(b)** Lyapunov function V and ΔV

Figure 4-7: Simulation of MMPS system with initial conditions $x_0 = [3, 171, 521]^T$

Using the method from lemma 4.4, we can find all possible additive eigenvalues. Using Matlab, the additive eigenvalue corresponding to the actual additive eigenvalue is $\lambda_g = 106.539$.

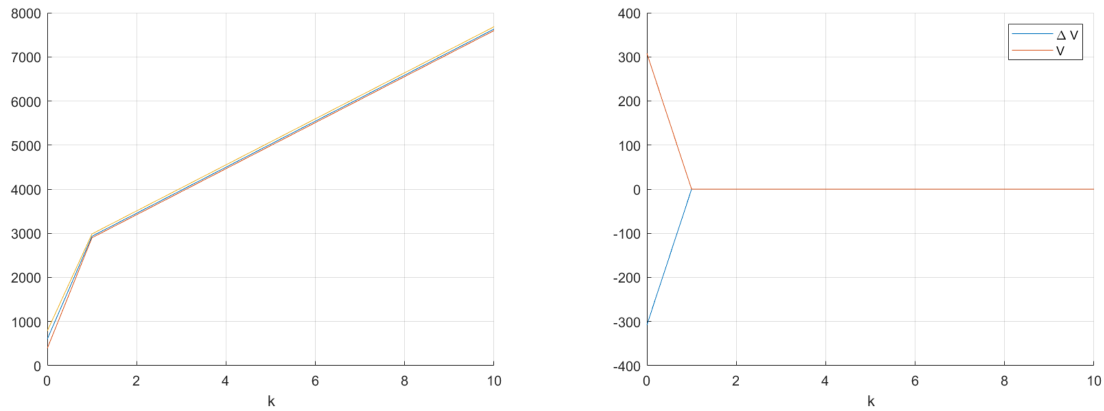
Example 4.12. *Consider a fully correlated, non-monotonic and time-invariant MMPS system in ABC canonical form with system matrices A, B and C:*

$$A = \begin{bmatrix} 3.5039 & -6.9609 & 4.4570 \\ 3.3739 & 6.4015 & -8.7754 \\ -1.4085 & -5.0195 & 7.4280 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 5 \\ 3 & \epsilon & \epsilon \\ 4 & 7 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 87 & 46 & \top \\ 83 & 12 & 84 \\ \top & 99 & 29 \end{bmatrix} \tag{4-39}$$

Using the method presented in algorithm 2, we can derive that the matrix K is equal to:

$$K = \begin{bmatrix} 0 & -34 & -53 \\ -34 & 0 & -87 \\ -53 & -87 & 0 \end{bmatrix} \tag{4-40}$$

The MMPS system is simulated using arbitrary initial conditions. Figure 4-8 shows the states and the max-plus Lyapunov function. It can be observed that after one event, the states get a linear drift with no oscillations. The difference between the states is bounded and equal to the corresponding values of matrix K. As a result, the max-plus Lyapunov function is positive semi-definite, and the derivative is negative semi-definite.



(a) States **(b)** Lyapunov function V and ΔV

Figure 4-8: Simulation of MMPS system with initial conditions $x_0 = [601, 392, 787]^T$

Using the method from lemma 4.4, we can find all possible additive eigenvalues. Using Matlab, the additive eigenvalue corresponding to the actual additive eigenvalue is $\lambda_g = 521.894$. *Something to note is that this system has two eigenvalues to which it can converge.*

Example 4.13. *Consider a fully correlated, monotonic and time-invariant MMPS system in ABC canonical form with system matrices A, B and C:*

$$A = \begin{bmatrix} 0.3342 & 0.0248 & 0.6410 \\ 0.1147 & 0.0482 & 0.8371 \\ 0.1544 & 0.4924 & 0.3532 \end{bmatrix}, \quad B = \begin{bmatrix} 82 & 89 & 36 \\ 42 & 17 & 77 \\ 52 & 72 & 64 \end{bmatrix}, \quad C = \begin{bmatrix} 83 & 52 & 43 \\ 80 & 6 & 12 \\ 45 & 7 & 68 \end{bmatrix} \tag{4-41}$$

Using the method presented in algorithm 2, we can derive that the final matrix K is equal to:

$$K = \begin{bmatrix} 0 & -34.3880 & -33.3880 \\ -34.3880 & 0 & -2.3880 \\ -33.3880 & -2.3880 & 0 \end{bmatrix} \tag{4-42}$$

Simulating the MMPS system with arbitrary initial conditions yields the intermediate and final results of set $\mathscr{K}$, as shown in figure 4-9. The system converges to the final set in four events, with a difference smaller or equal to the last buffer level. This results in a positive semi-definite max-plus Lyapunov function with a negative semi-definite derivative.

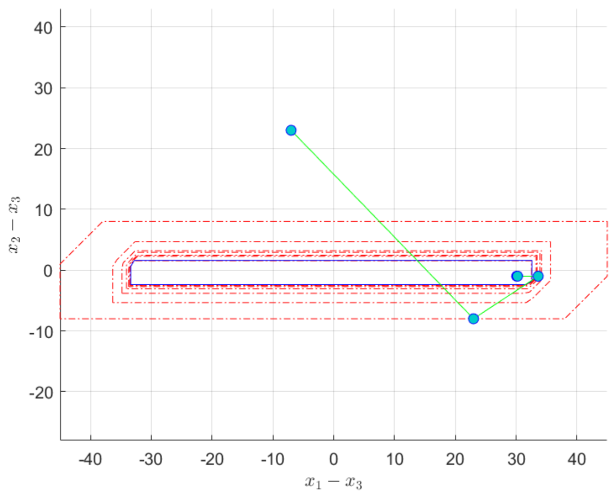


Figure 4-9: The final set $\mathscr{K}$ (blue line –), the intermediate results of set $\mathscr{K}$ (red dotted line – –), system dynamics of the MMPS system (green line –) with sky blue markers). All are plotted on the hyperplane$\{x \in \mathbb{R}^3 | x_3 = 0\}$.

4-5 Conclusion

This chapter has introduced and developed a framework for analyzing the stability of the buffer levels and the additive eigenvalues of MMP and MMPS. The chapter established the use of the max-plus Lyapunov function and proposed a method to determine the boundedness of MMP and MMPS systems. The fully correlated MMP and MMPS definitions are introduced to ensure a bounded difference between states. The proof is given for both systems to show that the buffer will be bounded. Additionally, the chapter presented an algorithm for constructing an attractive set for fully correlated MMP and MMPS systems based on their dynamics.

Methods to calculate the maximal buffer between states for bipartite and MMPS systems were derived analytically and demonstrated through examples. Extra conditions on stability were introduced for MMPS systems to ensure the bounded growth rate of fully correlated systems. A method to determine the growth rate of time-invariant monotonic MMPS systems is derived based on Markov chains.

We can draw meaningful conclusions from the provided theorems and lemmas on buffer boundedness and additive eigenvalues. Fully correlated time-invariant monotonic MMPS systems can have a buffer that converges to a steady state value, which depends on different α_i. This convergence occurs due to the Markov properties of each mode, eliminating the need for state cancellations to maintain a constant buffer.

Fully correlatedness ensures that time-invariant non-monotonic MMPS systems will always have a bounded growth rate, even though some modes may exhibit exponential growth. This stability arises because modes where the states cancel out will experience linear growth. The property of fully correlation gives a boundary condition on the maximal difference between states. If the system dynamics reach this bound, the additive eigenvalue becomes constant. Conversely, non-monotonic time-invariant systems that are not fully correlated can become unstable because this bound will not exist.

While it needs further confirmation, there is a hypothesis that fully correlated time-invariant monotonic MMPS systems always have one dominant mode and, thus, only one additive eigenvalue. Extensive simulations support this hypothesis. In contrast, fully correlated time-invariant non-monotonic MMPS systems can have multiple eigenvalues.

We could calculate each mode's additive eigenvalues using the methods presented for MMPS systems and the equivalences between bipartite systems and the ABC canonical form. Additionally, if we can prove the conjecture 4.1, we could conclude that a bipartite system that is fully correlated with initial conditions that are partly arbitrary will have one eigenvalue.

Lastly, analytical solutions for two-dimensional fully correlated MMPS systems were derived, revealing the downsides of analytical solutions.

The chapter presents analytical tools and methods for understanding the buffer stability and additive eigenvalues of MMP and MMPS systems. It offers insights into the behaviour of MMP and MMPS systems. It provides a foundation for further research in control strategies and stability analysis for discrete-event MMP and MMPS systems.

4-6 Technical proofs

Proof of theorem 4.1. Consider an MMP system rewritten in conjunctive normal form. Let $\omega_i \in \{\omega_1, \cdots, \omega_p\}$ be the power set[3] of set $\{x_1, \cdots, x_n\}$ present in max-plus functions f_i, $i \in \{1, \cdots, q\}$. Let $\{\xi_{1,i}, \cdots, \xi_{q,i}\}$ be the values added to the max-plus functions. Then, the state update can be written down as follows:

$$x_i(k+1) = \min(\xi_{1,i} + f_{1,i}(\omega_1), \xi_{2,i} + f_{2,i}(\omega_2), \cdots, \xi_{q,i} + f_{q,i}(\omega_q)) \tag{4-43}$$

After one iteration, we will prove that the difference between states i and j will be bounded. We consider two cases: the first case where for both states i and j, the minimal values are composed of max-plus functions dependent on the same ω_i. In the second case, for states i and j, the minimal values are composed of max-plus functions dependent on different ω_i

Case 1: Assume that the following inequalities are true:

$$\begin{aligned} \xi_{1,i} + f_{1,i}(\omega_1) \leq \xi_{2,i} + f_{2,i}(\omega_2) \leq \cdots \leq \xi_{q,i} + f_{q,i}(\omega_p) \\ \xi_{1,j} + f_{1,j}(\omega_1) \leq \xi_{2,j} + f_{2,j}(\omega_2) \leq \cdots \leq \xi_{q,j} + f_{q,j}(\omega_p) \end{aligned} \tag{4-44}$$

Let the set ω_i be equal to $\{x_{1,\omega_i}, \cdots, x_{n,\omega_i}\}$ and the corresponding finite variables $\{\theta_{1,f_{1,i}}, \cdots, \theta_{n,f_{1,i}}\}$. Then, the max-plus function is written in the following form:

$$f_{1,i}(\omega_i) = \max(x_{1,\omega_i} + \theta_{1,f_{1,i}}, x_{2,\omega_1} + \theta_{2,f_{1,i}}, \cdots, x_{n,\omega_i} + \theta_{n,f_{1,i}}) \tag{4-45}$$

Using inequalities (4-44) the difference between the updated states i and j will be equal to:

$$\begin{aligned} &x_i(k+1) - x_j(k+1) \\ =&\xi_{1,i} + f_{1,i}(\omega_1) - \xi_{1,j} + f_{1,j}(\omega_1) \\ =&\xi_{1,i} - \xi_{1,j} + \max(x_{1,\omega_i} + \theta_{1,f_{1,i}}, \cdots, x_{n,\omega_i} + \theta_{n,f_{1,i}}) - \max(x_{1,\omega_j} + \theta_{1,f_{1,j}}, \cdots, x_{n,\omega_j} + \theta_{n,f_{1,j}}) \end{aligned} \tag{4-46}$$

The first two terms $\xi_{1,i}$ and $\xi_{1,j}$ will be constant. Next, we prove that the difference between two max-plus functions in (4-46) will always be bounded. Assume that the following inequalities are true:

$$\begin{aligned} x_{1,\omega_1} + \theta_{1,f_{1,i}} \geq x_{2,\omega_1} + \theta_{2,f_{1,i}} \geq \cdots \geq x_{n,\omega_1} + \theta_{n,f_{1,i}} \\ x_{1,\omega_1} + \theta_{1,f_{1,j}} \geq x_{2,\omega_1} + \theta_{2,f_{1,j}} \geq \cdots \geq x_{n,\omega_1} + \theta_{n,f_{1,j}} \end{aligned} \tag{4-47}$$

Then, using inequalities (4-47), the difference between the states will be:

$$x_i(k+1) - x_j(k+1) = \xi_{1,i} - \xi_{1,j} + \theta_{1,f_{1,i}} - \theta_{1,f_{1,j}}, \tag{4-48}$$

which is constant. Now, consider a scenario where the following set of inequalities are true:

$$\begin{aligned} x_{1,\omega_1} + \theta_{1,f_{1,i}} \geq x_{2,\omega_1} + \theta_{2,f_{1,i}} \geq \cdots \geq x_{n,\omega_1} + \theta_{n,f_{1,i}}, \\ x_{2,\omega_1} + \theta_{2,f_{1,j}} \geq x_{1,\omega_1} + \theta_{1,f_{1,j}} \geq \cdots \geq x_{n,\omega_1} + \theta_{n,f_{1,j}} \end{aligned} \tag{4-49}$$

.

[3] Let A be a set; by the power set of A we mean the class of all the subsets of A[17].

Then, the difference between the states will be dependent on different states:

$$x_i(k+1) - x_j(k+1) = \xi_{1,i} - \xi_{1,j} + x_{1,\omega_1} + \theta_{1,f_{1,i}} - x_{2,\omega_1} - \theta_{2,f_{1,j}}. \tag{4-50}$$

Using the inequalities (4-49), we can derive that the difference (4-50) is upper and lower bounded:

$$\theta_{1,f_{1,i}} - \theta_{1,f_{1,j}} \geq x_{1,\omega_1} + \theta_{1,f_{1,i}} - x_{2,\omega_1} - \theta_{2,f_{1,j}} \geq \theta_{2,f_{1,i}} - \theta_{2,f_{1,j}} \tag{4-51}$$

Thus, the difference between the states is bounded by the following constants:

$$\theta_{1,f_{1,i}} - \theta_{1,f_{1,j}} + \xi_{1,i} - \xi_{1,j} \geq x_{1,\omega_1} + \theta_{1,f_{1,i}} - x_{2,\omega_1} - \theta_{2,f_{1,j}} + \xi_{1,i} - \xi_{1,j} \geq \theta_{2,f_{1,i}} - \theta_{2,f_{1,j}} + \xi_{1,i} - \xi_{1,j} \tag{4-52}$$

The same can be proved for all the possible combinations of inequalities (4-49).

Case 2: Now consider a situation where the states $x_i(k+1)$ and $x_j(k+1)$ depend on max-plus functions with different ω. Assume that the following set of inequalities are true:

$$\begin{aligned} \xi_{1,i} + f_{1,i}(\omega_1) \leq \xi_{2,i} + f_{2,i}(\omega_2) \leq \cdots \leq f_{q,i}(\omega_p) \\ \xi_{2,j} + f_{2,j}(\omega_2) \leq \xi_{1,j} + f_{1,j}(\omega_1) \leq \cdots \leq f_{q,j}(\omega_p) \end{aligned} \tag{4-53}$$

Then, the difference between the states will be equal to:

$$x_i(k+1) - x_j(k+1) = \xi_{1,i} + f_{1,i}(\omega_1) - \xi_{2,j} - f_{2,j}(\omega_2) \tag{4-54}$$

Using the inequalities (4-53), we can find that the upper bound will be the difference between two max-plus functions dependent on the same ω_i:

$$\xi_{1,i} + f_{1,i}(\omega_1) - \xi_{2,j} - f_{2,j}(\omega_2) \leq \xi_{2,i} + f_{2,i}(\omega_2) - \xi_{2,j} - f_{2,j}(\omega_2) \tag{4-55}$$

We have already derived that the difference between max-plus functions that depend on the same ω_i will be bounded. This means the difference between $f_{2,i}(\omega_2)$ and $f_{2,j}(\omega_2)$ is bounded, so the difference between the states will be upper bounded. Similarly, we can prove that the states are lower bounded:

$$\xi_{1,i} + f_{1,i}(\omega_1) - \xi_{2,j} - f_{2,j}(\omega_2) \geq \xi_{1,i} + f_{1,i}(\omega_1) - \xi_{1,j} - f_{1,j}(\omega_1). \tag{4-56}$$

This will hold for all possible combinations of inequalities (4-53). Thus, a fully correlated max-min-plus (MMP) system will have a bounded absolute difference between all the states. □

Proof of theorem 4.2. This proof is similar to that of theorem 4.1 except that we substitute the states x with α where $\alpha = A \cdot x$. □

Proof of theorem 4.3. If we rewrite the state update of a bipartite system, the system can be fully dependent on $x(k)$:

$$x(k+2) = B \otimes y(k+1) = B \otimes (C \otimes' x(k)) \tag{4-57}$$

As long as matrix C is regular, all the state updates will have a finite value. We can find the maximal buffer between each state using periodic behaviour. This is the result of using the partly arbitrary initial state:

$$\begin{aligned} x_0 &= x_0 & y_0 &= C \otimes' x_0 \\ x_1 &= B \otimes (C \otimes' x_0) & y_1 &= C \otimes' x_0 \\ x_2 &= B \otimes (C \otimes' x_0) & y_2 &= C \otimes' (B \otimes (C \otimes' x_0)) \\ x_3 &= B \otimes (C \otimes' (B \otimes (C \otimes' x_0))) & y_3 &= C \otimes' (B \otimes (C \otimes' x_0)) \end{aligned} \tag{4-58}$$

Each state will have the same value for two events due to the initial conditions. Consider β_i with $i \in \{1, \cdots, m\}$ as:

$$\beta_i = \min(c_{i,1} + x_1(k), c_{i,2} + x_2(k), \cdots, c_{i,n} + x_n(k)). \tag{4-59}$$

Now we can substitute the variable β, the difference between state i and j can be described as:

$$x_i(k+2) - x_j(k+2) = \max(b_{i,1}+\beta_1, b_{i,2}+\beta_2, \cdots, b_{i,m}+\beta_m) - \max(b_{j,1}+\beta_1, b_{j,2}+\beta_2, \cdots, b_{j,m}+\beta_m). \tag{4-60}$$

Assume the following inequalities are true:

$$\begin{aligned} b_{i,1} + \beta_1 \geq b_{i,2} + \beta_2 \geq \cdots \geq b_{i,m} + \beta_m, \\ b_{j,2} + \beta_2 \geq b_{j,1} + \beta_1 \geq \cdots \geq b_{j,m} + \beta_m. \end{aligned} \tag{4-61}$$

For the conditions, the difference between state i and state j will be equal to:

$$x_i(k+2) - x_j(k+2) = b_{i,1} + \beta_1 - b_{j,2} - \beta_2. \tag{4-62}$$

Using the inequalities (4-61), the upper and lower bound of the solution will be equal to:

$$\begin{aligned} b_{j,2} - b_{j,1} \geq \beta_1 - \beta_2 \geq b_{i,2} - b_{i,1}, \\ b_{i,1} - b_{j,1} \geq b_{i,1} + \beta_1 - b_{j,2} - \beta_2 \geq b_{i,2} - b_{j,2}. \end{aligned} \tag{4-63}$$

The difference will be bounded for all possible combinations of inequalities (4-61). In general, the following will hold:

$$b_{i,u} - b_{j,u} \geq b_{i,u} + \beta_u - b_{j,v} - \beta_v \geq b_{i,v} - b_{j,v}. \tag{4-64}$$

Therefore, the maximal value of the difference between state i and j will be equal to the difference between the entries of matrix B:

$$\max(x_i(k+2) - x_j(k+2)) = \max(b_{i,1} - b_{j,1}, b_{i,2} - b_{j,2}, \cdots, b_{i,m} - b_{j,m}) \tag{4-65}$$

□

Proof of lemma 4.4. If the MMPS system is monotonic, then $a_{i,j} \geq 0, \forall i, j$ [7, Lemma 12], and the time-invariance implies that the rows sum up to one[7, Lemma 11]. As a result, the system matrices $\{\Phi_1, \cdots, \Phi_p\}$ for each mode will be transposed Markov matrices. Using the properties of the Markov chain, we can derive that the growth rate of each state will converge to a constant if the system stays in the same mode. Consider two state updates for one mode $\alpha^{(i)}$ with system matrix Φ and with constant addition γ:

$$\begin{aligned} x(k+n) &= \Phi^n x(k) + \Phi^{(n-1)}\gamma + \Phi^{(n-2)}\gamma + \cdots + \Phi\gamma + \gamma \\ x(k+n-1) &= \Phi^{(n-1)}x(k) + \Phi^{(n-2)}c + \Phi^{(n-3)}c \cdots + \Phi\gamma + \gamma \end{aligned} \tag{4-66}$$

If we take the difference, most terms fall away, and we keep:

$$x(k+n) - x(k+n-1) = \Phi^n x(k) + \Phi^{(n-1)}\gamma - \Phi^{(n-1)}x(k) \tag{4-67}$$

Using the properties of Markov chains, we know that each of the terms will converge to the steady state. All three terms in equation 4-70 are Markov chains, so the growth rate will be a summation of the steady state values. For a sufficiently large value of n where each term is in a steady state, the growth rate becomes:

$$\lambda_g = \Phi^{(n-1)}\gamma. \tag{4-68}$$

If, for each state, this growth rate is equal, this would be an additive eigenvalue of the system. □

Proof of lemma 4.5. If the MMPS system in ABC canonical form is time-invariant, the rows sum up to one[7, Lemma 11]. If the mode with system matrix Φ is dependent on the same α_i, each row is identical, and the system matrix is idempotent. This means that the matrix $\Phi = \Phi \cdot \Phi$. Now consider two state updates for one mode with the addition of vector γ:

$$\begin{aligned} x(k+n) &= \Phi^n x(k) + \Phi^{(n-1)}\gamma + \Phi^{(n-2)}\gamma + \cdots + \Phi\gamma + \gamma \\ &= \Phi x(k) + \Phi\gamma + \Phi\gamma + \cdots + \Phi\gamma + \gamma \\ x(k+n-1) &= \Phi^{(n-1)}x(k) + \Phi^{(n-2)}\gamma + \Phi^{(n-3)}\gamma \cdots + \Phi\gamma + \gamma \\ &= \Phi x(k) + \Phi\gamma + \Phi\gamma \cdots + \Phi\gamma + \gamma \end{aligned} \tag{4-69}$$

If we take the difference, most terms fall away, and we keep:

$$x(k+n) - x(k+n-1) = \Phi\gamma \tag{4-70}$$

Each mode is identical; thus, the growth rate is equal for each state. So, for each mode that is dependent on the same α_i, the additive eigenvalue will be:

$$\lambda_g = \Phi_{1\cdot}\gamma. \tag{4-71}$$

□

Proof of theorem 4.6. The proof for the bounded buffer of a fully correlated MMPS system follows from theorem 4.2.

If the MMPS system is monotonic, then $a_{i,j} \geq 0$, $\forall i, j$ [7, Lemma 12]. As a result, all system matrices $\{\Phi_1, \cdots, \Phi_p\}$ for each mode will be transposed Markov matrices. We know that the eigenvalues of a transposed matrix will equal the eigenvalues of the original matrix. We also know that every Markov matrix will have an eigenvalue $\lambda_1 = 1$, and the other eigenvalues will satisfy $||\lambda_i|| \leq 1$. If A has all positive entries, the other $|\lambda_i| < 1$ [21]. Thus, we can conclude the system will not experience exponential growth in any of the modes, and the growth rate of each mode can be calculated as in lemma 4.4. If the states have different growth rates, the states will diverge until it reaches another mode. If the mode depends on the same α_i, the growth rate will become linear (lemma 4.5).

If an MMPS system is non-monotonic, it will have a system matrix Φ with positive and negative values. In modes that combine different α_i, the system dynamics may become unstable due to absolute eigenvalues greater than one, causing the states to diverge. Consequently, the system will reach a mode with the same α_i. Again, if the mode depends on the same α_i, the growth rate will become linear (lemma 4.5). Another possibility is that the system switches between modes, which creates small oscillations. □

Chapter 5

Max-plus Lyapunov Functions as Control Lyapunov Functions for Model-Predictive-Control (MPC)

This chapter explores using max-plus Lyapunov functions as cost functions for model predicitive control (MPC). This chapter introduces a promising approach that offers new control strategies.

The first section gives a brief introduction to MPC. It will discuss the importance of control Lyapunov functions (CLF's), and it proposes a control method using the max-plus Lyapunov functions as control Lyapunov functions. The second section provides two examples of their application in production systems. The first example will stabilize a max-plus-linear (MPL), and the second model will stabilize a max-min-plus (MMP) system.

5-1 model predicitive control (MPC)

Model predictive control is a control approach that makes use of optimization. Each iteration calculates the optimal control strategy over a prediction horizon N_p. The optimal control strategy is calculated using a cost function. The most commonly used cost function is of the form:

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)). \tag{5-1}$$

Where the stage cost is $l(x, u)$ and the terminal penalty is $V_f(x)$[22]. A graphical representation of the algorithm is shown in figure 5-1. The optimization is performed over the prediction horizon. If the control horizon is smaller than the prediction horizon, the input will remain constant for the rest of the prediction horizon.

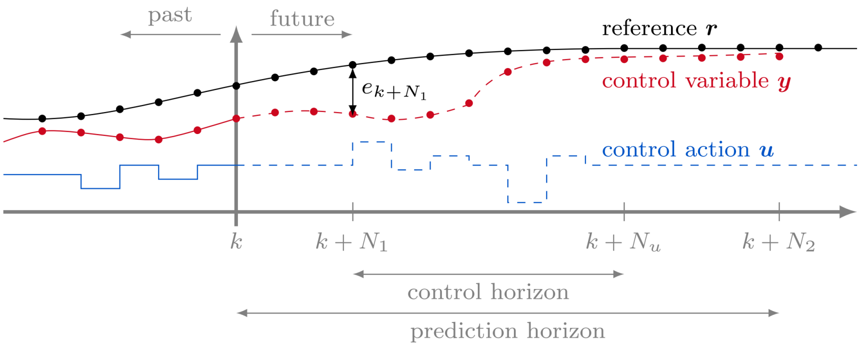


Figure 5-1: Function principle of a model-based predictive controller, with horizons N_1, N_2, N_u [23]

5-1-1 Control Lyapunov function (CLF)

Lyapunov functions are relevant for proving the stability of an autonomous system and can provide conditions for asymptotic stability. The importance of the concept of control Lyapunov function is that the existence of one of such functions is also a sufficient condition for the existence of stabilizing feedback[24].

Definition 5.1 (Global control Lyapunov function (CLF) [22]). *A function $V : \mathbb{R}^n \to \mathbb{R}_{z0}$ is a global control Lyapunov function for the system $x^+ = f(x,u)$ and closed set $\mathcal{A}$ if there exist $\mathcal{K}_w$ functions $\alpha_1(\cdot), \alpha_2(\cdot)$, $\alpha_3(\cdot)$ satisfying for all $x \in \mathbb{R}^n$:*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$\inf_{u\in\mathbb{U}} V(f(x,u)) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})$$

5-1-2 Max-plus Lyapunov function as CLF

Consider the max-plus Lyapunov function that denotes the minimum max-plus Hilbert projective distance between x and $\mathscr{K}$:

$$V(x) = -x^T \otimes K \otimes x \tag{5-2}$$

The max-plus Lyapunov function uses matrix K to incorporate the buffer. In chapter four, we derived a method to determine the maximal allowable buffer for fully correlated MMP and MMPS systems. Instead of finding the maximal buffer for the CLF, we will use the minimal possible buffer for the whole state space, optimizing over the states and the input. It makes sense to use the minimal buffer value. The buffer level of an unstable system will be unbounded, so there will be no maximal buffer. Using the minimal buffers, the matrix K is constructed. Now consider a discrete-event function:

$$x_i(k+1) = f_i(x(k), u(k)). \tag{5-3}$$

Using a dummy variable, the absolute optimization is again rewritten in a mixed-integer linear program (MILP) problem. For all the combinations of states, the minimal buffer is calculated. The variable N should be sufficiently large, and b is a binary variable [1].

$$\begin{aligned} \min_{x(k),u(k)} \quad & \delta_{i,j} \\ \text{s.t.} \quad & \delta_{i,j} \geq 0 \\ & \delta_{i,j} \geq f_i(x(k),u(k)) - f_j(x(k),u(k)) \\ & \delta_{i,j} \geq -f_i(x(k),u(k)) + f_j(x(k),u(k)) \\ & \delta_{i,j} \leq f_i(x(k)) - f_j(x(k)) + Nb \\ & \delta_{i,j} \leq -(f_i(x(k)) - f_j(x(k))) + N(b-1) \end{aligned} \tag{5-4}$$

The buffer levels are implemented using the same K matrix as in (4-12). Using the max-plus Lyapunov function as a control Lyapunov function, we can stabilize the buffer levels and keep it constant. Stabilizing the system is done using model-predictive control, using the discrete-time derivative of the max-plus Lyapunov function as a cost function that is minimized. The minimization can be presented as follows:

$$\min_{u(k)} \quad \Delta V(x(k),u(k)) = \min_{u(k)} \quad V(f(x,u)) - V(x) \tag{5-5}$$

After simulating various scenarios, it became clear that the length of the prediction horizon has no impact on the stability of the buffer levels. Therefore, we will use a prediction horizon of one for the following two examples, as this requires the least computational power. This may be because of the linear behaviour of the stabilized system.

5-2 MPC for max-plus-linear system

The first example of using max-plus Lyapunov functions as CLF will be on a max-plus-linear (MPL) system. The system will be a simple manufacturing system from [10]. Consider the manufacturing system in figure 5-2. The systems consist of three different processing units: P_1, P_2 and P_3. The raw material is fed to P_1 and P_2, processed and sent to P_3 where assembly occurs. The processing times for each processing unit are $d_1 = 11$, $d_2 = 12$ and $d_3 = 7$ respectively. The transportation time between the processing units (denoted in figure 5-2) are $t_1 = 2$, $t_2 = 0$, $t_3 = 1$, $t_4 = 0$ and $t_5 = 7$. The other transportation times and set-up times are assumed to be negligible. The system can be described using the MPL state space model:

$$x(k+1) = \begin{bmatrix} 11 & \varepsilon & \varepsilon \\ \varepsilon & 12 & \varepsilon \\ 23 & 24 & 7 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix} \otimes u(k), \tag{5-6}$$

$$y(k) = \begin{bmatrix} \varepsilon & \varepsilon & 7 \end{bmatrix} \otimes x(k) \tag{5-7}$$

With $u(k)$ the time at which a batch of raw material is fed to the system for the (k+1)th time, $x(k)$ the time at which P_i starts working for the kth time, and $y(k)$ the time at which

[1] Note that the max-plus functions need to be rewritten to make it a MILP problem.

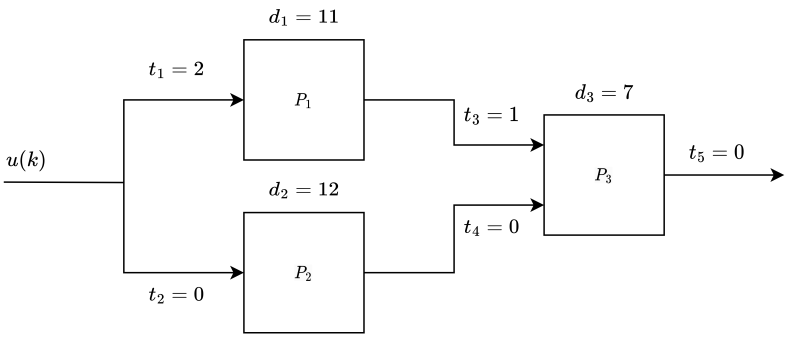
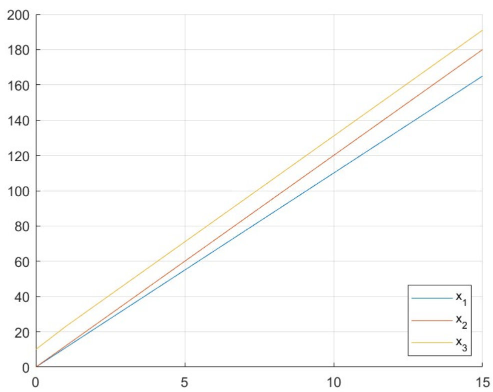


Figure 5-2: A simple manufacturing system

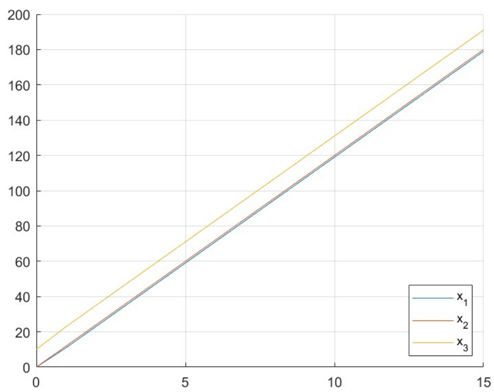
the kth finished product leaves the system. This example will focus on stabilizing the buffer using a max-plus Lyapunov function as a cost function. The input will be a delay to prevent buffer levels from increasing if the buffer level is not constant. First, we find the minimal buffer for all states and inputs. Using the optimization (5-4) for each pair of states, we can find the matrix K:

$$K = \begin{bmatrix} 0 & -2 & -12 \\ -2 & 0 & -11 \\ -12 & -11 & 0 \end{bmatrix} \tag{5-8}$$

In figure 5-3, the system is simulated with and without MPC. Figure 5-4a shows the uncontrolled system. The states are diverging. Over time, this difference will grow linearly. Processing unit P_1 is processing the raw material faster; thus, the product processed by P_1 will stack up until it can be assembled. The amount of product will keep growing, and the buffer will keep increasing; thus, the system is considered unstable. Figure 5-4b shows the controlled system. It can be seen that the buffer immediately converges to constant values and stays stable. Therefore, we can conclude that the system is stabilized. Now consider the



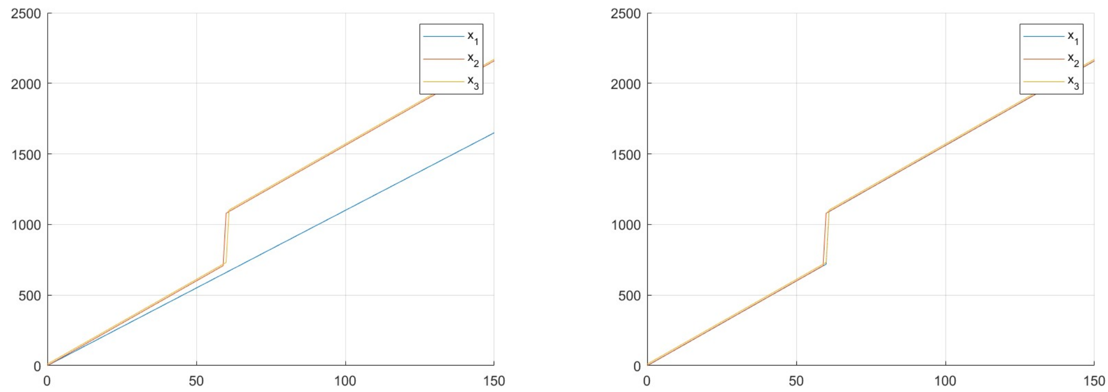
(a) Without control



(b) With MPC

Figure 5-3: Simulation of max-plus-linear manufacturing unit, with initial state $x_0 = [0, 0, 10]^T$.

same system where at $k = 60$, there is a delay due to a technical difficulty at processing unit P_2. It takes $d(60) = 360$ time units to repair the processing unit. We simulate the system with the same initial conditions in figure 5-4. The uncontrolled system has a buffer that grows even more. The model-predictive controlled system using the max-plus Lyapunov function stabilizes the system and keeps the buffer level constant by delaying the input of raw product. The Matlab code used for this simulation can be found in appendix B-0-2.



(a) Without control **(b)** With MPC

Figure 5-4: Simulation of max-plus-linear manufacturing unit with disturbance, with initial state $x_0 = [0, 0, 10]^T$.

5-3 MPC for MMP system

Consider the production system from figure 5-5:

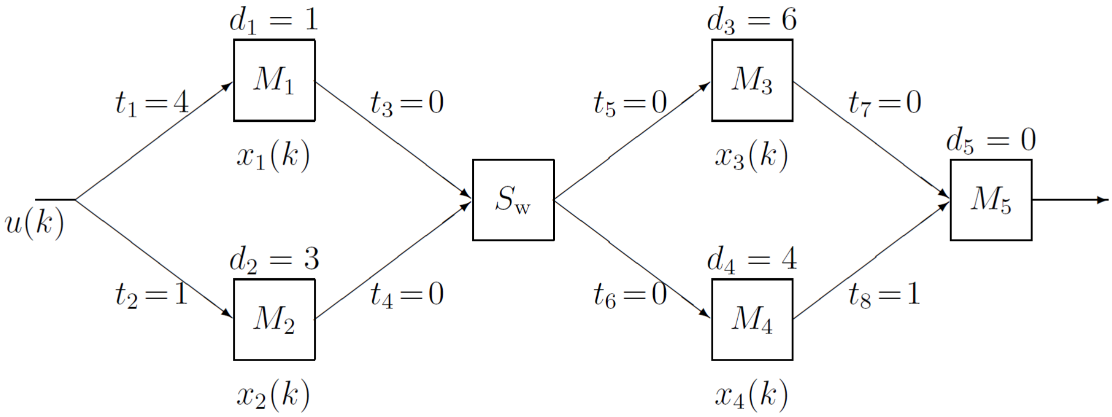


Figure 5-5: Production system with competition

The system has five different machines M_i, $i \in \{1, ..., 5\}$. M_1 and M_2 are the first machines that prepossess the raw product. From M_1 and M_2, the product is transported to the switching device S_W. The switching device feeds the first finished product in the k-th cycle to the slower machine M_3 and the second to the faster M_4 machine. Finally, the products are assembled in the last machine M_5. The following system equations can describe the system:

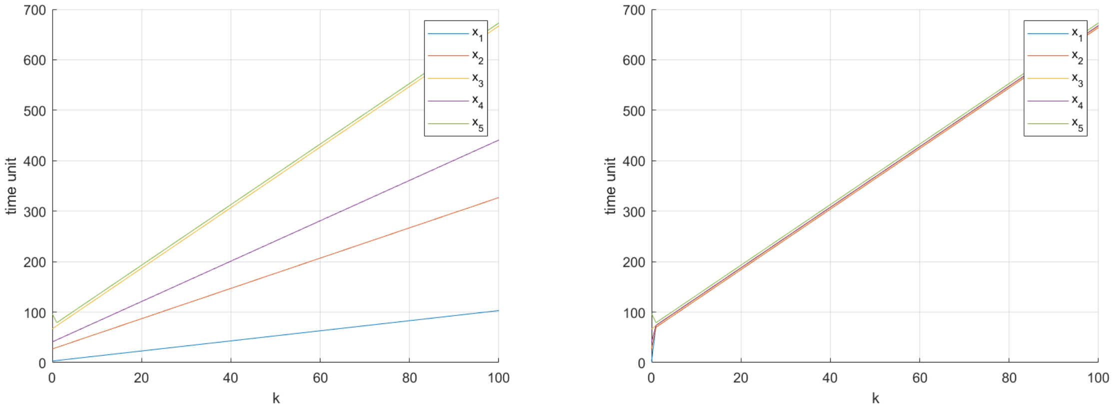
$$\begin{aligned}
x_1(k+1) &= \max\left(x_1(k) + d_1, u(k+1) + t_1\right),\\
x_2(k+1) &= \max\left(x_2(k) + d_2, u(k+1) + t_2\right),\\
x_3(k+1) &= \max\left(\min\left(x_1(k+1) + d_1, x_2(k+1) + d_2\right), x_3(k) + d_3\right)\\
x_4(k+1) &= \max(\max\left(x_1(k+1) + d_1, x_2(k+1) + d_2\right), x_4(k) + d_4)\\
x_5(k+1) &= \max\left(x_3(k+1) + d_3, x_4(k+1) + d_4 + t_8\right)
\end{aligned} \tag{5-9}$$

The states x_i describe the time the i-th machine starts processing. The input u describes the time the system is fed. The input can serve as a delay in supplying the raw product when the buffers between states are not constant. The states will diverge when simulating the system with zero input, as shown in figure 5-6a. When comparing the states, the first machine will process a hundred products in the same time that the last machine assembled ten products. As a result, there will already be ninety products waiting for processing. The system's buffer between machines is the amount of products it will be able to hold before processing between machines. As the growth rate of the last state is ten times greater than the first state's growth rate, this buffer will be violated at some point. The goal of using model-predictive control (MPC) is to find an optimal control input so that the amount of product is smaller or equal to the maximal buffer. The matrix K is derived using the optimisation (5-4), the result is:

$$K = \begin{bmatrix} 0 & -3 & 0 & -1 & -6 \\ -3 & 0 & -3 & -3 & -8 \\ 0 & -3 & 0 & -10000 & -6 \\ -1 & -3 & -10000 & 0 & -5 \\ -6 & -8 & -6 & -5 & 0 \end{bmatrix} \tag{5-10}$$

The dependence on different variables may cause the high $\delta_{3,4}$ value. The high value of a $\delta_{3,4}$ is not necessary. Therefore, we can tune the value; the value can be as small as one, but it does not improve the performance. The value of one can also be determined using the other buffers. Using the dependency between the states, we can derive that the minimal buffer will be equal to $|\delta_{1,3} - \delta_{1,4}| = |\delta_{3,5} - \delta_{4,5}| = 1$. The prediction horizon is set equal to $N_p = 1$. In figure 5-6, the uncontrolled and the model-predictive controlled systems are simulated. From the simulation, it is clear that the MPC-controlled system stays stable. The states do not diverge and have a limited growth rate.

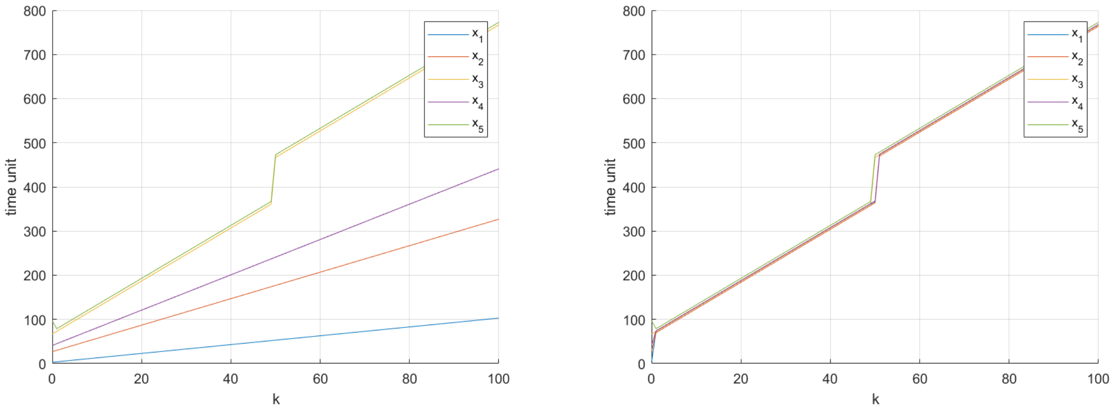
Now consider the same system where at $k = 50$, there is a delay due to a technical difficulty at processing unit M_4. It takes $d(50) = 100$ time units to repair the processing unit. We simulate the system with the same initial conditions in figure 4-8. The uncontrolled system has a buffer that grows even more. Using the max-plus Lyapunov function, the model-predictive controlled system stabilizes the system and keeps the buffer level constant. The Matlab code used for this simulation can be found in appendix B-0-3.



(a) States production unit without control

(b) States production unit with MPC

Figure 5-6: Simulation of MMP system with initial conditions $x_0 = [3, 27, 67, 41, 97]^T$



(a) States production unit without control

(b) States production unit with MPC

Figure 5-7: Simulation of MMP system with disturbance, initial conditions $x_0 = [3, 27, 67, 41, 97]^T$

5-4 Conclusion

This chapter explored a novel approach to control discrete-event systems with Max-Plus Lyapunov functions as Control Lyapunov functions. The investigation underscored the critical role of CLF as control functions in MPC and their implications for the existence of a stabilizing controller. Building on this foundation, we introduced max-plus Lyapunov functions as effective tools for stabilizing buffer levels in inherently unstable systems. The chapter presents a method to determine the buffer for the max-plus Lyapunov function as a CLF that is similar to the method presented in chapter four. Instead of using the maximal buffer, the absolute minimal buffer is used for matrix K.

We were able to show the effectiveness of this approach through analysis and simulation. For a MPL system, max-plus Lyapunov functions successfully prevented buffer divergence and maintained stability even with disturbance. This success was also replicated for a MMP system.

This chapter shows promising results using max-plus Lyapunov functions as cost functions for MPC. This approach ensured stable buffer levels even with uncertainties. This chapter contributes to advancing the field of control for discrete-event systems, setting the stage for future innovations in system stability for discrete-event systems.

Chapter 6

Conclusions and Contributions

This chapter summarizes the main conclusions drawn from the research in this thesis. Additionally, it provides a concise overview of the thesis' contribution to the fields of systems and control and discrete-event systems.

6-1 Conclusions

This section gives an overview of the main conclusions of this master thesis. It answers the research questions and subquestions.

- How can buffer stability be proven using Lyapunov or Lyapunov-like functions for discrete event max-min-plus (MMP) systems?
  - What kind of Lyapunov function can be used to determine the buffer stability of a max-min-plus (MMP) system?
  - Can we provide a generalized methodology to determine the buffer stability of max-min-plus (MMP) systems?

To establish the stability of discrete-event MMP systems, the aim was to use the concept of Lyapunov stability for the buffer. In the early stages of the research, it became clear that the max-plus Lyapunov function had the best potential for analyzing the buffer stability of MMP systems. The first approach made use of conventional Lyapunov functions and a converted system. To implement conventional Lyapunov theory for MMP systems, the system was rewritten such that every state represents the difference between the state and one reference state. MMP systems tend to oscillate, which can result in a slight divergence between the states. Using conventional Lyapunov functions such as the two-norm would not be sufficient because this would result in an oscillating Lyapunov derivative, which often would be positive and thus not considered stable. The max-plus Lyapunov function uses the K matrix to incorporate the allowable buffer. If the system dynamics converge to the max-plus C-set created by this matrix K, the max-plus Lyapunov function will be semi-positive definite and its derivative semi-negative definite. If the oscillations of the system remain within the buffer, the system is considered to be stable.

The definition of fully correlated MMP systems was introduced to determine the levels of MMP systems. Using this definition, it is possible to prove the boundedness of the buffer levels after one iteration. Analytical methods were derived for finding the maximal buffer of bipartite systems, dependent on special initial conditions. We derive that the buffers stay constant if a bipartite system has initial conditions equal to an eigenvector. If the initial conditions are partly arbitrary (dependent on the C matrix), it will result in periodic behaviour. We could use that information to analytically determine the maximal buffer for fully correlated bipartite systems.

Using the properties of a fully correlated MMP system, we provided an algorithm to determine the buffers for the max-plus Lyapunov function. Fully correlated MMP systems will have a bounded difference after one iteration; therefore, the states will never diverge more than the maximal absolute buffer. The algorithm finds the absolute maximal buffer after one iteration. The next iteration updates the domain of the optimisation and calculates the next maximal buffer until the optimisation does not provide a smaller set. Compared with the analytical method provided for bipartite systems, this algorithm results in a smaller buffer.

For bipartite systems, an additive eigenvalue exists if the system matrices are an irreducible pair (theorem 2.1). Fully correlated MMP systems are a bigger class. To get insights into the additive eigenvalues of bipartite systems with partly arbitrary initial conditions, we use the similarities between bipartite systems and MMPS systems in ABC canonical form. The bipartite system can be rewritten in the ABC canonical form using an identity matrix (conventional algebra) for the A matrix. From conjecture 4.1, we can derive that the bipartite system will have one additive eigenvalue.

- How can buffer stability be proven using Lyapunov or Lyapunov-like functions for the time signals of discrete event max-min-min-plus-scaling (MMPS) systems?
  - What kind of Lyapunov function can be used to determine the buffer stability of a max-min-min-plus-scaling (MMPS) system?
  - Can we provide a generalised methodology to determine the buffer stability of max-min-min-plus-scaling (MMPS) systems?
  - Is it possible to determine the additive eigenvalues for discrete-event max-min-min-plus-scaling (MMPS) systems?

Using the same motivation as for MMP systems, max-plus Lyapunov functions are used to determine the buffer stability of MMPS systems. To determine the buffers of max-min-min-plus-scaling (MMPS) systems, we introduced the definition of fully correlated MMPS systems. Using this definition, we are able to determine the boundedness of the buffer levels. The same algorithm can be applied to determine the buffer levels for MMPS systems, as is done with MMP systems.

Next to ensure buffer stability, the growth rate of MMPS systems should be bounded. It is possible to predict the behaviour of fully correlated MMPS systems using the properties of time-invariance and monotonicity. The properties of time-invariance and monotonicity will ensure that in each mode, the growth rate will converge to a steady state. If we write out the MMPS system, the mode of a system can be rewritten in an affine system. The system matrices Φ of each mode will be Markov matrices. Using this property, the absolute eigenvalues of the system matrices of each mode will be smaller or equal to one. Therefore, we can conclude that the system will not experience exponential growth in any of the modes. If the growth rate is equal for all the states, we can calculate the additive eigenvalue of each mode.

We could draw meaningful conclusions from the provided theorems and lemmas from chapter four. Fully correlated time-invariant monotonic MMPS systems can have a buffer that converges to a steady state value, which depends on different α_i. This convergence occurs due to the Markov properties of each mode, eliminating the need for state cancellations to maintain a constant buffer.

Fully correlatedness ensures that time-invariant non-monotonic MMPS systems will always have a bounded growth rate, even though some modes may exhibit exponential growth. This stability arises because modes, where the states cancel out, have eigenvalues smaller or equal to one. The maximal buffer forms a boundary for the difference between the states, and in such a mode, the system matrix will be dependent on the same α_i. In these modes, the system will experience linear growth (lemma 4.5). Conversely, non-monotonic time-invariant systems that are not fully correlated can become unstable because this bound will not exist.

While it needs further confirmation, there is a hypothesis that fully correlated time-invariant monotonic MMPS systems always have one dominant mode and, thus, only one additive eigenvalue. Extensive simulations support this hypothesis. In contrast, fully correlated time-invariant non-monotonic MMPS systems can have multiple additive eigenvalues, which are shown using simulation.

- Can Max-plus Lyapunov Functions serve as Control Lyapunov Functions for Model Predictive Control in Discrete-Event Systems?
  - Can a generalized approach be developed to stabilize the buffers of unstable discrete-event systems?
  - Is it possible to utilize max-plus Lyapunov functions as control Lyapunov function (CLF) to stabilize the buffer of unstable max-plus-linear (MPL) systems?
  - Is it possible to utilize max-plus Lyapunov functions as control Lyapunov function (CLF) to stabilize the buffer of unstable max-min-plus (MMP) systems?

The research conducted in chapter 5 explores the application of max-plus Lyapunov functions as CLF for stabilizing the buffer levels in discrete-event systems. By utilizing these functions in a MPC framework, a generalized approach to stabilize buffers of inherently unstable discrete-event systems is presented. The chapter demonstrates the effectiveness of this approach through the analysis and simulation of two different systems, including MPL and MMP systems. These simulations reveal that the MPC strategy employing max-plus Lyapunov functions successfully maintains buffer stability in the face of varying conditions and disturbances.

Through the case study of a manufacturing system modelled as a MPL system, this thesis demonstrates the viability of using max-plus Lyapunov functions as CLF. The analysis of the manufacturing system indicates that without control, the buffer levels between machines can diverge rapidly, leading to unstable buffer levels. However, the application of MPC with max-plus Lyapunov functions as cost functions effectively stabilize the buffer levels, ensuring they remain bounded and preventing system divergence with and without disturbance. The second example analyses a more complex scenario involving an MMP system. In this case, the controlled system successfully maintains stable buffer levels even when subjected to disturbances, such as delays in processing units. By effectively managing buffer levels between different machines, the MPC strategy using max-plus Lyapunov functions stabilizes an inherently unstable MMP system.

In summary, the research conducted in this chapter provides affirmative answers to the research questions. The chapter showcases that max-plus Lyapunov functions can serve as effective control Lyapunov functions for MPC in discrete-event systems. The chapter provides a generalized approach to stabilize buffers of unstable systems. The control method is successfully applied on both MPL and MMP systems, thereby contributing to the advancement of control strategies in the domain of discrete-event systems.

6-2 Contributions

This thesis contributes to the field of systems and control and discrete-event systems through the following results:

- This thesis demonstrates how to use max-plus Lyapunov functions to analyze buffer stability for both MMP and MMPS systems.
- A new definition, fully correlated, is introduced to assess buffer level boundedness for MMP and MMPS systems.
- A methodology is developed for determining the buffer across both fully correlated MMP and MMPS systems.
- A approach is presented for determining the additive eigenvalues of each mode of fully correlated time-invariant monotonic MMPS systems using Markov chains.
- The existence of multiple eigenvalues for time-invariant, non-monotonic fully correlated MMPS is shown.
- A control strategy is formulated for effectively stabilizing buffers within unstable discrete-event systems.
- The thesis demonstrates the potential of MPC by using max-plus Lyapunov functions as CLF for MPL and MMP systems.

Chapter 7

Recommendations for future work

- **Give a proof for the existence of a dominant mode/singular additive eigenvalue.**

  After conducting numerous simulations, it has become evident that fully correlated, time-invariant monotonic MMPS systems converge to only one additive eigenvalue consistently. However, further mathematical proof is required to confirm the validity of this observation.

- **Determine the dominant mode or modes of the MMPS system.**

  We suggest that fully correlated time-invariant monotonic MMPS systems will have only one additive eigenvalue and that fully correlated time-invariant non-monotonic systems can have multiple eigenvalues. We can now determine the eigenvalue of all possible modes, but it would be beneficial to know which mode or modes will be dominant. Identifying this mode will allow us to predict growth rates.

- **Determine stability for MMP and MMPS systems that are not fully correlated.**

  Because a lot of MMP and MMPS systems are fully correlated, this can be used to determine a system's buffer and ensure that the difference between the states will be bounded. But especially time invariant monotonic MMPS systems, there will be enough examples where MMPS systems will have a stable buffer even if the system is not fully correlated. This is probably due to the convergence of the growth rate, which is shown in lemma 4.4. It would be beneficial to prove for these systems that the buffer will always converge to a steady value, such that we can verify the buffer stability of more MMPS systems.

- **Find more real-world examples of MMPS systems**

  There aren't many real-world examples or practical implementations in the field of MMPS systems. The appeal of creating a new MMPS system is not only about filling this gap but also about potentially contributing to various fields, such as improving problem-solving methods, decision-making processes, and resource allocation strategies in real-world situations.

- **Apply the MPC method presented on MMPS system**

  The general approach for MPC control using CLF is worked out. It would be interesting to apply the method to many more different systems. It would be interesting to apply this method to an MMPS system and more discrete-event systems in general to validate the viability further.

Appendix A

Inequalities for a two-dimensional conjunctive MMPS system

$$\delta = |c_1 - c_3| \iff \begin{cases} b_1 + c_1 \leq b_3 + c_2 \\ b_2 + c_1 \leq b_4 + c_2 \\ b_1 + c_3 \leq b_3 + c_4 \\ b_2 + c_3 \leq b_4 + c_4 \end{cases} \quad \text{(A-1)}$$

$$\delta = |c_2 - c_4| \iff \begin{cases} b_1 + c_1 \geq b_3 + c_2 \\ b_2 + c_1 \geq b_4 + c_2 \\ b_1 + c_3 \geq b_3 + c_4 \\ b_2 + c_3 \geq b_4 + c_4 \end{cases} \quad \text{(A-2)}$$

$$\delta = \max(|c_1 - c_3|, |c_2 - c_4|) \iff \begin{cases} b_1 + c_1 \leq b_3 + c_2 \\ b_4 + c_2 \leq b_2 + c_1 \\ b_1 + c_3 \leq b_3 + c_4 \\ b_4 + c_4 \leq b_2 + c_3 \end{cases} \quad \text{(A-3)}$$

$$\delta = \max(|c_1 - c_3|, |c_2 - c_4|) \iff \begin{cases} b_1 + c_1 \geq b_3 + c_2 \\ b_4 + c_2 \geq b_2 + c_1 \\ b_1 + c_3 \geq b_3 + c_4 \\ b_4 + c_4 \geq b_2 + c_3 \end{cases} \quad \text{(A-4)}$$

$$\delta = \max(|c_1 - c_3|, |b_2 + c_1 - b_4 - c_4|) \iff \begin{cases} b_1 + c_1 \leq b_3 + c_2 \\ b_2 + c_1 \leq b_4 + c_2 \\ b_4 + c_4 \leq b_2 + c_3 \\ b_1 + c_3 \leq b_3 + c_4 \end{cases} \tag{A-5}$$

$$\delta = \max(|c_1 - c_3|, |b_1 + c_1 - b_3 - c_4|) \iff \begin{cases} b_1 + c_1 \leq b_3 + c_2 \\ b_2 + c_1 \leq b_4 + c_2 \\ b_3 + c_4 \leq b_1 + c_3 \\ b_1 + c_3 \geq b_3 + c_4 \end{cases} \tag{A-6}$$

$$\delta = \max(|c_2 - c_4|, |b_3 + c_2 - b_1 - c_3|) \iff \begin{cases} b_1 + c_1 \geq b_3 + c_2 \\ b_2 + c_1 \geq b_4 + c_2 \\ b_4 + c_4 \leq b_2 + c_3 \\ b_1 + c_3 \leq b_3 + c_4 \end{cases} \tag{A-7}$$

$$\delta = \max(|c_2 - c_4|, |b_4 + c_2 - b_2 - c_3|) \iff \begin{cases} b_1 + c_1 \geq b_3 + c_2 \\ b_2 + c_1 \geq b_4 + c_2 \\ b_4 + c_4 \leq b_2 + c_3 \\ b_3 + c_4 \leq b_1 + c_3 \end{cases} \tag{A-8}$$

$$\delta = \max(|c_1 - c_3|, |b_3 + c_2 - b_1 - c_3|) \iff \begin{cases} b_1 + c_1 \geq b_3 + c_2 \\ b_4 + c_2 \geq b_2 + c_1 \\ b_1 + c_3 \leq b_3 + c_4 \\ b_4 + c_4 \geq b_2 + c_3 \end{cases} \tag{A-9}$$

$$\delta = \max(|c_1 - c_3|, |b_4 + c_2 - b_2 - c_3|) \iff \begin{cases} b_1 + c_1 \leq b_3 + c_2 \\ b_4 + c_2 \leq b_2 + c_1 \\ b_1 + c_3 \leq b_3 + c_4 \\ b_4 + c_4 \geq b_2 + c_3 \end{cases} \tag{A-10}$$

$$\delta = \max(|c_2 - c_4|, |b_1 + c_1 - b_3 - c_4|) \iff \begin{cases} b_1 + c_1 \leq b_3 + c_2 \\ b_4 + c_2 \leq b_2 + c_1 \\ b_1 + c_3 \geq b_3 + c_4 \\ b_4 + c_4 \leq b_2 + c_3 \end{cases} \tag{A-11}$$

$$\delta = \max(|c_2 - c_4|, |b_2 + c_1 - b_4 - c_4|) \iff \begin{cases} b_1 + c_1 \geq b_3 + c_2 \\ b_4 + c_2 \geq b_2 + c_1 \\ b_1 + c_3 \geq b_3 + c_4 \\ b_4 + c_4 \leq b_2 + c_3 \end{cases} \tag{A-12}$$

$$\delta = \max(|b_1 + c_1 - b_3 - c_4|, |b_2 + c_1 - b_4 - c_4|) \iff \begin{cases} b_1 + c_1 \leq b_3 + c_2 \\ b_3 + c_4 \leq b_1 + c_3 \\ b_2 + c_1 \leq b_4 + c_2 \\ b_4 + c_4 \leq b_2 + c_3 \end{cases} \tag{A-13}$$

$$\delta = \max(|b_3 + c_2 - b_1 - c_3|, |b_4 + c_2 - b_2 - c_3|) \iff \begin{cases} b_1 + c_1 \geq b_3 + c_2 \\ b_3 + c_4 \geq b_1 + c_3 \\ b_2 + c_1 \geq b_4 + c_2 \\ b_4 + c_4 \geq b_2 + c_3 \end{cases} \tag{A-14}$$

Appendix B

Matlab

B-0-1 General construction K

```
%----------GENERAL OPTIMISATION - K -----------

x = sdpvar(n,1);     %DEFINE VARIABLES

x1 = ...             %DEFINE STATE UPDATE FROM 1,..,n
x2 = ...
xn = ...

xc = [x1;x2; ...  ;xn];      %COMBINING STATE UPDATES

options = sdpsettings('verbose',1,'solver','Gurobi');
options.gurobi.PoolSearchMode=0;
options.gurobi.PoolSolutions=1000;
options.savesolveroutput = 3;

n = length(xc);
z = ones(n)*inf;             %SET INITIAL DOMAIN FOR FIRST ITERATION
N = 1000;

K_save = [];                 %SAVE ALL ITERATIONS

for k = 1:100
    F = [];
    for i=1:n
        for j=1:n
            if i>=j
                f = [];
            else
                f = [x(j)<=x(i)+abs(z(i,j)),x(j)>=x(i)-abs(z(i,j))];     % INTRODUCE UPDATES CONSTRAINTS
            end
```

```
            F = [F,f];                                  %STACKS CONSTRAINTS
        end
    end
    for i = 1:n
        for j = 1:n
            if i>=j                                     %ABSOLUTE DIFFERENCE BETWEEN i,j AND j,i WILL BE EQUAL, MAKES SURE THAT IT OPTIMIZES ONCE
                d = 0;
            else
                obj = xc(i)-xc(j);                      %OBJECTIVE (DIFFERENCE BETWEEN STATES)

                delta = sdpvar(1);                      %INTRODUCE DUMMY VARIABLE
                y = binvar(1);                          %INTRODUCE BINARY VARIABLE

                con = [F,obj+N*y>=delta,-obj+N*(1-y)>=delta,-obj<=delta,-obj<=delta];   %INTRODUCE EXTRA CONSTRAINTS TO CONVERT THE OPTIMISATION PROBLEM

                sol = optimize(con,-delta,options);     %OPTIMISATION

                d_str = sol.solveroutput.result.pool;   %POOL OF SOLUTIONS TO STRING
                d_cell = struct2cell(d_str);            %CONVERT TO CELL
                sols = cell2mat(d_cell(1,1,:));         %COVERT TO MAT
                sols = reshape(sols,[],1)';             %RESHAPE
                d  =max(abs(sols));                     %USE MAXIMAL VALUE OF ALL SOLUTIONS

                disp('---------------------------------------------')
            end
            K(i,j) = d;
        end
    end
    K = -1*(K + flip(K',3));
    K_save = cat(3,K_save,K);
    if round(K,2) == round(z,2)         %CHECK IF CONVERGED
        break
    end
    z = K;
end
```

B-0-2 MPC for MPL system

```

% ---------- MPC MPL ----------
clc
clear all

```

```matlab
%% System matrices
e = -inf;

A = [11 e e;
     e 12 e;
     23 24 7];
B = [2;0;14];

%% Optimization

x = sdpvar(3,1);
u = sdpvar(1,1);

x1 = max(11+x(1),2+u);
x2 = max(12+x(2),u);
x3 = max([23+x(1),23+x(2),7+x(3),u+14]);

xc = [x1;x2;x3];

options = sdpsettings('verbose',2,'solver','Gurobi');
options.gurobi.PoolSearchMode=0;
options.gurobi.PoolSolutions=1000;
options.savesolveroutput = 3;

n = length(xc);
F = [];

N = 1000;

%   This part is very similar to the construnction of K but then using the
%   minimal value
for i = 1:n
    for j = 1:n
        if i>=j
            d = 0;
        else
            obj = xc(i)-xc(j);

            delta = sdpvar(1);
            y = binvar(1);

            con = [F,obj+N*y>=delta,-obj+N*(1-y)>=delta,-obj<=delta,-obj<=delta];

            sol1 = optimize(con,delta,options);

            d1_str = sol1.solveroutput.result.pool;
            d1_cell = struct2cell(d1_str);

            sols1 = cell2mat(d1_cell(1,1,:));
            sols1 = reshape(sols1,[],1)';
            d  =min(abs(sols1));
```

```
            disp('---------------------------------------------')
        end
        K(i,j) = d;
    end
end
K = -1*(K + flip(K',3));

%% Update
clearvars -except K A B

N     = 1;
evnts = 150;

% x0    = randi([1,100],[3,1])
x0 = [0;0;10];

dis = zeros(evnts);
dis(60) = 360;

x1 = [x0(1) zeros(1,evnts-1)];
x2 = [x0(2) zeros(1,evnts-1)];
x3 = [x0(3) zeros(1,evnts-1)];

x = [x1;x2;x3];

V(1) = normpK(x0,K);
for k = 1:evnts
    x0 = x(:,k);

    u(:,k) = MPC_mpl(K,x0);

    x1(k+1) = max(11+x1(k),2+u(k));
    x2(k+1) = max(12+x2(k),u(k))+dis(k);
    x3(k+1) = max([x1(k)+23,x2(k)+23,x3(k)+7,u(k)+14]);

    x = [x1;x2;x3];

    V(k+1) = normpK(x(:,k),K);
    DeltaV(k) = V(k+1) - V(k);
end

%% PLOTS

figure()
hold on
grid on
plot((0:evnts),x(1,:))
plot((0:evnts),x(2,:))
plot((0:evnts),x(3,:))
legend('x_1','x_2','x_3')
title('States - MPC')
```

```
figure()
hold on
plot((0:length(DeltaV)−1),DeltaV)
grid on

figure()
hold on
plot((0:evnts−1),u(1,:))
title('Input')
grid on
```

B-0-3 MPC for MMP system

```
% ---------- MPC MMP ----------
clear all
clc
close all
%% Calculate minimal set
d1 = 1;
d2 = 3;
d3 = 6;
d4 = 4;
d5 = 0;

t1 = 4;
t2 = 1;
t3 = 0;
t4 = 0;
t5 = 0;
t6 = 0;
t7 = 0;
t8 = 1;

%% Optimizations

x = sdpvar(5,1);
u = sdpvar(1,1);

x1 = max(x(1)+d1,u+t1);
x2 = max(x(2)+d2,u+t2);
x3 = max(min(x1+d1,x2+d2),x(3)+d3);
x4 = max(max(x1+d1,x2+d2),x(4)+d4);
x5 = max(x3+d3,x4+d4+t8);

save_sols = [];

xc = [x1;x2;x3;x4;x5];

options = sdpsettings('verbose',2,'solver','Gurobi');
options.gurobi.PoolSearchMode=0;
options.gurobi.PoolSolutions=1000;
options.savesolveroutput = 3;

```

```
n = length(xc);
F = [];

N = 1e6;

for i = 1:n
    for j = 1:n
        if i>=j
            d = 0;
        else
            obj = xc(i)-xc(j);

            delta = sdpvar(1);
            y = binvar(1);

            con = [F,obj+N*y>=delta,-obj+N*(1-y)>=delta,-obj<=delta,-obj<=delta];

            sol1 = optimize(con,delta,options);

            d1_str = sol1.solveroutput.result.pool;
            d1_cell = struct2cell(d1_str);

            sols1 = cell2mat(d1_cell(1,1,:));
            sols1 = reshape(sols1,[],1)';
            d  =min(abs(sols1));

            disp('--------------------------------------------')
        end
        K(i,j) = d;
    end
end

K = -(K + flip(K',3));
%% Update
clearvars -except K

N     = 1;
evnts = 100;
u0    = 0;
x0 = zeros(5,1);
% x0 = [3;27;67;41;97];

d1 = 1;
d2 = 3;
d3 = 6;
d4 = 4;
d5 = 0;

t1 = 4;
t2 = 1;
t3 = 0;
t4 = 0;
```

```
t5 = 0;
t6 = 0;
t7 = 0;
t8 = 1;

x1s(1) = x0(1);
x2s(1) = x0(2);
x3s(1) = x0(3);
x4s(1) = x0(4);
x5s(1) = x0(5);

dis = zeros(evnts,1);
% dis(50) = 100;
% ucheck = zeros(20,100);

for k = 1:evnts
    V(1) = normpK(x0,K);
    x0 = [x1s(k);x2s(k);x3s(k);x4s(k);x5s(k)];

    u_ = MPC_prod_unit_compK(x0,N,K);
    u(k+1) = value(u_(1));

    x1s(k+1) = max(x1s(k)+d1,u(k+1)+t1);
    x2s(k+1) = max(x2s(k)+d2,u(k+1)+t2);
    x3s(k+1) = max(min(x1s(k+1)+d1,x2s(k+1)+d2),x3s(k)+d3)+dis(k);
    x4s(k+1) = max(max(x1s(k+1)+d1,x2s(k+1)+d2),x4s(k)+d4);
    x5s(k+1) = max(x3s(k+1)+d3,x4s(k)+d4+t8);

    x = [x1s(k+1);x2s(k+1);x3s(k+1);x4s(k+1);x5s(k+1)];

    V(k+1) = normpK(x,K);
    DeltaV(k) = V(k+1) - V(k);
end

%% PLOTS

u = value(u);

figure()
% subplot(2,1,1)
hold on
grid on
plot((0:length(x1s)-1),x1s)
plot((0:length(x1s)-1),x2s)
plot((0:length(x1s)-1),x3s)
plot((0:length(x1s)-1),x4s)
plot((0:length(x1s)-1),x5s)
legend('x_1','x_2','x_3','x_4','x_5')
xlabel('k')
ylabel('time unit')

% subplot(2,1,2)
% plot((0:length(dis)-1),dis,'r--')
```

```
% grid on
% title('Disturbance on state x_2')
% xlabel('k')

figure()
hold on
plot((0:length(DeltaV)-1),DeltaV)
grid on

figure()
plot((0:size(u,2)-1),u)
title('Input')
grid on


%% Functions

function C = tplus(A,B)
C = max(A,B);
end

function C = ttimes(A,B)
n = size(A,1);
m = size(B,2);
X = kron(ones(m,1),A);
Y = kron(B',ones(n,1));
C = reshape(max(X+Y,[],2),n,m);
end
```

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Glossary

List of Acronyms

MMP	max-min-plus
MMPS	max-min-min-plus-scaling
MPC	model predicitive control
SMPL	switching-max-plus-linear
DES	discrete-event systems
CLF	control Lyapunov function
MILP	mixed-integer linear program
MPL	max-plus-linear

List of Symbols

ϵ	Max-plus zero element $-\infty$
λ_g	Additive eigenvalue
$\mathbb{R}_\epsilon$	Set of real numbers including ϵ
$\mathbb{R}_\top$	Set of real numbers including $\top$
$\mathbb{R}_c$	Set of real numbers including ϵ and $\top$ (complete)
$\mathbb{R}$	Set of real numbers
$\mathcal{R}$	Either $\mathbb{R}_\epsilon$, $\mathbb{R}_\top$, $\mathbb{R}$ or $\mathbb{R}_c$
$\oplus$	Max-plus addition operator ('oplus')
$\otimes'$	Min-plus addition operator
$\otimes$	Max-plus multiplication operator ('otimes')
$\top$	Min-plus zero element ∞
v_g	Additive eigenvector