Delft University of Technology

Document Version
Final published version

Licence
Dutch Copyright Act (Article 25fa)

Citation (APA)
Chen, Y., & Kraaikamp, C. (2025). Matching of orbits of certain N-expansions with a finite set of digits. *Tohoku Mathematical Journal*, *77*(3), 319-343. https://doi.org/10.2748/tmj.20230802

Tohoku Math. J.
77 (2025), 319–343
DOI: 10.2748/tmj.20230802

MATCHING OF ORBITS OF CERTAIN *N*-EXPANSIONS WITH A FINITE SET OF DIGITS

YUFEI CHEN* AND COR KRAAIKAMP

(Received January 18, 2023, revised August 1, 2023)

Abstract. In this paper we consider a class of continued fraction expansions: the so-called *N-expansions with a finite digit set*, where $N \geq 2$ is an integer. These *N-expansions with a finite digit set* were introduced in [13, 15], and further studied in [10, 23]. For N fixed they are steered by a parameter $\alpha \in (0, \sqrt{N} - 1]$. In [13], for $N = 2$ an explicit interval $[A, B]$ was determined, such that for all $\alpha \in [A, B]$ the entropy $h(T_\alpha)$ of the underlying Gauss-map T_α is equal. In this paper we show that for all $N \in \mathbb{N}$, $N \geq 2$, such plateaux exist. In order to show that the entropy is constant on such plateaux, we obtain the underlying planar natural extension of the maps T_α, the T_α-invariant measure, ergodicity, and we show that for any two α, α' from the same plateau, the natural extensions are metrically isomorphic, and the isomorphism is given explicitly. The plateaux are found by a property called *matching*.

1. Introduction.

It is well known that every real number x can be written as a finite (in case $x \in \mathbb{Q}$) or infinite (regular) continued fraction of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}} = [a_0; a_1, a_2, \dots, a_n, \dots]\,, \tag{1}$$

where $a_0 \in \mathbb{Z}$ such that $x - a_0 \in [0, 1)$, and $a_n \in \mathbb{N}$ for $n \geq 1$. Such a *regular continued fraction expansion* (RCF) of x is unique if and only if x is irrational; in case $x \in \mathbb{Q}$ one has two expansions of the form (1).

Apart from the regular continued fraction expansion algorithm there exist a bewildering number of other continued fraction expansion algorithms. In this paper we consider a recent algorithm, which was introduced by Edward Burger and some of his students in 2008 in [2].

Let $N \in \mathbb{N}$, $N \geq 2$, and define the map $T_N : [0, 1) \to [0, 1)$ by:

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \; x \neq 0; \quad T_N(0) = 0\,. \tag{2}$$

Setting $d_1 = d_1(x) = \lfloor N/x \rfloor$, and $d_n = d_n(x) = d_1\left(T_N^{n-1}(x)\right)$, whenever $T_N^{n-1}(x) \neq 0$, we find:

2020 *Mathematics Subject Classification*. Primary 11J70; Secondary 37E05.

Key words and phrases. Continued fractions, dynamical systems, gaps, matching.

[*] The research of the first author is supported by the *National Natural Science Foundation of China* (Nos. 12071148) and by the *Science and Technology Commission of Shanghai Municipality* (No. 22DZ2229014).

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \ddots + \cfrac{N}{d_n + T_N^n(x)}}} . \tag{3}$$

Taking finite truncations yield the convergents, which converge to x.

Burger *et al.* studied these N-expansions, as they could show that for every quadratic irrational number x there exists infinitely many $N \in \mathbb{N}$ for which the N-expansion of x is ultimately periodic with period length 1. In 2011, Anselm and Weintraub further studied N-expansions in [1]. They showed that every positive real number x always has an N-expansion, and for $N \geq 2$ even infinitely many, and that rationals always have finite and infinite expansions. Furthermore, in case $N \geq 2$ every quadratic irrational has both periodic and non-periodic expansions. In their algorithm to find an N-expansion of a real number x there is a *best choice* for the partial quotient (i.e., digit), and if one always makes this best choice for the partial quotients one finds what they call the *best expansion* of x. One can show that the N-expansions obtained via the Gauss-map T_N from (2) are always best expansions. Note that in [1] N-expansions are not introduced or studied via maps such as defined in (2). This was done in [7], where many properties of N-expansions (such as ergodicity, the form of the invariant measure, entropy) were obtained in a very easy way.

In his MSc-thesis [15] from 2015, and in a subsequent paper with the second author [13], Niels Langeveld considered N-expansions on an interval **not** containing 0. To be more precise: let $N \in \mathbb{N}$, $N \geq 2$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq \sqrt{N} - 1$, then we define $I_\alpha := [\alpha, \alpha + 1]$ and $I_\alpha^- := [\alpha, \alpha + 1)$ and investigate the continued fraction map $T_\alpha : I_\alpha \to I_\alpha^-$, defined as:

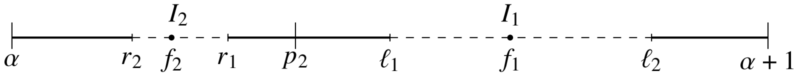
$$T_\alpha(x) := \frac{N}{x} - d(x) , \tag{4}$$

where $d : I_\alpha \to \mathbb{N}$ is defined by $d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor$.

Note that due to the fact that $\alpha > 0$, there are only finitely many values of partial quotients possible. Furthermore, *all* expansions are infinite. This new N-expansion (with a finite digit set) could be viewed as a small variation of the N-expansions with infinitely many digits, but actually the situation is suddenly dramatically different and more difficult as for certain values of N and α "*gaps*" in the interval I_α appear. These 'gaps" are intervals where the invariant density vanishes. Although "gaps" do not play a role in this paper, we briefly want to discuss them, as they are indicative of how different the dynamical systems underlying the maps T_α are from all other continued fraction dynamical systems we know. We mention here an example from [10]: take $N = 51$, $\alpha = 6$. In this case there are only 2 digits (viz. 1 and 2), and setting for $n \geq 0$: $r_n = T_\alpha^n(\alpha + 1)$, $\ell_n = T_\alpha(\alpha)$, and in general for a digit i:

$$f_i = f_i(N) = \frac{\sqrt{4N + i^2} - i}{2} ,$$

as the fixed point of T_α with digit i. Denoting in general the *cylinder* of a digit i by I_i, i.e., $I_i = \{x \in I_\alpha\, ;\, d(x) = i\}$. For the example $N = 51$, $\alpha = 6$ we have cylinders $I_2 = [\alpha, p_2]$ and $I_1 = (p_2, \alpha + 1]$, where $p_2 \in I_\alpha$ is such, that $p_2 \in I_2$ and $T_\alpha(p_2) = \alpha$. In general it follows from the fact that $|T'_\alpha(x)| > 1$ (i.e., the map T_α is expanding) that the orbits of all points $x \in I_2$ (except the fixed point f_2) will eventually leave I_2. Since $[p_2, \ell_1] = T_\alpha(I_2) \setminus I_2$, we see that when leaving I_2 these orbits enter I_1 in the interval $(p_2, \ell_1]$. Now $\ell_1 < f_1$, and $T_\alpha((p_2, \ell_1]) = [\ell_2, r_0)$, so in this example $\ell_1 < f_1 < \ell_2$. Note that $f_1 - \ell_1 < \ell_2 - f_1$ and $r_1 < p_2$, again due to the fact that T_α is expanding. For $N = 51$, $\alpha = 6$ we have that $\ell_3 < \ell_1$. Thus we see that the orbit of any point x from $I_2 \setminus \{f_2\}$ will *never* enter the interval (ℓ_1, ℓ_2). In a similar way we see that the orbit of any point x from $I_1 \setminus \{f_1\}$ will *never* enter the interval (r_2, r_1); two *gaps* are popping up:



In [10] it has been investigated when these “gaps” appear. In [11] it is shown that the number of gaps grows when $N \in \mathbb{N}$ increases. In spite of the gaps, in [10] the following results were obtained.

Since $\inf |T'_\alpha| > 1$, applying Theorem 1 from the classical 1973 paper by Lasota and Yorke (cf. [16], see also [19]) immediately yields the following assertion:

PROPOSITION 1.1. *If μ is an absolutely continuous invariant probability measure for T_α, then there exists a function h of bounded variation such that*

$$\mu(A) = \int_A h \, d\lambda, \ \lambda - a.e., \ \text{with } \lambda \text{ the Lebesgue measure},$$

i.e., any absolutely continuous invariant probability measure has a version of its density function of bounded variation.

We have the following result from [10].

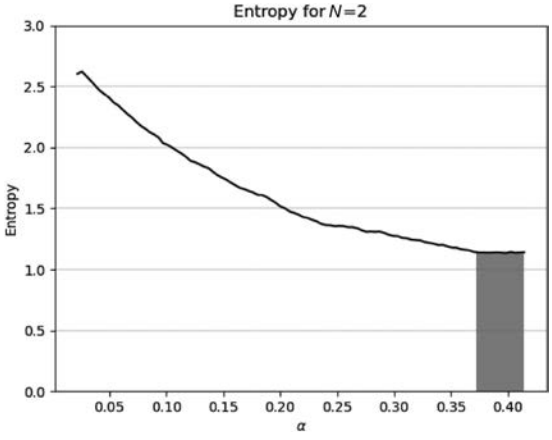


FIGURE 1. A simulation of the entropy of T_α when $N = 2$.

THEOREM 1.2. *Let $N \in \mathbb{N}_{\geq 2}$. Then there is a unique absolutely continuous invariant probability measure μ_α such that T_α is ergodic with respect to μ_α.*

With these results it was shown in [10], that if $|T'_\alpha(x)| > 2$ for all $x \in I_\alpha$, there will be no gaps in I_α.

In this paper we will not focus on gaps, but rather on 'plateaus' whether the entropy is constant. In [13], simulation of the entropy of T_α is given as a function of $\alpha \in (0, \sqrt{2}-1])$ in case $N = 2$; see Figure 1.

In Figure 1 the shadowed vertical stripe represents a 'plateau' from $\frac{\sqrt{33}-5}{2}$ to $\sqrt{2}-1$ where the entropy as a function of α is constant (which is the maximal possible value for α in case $N = 2$; for larger values of α some of the digits could be equal to 0). In [13] it was then showed that for these values of α the so-called natural extensions could be built using a technique called *quilting* (this technique will be explained in Section 3), and it could be shown that for $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$ these natural extensions are metrically isomorphic. In general, a *natural extension* is an almost surely minimal invertible system which has the original system (in this case $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$) as a factor. For continued fractions, the natural extension is (isomorphic to) some planar domain Ω_α, with an almost surely invertible map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$, given in the particular case of N-expansions by

$$\mathcal{T}_\alpha(x, y) = \left(T_\alpha(x), \frac{N}{d(x) + y}\right), \tag{5}$$

where $d(x) \in \mathbb{N}$ is such, that $T_\alpha(x) = \frac{N}{x} - d(x) \in I_\alpha = [\alpha, \alpha+1)$ (cf. (4)). In [13] the following result was obtained.

THEOREM 1.3. *If $N = 2$, for $\alpha \in \left[\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right]$ the natural extension can be build (see Figure 2 below). Moreover, the invariant density f_α is given by*:

$$\begin{aligned} f_\alpha(x) = H\Big(&\frac{D}{2+Dx}\mathbf{1}_{(\alpha, T(\alpha+1))} + \frac{E}{2+Ex}\mathbf{1}_{(T(\alpha+1), T^2(\alpha))} + \frac{F}{2+Fx}\mathbf{1}_{(T^2(\alpha), \alpha+1)} \\ &- \frac{A}{2+Ax}\mathbf{1}_{(\alpha, T^2(\alpha+1))} - \frac{B}{2+Bx}\mathbf{1}_{(T^2(\alpha+1), T(\alpha))} - \frac{C}{2+Cx}\mathbf{1}_{(T(\alpha), \alpha+1)}\Big), \end{aligned}$$

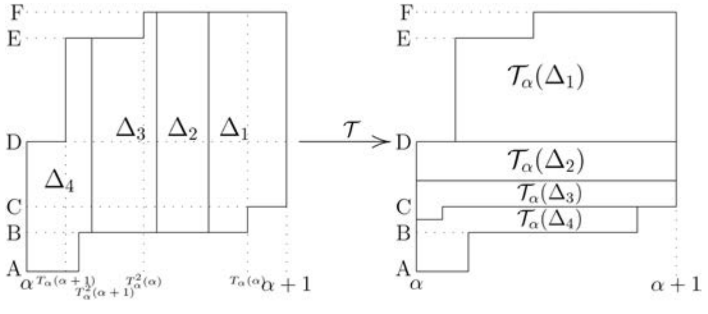


FIGURE 2. A planar natural extension in case $N = 2$ for $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$.

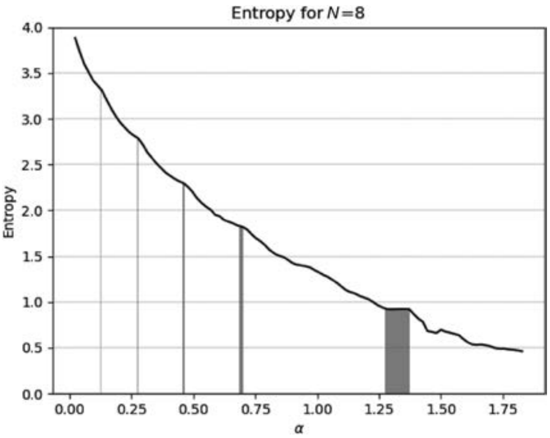


FIGURE 3. A simulation of the entropy of T_α when $N = 8$.

where $A = \frac{\sqrt{33}-5}{2}, B = \sqrt{2}-1, C = \frac{\sqrt{33}-3}{6}, D = 2\sqrt{2}-2, E = \frac{\sqrt{33}-3}{2}, F = \sqrt{2}$ *and* $H^{-1} = \log\left(\frac{1}{32}(3+2\sqrt{2})(7+\sqrt{33})(\sqrt{33}-5)^2\right) \approx 0.25$ *the normalizing constant.*

In this paper we will show, that for every integer $N \geq 2$ such plateaux exist, and give them explicitly. For example, five plateaux are found when $N = 8$; see also Figure 3. The number of such plateaux will be a function of N.

At this point we would like to thank the anonymous referee of our paper. Due to the many remarks and text suggestions of the referee the quality of the presentation of this paper has improved tremendously.

2. A set of α for which there is matching for T_α in 3 steps for every $N \in \mathbb{N}, N \geq 2$, and a related planar domain for $\mathcal{T}_\alpha$. Let $N \in \mathbb{N}$, $N \geq 2$ arbitrary but fixed and let[1] $0 \leq \alpha \leq \sqrt{N}-1$. Let $T_\alpha : [\alpha, \alpha+1] \to [\alpha, \alpha+1)$ be the Gauss map defined as in (4), and let d_{min} and d_{max} be the smallest resp. largest possible digit of T_α. Then, we take $d = d_{min}$ and $i = d_{max} - d_{min}$, it follows that $\{d, d+1, \ldots, d+i\}$ is the (finite) set of partial quotients (i.e., digits) for T_α (in [10] it was shown that $i \in \mathbb{N}, i \geq 1$, so the number of partial quotients $i+1$ is at least 2).

Here we briefly explain why α is limited as $\alpha \leq \sqrt{N}-1$ on T_α: Since $T_\alpha : [\alpha, \alpha+1) \to [\alpha, \alpha+1)$, we have $\frac{N}{\alpha+1} - d(k) \geq \alpha$, where $d(k) \in \{d, d+1, \ldots, d+i\}$. Then $N \geq (\alpha+1)(\alpha+d(k))$. From $d(k) \geq d \geq 1$, it has $N \geq (\alpha+1)^2$, so $\alpha \leq \sqrt{N}-1$.

Consider the partition $\mathcal{P} = \bigcup I_k$ of $[\alpha, \alpha+1]$, where $I_k = \{x \mid d_1(x) = k\}$. Note that we choose the largest digit $d+i$ always in such a way, that each partition element is an interval (see also [10] for a small discussion about this).

In [10] it was mentioned (cf. Lemma 1 in [10]), that given N and α, and d_{max} as the largest possible digit of T_α, one has

---

[1] Usually we suppress in our notation the dependence of the various maps and domains on N.

$$d_{max} \geq N - 1 \text{ if and only if } \alpha < 1\,.$$

We have the following, similar result on the smallest possible digit $d = d_{min}$ of T_α.

LEMMA 2.1. *Let $N \in \mathbb{N}$, $N \geq 2$, and $0 < \alpha \leq \sqrt{N} - 1$, then for the smallest possible digit d of T_α we have that $d \in \{1, 2, \ldots, N-1\}$ and $\lim_{\alpha \downarrow 0} d = N - 1$.*

PROOF. From $\alpha < N/(\alpha+1) - d$, it follows that $\alpha^2 + (d+1)\alpha + d - N < 0$. Since $\alpha, d > 0$ it follows that $d < N$. Furthermore, if α tends to 0 it follows that d tends to $N - 1$. Note that if $\alpha = 0$, we have that $d = N$ (cf. [7]). □

The following result gives bounds on the number $i + 1$ of possible digits.

LEMMA 2.2. *For all $N \in \mathbb{N}$, $N \geq 2$, and $0 < \alpha \leq \sqrt{N} - 1$, $d \geq 1$, one has $\frac{d}{\alpha} \leq i < \frac{d+1}{\alpha} + 2$, where $i + 1$ is the number of possible digits. Furthermore, $\lim_{\alpha \downarrow 0} i = +\infty$.*

PROOF. (i) Since T_α is a map from $[\alpha, \alpha+1]$ to $[\alpha, \alpha+1)$ we have that $\alpha \leq N/(\alpha+1) - d$ (which is the same as saying that $T_\alpha(\alpha+1) \geq \alpha$), it follows that $(\alpha+1)(\alpha+d) \leq N$; and from $N/\alpha - (d+i) \leq \alpha + 1$, one trivially has that $N/\alpha - (\alpha+1) \leq d + i$. Then $(\alpha+1)(\alpha+d)/\alpha - (\alpha+1) \leq d+i$, and one has that $\alpha + (d+1) + d/\alpha - \alpha - 1 \leq d + i$, yielding that $i \geq d/\alpha$.

(ii) Again since T_α is a map from $[\alpha, \alpha+1]$ to $[\alpha, \alpha+1)$ we have that $N/(\alpha+1) - d < \alpha+1$; one immediately sees that $N < (\alpha+1)(\alpha+d+1)$; from $\alpha \leq N/\alpha - (d+i)$, it immediately follows that $(d+i) \leq N/\alpha - \alpha$. Combining this yields that $d + i < (\alpha+1)(\alpha+d+1)/\alpha - \alpha = d + 2 + (d+1)/\alpha$; we find that $i < (d+1)/\alpha + 2$. □

Now define $\mathcal{A}_{N,d,i}$ be the set of all $\alpha \in (0, \sqrt{N} - 1]$ with digit set $\{d, d+1, \ldots, d+i\}$. For any set A let A^o denote the interior part of A. We define the sets $X_{N,d,i}$ and $X_{N,d,i,k}$ as follows:

$$X_{N,d,i} = \{\alpha \in \mathcal{A}_{N,d,i} \,\big|\, T_\alpha(\alpha) \in I_d^o, T_\alpha(\alpha+1) \in I_{d+i}^o\}\,; \tag{6}$$

$$X_{N,d,i,k} = \{\alpha \in X_{N,d,i} \,\big|\, T_\alpha^2(\alpha) \in I_k, T_\alpha^2(\alpha+1) \in I_{k+1}\}, \quad \text{for } k = d, \ldots, d+i-1\,. \tag{7}$$

Due to the fact that $|T_\alpha'(x)| > 1$ for $x \in [\alpha, \alpha+1)$ we have that

$$X_{N,d,i} = \left\{\alpha \in \mathcal{A}_{N,d,i} \;\Big|\; \frac{N}{d+1+\alpha} < \frac{N}{\alpha} - (d+i) < \alpha + 1,\;\; \alpha < \frac{N}{\alpha+1} - d < \frac{N}{d+i+\alpha}\right\}.$$

In the next theorem we show that for $N \in \mathbb{N}$, $N \geq 2$, for which there exist positive integers d and i such that $N = \frac{d(d+i)}{i-1}$, for $\alpha \in X_{N,d,i}$ the corresponding maps T_α *synchronize* in 3 steps; $T_\alpha^3(\alpha) = T_\alpha^3(\alpha+1)$. In several recent papers this property is called *matching*. This property is key for us, as it helps us to construct the *natural extensions* of the dynamical systems $([\alpha, \alpha+1), T_\alpha)$, but also to understand why for such values of α the entropy is constant. At first it might not be clear that for every integer $N \geq 2$ positive integers i and d exist for which $N = \frac{d(d+i)}{i-1}$; this will be investigated in Proposition 2.3.

Proposition 2.3. *Let $N \geq 2$ be an integer, and let $D(N)$ be the number of pairs of integers (d, i) with $d \geq 1$, $i \geq 2$, and $N = \frac{d(d+i)}{i-1}$. Then we have $D(2) = 1$, $D(3) = D(4) = 2$, and for all $N \geq 5$, one has that $3 \leq D(N) \leq M(N)$ with $M(N) = (\sigma_0(N) - 1)(\sigma_0(N+1) - 1)$. Here $\sigma_0 : \mathbb{N} \to \mathbb{N}$ is the divisor function of $n \in \mathbb{N}$, defined by:*

$$\sigma_0(n) = \sum_{d|n} d^0, \quad \textit{for } n \in \mathbb{N}\,. \tag{8}$$

Proof. Note that, setting $k = N - d$, the equation $N = \frac{d(d+i)}{i-1}$ can be rewritten as

$$i = \frac{d(d+1)}{k} + 1\,. \tag{9}$$

We are interested in the integer solutions of (9) and, by Lemma 2.1, we have to restrict to the cases $d, k \in \{1, 2, \ldots, N-1\}$.

Let alone the case $N = 2$ (for which $i = 3$, $k = 1 = d$ is the unique admissible solution), for $N \geq 3$ the values $k \in \{1, 2, \lfloor \frac{N+1}{2} \rfloor\}$ give rise to multiple admissible solutions which actually turn out to be 3 distinct values for $N \geq 5$, but reduce to 2 values for $N \in \{3, 4\}$ as one can check from the table below (the last two columns refer to a solution of (9) in case N is even or odd, respectively).

k	1	2	$\frac{N}{2}$	$\frac{N+1}{2}$
d	$N-1$	$N-2$	$\frac{N}{2}$	$\frac{N-1}{2}$
i	$d(d+1)+1$	$\frac{d(d+1)}{2}+1$	$\frac{N+4}{2}$	$\frac{N+1}{2}$

If we substitute $d = N - k$ in (9), we trivially find that:

$$i = \frac{N(N+1)}{k} - 2N + k\,. \tag{10}$$

Now for $N \in \mathbb{N}$ we have that N and $N+1$ are relative prime, so we have that $\sigma_0(N(N+1)) = \sigma_0(N)\sigma_0(N+1)$ as σ_0 is an arithmetic function. Note that k cannot be N nor $N+1$ (in the first case we would have that $d = 0$, and in the second case even $d = -1$; these are both impossible since digits d are at least 1), so we find from (10) and the fact that $i \in \mathbb{N}$, $i \geq 2$, that $D(N)$ is at most the number of divisors $k \in \{1, 2, \ldots, N-1\}$ of $N(N+1)$ for which i from (10) is an integer at least 2. □

Remarks 2.4. (i) For every real or complex x one can define the *sum of positive divisors function* σ_x as

$$\sigma_x(n) = \sum_{d|n} d^x, \quad \text{for } n \in \mathbb{N}\,.$$

In this paper we are only interested in $x = 0$, but e.g. $x = 1$ yields the sum of all positive divisors of n, and $s(n) = \sigma_1(n) - 1$ is the so-called *aliquot sum*, i.e., the sum of all proper divisors of $n \in \mathbb{N}$. Obviously we have $\sigma_0(p) = 2$ for all prime numbers p, and therefore

FIGURE 4. $D(N)$ for $N = 2, \dots, 200$ (left) and $N = 2, \dots, 10.000$ (right).

$\liminf_{n\to\infty} \sigma_0(n) = 2$. On the other hand, it was shown by Severin Wigert (cf. [8], pp. 342–347, Section 18.1) that

$$\limsup_{n\to\infty} \frac{\log \sigma_0(n)}{\log n / \log\log n} = \log 2\,.$$

(ii) Clearly $D(N) = 3$ if $N \geq 5$ and $(N+1)/2$ are prime, or if $N/2$ and $N+1$ are prime (e.g. for $N = 5, 6, 7, 10, 22, 37, 58, 61, 73, 82, 157, \dots, 613, \dots$). The right-hand side figure in Figure 4 seems to suggest that $\liminf\limits_{n\to\infty} D(n) = 3$, but we have no proof of this.

In general $D(N)$ is smaller than the maximum possible value $M(N) = (\sigma_0(N) - 1)(\sigma_0(N+1) - 1)$. The smallest N for which this happens is $N = 8$; one easily sees that $M(8) = 6$, while $D(8) = 5$ The reason is, that $4|8$ and $3|9$, so $k = 4 \times 3 = 12 | N(N+1) = 72$, but $k = 12 \geq N = 8$, and therefore we cannot find an admissible digit d (since $d = N - k = 8 - 12 = -4$, and we must have $d \geq 1$). See also Figure 4, where we display $D(N)$ in the left-hand figure for $N = 2, \dots, 200$, and in the right-hand figure for $N = 2, \dots, 10.000$. △

THEOREM 2.5. *Let $N \geq 2$ be an integer, and let $d, i \in \mathbb{N}$, $i \geq 2$, be such, that $N = \frac{d(d+i)}{i-1}$. Then for any $\alpha \in X_{N,d,i}$, one has that $T_\alpha^2(\alpha) \in I_k$ and $T_\alpha^2(\alpha+1) \in I_{k+1}$ for some $k \in \{d, \dots, d+i-1\}$. Moreover, $T_\alpha^3(\alpha) = T_\alpha^3(\alpha+1)$.*

PROOF. By definition of $X_{N,d,i}$ and T_α, one has for $\alpha \in X_{N,d,i}$ that $T_\alpha^2(\alpha) = \frac{N}{\frac{N}{\alpha} - (d+i)} - d$, and that $T_\alpha^2(\alpha+1) = \frac{N}{\frac{N}{\alpha+1} - d} - (d+i)$. Then,

$$\frac{N}{T_\alpha^2(\alpha)} = \frac{N}{\frac{N}{\frac{N}{\alpha}-(d+i)} - d} = -\frac{N(N-(d+i)\alpha)}{Nd - (d^2 + di + N)\alpha}\,,$$

$$\frac{N}{T_\alpha^2(\alpha+1)} = \frac{N}{\frac{N}{\frac{N}{\alpha+1}-d} - (d+i)} = -\frac{N(N - d(\alpha+1))}{(d+i-\alpha-1)N - d(d+i)(\alpha+1))}\,,$$

and using CAS (i.e., a *computer algebra system*) one easily finds that:

$$\frac{N}{T^2(\alpha)} - \left(\frac{N}{T^2(\alpha+1)} - 1\right) = \left(d^2 + di - N(i-1)\right) \cdot R_{N,d,i,\alpha}\,,$$

where $R_{N,d,i,\alpha}$ satisfies:

$$R_{N,d,i,\alpha} = \frac{\left((\alpha^2+\alpha)d^2 + ((-2\alpha-1)N + di\alpha(\alpha+1)) + (N - \alpha(i-\alpha-1))N\right)}{(-d^2\alpha + (-i\alpha+N)d - N\alpha)((-\alpha-1)d^2 + (-i\alpha+N-i)d + N(i-\alpha-1))}\,.$$

Note that if $d^2 + di - N(i-1) = 0$, so if $N = \frac{d(d+i)}{i-1}$, we have that:

$$\frac{N}{T_\alpha^2(\alpha)} = \frac{N}{T_\alpha^2(\alpha+1)} - 1 .$$

Since the length of the interval $[\alpha, \alpha+1)$ is 1, we see that for $N = \frac{d(d+i)}{i-1}$ we have *matching* in 3 steps: $T_\alpha^3(\alpha) = T_\alpha^3(\alpha+1)$. Furthermore, $T_\alpha^2(\alpha) \in I_k$ and $T_\alpha^2(\alpha+1) \in I_{k+1}$ for some $k \in \{d, \dots, d+i-1\}$. □

REMARKS 2.6. (i) For the case $N = 2$ the result of Theorem 2.5 were already obtained in Theorem 3.1 of [13].

(ii) Note that under the conditions of Theorem 2.5, an immediate consequence of the proof of Theorem 2.5 is that

$$X_{N,d,i} = \bigcup_{k=d}^{d+i-1} X_{N,d,i,k} .$$

(iii) The conditions of Theorem 2.5 which lead to matching in 3 steps were obtained using an extensive search using CAS. We did not find other relations, but that obviously does not imply these do not exist; see for a brief discussion Example 4.3.

(iv) In the definition (6) of $X_{N,d,i}$ we demanded that $T_\alpha(\alpha) \in I_d^o$, and that $T_\alpha(\alpha+1) \in I_{d+i}^o$; the reason is, that we must avoid the endpoints of the cylinders I_d and I_{d+i} in order to be able to draw the conclusions of Theorem 2.5. For example, if $T_\alpha(\alpha) = \alpha + 1$ and $T_\alpha(\alpha+1) = \alpha$, clearly all branches of T_α are full and there will be no matching in 3 steps (or any number of steps), but it is very easy to construct the natural extension; see Theorem 1 in [10, 23] where it is explicitly stated for which N and α one has that T_α only has full branches, and also [23] where the density of T_α (and of its dual map) is given. Although not explicitly stated, the driving idea behind these calculations is the concept of *planar natural extension*. Another example is, when N, d, i and α are such, that

$$T_\alpha(\alpha) = \frac{N}{\alpha+1+d}, \quad \text{or } T_\alpha(\alpha+1) = \alpha .$$

Here $N/(\alpha+1+d)$ is the left endpoint of the cylinder I_d and the right endpoint of the cylinder I_{d+1}. Note that by definition of the map T_α we have that $N/(\alpha+1+d) \in I_{d+1}$. So formally, $N/(\alpha+1+d) \notin I_d$, so certainly $N/(\alpha+1+d) \notin I_d^o$, and therefore $\alpha \notin X_{N,d,i}$ (so clearly the statement of Theorem 2.5 does not apply to this α. Indeed we don't have matching in 3 steps: $T_\alpha(\alpha) = N/(\alpha+1+d)$, and therefore $T_\alpha^2(\alpha) = \alpha = T_\alpha(\alpha+1)$. Still, this is an interesting case, as it is very easy to construct the planar natural extension, and once this has been obtained, to find the $\mathcal{T}_\alpha$-invariant measure, and by projecting the invariant measure of the dual algorithm. As the proof of Theorem 2.7 is similar to this construction, we decided to skip it here, but discuss this case briefly in Section 4. Note however, that for this example the dual algorithm exist, while in general (under the conditions of Theorem 2.5) there is no dual algorithm; see also [22], p. 58. △

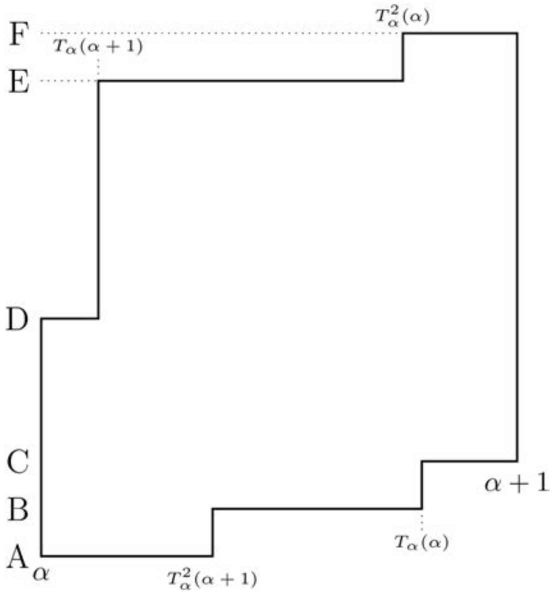


FIGURE 5. Ω_α.

Under the assumptions of Theorem 2.5 and using the matching in 3 steps guaranteed by Theorem 2.5, we will show how we can find the planar domains Ω_α of the natural extensions of the dynamical systems (I_α, T_α) for $\alpha \in X_{N,d,i}$. Recall the definition of the map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ from (5):

$$\mathcal{T}_\alpha(x,y) = \left(\frac{N}{x} - d(x), \frac{N}{d(x)+y}\right),$$

where $x \in I_\alpha$, $d(x) \in \{d, d+1, \ldots, d+i\}$ and N, d, i positive integers, with $N, i \geq 2$.

THEOREM 2.7. *Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha \in X_{N,d,i}$ arbitrary, and let the planar domain Ω_α be the polygon, bounded by the straight line segments between the vertices (in counterclockwise order) (α, A), $(T_\alpha^2(\alpha+1), A)$, $(T_\alpha^2(\alpha+1), B)$, $(T_\alpha(\alpha), B)$, $(T_\alpha(\alpha), C)$, $(\alpha+1, C)$, $(\alpha+1, F)$, $(T_\alpha^2(\alpha), F)$, $(T_\alpha^2(\alpha), E)$, $(T_\alpha(\alpha+1), E)$, $(T_\alpha(\alpha+1), D)$, (α, D), and finally 'back' to (α, A) (see Figure 5), where $0 < A < B < C < D < E < F$.*

Then if the map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ is bijective almost surely with respect to Lebesgue measure λ we have that

(11)
$$A = E - 1 = \frac{-(d+i+1) + \sqrt{(d+i+1)^2 + 4N}}{2}, \quad B = F - 1 = \frac{-(d+1) + \sqrt{(d-1)^2 + 4N}}{2},$$

and

$$C = \frac{N(-(d+i-1) + \sqrt{(d+i+1)^2 + 4N})}{2(d+i+N)}, \quad D = \frac{N(-(d+1) + \sqrt{(d-1)^2 + 4N})}{2(N-d)}, \tag{12}$$

and indeed we have that $0 < A < B < C < D < E < F$.

PROOF. As was the case in [13] for $N = 2$, we need to consider several cases, depending on the value of $k \in \{d, d+1, \ldots, d+i-1\}$ for which $\alpha \in X_{N,d,i,k}$. Since all these case are proved in a similar way, we only consider the case $k = d$ here; the other cases are left to the reader.

If $\alpha \in X_{N,d,i,k}$, we have by definition (7) of $X_{N,d,i,k}$ that

$$T_\alpha^2(\alpha) \in I_d, \quad T_\alpha^2(\alpha+1) \in I_{d+1}\,,$$

(and since $\alpha \in X_{N,d,i}$ we also have (by definition (6)) that $T_\alpha(\alpha) \in I_d$ and $T_\alpha(\alpha+1) \in I_{d+i}$), and there is matching, as $T_\alpha^3(\alpha) = T_\alpha^3(\alpha+1)$. We now will show that the polygon Ω_α satisfies the various values of A, B, et cetera, as mentioned in (11) and (12).

Define the two-dimensional 'cylinders' Δ_j as:

$$\Delta_j := \{(x,y) \in \Omega_\alpha \,|\, x \in I_j\}, \quad \text{for } j = d, d+1, \ldots, d+i\,, \tag{13}$$

and recall from [13] that we want the images of the various Δ_j under $\mathcal{T}_\alpha$ to '*laminate*'; there should not be horizontal 'gaps' between the images, as these will lead to infinitely many of such horizontal 'gaps' (see [13] for more details). So we must choose A, B, et cetera in such a way, that for $j = d, d+1, \ldots, d+i$ the polygon $\mathcal{T}_\alpha(\Delta_j)$ is mapped "seamlessly on top" of the polygon $\mathcal{T}_\alpha(\Delta_{j+1})$; see also Figures 6 and 7, and also [13] for more details. By definition of the second coordinate of the map $\mathcal{T}_\alpha$ this occurs when:

$$A = \frac{N}{d+i+E}, \quad B = \frac{N}{d+i+D}, \quad C = \frac{N}{d+i+A}, \quad C = \frac{N}{d+i-1+E}\,, \tag{14}$$

$$D = \frac{N}{d+1+B}, \quad D = \frac{N}{d+F}, \quad E = \frac{N}{d+C}, \quad F = \frac{N}{d+B}\,. \tag{15}$$

Finally, from Figures 6 and 7 we see we also need to have that

$$M = \frac{N}{k+E} = \frac{N}{k+1+A}\,,$$

which is equivalent with $E = A + 1$; we will see below that this also follows from (14). Note that due to the fact there is matching the set $\mathcal{T}_\alpha(\Delta_k)$ has a 'snug fit' on top of the set $\mathcal{T}_\alpha(\Delta_{k+1})$.

From (14) resp. (15) we see that:

$$\frac{N}{d+i+A} = \frac{N}{d+i-1+E}, \quad \text{resp.} \quad \frac{N}{d+1+B} = \frac{N}{d+F}\,,$$

and it follows that $1 + A = E$ resp. $1 + B = F$. But then we immediately have from (14) resp. (15) that

$$E - 1 = \frac{N}{d+i+E}, \qquad \text{and that } F = \frac{N}{d+F-1}\,. \tag{16}$$

From the first equation in (16) we find a quadratic equation with determinant $(d+i+1)^2+4N > 0$ and one positive root E:

$$E = \frac{-(d+i-1) + \sqrt{(d+i+1)^2 + 4N}}{2}\,,$$

hence A is also known (and positive). The second equation in (16) yields F (and therefore

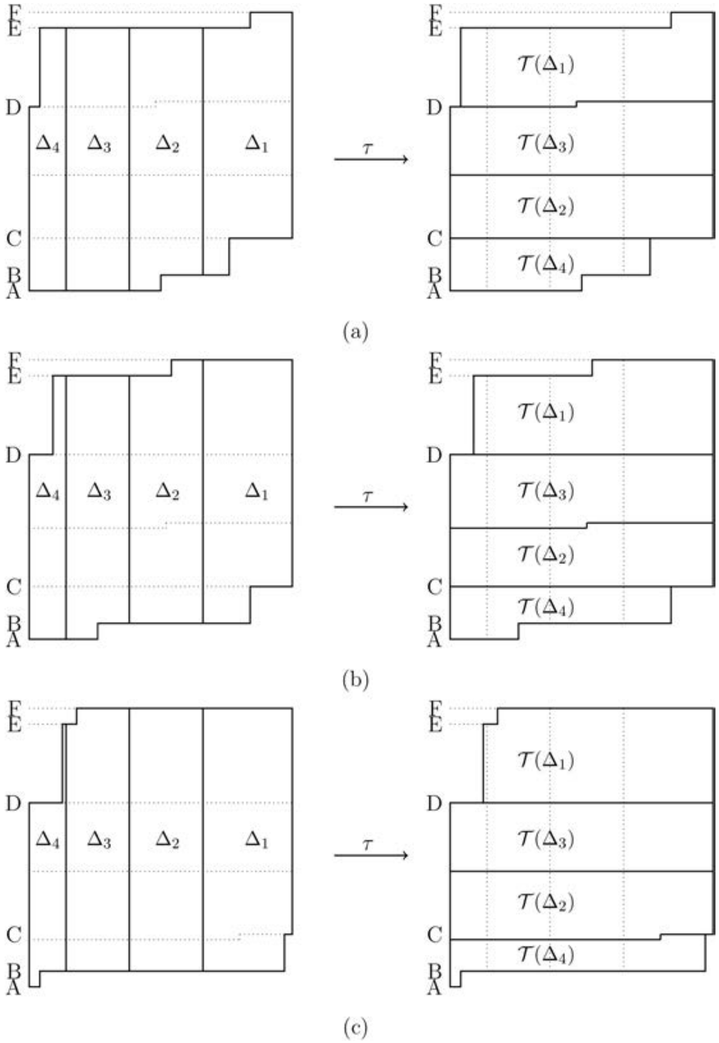
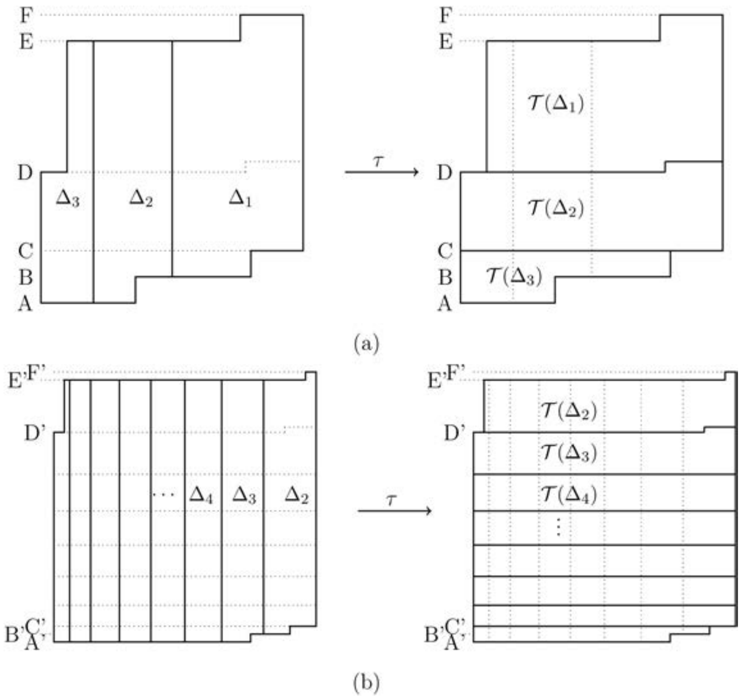


FIGURE 6. Ω_α and $T(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$; (b): $\alpha \in X_{N,d,i,2}$; (c): $\alpha \in X_{N,d,i,3}$, for $N = 2, d = 1, i = 3$.

also B) in a similar way:

$$F = \frac{-(d-1) + \sqrt{(d-1)^2 + 4N}}{2} > 0\,.$$

Since we know A (and E), from (14) also C immediately follows. Similarly, D immediately follows from (15) since B (and F) are known. Due to Lemma 2.1 we know that $d < N$, and therefore $(d-1)^2+4N > (d-1)^2+4d = (d+1)^2$, and we see that $D > 0$. We still need to show that $A < B < C < D < E < F$. This can be proved by contradiction. Suppose e.g. that $A \geq B$.



(a)

(b)

FIGURE 7. Ω_α and $\mathcal{T}_\alpha(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$, $d = 1, i = 2$; (b): $\alpha \in X_{N,d,i,1}$, $d = 2, i = 7$ for $N = 3$.

From (14) we immediately see that this is equivalent with $D \geq E$. From (15) we see that $D \geq E$ is equivalent with $C \geq F$, and from (14) and (15) it follows that $C \geq F$ is equivalent with $B \geq A + i$. Since $i \geq 2$, this last inequality immediately leads to $B \geq A + i \geq B + i > B$, which is impossible. So we must have that $A < B$ (and immediately that $D < E$). The other inequalities can be obtained in a similar way by contradiction. For example, from (14) and (15) we see that $B \geq C \Leftrightarrow A \geq D \Leftrightarrow F \geq E + i$. But also we have that $E \geq F \Leftrightarrow B \geq C$, so assuming $B \geq C$ we find that $F \geq E + i$ and that $E \geq F$, and again due to the fact that $i \geq 2$ we find $F \geq E + i \geq F + i > F$, which is impossible. The other two inequalities are left to the reader.

As $\mathcal{T}_\alpha : \Delta_j \to \mathcal{T}_\alpha(\Delta_j)$ is bijective for $j = d, d + 1, \ldots, d + i$, it now follows that $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ is bijective almost surely. □

REMARKS 2.8. Note that from (14) and (15) one finds that:

$$B = \frac{N}{d + i + \dfrac{N}{d + 1 + B}}, \tag{17}$$

from which we find that

$$B = \frac{-(d + 1)(d + i) + \sqrt{(d + 1)^2(d + i)^2 + N(d + 1)(d + i)}}{2(d + i)}. \tag{18}$$

It is not immediately apparent why this last expression (18) for B is equal to the one in (11). However, it is an easy exercise to see that they are equal if and only if $N = \frac{d(d+i)}{i-1}$. Although expression (18) for B is less attractive than the one from (11), it comes in handy in Remark 3.2(i). △

EXAMPLE 2.9. In Figures 2 and 6 the cases $\alpha \in X_{N,d,i,k}$, for $k = 1, 2, 3$, are illustrated when $N = 2, d = 1, i = 3$. Note that in this case we have: $A = \frac{\sqrt{33}-5}{2} = E-1, B = \sqrt{2}-1 = F-1$, $C = \frac{\sqrt{33}-3}{6}$ and $D = 2\sqrt{2} - 2$, which were also obtained on p. 121 in [13]. Figure 7 illustrates the cases $\alpha \in X_{d,i,k}$, for $k = 1, 2, 3$, when $N = 3$, $d = 1, i = 2$, and $d = 2, i = 7$. △

THEOREM 2.10. *Let $N \geq 2$ be an integer, and let $d, i \in \mathbb{N}$, $i \geq 2$, such that $N = \frac{d(d+1)}{i-1}$. Then $X_{N,d,i} = (A, B)$, where A and B are from (11).*

PROOF. In the proof of Theorem 2.7 we saw in (14) that $A = \frac{N}{d+i+E}$, and we derived that $E = 1 + A$; these combined yield that:

$$\frac{N}{A} - (d + i) = 1 + A, \quad \text{i.e., } T_A(A) = 1 + A\,. \tag{19}$$

Furthermore, in the proof of Theorem 2.7 we saw in (14) that $E = \frac{N}{d+C}$, and from (14) we derived that $C = \frac{N}{d+i+A}$ and $E = 1 + A$. These yield that

$$\frac{N}{A+1} - d = \frac{N}{A+d+i}\,, \tag{20}$$

i.e., $T_A(A + 1)$ is equal to the right endpoint of the cylinder I_{d+i}. From (19) and (20) we see that A is an endpoint of $X_{N,d,i}$, which (by definition (6) of $X_{N,d,i}$) does not belong to $X_{N,d,i}$.

Also we saw in (14) that $B = \frac{N}{d+i+D}$, and we obtained that $D = \frac{N}{d+1+B}$, yielding that

$$\frac{N}{B+d+1} = \frac{N}{B} - (d+i), \quad \text{i.e., } T_B(B) \text{ is the left endpoint of the cylinder } I_d\,. \tag{21}$$

In (15) we also saw that $F = \frac{N}{d+B}$, and we found that $F = 1 + B$. From these we see that

$$B = \frac{N}{B+1} - d, \quad \text{i.e., } T_B(B) = 1 + B\,. \tag{22}$$

Now (21) and (22) yield that B is an endpoint of $X_{N,d,i}$, which (again by definition (6) of $X_{N,d,i}$) does not belong to $X_{N,d,i}$.

Moreover, from (22) one obviously has $(B + d)(B + 1) = N$. From the fact that $d \geq 1$ it then follows that $B \leq \sqrt{N} - 1$.

Since $A < B$, we find that $(A, B) = X_{N,d,i}$. □

THEOREM 2.11. *Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha \in \overline{X}_{N,d,i}$ arbitrary, and let the planar domain Ω_α be the polygon, as given in the statement of Theorem 2.7, where the values of A, B, C, D and E are also given in Theorem 2.7.*

Consider the probability measure $\bar{\mu}_\alpha$ on Ω_α, with density d_α given by

$$d_\alpha(x, y) = H \cdot \frac{N}{(N + xy)^2} 1_{\Omega_\alpha}(x, y)\,,$$

where

$$(23)\qquad H^{-1} = 2\log A + 2\log(B+1) - \log\big(N-(A+1)d\big) - \log\big(N-(d+i)B\big)\,,$$

is the normalising constant of $\bar{\mu}_\alpha$. *Then one easily sees that* $\bar{\mu}_\alpha$ *is* $\mathcal{T}_\alpha$*-invariant. Let* $\bar{\mathcal{B}}_\alpha$ *be the collection of Borel sets of* Ω_α. *Then the dynamical system* $(\Omega_\alpha, \bar{\mathcal{B}}_\alpha, \bar{\mu}, \mathcal{T}_\alpha)$ *is ergodic. It is also the natural extension of the ergodic system* $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$, *where* $\mathcal{B}_\alpha$ *is the collection of Borel sets of* I_α *and* μ_α *is the projection of* $\bar{\mu}_\alpha$ *on the first coordinate (i.e., on* I_α*).*

Furthermore, the density $f_\alpha(x)$ *of the* T_α*-invariant measure* μ_α *is given by*

$$\begin{aligned} f_\alpha(x) = H\Big(&\frac{D}{N+Dx}\mathbf{1}_{(\alpha,T(\alpha)+1)}(x) + \frac{E}{N+Ex}\mathbf{1}_{(T(\alpha)+1,T^2(\alpha))}(x) + \frac{F}{N+Fx}\mathbf{1}_{(T^2(\alpha),\alpha+1)}(x) \\ &-\frac{A}{N+Ax}\mathbf{1}_{(\alpha,T^2(\alpha)+1)}(x) - \frac{B}{N+Bx}\mathbf{1}_{(T^2(\alpha)+1,T(\alpha))}(x) - \frac{C}{N+Cx}\mathbf{1}_{(T(\alpha),\alpha+1)}(x)\Big)\,, \end{aligned}$$

where H *is given by (*23*).*

Proof. The proof of this Theorem is nowadays largely routine; that $\bar{\mu}$ is a $\mathcal{T}_\alpha$-invariant probability measure is a Jacobian calculation (c.f. p. 3189 of [7] and pp. 90 and 136 of [5]): for $j \in \{d,\dots,d+i\}$, let $(x,y) \in \Delta_j$, where Δ_j is defined as in (13). We already saw that $\Delta_k \cap \Delta_\ell = \emptyset$ as $k,\ell \in \{d,\dots,d+i\}$ and $k \neq \ell$. Furthermore, apart from a set of measure zero we have that $\Omega_\alpha = \bigcup_{j=d}^{d+i} \Delta_j = \bigcup_{j=d}^{d+i} \mathcal{T}_\alpha(\Delta_j)$. Now let $R \subset \Delta_j$, for some $j \in \{d,\dots,d+i\}$, and set $S = \mathcal{T}_\alpha(R)$. Setting

$$u = \frac{N}{x} - j,\ v = \frac{N}{j+y},\quad \text{we have that}\ \ x = \frac{N}{u+j},\ y = \frac{N}{v} - j\,,$$

from which we find that the Jacobian of $\mathcal{T}_\alpha$ is given by:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{N^2}{v^2(n+j)^2}\,.$$

So we see that $\bar{\mu}$ is $\mathcal{T}_\alpha$-invariant, as

$$\iint_R \frac{N}{(N+xy)^2}\,\mathrm{d}x\mathrm{d}y = \iint_S \frac{N}{(N+x(u,v)y(u,v))^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v = \iint_S \frac{N}{(N+uv)^2}\,\mathrm{d}u\mathrm{d}v\,.$$

In Theorem 2.7 we obtained that $\mathcal{T}_\alpha \longrightarrow \Omega_\alpha$ is almost surely bijective, and in Theorem 1.2 we already saw that $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ is an ergodic system. That $(\Omega_\alpha, \bar{\mathcal{B}}_\alpha, \bar{\mu}, \mathcal{T}_\alpha)$ is the natural extension of $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ can be seen by adapting the proof from [7] or from applying Definition 5.3.1 from [5]. Due to Theorem 1.2 and Theorem 5.3.1(iii) from [5] we now have that the natural extension $(\Omega_\alpha, \bar{\mathcal{B}}_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$ is ergodic. In fact, stronger mixing properties hold, but we do not investigate these here.

Perhaps the most surprising fact is, is that the normalising constant H is *constant* for $\alpha \in \overline{X}_{N,d,i}$. One way to see this is by brute force calculations, as we will do in the rest of this proof. However, in Section 3 we will see that for any two $\alpha, \alpha' \in X_{N,d,i}$ we have that the planar natural extensions of the underlying dynamical systems are metrically isomorphic. This will not only yield that the normalizing constant H is constant for $\alpha \in X_{N,d,i}$, but also that the entropy for all these dynamical systems is equal.

To obtain the normalising constant, we project the density $\bar{\mu}$ of the planar natural extension Ω_α on the first coordinate by integrating out the second coordinate:

$$\begin{aligned} H^{-1} = & \int_\alpha^{\alpha+1} \frac{D}{N+Dx} \mathbf{1}_{(\alpha, T(\alpha+1))} + \int_\alpha^{\alpha+1} \frac{E}{N+Ex} \mathbf{1}_{(T(\alpha+1)), T^2(\alpha))} \\ & + \int_\alpha^{\alpha+1} \frac{F}{N+Fx} \mathbf{1}_{(T^2(\alpha), \alpha+1)} - \int_\alpha^{\alpha+1} \frac{A}{N+Ax} \mathbf{1}_{(\alpha, T^2(\alpha+1))} \\ & - \int_\alpha^{\alpha+1} \frac{B}{N+Bx} \mathbf{1}_{(T^2(\alpha+1)), T(\alpha))} - \int_\alpha^{\alpha+1} \frac{C}{N+Cx} \mathbf{1}_{(T(\alpha), \alpha+1)} \,. \end{aligned}$$

After an elementary but tedious calculation, one finds that

$$\begin{aligned} H^{-1} = & \log\left(\frac{(N-Dd)\alpha + N + D(N-d)}{N+D\alpha}\right) \\ & + \log\left(\frac{\big((E-d-i)N + Ed(d+i)\big)\alpha + N^2 - ENd}{(N-Ed)\alpha + E(N-d) + N}\right) \\ & + \log\left(\frac{F\alpha + N + F}{\big(N(F-d-i) + F(d^2+di)\big)\alpha - NFd + N^2}\right) \\ & - \log\left(\frac{\big(N(-d+A) + Ad(d+i)\big)\alpha + N^2 - \big((A+1)d + A(i-1)\big)N + Ad(d+i)}{A\alpha + N}\right) \\ & - \log\left(\frac{\big(B(d+i) - N\big)\alpha - NB}{(Nd - (N + d(d+i))B)\alpha + ((d+i-1)N - d(d+i))B - (N-d)N}\right) \\ & - \log\left(\frac{C\alpha + C + N}{\big(N - (d+i)C\big)\alpha + NC}\right) \,. \end{aligned}$$

We will show now that the first and second terms are constants; the other terms have similar proofs of being constant.

(i): If we can show that $\frac{N-Dd}{D} = \frac{N+D(N-d)}{N}$, then $\log\big(\frac{(N-Dd)\alpha+N+D(N-d)}{N+D\alpha}\big)$ is a constant.

To see why this last statement holds, assume that $\frac{N-Dd}{D} = \frac{N+D(N-d)}{N}$. Now,

$$\frac{(N-Dd)\alpha + N + D(N-d)}{N+D\alpha} = \frac{D \cdot \frac{(N-D\alpha)}{D}\alpha + N \cdot \frac{N+D(N-D)}{N}}{N+D\alpha} = \frac{D\alpha+N}{N+D\alpha} \cdot \frac{N-Dd}{D} \,,$$

and for $\alpha \in X_{N,d,i}$ one has that $\frac{N-Dd}{D}$ is a constant. By substituting $D = \frac{N}{d+1+B}$ (c.f. (15)), the following statements are equivalent:

$$\begin{aligned} & \frac{N-Dd}{D} = \frac{N+D(N-d)}{N} \iff \frac{N}{D} - d = 1 + D - \frac{Dd}{N} \\ \iff & \frac{N}{\frac{N}{d+1+B}} - d = 1 + \frac{N}{d+1+B} - \frac{\frac{N}{d+1+B}d}{N} \iff B = \frac{N-d}{d+1+B} \,. \end{aligned}$$

Meanwhile, from (11) we know that $B = \frac{N}{B+1} - d$, so $(B+d)(B+1) = N$, and from this we see that $B(d+1+B) = N-d$; i.e., we find that $B = \frac{N-d}{d+1+B}$. Therefore, $\frac{N-Dd}{D} = \frac{N+D(N-d)}{N}$, and $\log\big(\frac{(N-Dd)\alpha+N+D(N-d)}{N+D\alpha}\big)$ is a constant. Note that $\frac{N-Dd}{D} = \frac{N}{D} - d = 1 + B$.

(ii): The proof that the second term is constant is similar to case (i). Here we show that *if* $\frac{(E-d-i)N+Ed(d+i)}{N-Ed} = \frac{N^2-ENd}{E(N-d)+N}$, *then* $\log\big(\frac{((E-d-i)N+Ed(d+i))\alpha+N^2-ENd}{(N-Ed)\alpha+(E+1)N-Ed}\big)$ is a constant. To see this last statement, note that *if* we assume that $\frac{(E-d-i)N+Ed(d+i)}{N-Ed} = \frac{N^2-ENd}{E(N-d)+N}$, we have that $\frac{((E-d-i)N+Ed(d+i))\alpha+N^2-ENd}{(N-Ed)\alpha+(E+1)N-Ed}$ is equal to

$$\frac{(N-Ed)\cdot\frac{(E-d-i)N+Ed(d+i))}{N-Ed}\cdot\alpha+\frac{N^2-ENd}{(E(N-d)+N}\cdot(E(N-d)+N)}{(N-Ed)\alpha+(E+1)N-Ed}$$
$$=\frac{(N-Ed)\alpha+E(N-d)+N}{(N-ED)\alpha+(E+1)N-Ed}\cdot\frac{N^2-ENd}{E(N-d)+N},$$

and for $\alpha\in X_{N,d,i}$ we have that $\frac{N^2-ENd}{E(N-d)+N}$ is a constant. In order to see that we indeed have $\frac{(E-d-i)N+Ed(d+i)}{N-Ed} = \frac{N^2-ENd}{E(N-d)+N}$, note that from (20) we know that $\frac{N}{A+1}-d=\frac{N}{A+d+i}$, and therefore that we have that $d(A+1)(d+i)=N(d+i)-N-dA(A+1)$.

From this last expression, and since $E=A+1$,

$$\frac{(E-d-i)N+Ed(d+i)}{N-Ed}=\frac{(1+A-d-i)N+d(1+A)(d+i)}{N-(1+A)d}$$
$$=\frac{(1+A-d-i)N+N(d+i)-N-dA(A+1)}{N-(A+1)d}$$
$$=\frac{NA-dA(A+1)}{N-(A+1)d})=A\,.$$

Now from (19) it immediately follows that $\frac{N}{d+i+A+1}=A$, and once more using that $\frac{N}{A+1}-d=\frac{N}{A+d+i}$, we find that:

$$\frac{N^2-ENd}{E(N-d)+N}=\frac{N(\frac{N}{E}-d)}{N-d+\frac{N}{E}}=\frac{N(\frac{N}{A+1}-d)}{N+\frac{N}{A+1}-d}=N\cdot\frac{\frac{N}{A+d+i}}{N+\frac{N}{A+d+i}}=\frac{N}{d+i+A+1}=A\,.$$

Therefore, $\frac{(E-d-i)N+Ed(d+i)}{N-Ed}=\frac{N^2-ENd}{E(N-d)+N}$; the second term $\log\big(\frac{((E-d-i)N+Ed(d+i))\alpha+N^2-ENd}{(N-Ed)\alpha+(E+1)N-Ed}\big)$ is a constant.

Similarly, one can show that the other terms in H^{-1} are also constant. In doing so one finds that:

$$H^{-1}=\log(1+B)+\log A+\log\left(\frac{1}{N-(d+i)B}\right)-\log\big(N-(A+1)d\big)-\log\left(\frac{1}{B+1}\right)-\log\left(\frac{1}{A}\right),$$

which can be simplified into (23):

$$H^{-1}=2\log A+2\log(B+1)-\log\big(N-(A+1)d\big)-\log\big(N-(d+i)B\big)\,.$$

□

3. Quilting. In this section we will use a technique called *quilting*, which was first used in [14], and which was originally based on the use of ‘insertions’ and ‘singularizations’ (which are operations on the partial quotients of the regular continued fraction expansion of any real number x) as investigated in [6, 10, 12]. In these last three papers quilting was steered by the insertion and singularization operations, while quilting was 'blind' in [13, 14], in the sense that the quilting maps were 'guessed'. For N-expansions with finitely many digits quilting was already used in [13] for the case $N = 2$, in order to show the occurrence of an 'entropy plateau'; i.e., a set of values of α for which the dynamical systems $(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ all have the same entropy. In this section we will see that the approach from [13] can be generalized to the intervals (A, B) we obtained in Theorem 2.10, where A and B are given in (11). Recently, quilting has received a thorough theoretic founding in [3], but here we follow the more 'hands-on' approach from [13, 14].

Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha, \beta \in X_{N,d,i,k}$, $\alpha < \beta$ arbitrary, with $k \in \{d, \dots, d+i-1\}$, and let the planar domains Ω_α and Ω_β be the polygon, as given in the statement of Theorem 2.7, where the values of A, B, C, D and E are also given in Theorem 2.7. Since $\alpha < \beta$ both maps $T_\alpha : I_\alpha \longrightarrow I_\alpha^-$ and $T_\beta : I_\beta \longrightarrow I_\beta^-$ have cylinders for the digits $d, \dots, d+i$, but that these cylinder are not the same (but overlapping) for the same digit k (with $k \in \{d, \dots, d+i\}$). For this reason we define the cylinders of T_α by $I_k(\alpha)$, and similarly those for the map T_β by $I_k(\beta)$. We first assume that $T_\alpha^2(\alpha) \in I_k^o(\alpha), T_\alpha^2(\alpha+1) \in I_{k+1}^o(\alpha)$, and similarly $T_\beta^2(\beta) \in I_k^o(\beta), T_\beta^2(\beta+1) \in I_{k+1}^o(\beta)$; c.f. (7).

As in [13], we define sets A_0, $A_1 = \mathcal{T}_\alpha(A_0)$ and $A_2 = \mathcal{T}_\alpha(A_1)$, and sets D_0, $D_1 = \mathcal{T}_\beta(D_0)$ and $D_2 = \mathcal{T}_\beta(D_1)$, where:

$$A_0 = [\alpha, \beta] \times [A, D], \quad D_0 = [\alpha+1, \beta+1] \times [C, F]\,. \tag{24}$$

We have the following result.

PROPOSITION 3.1. *Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha, \beta \in X_{N,d,i,k}$, $\alpha < \beta$ arbitrary, with $k \in \{d, \dots, d+i-1\}$. Moreover, let $T_\alpha^2(\alpha) \in I_k^o(\alpha), T_\alpha^2(\alpha+1) \in I_{k+1}^o(\alpha)$, and similarly $T_\beta^2(\beta) \in I_k^o(\beta), T_\beta^2(\beta+1) \in I_{k+1}^o(\beta)$.*

Then we have that:

$$\mathcal{T}_\alpha(A_2) = \mathcal{T}_\beta(D_2)\,, \tag{25}$$

and the map $\mathcal{M}_{\beta,\alpha} : \Omega_\beta \longrightarrow \Omega_\alpha$, defined by:

$$\mathcal{M}_{\beta,\alpha}(x,y) = \begin{cases} (x,y), & \text{if } (x,y) \in \Omega_\beta \setminus (D_0 \cup D_1 \cup D_2)\,; \\ \mathcal{T}_\alpha^{-3}\Big(\mathcal{T}_\beta^{3}(x,y)\Big), & \text{if } (x,y) \in D_0\,; \\ \mathcal{T}_\alpha^{-2}\Big(\mathcal{T}_\beta^{2}(x,y)\Big), & \text{if } (x,y) \in D_1\,; \\ \mathcal{T}_\alpha^{-1}\Big(\mathcal{T}_\beta(x,y)\Big), & \text{if } (x,y) \in D_2\,, \end{cases} \tag{26}$$

is a metric isomorphism from Ω_β to Ω_α; cf. Figure 8.

PROOF. Note that from Theorem 2.7 it follows that $A_1 = \mathcal{T}_\alpha(A_0) = [T_\beta(\beta), T_\alpha(\alpha)] \times [B, C]$, so due to the construction of the planar natural extension Ω_α we *must* have that $T_\alpha(\beta) = T_\beta(\beta)$; i.e., that $\beta \in I_{d+i}(\alpha)$ (if $\beta \notin I_{d+i}(\alpha)$ then the "α-digit" of β is at most $d+i-1$, and we would have that $T_\alpha(\beta) \geq T_\beta(\beta) + 1 \notin I_\alpha$. In a similar way we find that

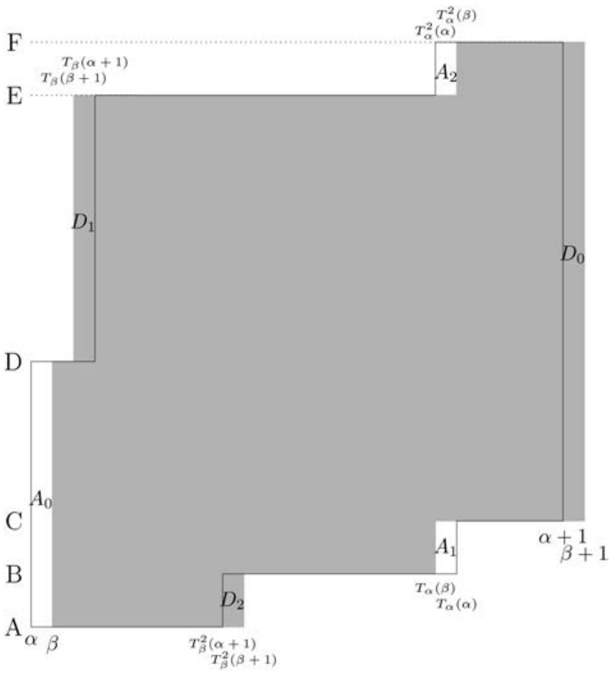


FIGURE 8. Ω_α and Ω_β for $\alpha, \beta \in X_{N,d,i,k}$, where $\alpha < \beta$.

$$A_2 = \mathcal{T}_\alpha^2(A_0) = [T_\alpha^2(\alpha), T_\beta^2(\beta)] \times [E, F],$$

that

$$D_1 = \mathcal{T}_\beta(D_0) = [T_\beta(\beta+1), T_\alpha(\alpha+1)] \times [D, E], \ \ D_2 = \mathcal{T}_\beta(D_0) = [T_\alpha^2(\alpha+1), T_\beta^2(\beta+1)] \times [A, B].$$

From this we see that

$$\mathcal{T}_\alpha(A_2) = [T_\beta^3(\beta), T_\alpha^3(\alpha)] \times \left[\frac{N}{k+F}, \frac{N}{k+E}\right], \tag{27}$$

and that

$$\mathcal{T}_\beta(D_2) = [T_\beta^3(\beta+1), T_\alpha^3(\alpha+1)] \times \left[\frac{N}{k+1+B}, \frac{N}{k+1+A}\right]. \tag{28}$$

Since both $\alpha, \beta \in X_{N,d,i,k}$, it follows from Theorem 2.5 that $T_\alpha^3(\alpha) = T_\alpha^3(\alpha+1)$ and that $T_\beta^3(\beta) = T_\beta^3(\beta+1)$. So from (27), (28) and the fact that in (11) we saw that $A+1 = E$ and $B+1 = F$, we immediately find that $\mathcal{T}_\alpha^3(A_0) = \mathcal{T}_\beta^3(D_0)$.

To see that $\mathcal{M} : \Omega_\beta \to \Omega_\alpha$ is a metric isomorphism, note that the sets A_0, A_1 and A_2 are disjoint a.s. from Ω_β, and that the sets D_0, D_1 and D_2 are disjoint a.s. from Ω_α. Furthermore, all four maps $\mathcal{T}_\alpha$, $\mathcal{T}_\alpha^{-1}$, $\mathcal{T}_\beta$ and $\mathcal{T}_\beta^{-1}$ preserve any measure with density $\frac{1}{H}\frac{N}{(N+xy)^2}$, where H is the normalizing constant given in (23), and the maps $\mathcal{T}_\alpha$, $\mathcal{T}_\alpha^{-1}$ are a.s. bijective on Ω_α, while the maps $\mathcal{T}_\beta$ and $\mathcal{T}_\beta^{-1}$ are a.s. bijective on Ω_β.

Thus we see that for $\alpha, \beta \in X_{N,d,i,k}$, $\alpha < \beta$, for $k \in \{d, d+1, \dots, d+i\}$, the ergodic dynamical systems $(\Omega_\alpha, \mathcal{B}_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$ and $(\Omega_\beta, \mathcal{B}_\beta, \bar{\mu}_\beta, \mathcal{T}_\beta)$ are metrically isomorphic. $\square$

REMARKS 3.2. (i) To see that for any $\alpha, \beta \in \bar{X}_{N,d,i}$ (say with $\alpha < \beta$) we have that their corresponding dynamical systems are metrically isomorphic it is enough to show that for $\alpha, \beta \in \bar{X}_{N,d,i,k}$ (for some $k \in \{d, d+1, \dots, d+i\}$) with $\alpha < \beta$, where either $T_\alpha(\alpha) = \frac{N}{\alpha+1+d}$, or $T_\alpha(\alpha+1) = \alpha$, or $T_\alpha^2(\alpha)$ is a boundary point of I_d, or $T_\alpha^2(\alpha+1)$ is a boundary point of I_{d+i}, and similarly for T_β, that the corresponding dynamical systems are isomorphic. In Remarks 2.6(iv) we mentioned as special cases when either $T_\alpha(\alpha) = \frac{N}{\alpha+1+d}$, or $T_\alpha(\alpha+1) = \alpha$. We will show in these last two cases that it is very easy to construct a natural extension. All other cases mentioned here are similar to these two cases, and therefore omitted.

Let us first assume that $T_\alpha(\alpha) = \frac{N}{\alpha+1+d}$. As we already remarked in Remarks 2.6(iv), $\frac{N}{\alpha+1+d}$ is the dividing point between the cylinders I_d and I_{d+1}. But then we have, that $T_\alpha(\alpha) = \frac{N}{\alpha+d+1}$, and from (17) we immediately find that $\alpha = B$, and due to $F = B + 1$ (cf. (11)) and $F = \frac{N}{d+B}$ (cf. (12)), from which we see that $B = \frac{N}{F} - d$ (i.e., $T_B(B+1) = B$), we see that T_B is full on the right-most cylinder I_d. Consequently, the domain of the planar natural extension for this particular value of α is the left-hand side polygon given in Figure 9.

Next, let us assume that $T_\alpha(\alpha+1) = \frac{N}{\alpha+d+i}$. So the map T_α sends $\alpha + 1$ to the dividing point of the most left-hand cylinder I_{d+i} and I_{d+i-1}. From $T_\alpha(\alpha+1) = \frac{N}{\alpha+d+i}$ it follows that α is the positive root of $d\alpha^2 + d(d+i+1)\alpha + d(d+i) + (1-(d+i))N = 0$, which is

$$\alpha = \frac{-d(d+i+1) + \sqrt{d^2(d+i+1)^2 - 4d(d(d+i) + (1-(d+i))N)}}{2d}. \tag{29}$$

Now a trivial but somewhat tedious calculation shows that the expression for A from (11) and for α from (29) are the same whenever $N = \frac{(d+i)}{i-1}$; i.e., we find that $T_A(A+1) = \frac{N}{A+d+i}$. In this case we have that $T_A(A) = A + 1$ if and only if $\frac{N}{A} - (d+i) = A + 1$, which is equivalent to A satisfying $A^2 + (d+i+1)A - N = 0$. The positive solution to this equation is A from (11). We see that T_A is full on the left-most cylinder I_{d+i}. Consequently, the domain of the planar natural extension for this particular value of α is the right-hand side polygon given in Figure 9.

(ii) The map $\mathcal{M}_{\beta,\alpha}$ from (26) is the so-called *quilting map* from Ω_β to Ω_α. Note that $\mathcal{M}_{\beta,\alpha}$ maps D_ℓ bijectively (a.s.) to A_ℓ for $\ell = 0, 1, 2$. That we *quilt* can be seen as follows:

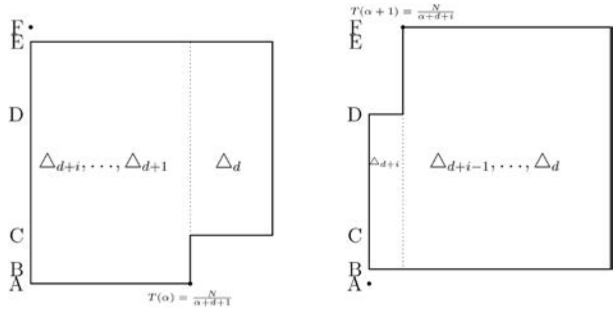


FIGURE 9. Ω_B (left) and Ω_A (right).

we map D_0 to A_0, so we add A_0 to Ω_β as a 'patch'. We then remove D_0, D_1 and D_2 from Ω_β, and add to Ω_β the sets A_1 and A_2, thus finding Ω_α. Note that we also should remove $\mathcal{T}_\beta(D_2)$ from Ω_β, but the 'gap' thus created is filled by $\mathcal{T}_\alpha(A_2)$, and the *quilting process* stops.

(iii) In [9, 10], the quilting is steered by operations on the partial quotients called insertions and singularizations, and in these cases the map $\mathcal{M}_{\beta,\alpha}$ can be more explicitly given. In fact one only needs to know $\mathcal{M}_{\beta,\alpha} : D_0 \to A_0$, remove all forward images $D_\ell = \mathcal{T}_\beta^\ell(D_0)$ from Ω_β, and add all forward images $\mathcal{T}_\alpha^\ell(A_0)$ to Ω_β (where $\ell = 0, 1, 2, \dots$), yielding Ω_α. One easily sees that the first coordinate map of $\mathcal{M}_{\beta,\alpha} : D_0 \to A_0$ must be $x \mapsto x - 1$, but the second coordinate map is usually more complicated. △

From Proposition 3.1 and Remarks 3.2(i) we have the following result.

THEOREM 3.3. *Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha, \beta \in [A, B] = \overline{X}_{N,d,i}$, $\alpha < \beta$ arbitrary. Then the dynamical systems $(\Omega_\alpha, \bar{\mathcal{B}}_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$ and $(\Omega_\beta, \bar{\mathcal{B}}_\beta, \bar{\mu}_\beta, \mathcal{T}_\beta)$ are metrically isomorphic.*

4. Plateaux with the same entropy for every $N \in \mathbb{N}$, $N \geq 2$.

Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha \in X_{N,d,i}$, then a direct corollary of Theorem 3.3 is that not only the normalizing constant H is the same for all α, but also that that the entropy is $h(T_\alpha)$ is constant for all $\alpha \in X_{N,d,i}$. This is exactly the statement of Theorem 4.2. In the statement (and proof) of Theorem 4.2 the dilogarithm function Li_2 appears at various places, and therefore we first recall some facts about the dilogarithm, which for $z \in \mathbb{C}$ can be defined by the sum

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \text{ for } |z| \leq 1, \qquad \text{or by the integral} \quad \mathrm{Li}_2(z) = \int_z^0 \frac{\log(1-t)}{t}\,\mathrm{d}t\,; \tag{30}$$

see also [17, 18] for more information on the dilogarithm function (and polylogarithm functions in general).

LEMMA 4.1. *For any $m > n > 0$, and $M, N > 0$,*

$$\int_n^m (\log x) \frac{M}{N + Mx}\,\mathrm{d}x = \Big(\mathrm{Li}_2(-\tfrac{M}{N}x) + (\log x)\log(1 + \tfrac{M}{N}x)\Big)\Big|_n^m,$$

where $\mathrm{Li}_2(.)$ *is the dilogarithm function from (*30*).*

PROOF. Note that integration by parts yields that,

$$\begin{aligned}\int (\log x) \frac{M}{N + Mx}\,\mathrm{d}x &= (\log x)\log(\tfrac{N}{M} + x) - \int \frac{\log(\frac{N}{M} + x)}{x}\,\mathrm{d}x \\ &= (\log x)\log(\tfrac{N}{M} + x) - \left(\int \frac{\log(1 + \frac{M}{N}x)}{x}\,\mathrm{d}x + \int \frac{\log\frac{N}{M}}{x}\,\mathrm{d}x\right) \\ &= -\int \frac{\log(1 + \frac{M}{N}x)}{x}\,\mathrm{d}x + (\log x)\log(1 + \tfrac{M}{N}x)\,.\end{aligned}$$

Setting $\frac{M}{N}x = -t$ one easily sees that, for $m > 0$,

$$-\int_0^m \frac{\log(1+\frac{M}{N}x)}{x}\,\mathrm{d}x = \int_{-\frac{M}{N}m}^0 \frac{\log(1-t)}{t}\,\mathrm{d}t = \mathrm{Li}_2(-\tfrac{M}{N}m)\,.$$

Therefore,

$$\int_n^m (\log x)\frac{M}{N+Mx}\,\mathrm{d}x = \Big(\mathrm{Li}_2(-\tfrac{M}{N}x) + (\log x)\log(1+\tfrac{M}{N}x)\Big)\Big|_n^m\,.$$

□

We have the following result.

THEOREM 4.2. *Let $N \geq 2$ be an integer, and let $d, i \in \mathbb{N}$, $i \geq 2$, be such, that $N = \frac{d(d+i)}{i-1}$. Then for any $\alpha \in [A,B] = \overline{X}_{N,d,i}$, one has that the entropy function $h(T_\alpha)$ is constant on $[A,B] = \overline{X}_{N,d,i}$, and is given by:*

$$h(T_\alpha) = \log N - 2H\bigg(\Big(\mathrm{Li}_2(-\tfrac{Ex}{N}) + (\log x)\log(\tfrac{Ex}{N}+1)\Big)\Big|_B^{B+1}$$
$$-\Big(\mathrm{Li}_2(-\tfrac{Ax}{N}) + (\log x)\log(\tfrac{Ax}{N}+1)\Big)\Big|_B^{D} - \Big(\mathrm{Li}_2(-\tfrac{Cx}{N}) + (\log x)\log(\tfrac{Cx}{N}+1)\Big)\Big|_D^{B+1}\bigg),$$

where $H^{-1} = 2\log A + \log(A+1) + \log(B+1) - \log\big(N-(A+1)d\big) - \log\big(N-(d+i)B\big)$ is the normalising constant for the T_α-invariant measure μ_α for $\alpha \in X_{N,d,i}$.

TABLE 1. The pairs of integers $d \geq 1, i \geq 2$, the related plateau intervals $[A,B]$ and constant entropy $h(T_\alpha)$ for $\alpha \in [A,B]$. Here $N = 8$.

(d,i)	Plateau intervals	H_α	$h(T_\alpha)$
(2, 2)	$\left[\frac{\sqrt{57}-5}{2}, \frac{\sqrt{33}-3}{2}\right] = [1.2749, 1.3723]$	18.377877038370	0.9212748062044
(4, 6)	$\left[\frac{3\sqrt{17}-11}{2}, \frac{\sqrt{41}-5}{2}\right] = [0.6847, 0.7016]$	11.239480662654	1.8212263472923
(5, 11)	$\left[\frac{3\sqrt{97}-29}{2}, \frac{\sqrt{57}-7}{2}\right] = [0.2733, 0.2749]$	9.9626774452815	2.7933207303296
(6, 22)	$\left[\frac{\sqrt{321}-17}{2}, \frac{2\sqrt{3}-3}{2}\right] = [0.4582, 0.4641]$	9.2212359716540	2.2547418855378
(7, 57)	$\left[\frac{3\sqrt{473}-65}{2}, \frac{\sqrt{17}-4}{2}\right] = [0.1228, 0.1231]$	8.7715446381451	3.3495778601659

PROOF. From Theorem 3.3 we know for any $\alpha, \beta \in [A,B] = \overline{X}_{N,d,i}$ we have that the dynamical systems $(\Omega_\alpha, \mathcal{B}_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$ and $(\Omega_\beta, \mathcal{B}_\beta, \bar{\mu}_\beta, \mathcal{T}_\beta)$ are metrically isomorphic, and due to Theorem 9.22 from [5] all $\alpha, \beta \in [A,B]$ have the same entropy.

So to know the entropy $h(T_\alpha)$ it suffices to calculate it for just one $\alpha \in [A,B]$; we choose

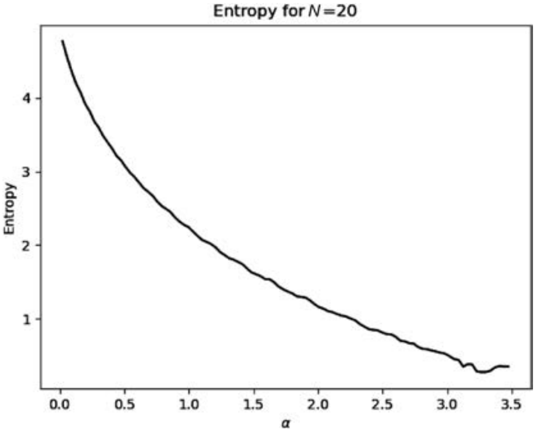


FIGURE 10. A simulation of the entropy of T_α when $N = 20$.

$\alpha = B$, as in this case (or in the case $\alpha = A$) the shape of Ω_α is the easiest (cf. Figures 2, 6, 7, and 9). In this case we also have that $T_B(B) = \frac{N}{B+d+1} = D$; cf. (15).

By using Rohlin's formula (see [5]), and with T_B-invariant density f_B from Theorem 2.11, we have for $\alpha \in [A, B]$, and in particular for $\alpha = B$ that:

$$\begin{aligned}
h(T_\alpha) &= \int_B^{B+1} \log|T_B'(x)|\,\mathrm{d}\mu_B(x) \;=\; \int_B^{B+1} \log|T_B'(x)| f_B(x)\,\mathrm{d}x \\
&= \int_B^{B+1} (\log N - 2\log x) f_B(x)\,\mathrm{d}x \;=\; \log N - 2\cdot\int_B^{B+1} (\log x)\, f_B(x)\,\mathrm{d}x \\
&= \log N - 2H\cdot \int_B^{B+1} (\log x)\left(\Big(\frac{E}{N+Ex} - \frac{A}{N+Ax}\Big)\mathbf{1}_{(B,D)}(x)\,\mathrm{d}x\right. \\
&\quad \left. + \Big(\frac{E}{N+Ex} - \frac{C}{N+Cx}\Big)\mathbf{1}_{(D,B+1)(x)}\,\mathrm{d}x\right) \\
&= \log N - 2H\bigg(\Big(\mathrm{Li}_2(-\tfrac{Ex}{N}) + (\log x)\log(\tfrac{Ex}{N}+1)\Big)\Big|_B^{B+1} \\
&\quad - \Big(\mathrm{Li}_2(-\tfrac{Ax}{N}) + (\log x)\log(\tfrac{Ax}{N}+1)\Big)\Big|_B^{D} - \Big(\mathrm{Li}_2(-\tfrac{Cx}{N}) + (\log x)\log(\tfrac{Cx}{N}+1)\Big)\Big|_D^{B+1}\bigg),
\end{aligned}$$

where $H^{-1} = 2\log A + \log(A+1) + \log(B+1) - \log\big(N - (A+1)d\big) - \log\big(N - (d+i)B\big)$ is the normalising constant; cf. (23) in Theorem 2.11. □

EXAMPLE 4.3. In case $N = 2$ our method yields only one plateau with equal entropy which follows from our method. This is the interval $[A, B] = [\frac{\sqrt{33}-5}{2}, \sqrt{2} - 1] = [0.3722813\ldots, 0.4142136\ldots]$, which was already found in [13], where it was also determined that for $\alpha \in [A, B]$ we have that $h(T_\alpha) = 1.137779584292255\ldots$ and $H = 3.965116120651161\ldots$.

In case $N = 8$ it follows from our method that there are five plateaux of equal entropy;

see Table 1. In Figure 3 a simulation of the entropy for $N = 8$ is given as function of $\alpha \in (0, 2\sqrt{2} - 1]$. The largest of these plateaux of equal entropy from Table 1 is clearly visible, but the simulation seems to suggest there are other plateaux as well (which can be visualized by drawing shadowed vertical stripes). In general it seems that the entropy is decreasing when α is increasing, but in Figure 3 there is clearly also an interval where the entropy increases. Our method yields one plateau in case $N = 2$, but Figures 1 and 10 seem to indicate that there are intervals where the entropy is increasing or decreasing. In case $N = 20$ our method yields 11 plateaux, the most right-hand one being $[1.844288770, 1.898979486]$. However, the simulation in Figure 10 seem to indicate there are other plateaux.

Clearly what we know about N-expansions with a finite set of digits is still in its infancy, certainly when compared to the vast body of knowledge about Nakada's α-expansions. For these Nakada α-expansions Laura Luzzi and Stefano Marmi first showed in [20] using simulations that for $\alpha \in [0, g^2]$ there are intervals where the entropy either increases, is constant, or decreases. In [21], Hitoshi Nakada and Rie Natsui showed that there exist decreasing sequences of intervals of α, denoted by (I_n), (J_n), (K_n) and (L_n), such that the entropy of T_α is increasing on I_n, constant on J_n and L_n, and decreasing on K_n. Furthermore, for $n \in \mathbb{N}$ we have $\frac{1}{n} \in I_n$, and the various intervals are ordered[2] by $I_{n+1} < J_n < L_n < I_n$. See also the paper by Carlo Carminati and Giulio Tiozzo [4], and in particular Tiozzo's PhD-thesis [24]. △

References

[1] M. Anselm, and S. H. Weintraub, A Generalization of continued fractions, J. Number Theory 131 (2011), no. 12, 2442–2460.

[2] E. B. Burger, J. Gell-Redman, R. Kravitz, D. Walton and N. Yates, Shrinking the period lengths of continued fractions while still capturing convergents, J. Number Theory 128 (2008), no. 1, 144–153.

[3] K. Calta, C. Kraaikamp and T. A. Schmidt, Proofs of ergodicity of piecewise Möbius interval maps using planar extensions, Expo. Math. 42 (2024), no. 4, Paper No. 125575, 39 pp.

[4] C. Carminati and G. Tiozzo, Tuning and plateaux for the entropy of α-continued fractions, Nonlinearity 26 (2013), no. 4, 1049–1070.

[5] K. Dajani and C. Kalle, A First Course in Ergodic Theory, (1st ed.), Chapman and Hall/CRC (2021).

[6] K. Dajani and C. Kraaikamp, The mother of all continued fractions, Colloq. Math. 84/85 (2000), part 1, 109–123.

[7] K. Dajani, C. Kraaikamp and N. van der Wekken, Ergodicity of N-continued fraction expansions, J. Number Theory 133 (2013), no. 9, 3183–3204.

[8] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Revised by D.R. Heath-Brown and J. H. Silverman. Foreword by Andrew Wiles. (6th ed.), Oxford, at the Clarendon Press, 1954, xvi+419 ppm, [1938].

[9] Y. Hartono, C. Kraaikamp, N. Langeveld and C. Merriman, Natural extensions and entropy of α-continued fraction expansion maps with odd partial quotients, Discrete Contin. Dyn. Syst. 43 (2023), no. 8, 2852–2888.

[10] J. de Jonge, C. Kraaikamp and H. Nakada, Orbits of N-expansions with a finite set of digits, Monatsh. Math. 198 (2022), no. 1, 79–119.

[11] J. de Jonge and C. Kraaikamp, Gaps in intervals of N-expansions, Integers 23, Paper A42, 17 p. (2023).

---

[2] Here $I < J$ means that $I \cap J = \emptyset$, and $i < j$ for all $i \in I$ and all $j \in J$.

[12] C. KRAAIKAMP, A new class of continued fraction expansions, Acta Arith. 57 (1991), no. 1, 1–39.

[13] C. KRAAIKAMP AND N. LANGEVELD, Invariant measures for continued fraction algorithms with finitely many digits, J. Math. Anal. Appl. 454 (2017), no. 1, 106–126.

[14] C. KRAAIKAMP, T. A. SCHMIDT AND I. SMEETS, Natural extensions for α-Rosen continued fractions, J. Math. Soc. Japan 62 (2010), no. 2, 649–671.

[15] N. LANGEVELD, Finding infintely many even or odd continued fractions by introducing a new family of maps, Master thesis TU Delft, 2015.

[16] A. LASOTA AND J. A. YORKE, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc. 186 (1973), 481–488.

[17] L. LEWIN, Dilogarithms and associated functions. Foreword by J. C. P. Miller Macdonald, London, 1958.

[18] L. LEWIN, Polylogarithms and associated functions, With a foreword by A. J. Van der Poorten, North-Holland Publishing Co., New York-Amsterdam, 1981.

[19] T.Y. LI AND J. A. YORKE, Ergodic transformations from an interval into itself, Trans. Amer. Math. Soc. 235 (1978), 183–192.

[20] L. LUZZI AND S. MARMI, On the entropy of Japanese continued fractions, Discrete Contin. Dyn. Syst. 20 (2008), no. 3, 673–711.

[21] H. NAKADA AND R. NATSUI, The non-monotonicity of the entropy of α-continued fraction transformations, Nonlinearity 21 (2008), 1207–1225.

[22] G. PANTI, Attractors of dual continued fractions, J. Number Theory 240 (2022), 50–73.

[23] F. SCHWEIGER, Remarks on continued fractions with finitely many digits, Acta Arith. 203 (2022), no. 3, 239–250.

[24] G. TIOZZO, Entropy, dimension and combinatorial moduli for one-dimensional dynamical systems, Thesis (Ph.D.) Harvard University, 2013.

JIANGSU UNIVERSITY
SCHOOL OF MATHEMATICAL SCIENCES
XUEFU 301
ZHENJIANG, 212013
P.R. CHINA

E-mail address: yufechen@163.com

DELFT UNIVERSITY OF TECHNOLOGY
EWI (DIAM)
MEKELWEG 4
2628 CD DELFT
THE NETHERLANDS

E-mail address: c.kraaikamp@tudelft.nl