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Adaptive and efficient isotonic estimation in Wicksell's problem

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ABSTRACT

We consider nonparametric estimation in Wicksell's problem, which has applications in astronomy for estimating the distribution of star positions in a galaxy and in material sciences for determining a material's 3D microstructure from 2D cross sections. We focus on the isotonised version of the plug-in estimator (IIE) for the cdf F of the spheres' squared radii. This estimator is fully automatic, requiring no tuning parameters, and we show it is adaptive to local smoothness properties of the distribution function F to be estimated. We also prove a local asymptotic minimax lower bound in this non-standard setting, with $\sqrt{\log n/n}$ -asymptotics and where the functional F to be estimated is not regular. Combined, our results prove that the isotonic estimator (IIE) is an adaptive, easy-to-compute, and efficient estimator.

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1. Introduction

In the field of stereology, scientists are usually concerned with the study of three-dimensional properties of materials and objects by interpreting their two-dimensional cross-sections. This differs from three-dimensional reconstructions in that some 3D quantities can be estimated without expensive 3D reconstructions. In 1925, Sven Wicksell published an article (Wicksell 1925) that studied the following classical stereological inverse problem. Suppose that a number of spheres are embedded in an opaque three-dimensional medium. Because the medium is opaque, we are not able to observe the spheres directly; however, we can observe cross-sections of the medium (see Figure 1), which show circular sections of the spheres that happen to be cut by the plane. This model has present-day relevant applications in astronomy for estimating the positions of stars that are spherically symmetrically distributed in galactic clusters, given projected stellar positions (see Sen and Woodrooffe 2012), or to determine the 3D microstructure of a material given its cross-sections, useful to assess its structural properties like resistance and fracture toughness (see Lopez-Sanchez and LLana-Fúnez 2016; Cuzzi and Olson 2017 for applications of this setting).

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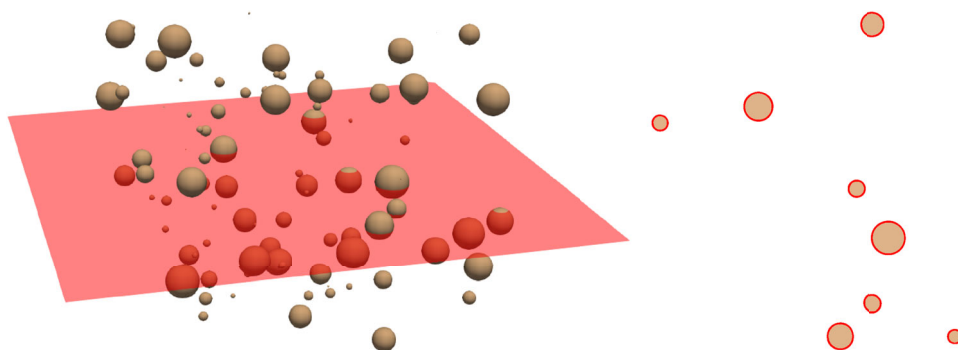


Figure 1. Sampling in Wicksell's problem. Left panel: 3D configuration with spheres and the intersecting plane. Right panel: Observable circular profiles.

Assume the spheres' squared radii are i.i.d. realisations from a random variable X with distribution function F , the object to be estimated. Following the same notation as in Groeneboom and Jongbloed (1995), we have that a version of the density g (c.f. Watson 1971) of the observable squared circle radii Z (thus $Z \sim g$) is given by

$$g(z) = \frac{1}{2m_0} \int_z^\infty \frac{dF(x)}{\sqrt{x-z}}, \quad (1)$$

where $0 < m_0 = \int_0^\infty \sqrt{y} dF(y) < \infty$ is the expected sphere radius under F .

Wicksell (1925) inverted this equation, by recognising an Abel-type integral, and found an expression for F in terms of g :

$$F(x) = 1 - \frac{\int_x^\infty (z-x)^{-1/2} g(z) dz}{\int_0^\infty z^{-1/2} g(z) dz} = 1 - \frac{V(x)}{V(0)}, \quad x > 0, \quad (2)$$

where

$$V(x) = \int_x^\infty \frac{g(z)}{\sqrt{z-x}} dz,$$

and $F(x) = 0$ for all $x \leq 0$. Therefore, to estimate F at a point $x > 0$, the essential object to be estimated is the function V at x and at 0. Given a sample $Z_1, \dots, Z_n$ from the density g and for $\mathbb{G}_n$ the empirical distribution function based on this sample, a naive (empirical plug-in) estimator of V is

$$V_n(x) = \int_x^\infty \frac{d\mathbb{G}_n(z)}{\sqrt{z-x}}. \quad (3)$$

The estimator for F is then defined by $F_n(x) = 1 - V_n(x)/V_n(0)$. As a function of x , this estimator has some undesirable properties, being non-monotone and having infinite discontinuities at a set of points becoming dense in the support of the density g as $n \rightarrow \infty$. This led in Groeneboom and Jongbloed (1995) to the definition of the so-called 'isotonic inverse estimator' (IIE) $\hat{F}_n$ for F , which is obtained by replacing V_n by its isotonic projection. This can be calculated as the right continuous derivative of the least concave majorant

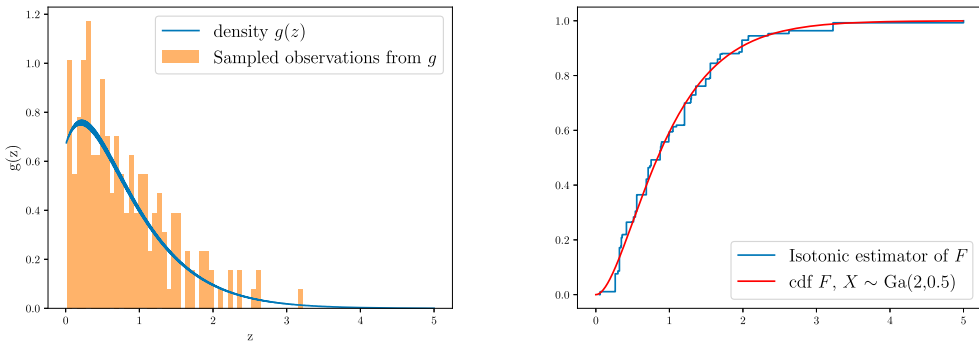


Figure 2. 200 observed draws from g and isotonic estimator for $F \sim \text{Ga}(2, 0.5)$.

of the primitive function of V_n , which is

$$U_n(x) = \int_0^x V_n(y) dy = 2 \int_0^\infty \sqrt{z} d\mathbb{G}_n(z) - 2 \int_x^\infty \sqrt{z-x} d\mathbb{G}_n(z).$$

Note that $x \mapsto U(x) = \int_0^x V(y) dy$ is a concave and increasing function on $[0, \infty)$, whereas U_n is not. Taking the least concave majorant of U_n on $[0, \infty)$ yields an increasing concave function. Denote by $\hat{V}_n$ the right derivative of this least concave majorant of U_n , and note that this will be a decreasing function on $[0, \infty)$. Then $\hat{F}_n$ is given by

$$\hat{F}_n(x) := 1 - \hat{V}_n(x)/\hat{V}_n(0). \quad (4)$$

This estimator has the important properties of being monotone (see Figure 2), right-continuous and does not depend on tuning parameters, unlike other estimators in the literature (cf. Golubev and Levit 1998).

This work is closely related to Groeneboom and Jongbloed (1995) and Golubev and Levit (1998). In Groeneboom and Jongbloed (1995), the authors observed a peculiar behaviour of the isotonic estimator (IIE) compared to the naive estimator $F_n(x) = 1 - V_n(x)/V_n(0)$. Both estimators exhibit the same rate of estimation, $(\log n)^{-1/2}\sqrt{n}$, and normal limiting behaviour. However, the naive estimator's asymptotic variance $\sigma^2(F) = \frac{4m_0^2}{\pi^2}(g(x) + g(0)(1 - F(x))^2)$ is exactly twice that of the IIE. The authors of Golubev and Levit (1998) propose an explanation, showing that for F in the Hölder class Λ_γ with $\gamma > \frac{1}{2}$, any efficient estimator attains the variance $\sigma^2(F)/(2\gamma)$. In Groeneboom and Jongbloed (1995), it is assumed that F has a bounded density f , which corresponds to $\gamma = 1$, thus explaining the variance reduction by half.

Moreover, the authors of Golubev and Levit (1998) propose an estimator based on Kernel Density Estimation (KDE) that achieves the variance $\sigma^2(F)/(2\gamma)$ and which is, to the best of our knowledge, the only other efficient estimator in the literature. However, the estimator they propose depends directly on the degree γ of smoothness, which should be guesstimated before using the estimator, making it non-adaptive. Furthermore, under local conditions at x and 0 (cf. (24) and (25)), they derive an asymptotic minimax lower bound assuming F is in the Hölder class Λ_γ , for $\gamma > \frac{1}{2}$.

This paper has two main contributions. On the one hand, we show that the IIE $\hat{F}_n$ automatically adapts to the *local* smoothness of F , without requiring F to belong to a *global*

Hölder class. It suffices to consider the *local* smoothness at points x and 0 , measured by constants $\gamma_x > \frac{1}{2}$ and $\gamma_0 > \frac{1}{2}$ (see conditions (5) and (38)). These can be interpreted as the tight degree of Hölder smoothness at the points x and 0 . For F with density f (i.e. differentiable at x and 0), we retrieve the $\gamma_x = \gamma_0 = 1$ that explains the behaviour observed in Groeneboom and Jongbloed (1995). On the other hand, we prove a lower bound on the minimax risk for any estimator sequence. The minimax theorem holds as well under local conditions of smoothness of F around x and 0 . If all conditions are met, this shows that the isotonic estimator is asymptotically efficient. No other estimator in the literature simultaneously has these properties. Consequently, the IIE emerges as the unequivocal choice, surpassing alternative estimators previously proposed in Wicksell's problem.

First, we prove that the IIE is asymptotically normal and the smoother the cdf F is at x and 0 , the smaller the asymptotic variance automatically attained by the IIE (Theorem 1.1 a). Here, the smoothness of F at a point p is measured by considering constants $K_p > 0$ and $\gamma_p > \frac{1}{2}$ such that

$$H_p(\delta) = \int_0^1 (F(p + u\delta) - F(p)) \, du \sim \text{sgn}(\delta) |\delta|^{\gamma_p} K_p, \quad \text{as } \delta \rightarrow 0. \quad (5)$$

Note that for $p=0$ we can only allow $\delta > 0$, because $\delta < 0$ would lead to $K_0 = 0$.

The adaptivity extends also to the situation where F is constant around x and 0 . Then a faster $\sqrt{n}$ -rate of convergence is attained (Theorem 1.1 b). Note that previously there were no results in the literature for nonparametric estimators that attained $\sqrt{n}$ rate in Wicksell's problem. In the case in which F is constant only around x but not around 0 , or vice versa, the rate will be again $(\log n)^{-1/2} \sqrt{n}$. We now state the first main result and its proof is given in Appendix 1.

Theorem 1.1: *Let F be the distribution function of the squared sphere radii and g the corresponding density of the squared circle radii according to (1). Assume $\int_0^\infty y^{\frac{3}{2}} \, dF(y) < \infty$ and choose $x > 0$.*

- (a) *If F satisfies condition (5) both at 0 and at x for $\gamma_0 > \frac{1}{2}$ and $\gamma_x > \frac{1}{2}$ respectively and $g(x) < \infty$ holds true, then, as $n \rightarrow \infty$:*

$$\sqrt{\frac{n}{\log n}} \left(\hat{F}_n(x) - F(x) \right) \rightsquigarrow N \left(0, \frac{4m_0^2}{\pi^2} \left(\frac{g(x)}{2\gamma_x} + (1 - F(x))^2 \frac{g(0)}{2\gamma_0} \right) \right).$$

- (b) *If F is constant around x and 0 , with K_x and K_0 the biggest compact intervals in $[0, \infty)$ that contain x and 0 , respectively, so that F is constant, on K_x and K_0 , with $K_x \cap K_0 = \emptyset$, then, as $n \rightarrow \infty$:*

$$\sqrt{n} \left(\hat{F}_n(x) - F(x) \right) \rightsquigarrow \frac{2m_0}{\pi} \left(L_x + (1 - F(x)) L_0 \right),$$

where, for any $a, b \in \mathbb{R}$, the joint distribution of L_x and L_0 is given by

$$\begin{aligned} & \mathbb{P}(L_x \leq a, L_0 \leq b) \\ &= \mathbb{P} \left(\arg \max_{s \in K_x} \{ \mathbb{Z}_x(s) - as \} \leq x, \arg \max_{s \in K_0} \{ \mathbb{Z}_0(s) - bs \} \leq 0 \right). \end{aligned}$$

Here $\mathbb{Z}_p(s) = \mathbb{Z}(s) - \mathbb{Z}(p)$, for $s \geq 0$ and $p \in \{0, x\}$. $\mathbb{Z}$ is a continuous centred Gaussian process with covariance structure given by

$$\text{Cov}(\mathbb{Z}(t), \mathbb{Z}(s)) = \text{Cov}\left(2(\sqrt{Z} - \sqrt{(Z_t)_+}), 2(\sqrt{Z} - \sqrt{(Z_s)_+})\right), \quad (6)$$

where $Z_t = Z - t$ and $Z_s = Z - s$.

The second main result of this paper consists of showing that the IIE is asymptotically efficient in this setting, i.e. the attained asymptotic variance is the smallest attainable. This kind of result is obtained by proving a minimax lower bound for any estimator sequence estimating $F(x)$. Here we give a weaker (but perhaps more insightful) version of this result and its proof is given in Appendix 2.

Theorem 1.2: *Let the conditions of Theorem 1.1(a) hold, assume that F for such γ_0, γ_x satisfies (38) and that g is continuous at x and 0. Let ℓ be a bowl-shaped loss function, i.e. ℓ is non-negative, symmetric, and subconvex, then for any estimator sequence $(F_n(x))_{n \in \mathbb{N}}$:*

$$\lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \sup_{\|F' - F\|_\infty < \delta} \mathbb{E}_{G_{F'}} \ell \left(\sqrt{\frac{n}{\log n}} (F_n(x) - F'(x)) \right) \geq \mathbb{E} \ell(L), \quad (7)$$

where L possesses the normal distribution in Theorem 1.1(a). $G_{F'}$ is the probability measure associated to the density g in (1) with F replaced by F' .

Combining the above results shows that the IIE is an easy-to-compute, adaptive, and efficient estimator for F .

Review of other studies on Wicksell's problem

There is a vast literature on Wicksell's problem, see for instance Groeneboom and Jongbloed (1995, 2014), Jongbloed (2001), Stoyan and Kendall (1987), Golubev and Levit (1998), Watson (1971), Hall and Smith (1988), Golubev and Enikeeva (2001), Chan and Qin (2016), Sen and Woodroffe (2012), and Deng et al. (2021). For a detailed introduction to Wicksell's problem, we refer the reader to Stoyan and Kendall (1987). The setting of Wicksell's problem has several modern applications, as mentioned above. In these applications, it is common to see the use of the so-called Saltykov method or methods based on numerical discretisation. These methods are far from being efficient and in some cases not even consistent, making our work of practical relevance. The estimation of F given data from g is of interest not only for stereological procedures but also from the mathematical point of view. Consider the class of inverse problems that entails the estimation of a functional: $\Psi_\alpha(x; g) = \int_x^\infty (z - x)^{-\alpha} g(z) dz$, $0 < \alpha < 1$ based on samples from g . This type of functional estimation emerges in a vast variety of applications. For $\alpha < \frac{1}{2}$, the above gives a regular statistical functional (cf. Theorem 4 in Koshevnik and Levit 2023), whereas for $\alpha > \frac{1}{2}$ gives instances of irregular functionals. Wicksell's problem entails the boundary case for $\alpha = 1/2$, which makes the problem mathematically intriguing, both because of the unusual rate of convergence and because of the non-standard efficiency theory that in this paper is developed.

Before continuing, we want to mention some other works related to the present paper. The first time the naive plug-in estimator was studied by Watson (1971). Later in Groeneboom and Jongbloed (1995) the proof of the asymptotic normality of this estimator was simplified considerably. However, viewed as a function, the naive plug-in estimator is severely ill-behaved as we already pointed out, and this led to the definition of the IIE. Naturally, one may wonder why not use the Nonparametric Maximum Likelihood Estimator. For Wicksell's problem, the usual definition of the NPMLE is also ill-posed, as the log-likelihood can attain value infinity. However, work has been done, for instance in Jongbloed (2001) and Chan and Qin (2016), to make the NPMLE an employable estimator. For results about confidence intervals in Wicksell's problem, we mention Sen and Woodroffe (2012) and Deng et al. (2021). On the other hand for KDE-based estimators, we refer to Hall and Smith (1988) and Golubev and Levit (1998). For extensions to convex bodies other than spheres, see van der Jagt et al. (2023). Finally, regarding efficient estimation, it is worth mentioning that one of the authors of Golubev and Levit (1998) did some further works on Wicksell's problem in Golubev and Enikeeva (2001) where it was shown how to attain efficient smoothing for the plug-in estimator under square losses.

2. Asymptotics of the isotonic inverse estimator

In this section, we prove two main theorems that will imply Theorem 1.1. To prove Theorem 1.1(a), it is enough to look at the behaviour of $\hat{V}_n(x)$ and then apply the same result also to $\hat{V}_n(0)$. We now state the conditions needed for Theorems 1.1 and 2.1. First, a natural assumption is a moment constraint for Z , which can be translated into a moment condition on X (for $0 < m_0 < \infty$):

$$\int_0^\infty zg(z) dz = \frac{2}{3m_0} \int_0^\infty y^{\frac{3}{2}} dF(y) < \infty. \quad (8)$$

Additionally, for $x \geq 0$, we need our version of g to be well defined, i.e.

$$\frac{1}{2m_0} \int_x^\infty \frac{dF(s)}{\sqrt{s-x}} < \infty. \quad (9)$$

Furthermore, we measure the smoothness of F at x ; $\exists K > 0$ and $\gamma_x > \frac{1}{2}$ such that (5) is satisfied.

Before proving Theorem 1.1(a), we show that condition (5) entails the local smoothness of the distribution function F by looking at the following examples 2.1 and 2.2.

Example 2.1: For $\gamma_x > \frac{1}{2}$ consider the class of distribution functions locally behaved as follows. Let F be so that there exists a neighbourhood U_x of $x > 0$ and some $K > 0$:

$$F(u) = \begin{cases} F(x) + K(u-x)^{\gamma_x}, & \text{for } u \geq x, u \in U_x \\ F(x) - K(x-u)^{\gamma_x}, & \text{for } u < x, u \in U_x \end{cases}$$

One can think of such distribution functions as tightly Hölder smooth of degree γ_x at the point x . This class is of particular importance because it illustrates what type of functions we are thinking of when we derive results under smoothness conditions at the point x and

0. Such functions satisfy both conditions (5) and (22). Condition (22) follows immediately. For condition (5) note that as $|\delta| \downarrow 0$, small enough so that $[x - |\delta|, x + |\delta|] \subset U_x$:

$$H_x(\delta) = \int_0^1 (F(x + u\delta) - F(x)) \, du = \operatorname{sgn}(\delta) |\delta|^{\gamma_x} K \int_0^1 u^{\gamma_x} \, du + o(\delta).$$

Example 2.2: For any $x > 0$, let μ_x be the discrete probability measure given by

$$\mu_x = \frac{1}{2} \sum_{i=1}^{\infty} p_i (\delta_{x+t_i} + \delta_{x-t_i}),$$

for sequences p_i of probability weights and $t_i \downarrow 0$. In particular set: $p_i = \frac{6}{\pi^2} i^{-2}$ and $t_i = i^{-\frac{1}{\gamma_x}}$ where $\gamma_x > 1/2$ and $\delta_x(\cdot)$ indicates the Dirac measure at x . Denote by F the associated cdf, so that, because $\gamma_x > 1/2$, we have

$$\int_x^{\infty} \frac{dF(s)}{\sqrt{s-x}} = \frac{3}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \frac{1}{\sqrt{t_i}} = \frac{3}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^{2-\frac{1}{\gamma_x}}} < \infty.$$

Moreover, we have for $\delta \downarrow 0$ (similar reasonings hold also for a generic sequence $|\delta| \downarrow 0$, we consider $\delta > 0$ for simplicity):

$$H_x(\delta) = \int_0^1 (F(x + u\delta) - F(x)) \, du = \sum_i \frac{p_i}{2} \int_0^1 \mathbf{1}_{t_i \leq u\delta} \, du = \sum_i \frac{p_i(\delta - t_i)_+}{2\delta},$$

which means $t_i < \delta$, thus $i > \frac{1}{\delta^{\gamma_x}}$. Therefore $H_x(\delta) = \sum_{i > 1/\delta^{\gamma_x}} \frac{p_i(\delta - i^{-1/\gamma_x})}{2\delta}$. Now we prove the following fact, as $\delta \downarrow 0$ and $k \geq 2$:

$$\frac{1}{(\delta^{\gamma_x})^{k-1}} \sum_{i > \frac{1}{\delta^{\gamma_x}}} \frac{1}{i^k} \xrightarrow{\delta \downarrow 0} \frac{1}{k-1}.$$

This follows directly from the fact that for $\alpha \rightarrow \infty$ (e.g. $\alpha = 1/\delta^{\gamma_x}$), we have

$$\frac{1}{k-1} \frac{1}{(\alpha+1)^{k-1}} = \int_{\alpha+1}^{\infty} \frac{1}{x^k} \, dx \leq \sum_{i > \alpha} \frac{1}{i^k} \leq \int_{\alpha-1}^{\infty} \frac{1}{x^k} \, dx = \frac{1}{k-1} \frac{1}{(\alpha-1)^{k-1}}.$$

Moreover $(\alpha/(\alpha \pm 1))^{k-1} \rightarrow 1$. Therefore as $\delta \downarrow 0$ we have

$$H_x(\delta) = \frac{6}{\pi^2} \left(\sum_{i > 1/\delta^{\gamma_x}} \frac{1}{i^2} - \frac{1}{\delta} \sum_{i > 1/\delta^{\gamma_x}} \frac{1}{i^{2+\frac{1}{\gamma_x}}} \right) \sim \frac{6}{\pi^2(1+\gamma)} \delta^{\gamma_x}.$$

This demonstrates that the required condition for Theorem 2.1 can also be satisfied for a discrete cumulative distribution function.

Remark 2.1: Example 2.1 shows that *tight* Hölder smoothness of degree γ_x at the point x implies (5). The implication in the other direction is not true as it is shown by Example 2.2, where a distribution that is not Hölder continuous at x satisfies (5). This example also

shows that (5) does not imply Hölder continuity of F at x (i.e. $|F(x + \delta) - F(x)| = O(|\delta|^{\gamma_x})$ as $\delta \rightarrow 0$). The condition $|F(x + \delta) - F(x)| = O(|\delta|^{\gamma_x})$ as $\delta \rightarrow 0$ is also not sufficient for Theorem 2.1 because $|F(x + \delta) - F(x)| = O(|\delta|^{\gamma_x})$ only gives information about the upper bound of the difference, but not about the lower bound, which is also needed to be tight Hölder of degree γ_x , as can be seen by inspection of the proof of Theorem 2.1.

Theorem 2.1: *Let $x \geq 0$, suppose that (5), (8) and (9) hold true and $g(x)$ is given by (1). Then*

$$\sqrt{\frac{n}{\log n}} \left(\hat{V}_n(x) - V(x) \right) \rightsquigarrow N \left(0, \frac{1}{2\gamma_x} g(x) \right) \quad \text{as } n \rightarrow \infty.$$

Proof: We generalise the proof of Groeneboom and Jongbloed (1995), for $\gamma_x \neq 1$.

First note that (5) implies that F is not constant in a neighbourhood of x (c.f. Lemma A.2). Fix $x \geq 0$ such that x satisfies the conditions stated above. Define, for $a > 0$, the process T_n is as follows:

$$T_n(a) = \inf \{ t \geq 0 : U_n(t) - at \text{ is maximal} \}.$$

The relation between T_n and $\hat{V}_n$ is given by the switch relation (see Groeneboom and Jongbloed 2014):

$$T_n(a) \leq t \quad \Leftrightarrow \quad \hat{V}_n(t) \leq a. \quad (10)$$

From this relation (Lemma 3.2, Groeneboom and Jongbloed 2014), it follows that for each vanishing sequence (δ_n) of positive numbers, for $a \in \mathbb{R}$, for $\delta_n^* = \delta_n^{1/\gamma_x}$ (γ_x as in (5)), and all n sufficiently large:

$$\begin{aligned} \delta_n^{-1} \left(\hat{V}_n(x) - V(x) \right) \leq a &\Leftrightarrow \delta_n^{-1} (T_n(a_0 + \delta_n a) - x) \leq 0 \\ &\Leftrightarrow (\delta_n^*)^{-1} (T_n(a_0 + \delta_n a) - x) \leq 0. \end{aligned}$$

We aim to study the probability of the event on the left-hand side. We write a_0 for $V(x)$, which is fixed. The introduction of $\delta_n^* = \delta_n^{1/\gamma_x}$ is one of the main steps that differentiates the current proof from the one in Groeneboom and Jongbloed (1995). Now denote by

$$I_x := \begin{cases} [0, \infty), & \text{if } x = 0 \\ \mathbb{R}, & \text{if } x > 0. \end{cases}$$

Define the localised process $\tilde{Z}_n$ in $C(I_x)$, the space of continuous functions on I_x , as follows (recall $a_0 = V(x)$):

$$\tilde{Z}_n(t) = \delta_n^{-1} (\delta_n^*)^{-1} [U_n(x + \delta_n^* t) - U_n(x) - a_0 \delta_n^* t].$$

For each fixed $a \in \mathbb{R}$ we can, for all n sufficiently large, write

$$\begin{aligned} &(\delta_n^*)^{-1} (T_n(a_0 + \delta_n a) - x) \\ &= (\delta_n^*)^{-1} \left(\inf \{ x + t \geq 0 : U_n(x + t) - a_0 x \right. \\ &\quad \left. - a_0 t - \delta_n a x - \delta_n at \text{ is maximal} \} - x \right) \end{aligned}$$

$$\begin{aligned}
&= (\delta_n^*)^{-1} \inf \{t \geq -x : U_n(x+t) - U_n(x) - a_0 t - \delta_n a t \text{ is maximal}\} \\
&= \inf \left\{ t \geq -(\delta_n^*)^{-1} x : U_n(x + \delta_n^* t) - U_n(x) - a_0 \delta_n^* t - \delta_n \delta_n^* a t \text{ is maximal} \right\} \\
&= \inf \left\{ t \geq -(\delta_n^*)^{-1} x : \tilde{Z}_n(t) - at \text{ is maximal} \right\},
\end{aligned}$$

where we use that the location of a maximum of a function is equivariant under translations and invariant under multiplication by a positive number and addition of a constant. Thus we analyse the asymptotic behaviour of the process:

$$\tilde{Z}_n(t) - at = \delta_n^{-1} (\delta_n^*)^{-1} \left\{ U_n(x + \delta_n^* t) - U_n(x) - U(x + \delta_n^* t) + U(x) \right\} \quad (11)$$

$$+ \delta_n^{-1} (\delta_n^*)^{-1} \left\{ U(x + \delta_n^* t) - U(x) - V(x) \delta_n^* t \right\} - at. \quad (12)$$

We will now analyse the behaviour of the stochastic and deterministic terms (11) and (12) separately. Let us start with (12). The Taylor expansion of U with integral form of the remainder yields

$$\begin{aligned}
U(x + \delta_n^* t) - U(x) - V(x) \delta_n^* t &= \delta_n^* t \int_0^1 V(x + u \delta_n^* t) - V(x) du \\
&= -\frac{\delta_n^* t \pi}{2m_0} \int_0^1 F(x + u \delta_n^* t) - F(x) du.
\end{aligned}$$

When we look at the limit of $\tilde{Z}_n(t) - at$ we multiply the above expression by $\delta_n^{-1} (\delta_n^*)^{-1} = (\delta_n^*)^{-\gamma_x - 1}$. Therefore we have to look at the limit behaviour of

$$-(\delta_n^*)^{-\gamma_x - 1} \frac{\delta_n^* t \pi}{2m_0} \int_0^1 F(x + u \delta_n^* t) - F(x) du = -\frac{|t|^{\gamma_x + 1} \pi}{2m_0} (K_x + o(1)), \quad (13)$$

where we use assumption (5).

Now we consider (11). Defining the processes (Z_n) in $C(I_x)$ as

$$\begin{aligned}
Z_n(t) &= \delta_n^{-1} (\delta_n^*)^{-1} \left\{ U_n(x + \delta_n^* t) - U_n(x) - U(x + \delta_n^* t) + U(x) \right\} \\
&= 2\delta_n^{-1} (\delta_n^*)^{-1} \int \left\{ \sqrt{z-x} \mathbf{1}_{[x, \infty)}(z) - \sqrt{z-x-\delta_n^* t} \mathbf{1}_{[x+\delta_n^* t, \infty)}(z) \right\} d(\mathbb{G}_n - G)(z),
\end{aligned} \quad (14)$$

we will now establish convergence in distribution of the process Z_n . To do so, we need that $g(x) < \infty$. This follows directly from (9). Moreover, we need the asymptotic covariance structure of the process Z_n , which is given by

$$\text{Cov}(Z_n(s), Z_n(t)) = \frac{1}{2\gamma_x} g(x) st \left(1 - \frac{\log \log n}{\log n} \right) + O\left(\frac{1}{\log n} \right), \quad (15)$$

for each fixed $s, t \in I_x$, provided that $\delta_n = n^{-1/2} \sqrt{\log n}$. Therefore, using (15) and the Lindeberg–Feller Central Limit theorem we have that

$$Z_n(1) \rightsquigarrow N(0, g(x)/(2\gamma_x)), \quad (16)$$

(the proof of (15) and the proof of (16) are provided in Appendix 1). On the other hand, by Chebyshev's inequality and (15), we have, $\forall \varepsilon > 0$:

$$\mathbb{P}(|Z_n(t) - tZ_n(1)| > \varepsilon) \leq \frac{\mathbb{V}(Z_n(t) - tZ_n(1))}{\varepsilon^2} \rightarrow 0.$$

Therefore, the finite-dimensional distributions of Z_n converge weakly to the finite-dimensional distributions of the process: $Z(t) = tX$, in $C(I_x)$, where $X \sim N(0, g(x)/(2\gamma_x))$. We now need a stochastic equicontinuity condition to apply Theorem 2.3 from Kim and Pollard (1990). Take the class of functions for $\varepsilon > 0, M > 0$:

$$\begin{aligned} \mathcal{F}_{n,\varepsilon}^M = \{ & \sqrt{z - x - \delta_n^* t} \mathbf{1}_{[x + \delta_n^* t, \infty)}(z) - \sqrt{z - x - \delta_n^* s} \mathbf{1}_{[x + \delta_n^* s, \infty)}(z) \\ & : s, t \in I_x, |s - t| < \varepsilon, \max(|s|, |t|) < M \}. \end{aligned} \quad (17)$$

This class has square integrable natural envelope H_ε that satisfies for all $\varepsilon > 0$ small (cf. Lemma A.1): $\int H_\varepsilon^2(z)g(z)dz \leq g(x + \delta_n^* s \wedge t)(\varepsilon\delta_n^*)^2 \log \frac{1}{(\varepsilon\delta_n^*)^2}$ (as proved in Lemma 2.1 this class of functions is also VC). Therefore, by using maximal inequality 3.1 (ii) from Kim and Pollard (1990) we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\mathcal{F}_{n,\varepsilon}^M} |Z_n(s) - Z_n(t)| > \eta \right\} & \leq \limsup_{n \rightarrow \infty} \frac{\mathbb{E} \sup_{\mathcal{F}_{n,\varepsilon}^M} |Z_n(s) - Z_n(t)|^2}{\eta^2} \\ & \leq \limsup_{n \rightarrow \infty} \frac{J(1)^2 g(x + \delta_n^* s \wedge t)}{(\delta_n^*)^2 \log n} (\varepsilon\delta_n^*)^2 \log (\varepsilon\delta_n^*)^{-2} = J(1)^2 \gamma_x^{-1} \eta^{-2} g(x) \varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} 0, \end{aligned}$$

where the reason why the uniform entropy integral is finite is as in the proof of Lemma 2.1. The stochastic equicontinuity condition, together with the convergence in distribution of the finite-dimensional distributions, implies convergence in distribution of Z_n to Z in $(C(I_x), d)$ where d is the metric of uniform convergence on compacta. Next, we need the stochastic boundedness of the sequence T_n given below in (18) (the proof of which is given in Appendix 1). For $a_0 = V(x)$ and $n \rightarrow \infty$:

$$T_n(a_0 + \delta_n a) = V^{-1}(a_0) + O_p(\delta_n^*). \quad (18)$$

One can now apply Theorem 2.7 from Kim and Pollard (1990) to the process $\tilde{Z}_n$.

$$(\delta_n^*)^{-1} (T_n(a_0 + \delta_n a) - x) \rightsquigarrow \arg \max_{t \in I_x} \left\{ tX - \frac{|t|^{\gamma_x+1} \pi}{2m_0} K_x - at \right\}. \quad (19)$$

But it is immediate to see that, almost surely:

$$\arg \max_{t \in I_x} \left\{ tX - \frac{|t|^{\gamma_x+1} \pi}{2m_0} K_x - at \right\} \leq 0 \iff X \leq a. \quad (20)$$

Hence, using the two relations above:

$$\begin{aligned} \mathbb{P} \left\{ \delta_n^{-1} \left(\hat{V}_n(x) - V(x) \right) \leq a \right\} &= \mathbb{P} \left\{ (\delta_n^*)^{-1} (T_n(a_0 + \delta_n a) - x) \leq 0 \right\} \xrightarrow{n \rightarrow \infty} \\ &\stackrel{(19)}{\longrightarrow} \mathbb{P} \left\{ \arg \max_{t \in I_x} \left\{ tX - \frac{|t|^{\gamma_x+1} \pi}{2m_0} K_x - at \right\} \leq 0 \right\} \stackrel{(20)}{=} \mathbb{P}\{X \leq a\}. \end{aligned}$$

■

Example 2.3: As in Golubev and Levit (1998), it is possible to conduct the analysis in accordance with the global smoothness properties of the underlying distribution function F . Consider, for arbitrary $\gamma = \alpha + \beta, \alpha = 0, 1, \dots, 0 < \beta \leq 1$, the Hölder class C^γ of functions $f(y), y \geq 0$ having finite norm:

$$\|f\|_\gamma = \sup_{y \geq 0} |f(y)| + \sup_{x, y \geq 0} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|}{|x - y|^\beta}. \quad (21)$$

Then assume $F \in C^\gamma$ for $\gamma > \frac{1}{2}$. Moreover, assume

$$\lim_{t \rightarrow 0} \frac{|F(x+t) - F(x)|}{|t|^\gamma} = K > 0. \quad (22)$$

Note that it is possible to prove that if $F \in C^\gamma$ for $\gamma > 1/2$, then the density of the observations g is bounded and continuous (see Lemma A.3). We show that the combination of (21) and (22) implies (5). Using the Dominated Convergence Theorem in combination with the assumption of the Hölder condition, and (22), we obtain as $\delta \rightarrow 0$:

$$\int_0^1 \frac{F(x+u\delta) - F(x)}{\text{sgn}(\delta)|\delta|^\gamma u^\gamma} u^\gamma \, du \rightarrow K. \quad (23)$$

Also for the case of Theorem 1.1(b), we look first at the behaviour of $\hat{V}_n(x)$.

Theorem 2.2: Let $x \geq 0$, and $K_x := [\underline{x}, \bar{x}] \subset [0, \infty)$ the biggest interval that contains x on which F is constant. Assume that (8) holds true. Then

$$\sqrt{n} \left(\hat{V}_n(x) - V(x) \right) \rightsquigarrow L_x,$$

where, for any $a \in \mathbb{R}$:

$$\mathbb{P}(L_x \leq a) = \mathbb{P} \left(\arg \max_{s \in K_x} \{ \mathbb{Z}_x(s) - as \} \leq x \right),$$

for $\mathbb{Z}_x$ a centred continuous Gaussian Process with covariance structure given by

$$\text{Cov}(\mathbb{Z}_x(t), \mathbb{Z}_x(s)) = \text{Cov} \left(2(\sqrt{(Z_x)_+} - \sqrt{(Z_t)_+}), 2(\sqrt{(Z_x)_+} - \sqrt{(Z_s)_+}) \right), \quad (24)$$

where $Z_x := Z - x$ and analogously Z_t and Z_s .

Proof: Using (10), we obtain for each fixed $a \in \mathbb{R}$, for all n sufficiently large:

$$\sqrt{n} \left(\hat{V}_n(x) - V(x) \right) \leq a$$

$$\begin{aligned}
&\Leftrightarrow \inf \{s \geq 0 : U_n(s) - (V(x) + a/\sqrt{n})s \text{ is maximal}\} \leq x \\
&\Leftrightarrow \inf \{s \geq 0 : (U_n(s) - U_n(x) - U(s) + U(x)) \\
&\quad + (U(s) - U(x) - V(x)(s - x)) - (as)/\sqrt{n} \text{ is maximal}\} \leq x,
\end{aligned}$$

where we use that the location of a maximum of a function is equivariant under translations. Now using the invariance under multiplication by a positive number and addition of a constant, and defining

$$Z_n(s) := \sqrt{n}(U_n(s) - U_n(x) - U(s) + U(x)) \quad \text{and} \quad (25)$$

$$h_x(s) := U(x) - U(s) - V(x)(x - s), \quad (26)$$

we obtain that the above event can be rewritten as

$$\inf \{s \geq 0 : Z_n(s) - as - \sqrt{n}h_x(s) \text{ is maximal}\} \leq x.$$

Lemma 2.1: *Let ℓ^∞ be endowed with uniform norm. Define*

$$Z_n^*(s) := \sqrt{n}(U_n(s) - U(s)),$$

then

$$Z_n^* \rightsquigarrow \mathbb{Z} \quad \text{in } \ell^\infty[0, \infty), \quad (27)$$

where $\mathbb{Z}$ is a Gaussian Process with a.s. continuous sample paths relative to the Euclidean metric. In particular

$$\|U_n - U\|_\infty = \sup_{s \geq 0} |U_n(s) - U(s)| \xrightarrow{\mathbb{P}} 0. \quad (28)$$

The proof of this lemma is given in Appendix 1. Consequently:

$$Z_n = Z_n^* - Z_n^*(x) \rightsquigarrow \mathbb{Z} - \mathbb{Z}(x) =: \mathbb{Z}_x \quad \text{in } \ell^\infty[0, \infty), \quad (29)$$

where also $\mathbb{Z}_x$ is a Gaussian Process with a.s. continuous sample paths relative to the Euclidean metric. By the above computations, we conclude that the limiting behaviour of

$$\tilde{Z}_n(s) := Z_n(s) - as - \sqrt{n}h_x(s)$$

is determined by the process $\mathbb{W}$ which is defined by

$$\mathbb{W}(s) = \begin{cases} \mathbb{Z}_x(s) - as, & \text{for } s \in [\underline{x}, \bar{x}] \\ -\infty, & \text{for } s \notin [\underline{x}, \bar{x}]. \end{cases}$$

This is because $h_x(s) \geq 0$ for all $s \in (0, \underline{x}) \cup (\bar{x}, \infty)$, and therefore multiplied by $-\sqrt{n}$ will go to $-\infty$ in case the inequality is strict; however $h_x(s) = 0, \forall s \in [\underline{x}, \bar{x}]$ and thus on that interval the limiting behaviour is completely determined by $\mathbb{Z}_x(s) - as$ (using the limiting behaviour of Z_n derived above). Let

$$\hat{s}_n := \inf \{s \geq 0 : Z_n(s) - as - \sqrt{n}h_x(s) \text{ is maximal}\} \quad (30)$$

$$\hat{s} := \arg \max_{s \geq 0} \{\mathbb{W}(s)\} = \arg \max_{s \in [\underline{x}, \bar{x}]} \{\mathbb{Z}_x(s) - as\}. \quad (31)$$

To prove convergence in distribution, we use the Portmanteau Lemma and show for every closed subset F of $[0, \infty)$: $\limsup_{n \rightarrow \infty} \mathbb{P}(\hat{s}_n \in F) \leq \mathbb{P}(\hat{s} \in F)$.

Let us start by choosing a sequence $\varepsilon_n \downarrow 0$ such that $\sqrt{n}h_x(\underline{x} - \varepsilon_n) \rightarrow \infty$ and $\sqrt{n}h_x(\bar{x} + \varepsilon_n) \rightarrow \infty$. This exists since $\forall \varepsilon > 0$, we have $\sqrt{n}h_x(\underline{x} - \varepsilon) \wedge h_x(\bar{x} + \varepsilon) \rightarrow \infty$. Moreover, $h_x(\bar{x} + \varepsilon)$ is an increasing function in ε that has a zero in $\varepsilon = 0$. The behaviour is analogous for $h_x(\underline{x} - \varepsilon)$. Therefore it is sufficient to construct a sequence of ε_n such that $h_x(\underline{x} - \varepsilon_n) \wedge h_x(\bar{x} + \varepsilon_n) = O(1/n^\alpha)$ for some $\alpha < 1/2$. For $K_n := [\underline{x} - \varepsilon_n, \bar{x} + \varepsilon_n]$ and for $K := [\underline{x}, \bar{x}]$ (denoted by K_x in the statement):

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}(\hat{s}_n \in F) &\leq \limsup_{n \rightarrow \infty} \mathbb{P}(\hat{s}_n \in F \cap K_n) + \limsup_{n \rightarrow \infty} \mathbb{P}(\hat{s}_n \in K_n^c) \\ &\leq \underbrace{\limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{s \in F \cap K_n} \tilde{Z}_n(s) \geq \sup_{s \in K_n} \tilde{Z}_n(s)\right)}_{(1)} + \underbrace{\limsup_{n \rightarrow \infty} \mathbb{P}(\hat{s}_n \in K_n^c)}_{(2)}. \end{aligned} \quad (32)$$

We look at (2) in (32) and show it goes to zero. To do this, we need relation (33) and its proof is given in Appendix 1. Thus

$$\sup_{s \in K_n^c} \tilde{Z}_n(s) = \sup_{s \in K_n^c} \{Z_n(s) - as - \sqrt{n}h_x(s)\} \xrightarrow{\mathbb{P}} -\infty \quad (33)$$

implies that

$$\begin{aligned} \mathbb{P}(\hat{s}_n \in K_n^c) &\leq \mathbb{P}\left(\sup_{s \in K_n^c} \tilde{Z}_n(s) \geq \sup_{s \in K_n} \tilde{Z}_n(s)\right) \\ &\leq \mathbb{P}\left(\sup_{s \in K_n^c} \tilde{Z}_n(s) \geq \sup_{s \in K} \{Z_n(s) - as\}\right) \rightarrow 0. \end{aligned} \quad (34)$$

The last convergence is a consequence of (33) and that $Z_n \rightsquigarrow \mathbb{Z}$ on $\ell^\infty(K)$, combined with the continuous mapping theorem (i.e. the term on the left-hand side of the inequality is $O_p(1)$).

Now we argue the behaviour of (1). For that, we need an additional convergence in probability given in (35), and its proof is provided in Appendix 1.

$$\sup_{s \in F \cap K_n} \{Z_n(s) - \sqrt{n}h_x(s) - as\} - \sup_{s \in F \cap K} \{Z_n(s) - as\} \xrightarrow{\mathbb{P}} 0. \quad (35)$$

Using that

$$\sup_{s \in F \cap K} \{Z_n(s) - as\} \rightsquigarrow \sup_{s \in F \cap K} \{\mathbb{Z}_x(s) - as\},$$

we conclude using Theorem 2.7(iv) from van der Vaart (1998) and (35):

$$\sup_{s \in F \cap K_n} \{Z_n(s) - \sqrt{n}h_x(s) - as\} \rightsquigarrow \sup_{s \in F \cap K} \{\mathbb{Z}_x(s) - as\}.$$

We derive the behaviour of (1) in (32). This term can be upper bounded by

$$\begin{aligned} & \limsup \mathbb{P} \left(\sup_{s \in F \cap K_n} \{Z_n(s) - \sqrt{n}h_x(s) - as\} \geq \sup_{s \in K} \{Z_n(s) - as\} \right) \\ & \leq \mathbb{P} \left(\sup_{F \cap K} \{\mathbb{Z}_x(s) - as\} \geq \sup_{s \in K} \{\mathbb{Z}_x(s) - as\} \right) \leq \underbrace{\mathbb{P}(\hat{s} \in K^c)}_{=0} + \mathbb{P}(\hat{s} \in F), \end{aligned} \quad (36)$$

where we used the above derivations, together with Lemma 2.1 and Theorem 2.7(v) from van der Vaart (1998). Therefore by (32) combined with (34), (36) and Portmanteau Lemma, we obtain the desired convergence in distribution, which can be restated:

$$\begin{aligned} & \mathbb{P} \left(\sqrt{n}(\hat{V}_n(x) - V(x)) \leq a \right) = \mathbb{P} \left(\inf_{s \geq 0} \{Z_n(s) - as - \sqrt{n}h_x(s) \text{ is maximal}\} \leq x \right) \\ & \xrightarrow{n \rightarrow \infty} \mathbb{P} \left(\arg \max_{s \in K} \{\mathbb{Z}_x(s) - as\} \leq x \right). \end{aligned}$$

Note that the last expression defines the cdf of a random variable L_x , which is the right derivative of the least concave majorant of the process $\mathbb{Z}_x$ evaluated at x . Indeed, by the switch relation, we have $\mathbb{P}(\arg \max_{s \in K} \{\mathbb{Z}_x(s) - as\} \leq x) = \mathbb{P}(L_x \leq a)$. ■

Note that the proof of Theorem 1.1(a) (included in Appendix 1) is a consequence of Slutsky's Lemma, Theorems 2.1 and 2.2, the fact that $1 - F(x) = V(x)/V(0)$ together with the asymptotic independence of $\hat{V}_n(x)$ and $\hat{V}_n(0)$. The proof of Theorem 1.1(b) follows the same steps as the proof of Theorem 2.2.

Remark 2.2: Let $x > 0$, and suppose that F satisfies condition (5) both at x and at 0 for a common $\gamma_x > \frac{1}{2}$ and (8) holds true. Then, by Theorem 1.1(a):

$$\sqrt{\frac{n}{\log n}} \left(\hat{F}_n(x) - F(x) \right) \rightsquigarrow N \left(0, \frac{g(x)V(0)^2 + g(0)V(x)^2}{2\gamma_x V(0)^4} \right). \quad (37)$$

The asymptotic variance obtained here coincides with the one obtained in Golubev and Levit (1998) where it is given a unique γ and the one in Groeneboom and Jongbloed (1995) with $\gamma_x = 1$.

Simulation study limiting process Theorem 2.2

In this section, we present some simulations about the distribution of L_x of Theorem 2.2; in particular we show how to constructively sample from L_x and that, even if L_x is not normal, its behaviour is close to being normal. First, we consider an underlying distribution function F of the squared radii as displayed in the left panel of Figure 3, which is constant on $[2, 3]$; observe in light blue a histogram of 1000 gathered samples from the associated density of the observations g (note that we obtain observations also on the interval where F is constant).

In this setting, we show some simulations of the distribution of the limiting random variable of Theorem 2.2, above denoted by L_x . To do that, we compute the kernel given

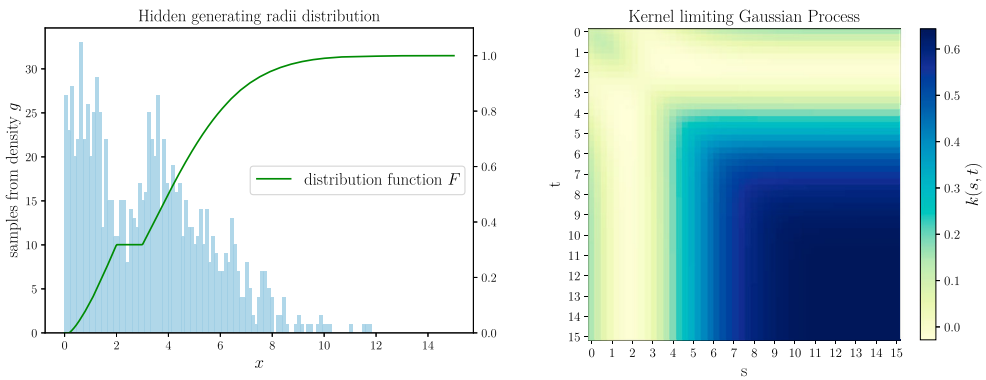


Figure 3. Left panel: Underlying cdf F with samples from g . Right panel: Representation of covariance structure of the limiting Gaussian Process $\mathbb{Z}_x$.

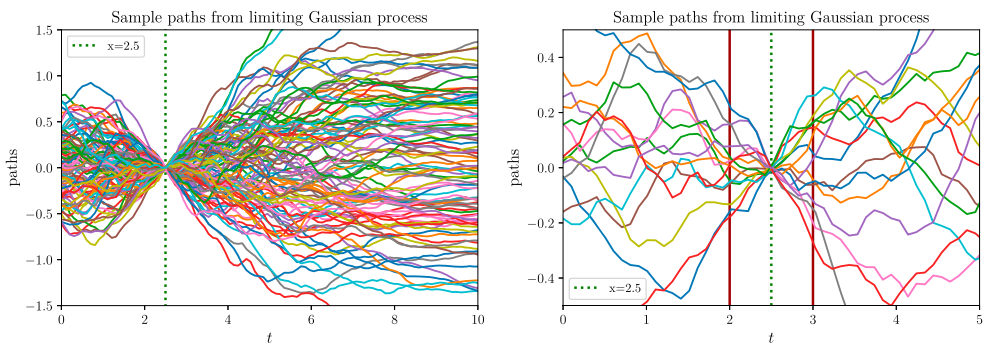


Figure 4. Left panel: 200 samples from limiting Gaussian Process $\mathbb{Z}_x$. Right panel: Zoom in on 15 sample paths, putting into evidence the interval $[2, 3]$.

in (6) of the limiting Gaussian process $\mathbb{Z}_x$. For the F given in the left panel of Figure 3, the corresponding kernel is given in the right panel of Figure 3. Subsequently, we simulate 200 sample paths from the limiting Gaussian Process $\mathbb{Z}_x$ with the given kernel. These sample paths are shown in Figure 4. In the right panel of Figure 4, we zoom in on the interval $[2, 3]$ of interest and look at the behaviour of 15 selected sample paths. As we would expect, all sample paths pass through $(2.5, 0)$.

To simulate the distribution of L_x , we use the fact that such random variable, by the proof of the theorem, is the right derivative of the least concave majorant of the Gaussian Process $\mathbb{Z}_x$ evaluated at $x=2.5$. With all the gathered samples from L_x , we looked at its estimated distribution and below we show some tests about the normality of the distribution of L_x . With the gathered samples, we see in Figure 5 that we could not reject the null hypothesis that L_x is normally distributed.

3. Local asymptotic minimax theorem

In this section, we show that the isotonic estimator (IIE) is asymptotically efficient, in the sense that its asymptotic variance is the smallest attainable. For this reason, we establish a statement about a lower bound of the minimax risk for any estimator sequence. We do

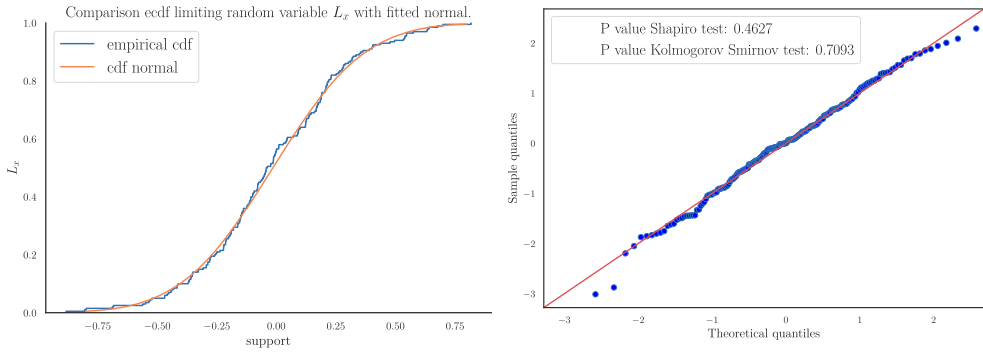


Figure 5. Left panel: ecdf of limiting random variable L_x compared to a suitably fitted normal. Right panel: qq-plot of L_x compared to the fitted normal and p -values of the standard tests.

this by constructing a parametric perturbation along the least favourable direction in the proximity of the true distribution function F .

Let $x > 0$ be the point at which we want to estimate F . The existence of a least favourable path is a non-trivial matter. We will be guided by intuition in the construction of this path. We want to estimate $F(x) = 1 - V(x)/V(0)$. The local behaviour at x and 0 of the kernels in the definitions of $V(x)$ and $V(0)$ is what essentially characterises the estimation in Wicksell's problem and leads to the unusual rate $n^{-1/2} \sqrt{\log n}$. Thus it should be enough to introduce a local perturbation of F around x and 0. However, such perturbation must be tailored according to the local level of smoothness of the true F near x and 0. Therefore, we need to identify such a level, and this is done in the following condition (38).

The level of locality of the condition must be tied to the smoothness itself, because if the true F is very rough around x or 0, then we expect a more stringent level of locality (and vice versa). Therefore we define the following intervals close to $p \in \{0, x\}$ for constants $\gamma_0 > \frac{1}{2}$, $\gamma_x > \frac{1}{2}$ to be determined later:

$$rK_n^0 := \left[n^{-\frac{1}{2\gamma_0}}, \eta_n \right],$$

$$lK_n^x := \left[x - \eta_n, x - n^{-\frac{1}{2\gamma_x}} \right], \quad rK_n^x := \left[x + n^{-\frac{1}{2\gamma_x}}, x + \eta_n \right],$$

where $\eta_n \downarrow 0$, $0 < \eta_n \leq 1$, $\forall n$ and $\exists \alpha \in (0, 1)$ such that $\eta_n \gtrsim \frac{1}{\log^\alpha(n)}$. Now, let γ_0, γ_x be the smallest constants greater than 1/2, such that the true F , for $K_n^p \in \{rK_n^x, lK_n^x, rK_n^0\}$ and therefore $p \in \{0, x\}$ satisfies

$$\liminf_{n \rightarrow \infty} \inf_{x_1, x_2 \in K_n^p} \sqrt{n \log n} \frac{|F(x_2) - F(x_1)|}{\left| \log \left\{ \frac{p - x_1}{p - x_2} \right\} \right|} = \infty. \quad (38)$$

With condition (38), our path can be shown to be well-defined and it is the least restrictive we can require. As we naturally would expect, the γ_0, γ_x introduced above contain information about the local roughness of F near x and 0. We illustrate this with our previous Example 2.1 and an additional example.

Example 3.1: Let us consider the same class of functions as in Example 2.1. These functions can be thought to be tightly Hölder smooth at x and satisfy (38). Let us consider the

interval $rK_n^x = [x + n^{-1/(2\gamma_x)}, x + \log(n)^{-\alpha}]$. If $F(u) = F(x) + K(u - x)^{\gamma_x}$ for $u \in rK_n^x$, then

$$\begin{aligned} \inf_{x_1, x_2 \in rK_n^x} \frac{|F(x_2) - F(x_1)|}{|\log \{(x - x_1) / (x - x_2)\}|} &= K \inf_{\substack{x_1, x_2 \in rK_n^x \\ x_1 < x_2}} \frac{(x_2 - x)^{\gamma_x} - (x_1 - x)^{\gamma_x}}{\log \{1 + (x_2 - x_1) / (x_1 - x)\}} \\ &= K \inf_{\substack{x_1, x_2 \in rK_n^x \\ x_1 < x_2}} (x_1 - x)^{\gamma_x} \frac{[1 + (x_2 - x_1) / (x_1 - x)]^{\gamma_x} - 1}{\log [1 + (x_2 - x_1) / (x_1 - x)]} = K\gamma_x \inf_{x_1 \in rK_n^x} (x_1 - x)^{\gamma_x} \\ &= K\gamma_x n^{-1/2}, \end{aligned}$$

because $\frac{(1+y)^\gamma - 1}{\log(1+y)}$ is increasing in $y > 0$ with limit γ as $y \downarrow 0$.

Example 3.2: Condition (38), which is similar to the conditions of Golubev and Levit (1998) (cf. (24) and (25)), is a consequence of a stronger one, namely:

$$\liminf_{n \rightarrow \infty} \inf \frac{|F(x_2) - F(x_1)|}{|x_2 - x_1|^{\gamma_x}} > 0,$$

where the infimum is as in (38). We show that it implies (38) on rK_n^x :

$$\inf_{x_1, x_2 \in rK_n^x} \frac{F(x_2) - F(x_1)}{\log \left\{ \frac{x - x_1}{x - x_2} \right\}} = \inf_{x_1, x_2 \in rK_n^x} \frac{|F(x_2) - F(x_1)|}{|x_2 - x_1|^{\gamma_x}} \frac{|x_2 - x_1|^{\gamma_x}}{\log \left(1 - \frac{x_2 - x_1}{x_2 - x} \right)},$$

where $x_1 < x_2$, both $x_1, x_2 \in rK_n^x$ and thus $0 < \frac{x_2 - x_1}{x_2 - x} < 1$ going to zero as $n \rightarrow \infty$. The above expression, for all n big enough behaves like in the previous example. Multiplied by $\sqrt{n \log n}$ (38) diverges to ∞ .

The second fundamental assumption to obtain our minimax lower bound is that g is continuous in a neighbourhood of x and in a right neighbourhood of 0. We will assume this all along. A sufficient condition to obtain this is that F is locally Hölder continuous around x and 0 of degrees greater than $\frac{1}{2}$ (see Lemma A.3 in Appendix 1). We also note that from (38) we have $F(x) < 1$.

We will now proceed with the construction of the path on the space of distribution functions and verify that it is well defined. The construction of our path is based on ideas from Golubev and Levit (1998). Condition (38) gives us the appropriate γ_0, γ_x that we use for the definition of the perturbation. For $u \geq 0$ define

$$F_{h_n}(u) = \frac{1}{D_{h_n}} \left(F(u) + h_{1,n} \int_0^u \chi_{1,n}(v) dv + h_{2,n} \int_0^u \chi_{2,n}(v - x) dv \right), \quad (39)$$

where for $\gamma_0 > \frac{1}{2}, \gamma_x > \frac{1}{2}$, using that $\int_0^\infty \chi_{2,n}(v - x) dv = 0$:

$$\begin{aligned}
\chi_{1,n}(v) &= (v)^{-1} \mathbf{1}_{\{n^{-1/(2\gamma_0)} \leq v \leq \eta_n\}}, \\
\chi_{2,n}(v-x) &= (v-x)^{-1} \mathbf{1}_{\{n^{-1/(2\gamma_x)} \leq |v-x| \leq \eta_n\}}, \\
D_{h_n} &= 1 + h_{1,n} \int_0^\infty \chi_{1,n}(v) \, dv + h_{2,n} \int_0^\infty \chi_{2,n}(v-x) \, dv \\
&= 1 + h_{1,n} \int_0^\infty \chi_{1,n}(v) \, dv.
\end{aligned}$$

We will see that the truncation levels of the function $1/(v-x)$ proposed here above work, but they may not be unique. The function $\chi_{i,n}$, with similar truncation levels, was introduced for the first time in Golubev and Levit (1998), however, there it was not used directly for the perturbation as done here and the assumption $\gamma_0 = \gamma_x$ was used.

We will check that the perturbation F_{h_n} is well defined for

$$h_{i,n} = \frac{h_i}{\sqrt{n \log n}}, \quad h_i \in \mathbb{R}, \quad i = 1, 2. \quad (40)$$

First, we compute the case of $\chi_{2,n}$ to show how the perturbation looks like:

$$\begin{aligned}
\int_0^u \chi_{2,n}(v-x) \, dv &= \log(\eta_n^{-1}(x-u)) \mathbf{1}_{\{x-\eta_n \leq u \leq x-n^{-\frac{1}{2\gamma_x}}\}} \\
&+ \log(\eta_n^{-1}n^{-\frac{1}{2\gamma_x}}) \left(\mathbf{1}_{\{u > x-n^{-\frac{1}{2\gamma_x}}\}} - \mathbf{1}_{\{u > x+\eta_n\}} \right) \\
&+ \log(\eta_n^{-1}(u-x)) \mathbf{1}_{\{x+n^{-\frac{1}{2\gamma_x}} \leq u \leq x+\eta_n\}}.
\end{aligned} \quad (41)$$

Figure 6 visualises the perturbation $\int_0^u \chi_{2,n}(v-x) \, dv$ in (41). It is immediate from (41) that this perturbation is continuous and therefore the summation with F results in a function F_{h_n} that is right-continuous. Moreover, thanks to the normalisation D_{h_n} , $\lim_{u \rightarrow +\infty} F_{h_n}(u) = 1$ and, from the definition of F_{h_n} , note that $\lim_{u \rightarrow 0} F_{h_n}(u) = 0$. What is not evident is if F_{h_n} is non-decreasing. Intuitively, for this to be true, we need F to be ‘increasing enough’ to compensate for the negative drop introduced by the perturbation. Condition (38) ensures that this is the case.

Lemma 3.1: *For all n sufficiently large, F_{h_n} is nondecreasing.*

The proof of this lemma is provided in Appendix 2. We start by stating the most interesting result of this section which is the so-called LAM (locally asymptotically minimax) theorem and it entails the minimax risk for any estimator sequence. The proof of this theorem requires a few additional technical statements (the LAN expansion and the expression of the derivative of the functional of interest), which will be provided right after.

Theorem 3.1 (LAM): *Let γ_0, γ_x be the smallest constants above $\frac{1}{2}$ such that F satisfies (38). Let $J = \frac{\pi^2}{8m_0^2} \text{diag}(\frac{1}{\gamma_0 g(0)}, \frac{1}{\gamma_x g(x)}), \dot{\psi} = \left[\frac{1}{2\gamma_0}(1-F(x)) - \frac{1}{2\gamma_x} \right]^\top$, g as in (1), and ℓ be*

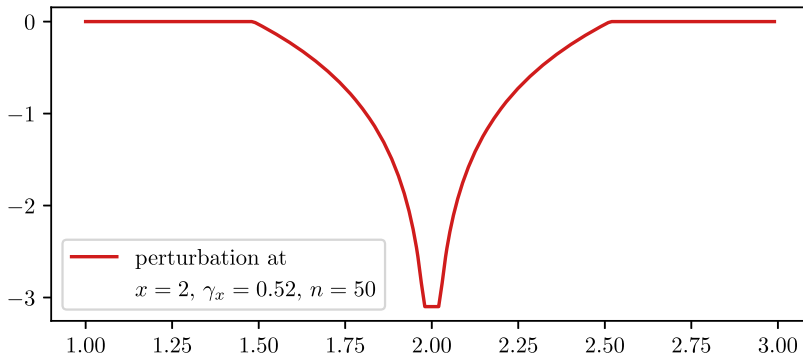


Figure 6. Visualisation of $\int_0^u \chi_{2,n}(v-x) dv$ for $x=2$, $n=50$ and $\gamma_x = 0.52$, $\eta_n = 1/\sqrt{\log n}$.

a bowl-shaped loss function, i.e. ℓ is non-negative, symmetric and subconvex. Then for every estimator sequence $(F_n(x))_{n \in \mathbb{N}}$:

$$\sup_{\substack{I \subset \mathbb{R}^2 \\ I \text{ finite}}} \liminf_{n \rightarrow \infty} \sup_{h \in I} \mathbb{E}_h \ell \left(\sqrt{\frac{n}{\log n}} (F_n(x) - F_{h_n}(x)) \right) \geq \mathbb{E} \ell(L). \quad (42)$$

The first supremum is taken over all finite subsets I of $\mathbb{R}^2$, the expectation $\mathbb{E}_h$ is under $G_{F_{h_n}}$, the probability measures associated to the perturbed densities g_n in (43), and $L \sim N(0, \dot{\psi}^\top J^{-1} \dot{\psi})$;

$$\dot{\psi}^\top J^{-1} \dot{\psi} = \frac{4m_0^2}{\pi^2} \left(\frac{g(x)}{2\gamma_x} + (1-F(x))^2 \frac{g(0)}{2\gamma_0} \right).$$

Proof: The proof is an immediate application of Theorem 3.11.5 from van der Vaart and Wellner (1996), using Theorem 3.2 and Lemma 3.3 given below. ■

Now we proceed by proving the technical results needed for the proof of Theorem 3.1. First, we show that given the path (39), we can obtain an LAN expansion (see van der Vaart 1998, Chapter 7). In the statement of the LAN Theorem, the perturbed density of the observations Z_i appears. Therefore, using (39), we derive it before proceeding:

$$\begin{aligned} g_n(z) &= \frac{1}{2m_{h_n}} \int_z^\infty \frac{dF_{h_n}(v)}{\sqrt{v-z}} \\ &= \frac{1}{2m_{h_n} D_{h_n}} \left\{ \int_z^\infty \frac{dF(v)}{\sqrt{v-z}} + h_{1,n} \int_z^\infty \frac{\chi_{1,n}(v)}{\sqrt{v-z}} dv + h_{2,n} \int_z^\infty \frac{\chi_{2,n}(v-x)}{\sqrt{v-z}} dv \right\}, \end{aligned} \quad (43)$$

where $m_{h_n} = \int_0^\infty \sqrt{s} dF_{h_n}(s)$, analogously to m_0 .

Theorem 3.2 (LAN expansion): Let γ_0, γ_x be constants above $\frac{1}{2}$ such that F satisfies (38). Let g be continuous in a neighbourhood of x and in a right-neighbourhood of 0. Then for

$h^\top = [h_1, h_2] \in \mathbb{R}^2$, $h_n = \frac{h}{\sqrt{n \log n}}$, F_{h_n} as in (39) and accordingly g_n in (43):

$$\sum_{i=1}^n \log \frac{g_n(Z_i)}{g(Z_i)} = h^\top \Delta_n - \frac{1}{2} h^\top J h + o_p(1),$$

where the $o_p(1)$ is under $Z_1, \dots, Z_n \stackrel{i.i.d.}{\sim} g$, for $J = \frac{\pi^2}{8m_0^2} \text{diag}(\frac{1}{\gamma_0 g(0)}, \frac{1}{\gamma_x g(x)})$ and $\Delta_n \rightsquigarrow N(0, J)$, where

$$\Delta_n = \begin{bmatrix} \sum_{i=1}^n \frac{1}{m_{h_n} D_{h_n} \sqrt{n \log n}} \left(\frac{1}{2g(Z_i)} \int_{Z_i}^{\infty} \frac{\chi_{1,n}(v)}{\sqrt{v-Z_i}} dv - \int_0^{\infty} \sqrt{v} \chi_{1,n}(v) dv \right) \\ \sum_{i=1}^n \frac{1}{m_{h_n} D_{h_n} \sqrt{n \log n}} \left(\frac{1}{2g(Z_i)} \int_{Z_i}^{\infty} \frac{\chi_{2,n}(v-x)}{\sqrt{v-Z_i}} dv - \int_0^{\infty} \sqrt{v} \chi_{2,n}(v-x) dv \right) \end{bmatrix}.$$

Proof: The structure of the proof of this result is as follows. First, we give the expansions of the functions $\int_z^{\infty} \frac{\chi_{2,n}(v-x)}{\sqrt{v-z}} dv$, $\int_z^{\infty} \frac{\chi_{1,n}(v)}{\sqrt{v-z}} dv$ that will highlight the leading terms that contribute to the limiting behaviour.

Next, we use the obtained approximations to write out the Taylor approximation of the logarithm and prove some additional relations that will eventually give the LAN. Finally, we show that the remaining terms are negligible. First of all, using the definition of F_{h_n} in (39), we obtain

$$\frac{g_n(z)}{g(z)} = \frac{1}{m_{h_n} D_{h_n}} \left\{ m_0 + \frac{h_{1,n}}{2g(z)} \int_z^{\infty} \frac{\chi_{1,n}(v)}{\sqrt{v-z}} dv + \frac{h_{2,n}}{2g(z)} \int_z^{\infty} \frac{\chi_{2,n}(v-x)}{\sqrt{v-z}} dv \right\}.$$

Then using (A16) (see Lemma A.4 proved in Appendix 2):

$$\begin{aligned} \int_z^{\infty} \frac{\chi_{2,n}(v-x)}{\sqrt{v-z}} dv &= \left\{ \frac{\pi}{\sqrt{z-x}} + O\left(\eta_n^{-\frac{1}{2}}\right) \right\} \mathbf{1}_{\left\{n^{-\frac{1}{2\gamma x}} \leq z-x \leq \eta_n\right\}} \\ &+ O\left(n^{\frac{1}{4\gamma x}}\right) \mathbf{1}_{\left\{-n^{-\frac{1}{2\gamma x}} \leq z-x \leq n^{-\frac{1}{2\gamma x}}\right\}} + O\left(\eta_n^{-\frac{1}{2}}\right) \mathbf{1}_{\{z-x \leq -\eta_n\}} \end{aligned} \quad (44)$$

$$+ \left\{ O\left(n^{-\frac{1}{2\gamma x}} (x-z)^{-\frac{3}{2}}\right) + O\left(\eta_n^{-\frac{1}{2}}\right) \right\} \mathbf{1}_{\left\{-\eta_n \leq z-x \leq -n^{-\frac{1}{2\gamma x}}\right\}}. \quad (45)$$

The proof of Lemma A.4 has the following relation as an important ingredient:

$$\int_u^{\infty} \frac{1}{t\sqrt{t-u}} dt = \pi \mathbf{1}_{\{u \geq 0\}} u^{-1/2}, \quad (46)$$

which follows from $\int_1^{\infty} \frac{1}{t\sqrt{t-1}} dt = \pi$. We will see that asymptotically the only term that matters is the first one in (44). It is immediate to see that $\sup_u \int_0^u \chi_{2,n}(v-x) dv = O(\log(\eta_n^{-1} n^{1/2\gamma x}))$, once multiplied by $h_2/\sqrt{n \log n}$ goes to zero as $n \rightarrow \infty$. Note also that using the same computations as in Equation (41) also for $\chi_{1,n}$, it follows that $D_{h_n} = 1 + o(1)$.

Denote by $A_n^{i,x} := \{n^{-\frac{1}{2\gamma x}} \leq Z_i^x \leq \eta_n\}$, where $Z_i^x := Z_i - x$. As will be seen, the only contributing term is $\pi \mathbf{1}_{A_n^{i,x}} / \sqrt{Z_i - x}$, we denote all the remainder terms by $R_i^{n,x}$ and it is immediate to see that

$$R_i^{n,x} = O\left(\eta_n^{-\frac{1}{2}}\right) \left\{ \mathbf{1}_{\{-\eta_n \leq Z_i^x \leq -n^{-\frac{1}{2\gamma x}}\}} + \mathbf{1}_{\{Z_i^x \leq -\eta_n\}} + \mathbf{1}_{A_n^{i,x}} \right\} \\ + O\left(n^{-\frac{1}{2\gamma x}}(x - Z_i)^{-\frac{3}{2}}\right) \mathbf{1}_{\{-\eta_n \leq Z_i^x \leq -n^{-\frac{1}{2\gamma x}}\}} + O\left(n^{\frac{1}{4\gamma x}}\right) \mathbf{1}_{\{-n^{-\frac{1}{2\gamma x}} \leq Z_i^x \leq n^{-\frac{1}{2\gamma x}}\}}.$$

Analogously, we define $R_i^{n,0}$. Using the Taylor expansion of the logarithm with Peano residual, and (45):

$$\sum_{i=1}^n \log \frac{g_n(Z_i)}{g(Z_i)} = \sum_{i=1}^n \left\{ \frac{1}{m_{h_n} D_{h_n}} \left[m_0 - m_{h_n} D_{h_n} + \frac{h_{1,n}}{2g(Z_i)} \int_{Z_i}^{\infty} \frac{\chi_{1,n}(v)}{\sqrt{v - Z_i}} dv + \right. \right. \\ \left. \left. + \frac{h_{2,n}}{2g(Z_i)} \int_{Z_i}^{\infty} \frac{\chi_{2,n}(v - x)}{\sqrt{v - Z_i}} dv \right] - \frac{1}{2} \left[\frac{m_0 - m_{h_n} D_{h_n}}{m_{h_n} D_{h_n}} + \frac{h_{1,n}}{2m_{h_n} D_{h_n} g(Z_i)} \left(\frac{\pi \mathbf{1}_{A_n^{i,0}}}{\sqrt{Z_i}} \right. \right. \right. \\ \left. \left. + R_n^{i,0} \right) + \frac{h_{2,n}}{2m_{h_n} D_{h_n} g(Z_i)} \left(\frac{\pi \mathbf{1}_{A_n^{i,x}}}{\sqrt{Z_i - x}} + R_n^{i,x} \right) \right]^2 + o_p \left(\left[\frac{m_0 - m_{h_n} D_{h_n}}{m_{h_n} D_{h_n}} + \right. \right. \\ \left. \left. + \frac{h_{1,n}}{2m_{h_n} D_{h_n} g(Z_i)} \left(\frac{\pi \mathbf{1}_{A_n^{i,0}}}{\sqrt{Z_i}} + R_n^{i,0} \right) + \frac{h_{2,n}}{2m_{h_n} D_{h_n} g(Z_i)} \left(\frac{\pi \mathbf{1}_{A_n^{i,x}}}{\sqrt{Z_i - x}} + R_n^{i,x} \right) \right]^2 \right) \right\}. \quad (47)$$

The Taylor expansion of the logarithm is granted by relation (48) (see Appendix 2 for its proof). For $\zeta_{1,n}(z) := \int_z^{\infty} \frac{\chi_{1,n}(v)}{\sqrt{v-z}} dv$, and for $\zeta_{2,n}(z-x) := \int_z^{\infty} \frac{\chi_{2,n}(v-x)}{\sqrt{v-z}} dv$ and F as in (38); then in probability:

$$\max_{1 \leq i \leq n} \left| \frac{h_{1,n} \zeta_{1,n}(Z_i)}{g(Z_i)} + \frac{h_{2,n} \zeta_{2,n}(Z_i - x)}{g(Z_i)} \right| \rightarrow 0. \quad (48)$$

Now we provide a lemma on the normal behaviour of the term inside the first square parenthesis in (47).

Lemma 3.2: For F as in (38) and $h_{i,n}$, $i = 1, 2$ as in (40):

$$\frac{1}{m_{h_n} D_{h_n}} \sum_{i=1}^n \left\{ \frac{h_{1,n}}{2g(Z_i)} \int_{Z_i}^{\infty} \frac{\chi_{1,n}(v)}{\sqrt{v - Z_i}} dv + \frac{h_{2,n}}{2g(Z_i)} \int_{Z_i}^{\infty} \frac{\chi_{2,n}(v - x)}{\sqrt{v - Z_i}} dv \right. \\ \left. - m_{h_n} D_{h_n} + m_0 \right\} \rightsquigarrow N \left(0, \frac{\pi^2}{8m_0^2} \left(\frac{h_1^2}{\gamma_0 g(0)} + \frac{h_2^2}{\gamma_x g(x)} \right) \right). \quad (49)$$

The proof of this lemma is provided in Appendix 2. Now we can partition $m_0 - m_{h_n} D_{h_n}$ into the means showed in the definition of Δ_n (use (??)). The terms appearing in Lemma 3.2 are a linear combination of the components of Δ_n . By Lemma 3.2 all such linear combinations have normal behaviour. But since we can show the normal convergence for any pair of h_i , $i = 1, 2$, the Cramér–Wold device allows us to obtain the joint asymptotic normality claim for the components of Δ_n .

To determine the behaviour of the leading terms inside the second square parenthesis in (47), i.e.

$$\sum_{i=1}^n \frac{h_{1,n}^2}{4m_{h_n}^2 D_{h_n}^2 g^2(Z_i)} \frac{\pi^2 \mathbf{1}_{A_n^{i,0}}}{Z_i}, \quad \sum_{i=1}^n \frac{h_{2,n}^2}{4m_{h_n}^2 D_{h_n}^2 g^2(Z_i)} \frac{\pi^2 \mathbf{1}_{A_n^{i,x}}}{Z_i - x}, \quad (50)$$

it is enough to use that (see Appendix 2 for its proof):

$$\frac{1}{n \log n} \sum_{i=1}^n \frac{\pi^2}{4m_{h_n}^2 g^2(Z_i)(Z_i - x)} \mathbf{1}_{\left\{\eta_n^{-1} \leq \frac{1}{Z_i - x} \leq n^{\frac{1}{2\gamma_x}}\right\}} \xrightarrow{\mathbb{P}} \frac{\pi^2}{8\gamma_x m_0^2 g(x)}. \quad (51)$$

The same result also applies for $A_n^{i,0}$ around 0, and this gives convergence in probability of the terms in (50). Now by grouping out properly the $h^\top = [h_1, h_2]$, we get the claim, as convergence in probability of the matrix is implied by the convergence of its components.

We now show that all the other terms converge in probability to zero. It is enough to look at multiples of the squared terms and show that these go to zero. Relation (52) gives that these remaining terms are negligible and its proof is given in Appendix 2. For $h_{j,n}, j = 1, 2$ as in (40) and accordingly $s \in \{0, x\}$:

$$\left(\frac{m_0 - m_{h_n} D_{h_n}}{m_{h_n} D_{h_n}}\right)^2 = o\left(\frac{1}{n}\right), \quad \sum_{i=1}^n \left(\frac{h_{j,n} R_n^{i,s}}{2m_{h_n} D_{h_n} g(Z_i)}\right)^2 = o_p(1). \quad (52)$$

We obtain also

$$o_p\left(\sum_{i=1}^n \left[\frac{m_0 - m_{h_n} D_{h_n}}{m_{h_n} D_{h_n}} + \frac{h_{1,n}}{2m_{h_n} D_{h_n} g(Z_i)} \left(\frac{\pi \mathbf{1}_{A_n^{i,0}}}{\sqrt{Z_i}} + R_n^{i,0}\right) + \frac{h_{2,n}}{2m_{h_n} D_{h_n} g(Z_i)} \left(\frac{\pi \mathbf{1}_{A_n^{i,x}}}{\sqrt{Z_i - x}} + R_n^{i,x}\right)\right]^2\right) = o_p(1),$$

by combining (51) and (52). ■

Remark 3.1: If we take $\gamma = \gamma_0 = \gamma_x$, the obtained asymptotic covariance looks the same as the one obtained in Golubev and Levit (1998) (cf. page 2418), but our γ_0, γ_x have a different meaning.

For the proof of the LAM (locally asymptotically minimax) theorem, the derivative of the map we are interested in estimating (cf. definition 1.10 in Bolthausen et al. 1999) is needed. In particular, let $\psi : \mathcal{G} \mapsto \mathbb{R}$, where $\mathcal{G}$ is the space of the probability measures G_F associated to the random variable Z (the observations). In our case, for our fixed estimation point $x > 0$, $\psi(G_F) = F(x) \in \mathbb{R}$.

Lemma 3.3: For γ_0, γ_x as in Theorem 3.2 greater than $\frac{1}{2}$, $h^\top = [h_1, h_2] \in \mathbb{R}^2$, $h_n = \frac{h}{\sqrt{n \log n}}$, F_{h_n} as in (39), $\dot{\psi} = \begin{bmatrix} \frac{1}{2\gamma_0}(1 - F(x)) & -\frac{1}{2\gamma_x} \end{bmatrix}^\top$, as $n \rightarrow \infty$:

$$\sqrt{\frac{n}{\log n}} (\psi(G_{F_{h_n}}) - \psi(G_F)) = \sqrt{\frac{n}{\log n}} (F_{h_n}(x) - F(x)) \rightarrow h^\top \dot{\psi}.$$

The proof of this lemma is provided in Appendix 2. The combination of Theorem 3.2 and Lemma 3.3 yields the main result Theorem 3.1.

4. Conclusion

We conclude by stressing the fact that the local asymptotically minimax variance in Theorem 3.1 coincides with the one obtained in Theorem 1.1(a). Moreover, as we naturally expect, both the estimator's behaviour and the minimax theorem are obtained under some local conditions at x and 0 of the smoothness of F . The class of functions that are tightly Hölder continuous at these points as in Example 2.1 satisfy the key conditions needed for the estimator and for the minimax theorem. In general, whenever the underlying F satisfies both (5) and (38), then the isotonic estimator (IIE) will attain the minimax asymptotic variance in an adaptive way and without the need of external tuning.

Regarding possible future works, Theorem 2.2 shows that if F is assumed to be constant on an interval containing x , the optimal rate of convergence increases to $\sqrt{n}$ and the IIE attains this rate adaptively, i.e. without explicitly using the assumption of local constancy. However, the asymptotic distribution in this case is not normal, thus the IIE is not efficient in the classical sense. An interesting future work would be to study other estimators that in this setting attain $\sqrt{n}$ rate of convergence and find which one is efficient. Another interesting but different research direction would be to study this problem in the Bayesian nonparametric setting and to compare Bayesian estimators to the state-of-the-art frequentist estimators. The authors of this paper are currently exploring these two research directions.

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Appendices

Appendix 1. Proofs of complementary statements for Theorem 2.1

Now we provide some proofs about the fundamental behaviour of the covariance structure of the process $Z_n(t)$. We start by showing the case of the variance and we will proceed by giving a proof along the same lines of (15). We will use several times how samples of Z with density g (see (1)) are obtained in Wicksell's problem. First, note that the observable radii in the cross-sections cannot exceed the radii of the associated sphere; the spheres with bigger radius are more likely to be sampled. Thus the distribution of the squared radii of the spheres actually cut is not F but a biased version of it; thus we write X^b with distribution function F^b for the squared radius of the spheres actually cut. Since we assume the samples are gathered uniformly, the observed squared circle radius Z is related to X^b via

$$Z = (1 - U^2)X^b = VX^b, \quad U \sim U[0, 1], \quad V \sim Be(1, 1/2). \quad (\text{A1})$$

Lemma A.1: Fix $x > 0$. If $g(x) < \infty$, then

$$\mathbb{E} \left(\sqrt{Z-x} \mathbf{1}_{\{Z>x\}} - \sqrt{Z-x-\varepsilon} \mathbf{1}_{\{Z>x+\varepsilon\}} \right)^2 \sim \varepsilon^2 \log \left(\frac{1}{\varepsilon} \right) g(x), \quad \text{as } \varepsilon \downarrow 0.$$

Proof: In the proof, to make the notation lighter, we will denote X^b by X (see (A1)). Define $Z_x := Z - x$. The left side is

$$\underbrace{\mathbb{E} Z_x \mathbf{1}_{\{Z_x < \varepsilon\}}}_{(1)} + \underbrace{\mathbb{E} \left(\sqrt{Z_x} - \sqrt{Z_x - \varepsilon} \right)^2 \mathbf{1}_{\{Z_x > \varepsilon\}}}_{(2)}.$$

We start with (1). By making a substitution below with $vX - x = y(X - x)$, which means $1 - v = (1 - y) \frac{X-x}{X}$, we obtain

$$\begin{aligned} \mathbb{E} Z_x \mathbf{1}_{Z_x < \varepsilon} &\stackrel{(A1)}{=} \mathbb{E} \left\{ \mathbf{1}_{X>x} \int_{0 < vX-x < \varepsilon} (vX - x) \frac{\mathbf{1}_{0 < v < 1}}{2\sqrt{1-v}} dv \right\} \\ &= \mathbb{E} \left\{ \mathbf{1}_{X>x} \int_0^{\frac{\varepsilon}{X-x} \wedge 1} \frac{y}{2\sqrt{1-y}} \frac{(X-x)^{3/2}}{\sqrt{X}} dy \right\}. \end{aligned}$$

If we define $y(z) = \int_0^z \frac{y}{2\sqrt{1-y}} dy$, then the above equals to

$$\mathbb{E} \left\{ y \left(\frac{\varepsilon}{X-x} \wedge 1 \right) \mathbf{1}_{X>x} \frac{(X-x)^{3/2}}{\sqrt{X}} \right\}.$$

Using $\int \frac{y}{2\sqrt{1-y}} dy = \frac{(1-y)^{3/2}}{3} - \sqrt{1-y} + c$, we can see that

$$y \left(\frac{\varepsilon}{X-x} \wedge 1 \right) = \int_0^{\frac{\varepsilon}{X-x} \wedge 1} \frac{y}{2\sqrt{1-y}} dy = \frac{1}{4} \left(\frac{\varepsilon}{X-x} \right)^2 + O \left(\left(\frac{\varepsilon}{X-x} \right)^3 \right).$$

Moreover, it is also true $y(z) \leq z^2, \forall z \in (0, 1)$. The above proves

$$\mathbb{E} Z_x \mathbf{1}_{Z_x < \varepsilon} = O(\varepsilon^2). \quad (A2)$$

Therefore (1) is negligible. Now we proceed with (2). Using Taylor's theorem with integral form of the remainder, we obtain

$$\begin{aligned} \mathbb{E} \left(\sqrt{Z_x} - \sqrt{Z_x - \varepsilon} \right)^2 \mathbf{1}_{\{Z_x > \varepsilon\}} &= \mathbb{E} \int_0^1 \int_0^1 \frac{\varepsilon}{2\sqrt{Z_x - \varepsilon s}} \frac{\varepsilon}{2\sqrt{Z_x - \varepsilon t}} ds dt \mathbf{1}_{\{Z_x > \varepsilon\}} \\ &\stackrel{(A1)}{=} \mathbb{E} \int_0^1 \int_0^1 \int_0^1 \frac{\varepsilon}{2\sqrt{vX - x - \varepsilon s}} \frac{\varepsilon}{2\sqrt{vX - x - \varepsilon t}} \frac{1}{2\sqrt{1-v}} \mathbf{1}_{\{vX-x>\varepsilon\}} dv ds dt. \end{aligned}$$

Now we use again the substitution $vX - x = y(X - x)$ which implies $1 - v = (1 - y) \frac{X-x}{X}$ and $y(\varepsilon, s, t) = \int_\varepsilon^1 \frac{1}{\sqrt{y-\varepsilon s}} \frac{1}{\sqrt{y-\varepsilon t}} \frac{1}{\sqrt{1-y}} dy$ to obtain

$$\begin{aligned} &= \frac{\varepsilon^2}{8} \mathbb{E} \int_0^1 \int_0^1 \int_0^1 \frac{1}{\sqrt{y - \frac{\varepsilon s}{X-x}}} \frac{1}{\sqrt{y - \frac{\varepsilon t}{X-x}}} \frac{1}{\sqrt{1-y}} \frac{\mathbf{1}_{\{y(X-x)>\varepsilon\}}}{\sqrt{X-x}\sqrt{X}} dy ds dt \\ &= \frac{\varepsilon^2}{8} \mathbb{E} \int_0^1 \int_0^1 y \left(\frac{\varepsilon}{X-x}, s, t \right) \frac{\mathbf{1}_{\{X-x>\varepsilon\}}}{\sqrt{X-x}\sqrt{X}} ds dt. \end{aligned}$$

Now we show the two following relations $\forall 0 < \varepsilon, s, t < 1$:

$$y(\varepsilon, s, t) \sim \log \frac{1}{\varepsilon} - \frac{1}{2} \log \log \frac{1}{\varepsilon} + O(1) \quad \text{as } \varepsilon \downarrow 0 \quad (A3)$$

$$y(\varepsilon, s, t) \leq \left(\log \frac{1}{\varepsilon} + \log \frac{e^2}{1-s \vee t} \right) \frac{1}{\sqrt{1-s \vee t}} + 1. \quad (\text{A4})$$

From this it will follow that

$$\mathbb{E} \left(\sqrt{Z_x} - \sqrt{Z_x - \varepsilon} \right)^2 \mathbf{1}_{\{Z_x > \varepsilon\}} \sim \frac{\varepsilon^2}{4} g(x) \log \frac{1}{\varepsilon}. \quad (\text{A5})$$

Fix $\varepsilon < b < 1$, then we can split the integral in the definition of $y(\varepsilon, s, t)$ into two integrals: one from ε to b and another one from b to 1. Let us look at the second one:

$$\int_b^1 \frac{(1-y)^{-\frac{1}{2}}}{y\sqrt{y-\varepsilon s}} \frac{y}{\sqrt{y-\varepsilon t}} dy \leq \frac{b}{b-\varepsilon} \int_b^1 \frac{1}{y\sqrt{1-y}} dy = \frac{b}{b-\varepsilon} 2 \tanh^{-1}(\sqrt{1-b}).$$

Here we used the fact that the mapping $y \mapsto \frac{y}{y-\varepsilon s}$ is decreasing on $[\varepsilon, 1]$, that $\frac{b}{b-\varepsilon s} \leq \frac{b}{b-\varepsilon}$ and $\int \frac{1}{y\sqrt{1-y}} dy = -2 \tanh^{-1}(\sqrt{1-y}) + c$. This shows that the asymptotic behaviour will be determined by the integral between ε and b ;

$$\begin{aligned} & \left| \int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon s}} \frac{1}{\sqrt{y-\varepsilon t}} \frac{1}{\sqrt{1-y}} dy - \int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon s}} \frac{1}{\sqrt{y-\varepsilon t}} dy \right| \\ & \leq \int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon s}} \frac{1}{\sqrt{y-\varepsilon t}} dy \left(\frac{1}{\sqrt{1-b}} - 1 \right). \end{aligned}$$

On the other hand, since

$$\int_\varepsilon^b \frac{1}{y-\varepsilon s \wedge t} dy \leq \int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon s}} \frac{1}{\sqrt{y-\varepsilon t}} dy \leq \int_\varepsilon^b \frac{1}{y-\varepsilon s \vee t} dy.$$

Therefore we have

$$\int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon s}} \frac{1}{\sqrt{y-\varepsilon t}} \frac{1}{\sqrt{1-y}} dy \begin{cases} \leq \log \left\{ \frac{b-\varepsilon s \vee t}{\varepsilon(1-s \vee t)} \right\} \frac{1}{\sqrt{1-b}}, \\ \geq \log \left\{ \frac{b-\varepsilon s \wedge t}{\varepsilon(1-s \wedge t)} \right\} \left(2 - \frac{1}{\sqrt{1-b}} \right). \end{cases}$$

Using $b = \frac{1}{\log \frac{1}{\varepsilon}}$, we obtain for fixed c :

$$\begin{aligned} \log \left\{ \frac{1/\sqrt{\log \frac{1}{\varepsilon}} - \varepsilon c}{\varepsilon(1-c)} \right\} &= \log \left\{ 1 - \varepsilon \sqrt{\log \frac{1}{\varepsilon}} c \right\} + \log \left\{ \frac{1}{\sqrt{\log \frac{1}{\varepsilon}}} \right\} + \log \frac{1}{\varepsilon} \\ &+ \log \frac{1}{1-c} \sim \log \frac{1}{\varepsilon} - \frac{1}{2} \log \log \frac{1}{\varepsilon} + O(1), \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$

This proves the first relation (A3). Now we proceed with relation (A4). Assume $s \leq t$. By performing a substitution $y - \varepsilon t = z(1 - \varepsilon t)$ which means $1 - y = (1 - \varepsilon t)(1 - z)$, we obtain

$$y(\varepsilon, s, t) \leq \int_\varepsilon^1 \frac{1}{y - \varepsilon t} \frac{1}{\sqrt{1-y}} dy = \frac{1}{\sqrt{1-\varepsilon t}} \int_{\frac{\varepsilon(1-t)}{1-\varepsilon t}}^1 \frac{1}{z} \frac{1}{\sqrt{1-z}} dz.$$

We used the fact that we can compute the following primitive: $\int \frac{1}{y\sqrt{1-y}} dy = -2 \tanh^{-1}(\sqrt{1-y}) + c$. Therefore

$$\frac{1}{\sqrt{1-\varepsilon t}} \int_{\frac{\varepsilon(1-t)}{1-\varepsilon t}}^1 \frac{1}{z} \frac{1}{\sqrt{1-z}} dz = 2 \tanh^{-1} \left(\sqrt{1 - \frac{\varepsilon(1-t)}{1-\varepsilon t}} \right) \frac{1}{\sqrt{1-\varepsilon t}}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{1-\varepsilon t}} \log \left\{ \frac{1 + \sqrt{\frac{1-\varepsilon}{1-\varepsilon t}}}{1 - \sqrt{\frac{1-\varepsilon}{1-\varepsilon t}}} \right\} = \log \left\{ \frac{(\sqrt{1-\varepsilon t} + \sqrt{1-\varepsilon})^2}{\varepsilon - \varepsilon t} \right\} \frac{1}{\sqrt{1-\varepsilon t}} \\
&\leq \left(\log \frac{1}{\varepsilon} + \log \frac{1-\varepsilon t}{1-t} + 2 \right) \frac{1}{\sqrt{1-\varepsilon t}} \leq \left(\log \frac{1}{\varepsilon} + \log \frac{e^2}{1-t} \right) \frac{1}{\sqrt{1-\varepsilon t}} + 1.
\end{aligned}$$

And this proves (A4). Now we proceed by using the two relations, we just proved. First of all, (A3) as $\varepsilon \downarrow 0$:

$$\frac{y\left(\frac{\varepsilon}{X-x}, s, t\right) \mathbf{1}_{\{X_x > \varepsilon\}}}{\log \frac{1}{\varepsilon}} = \frac{y\left(\frac{\varepsilon}{X_x}, s, t\right) \mathbf{1}_{\{X_x > \varepsilon\}}}{\log \frac{X_x}{\varepsilon}} \cdot \frac{\log X_x + \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}} \rightarrow \mathbf{1}_{\{X > x\}}.$$

Now using $\log xy \leq \log x \mathbf{1}_{x > 1} + \log y$ and (A4), we obtain

$$\begin{aligned}
\frac{y\left(\frac{\varepsilon}{X-x}, s, t\right) \mathbf{1}_{\{X-x > \varepsilon\}}}{\log \frac{1}{\varepsilon}} &\leq \frac{\left(\log \frac{X_x}{\varepsilon} + \log \frac{e^2}{1-s\sqrt{t}}\right) \frac{1}{\sqrt{1-s\sqrt{t}}} + 1}{\log \frac{1}{\varepsilon}} \\
&\lesssim \left(\log X_x \mathbf{1}_{X_x > 1} + \log \frac{e^2}{\sqrt{1-s\sqrt{t}}}\right) \frac{1}{\sqrt{1-s\sqrt{t}}} + 1.
\end{aligned}$$

Since the idea is to apply the Dominated Convergence Theorem to

$$\frac{1}{\log \frac{1}{\varepsilon}} \mathbb{E} \int_0^1 \int_0^1 y\left(\frac{\varepsilon}{X-x}, s, t\right) \frac{1}{2\sqrt{X-x}\sqrt{X}} \mathbf{1}_{\{X-x > \varepsilon\}} ds dt,$$

we need to verify that all the upper bounds have finite expectations. First $\frac{\log X_x \mathbf{1}_{\{X_x > 1\}}}{\sqrt{X}} \lesssim 1$ which means

$$\int_0^1 \int_0^1 \frac{\log X_x \mathbf{1}_{\{X_x > 1\}}}{\sqrt{X}} \frac{1}{\sqrt{1-s\sqrt{t}}} ds dt \lesssim 2 \int_0^1 \int_0^t \frac{1}{\sqrt{1-t}} ds dt < \infty.$$

On the other hand

$$\begin{aligned}
&\int_0^1 \int_0^1 \left(\log \frac{1}{1-s\sqrt{t}}\right) \frac{1}{\sqrt{1-s\sqrt{t}}} ds dt \lesssim 2 \int_0^1 \int_0^t \left(\log \frac{1}{1-t}\right) \frac{1}{\sqrt{1-t}} ds dt \\
&= 2 \int_0^1 t \left(\log \frac{1}{1-t}\right) \frac{1}{\sqrt{1-t}} dt < \infty.
\end{aligned}$$

So by the Dominated Convergence Theorem, if $g(x) = \mathbb{E} \frac{1}{2\sqrt{X-x}} \frac{1}{\sqrt{X}} \mathbf{1}_{\{X > x\}} < \infty$ we obtain the claim. ■

Proof of (15): In the proof, to make the notation lighter, we will denote X^b by X (see (A1)). Fix $s, t \in I_x$. Take the case $0 \leq s \leq t$. Fix μ ; we define the function $\phi(\cdot, \mu)$ as follows:

$$\phi(z, \mu) = \sqrt{z-x} \mathbf{1}_{[x, \infty)}(z) - \sqrt{z-x-\mu} \mathbf{1}_{[x+\mu, \infty)}(z).$$

Using the i.i.d. property, we can write

$$\text{Cov}(Z_n(s), Z_n(t)) = 4\delta_n^{-2} (\delta_n^*)^{-2} n^{-1} \text{Cov}(\phi(Z, \delta_n^* t), \phi(Z, \delta_n^* s)), \quad (\text{A6})$$

where Z has density g . Since $g(x) < \infty$, we will prove

$$\mathbb{E} \phi(Z, \delta_n^* s) \phi(Z, \delta_n^* t) = -\frac{1}{4} s t g(x) (\delta_n^*)^2 \log \delta_n^* + O((\delta_n^*)^2), \quad (\text{A7})$$

which implies (15) for $\delta_n = n^{-1/2} \sqrt{\log n}$ and $\delta_n^* = (\delta_n)^{1/\gamma_x}$. Define $Z_x := Z - x$:

$$\mathbb{E} \left[\sqrt{Z_x} \mathbf{1}_{\{Z_x > 0\}} - \sqrt{Z_x - \delta_n^* t} \mathbf{1}_{\{Z_x > \delta_n^* t\}} \right] \left[\sqrt{Z_x} \mathbf{1}_{\{Z_x > 0\}} - \sqrt{Z_x - \delta_n^* s} \mathbf{1}_{\{Z_x > \delta_n^* s\}} \right]$$

$$\begin{aligned}
&= \underbrace{\mathbb{E} Z_x \mathbf{1}_{\{0 < Z_x < \delta_n^* s\}}}_{(1)} + \underbrace{\mathbb{E} \left\{ \sqrt{Z_x} \left(\sqrt{Z_x} - \sqrt{Z_x - \delta_n^* s} \right) \mathbf{1}_{\{\delta_n^* s \leq Z_x \leq \delta_n^* t\}} \right\}}_{(2)} \\
&\quad + \underbrace{\mathbb{E} \left\{ \left(\sqrt{Z_x} - \sqrt{Z_x - \delta_n^* t} \right) \left(\sqrt{Z_x} - \sqrt{Z_x - \delta_n^* s} \right) \mathbf{1}_{\{Z_x > \delta_n^* t\}} \right\}}_{(3)}.
\end{aligned}$$

Let us start with (1). We can use (A2) to see that for δ_n^* going to zero:

$$\mathbb{E} Z_x \mathbf{1}_{\{0 < Z_x < \delta_n^* s\}} \leq (\delta_n^* s)^2 \mathbb{E} \frac{1}{\sqrt{X-x}} \frac{1}{\sqrt{X}} \mathbf{1}_{\{X > x\}} = O((\delta_n^* s)^2).$$

This shows that this term is negligible as for the covariance it will be multiplied by $4\delta_n^{-2}(\delta_n^*)^{-2}n^{-1} = 4\frac{1}{\log n}(\delta_n^*)^{-2}$, will be of order $\frac{1}{\log n}$ and it will go to zero.

Let us continue with (2). First note one can upper bound (2) using Hölder's inequality and note that by (A5) we have

$$\mathbb{E} \left(\sqrt{Z_x} - \sqrt{Z_x - \delta_n^* s} \right)^2 \mathbf{1}_{\{Z_x > \delta_n^* s\}} \sim \frac{(\delta_n^* s)^2}{4} \log \frac{1}{\delta_n^* s} g(x), \quad (\text{A8})$$

which implies that the square root of this term is of the order $O(\delta_n^* \sqrt{\log \frac{1}{\delta_n^* s}})$. Moreover, note that by (A2):

$$\left\{ \mathbb{E} Z_x \mathbf{1}_{\{\delta_n^* s < Z_x < \delta_n^* t\}} \right\}^{1/2} \leq \left\{ \mathbb{E} Z_x \mathbf{1}_{\{Z_x > \delta_n^* s\}} \right\}^{1/2} = O(\delta_n^*).$$

All together we obtain that (2) is of the order $O((\delta_n^*)^2 \sqrt{\log (\delta_n^*)^{-1}})$. This quantity for the covariance structure will be multiplied by $4\delta_n^{-2}(\delta_n^*)^{-2}n^{-1}$ and will be of order $O(\frac{1}{\sqrt{\log n}})$ thus it will go to zero

because $\sqrt{\log \frac{1}{\delta_n^* s}} \sim \sqrt{\log n}$.

Now we argue the order of (3). We argue like in the proof of (A5):

$$\begin{aligned}
(3) &= (\delta_n^*)^2 st \mathbb{E} \int_0^1 \int_0^1 \frac{1}{2\sqrt{Z_x - \delta_n^* sp}} \frac{1}{2\sqrt{Z_x - \delta_n^* tq}} dp dq \mathbf{1}_{\{Z_x > \delta_n^* t\}} \\
&= \mathbb{E} \int_0^1 \int_0^1 \int_0^1 \frac{(\delta_n^*)^2 st}{2\sqrt{vX - x - \delta_n^* sp}} \frac{\mathbf{1}_{\{vX - x > \delta_n^* t\}}}{2\sqrt{vX - x - \delta_n^* tq}} \frac{1}{2\sqrt{1-v}} dv dp dq.
\end{aligned}$$

Now we use again the substitution $vX - x = y(X - x)$ which implies $1 - v = (1 - y)\frac{X-x}{X}$ and $y(\delta_n^* s, t) = \int_{\delta_n^*}^1 \frac{1}{\sqrt{y - \delta_n^* s}} \frac{1}{\sqrt{y - \delta_n^* t}} \frac{1}{\sqrt{1-y}} dy$ to obtain

$$\begin{aligned}
&= \mathbb{E} \int_0^1 \int_0^1 \int_0^1 \frac{(\delta_n^*)^2 st}{8\sqrt{y - \frac{\delta_n^* t}{X-x}q}} \frac{\mathbf{1}_{\{y(X-x) > \delta_n^* t\}}}{\sqrt{y - \frac{\delta_n^* s}{X-x}p}} \frac{1}{\sqrt{1-y}} \frac{1}{\sqrt{X-x}\sqrt{X}} dy dp dq \\
&= \frac{(\delta_n^*)^2}{8} st \mathbb{E} \int_0^1 \int_0^1 y \left(\frac{\delta_n^*}{X-x}, sp, tq \right) \frac{1}{\sqrt{X-x}\sqrt{X}} \mathbf{1}_{\{X-x > \delta_n^* t\}} dp dq.
\end{aligned}$$

Then one can obtain the claim by using (A3) and (A4). The reason why we can apply the Dominated Convergence theorem is as in the proof of Lemma A.1. The cases $s \leq 0 \leq t$ and $s \leq t \leq 0$ can be argued in a similar way. ■

Proof of (16): We verify the Lindeberg condition for $Z_n(1)$. Recall from (14) the definition of $Z_n(1)$. Then set $Z_n(1) = \sum_{k=1}^n (X_k^n - \mathbb{E} X_k^n)$ and thus $s_n^2 = \sum_{k=1}^n \mathbb{V}(X_k^n)$, which on the other hand means

$$X_k^n = \frac{2\delta_n^{-1}}{n} \frac{\sqrt{Z_k - x} \mathbf{1}_{[x, \infty)}(Z_k) - \sqrt{Z_k - x - \delta_n^*} \mathbf{1}_{[x + \delta_n^*, \infty)}(Z_k)}{\delta_n^*}.$$

Therefore by (15): $s_n^2 = \mathbb{V}(Z_n(1)) = \frac{1}{2\gamma_x} g(x) \left(1 - \frac{\log \log n}{\log n}\right) + O\left(\frac{1}{\log n}\right)$. We show $\mathbb{E}X_k^n = \frac{(\delta_n^*)^{-1} \delta_n^{-1}}{n}$ $O(\delta_n^*) = O\left(\frac{1}{\sqrt{n \log n}}\right)$. It is enough to prove for $\delta_n^* \downarrow 0$:

$$\int \left\{ \sqrt{z-x} \mathbf{1}_{[x, \infty)}(z) - \sqrt{z-x-\delta_n^*} \mathbf{1}_{[x+\delta_n^*, \infty)}(z) \right\} g(z) dz = O(\delta_n^*). \quad (\text{A9})$$

Let $Z_x := Z - x$ therefore the right-hand side of the above can be written as

$$\underbrace{\mathbb{E} \left[\sqrt{Z_x} \mathbf{1}_{\{0 < Z_x < \delta_n^*\}} \right]}_{=(a)} + \underbrace{\mathbb{E} \left(\sqrt{Z_x} - \sqrt{Z_x - \delta_n^*} \right) \mathbf{1}_{\{Z_x > \delta_n^*\}}}_{=(b)}.$$

We show $(a) = O(\delta_n^*)$. By using (A1) (we denote X^b by X for brevity) and making a substitution below with $\nu X - x = y(X - x)$, which means $1 - \nu = (1 - y) \frac{X-x}{X}$, we obtain

$$\begin{aligned} \mathbb{E} \sqrt{Z_x} \mathbf{1}_{Z_x < \delta_n^*} &= \mathbb{E} \left\{ \mathbf{1}_{X > x} \int_{0 < \nu X - x < \delta_n^*} \sqrt{\nu X - x} \frac{\mathbf{1}_{0 < \nu < 1}}{2\sqrt{1-\nu}} d\nu \right\} \\ &= \mathbb{E} \left\{ \mathbf{1}_{X > x} \int_0^{\frac{\delta_n^*}{X-x} \wedge 1} \frac{\sqrt{y}}{2\sqrt{1-y}} \frac{(X-x)}{\sqrt{X}} dy \right\}. \end{aligned}$$

For $y(z) = \int_0^z \frac{\sqrt{y}}{2\sqrt{1-y}} dy$, the above can be upper bounded for all δ_n^* small:

$$\mathbb{E} \left\{ y \left(\frac{\delta_n^*}{X-x} \wedge 1 \right) \mathbf{1}_{X > x} \frac{X-x}{\sqrt{X}} \right\} \lesssim \mathbb{E} \delta_n^* \frac{1}{\sqrt{X}} \mathbf{1}_{\{X > x\}} = O(\delta_n^*). \quad (\text{A10})$$

That is because: $\int_0^z \frac{\sqrt{y}}{2\sqrt{1-y}} dy = \frac{1}{2} \sin^{-1}(\sqrt{z}) - \frac{1}{2} \sqrt{z(1-z)} \leq z$ for $z \in (0, 1)$. Now we proceed with (b) . By using again (A1) and by making a substitution below with $\nu X - x = y(X - x)$, which means $1 - \nu = (1 - y) \frac{X-x}{X}$, we obtain

$$\begin{aligned} \mathbb{E} \left(\sqrt{Z_x} - \sqrt{Z_x - \delta_n^*} \right) \mathbf{1}_{\{Z_x > \delta_n^*\}} &= \mathbb{E} \int_0^1 \frac{\delta_n^*}{2\sqrt{Z_x - \delta_n^*} s} ds \mathbf{1}_{\{Z_x > \delta_n^*\}} \\ &= \mathbb{E} \int_0^1 \int_0^1 \frac{\delta_n^*}{2\sqrt{\nu X - x - \delta_n^*} s} \frac{1}{2\sqrt{1-\nu}} \mathbf{1}_{\{Z_x > \delta_n^*\}} d\nu ds \\ &= \frac{1}{4} \mathbb{E} \int_0^1 \int_0^1 \frac{\delta_n^*}{\sqrt{y(X-x) - \delta_n^*} s} \frac{1}{\sqrt{1-y}} \frac{\sqrt{X}}{\sqrt{X-x}} \frac{X-x}{X} dy ds \\ &= \frac{\delta_n^*}{4} \mathbb{E} \int_0^1 y \left(\frac{\delta_n^*}{X-x}, s \right) \frac{1}{\sqrt{X}} \mathbf{1}_{\{X > x + \delta_n^*\}} ds, \end{aligned}$$

where $y(\varepsilon, s) = \int_\varepsilon^1 \frac{1}{\sqrt{y-\varepsilon} s} \frac{1}{\sqrt{1-y}} dy$. For $b \in (\varepsilon, 1)$, as done previously, we split the integral between ε and b and between b and 1. For the first integral, we have

$$\left| \int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon} s} \frac{1}{\sqrt{1-y}} dy - \int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon} s} dy \right| \leq \int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon} s} dy \left(\frac{1}{\sqrt{1-b}} - 1 \right).$$

From which we conclude

$$\int_\varepsilon^b \frac{1}{\sqrt{y-\varepsilon} s} \frac{1}{\sqrt{1-y}} dy = \begin{cases} \leq (2\sqrt{b-\varepsilon s} - 2\sqrt{\varepsilon(1-s)}) \left(\frac{1}{\sqrt{1-b}} - 1 \right), \\ \geq (2\sqrt{b-\varepsilon s} - 2\sqrt{\varepsilon(1-s)}) \left(2 - \frac{1}{\sqrt{1-b}} \right). \end{cases}$$

For the second integral, we have

$$\int_b^1 \frac{1}{\sqrt{y-\varepsilon s}} \frac{1}{\sqrt{1-y}} dy \leq \frac{1}{\sqrt{b^2-b\varepsilon}} \left\{ \sqrt{b(1-b)} - \sin^{-1}(\sqrt{1-b}) \right\}.$$

Putting together the above computations, for all δ_n^* small and $b \sim \sqrt{\frac{1}{\log \frac{1}{\delta_n^*}}}$:

$$\mathbb{E} \left(\sqrt{Z_x} - \sqrt{Z_x - \delta_n^*} \right) \mathbf{1}_{\{Z_x > \delta_n^*\}} \lesssim \delta_n^* \mathbb{E} \frac{\mathbf{1}_{\{X > x + \delta_n^*\}}}{\sqrt{X}} = O(\delta_n^*).$$

This proves that (A9) is true.

Let us now look at the indicator $\mathbf{1}_{\{|X_k^n - \mathbb{E}X_k^n| > \varepsilon s_n\}}$. We look separately at the cases: $X_k^n - \mathbb{E}X_k^n > \varepsilon s_n$ and $X_k^n - \mathbb{E}X_k^n < -\varepsilon s_n$. Using the following facts:

$$\begin{aligned} \sqrt{Z_x} \mathbf{1}_{\{Z_x > 0\}} - \sqrt{Z_x - \delta_n^*} \mathbf{1}_{\{Z_x > \delta_n^*\}} &= \sqrt{Z_x} \mathbf{1}_{\{0 < Z_x < \delta_n^*\}} + \left(\sqrt{Z_x} \right. \\ &\quad \left. - \sqrt{Z_x - \delta_n^*} \right) \mathbf{1}_{\{Z_x > \delta_n^*\}} \leq \sqrt{\delta_n^*} + \int_0^1 \frac{\delta_n^* \mathbf{1}_{\{Z_x > \delta_n^*\}}}{2\sqrt{Z_x - \delta_n^*} s} ds \leq \sqrt{\delta_n^*} + \delta_n^* \frac{\mathbf{1}_{\{Z_x > \delta_n^*\}}}{2\sqrt{Z_x - \delta_n^*}}. \end{aligned}$$

So we can rewrite the set $\{X_k^n - \mathbb{E}X_k^n > \varepsilon s_n\}$, using the fact that $\mathbb{E}X_k^n = O(\frac{1}{\sqrt{n \log n}})$, and noting that the last inclusion is valid for all n big enough:

$$\begin{aligned} &\left\{ \frac{\sqrt{Z_x} \mathbf{1}_{\{Z_x > 0\}} - \sqrt{Z_x - \delta_n^*} \mathbf{1}_{\{Z_x > \delta_n^*\}}}{\delta_n^*} > \varepsilon s_n \sqrt{n \log n} + O(1) \right\} \\ &\subseteq \left\{ \frac{\mathbf{1}_{\{Z_x > \delta_n^*\}}}{2\sqrt{Z_x - \delta_n^*}} > \varepsilon s_n \sqrt{n \log n} - \frac{1}{\sqrt{\delta_n^*}} + O(1) \right\} \\ &\subseteq \left\{ Z_x < \frac{\delta_n^*}{4(\sqrt{\delta_n^* n \log n \varepsilon s_n} - 1 + O(\sqrt{\delta_n^*}))^2} + \delta_n^* \right\}. \end{aligned}$$

Since $\frac{\delta_n^*}{4(\sqrt{\delta_n^* n \log n \varepsilon s_n} - 1 + O(\sqrt{\delta_n^*}))^2} \sim \frac{1}{n \log n}$, for all n big enough we conclude

$$\{X_k^n - \mathbb{E}X_k^n > \varepsilon s_n\} \subseteq \left\{ Z_x < \frac{1}{n \log n} + \delta_n^* \right\} =: A_n.$$

Now we see that the set $\{X_k^n - \mathbb{E}X_k^n < -\varepsilon s_n\}$ is negligible, because the object on the left-hand side in the following expression is nonnegative:

$$\left\{ \frac{\sqrt{Z_x} \mathbf{1}_{\{Z_x > 0\}} - \sqrt{Z_x - \delta_n^*} \mathbf{1}_{\{Z_x > \delta_n^*\}}}{\delta_n^*} < -\varepsilon s_n \sqrt{n \log n} + O(1) \right\}.$$

Therefore we see that the integral over the set $|X_k^n - \mathbb{E}X_k^n| > \varepsilon s_n$ for all n big enough has the same behaviour as integrating over the set: A_n . Using i.i.d.:

$$\sum_{k=1}^n \mathbb{E}[(X_k^n - \mathbb{E}X_k^n)^2 \mathbf{1}_{A_n}] \leq \underbrace{2n \mathbb{E}[(X_k^n)^2 \mathbf{1}_{A_n}]}_{(1)} + \underbrace{2n(\mathbb{E}X_k^n)^2 \mathbb{P}(A_n)}_{(2)} \rightarrow 0.$$

Since $\mathbb{E}X_k^n = O(\frac{1}{\sqrt{n \log n}})$ we see immediately that (2) goes to 0. Now we prove that (1) also does.

We need to look at the following upper bound:

$$\frac{(\delta_n^*)^{-2}}{\log n} \mathbb{E} \left[\left\{ Z_x \mathbf{1}_{\{Z_x > 0\}} + (Z_x - \delta_n^*) \mathbf{1}_{\{Z_x > \delta_n^*\}} \right\} \mathbf{1}_{A_n} \right] \leq \frac{2(\delta_n^*)^{-2}}{\log n} \mathbb{E} \left[Z_x \mathbf{1}_{\{Z_x > 0\} \cap A_n} \right].$$

From (A2), we recall that $\mathbb{E}Z_x \mathbf{1}_{0 < Z_x < \varepsilon} = O(\varepsilon^2)$ for $\varepsilon \downarrow 0$. We conclude that

$$\begin{aligned} \frac{(\delta_n^*)^{-2}}{\log n} \mathbb{E} \left[Z_x \mathbf{1}_{\{0 < Z_x < \delta_n^* + \frac{1}{n \log n}\}} \right] &\sim \frac{1}{\log n} \left(\frac{n}{\log n} \right)^{\frac{1}{\gamma_x}} \left(\left(\frac{\log n}{n} \right)^{\frac{1}{2\gamma_x}} + \frac{1}{n \log n} \right)^2 \\ &\sim \frac{1}{\log n} + \frac{1}{\log n} \left(\frac{n}{\log n} \right)^{1/2\gamma_x} \frac{1}{n \log n} + \frac{1}{\log n} \left(\frac{n}{\log n} \right)^{1/\gamma_x} \frac{1}{n^2 \log^2 n} \rightarrow 0. \end{aligned}$$

For $\gamma_x > \frac{1}{2}$. Therefore the Lindeberg condition is satisfied for $Z_n(1)$. ■

Proof of (18): We start with the class of functions $h_\theta : [0, \infty) \rightarrow \mathbb{R}$:

$$\mathcal{H} = \left\{ h_\theta : h_\theta(z) = 2\sqrt{z-x} \mathbf{1}_{[x, \infty)}(z) - 2\sqrt{z-\theta} \mathbf{1}_{[\theta, \infty)}(z) - a_0(x-\theta), \theta \geq 0 \right\}$$

and, for $\eta > 0$, its subclass $\mathcal{H}_\eta = \{h_\theta \in \mathcal{H} : |\theta - x| \leq \eta\}$. In Groeneboom and Jongbloed (1995), they show that the envelope of $\mathcal{H}_\eta$, $H_\eta(z) = 2(\sqrt{z-x} \mathbf{1}_{[x, \infty)}(z) - \sqrt{z-x-\eta} \mathbf{1}_{[x-\eta, \infty)}(z)) + a_0\eta$, satisfies for all η small (cf. (11) in Groeneboom and Jongbloed 1995):

$$\int H_\eta^2(z) g(z) dz \leq -2g(x)\eta^2 \log \eta. \quad (\text{A11})$$

The first thing that needs to be proved is that for $|\theta - x|$ small, we have

$$\int h_\theta(z) - h_x(z) dG(z) \lesssim -|\theta - x|^{\gamma_x+1}.$$

Using a Taylor expansion for $\int h_\theta(z) dG(z)$ around x (note $h_x(z) = 0$), we get, using the definition of U : $U(x) = \frac{\pi}{2m_0} \int_0^x (1 - F(y)) dy = \int_0^x V(y) dy$, that there exists $\eta_0 > 0$ such that

$$\begin{aligned} \int h_\theta(z) dG(z) &= 2 \int \sqrt{z-x} \mathbf{1}_{[x, \infty)}(z) - \sqrt{z-\theta} \mathbf{1}_{[\theta, \infty)}(z) dG(z) - V(x)(\theta - x) \\ &= U(x) - U(\theta) - V(x)(x - \theta) \stackrel{(T)}{=} \frac{(\theta - x)\pi}{2m_0} \int_0^1 F(x + u(\theta - x)) - F(x) du \\ &\stackrel{(5)}{=} -\frac{\pi}{2m_0} |\theta - x|^{\gamma_x+1} (K_x + o(1)), \end{aligned}$$

for all θ with $|\theta - x| < \eta_0$ and for some constant $K > 0$. Note that here we used the same reasoning as in (13) and we also used assumption (5).

Now we show the weak consistency of $T_n(a_0 + \delta_n a)$ for $x = V^{-1}(a_0)$. This will follow from Theorems 2.1 and 2.2 in Groeneboom and Jongbloed (1995) and the switch relation (10). In other words, we prove: $\forall \varepsilon > 0$ then $\mathbb{P}(|T_n(a_0 + \delta_n a) - x| > \varepsilon) \rightarrow 0$. Indeed using the switch relation and consistency of $\hat{V}_n$ we get

$$\begin{aligned} &\mathbb{P}(|T_n(a_0 + \delta_n a) - x| > \varepsilon) \\ &\leq \mathbb{P}(T_n(V(x) + \delta_n a) > \varepsilon + x) + \mathbb{P}(T_n(V(x) + \delta_n a) < x - \varepsilon) \\ &= \mathbb{P}(\hat{V}_n(x + \varepsilon) > V(x) + \delta_n a) + \mathbb{P}(\hat{V}_n(x - \varepsilon) < V(x) + \delta_n a) \rightarrow 0. \end{aligned}$$

As by consistency $\hat{V}_n(x + \varepsilon) \rightarrow V(x + \varepsilon)$, $\delta_n \downarrow 0$ and V is strictly decreasing at x , it will never be the case that $V(x + \varepsilon) \geq V(x)$, nor $V(x - \varepsilon) \leq V(x)$. Note that V is already decreasing by definition, and it cannot be constant in a neighbourhood of x , as we would get a contradiction with the fact that F is not constant in a neighbourhood of x .

Now we will proceed by deriving the rate of convergence of $T_n(a_0 + \delta_n a)$ to $x = V^{-1}(a_0)$. We will do it by a standard technique using a peeling argument, combined later with what we showed above.

First it needs to be proved that $\forall \varepsilon > 0$ there exists $\{M_n\}$ sequence of stochastically bounded random variables and a positive constant η_1 such that

$$2 \left| \int \left\{ \sqrt{z-x} \mathbf{1}_{[x,\infty)}(z) - \sqrt{z-\theta} \mathbf{1}_{[\theta,\infty)}(z) \right\} d(\mathbb{G}_n - G)(z) \right| \quad (\text{A12})$$

$$\leq \varepsilon |x - \theta|^{\gamma_x+1} + (\delta_n^*)^{\gamma_x+1} M_n, \quad (\text{A13})$$

for all θ with $|\theta - x| < \eta_1$, and γ as in assumption (5). Remember that x has been fixed and that $\delta_n = n^{-1/2} \sqrt{\log n}$, whereas $\delta_n^* = (\delta_n)^{1/\gamma_x}$. Now we proceed with the proof of (A12). Fix $\varepsilon > 0$, and let $\{M_n(\omega)\}$ be the random variable defined omega-wise as the infimum of all positive numbers η so that

$$\left| \int h_\theta(z) d(\mathbb{G}_n - G)(z; \omega) \right| \leq \varepsilon |x - \theta|^{\gamma_x+1} + (\delta_n^*)^{\gamma_x+1} \eta, \quad (\text{A14})$$

holds for all $\theta \in [x - \eta_1, x + \eta_1]$. If we define, for $n, j \geq 1$,

$$A(n, j) = \{\theta \geq 0 : (j-1)\delta_n^* \leq |\theta - x| \leq j\delta_n^*\},$$

we can write, for all $n \geq 1$ and ν constant:

$$\begin{aligned} & \mathbb{P}\{M_n > \nu\} \\ & \leq \sum_{j=1}^{\eta_1/\delta_n^*} \mathbb{P}\left\{ \sup_{\theta \in A(n, j)} (\delta_n^*)^{-(\gamma_x+1)} \left| \int h_\theta(z) d(\mathbb{G}_n - G)(z) \right| > \varepsilon (j-1)^{\gamma_x+1} + \nu \right\} \\ & \leq \sum_{j=1}^{\eta_1/\delta_n^*} \frac{\mathbb{E}\left\{ \sup_{|\theta-x| \leq j\delta_n^*} (\delta_n^*)^{-2(\gamma_x+1)} \left[\int h_\theta(z) d(\mathbb{G}_n - G)(z) \right]^2 \right\}}{[\varepsilon (j-1)^{\gamma_x+1} + \nu]^2}. \end{aligned} \quad (\text{A15})$$

By maximal inequality 3.1(ii) in Kim and Pollard (1990) and relation (A11) (with $\eta = \delta_n^* j$), $\exists$ a $C > 0$ such that, for each $j \leq \eta_1/\delta_n^*$ and using that $(\delta_n^*)^{-2(\gamma_x+1)} = (\delta_n^*)^{-2} \delta_n^{-2}$:

$$\begin{aligned} & \mathbb{E}\left\{ \sup_{|\theta-x| \leq j\delta_n^*} (\delta_n^*)^{-2(\gamma_x+1)} \left[\int h_\theta(z) d(\mathbb{G}_n - G)(z) \right]^2 \right\} \\ & = (\delta_n^*)^{-2} \frac{n}{\log n} \mathbb{E}\left\{ \sup_{|\theta-x| \leq j\delta_n^*} \left[\int h_\theta(z) d(\mathbb{G}_n - G)(z) \right]^2 \right\} \\ & \leq (\delta_n^*)^{-2} \frac{J(1)^2}{\log n} (-2g(x)j^2 (\delta_n^*)^2 \log(\delta_n^* \cdot j)) = Cg(x)j^2 + O((\log n)^{-1}), \end{aligned}$$

where $J(1)$ is the uniform entropy integral appearing from the application of the maximal inequality 3.1(ii) in Kim and Pollard (1990). We argue that $J(1)$ is finite in the proof of Lemma 2.1. Applying this inequality to (A15), it follows that we can upper bound it by: $\sum_{j=1}^{\eta_1/\delta_n^*} \frac{Cg(x)j^2 + o(1)}{[\varepsilon(j-1)^{\gamma_x+1} + \nu]^2}$, by choosing ν sufficiently large, $\mathbb{P}\{M_n > \nu\}$ can be made arbitrarily small uniformly in n , which proves (A12).

We are now ready to conclude that $T_n(a_0 + \delta_n a)$ is a δ_n^* -consistent estimator for $x = V^{-1}(a_0)$. From the above we know with probability tending to 1, $|T_n(a_0 + \delta_n a) - x| < \eta_0 \wedge \eta_1$, for $n \rightarrow \infty$.

If $|T_n(a_0 + \delta_n a) - x| < \eta_0 \wedge \eta_1$, take ε in relation (A14) equal to $\frac{\pi K}{4m_0}$ and obtain that

$$\begin{aligned} 0 &= \int h_x(z) d\mathbb{G}_n(z) \leq \int h_{T_n(a_0 + \delta_n a)}(z) d\mathbb{G}_n(z) \\ &\leq \int h_{T_n(a_0 + \delta_n a)}(z) dG(z) + \frac{\pi K_x}{4m_0} |x - T_n(a_0 + \delta_n a)|^{\gamma_x+1} + (\delta_n^*)^{\gamma_x+1} M_n \\ &= \left(\frac{\pi K_x}{4m_0} - \frac{\pi K_x}{2m_0} + o(1) \right) |x - T_n(a_0 + \delta_n a)|^{\gamma_x+1} + (\delta_n^*)^{\gamma_x+1} M_n, \end{aligned}$$

where the sequence $\{M_n\}$ is stochastically bounded by the above part. This proves that $T_n(a_0 + \delta_n a) - x = O_p(\delta_n^*)$ for $n \rightarrow \infty$. ■

Lemma A.2: Condition (22) implies that F is not constant in a neighbourhood of x and that $\int_x^\infty \frac{dF(s)}{\sqrt{s-x}} < \infty$.

Proof: Assume (22). Suppose F is constant in a neighbourhood of x then the limit in (22) is zero, a contradiction. From condition (22), it follows that we can fix $K > \epsilon > 0$ knowing that there exists $\delta > 0$ such that $\forall t \in \mathbb{R}^+ : t \leq \delta$ then $(K - \epsilon)t^{\gamma_x} \leq F(x+t) - F(x) \leq (K + \epsilon)t^{\gamma_x}$. Let us set a grid that goes from x to $x + \delta$ as: $x_0 = x + \delta \geq x_1 \geq \dots \geq x$, $x_j = x + 2^{-j}\delta$. It is enough to look at

$$\int_x^{x+\delta} \frac{dF(s)}{\sqrt{s-x}} \leq \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{dF(s)}{\sqrt{s-x}} \leq \sum_{j=0}^{\infty} \frac{(K + \epsilon)2^{-\gamma_x(j+1)}}{\sqrt{2^{-(j+1)}\delta}} - \sum_{j=0}^{\infty} \frac{(K - \epsilon)2^{-\gamma_x j}}{\sqrt{2^{-(j+1)}\delta}},$$

where the last expression is finite as $\gamma_x > \frac{1}{2}$. ■

Lemma A.3: If F is locally Hölder continuous of degree $\gamma_x > 1/2$ around x_0 , then the density of the observations g is bounded and continuous around x_0 .

Proof: Fix any z in a neighbourhood of x_0 , so that $\exists \eta > 0 : F$ is Hölder continuous on $(z - \eta, z + \eta)$. Set $z_j = z + \eta 2^{-j}$ for $j = 0, 1, 2, \dots$ so that $z_0 = z + \eta \geq z_1 \geq \dots \geq z$ and therefore $z_j - z_{j+1} = \eta 2^{-j-1}$. We have (note $m_0 < \infty$ by (8)):

$$\begin{aligned} g(z)m_0 &= \int_z^\infty \frac{1}{\sqrt{x-z}} dF(x) = \sum_{j=0}^{\infty} \int_{z_{j+1}}^{z_j} \frac{1}{\sqrt{x-z}} dF(x) + \int_{z+\eta}^\infty \frac{1}{\sqrt{x-z}} dF(x) \\ &\leq \sum_{j=0}^{\infty} \frac{(F(z_j) - F(z_{j+1}))}{\sqrt{z_{j+1} - z}} + \frac{1}{\sqrt{\eta}} \leq \frac{1}{\sqrt{\eta}} \left(\sum_{j=0}^{\infty} \frac{2^{-\gamma(j+1)}}{2^{-\frac{1}{2}(j+1)}} + 1 \right), \end{aligned}$$

where the last expression is $< \infty$ if and only if $\gamma_x > \frac{1}{2}$. Now we know that the function g is bounded around x_0 . We have to prove that it is continuous. Using the same reasoning as in the previous part we have that

$$g(z) = \underbrace{\frac{1}{m_0} \sum_{j=0}^{\infty} \int_{z_{j+1}}^{z_j} \frac{1}{\sqrt{x-z}} dF(x)}_{(1)} + \underbrace{\frac{1}{m_0} \int_{z+\eta}^\infty \frac{1}{\sqrt{x-z}} dF(x)}_{(2)}.$$

Since in the previous part of the proof we have been able to bound (1) with the convergent series independent of z : $\sum_{j=0}^{\infty} \frac{2^{-\gamma(j+1)}}{2^{-\frac{1}{2}(j+1)}}$, then the Weierstrass criterion for uniform convergence of series of functions (M-test) yields that the series (1) converges uniformly. Now we need to show that $z \mapsto \int_{z_{j+1}}^{z_j} \frac{1}{\sqrt{x-z}} dF(x)$ are continuous $\forall j$. We also need to show that (2) is continuous. First of all, note that the following function: $f(s) = \frac{1_{[\eta 2^{-j-1}, \eta 2^{-j}]}(s)}{\sqrt{s}}$ is continuous on the compact set $[\eta 2^{-j-1}, \eta 2^{-j}]$,

thus uniformly continuous. By taking $s = x - z$, one can retrieve $\frac{\mathbf{1}_{[z+\eta 2^{-j-1}, z+\eta 2^{-j}]}(x)}{\sqrt{x-z}}$. Now take δ so that whenever $|z_1 - z_2| < \delta$, by uniform continuity we have

$$\left| \frac{\mathbf{1}_{[z_1+\eta 2^{-j-1}, z_1+\eta 2^{-j}]}(x)}{\sqrt{x-z_1}} - \frac{\mathbf{1}_{[z_2+\eta 2^{-j-1}, z_2+\eta 2^{-j}]}(x)}{\sqrt{x-z_2}} \right| < \varepsilon.$$

Then we have, using the triangle inequality for integrals:

$$\left| \int \frac{\mathbf{1}_{[z_1+\eta 2^{-j-1}, z_1+\eta 2^{-j}]}(x)}{\sqrt{x-z_1}} - \frac{\mathbf{1}_{[z_2+\eta 2^{-j-1}, z_2+\eta 2^{-j}]}(x)}{\sqrt{x-z_2}} dF(x) \right| \leq \varepsilon \int dF(x) = \varepsilon.$$

Finally, (2) is continuous because $\frac{\mathbf{1}_{[z+\eta, \infty)}(x)}{\sqrt{x-z}} = \frac{\mathbf{1}_{[\eta, \infty)}(x-z)}{\sqrt{x-z}}$ is uniformly continuous on $[\eta, \infty)$. The Weierstrass M-test together with the uniform limit theorem yield the continuity of g . ■

Proof of Lemma 2.1: Because

$$U_n(s) := \frac{2}{n} \sum_{i=1}^n \left\{ \sqrt{(Z_i)_+} - \sqrt{(Z_i - s)_+} \right\},$$

the stated convergence is equivalent to saying that the class of functions $z \rightarrow \sqrt{(z)_+} - \sqrt{(z-s)_+}, s \geq 0$ is Donsker. From Lemma 2.6.16 in van der Vaart and Wellner (1996), it follows that the class of functions $\{z - s : s \geq 0\}$ is VC, next by Lemma 2.6.18 (ii) in van der Vaart and Wellner (1996) we see that the class $\{(z-s)_+ : s \geq 0\}$ is VC. Again by Lemma 2.6.18 (vii), (iv) and (v) in van der Vaart and Wellner (1996) we conclude that the class of functions $\{\sqrt{(z)_+} - \sqrt{(z-s)_+} : s \geq 0\}$ is VC. Such class has envelope: $0 \leq \sqrt{(z)_+} - \sqrt{(z-s)_+} \leq \sqrt{(z)_+}$. Therefore the class is P -Donsker for any P with $\int_0^\infty z dP(z) < \infty$, which is the case here by assumption (8) with $k = \frac{3}{2}$. Now we prove that $\mathbb{Z}$ is a Gaussian process with a.s. continuous sample paths. Consider:

$$\mathbb{E} (\mathbb{Z}(t_j) - \mathbb{Z}(t_i))^2 = \mathbb{E} \left| \sqrt{(Z - t_j)_+} - \sqrt{(Z - t_i)_+} \right|^2 \leq |t_j - t_i|,$$

where we use the fact that the square root is Hölder continuous of degree 1/2.

Since the intrinsic semimetric $d(s, t) := (\mathbb{E}(\mathbb{Z}(s) - \mathbb{Z}(t))^2)^{1/2} \leq \sqrt{|s - t|}$, we obtain sample path continuity of $\mathbb{Z}$.

By the very same computation as above, we deduce that $\mathbb{V}(\mathbb{Z}(t) - \mathbb{Z}(s)) \neq 0$ for $s \neq t$ and since $\mathbb{Z}$ is indexed by a σ -compact metric space, we obtain from Lemma 2.6 in Kim and Pollard (1990) that the maxima of the sample paths of $\mathbb{Z}$ are a.s. unique. Finally the covariance structure of $\mathbb{Z}$ is given by direct computation:

$$\text{Cov}(\mathbb{Z}(t), \mathbb{Z}(s)) = \text{Cov}(\sqrt{(Z)_+} - \sqrt{(Z_t)_+}, \sqrt{(Z)_+} - \sqrt{(Z_s)_+}).$$

■

Proof of (33): Note that

$$\sup_{s \in K_n^c} \{Z_n(s) - \sqrt{n}h_x(s) - as\} \leq \sup_{s \geq 0} Z_n(s) - \inf_{\substack{s < \underline{x} - \varepsilon_n \\ s > \bar{x} + \varepsilon_n}} \{\sqrt{n}h_x(s) + as\},$$

where the first term on the right-hand side is $O_p(1)$ by the convergence given in Lemma 2.1. The second infimum is the minimum over the infima over the intervals $[0, \underline{x} - c]$, $[\underline{x} - c, \underline{x} - \varepsilon_n]$, $[\bar{x} + \varepsilon_n, \bar{x} + 1]$ and $[\bar{x} + 1, \infty)$ for a constant $\varepsilon_1 < c < \underline{x}$. Because F is constant in $[\underline{x}, \bar{x}]$, we have $U(s) - U(x) - V(x)(s - x) = 0$ for $s \in [\underline{x}, \bar{x}]$ and from the concavity of U :

$$U(s) - U(x) - V(x)(s - x) < 0, \quad \text{for } s \notin [\underline{x}, \bar{x}].$$

Again from the fact that U is concave and because $[\underline{x}, \bar{x}]$ is maximal:

$$\forall \delta > 0, \exists \eta > 0 : U(s) - U(x) - V(x)(s - x) < -\eta|s - \underline{x}| \quad \text{for } s < \underline{x} - \delta,$$

$$\forall \delta > 0, \exists \eta > 0 : U(s) - U(x) - V(x)(s - x) < -\eta|s - \bar{x}| \quad \text{for } s > \bar{x} + \delta.$$

Therefore using these three facts stated above we have: on $[\bar{x} + 1, \infty)$,

$$\sqrt{nh_x}(s) + as \geq \sqrt{n\eta}|s - \bar{x}| - |a|s \gtrsim \sqrt{n} \rightarrow \infty$$

and on $[\bar{x} + \varepsilon_n, \bar{x} + 1]$,

$$\sqrt{nh_x}(s) + as \geq \sqrt{nh_x}(\bar{x} + \varepsilon_n) - |a| \rightarrow \infty$$

by the construction of ε_n . The other two intervals can be handled similarly. ■

Proof of (35): First note

$$\begin{aligned} 0 &\leq \sup_{s \in F \cap K_n} \{Z_n(s) - \sqrt{nh_x}(s) - as\} - \sup_{s \in F \cap K} \{Z_n(s) - as\} \\ &\leq \sup_{s \in F \cap K_n} \{Z_n(s) - as\} - \sup_{s \in F \cap K} \{Z_n(s) - as\} \end{aligned}$$

Define the process:

$$\tilde{Z}_n(s) := \begin{cases} Z_n(s) - as, & \text{if } s \in [\underline{x}, \bar{x}] \\ Z_n(\underline{x}) - a\underline{x}, & \text{if } s \in [0, \underline{x}] \\ Z_n(\bar{x}) - a\bar{x}, & \text{if } s \in [\bar{x}, \infty) \end{cases}$$

because $\sup_{s \in F \cap K_n} \tilde{Z}_n(s) = \sup_{s \in F \cap K} \tilde{Z}_n(s) = \sup_{s \in F \cap K} \{Z_n(s) - as\}$, then

$$\begin{aligned} &\left| \sup_{s \in F \cap K_n} \{Z_n(s) - as\} - \sup_{s \in F \cap K} \{Z_n(s) - as\} \right| \\ &= \left| \sup_{s \in F \cap K_n} \{Z_n(s) - as\} - \sup_{s \in F \cap K_n} \tilde{Z}_n(s) \right| \leq \sup_{s \in F \cap K_n} |Z_n(s) - as - \tilde{Z}_n(s)| \\ &\leq \sup_{s \in F \cap K_n \cap [0, \underline{x}]} |Z_n(s) - as - \tilde{Z}_n(\underline{x})| \vee \sup_{s \in F \cap K_n \cap [\bar{x}, \infty)} |Z_n(s) - as - \tilde{Z}_n(\bar{x})| \rightarrow 0, \end{aligned}$$

where we used the asymptotic equicontinuity of the Z_n . ■

Proof of Theorem 1.1: The claim is a consequence of $1 - F(x) = V(x)/V(0)$ and by analogy, we construct the estimator using the adaptive isotonic estimator. Apply Slutsky's Lemma together with Theorem 2.1 (cf. Corollary 1 in Groeneboom and Jongbloed 1995) to get Theorem 1.1(a). To prove Theorem 1.1(b), it is possible to follow the same steps as in the proof of Theorem 2.2 and show the joint convergence, for all $a, b \in \mathbb{R}$:

$$\begin{aligned} &\mathbb{P}\left(\sqrt{n}\left(\hat{V}_n(x) - V(x)\right) \leq a, \sqrt{n}\left(\hat{V}_n(0) - V(0)\right) \leq b\right) \\ &\rightarrow \mathbb{P}\left(\arg \max_{s \in K_x} \{\mathbb{Z}_x(s) - as\} \leq x, \arg \max_{s \in K_0} \{\mathbb{Z}_0(s) - bs\} \leq 0\right) \\ &= \mathbb{P}(L_x \leq a, L_0 \leq b), \end{aligned}$$

where in the last step we used the switch relation. Using again Slutsky's Lemma with the above convergence and the asymptotic independence of $\hat{V}_n(x)$ and $\hat{V}_n(0)$ (which follows from the fact that $K_x \cap K_0 = \emptyset$), we get claim. ■

Appendix 2. Proofs complementary statements for the LAN expansion

Proof of Lemma 3.1: It is enough to consider the case $h_1 > 0$ and $h_2 > 0$ (depicted in Figure 6 around x) and we prove in this case that F_{h_n} is non-decreasing for all n big enough. We check only

around x on IK_n^x , as around 0 is the same. From (38), for any $h_2 \geq 0$, $\exists N^*$ such that $\forall n \geq N^*$, all $x_1, x_2 \in K_n^x$, $x_1 < x_2$: $F(x_2) - F(x_1) \geq \frac{h_2}{\sqrt{n \log n}} \log \left\{ \frac{x-x_2}{x-x_1} \right\}$, and therefore eventually:

$$D_{h_n} (F_{h_n}(x_2) - F_{h_n}(x_1)) = F(x_2) - F(x_1) - \frac{h_2}{\sqrt{n \log n}} \log \left\{ \frac{x-x_2}{x-x_1} \right\} \geq 0.$$

■

Lemma A.4: Let $\zeta_n(x) := \int_x^\infty \frac{\mathbf{1}_{\{\delta_n \leq |v| \leq \eta_n\}}}{v\sqrt{v-x}} dv$, then

$$\begin{aligned} \zeta_n(x) &= 0 \quad \text{if } x > \eta_n, \quad \left| \zeta_n(x) \right| \lesssim \frac{\delta_n}{(-x)^{3/2}} + \frac{1}{\sqrt{\eta_n}} \quad \text{if } -\eta_n \leq x \leq -\delta_n, \\ \sup_{[\delta_n, \eta_n]} \left| \zeta_n(x) - \frac{\pi}{\sqrt{x}} \right| &\leq \frac{\pi}{\sqrt{\eta_n}}, \quad \sup_{[-\delta_n, \delta_n]} \left| \zeta_n(x) \right| \leq \frac{\pi}{\sqrt{\delta_n}}, \quad \sup_{x \leq -\eta_n} \left| \zeta_n(x) \right| \lesssim \frac{1}{\sqrt{\eta_n}}. \end{aligned}$$

Remark A.1: Another more compact way of stating the above lemma is (cf. Golubev and Levit 1998):

$$\zeta_n(x) = \begin{cases} 0, & \text{if } x > \eta_n, \\ \pi x^{-1/2} + O\left(\eta_n^{-1/2}\right), & \text{if } \delta_n \leq x \leq \eta_n, \\ O\left(\delta_n^{-1/2}\right), & \text{if } -\delta_n \leq x \leq \delta_n, \\ O\left(\delta_n(-x)^{-3/2}\right) + O\left(\eta_n^{-1/2}\right), & \text{if } -\eta_n \leq x \leq -\delta_n, \\ O\left(\eta_n^{-1/2}\right), & \text{if } x \leq -\eta_n. \end{cases} \quad (\text{A16})$$

Proof: (1) If $x > \eta_n$, then clearly $\zeta_n(x) = 0$, due to the indicator function.

(2) If $\delta_n \leq x \leq \eta_n$, using (46) and the triangle inequality:

$$\begin{aligned} \left| \zeta_n(x) - \frac{\pi}{\sqrt{x}} \mathbf{1}_{x \geq 0} \right| &= \left| \int_x^{\eta_n} \frac{1}{v\sqrt{v-x}} dv - \int_x^\infty \frac{1}{v\sqrt{v-x}} dv \right| \\ &= \left| \int_{\eta_n}^\infty \frac{1}{v\sqrt{v-x}} dv \right| \leq \int_{\eta_n}^\infty \frac{1}{v\sqrt{v-\eta_n}} dv = \frac{\pi}{\sqrt{\eta_n}}. \end{aligned}$$

(3) If $-\delta_n \leq x \leq \delta_n$, using (46):

$$|\zeta_n(x)| = \int_{\delta_n}^{\eta_n} \frac{1}{v\sqrt{v-x}} dv \leq \int_{\delta_n}^\infty \frac{1}{v\sqrt{v-x}} dv \leq \int_{\delta_n}^\infty \frac{1}{v\sqrt{v-\delta_n}} dv = \frac{\pi}{\sqrt{\delta_n}}.$$

(4) If $-\eta_n \leq x \leq -\delta_n$, $J(y) := \int_y^1 \frac{1}{v} \left(\frac{1}{\sqrt{v+1}} - \frac{1}{\sqrt{1-v}} \right) dv$, $H(y) := \int_1^y \frac{1}{v\sqrt{v+1}} dv$:

$$\begin{aligned} \zeta_n(x) &= \int_x^{-\delta_n} \frac{1}{v\sqrt{v-x}} dv + \int_{\delta_n}^{\eta_n} \frac{1}{v\sqrt{v-x}} dv \\ &= \int_{\delta_n}^{-x} \frac{1}{v} \left(\frac{1}{\sqrt{v-x}} - \frac{1}{\sqrt{-v-x}} \right) dv + \int_{-x}^{\eta_n} \frac{1}{v\sqrt{v-x}} dv \\ &= \frac{1}{\sqrt{-x}} \left(J\left(\frac{\delta_n}{-x}\right) + H\left(\frac{\eta_n}{-x}\right) \right). \end{aligned}$$

Because $H(\infty) = -J(0) = \log(3 + 2\sqrt{2})$, we obtain

$$|\zeta_n(x)| = \left| \frac{1}{\sqrt{-x}} \left(J\left(\frac{\delta_n}{-x}\right) - J(0) - H(\infty) + H\left(\frac{\eta_n}{-x}\right) \right) \right|$$

$$\begin{aligned}
&= \left| \frac{1}{\sqrt{-x}} \left(\int_0^{\delta_n/-x} \frac{1}{v} \left(\frac{1}{\sqrt{v+1}} - \frac{1}{\sqrt{1-v}} \right) dv + \int_{\eta_n/-x}^{\infty} \frac{1}{v\sqrt{v+1}} dv \right) \right| \\
&\lesssim \frac{\delta_n}{(-x)^{3/2}} + \frac{1}{\sqrt{\eta_n}}.
\end{aligned}$$

(5) If $x \leq -\eta_n$, using the same reasoning as above:

$$|\zeta_n(x)| = \frac{1}{\sqrt{-x}} \left| \int_{\delta_n/-x}^{\eta_n/-x} \frac{1}{v} \left(\frac{1}{\sqrt{v+1}} - \frac{1}{\sqrt{1-v}} \right) dv \right| \leq \frac{\log(3+2\sqrt{2})}{\sqrt{\eta_n}}.$$

■

Proof of (48): Fix $\epsilon > 0$, using the triangle inequality, we can just look separately at $p \in \{0, x\}$ (where for $j=1, p=0$; for $j=2, p=x$ and recall $h_{j,n}, j=1, 2$ as in (40)):

$$\begin{aligned}
&\mathbb{P} \left(\max_{1 \leq i \leq n} \left| \frac{h_{j,n} \zeta_{j,n}(Z_i - p)}{g(Z_i)} \right| > \epsilon \right) \leq \sum_{i=1}^n \mathbb{P} \left(\left| \frac{h_{j,n} \zeta_{j,n}(Z_i - p)}{g(Z_i)} \right| > \epsilon \right) \\
&\leq \sum_{i=1}^n \mathbb{P} \left(\left| \frac{\zeta_{j,n}(Z_i - p)}{g(Z_i)} \right| > \epsilon \frac{\sqrt{n \log n}}{h_j} \right), \quad j=1, 2.
\end{aligned}$$

Consider now the expansion in (45). Define

$$A_n^{i,x} := \left\{ n^{-\frac{1}{2\gamma_x}} \leq Z_i - x \leq \eta_n \right\}, \quad (\text{A17})$$

and analogously $A_n^{i,0}$. Now we can use (45) and, in particular, note that the term in (44) is the one of greatest order. So if we show that for $j=1, 2$ and accordingly $p \in \{0, x\}$:

$$\sum_{i=1}^n \mathbb{P} \left(\frac{\pi}{g(Z_i) \sqrt{Z_i - p}} \mathbf{1}_{A_n^{i,p}} > \epsilon \frac{\sqrt{n \log n}}{h_j} \right) \rightarrow 0,$$

where $A_n^{i,p}$ is as in (A17), we obtain the claim. The above summation is zero for all n big enough because the indicator on $A_n^{i,p}$ requires $1/\sqrt{Z_i - p}$ to be bounded by $n^{\frac{1}{4\gamma_p}} \leq \sqrt{n \log n}(C + o(1))$, for $\gamma_p > \frac{1}{2}$, $C > 0$ and n big enough. Moreover the density of the observations on $A_n^{i,p}$ is bounded away from zero. ■

Proof of Lemma 3.2: We will structure the proof as follows: first, we show that the expression inside the parenthesis in (49) is mean zero, second, we show that by using the expansion (45) the term in line (44) is the one that contributes to the normal behaviour (both around x and around 0) and lastly we show that all the other terms converge to zero in probability. First note:

$$\int_0^\infty \int_z^\infty \frac{\chi_{2,n}(v-x) dv}{2g(z)\sqrt{v-z}} g(z) dz = \int_0^\infty \int_0^v \frac{\chi_{2,n}(v-x)}{2\sqrt{v-z}} dz dv = \int_0^\infty \frac{\chi_{2,n}(v-x)}{v^{-\frac{1}{2}}} dv$$

and it is immediate to see, using the definition of m_{h_n} and m_0 , that

$$D_{h_n} m_{h_n} = m_0 + h_{1,n} \int_0^\infty \sqrt{v} \chi_{1,n}(v) dv + h_{2,n} \int_0^\infty \sqrt{v} \chi_{2,n}(v-x) dv. \quad (\text{A18})$$

This proves that the expression inside the parenthesis in the statement of Lemma 3.2 is mean zero. Let $\bar{k}_n^x$ be the mean of the term first term on the right-hand side of the equality in (44) multiplied $1/(2m_{h_n} \sqrt{n \log n})$ and analogously $\bar{k}_n^0$ the same mean with 0 instead of x . Define (independently of

i and recall (40):

$$\bar{k}_n := \bar{k}_n^x + \bar{k}_n^0 = \mathbb{E} \left\{ \frac{\pi}{2m_{h_n} g(Z_i) \sqrt{Z_i - x}} \mathbf{1}_{A_n^{i,x}} \right\} + \mathbb{E} \left\{ \frac{\pi}{2m_{h_n} g(Z_i) \sqrt{Z_i}} \mathbf{1}_{A_n^{i,0}} \right\},$$

where we used again the definition of $A_n^{i,p}$ in (A17). Note that we just isolate the term $\bar{k}_n$ from the centring $(m_{h_n} D_{h_n} - m_0)/(m_{h_n})$ which already contained it. We need to isolate $\bar{k}_n$ for the next part (note $k_n = o(1)$). Finally, denote by Σ_h the asymptotic variance in (49). Using the fact that $D_{h_n} = 1 + o(1)$, we can prove that for $h_{i,n}$, $i = 1, 2$ as in (40):

$$\frac{\pi}{2m_{h_n}} \sum_{i=1}^n \left\{ \frac{h_{1,n}}{g(Z_i) \sqrt{Z_i}} \mathbf{1}_{A_n^{i,0}} + \frac{h_{2,n}}{g(Z_i) \sqrt{Z_i - x}} \mathbf{1}_{A_n^{i,x}} - \bar{k}_n \right\} \rightsquigarrow N(0, \Sigma_h).$$

We apply the Lindeberg Central Limit Theorem. If we look at

$$s_n^2 := \frac{\pi^2}{4m_{h_n}^2} \sum_{i=1}^n \mathbb{V} \left\{ \frac{h_{1,n}}{g(Z_i) \sqrt{Z_i}} \mathbf{1}_{A_n^{i,0}} + \frac{h_{2,n}}{g(Z_i) \sqrt{Z_i - x}} \mathbf{1}_{A_n^{i,x}} \right\},$$

the covariance of the two terms above inside the parenthesis goes to zero because for all n sufficiently large $\mathbf{1}_{A_i^0} \mathbf{1}_{A_i^x} = 0$. Now we focus on the variance

$$\sum_{i=1}^n \mathbb{V} \left\{ \frac{h_2}{\sqrt{n \log n}} \frac{\pi}{2m_{h_n} g(Z_i) \sqrt{Z_i - x}} \mathbf{1}_{A_n^{i,x}} \right\} = \frac{h_2^2 \pi^2}{8\gamma_x m_0^2 g(x)} + o(1). \quad (\text{A19})$$

Indeed using the i.i.d. assumption and leaving out the constants $(h_i^2 \pi^2)/(4m_0^2)$ (recall $m_{h_n}/m_0 \rightarrow 1$), we obtain that the left-hand side of (A19) is equivalent to

$$\frac{1}{\log n} \mathbb{E} \left[\frac{1}{g^2(Z)(Z-x)} \mathbf{1}_{A_n^x} \right] - \left(\frac{1}{\sqrt{\log n}} \mathbb{E} \left[\frac{1}{g(Z) \sqrt{Z-x}} \mathbf{1}_{A_n^x} \right] \right)^2.$$

Now analyse the first term by taking limit in n . Using the continuity of g at x and that $g(x) > 0$, together with $0 < \eta_n < 1$, and $\eta_n \gtrsim \frac{1}{\log^\alpha(n)}$:

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \int_{x+\frac{1}{n^{1/2\gamma_x}}}^{x+\eta_n} \frac{1}{z-x} \frac{1}{g(z)} dz = \frac{1}{2\gamma_x g(x)}. \quad (\text{A20})$$

The second term converges to zero because: $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\log n}} \int_{x+\frac{1}{n^{1/2\gamma_x}}}^{x+\eta_n} \frac{1}{\sqrt{z-x}} dz = 0$ just by integrating and taking the limit. This proves the claim about the variance. Using the same relation (A19) with 0 instead of x , we obtain

$$s_n^2 = \frac{h_1^2 \pi^2}{8\gamma_0 m_0^2 g(0)} + \frac{h_2^2 \pi^2}{8\gamma_x m_0^2 g(x)} + o(1). \quad (\text{A21})$$

Now we look at Lindeberg condition; define

$$B_n^\epsilon := \left\{ \left| \frac{\pi}{2m_{h_n} \sqrt{n \log n}} \left(\frac{\mathbf{1}_{A_n^{i,x}}}{g(Z_i) \sqrt{Z_i - x}} + \frac{\mathbf{1}_{A_n^{i,0}}}{g(Z_i) \sqrt{Z_i}} \right) - \bar{k}_n \right| > \epsilon s_n \right\}.$$

Then $\forall \epsilon > 0$, as $n \rightarrow \infty$, we show

$$\frac{1}{s_n^2} \sum_{i=1}^n \mathbb{E} \left[\left(\frac{\pi h_{1,n} \mathbf{1}_{A_n^{i,0}}}{2g(Z_i) m_{h_n} \sqrt{Z_i}} + \frac{\pi h_{2,n} \mathbf{1}_{A_n^{i,x}}}{2g(Z_i) m_{h_n} \sqrt{Z_i - x}} - \bar{k}_n \right)^2 \mathbf{1}_{B_n^\epsilon} \right] \rightarrow 0. \quad (\text{A22})$$

By using the triangle inequality, the fact that $\bar{k}_n = O(1/\sqrt{n \log n})$, that from (A21) s_n converges to a constant, and that $\mathbf{1}_{A_n^{i,p}}/g(Z_i) = \mathbf{1}_{A_n^{i,p}}/g(p) + o(1)$, for $p \in \{0, x\}$ and all n big enough, we can work

with the superset of B_n^ϵ :

$$C_{n,i}^\epsilon := \left\{ \frac{\mathbf{1}_{A_n^{i,x}}}{(g(x) + o(1))\sqrt{Z_i - x}} + \frac{\mathbf{1}_{A_n^{i,0}}}{(g(0) + o(1))\sqrt{Z_i}} > \frac{\epsilon 2m_0 s_n}{\pi(n \log n)^{-1/2}} + O(1) \right\}.$$

Since $(\bar{k}_n)^2 = o(1)$, the term in (A22) involving $(\bar{k}_n)^2$ goes to zero. Alike, the cross-product terms go to zero. So the only terms that matter are the squared terms. Thus we look at

$$\frac{\pi^2}{4m_{h_n}^2 s_n^2 n \log n} \sum_{i=1}^n \mathbb{E} \left[\frac{1}{g^2(Z_i)(Z_i - p)} \mathbf{1}_{A_n^{i,p}} \mathbf{1}_{C_{n,i}^\epsilon} \right], \quad \text{for } p \in \{0, x\},$$

which are zero for all n sufficiently large as for all such n we have $\mathbf{1}_{A_n^{i,p}} \mathbf{1}_{C_{n,i}^\epsilon} = 0$.

Now we proceed with the last part of the proof and we show that all the terms that we did not take into account in the computations above converge to zero by using the Bernstein inequality (see for instance Pollard 1984 page 193). We will do it only for the terms deriving from (45) involving the perturbation around x . The analogous terms deriving from the perturbation around 0 can be treated in the same way.

To apply Bernstein's inequality, we need to centre the random variables of interest. The means of all the terms below are already contained in $(m_{h_n} D_{h_n} - m_0)/(m_{h_n})$ (as it was for $\bar{k}_n$), thus we only need to isolate them for the next part. Recall also the assumptions on F and g (continuous at x and 0 and $g(x) > 0, g(0) > 0$).

- (1) Let $A_n^i := \{\frac{1}{n^{1/(2\gamma_x)}} \leq Z_i - x \leq \eta_n\}$ and $Y_n^i = O(\eta_n^{-1/2})(\mathbf{1}_{A_n^i}/g(Z_i) - \mathbb{E}\mathbf{1}_{A_n^i}/g(Z_i))$ then $\mathbb{V}(Y_n^i) = O(1)$. Define

$$\tilde{Y}_n^i = \eta_n^{1/2} Y_n^i.$$

Then $\sqrt{\mathbb{V}(Y_n^i)} = \eta_n^{-1/2} \sqrt{\mathbb{V}(\tilde{Y}_n^i)}$ and by the assumptions on g we have $|\tilde{Y}_n^i| \leq L$ bounded for all n big enough, therefore $\forall \epsilon > 0$:

$$\begin{aligned} \mathbb{P}\left(\frac{1}{\sqrt{n \log n}} \left| \sum_{i=1}^n Y_n^i \right| > \epsilon\right) &= \mathbb{P}\left(\frac{\sqrt{\mathbb{V}(Y_n^i)}}{\sqrt{\mathbb{V}(Y_n^i)}\sqrt{n \log n}} \left| \sum_{i=1}^n Y_n^i \right| > \epsilon\right) \\ &\leq \mathbb{P}\left(\frac{O(1)}{\sqrt{\mathbb{V}(\tilde{Y}_n^i)}\sqrt{n \log n}} \left| \sum_{i=1}^n \tilde{Y}_n^i \right| > \epsilon\right) \lesssim 2e^{-\frac{\epsilon^2 \log n}{2} \left(1 + \frac{\epsilon L \sqrt{\log n}}{O(\sqrt{\eta_n})\sqrt{n}}\right)^{-1}} \rightarrow 0. \end{aligned} \quad (\text{A23})$$

- (2) Let $A_n^i := \{-\frac{1}{n^{1/(2\gamma_x)}} \leq Z_i - x \leq \frac{1}{n^{1/(2\gamma_x)}}\}$, $Y_n^i = O(n^{\frac{1}{4\gamma_x}})(\mathbf{1}_{A_n^i}/g(Z_i) - \mathbb{E}\mathbf{1}_{A_n^i}/g(Z_i))$ then $\mathbb{V}(Y_n^i) = O(1)$. Define

$$\tilde{Y}_n^i = n^{-\frac{1}{4\gamma_x}} Y_n^i.$$

Then $\sqrt{\mathbb{V}(Y_n^i)} = n^{\frac{1}{4\gamma_x}} \sqrt{\mathbb{V}(\tilde{Y}_n^i)}$ and by the assumptions on g we have $|\tilde{Y}_n^i| \leq L$ bounded for all n big enough, using the same steps as for the previous term in (A23) and because $\gamma_x > \frac{1}{2}$, $\forall \epsilon > 0$:

$$\mathbb{P}\left(\frac{1}{\sqrt{n \log n}} \left| \sum_{i=1}^n Y_n^i \right| > \epsilon\right) \lesssim 2e^{-\frac{\epsilon^2 \log n}{2} \left(1 + \frac{\epsilon L \sqrt{\log n}}{O(n^{-1/4\gamma_x})\sqrt{n}}\right)^{-1}} \rightarrow 0.$$

- (3) Let $A_n^i := \{-\eta_n \leq Z_i - x \leq -\frac{1}{n^{1/(2\gamma_x)}}\}$, $Y_n^i = \frac{O(n^{-1/2\gamma_x})}{g(Z_i)(x-Z_i)^{3/2}} \mathbf{1}_{A_n^i} - \mathbb{E} \frac{O(n^{-1/2\gamma_x})}{g(Z_i)(x-Z_i)^{3/2}} \mathbf{1}_{A_n^i}$, then $\mathbb{V}(Y_n^i) = O(1)$. Define

$$\tilde{Y}_n^i = n^{-\frac{1}{4\gamma_x}} Y_n^i.$$

Then $\sqrt{\mathbb{V}(Y_n^i)} = n^{\frac{1}{4\gamma_x}} \sqrt{\mathbb{V}(\tilde{Y}_n^i)}$ and by the assumptions on g we have $|\tilde{Y}_n^i| \leq L$ bounded for all n big enough, using the same steps as for the previous term in (A23) and because $\gamma_x > \frac{1}{2}$, $\forall$

$\text{varepsilon} > 0$:

$$\mathbb{P}\left(\frac{1}{\sqrt{n \log n}} \left| \sum_{i=1}^n Y_n^i \right| > \varepsilon\right) \lesssim 2e^{-\frac{\varepsilon^2 \log n}{2} \left(1 + \frac{\varepsilon L \sqrt{\log n}}{O(n^{-1/4} \gamma_x) \sqrt{n}}\right)^{-1}} \rightarrow 0.$$

(4) Let $A_n^i := \{Z_i - x \leq -\eta_n\}$, and $Y_n^i = O(\eta_n^{-\frac{1}{2}})(\mathbf{1}_{A_n^i}/g(Z_i) - \mathbb{E}\mathbf{1}_{A_n^i}/g(Z_i))$ then $\mathbb{V}(Y_n^i) = O(\eta_n^{-1})(\int_0^x 1/g(z) dz - x^2)$ (using the assumptions on g we have $\int_0^x 1/g(z) dz > 0$). Define

$$\tilde{Y}_n^i = \eta_n^{1/2} Y_n^i.$$

Then $\sqrt{\mathbb{V}(Y_n^i)} = \eta_n^{-1/2} \sqrt{\mathbb{V}(\tilde{Y}_n^i)}$ and by the assumptions on g we have $|\tilde{Y}_n^i| \leq L$ bounded for all n big enough, therefore $\forall \varepsilon > 0$:

$$\begin{aligned} \mathbb{P}\left(\frac{1}{\sqrt{n \log n}} \left| \sum_{i=1}^n Y_n^i \right| > \varepsilon\right) &\leq \mathbb{P}\left(\frac{O(\eta_n^{-\frac{1}{2}})}{\sqrt{\mathbb{V}(Y_n^i)} \sqrt{n \log n}} \left| \sum_{i=1}^n Y_n^i \right| > \varepsilon\right) \\ &= \mathbb{P}\left(\frac{O(1)}{\sqrt{n} \sqrt{\mathbb{V}(\tilde{Y}_n^i)}} \left| \sum_{i=1}^n \tilde{Y}_n^i \right| > \varepsilon \sqrt{n \log n}\right) \lesssim 2e^{-\frac{\varepsilon^2 \log n}{2(\log n)^\alpha} \left(1 + \frac{\varepsilon L \sqrt{\log n}}{\sigma \eta_n^{-1/2} \sqrt{n}}\right)^{-1}} \rightarrow 0, \end{aligned}$$

where $\alpha \in (0, 1)$ and $\sigma^2 := \int_0^x 1/g(z) dz - x^2$.

(5) The last term can be treated identically to 1. ■

Proof of (51): Let A_n^i as in (A17). We are going to verify the conditions of Theorem 2.2.6. from Durrett (2010) to prove

$$\frac{1}{n \log n} \left\{ \sum_{i=1}^n \frac{1}{g^2(Z_i)(Z_i - x)} \mathbf{1}_{A_n^i} - n \int_{x + \frac{1}{n^{1/2} \gamma_x}}^{x + \eta_n} \frac{1}{z - x} \frac{1}{g(z)} dz \right\} \rightarrow 0. \quad (\text{A24})$$

By (A20) we know how $\frac{1}{\log n} \int_{x + \frac{1}{n^{1/2} \gamma_x}}^{x + \eta_n} \frac{1}{z - x} \frac{1}{g(z)} dz$ behaves; for this reason, if we show (A24), we get the claim. We verify condition (i) of Theorem 2.2.6. in Durrett (2010):

$$n \int_{\left\{ \frac{1}{g^2(z)(z-x)} \mathbf{1}_{\left\{ \eta_n^{-1} \leq \frac{1}{z-x} \leq n^{1/2} \gamma_x \right\}} \right\} > n \log n} g(z) dz = 0,$$

because the set over which we are integrating is empty for all n big enough. Now we check condition (ii) of Theorem 2.2.6. from Durrett (2010); we have, using Leibniz integral rule together with de L'Hôpital rule and the fact that $\frac{1}{2\gamma_x} + 1 > \frac{1}{\gamma_x}$:

$$\lim_{n \rightarrow \infty} \frac{1}{n^2 \log^2 n} n \int_{x + \frac{1}{n^{1/2} \gamma_x}}^{x + \eta_n} \frac{1}{(z - x)^2} \frac{1}{g^3(z)} dz = 0.$$

The above computations now yield (A24). By combining (A24) and (A20), together with $\frac{\pi^2}{4m_{hn}^2} \rightarrow \frac{\pi^2}{4m_0^2}$, we obtain the claim. ■

Proof of (52): Using (A18), the definition of $h_{i,n}$, $i = 1, 2$ in (40), the fact that $\int_0^\infty \sqrt{v} \chi_n(v) dv = O(\sqrt{\eta_n})$ and that $\int_0^\infty \sqrt{v} \chi_n(v - x) dv = O(\eta_n)$:

$$\left(\frac{m_0 - m_{h_n} D_{h_n}}{m_{h_n} D_{h_n}} \right)^2 = O\left(\frac{\eta_n}{n \log n} \right) = O\left(\frac{1}{n \log^{1+\alpha}(n)} \right).$$

Now we look only at the terms derived from the perturbation around x (i.e. the ones for $p = x$), as the one derived from the perturbation around 0 behave alike. It is enough to consider all the squared terms in $(R_n^{i,x})^2$. The ones that are at most $O(\eta_n^{-1})$, go to zero as $\frac{nO(\log^\alpha n)}{n \log n} \rightarrow 0$, $\alpha \in (0, 1)$. The remaining terms converge in probability to 0 by Bernstein's inequality as in the proof of Lemma 3.2. ■

Proof of Lemma 3.3:

$$\begin{aligned} \sqrt{\frac{n}{\log n}} (F_{h_n}(x) - F(x)) &= \frac{\sqrt{n}}{D_{h_n} \sqrt{\log n}} ((1 - D_{h_n})F(x)) + \frac{h_1 \int_0^x \chi_{1,n}(v) dv}{D_{h_n} \log n} \\ &+ \frac{h_2 \int_0^x \chi_{2,n}(v - x) dv}{D_{h_n} \log n} = \left(-\frac{h_1 \int_0^\infty \chi_{1,n}(v) dv}{D_{h_n} \log n} - \frac{h_2 \int_0^\infty \chi_{2,n}(v - x) dv}{D_{h_n} \log n} \right) F(x) \\ &+ \frac{h_1 \int_0^x \chi_{1,n}(v) dv}{D_{h_n} \log n} + \frac{h_2 \int_0^x \chi_{2,n}(v - x) dv}{D_{h_n} \log n}. \end{aligned}$$

Now by using: $\int_0^\infty \chi_{1,n}(v) dv = -(1 + o(1)) \log n^{-1/(2\gamma_0)}$, $\int_0^\infty \chi_{2,n}(v - x) dv = o(1)$, $\int_0^x \chi_{1,n}(v) dv = -(1 + o(1)) \log n^{-1/(2\gamma_0)}$, and that $\int_0^x \chi_{2,n}(v - x) dv = (1 + o(1)) \log n^{-1/(2\gamma_x)}$, the above for all n big is equal to (using also $1/D_{h_n} \rightarrow 1$)

$$\begin{aligned} &= \frac{h_1}{D_{h_n} \log n} (1 + o(1)) (F(x) - 1) \log n^{-\frac{1}{2\gamma_0}} - \frac{h_2}{D_{h_n} \log n} (o(1)) F(x) \\ &+ \frac{h_2}{D_{h_n} \log n} (1 + o(1)) \log n^{-\frac{1}{2\gamma_x}} \rightarrow \frac{h_1}{2\gamma_0} (1 - F(x)) - \frac{h_2}{2\gamma_x} = h^\top \begin{bmatrix} \frac{1}{2\gamma_0} (1 - F(x)) \\ -\frac{1}{2\gamma_x} \end{bmatrix}. \end{aligned}$$

Proof of Theorem 1.2: Because for any neighbourhood in supremum norm of F there exists $n \in \mathbb{N}$ such that the 'least favourable' perturbation F_{h_n} (cf. (39)) is in such neighbourhood, the expression in (7) is lower-bounded by the expression in (42) (cf. p. 117 in van der Vaart 1998). ■