On Cohomology and Ext-groups

by

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Contents

Summary

This thesis, called 'on Cohomology and Ext-groups', would fit best under the mathematical research area called algebraic topology. As the name suggests, this thesis will contain some algebra, and some topology. Accordingly this thesis will contain two chapters, treating the different subjects separately. However, in chapter 2 we will use the proven algebraic tools from chapter 1 to obtain useful theorems and constructions.

In chapter 1 we will treat homological algebra. In subsection 1.1 we will start by defining the notion of an exact sequence of abelian groups. Then we will define the less restrictive concept of a complex of abelian groups. Using these complexes we will define their (co)homology groups in subsection 1.2. Thereafter, in subsection 1.3 we will define injective and projective groups and give useful equivalent definitions. In particular, we will prove that an abelian group A is injective if and only if A is a divisible group. Next up a special type of exact sequence, called an injective resolution, will be defined. Using these injective resolutions we will define Ext-groups in subsection 1.4 which play a central role in this thesis. We will derive various properties of these groups and characterize $\mathrm{Ext}(A, B)$ for any finitely generated abelian groups A and B. In subsection 1.5 we will prove that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{A}/\mathbb{Q}$ where $\mathbb{A}$ is the Adèle group consisting of all sequences $(x_2, x_3, x_5, \dots)$ such that all $x_p \in \mathbb{Q}_p$, the p-adic rationals and $x_p \in \mathbb{Z}_p$, the p-adic integers for all but finitely many primes p. Here $\mathbb{Q} \subset \mathbb{A}$ denotes the subgroup of constant sequences $(q, q, q, \dots)$ with $q \in \mathbb{Q}$. Finally, in subsection 1.6 we will state and prove the universal coefficient theorem which relates the homology and cohomology groups of a complex of free abelian groups. This allows one to calculate the cohomology groups of a complex of free abelian groups using the homology groups and sufficient knowledge of Ext-groups and Hom-groups.

In chapter 2 we will treat the topology. In subsection 2.1 we will introduce a specific kind of complex of abelian groups called a chain complex. This is a way to assign a complex of abelian groups to a topological space X. We will construct the boundary group homomorphisms and prove two properties. Firstly that the composition $\partial \circ \partial = 0$. Secondly, for a continuous function $f : X \to Y$ between topological spaces, that the induced map $f_\# : C_p(X) \to C_p(Y)$ commutes with the boundary group homomorphisms. That is, $\partial \circ f_\# = f_\# \circ \partial$. In subsection 2.2 we define singular homology as the homology groups of these chain complexes. We prove that homeomorphic topological spaces, and even better homotopy equivalent spaces, have isomorphic singular homology groups. We finish off this subsection by calculating the singular homology groups of a contractible space. In subsection 2.3 we will describe $H_0(X)$ and $H_1(X)$ explicitly for any topological space X. We prove that $H_0(X) \cong \oplus_{a \in A} \mathbb{Z}$ where A denotes the set of (disjoint) path-connected components. And we will state that for a path-connected topological space X, $H_1(X) \cong \pi_1(X, x)_{\mathrm{ab}}$ which is called Hurewicz' theorem. In subsection 2.4 we will use the tools developed in chapter 1 to prove the long exact sequence of Mayer-Vietoris. For a topological space X and an open covering $\mathcal{U} = \{U, V\}$ of X the Mayer-Vietoris long exact sequence allows one to calculate the singular homology groups of X in terms of the singular cohomology groups of U, V and $U \cap V$. This theorem will be used extensively in subsection 2.5 to calculate the singular homology groups of the sphere S^n and real projective n-space $\mathbb{P}^n(\mathbb{R})$. Finally, in subsection 2.6 we will combine the results of subsections 1.6 and 2.5 to calculate the singular cohomology groups of the spheres S^n and real projective n-space $\mathbb{P}^n(\mathbb{R})$. We will also give an explicit description of the 0-th singular cohomology group of a topological space X with coefficients in an abelian group A. We have proven that $H^0(X, A) \cong A^{\pi_0(X)}$ where $\pi_0(X)$ is the set of path-connected components of X.

1 Homological algebra

In this first section the notion of exact sequences and chain complexes of abelian groups will be presented. Thereafter injective and projective objects will be defined, and a useful equivalent definition will be given. Using these groups we will define injective resolutions and using these also Ext-groups. Finally the universal coefficient theorem will be stated and proven which relates homology and cohomology of a complex of free abelian groups. Various Ext-groups will be calculated, and in particular the group $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ will be thoroughly treated resulting in a very interesting isomorphism.

1.1 Sequences of abelian groups

Definition 1.1. Let A, B and C be abelian groups. Let $f : A \to B$ and $g : B \to C$ be group homomorphisms. We call

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

a *short exact sequence of abelian groups* when f is injective, g is surjective and $\mathrm{Im}(f) = \mathrm{Ker}(g)$. More generally a sequence

$$\dashrightarrow A^{i-1} \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \dashrightarrow$$

of abelian groups is called *exact* if for each $i \in \mathbb{Z}$ we have $\mathrm{Im}(d_{i-1}) = \mathrm{Ker}(d_i)$.

Example 1.2. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

where π is the usual projection. Furthermore, note that for any abelian groups A and B and injective $f : A \to B$ we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{\pi} B/f(A) \longrightarrow 0$$

Example 1.3. Another interesting example, which is not of the previous form is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{e^{2\pi i -}} S^1 \longrightarrow 0$$

where i is the inclusion of $\mathbb{Z}$ into $\mathbb{R}$. Clearly the inclusion is injective and the exponential map is surjective onto the circle S^1, embedded in the complex plane $\mathbb{C}$. Then one shows that $e^{2\pi i x} = 1$ if and only if $x \in \mathbb{Z}$ proving the exactness of the given sequence.

Definition 1.4. A short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called *split exact*, if it is isomorphic as a short exact sequence, to the trivial extension of C over A. That is, there exists some group homomorphism $\phi : B \to A \oplus C$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & A \oplus C & \xrightarrow{\pi_2} & C & \longrightarrow & 0
\end{array}$$

commutes.

We remark that the homomorphism $\phi : B \to A \oplus C$ is in fact an isomorphism, this can be shown by simple diagram chasing.

Lemma 1.5 (Splitting lemma)**.** *Given an exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the following are equivalent:
(i) The exact sequence is split exact;
(ii) There exists a section $s : C \to B$, a group homomorphism such that $g \circ s = \mathrm{id}_C$;
(iii) There exists a retract $r : B \to A$, a group homomorphism such that $r \circ f = \mathrm{id}_A$.

See for example [1, p. 147] for a proof.

Example 1.6. The trivial example of a split exact sequence is given by

$$0 \longrightarrow A \xrightarrow{i_1} A \oplus C \xrightarrow{\pi_2} C \longrightarrow 0$$

where i_1 is the inclusion in the first coordinate, and π_2 projection on the second.

Example 1.7. On the other hand not all exact sequences split, for example in 1.2:

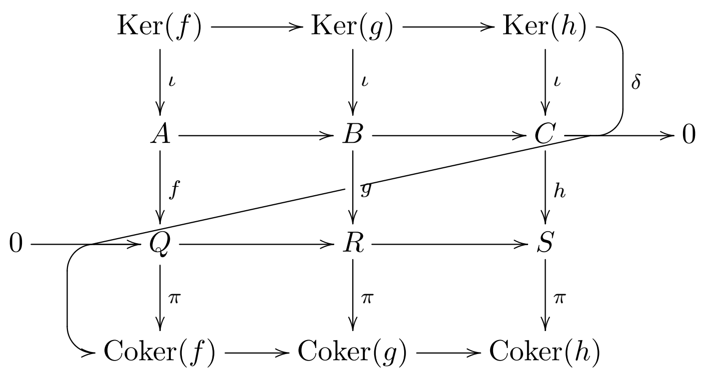
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

it is clear that $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ thus the given short exact sequence does not split.

Lemma 1.8 (Snake lemma)**.** *Given a commutative diagram of abelian groups*

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & S & & \end{array}$$

where the rows are exact, there exists an extended commutative diagram



such that the sequence

$$\mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(g) \longrightarrow \mathrm{Ker}(h) \xrightarrow{\delta} \mathrm{Coker}(f) \longrightarrow \mathrm{Coker}(g) \longrightarrow \mathrm{Coker}(h)$$

is exact. We call δ the connecting homomorphism.

We will give the construction of δ. The reader can either check well-definedness and exactness themselves or refer to e.g. [3, Lemma 1.3.2]. Let $c \in \operatorname{Ker}(h)$, then due to exactness c can be lifted to some $b \in B$. Because $h(c) = 0$ in S the image of $g(b)$ in S is also zero and can thus be lifted to an element $q \in Q$. We then define $\delta(c) = \pi(q) \in \operatorname{Coker}(f)$. This map is well-defined and does not depend on the chosen lifts of c.

The snake lemma plays a central role in the construction of connecting homomorphisms in long exact sequences. It is used in the proof of theorem 1.12, which is a very helpful tool. For example, the long exact sequences of Mayer-Vietoris will be proven using this theorem. Another interesting application is the associated long exact sequence for right derived objects, which can sometimes be used to calculate the right derived objects themselves.

1.2 (Co)homology of a complex

One can introduce a slightly weaker variant of an exact sequence.

Definition 1.9. Consider a sequence of abelian groups and group homomorphisms with increasing index

$$A^\bullet : \qquad \dashrightarrow A^{i-1} \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \dashrightarrow$$

we call $A^\bullet$ a *complex* if $\operatorname{Im}(d_{i-1}) \subset \operatorname{Ker}(d_i)$. An equivalent condition is that the composition of any two arrows results in the trivial group homomorphism.

The required inclusions allow us to consider the quotients $\operatorname{Ker}(d_i)/\operatorname{Im}(d_{i-1})$, this leads to the most important definition of this chapter.

Definition 1.10. For a given complex of abelian groups $A^\bullet$, we define the cohomology groups of $A^\bullet$ to be $H^i(A^\bullet) = \operatorname{Ker}(d_i)/\operatorname{Im}(d_{i-1})$ for $i \in \mathbb{Z}$.

If the indexing of a complex $A_\bullet$ goes down one can define the homology groups of $A_\bullet$ as $H_i(A_\bullet) = \operatorname{Ker}(d_i)/\operatorname{Im}(d_{i+1})$. In the second chapter we will discuss singular (co)homology of a topological space X and in section 1.6 prove the universal coefficient theorem which relates them. The term singular specifies the way in which we transform X into a complex, in this case called a chain complex $C_\bullet$.

Definition 1.11. Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be increasing complexes of abelian groups. We will denote the groups of these complexes by A^i, B^i and C^i respectively. We call a collection of short exact sequences of abelian groups

$$0 \longrightarrow A^i \longrightarrow B^i \longrightarrow C^i \longrightarrow 0$$

for $i \in \mathbb{Z}$, such that

$$\begin{array}{ccccc} A^i & \longrightarrow & B^i & \longrightarrow & C^i \\ \downarrow & & \downarrow & & \downarrow \\ A^{i+1} & \longrightarrow & B^{i+1} & \longrightarrow & C^{i+1} \end{array}$$

commutes for all $i \in \mathbb{Z}$ an *exact sequence of complexes* denoted by

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

The next theorem will be given without proof, for a proof one can read e.g. [3, Theorem 1.3.1].

Theorem 1.12 (Long exact sequence of cohomology)**.** *Any short exact sequence of complexes*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

gives rise to a long exact sequence of cohomology groups with natural group homomorphisms

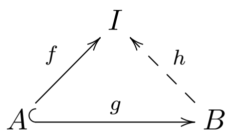
$$\dashrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \xrightarrow{\delta^i} H^{i+1}(A^\bullet) \dashrightarrow$$

We will give a description of the connecting homomorphism δ^i. This will become particularly important when proving the Mayer-Vietoris long exact sequence of homology and deriving the long exact sequence of right derived functors. Take an element $x \in H^i(C^\bullet)$. This can be represented by an element $x' \in \operatorname{Ker}(C^i \to C^{i+1})$ and this can be lifted to an $y \in B^i$. Then the image of y in B^{i+1} denoted by y' has $y' \in \operatorname{Ker}(B^{i+1} \to C^{i+1}) = \operatorname{Im}(A^{i+1} \to B^{i+1})$. Because of this and the fact that the group homomorphism $A^{i+1} \to B^{i+1}$ is injective we can identify y' with an element $z \in A^{i+1}$. Because $y' \in \operatorname{Ker}(B^{i+1} \to B^{i+2})$ and the injectivity of the group homomorphism $A^{i+2} \to B^{i+2}$ it must be that $z \in \operatorname{Ker}(A^{i+1} \to A^{i+2})$. So let $\delta^i(x) = [z]$ the class of z in $H^{i+1}(A^\bullet)$.

Note that after some reordering there exists also an associated long exact sequence of homology groups if the initial complexes had a downward indexing.

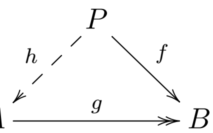
1.3 Injective and projective groups

Definition 1.13. We call an abelian group I injective if for any abelian groups A and B and any $f : A \to I$ and an injective $g : A \to B$ there exists a group homomorphism $h : B \to I$ such that



commutes.

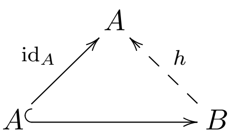
Definition 1.14. In similar fashion we call a group P projective if for any abelian groups A and B and any $f : P \to B$ and a surjective $g : A \to B$ there exists a group homomorphism $h : P \to A$ such that



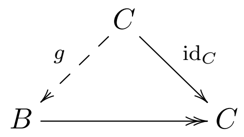
commutes.

Corollary 1.15. *A short exact sequence* $0 \to A \to B \to C \to 0$ *splits if* A *is injective or* C *is projective.*

Proof. In the case where A is injective the injective homomorphism $A \to B$ and the identity $\mathrm{id}_A : A \to A$ can fit in the commutative diagram



The injectivity of A guarantees the existence of such an h, which is a retract, as in 1.5 (ii). When C is projective the surjective homomorphism $B \to C$ and the identity $id_C : C \to C$ fit in the commutative diagram



The projectivity of C guarantees the existence of such a g, which is a section, as in 1.5 (iii). Hence in both cases the exact sequence splits. □

Definition 1.16. Let A be an abelian group, we call A *divisible* if for all $x \in A$ and $n \in \mathbb{Z}_{\geq 1}$ there exists an $y \in A$ such that $x = ny$.

Proposition 1.17. *An abelian group A is injective if and only if A is divisible.*

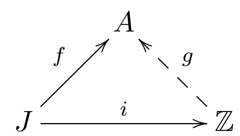
Before we will prove this proposition we will formulate and prove a useful lemma.

Lemma 1.18. *An abelian group A is divisible if and only if for all subgroups $J \subset \mathbb{Z}$ and homomorphisms $f : J \to A$, there exists a homomorphism $g : \mathbb{Z} \to A$ such that $g|_J = f$. We call this the integer extension property.*

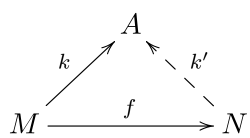
Proof. Suppose A is divisible, let $J \subset \mathbb{Z}$ a subgroup and $f : J \to A$ a group homomorphism. If $J = 0$ we can choose an arbitrary group homomorphism $g : \mathbb{Z} \to A$. If $J \neq 0$, then $J = n\mathbb{Z}$ for some $n \in \mathbb{Z}_{\geq 1}$. We write $f(n) = x$, and hence $f(kn) = kx$. Because A is divisible there exists a $y \in A$ such that $x = yn$. Now define $g : \mathbb{Z} \to A$ by $g(1) = y$. Then $g(n) = ny = x$ and $g(kn) = kny = kx$ so $g|_J = f$. This shows the first implication. Conversely, let $x \in A$ and $n \in \mathbb{Z}_{\geq 1}$ arbitrary. Define $f : n\mathbb{Z} \to A$ by $f(n) = x$, then there exists a $g : \mathbb{Z} \to A$ such that $g|_{n\mathbb{Z}} = f$. Thus $g(n) = ng(1) = x$, and as $g(1) \in A$ we have shown that $x = ny$ for some $y \in A$. □

Using the equivalent characterization of divisibility given in lemma 1.18 we will prove proposition 1.17.

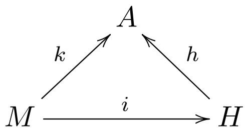
Proof. Suppose A is injective, let $J \subset \mathbb{Z}$ be a subgroup and let $i : J \to \mathbb{Z}$ denote the inclusion. Let $f : J \to A$ be any homomorphism, this fits in the commutative diagram:



because i is injective such a g exists by the injectivity of A. Hence A has the integer extension property. Conversely let A have the integer extension property. Let $f : M \to N$ be an injective group homomorphism. Let $k : M \to A$ be a group homomorphism. We want to show that k extends to a group homomorphism $k' : N \to A$ such that the diagram



commutes. Now consider the set $\{(H,h) | M \subseteq H \subseteq N, h : H \to A\}$ such that the following diagram



commutes. This set is nonempty as (M,k) is one such element. The set is also partially ordered, we say $(H_1,h_1) \leq (H_2,h_2)$ if and only if $H_1 \subseteq H_2$ and $h_2|_{H_1} = h_1$. Lastly, every chain of ordered elements has an upper bound. Namely, suppose

$$\cdots \leq (H_i,h_i) \leq (H_{i+1},h_{i+1}) \leq \ldots$$

is such a chain. Then let $(H,h) = (\bigcup_{i=1}^{\infty} H_i, h)$ where $h : H \to A$ is defined as $h(x) = h_i(x)$ for some i such that $x \in H_i$. We leave it to the reader to check that an uncountable chain or ordered elements also has an upper bound. Thus we may apply Zorn's lemma, stating that there exists a maximal element which we call (B,h), we now want to show that $B = N$. For the sake of a contradiction, suppose $N \setminus B \neq \varnothing$ and let $x \in N \setminus B$. Now $B \cap x \cdot \mathbb{Z} = n\mathbb{Z}$ for some $n \in \mathbb{Z}$. Now we have a group homomorphism $h : B \to A$ so in particular a group homomorphism $h : n\mathbb{Z} \to A$. Due to the integer extension property this homomorphism extends to a group homomorphism $l : \mathbb{Z} \to A$. Now we have a group homomorphism $B \oplus x \cdot \mathbb{Z} \to A$ given by $(b,z) \mapsto h(b) + l(z)$, and we have a natural group homomorphism $i : B \oplus x \cdot \mathbb{Z} \to B + x \cdot \mathbb{Z}$ given by $(b,z) \mapsto b + z$. Now consider the group homomorphism $k' : B + x \cdot \mathbb{Z} \to A$ sending an element y, that can be decomposed as $y = b + z$ for $b \in B$ and $z \in x \cdot \mathbb{Z}$ to $h(b) + l(z)$. This group homomorphism is well defined because $h = l$ on $B \cap x \cdot \mathbb{Z}$ and thus an extension of (B,h) which is a contradiction. Thus indeed, any maximal element is of the form (N,k') so A is injective. ☐

Example 1.19. The group $\mathbb{Q}$ is injective, because it is divisible. Furthermore, any quotient of an injective group, e.g. $\mathbb{Q}/\mathbb{Z}$ is also injective.

The following proposition will be given without proof, for a proof one can refer to [2, Theorem 11.6].

Proposition 1.20. *An abelian group A is projective if and only if A is free.*

1.4 Ext-groups

For abelian groups A and B we will define $\mathrm{Ext}(A,B)$, another abelian group. These groups are in particular useful when calculating the singular cohomology groups of some topological space X. In the next section this relation will be made precise as stated in the universal coefficient theorem. There are two main ways to define $\mathrm{Ext}(A,B)$, which use either injective or projective objects. In this section we will use the injective resolutions, and start with giving their definition.

Definition 1.21. Let A be an abelian group, let $I^0, I^1, \ldots$ be injective groups. We call any exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

an injective resolution of A.

One may note that there can exist many injective resolutions for a given group A. However, the injectivity of the I^k allows us to construct a chain map between different resolutions. It turns out that any two extensions are homotopic.

Lemma 1.22. *Let A be an abelian group, then any two injective resolutions $0 \to A \to I^\bullet$ and $0 \to A \to J^\bullet$ are homotopy equivalent. That is, the injective resolution of an abelian group is unique up to homotopy equivalence.*

Note that taking homology groups of an exact sequence is a rather trivial operation. Every (co)homology group will just be the zero group. However, given an exact sequence $I^\bullet$

$$0 \longrightarrow I^0 \xrightarrow{f_0} I^1 \xrightarrow{f_1} \cdots$$

apply the $\mathrm{Hom}(A, -)$ functor. That is, change every group I^k to $\mathrm{Hom}(A, I^k)$ and any $f_k : I^k \to I^{k+1}$ to the map $\varphi_k : \mathrm{Hom}(A, I^k) \to \mathrm{Hom}(A, I^{k+1})$ by sending an element $\phi_k \in \mathrm{Hom}(A, I^k)$ to $f_k \circ \phi_k$. Then one obtains a complex denoted by $\mathrm{Hom}(A, I^\bullet)$

$$0 \longrightarrow \mathrm{Hom}(A, I^0) \xrightarrow{\varphi_0} \mathrm{Hom}(A, I^1) \xrightarrow{\varphi_1} \cdots$$

which is not necessarily exact again. This gives rise to possibly interesting homology groups and leads us to the following definition.

Definition 1.23. Let B be an abelian group and $0 \to B \to I^\bullet$ an injective resolution. Then we define $\mathrm{Ext}^i(A, B)$ to be the i-th cohomology group of the complex $\mathrm{Hom}(A, I^\bullet)$.

Remark 1.24. One of the many fascinating properties of Ext-groups is that there is an equivalent definition given as follows. Let A be an abelian group and let

$$\cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow A \longrightarrow 0$$

be a projective resolution of A, that is the sequence is exact and all P^i are projective. Then apply the contra variant functor $\mathrm{Hom}(-, B)$ to the reduced complex, note the difference with $\mathrm{Hom}(A, -)$ used earlier. Giving a new complex with induced arrows

$$0 \longrightarrow \mathrm{Hom}(P^0, B) \longrightarrow \mathrm{Hom}(P^1, B) \longrightarrow \cdots$$

Then also $\mathrm{Ext}^i(A, B) = h^i(\mathrm{Hom}(P^\bullet, B))$. This gives two explicit methods of calculating different Ext-groups which can be very helpful. A proof that these two definitions are equivalent can be found in e.g. [3, Section 2.5].

Proposition 1.25. *For any abelian groups A and B we have* $\mathrm{Ext}^0(A, B) \cong \mathrm{Hom}(A, B)$.

Proof. Let $0 \to B \to I^\bullet$ be an injective resolution. Then in particular

$$0 \longrightarrow B \xrightarrow{i} I^0 \xrightarrow{f} I^1$$

is exact. I claim that the induced complex

$$0 \longrightarrow \mathrm{Hom}(A, B) \xrightarrow{i_*} \mathrm{Hom}(A, I^0) \xrightarrow{f_*} \mathrm{Hom}(A, I^1)$$

is also exact. First we show that i_* is injective. Suppose that $\phi : A \to B$ has $i_*(\phi) = 0$, that is $i \circ \phi = 0$. Because i is injective it must be that $\phi = 0$. Next up, because $\mathrm{Hom}(A, I^\bullet)$

is a complex, it is clear that $\operatorname{Im}(i_*) \subset \operatorname{Ker}(f_*)$. We will now also show the other inclusion. Suppose $\phi : A \to I^0$ has $f_*(\phi) = f \circ \phi = 0$. Let $a \in A$ be arbitrary, as $f(\phi(a)) = 0$ we can write $\phi(a) = i(b)$ for some $b \in B$. As a was arbitrary we can thus write this ϕ as $i \circ \phi^* = i_*(\phi^*)$ for some ϕ^*. Hence the derived complex is in fact exact. Now $\operatorname{Hom}(A, I^\bullet)$ is given by

$$0 \longrightarrow \operatorname{Hom}(A, I^0) \xrightarrow{f_*} \operatorname{Hom}(A, I^1) \longrightarrow \cdots$$

Thus $\operatorname{Ext}^0(A,B) = h^0(\operatorname{Hom}(A, I^\bullet)) \cong \operatorname{Ker}(f_*) = \operatorname{Im}(i_*) \cong \operatorname{Hom}(A,B)$ because of the shown exactness. □

Proposition 1.26. *For any abelian groups A, B we have $\operatorname{Ext}^i(A,B) = 0$ for $i \geq 2$.*

Proof. Let F be the free abelian group generated by the elements of B, thus $F = \bigoplus_{b \in B} b\mathbb{Z}$. Then clearly $F \subset \bigoplus_{b \in B} b\mathbb{Q}$. Let $\pi : F \to B$ be the associated map from the free abelian group F. Because π is surjective by the third isomorphism theorem $B \cong F/\operatorname{Ker}(\pi)$. So we have the inclusion

$$0 \longrightarrow B \longrightarrow \left(\bigoplus_{b \in B} b\mathbb{Q}\right) / \operatorname{Ker}(\pi)$$

Then we can project each $b\mathbb{Q} \to b\mathbb{Q}/\mathbb{Z}$ sending $x \to \bar{x}$. Thus we have the sequence

$$0 \longrightarrow B \longrightarrow \left(\bigoplus_{b \in B} b\mathbb{Q}\right) / \operatorname{Ker}(\pi) \longrightarrow \left(\bigoplus_{b \in B} b\mathbb{Q}/\mathbb{Z}\right) / \operatorname{Ker}(\pi) \longrightarrow 0$$

which is in fact exact and thus an injective resolution of B, write as $0 \to B \to I^\bullet$. Then $\operatorname{Hom}(A, I^\bullet)$ looks like

$$0 \longrightarrow \operatorname{Hom}\left(A, \left(\bigoplus_{b \in B} b\mathbb{Q}\right) / \operatorname{Ker}(\pi)\right) \longrightarrow \operatorname{Hom}\left(A, \left(\bigoplus_{b \in B} b\mathbb{Q}/\mathbb{Z}\right) / \operatorname{Ker}(\pi)\right) \longrightarrow 0$$

and hence $\operatorname{Ext}^i(A,B) = h^i(\operatorname{Hom}(A, I^\bullet)) = 0$ for $i \geq 2$. □

For abelian groups A and B we have seen that $\operatorname{Ext}^0(A,B) \cong \operatorname{Hom}(A,B)$ and that $\operatorname{Ext}^i(A,B) = 0$ for $i \geq 2$. Hence the only interesting Ext-group is the first one. From now on we will refer to $\operatorname{Ext}^1(A,B)$ as $\operatorname{Ext}(A,B)$.

Proposition 1.27. *Let B be an injective group, then for any abelian group A we have $\operatorname{Ext}^i(A,B) = 0$ for $i \geq 1$.*

Proof. We have the injective resolution $0 \to B \to B \to 0$, so then $\operatorname{Hom}(A, I^\bullet)$ is $0 \to \operatorname{Hom}(A,B) \to 0$. Hence the cohomology groups $h^i(\operatorname{Hom}(A, I^\bullet)) = 0$ for $i \geq 1$. □

Proposition 1.28. *Let A be a projective group, then for any abelian group B we have $\operatorname{Ext}^i(A,B) = 0$ for $i \geq 1$.*

For a proof one can read e.g. [3, Calculation 3.3.2].

Proposition 1.29. *We have the following isomorphisms of groups:*

$$\begin{aligned} \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) &= 0 \\ \operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) &\cong \mathbb{Z}/n\mathbb{Z} \\ \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) &= 0 \\ \operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) &\cong \mathbb{Z}/d\mathbb{Z} \end{aligned}$$

where $d = \gcd(m,n)$.

Proof. We will start by proving the first claim. For this we will use the injective resolution

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

After applying $\text{Hom}(\mathbb{Z}, -)$ the reduced complex is given by

$$0 \longrightarrow \text{Hom}(\mathbb{Z}, \mathbb{Q}) \longrightarrow \text{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

Which is canonically isomorphic, as a complex, to

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

So $\text{Ext}(\mathbb{Z}, \mathbb{Z}) = h^1(\text{Hom}(\mathbb{Z}, I^\bullet)) = \frac{\mathbb{Q}/\mathbb{Z}}{\mathbb{Q}/\mathbb{Z}} = 0$. For the second claim we can use the same injective resolution, but instead we apply $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, -)$ resulting in

$$0 \longrightarrow \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \longrightarrow \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

Now note that $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0$ and $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. The latter can be easily observed when viewing $\mathbb{Q}/\mathbb{Z}$ as $\bigcup_{n=1}^{\infty}(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$ revealing that the only options for $\overline{1}$ viewed as an element of $\mathbb{Z}/n\mathbb{Z}$ are the elements of the form $\frac{i}{n}$ for $0 \leq i \leq n-1$. Thus the reduced complex is isomorphic to

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

and hence $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. For the third and fourth case we will use the injective resolution of $\mathbb{Z}/m\mathbb{Z}$ given by

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Q}/m\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Q}/m\mathbb{Z} \longrightarrow 0$$

which we will denote by the following, equivalent, injective resolution

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow \bigcup_{n=1}^{\infty}(\tfrac{1}{mn}\mathbb{Z})/\mathbb{Z} \xrightarrow{\cdot m} \bigcup_{n=1}^{\infty}(\tfrac{1}{mn}\mathbb{Z})/\mathbb{Z} \longrightarrow 0$$

where $\overline{i} \mapsto \frac{i}{m}$ in the first map. Now applying $\text{Hom}(\mathbb{Z}, -)$ and considering the reduced complex we get

$$0 \longrightarrow \text{Hom}\left(\mathbb{Z}, \bigcup_{n=1}^{\infty}(\tfrac{1}{mn}\mathbb{Z})/\mathbb{Z}\right) \xrightarrow{(\cdot m)_*} \text{Hom}\left(\mathbb{Z}, \bigcup_{n=1}^{\infty}(\tfrac{1}{mn}\mathbb{Z})/\mathbb{Z}\right) \longrightarrow 0$$

which is again canonically isomorphic to

$$0 \longrightarrow \bigcup_{n=1}^{\infty}(\tfrac{1}{mn}\mathbb{Z})/\mathbb{Z} \xrightarrow{\cdot m} \bigcup_{n=1}^{\infty}(\tfrac{1}{mn}\mathbb{Z})/\mathbb{Z} \longrightarrow 0$$

Hence $\text{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = h^1(\text{Hom}(\mathbb{Z}, I^\bullet)) = \frac{\mathbb{Q}/m\mathbb{Z}}{\mathbb{Q}/m\mathbb{Z}} = 0$. Lastly, we will apply $\text{Hom}(\mathbb{Z}/n\mathbb{Z})$ to the injective resolution of $\mathbb{Z}/m\mathbb{Z}$ and considering the reduced complex we get

$$0 \longrightarrow \text{Hom}\left(\mathbb{Z}/n\mathbb{Z}, \bigcup_{k=1}^{\infty}(\tfrac{1}{mk}\mathbb{Z})/\mathbb{Z}\right) \xrightarrow{(\cdot m)_*} \text{Hom}\left(\mathbb{Z}/n\mathbb{Z}, \bigcup_{k=1}^{\infty}(\tfrac{1}{mk}\mathbb{Z})/\mathbb{Z}\right) \longrightarrow 0$$

A group homomorphism from $\mathbb{Z}/n\mathbb{Z}$ is completely determined by the image of the generator 1. Because 1 has order n in the group $\mathbb{Z}/n\mathbb{Z}$ it must be mapped to an element whose order divides n, which is exactly the n-torsion subgroup. Now for $\mathbb{Z}/m\mathbb{Z}$ we have $\mathbb{Z}/m\mathbb{Z}[n] = \frac{m}{d}\mathbb{Z}/m\mathbb{Z}$ where

$d = \gcd(m, n)$, because $\frac{m}{d}$ is the smallest integer dividing m such that $n \cdot \frac{m}{d}$ is a multiple of m. Then we have isomorphisms

$$\tfrac{m}{d}\mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} \tfrac{1}{d}\mathbb{Z}/\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}$$

So $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \frac{1}{m}\mathbb{Z}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$. For $k \geq 1$ we have $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \frac{1}{km}\mathbb{Z}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/mk\mathbb{Z}) \cong \mathbb{Z}/d_k\mathbb{Z}$, where $d_k = \gcd(n, km)$. Now taking $k = n$ yields $d_n = \gcd(n, nm) = n$ so $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \bigcup_{k=1}^{\infty}(\frac{1}{mk}\mathbb{Z})/\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Thus the reduced complex simplifies to

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

where. Hence

$$\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = h^1(\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, I^\bullet)) = \frac{\mathbb{Z}/n\mathbb{Z}}{m(\mathbb{Z}/n\mathbb{Z})} \cong \mathbb{Z}/\gcd(n, m)\mathbb{Z}.$$

□

Proposition 1.30. *Let A be an abelian group, then* $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$.

For a proof of proposition 1.30 one can refer to e.g. [3, Calculation 3.3.2]. This makes use of a projective resolution of $\mathbb{Z}/n\mathbb{Z}$.

Proposition 1.31. *Let A_i, B_i be abelian groups for all $i \in \mathbb{N}$, then*

$$\mathrm{Ext}^i\left(\bigoplus_j A_j, B_1\right) \cong \prod_j \mathrm{Ext}^i(A_j, B_1)$$

$$\mathrm{Ext}^i\left(A_1, \prod_j B_j\right) \cong \prod_j \mathrm{Ext}^i(A_1, B_j)$$

The key observation for the proof of proposition 1.31 is the fact that if

$$\ldots \longrightarrow P_i^1 \longrightarrow P_i^0 \longrightarrow A_i \longrightarrow 0$$

are projective resolutions of A_i for all i, then

$$\ldots \longrightarrow \bigoplus_{i=1}^{\infty} P_i^1 \longrightarrow \bigoplus_{i=1}^{\infty} P_i^0 \longrightarrow \bigoplus_{i=1}^{\infty} A_i \longrightarrow 0$$

is a projective resolution of $\bigoplus_i A_i$. Similarly, if

$$0 \longrightarrow B_i \longrightarrow I_i^0 \longrightarrow I_i^1 \longrightarrow \cdots$$

are injective resolutions of B_i for all i, then

$$0 \longrightarrow \prod_{i=1}^{\infty} B_i \longrightarrow \prod_{i=1}^{\infty} I_i^0 \longrightarrow \prod_{i=1}^{\infty} I_i^1 \longrightarrow \cdots$$

is an injective resolution of $\prod_{i=1}^{\infty} B_i$. Furthermore $\mathrm{Hom}(A_1, \prod_{i=1}^{\infty} B_i) \cong \prod_{i=1}^{\infty} \mathrm{Hom}(A_1, B_i)$ and $\mathrm{Hom}(\bigoplus_{i=1}^{\infty} A_i, B_1) \cong \prod_{i=1}^{\infty} \mathrm{Hom}(A_i, B_1)$. A complete proof can be found in e.g. [3, Proposition 3.3.4].

So combining prop. 1.29 and prop. 1.31 we can calculate the Ext-group for any two finitely generated abelian groups, because they can be decomposed as a direct sum of finitely many infinite and finite cyclic groups.

Remark 1.32. For abelian groups A, B, C, G and a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there exists an associated long exact sequence

$$0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow \mathrm{Ext}(G, A) \longrightarrow \mathrm{Ext}(G, B) \longrightarrow \mathrm{Ext}(G, C) \longrightarrow 0$$

where A^*, B^* and C^* denote $\mathrm{Hom}(A, G), \mathrm{Hom}(B, G)$ and $\mathrm{Hom}(C, G)$ respectively. This can be very useful in computing an Ext group, which will be illustrated in the proof of theorem 1.33. This is a special case of the long exact sequence of right derived functors, constructed as in [3, Section 2.5].

1.5 $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$

In this section we will compute a more exotic Ext-group, namely $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$. First we will state the main result.

Theorem 1.33. *$\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{A}/\mathbb{Q}$ where $\mathbb{A}$ is the Adèle group consisting of all sequences $(x_2, x_3, x_5, \dots)$ such that all $x_p \in \mathbb{Q}_p$, the p-adic rationals and $x_p \in \mathbb{Z}_p$, the p-adic integers for all but finitely many primes p, here $\mathbb{Q} \subset \mathbb{A}$ denotes the subgroup of constant sequences $(q, q, q, \dots)$ with $q \in \mathbb{Q}$.*

In order to prove this theorem we will prove some useful lemmas first. Before stating and proving those lemmas we will properly introduce the relevant abelian groups which will be used throughout this section. Let $\mathbb{Z}/p^\infty := \mathbb{Z}[p^{-1}]/\mathbb{Z}$, where $\mathbb{Z}[p^{-1}] = \{\frac{z}{p^k} : z \in \mathbb{Z}, k \geq 0\}$. Furthermore, let us define the p-adic integers $\mathbb{Z}_p$ for some prime p as follows:

$$\mathbb{Z}_p = \left\{ (a_1, a_2, a_3, \dots) \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} \,\middle|\, \text{for } n \geq 1 : a_{n-1} = a_n \left(\bmod p^{n-1} \right) \right\}$$

Here we will refer to the extra condition as the congruence conditions. Now we can define $\mathbb{Q}_p := \mathbb{Z}_p[p^{-1}]$. Lastly, for any abelian group A, we will denote the n-torsion subgroup of A by $A[n]$, which is exactly the subgroup of A consisting of the elements $x \in A$ such that $nx = 0$.

Lemma 1.34. *There is an isomorphism of abelian groups $\mathrm{End}(\mathbb{Z}/p^\infty) \cong \mathbb{Z}_p$.*

Proof. Note that we can write $\mathbb{Z}[p^{-1}] = \bigcup_{k=1}^{\infty} \mathbb{Z}\frac{1}{p^k}$, and thus $\mathbb{Z}/p^\infty = \bigcup_{k=1}^{\infty} \mathbb{Z}\frac{1}{p^k}/\mathbb{Z}$. Now an endomorphism $\varphi \in \mathrm{End}(\mathbb{Z}/p^\infty)$ must map the subgroup $\mathbb{Z}\frac{1}{p^k}/\mathbb{Z}$ into itself, because $\mathbb{Z}\frac{1}{p^k}/\mathbb{Z} = \mathbb{Z}/p^\infty[p^k]$ the p^k-torsion subgroup. Furthermore, the group $\mathbb{Z}/p^\infty$ is generated by $\frac{1}{p}, \frac{1}{p^2}, \dots$ hence the endomorphism φ is determined completely by the image of the generators under φ. Now, due to the earlier observation, $\varphi(\frac{1}{p^k}) \in \mathbb{Z}\frac{1}{p^k}/\mathbb{Z}$. So we can write $\varphi(\frac{1}{p^k}) = \frac{x_k}{p^k}$ for some $x_k \in \mathbb{Z}/p^k\mathbb{Z}$. Now consider the group homomorphism from $\mathrm{End}(\mathbb{Z}/p^\infty)$ to $\mathbb{Z}_p$ given by

$$\varphi \mapsto \left(\varphi\left(\frac{1}{p}\right)p, \varphi\left(\frac{1}{p^2}\right)p^2, \dots \right) \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$$

We will show that the obtained sequence satisfies the congruence conditions. Because $\varphi\left(\frac{p}{p^n}\right) = \varphi\left(\frac{1}{p^{n-1}}\right)$ we have that $\varphi\left(\frac{1}{p^n}\right)p^n = \varphi\left(\frac{1}{p^{n-1}}\right)p^{n-1}(\bmod (p^{n-1}))$. So this is indeed a group homomorphism from $\mathrm{End}(\mathbb{Z}/p^\infty) \to \mathbb{Z}_p$. It is clearly injective. For surjectivity let $(x_1, x_2, x_3, \cdots)$

be a p-adic integer, define $\varphi : \mathbb{Z}/p^\infty \to \mathbb{Z}/p^\infty$ by $\varphi\left(\frac{1}{p^n}\right) = \frac{x_n}{p^n}$. It is left to the reader to check that this is a well-defined endomorphism of $\mathbb{Z}/p^\infty$ and conclude that the constructed map is an isomorphism. $\square$

Lemma 1.35. *There are isomorphisms of abelian groups*

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/p^\infty) \cong \mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}/p^\infty) \cong \mathbb{Q}_p$$

Where, under the composed isomorphism $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/p^\infty) \xrightarrow{\sim} \mathbb{Q}_p$ *an element* $\varphi : \mathbb{Q} \to \mathbb{Z}/p^\infty$ *corresponds to a p-adic integer if and only if* $\varphi(1) = 0$.

Proof. For the first isomorphism we will construct an inverse, denoted by $\sim$, to the restriction map. Given a group homomorphism $\varphi : \mathbb{Z}[p^{-1}] \to \mathbb{Z}/p^\infty$ we will extend this to $\tilde{\varphi} : \mathbb{Q} \to \mathbb{Z}/p^\infty$ such that $\tilde{\varphi}|_{\mathbb{Z}[p^{-1}]} = \varphi$. Write $x \in \mathbb{Z}$ uniquely as $x = p^k m$ for some $k \geq 0$ and m coprime to p. Then let

$$\widetilde{\varphi}\left(\frac{1}{x}\right) = \varphi\left(\frac{1}{p^k}\right) \cdot \frac{1}{m}$$

We leave it to the reader to check that $\mathbb{Z}/p^\infty$ has unique divisibility by integers coprime to p, that $\tilde{\varphi}$ is a group homomorphism and that $\sim$ is indeed the inverse to the restriction map.

For the second isomorphism, consider a group homomorphism $\varphi : \mathbb{Z}[p^{-1}] \to \mathbb{Z}/p^\infty$. Then the image of $\mathbb{Z}$ in $\mathbb{Z}/p^\infty$ is $\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$ for some $k \geq 0$. Let $p^k : \mathbb{Z}[p^{-1}] \to \mathbb{Z}[p^{-1}]$ be given by multiplication with p^k. Then $\varphi \circ p^k : \mathbb{Z}[p^{-1}] \to \mathbb{Z}/p^\infty$ factorises through $\mathbb{Z}/p^\infty$ because the image of $\mathbb{Z}$ is 0 in $\mathbb{Z}/p^\infty$.

$$\begin{array}{ccc} \mathbb{Z}[p^{-1}] & \xrightarrow{\varphi \circ p^k} & \mathbb{Z}/p^\infty \\ \big\downarrow{\scriptstyle mod\,\mathbb{Z}} & \nearrow{\scriptstyle \overline{\varphi \circ p^k}} & \\ \mathbb{Z}/p^\infty & & \end{array}$$

This yields a well-defined map, which is not yet a group homomorphism, from $\mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}/p^\infty) \to \mathrm{End}(\mathbb{Z}/p^\infty)$. Using the isomorphism of Lemma 1.34 we can identify the obtained endomorphism with a p-adic integer. Finally we divide by p^k in $\mathbb{Z}_p$ resulting in a p-adic rational.

$$\begin{array}{ccccc} \mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}/p^\infty) & \longrightarrow & \mathrm{End}(\mathbb{Z}/p^\infty) & \xrightarrow{\sim} & \mathbb{Z}_p \\ & \searrow & & & \big\downarrow{\scriptstyle \cdot \frac{1}{p^k}} \\ & & & & \mathbb{Q}_p \end{array}$$

The obtained arrow is in fact a group homomorphism $\mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}/p^\infty) \to \mathbb{Q}_p$ where the maps with $\varphi(1) = 0$ correspond exactly to the p-adic integers. It is left to the reader to show that this group homomorphism is an isomorphism. $\square$

Now we can prove theorem 1.33.

proof of theorem 1.33. Consider the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Then the long exact sequence of $Ext(\mathbb{Q}, -)$ groups associated to this short exact sequence is

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \longrightarrow \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \longrightarrow \mathrm{Ext}(\mathbb{Q}, \mathbb{Q})$$

Note that $\mathrm{Hom}(\mathbb{Q},\mathbb{Z}) = 0$ and $\mathrm{Ext}(\mathbb{Q},\mathbb{Q}) = 0$, the latter because $\mathbb{Q}$ is injective. Hence from the relevant part of the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Q},\mathbb{Q}) \longrightarrow \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \longrightarrow 0$$

we can conclude that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})/\,\mathrm{Im}(\mathrm{Hom}(\mathbb{Q},\mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}))$. Now, note $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}/p^\infty \subset \prod_p \mathbb{Z}/p^\infty$. This inclusion induces an injective group homomorphism

$$\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \to \prod_p \mathrm{Hom}(\mathbb{Q},\mathbb{Z}/p^\infty) \cong \prod_p \mathbb{Q}_p$$

The image of this injection is exactly the group

$$\left\{ (\varphi_2, \varphi_3, \varphi_5, \dots) \in \prod_p \mathrm{Hom}(\mathbb{Q},\mathbb{Z}/p^\infty) \middle| \varphi_p = 0 \text{ for all but finitely many } p \right\}$$

which, under the second isomorphism, corresponds exactly with the defined adèles group $\mathbb{A}$. Then the image of $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}) \cong \mathbb{Q}$ in $\prod_p \mathbb{Q}_p$ are exactly the constant sequences of rational numbers $(q, q, q, \dots)$. □

1.6 Universal coefficient theorem

In this section we will state and prove the universal coefficient theorem, which is a very powerful abstract tool relating the homology and cohomology of complexes. For a chain complex of free abelian groups $C_\bullet$:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

We denote $\mathrm{Hom}(C_n, A)$ by C_n^*, so after applying the left exact functor $\mathrm{Hom}(-, A)$ to $C_\bullet$ we achieve the dual complex

$$\cdots \longleftarrow C_{n+1}^* \xleftarrow{\delta_n} C_n^* \xleftarrow{\delta_{n-1}} C_{n-1}^* \longleftarrow \cdots$$

We will construct a surjective group homomorphism $h : H^n(C_\bullet, A) \to \mathrm{Hom}(H_n(C_\bullet), A)$. For clarity define cycles $Z_n := \mathrm{Ker}(\partial_n) \subset C_n$ and boundaries $B_n := \mathrm{Im}(\partial_{n+1}) \subset C_n$ as subgroups of the groups C_n. Similarly, define cocycles $Z_n^* := \mathrm{Ker}(\delta_n) \subset C_n^*$ and coboundaries $B_n^* := \mathrm{Im}(\delta_{n-1}) \subset C_n^*$. An element $[\varphi] \in H^n(C_\bullet, A)$ is represented by a group homomorphism $\varphi : C_n \to A$ such that $\delta_n \circ \varphi = 0$. Thus $\varphi \circ \partial_{n+1} = 0$ so $\varphi|_{B_n} = 0$, that is φ vanishes on boundaries. Because of which the restriction $\varphi_0 = \varphi|_{Z_n}$ factorises via B_n yielding a group homomorphism $\overline{\varphi_0} : Z_n/B_n \to A$, note that $\overline{\varphi_0} \in \mathrm{Hom}(H_n(C_\bullet), A)$.

Theorem 1.36. *Let $C_\bullet$ be a chain complex of free abelian groups, in the downward sense. Then the homology groups $H_n(C_\bullet)$ and the cohomology groups $H^n(C_\bullet, A)$ of $\mathrm{Hom}(C_\bullet, A)$ relate via the split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C_\bullet), A) \longrightarrow H^n(C_\bullet, A) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), A) \longrightarrow 0$$

Proof. We will show that h is well defined. Suppose an element $[\varphi] \in H^n(C_\bullet, A)$ is a coboundary. Then $\varphi = \delta_{n-1} \circ \psi = \psi \circ \partial_n$ for some $\psi : C_{n-1} \to A$. Now $\varphi|_{Z_n} = \psi \circ \partial_n|_{Z_n} = 0$ as $\partial_n \circ \partial_{n+1} = 0$ thus $h([\varphi]) = 0$. This shows that h is well defined.

For surjectivity consider the short exact sequence of abelian groups

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0$$

which splits because B_{n-1} is free as it is a subgroup of the free group C_{n-1}. This means there exists a retraction $p : C_n \to Z_n$ such that p restricts to the identity on Z_n. This gives a way to extend a group homomorphism $\varphi_0 : Z_n \to A$ to $\varphi = \varphi_0 \circ p : C_n \to A$. Note that if φ_0 vanishes on B_n, that also φ vanishes on B_n. Composing with p is a group homomorphism $\mathrm{Hom}(H_n(C_\bullet), A) \to Z_n^*$. By taking the quotient to B_n^* we get a group homomorphism, denoted by $\theta : \mathrm{Hom}(H_n(C_\bullet), A) \to H^n(C_\bullet, A)$. We will show that $h \circ \theta$ is the identity on $\mathrm{Hom}(H_n(C_\bullet), A)$, and hence conclude that h is surjective. Now for $\varphi : H_n(C_\bullet) \to A$ note that $\theta(\varphi) = [\varphi \circ p] \in H^n(C_\bullet, A)$. But then $h(\theta(\varphi)) = h([\varphi \circ p]) = \varphi \circ p \circ \iota = \varphi$ where $\iota : Z_n \to C_n$ is the inclusion.

Because $\theta : \mathrm{Hom}(H_n(C_\bullet), A) \to H^n(C_\bullet, A)$ is a section this shows that we have a split exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Ker}(h) \longrightarrow H^n(C_\bullet, A) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), A) \longrightarrow 0$$

We will now analyse $\mathrm{Ker}(h)$. Note that the boundary map $\partial_n : C_n \to C_{n-1}$ restricts to the zero map on Z_n because $\partial_n \circ \partial_{n+1} = 0$. Hence we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \end{array}$$

Applying the left exact functor $\mathrm{Hom}(-, A)$ yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longleftarrow & Z_{n+1}^* & \longleftarrow & C_{n+1}^* & \longleftarrow & B_n^* & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle 0} & & \\ 0 & \longleftarrow & Z_n^* & \longleftarrow & C_n^* & \longleftarrow & B_{n-1}^* & \longleftarrow & 0 \end{array}$$

where the rows are still exact because the functor $\mathrm{Hom}(-, A)$ takes split short exact sequences to split short exact sequences. So in fact this gives rise to a short exact sequence of complexes, hence we can consider the associated long exact sequence of cohomology. We will denote the cohomology groups of the complexes by $H^n(Z^*)$, $H^n(C^*)$ and $H^n(B^*)$ respectively.

$$\cdots \longleftarrow H^n(B^*) \longleftarrow H^n(Z^*) \longleftarrow H^n(C^*) \longleftarrow H^{n-1}(B^*) \longleftarrow \cdots$$

because of the zero maps, and the definition of $H^n(C^*)$ this is really

$$\cdots \longleftarrow B_n^* \longleftarrow Z_n^* \longleftarrow H^n(C_\bullet, A) \longleftarrow B_{n-1}^* \longleftarrow \cdots$$

I will now show that the connecting homomorphisms $Z_n^* \to B_n^*$ are in fact the induced dual maps of the inclusions $i_n : B_n \to Z_n$. The connecting homomorphisms are defined as follows, take an element $\varphi_0 \in Z_n^*$ and pull back to an element $\varphi \in C_n^*$. Then apply δ_n to φ and finally pull back to B_n^*. This takes a group homomorphism $\varphi_0 : Z_n \to A$ and extends it to $\varphi : C_n \to A$. Then $\delta_n(\varphi) = \varphi \circ \partial_{n+1} : C_{n+1} \to A$. Then pulling back to B_n^* undoes this composition and restricts to B_n yielding $\varphi|_{B_n} = \varphi_0|_{B_n}$. Which is indeed the induced dual map of the inclusions $\iota : B_n \to Z_n$. The long exact sequence can be broken up into small exact sequences of abelian groups

$$0 \longrightarrow \mathrm{Coker}(\iota_{n-1}^*) \longrightarrow H^n(C_\bullet, A) \longrightarrow \mathrm{Ker}(\iota_n^*) \longrightarrow 0$$

Here $\mathrm{Ker}(\iota_n^*)$ consists of exactly the group homomorphisms $\varphi : Z_n \to A$ that vanish on B_n so they can be identified with elements of $\mathrm{Hom}(H_n(C_\bullet, A))$ using the constructed group homomorphism h. Hence the short exact sequence simplifies to

$$0 \longrightarrow \mathrm{Coker}(\iota_{n-1}^*) \longrightarrow H^n(C_\bullet, A) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), A) \longrightarrow 0$$

Lastly we will examine $\mathrm{Coker}(\iota_{n-1}^*)$. Note that the short exact sequence

$$0 \longrightarrow B_{n-1} \xrightarrow{\iota_{n-1}} Z_n \longrightarrow H_{n-1}(C_\bullet) \longrightarrow 0$$

is in fact a free resolution of $H_{n-1}(C_\bullet)$, denoted by $F_\bullet$. Applying the contravariant functor $\mathrm{Hom}(-, A)$ yields

$$0 \longrightarrow \mathrm{Hom}(H_{n-1}(C_\bullet), A) \longrightarrow Z_n^* \xrightarrow{\iota_{n-1}^*} B_{n-1}^* \longrightarrow 0$$

so the reduced complex $\mathrm{Hom}(F_\bullet, A)$ is

$$0 \longrightarrow Z_n^* \xrightarrow{\iota_{n-1}^*} B_{n-1}^* \longrightarrow 0$$

Now $\mathrm{Coker}(\iota_{n-1}^*) = h^1(\mathrm{Hom}(F_\bullet, A)) = \mathrm{Ext}(H_{n-1}(C_\bullet), A)$ proving the desired result. □

2 Singular homology

In this chapter I will define a special kind of homology, alongside tools with which the homology groups can be calculated. In particular I will prove the Mayer-Vietoris long exact sequence and use it to calculate the singular homology groups of the spheres and real projective n-space. The chapter will also include an explicit interpretation for the zero'th and first singular homology group. There Hurewicz' theorem will give a relation between the fundamental group of a topological space, and its first homology group. The adjective singular refers to the complex of abelian groups of which we will be taking the homology groups.

2.1 Chain complex

The central structure in the construction of singular homology is called a singular p-simplex.

Definition 2.1. For $p \in \mathbb{Z}_{\geq 0}$ we define a *standard p-simplex* Δ_p to be the convex hull in $\mathbb{R}^p$ of the unit vectors $e_1, \ldots, e_p$ and the origin which we denote by e_0. More concretely

$$\Delta_p = \left\{ (x_1, \ldots, x_p) \in \mathbb{R}^p : x_i \geq 0 \text{ and } \sum_{i=1}^{p} x_i \leq 1. \right\}$$

Definition 2.2. Let X be a topological space. For $p \in \mathbb{Z}_{\geq 0}$ we define a *singular p-simplex in X* to be a continuous map $\sigma : \Delta_p \to X$. Furthermore, we define the group of *singular p-chains* $C_p(X)$ to be the free abelian group generated by all singular p-simplices in X, i.e.

$$C_p(X) = \bigoplus_{\sigma:\Delta_p \to X} \mathbb{Z} \cdot \sigma$$

Note that Δ_0 is just a point, more precisely the point $* \in \mathbb{R}^0$. That means that a singular 0-simplex in X for a given topological space X is just a constant map $\sigma_x : \Delta_0 \to X$ mapping the unique point $*$ to x for some $x \in X$. This means that $C_0(X)$ is canonically isomorphic to the free abelian group generated by points $x \in X$. Under this correspondence singular 0-chains can be identified with $\mathbb{Z}$-linear combinations of points $x \in X$.

Next up the boundary maps $\partial_p : C_p(X) \to C_{p-1}(X)$ will be constructed. There are some properties these boundary maps should have. First of all we must have $\partial_{p-1} \circ \partial_p = 0$ for all $p \geq 1$, as this actually makes sure the constructed sequence of abelian groups and maps is a complex. Secondly, we want the boundary maps to be natural in the following sense: suppose $f : X \to Y$ is a continuous map. Then f induces a map $f_\# : C_p(X) \to C_p(Y)$ where $n_1\sigma_1 + \cdots + n_s\sigma_s \mapsto n_1(f \circ \sigma_1) + \cdots + n_s(f \circ \sigma_s)$. We want $\partial_p \circ f_\# = f_\# \circ \partial_p$ for all $p \in \mathbb{Z}_{\geq 0}$, i.e. the following diagram

$$\begin{array}{ccc} C_p(X) & \xrightarrow{f_\#} & C_p(Y) \\ \downarrow \partial_p & & \downarrow \partial_p \\ C_{p-1}(X) & \xrightarrow{f_\#} & C_{p-1}(Y) \end{array}$$

commutes. I will present a construction of the boundary map ∂. Thereafter I will show that the constructed group homomorphism satisfies both desired properties.

Definition 2.3. For $p \geq 1$ let $F_{i,p} : \Delta_{p-1} \to \Delta_p$ for $0 \leq i \leq p$ be the embedding of Δ_{p-1} in Δ_p on the face opposite of $e_i \in \Delta_p$. This embedding is given by

$$(x_1, \dots, x_{p-1}) \mapsto (1 - x_1 - \dots - x_{p-1})e_0 + x_1 e_1 + \dots + x_{i-1}e_{i-1} + x_i e_{i+1} \dots + x_{p-1}e_p$$

Then for a singular p-simplex on X define

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i (\sigma \circ F_{i,p}) \in C_{p-1}(X)$$

and define $\partial_0 : C_0(X) \to 0$ the trivial map.

This now gives rise to a sequence of abelian groups (which is in fact a complex which will be shown in this chapter) and maps called the chain complex of a topological space X.

Definition 2.4. Let X be a topological space. For $p \in \mathbb{Z}_{\geq 0}$ let $C_p(X)$ denote the groups of *singular p-chains* and let $\partial_p : C_p(X) \to C_{p-1}(X)$ be the boundary maps given as in def 2.3. Then we call

$$\dashrightarrow C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \dashrightarrow$$

with $C_i(X) = 0$ for $i < 0$ and $\partial_j = 0$ for $j \leq 0$ the chain complex of X.

Next up, the two desired properties of a chain complex will be proven.

Proposition 2.5. *Let $p \in \mathbb{Z}_{\geq 1}$, the composition $\partial_{p-1}\partial_p : C_p(X) \to C_{p-2}(X)$ is the zero map.*

Proof. We will prove that $\partial_{p-1}\partial_p(\sigma) = 0$ for any singular p-simplex $\sigma : \Delta_p \to X$. Because these generate the group $C_p(X)$, it follows that $\partial_{p-1}\partial_p = 0$. Now

$$\begin{aligned}\partial_{p-1}\partial_p(\sigma) &= \sum_{i=0}^{p} (-1)^i \partial_{p-1}(\sigma \circ F_{i,p}) \\ &= \sum_{i=0}^{p} (-1)^i \left(\sum_{j<i} (-1)^j (\sigma \circ F_{i,p} \circ F_{j,p-1}) + \sum_{j>i} (-1)^{j-1} (\sigma \circ F_{i,p} \circ F_{j,p-1}) \right)\end{aligned}$$

Note that in the second inner sum we raise to the $(-1)^{j-1}$ power instead of $(-1)^j$ to account for the missing vertex e_i. Now for fixed $0 \leq k < l \leq p$ we can count how many times $(\sigma \circ F_{k,p} \circ F_{l,p-1}) = (\sigma \circ F_{l,p} \circ F_{k,p-1})$ occurs.

$$(-1)^k \big((-1)^{l-1} (\sigma \circ F_{k,p} \circ F_{l,p-1}) \big) + (-1)^l \big((-1)^k (\sigma \circ F_{l,p} \circ F_{k,p-1}) \big) = 0$$

As k, l were arbitrary the total sum will vanish. □

Note that this proposition indeed verifies the fact that the chain complex of any topological space X is indeed a *complex of abelian groups.*

Proposition 2.6. *Let $f : X \to Y$ be a continuous map, for $p \in \mathbb{Z}_{\geq 1}$ let $f_\#$ be the induced map on the groups of singular chains. Then the following diagram*

$$\begin{array}{ccc} C_p(X) & \xrightarrow{f_\#} & C_p(Y) \\ \downarrow \partial_p & & \downarrow \partial_p \\ C_{p-1}(X) & \xrightarrow{f_\#} & C_{p-1}(Y) \end{array}$$

commutes.

Proof. Again, we will prove the claim for some singular p-simplex $\sigma : \Delta_p \to X$ as these generate the groups of singular p-chains. Now

$$\begin{aligned}\partial_p f_\#(\sigma) &= \partial_p(f \circ \sigma) \\ &= \sum_{i=0}^{p}(-1)^i(f \circ \sigma \circ F_{i,p}) \\ &= \sum_{i=0}^{p}(-1)^i f_\#(\sigma \circ F_{i,p}) \\ &= f_\#\left(\sum_{i=0}^{p}(-1)^i(\sigma \circ F_{i,p})\right) = f_\#\partial_p(\sigma).\end{aligned}$$

This shows that $\partial_p \circ f_\# = f_\# \circ \partial_p$. □

2.2 Singular homology

In this section we will define singular homology and prove that the homology groups of a topological space X are well-defined. That is, if two spaces X and Y are homeomorphic, or even homotopy equivalent, the derived homology groups $H_p(X)$ and $H_p(Y)$ will be isomorphic for all $p \in \mathbb{Z}_{\geq 0}$.

Definition 2.7. Let X be a topological space. Let

$$\dashrightarrow C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \dashrightarrow$$

be the chain complex of X. For $p \in \mathbb{Z}_{\geq 0}$ we define the *p-th singular homology group* $H_p(X) = \operatorname{Ker}(\partial_p)/\operatorname{Im}(\partial_{p-1})$. For convenience purposes we call an element $c \in \operatorname{Ker}(\partial_p)$ a *p-cycle* and denote the group of *p-cycles* by $Z_p(X)$. We call an element $c \in \operatorname{Im}(\partial_{p+1})$ a *p-boundary* and denote the group of *p-boundaries* by $B_p(X)$. Hence for $p \in \mathbb{Z}_{\geq 0}$

$$H_p(X) = Z_p(X)/B_p(X).$$

Example 2.8. The newly defined concepts of p-cycles and p-boundaries gives rise to a short exact sequence

$$0 \longrightarrow B_p(X) \xrightarrow{i} Z_p(X) \xrightarrow{\pi} H_p(X) \longrightarrow 0$$

and also

$$0 \longrightarrow Z_{p+1}(X) \xrightarrow{i} C_{p+1}(X) \xrightarrow{\partial_{p+1}} B_p(X) \longrightarrow 0$$

Let $f : X \to Y$ be a continuous map. We have shown that f induces a map $f_\# : C_p(X) \to C_p(Y)$ for all $p \in \mathbb{Z}_{\geq 0}$. If $f_\#|_{Z_p(X)} : Z_p(X) \to Z_p(Y)$ and $f_\#|_{B_p(X)} : B_p(X) \to B_p(Y)$ then f also induces a map $f_* : H_p(X) \to H_p(Y)$. This is indeed the case and we will check both conditions. First of all, suppose $c \in Z_p(X)$, that is $\partial_p(c) = 0$. Then $\partial_p f_\#(c) = f_\#\partial_p(c) = 0$ hence $f_\#(c) \in Z_p(Y)$. Secondly, suppose $c \in B_p(X)$, that is $c = \partial_{p+1}(c')$ for some $c' \in C_{p+1}(X)$. Then $f_\#(c) = f_\#(\partial_{p+1}(c')) = \partial_{p+1}(f_\#(c'))$ hence $f_\#(c) \in B_p(Y)$. The reader can verify that this assignment is functorial, i.e. if $f : X \to Y$ and $g : Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{H_p(X)}$.

Theorem 2.9. *Suppose $f : X \to Y$ is a homeomorphism, then for $p \in \mathbb{Z}_{\geq 0}$ the induced map $f_* : H_p(X) \to H_p(Y)$ is an isomorphism of groups.*

Proof. Let $p \in \mathbb{Z}_{\geq 0}$. Because of the functoriality of $(-)_*$ we have $(\mathrm{id})_{H_p(X)} = (\mathrm{id}_X)_* = (f^{-1} \circ f)_* = f_*^{-1} \circ f_*$ and similarly $(\mathrm{id})_{H_p(Y)} = f_* \circ f_*^{-1}$. That is, $(f_*)^{-1} = (f^{-1})_*$ and hence f_* is an isomorphism of groups. □

This result is very important. For if it were not true the assignment $X \mapsto H_p(X)$ would not be well-defined. A topologist cannot distinguish homeomorphic spaces and hence wouldn't find any use for an algebraic assignment which does not respect this equivalence relation. Luckily enough, this theorem can be made even stronger. Next up I will present a theorem without proof because the verification is tedious and does not introduce a construction which is of importance in this report. A proof can be found in e.g. [1, p. 111-113].

Theorem 2.10. *Let $f, g : X \to Y$ be two homotopic continuous maps, i.e. $f \simeq g$. Then the induced maps $f_*, g_* : H_p(X) \to H_p(Y)$ are equal for all $p \in \mathbb{Z}_{\geq 0}$.*

Corollary 2.11. *Let X and Y be homotopy equivalent, then for all $p \in \mathbb{Z}_{\geq 0}$ the homology groups $H_p(X)$ and $H_p(Y)$ are isomorphic.*

Proof. Let $f : X \to Y$ be any homotopy equivalence. That means that there is some $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Then using lemma 2.10 and the functoriality of $(-)_*$ we have

$$\mathrm{id}_{H_p(X)} = (\mathrm{id}_X)_* = (g \circ f)_* = g_* \circ f_*$$

and similarly $\mathrm{id}_{H_p(Y)} = f_* \circ g_*$ hence $(f_*)^{-1} = g_*$ and thus f_* is an isomorphism of groups. □

Example 2.12. We will calculate all homology groups of a contractible space X. Because of corollary 2.11 we can assume X to be the one point space $\{*\}$. Then for all $p \in \mathbb{Z}_{\geq 0}$ the singular p-simplices consist of only the constant maps. Thus $C_p(X) \cong \mathbb{Z}$ generated by σ_p respectively for all p. We can calculate the boundary of these constant simplices

$$\partial_p \sigma_p = \sum_{i=0}^{p} (-1)^i (\sigma_p \circ F_{i,p}) = \sum_{i=0}^{p} (-1)^i \sigma_{p-1}.$$

Note that $\sigma_p \circ F_{i,p} = \sigma_{p-1}$ for all i only because all of those singular simplices are the constant map. This now gives us the special case where the various σ_{p-1} cancel each other in the second sum. That is, $\partial_p \sigma_p = 0$ if p is odd, and $\partial_p \sigma_p = \sigma_{p-1}$ if p is even. Now let $p \geq 1$ odd, then $\partial_{p+1} : C_{p+1}(X) \to C_p(X)$ is an isomorphism and $\partial_p : C_p(X) \to C_{p-1}(X)$ is the trivial map. Hence $B_p(X) = Z_p(X) = C_p(X) \cong \mathbb{Z}$. Similarly for $p \geq 1$ even we have $B_p(X) = Z_p(X) = 0$, in both cases $H_p(X) = Z_p(X)/B_p(X) = 0$. Furthermore, the map $\partial_1 : C_1(X) \to C_0(X)$ and $\partial_0 : C_0(X) \to 0$ are both the trivial maps. Hence $Z_0(X) = C_0(X) \cong \mathbb{Z}$ and $B_0(X) = 0$ so we finally get a non-trivial result that $H_0(X) = Z_0(X)/B_0(X) \cong \mathbb{Z}$.

$$H_p(X) \cong \begin{cases} \mathbb{Z}, & p = 0 \\ 0, & p \geq 1 \end{cases}$$

2.3 Explicit descriptions of $H_0(X)$ and $H_1(X)$

First we will describe the zero'th singular homology group. For this, we will present and prove two useful results.

Proposition 2.13. *Let X be a path-connected topological space. The map $f : C_0(X) \to \mathbb{Z}$ given by $n_1 x_1 + \cdots + n_s x_s \mapsto n_1 + \cdots + n_s$ induces an isomorphism $f_* : H_0(X) \xrightarrow{\sim} \mathbb{Z}$.*

Proof. First we will prove surjectivity. Let $n \in \mathbb{Z}$ arbitrary, for any $x \in X$ we have $f(nx) = n$. Note that $\partial_0 : C_0 \to 0$ is the trivial map thus x is in fact a 0-cycle, and hence $[x] \in H_0(X)$. Secondly we will prove that $\text{Ker}(f) = B_0(X)$. From the discussion earlier it is clear that $B_0(X) \subset \text{Ker}(f)$ as f_* is a well-defined map. Let $c \in \text{Ker}(f)$. We can write $c = n_1x_1+\cdots+n_sx_s$ with $n_1+\cdots+n_s = 0$. Choose $y \in X$ and let $\sigma_{x_i} : \Delta_1 \to X$ denote a path with $\sigma_{x_i}(1) = x_i$ and $\sigma_{x_i}(0) = y$. This can be done because X is path-connected. Now let $c' = n_1\sigma_{x_1} + \cdots + n_s\sigma_{x_s}$. Then

$$\partial_1 c' = \partial_1 \left(\sum_{i=1}^{s} n_1 \sigma_{x_i} \right) = \sum_{i=1}^{s} n_i(x_i - y) = \sum_{i=1}^{s} n_i x_i - y \sum_{i=1}^{s} n_i = c$$

Hence $c \in B_0(X)$ proving that $\text{Ker}(f) = B_0(X)$. Hence indeed the induced map $f_* : H_0(X) \to \mathbb{Z}$ is an isomorphism.

□

Observe that the image of any singular simplex must be contained in some path-connected component of X. The next proposition allows us to decompose some space X into its path-connected components and determine the singular homology groups completely in terms of the path-connected components.

Proposition 2.14. *Let X a topological space. Let $\{X_\alpha\}_{\alpha \in A}$ be the decomposition of X into path-connected (disjoint) components. The inclusions $i_\alpha : X_\alpha \to X$ induce an isomorphism*

$$\bigoplus_{\alpha \in A} H_p(X_\alpha) \xrightarrow{\sim} H_p(X)$$

for all $p \in \mathbb{Z}_{\geq 0}$.

Proof. First we will prove that $\sigma_p : \Delta_p \to X$ must be contained in some unique X_α. Now suppose there exists $x_1, x_2 \in \Delta_p$ such that $\sigma_p(x_1) \in X_\alpha$ and $\sigma_p(x_2) \in X_\beta$ where $\alpha \neq \beta$. Then consider the path $\gamma : [0, 1] \to X$ given by $\gamma(t) = \sigma_p(tx_2 + (1 - t)x_1)$. Because Δ_p is convex we have $tx_2 + (1 - t)x_1 \in \Delta_p$ for all $t \in [0, 1]$. The path γ is continuous with $\gamma(1) = x_2$ and $\gamma(0) = x_1$ hence a path from $X_\alpha \to X_\beta$ which is a contradiction. Now we can prove surjectivity. Let $p \in \mathbb{Z}_{\geq 0}$, let $[c] \in H_p(X)$ where $c = n_1\sigma_1 + \cdots + n_s\sigma_s$ where each σ_i is a singular p-simplex in X. Then we have $\sigma_i[\Delta_p] \subset X_{\alpha_i}$ for all i. Hence $\sigma_i \in H_p(X_{\alpha_i})$ thus $[n_1\sigma_1 + \cdots + n_s\sigma_s] \in \bigoplus_{\alpha \in A} H_p(X_\alpha)$. The injectivity is clear hence the induced map is indeed an isomorphism. □

Thus from propositions 2.13 and 2.14 we derive the following conclusion.

Corollary 2.15. *Let X a topological space. Let $\{X_\alpha\}_{\alpha \in A}$ be the decomposition of X into path-connected (disjoint) components. Then there exists an isomorphism*

$$H_0(X) \xrightarrow{\sim} \bigoplus_{\alpha \in A} \mathbb{Z}.$$

Next up we will give Hurewicz's theorem without proof. For a proof the reader can refer to [1, p. 369-373]. For this note that a path $\gamma : [0, 1] \to X$ can be interpreted as a 1-cycle if $\gamma(1) - \gamma(0) = 0$. In particular if $\gamma \in P(x; X)$, i.e. a continuous map $\gamma : [0, 1] \to X$ with start- (and end) point x there is a natural map $h : P(x; X) \to H_1(X)$.

Theorem 2.16 (Hurewicz)**.** *The natural map $h : P(x; X) \to H_1(X)$ induces a map*

$$h_* : \pi_1(X, x) \to H_1(X)$$

with kernel the commutator subgroup $[\pi_1(X, x), \pi_1(X, x)]$. If X is path-connected the map h_* is surjective so in particular there is an isomorphism

$$\bar{h}_* : \pi_1(X, x)_{\mathrm{ab}} \xrightarrow{\sim} H_1(X).$$

Both of these identifications will help us in the calculation of higher singular homology groups. The long exact sequence of Mayer-Vietoris will, in particular, relate different singular homology groups. Because of that it helps to already have a grasp on the first two homology groups.

2.4 Mayer-Vietoris long exact sequence

In this section we will prove a computational tool called the long exact sequence of Mayer-Vietoris. For this we will state, without proof, a result regarding the homology groups of some topological space X, and the homology groups of X where each singular simplex in X must be contained in some open $U \in \mathcal{U}$. Denote the chain groups by $C_p^{\mathcal{U}}(X)$, as $\partial_p : C_p^{\mathcal{U}}(X) \to C_{p-1}^{\mathcal{U}}(X)$ is well defined these form a complex. Denote their homology groups by $H_p^{\mathcal{U}}(X)$. A proof of the following theorem can be found in e.g. [1, p. 119-124].

Theorem 2.17. *The inclusion $i : C_p^{\mathcal{U}}(X) \to C_p(X)$ is a chain homotopy equivalence. In particular, i induces isomorphisms $i_* : H_p^{\mathcal{U}}(X) \xrightarrow{\sim} H_p(X)$ for all p.*

This is a very general and powerful result, we will only need the following corollary.

Corollary 2.18. *Let X be a topological space, let $\mathcal{U} = \{U, V\}$ be an open cover of X. Let $C_p^{\mathcal{U}}(X)$ denote the subgroup of $C_p(X)$ consisting of sums of elements in $C_p(U)$ and $C_p(V)$. Then the inclusion $C_p^{\mathcal{U}}(X) \to C_p(X)$ induces an isomorphism $H_p^{\mathcal{U}}(X) \xrightarrow{\sim} H_p(X)$.*

Theorem 2.19 (Mayer-Vietoris)**.** *Let X be a topological space. Let $\mathcal{U} = \{U, V\}$ be an open cover of X. The following long sequence is exact.*

$$\dashrightarrow H_{p+1}(X) \xrightarrow{\partial} H_p(U \cap V) \xrightarrow{\alpha^*} H_p(U) \oplus H_p(V) \xrightarrow{\beta^*} H_p(X) \dashrightarrow$$

where α^* is the induced map of the inclusion α given by $c \mapsto (c, c)$. And β^* is the induced map of β given by $(c, d) \mapsto c - d$. The connecting group homomorphism is given as follows: take an element $[z] \in H_p^{\mathcal{U}}(X)$ and represent it by $z = c_1 - c_2$ with $c_1 \in C_p(U)$ and $c_2 \in C_p(V)$. Then $\partial c_1 = \partial c_2$ and let $\partial(z) = [\partial c_1] \in H_{p-1}(U \cap V)$.

Proof. Consider the exact sequence of chain complexes

$$0 \longrightarrow C(U \cap V)_\bullet \xrightarrow{\alpha} C(U)_\bullet \oplus C(V)_\bullet \xrightarrow{\beta} C^{\mathcal{U}}(X)_\bullet \longrightarrow 0$$

with α and β defined as above. Then the associated long exact sequence, by theorem 1.12 is then given by

$$\dashrightarrow H_{p+1}^{\mathcal{U}}(X) \xrightarrow{\partial} H_p(U \cap V) \xrightarrow{\alpha} H_p(U) \oplus H_p(V) \xrightarrow{\beta} H_p^{\mathcal{U}}(X) \dashrightarrow$$

After applying corollary 2.18 we obtain the desired result. □

2.5 The spheres S^n and projective space $\mathbb{P}^n(\mathbb{R})$

In this last section we will use our constructed tools to calculate the singular homology groups of two important topological spaces, the spheres and real projective space.

Proposition 2.20. *The singular homology groups of the spheres are given by*

$$H_p(S^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 0, p = 0 \\ \mathbb{Z}, & n \geq 1, p \in \{0, n\} \\ 0, & \textit{otherwise} \end{cases}$$

Proof. First we will treat the case where $n = 0$. Observe that $S^0 \cong \{-1, 1\} \subset \mathbb{R}$, equipped with the discrete topology. So S^0 consists of two path-connected components $\{-1\}$ and $\{1\}$ so by proposition 2.14 $H_p(S^0) \cong H_p(\{1\}) \oplus H_p(\{-1\})$. Now note that $\{1\}$ and $\{-1\}$ are both 1 point spaces, and thus contractible, so using example 2.12 we see that $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_p(S^0) = 0$ for all $p \geq 1$.

Now let $n \geq 1$ and let us consider the sphere S^n. Let N denote the north pole of the sphere and S denote the south pole. We construct an open cover in the following way; let $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$. Then both U and V are contractible, because they are homeomorphic to $\mathbb{R}^n$ and $U \cap V$ is homotopy equivalent to S^{n-1}. We will prove the claim with induction.

Base case where $n = 1$. By our characterization of the zero'th and first singular homology group we know that $H_0(S^1) \cong \mathbb{Z}$ and $H_1(S^1) \cong \mathbb{Z}$ because $\pi_1(S^1, 1) \cong \mathbb{Z}$. Now for $p \geq 2$ the relevant part of the MV sequence is

$$H_p(U) \oplus H_p(V) \longrightarrow H_p(S^1) \longrightarrow H_{p-1}(U \cap V)$$

Note that because U and V are contractible $H_p(U) \oplus H_p(V) = 0$. Furthermore, $U \cap V \simeq * \sqcup *$ hence $H_p(U \cap V) \cong H_p(*) \oplus H_p(*) = 0$. Thus $H_p(S^1) = 0$ for $p \geq 2$.

Now assume that the calculation has been done for all $n \leq k$. First we will calculate $H_1(S^{k+1})$, for this the relevant part of the MV sequence is

$$H_1(U) \oplus H_1(V) \longrightarrow H_1(S^{k+1}) \longrightarrow H_0(S^k) \xrightarrow{\alpha} H_0(U) \oplus H_0(V)$$

Because U and V are contractible and S^n, U and V are path-connected this simplifies to

$$0 \longrightarrow H_1(S^{n+1}) \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z}$$

the map $\alpha : c \mapsto (c, c)$ is injective and hence $\mathrm{Ker}(\alpha) = \mathrm{Im}(H_1(S^{n+1}) \to \mathbb{Z}) = 0$. However, the map $H_1(S^{n+1}) \to \mathbb{Z}$ is also injective showing that $H_1(S^{n+1}) = 0$. For $p \geq 2$ the relevant part of the MV sequence is

$$H_p(U) \oplus H_p(V) \longrightarrow H_p(S^{k+1}) \longrightarrow H_{p-1}(S^k) \longrightarrow H_{p-1}(U) \oplus H_{p-1}(V)$$

due to the contractibility of U and V this simplifies to

$$0 \longrightarrow H_p(S^{k+1}) \longrightarrow H_{p-1}(S^k) \longrightarrow 0$$

Thus $H_p(S^{k+1}) \cong H_{p-1}(S^k)$ proving that indeed $H_p(S^{k+1}) \cong \mathbb{Z}$ if $p = 0$ or $p = k + 1$.

□

Next up we will calculate the singular homology groups of real projective space. This will take a similar approach where we first, explicitly, calculate the homology groups of $\mathbb{P}^2(\mathbb{R})$ and $\mathbb{P}^3(\mathbb{R})$ and then conclude with an inductive argument.

Lemma 2.21. *The singular homology groups of* $\mathbb{P}^2(\mathbb{R})$ *are*

$$H_p(\mathbb{P}^2(\mathbb{R})) \cong \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}/2\mathbb{Z}, & p = 1 \\ 0, & p \geq 2 \end{cases}$$

Proof. Think of $\mathbb{P}^2(\mathbb{R})$ as a disk D^2 where antipodal points on the boundary S^1 are identified. We will construct an open cover of $\mathbb{P}^2(\mathbb{R})$. Let $U = D^2 \setminus S^1$ a contractible subspace. Let V the complement of some small closed disk around the center. Then V deformation retracts onto the boundary S^1 where antipodal points are identified. Lastly, $U \cap V \cong S^1 \times I$ and thus deformation retracts onto S^1. An important observation is that a loop γ generating this circle must loop twice around the boundary S^1. This is because antipodal points are the same in our original space, so at some point during the loop γ the projection of this loop onto the boundary S^1 has reached the opposite point of where the loop started. Hence these two points must be considered equal. This means that our S^1 wraps twice around the boundary S^1.

Because $\mathbb{P}^2(\mathbb{R})$ is path-connected we know that $H_0(\mathbb{P}^2(\mathbb{R})) \cong \mathbb{Z}$. Let us now calculate $H_1(\mathbb{P}^2(\mathbb{R}))$, consider the relevant part of the MV sequence

$$H_1(U \cap V) \xrightarrow{\alpha_1} H_1(U) \oplus H_1(V) \xrightarrow{\beta_1} H_1(\mathbb{P}^2(\mathbb{R})) \xrightarrow{\partial} H_0(U \cap V) \xrightarrow{\alpha_0} H_0(U) \oplus H_0(V)$$

which simplifies to

$$\mathbb{Z} \xrightarrow{\alpha_1} 0 \oplus \mathbb{Z} \xrightarrow{\beta_1} H_1(\mathbb{P}^2(\mathbb{R})) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\alpha_0} \mathbb{Z} \oplus \mathbb{Z}$$

The induced map α_0 is given by$c \mapsto (c, c)$ so in particular α_0 is injective. Thus ∂ is the zero homomorphism and hence β_1 surjects onto $H_1(\mathbb{P}^2(\mathbb{R}))$. Because β_1 is a homomorphism from $\mathbb{Z}$ the image will be $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{Z}$. That is $H_1(\mathbb{P}^2(\mathbb{R})) \cong \mathbb{Z}/\operatorname{Ker}(\beta_1) = \mathbb{Z}/\operatorname{Im}(\alpha_1)$. By our previous discussion a generating loop of $U \cap V$ wraps twice around V, thus $\alpha_1 : c \mapsto (0, 2c)$. Thus $\mathbb{Z}/\operatorname{Im}(\alpha_1) \cong \mathbb{Z}/2\mathbb{Z} \cong H_1(\mathbb{P}^2(\mathbb{R}))$.

Next up we will calculate $H_2(\mathbb{P}^2(\mathbb{R}))$. The relevant part of the MV sequence

$$H_2(U) \oplus H_2(V) \xrightarrow{\beta_2} H_2(\mathbb{P}^2(\mathbb{R})) \xrightarrow{\partial} H_1(U \cap V) \xrightarrow{\alpha_1} H_1(U) \oplus H_1(V)$$

simplifies to

$$0 \xrightarrow{\beta_2} H_2(\mathbb{P}^2(\mathbb{R})) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\alpha_1} 0 \oplus \mathbb{Z}$$

where again $\alpha_1 : c \mapsto (0, 2c)$. So in particular, α_1 is injective so ∂ is the zero homomorphism. So β_2 surjects onto $H_2(\mathbb{P}^2(\mathbb{R}))$ showing that $H_2(\mathbb{P}^2(\mathbb{R})) = 0$. For $p \geq 3$ each of $H_p(U)$, $H_p(V)$ and $H_{p-1}(U \cap V)$ are all 0, so $H_p(\mathbb{P}^2(\mathbb{R})) = 0$ proving the claim. □

Lemma 2.22. *The singular homology groups of* $\mathbb{P}^3(\mathbb{R})$ *are*

$$H_p(\mathbb{P}^3(\mathbb{R})) \cong \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}/2\mathbb{Z}, & p = 1 \\ 0, & p = 2 \\ \mathbb{Z}, & p = 3 \\ 0, & p \geq 4 \end{cases}$$

Proof. Think of $\mathbb{P}^3(\mathbb{R})$ as the identification space obtained from D^3 where antipodal points on the boundary S^2 are identified. Let U be an open neighborhood of the center. Let V be the complement of a smaller closed neighborhood of the center. Then $U \cup V = \mathbb{P}^3(\mathbb{R})$, U is contractible and V deformation retracts onto $S^2/\pm 1 \simeq \mathbb{P}^2(\mathbb{R})$. Furthermore, $U \cap V \cong S^2 \times I$ and hence deformation retracts onto S^2.

Again, $H_0(\mathbb{P}^3(\mathbb{R})) = \mathbb{Z}$ because $\mathbb{P}^3(\mathbb{R})$ is path-connected. For $H_1(\mathbb{P}^3(\mathbb{R}))$ consider

$$H_1(U \cap V) \xrightarrow{\alpha_1} H_1(U) \oplus H_1(V) \xrightarrow{\beta_1} H_1(\mathbb{P}^3(\mathbb{R})) \xrightarrow{\partial} H_0(U \cap V) \xrightarrow{\alpha_0} H_0(U) \oplus H_0(V)$$

which simplifies to

$$0 \longrightarrow 0 \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\beta_1} H_1(\mathbb{P}^3(\mathbb{R})) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\alpha_0} \mathbb{Z} \oplus \mathbb{Z}$$

Now the induced map α_0 given by$c \mapsto (c, c)$ is injective, thus ∂ is the trivial map and hence β_1 surjects onto $H_1(\mathbb{P}^3(\mathbb{R}))$. Because of the start of this exact sequence β_1 is injective and hence an isomorphism. Thus $H_1(\mathbb{P}^3(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. For $H_2(\mathbb{P}^3(\mathbb{R}))$ note that each of $H_2(U)$, $H_2(V)$ and $H_1(U \cap V)$ are zero. The fact that $H_2(V) = 0$ follows from lemma 2.21. The exactness of

$$H_2(U) \oplus H_2(V) \longrightarrow H_2(\mathbb{P}^3(\mathbb{R})) \longrightarrow H_1(U \cap V)$$

implies that $H_2(\mathbb{P}^3(\mathbb{R})) = 0$. For $H_3(\mathbb{P}^3(\mathbb{R}))$ consider the part of the MV sequence

$$H_3(U) \oplus H_3(V) \xrightarrow{\beta_3} H_3(\mathbb{P}^3(\mathbb{R})) \xrightarrow{\partial} H_2(U \cap V) \xrightarrow{\alpha_2} H_2(U) \oplus H_2(V)$$

which is

$$0 \oplus 0 \xrightarrow{\beta_3} H_3(\mathbb{P}^3(\mathbb{R})) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\alpha_2} 0 \oplus 0$$

So $H_3(\mathbb{P}^3(\mathbb{R})) \cong \mathbb{Z}$. Lastly for $p \geq 4$ note that $H_p(U) = H_p(V) = H_{p-1}(U \cap V) = 0$ so $H_p(\mathbb{P}^3(\mathbb{R})) = 0$. □

Now we are finally ready for the last proposition of this chapter.

Proposition 2.23. *The singular homology groups of* $\mathbb{P}^n(\mathbb{R})$ *are*

$$H_p(\mathbb{P}^n(\mathbb{R})) \cong \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z}, & p = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z}, & 0 < p < n \text{ odd} \\ 0, & \text{else} \end{cases}$$

Proof. We will use the method of induction. We have shown the claim holds for $n = 2$ and $n = 3$ in lemma 2.21 and lemma 2.22. Now assume that $n > 3$ and the claim holds for $n - 1$, we will prove that it also holds for n.

Think of $\mathbb{P}^n(\mathbb{R})$ as the identification space obtained from D^n where antipodal points on the boundary S^{n-1} are identified. Let U be an open neighborhood of the center. Let V be the complement of a smaller closed neighborhood of the center. Then $U \cup V = \mathbb{P}^n(\mathbb{R})$, U is contractible and V deformation retracts onto $S^{n-1}/\pm 1 \simeq \mathbb{P}^{n-1}(\mathbb{R})$. Furthermore, $U \cap V \cong S^{n-1} \times I$ and hence deformation retracts onto S^{n-1}.

Because $\mathbb{P}^n(\mathbb{R})$ is path-connected $H_0(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}$. For the first singular homology group consider the part of the Mayer-Vietoris sequence

$$H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} H_0(U \cap V)$$

Here ∂ is again the zero group homomorphism because of the same reason used in the previous proof. Also, as $n > 3$ we have $H_1(U \cap V) \cong H_1(S^{n-1}) = 0$, $H_1(U) \cong H_1(\{*\}) = 0$ and $H_1(V) \cong H_1(\mathbb{P}^{n-1}(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ by our induction hypothesis. Thus the exact sequence

$$H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} H_0(U \cap V)$$

simplifies to

$$0 \longrightarrow 0 \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow H_1(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} 0$$

Hence $H_1(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$. For $1 < p < n - 1$ we have

$$H_p(U \cap V) \longrightarrow H_p(U) \oplus H_p(V) \longrightarrow H_p(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} H_{p-1}(U \cap V)$$

which simplifies to

$$0 \longrightarrow 0 \oplus H_p(\mathbb{P}^{n-1}(\mathbb{R})) \longrightarrow H_p(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} 0$$

Meaning that $H_p(\mathbb{P}^n(\mathbb{R})) \cong H_p(\mathbb{P}^{n-1}(\mathbb{R}))$ hence $H_p(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ if p is odd and $H_p(\mathbb{P}^n(\mathbb{R})) = 0$ if p is even. Lastly, we will determine $H_n(\mathbb{P}^n(\mathbb{R}))$ and $H_{n-1}(\mathbb{P}^n(\mathbb{R}))$. For this consider the relevant part of the Mayer-Vietoris sequence

$$0 \longrightarrow H_n(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} H_{n-1}(U \cap V) \longrightarrow H_{n-1}(U) \oplus H_{n-1}(V) \longrightarrow H_{n-1}(\mathbb{P}^n(\mathbb{R})) \longrightarrow 0$$

If n is odd, then $n - 1$ is even and thus this simplifies to

$$0 \longrightarrow H_n(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0 \longrightarrow H_{n-1}(\mathbb{P}^n(\mathbb{R})) \longrightarrow 0$$

so $H_n(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}$ and $H_{n-1}(\mathbb{P}^n(\mathbb{R})) = 0$. Conversely, suppose n is even meaning $n-1$ is odd. Then $H_{n-1}(\mathbb{P}^{n-1}(\mathbb{R})) \cong \mathbb{Z}$ so we get

$$0 \longrightarrow H_n(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\alpha} 0 \oplus \mathbb{Z} \xrightarrow{\beta} H_{n-1}(\mathbb{P}^n(\mathbb{R})) \longrightarrow 0$$

Note that under the isomorphisms $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and $H_{n-1}(\mathbb{P}^{n-1}(\mathbb{R})) \cong \mathbb{Z}$ a generator of S^{n-1} wraps twice around the $\mathbb{P}^{n-1}(\mathbb{R})$ resulting in the induced group homomorphism on the level of homology groups given by $\alpha : x \mapsto (0, 2x)$. Which is injective meaning that ∂ is the zero group homomorphism. Thus $H_n(\mathbb{P}^n(\mathbb{R})) = 0$ and $H_{n-1}(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/\operatorname{Ker}(\beta) \cong \mathbb{Z}/\operatorname{Im}(\alpha) \cong \mathbb{Z}/2\mathbb{Z}$. For $p > n$ we have an exact sequence

$$H_p(\{*\}) \oplus H_p(\mathbb{P}^{n-1}(\mathbb{R})) \longrightarrow H_p(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} H_{p-1}(S^{n-1})$$

which simplifies to

$$0 \longrightarrow H_p(\mathbb{P}^n(\mathbb{R})) \xrightarrow{\partial} 0$$

showing that $H_p(\mathbb{P}^n(\mathbb{R})) = 0$ for $p > n$, this proves the claim by induction. □

2.6 Singular Cohomology

In this final subsection I will define singular cohomology and give an explicit identification for the 0-th singular cohomology group given any topological space X and abelian group A. Lastly, we will combine the universal coefficient theorem 1.36 and the singular homology groups of $\mathbb{P}^n(\mathbb{R})$ as given in proposition 2.23 to compute the singular cohomology groups for any abelian group A of projective n-space.

Definition 2.24. Let X be a topological space and let A be an abelian group. Let $C_\bullet$ denote the chain complex

$$\cdots \longrightarrow C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \longrightarrow \cdots$$

Then apply the functor $\operatorname{Hom}(-, A)$ to this complex to achieve the *cochain complex*

$$\cdots \longleftarrow \operatorname{Hom}(C_{p+1}(X), A) \xleftarrow{\delta_p} \operatorname{Hom}(C_p(X), A) \xleftarrow{\delta_{p-1}} \operatorname{Hom}(C_{p-1}(X), A) \longleftarrow \cdots$$

with induced arrows δ_p. For simplicity we will denote $\operatorname{Hom}(C_p(X), A)$ by $C^p(X, A)$. We now define the p-th singular cohomology group $H^p(X, A)$ to be the p-th cohomology group of the complex $\operatorname{Hom}(C_\bullet, A)$.

We will now formulate and prove two useful lemmas which will be used to explicitly identify the 0-th singular cohomology group for a given topological space X and abelian group A.

Lemma 2.25. *Let X be a topological space. Recall $C_0(X) \cong \bigoplus_{x \in X} x \cdot \mathbb{Z}$. Let A be an abelian group, then $C^0(X, A) \cong \operatorname{Map}(X, A)$.*

Proof. We have the following natural isomorphisms

$$C^0(X, A) \cong \operatorname{Hom}(C_0(X), A) \cong \operatorname{Hom}(\bigoplus_{x \in X} x \cdot \mathbb{Z}, A) \cong \prod_{x \in X} \operatorname{Hom}(x \cdot \mathbb{Z}, A) \cong \prod_{x \in X} A \cong \operatorname{Map}(X, A)$$

Explicitly, a group homomorphism $f : C_0(X) \to A$ gets mapped to the set theoretic map $\overline{f} : X \to A$ given by $x \mapsto f(1_x)$ where $1_x = (0, 0, \ldots, 0, 1, 0, \ldots)$ with the 1 exactly on the index corresponding to $x \cdot \mathbb{Z}$. This is an isomorphism of abelian groups. □

Lemma 2.26. *Let $\delta_0 : C^0(X, A) \to C^1(X, A)$ be the induced boundary map in the cochain complex. Then $f \in \operatorname{Map}(X, A)$ has $\delta_0(f) = 0$ if and only if f is constant on each path connected component of X.*

Proof. Suppose f is constant on each path connected component. Consider an arbitrary element $z \in C_1(X)$, we can write $z = n_1\gamma_1 + \cdots + n_s\gamma_s$ for some $n_1, \ldots, n_s \in \mathbb{Z}$ and $\gamma_1, \ldots, \gamma_s$ continuous maps $\gamma_i : [0,1] \to X$. Now $\delta_0(f) = f \circ \partial_1$ so we will show that the latter is the zero homomorphism.

$$\begin{aligned} f \circ \partial_1(z) &= \partial_1(n_1(\gamma_1(1) - \gamma_1(0)) + \cdots + n_s(\gamma_s(1) - \gamma_s(0))) \\ &= n_1(f(\gamma_1(1)) - f(\gamma_1(0))) + \cdots + n_2(f(\gamma_2(1)) - f(\gamma_2(0))) = 0 \end{aligned}$$

as for all $1 \leq i \leq s$ we have that $\gamma_i(1)$ and $\gamma_i(0)$ are contained in the same path-connected component. Hence for all $z \in C_1(X)$ we have $\partial_1 \circ f(z) = 0$ thus $f \circ \partial_1 = \delta_0 \circ f = 0$.
Conversely, suppose that $\delta_0(f) = 0$. Choose two elements x and y in a path-connected component of X. Then there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Now $\delta_0(f) = f \circ \partial_1 = 0$ so $f \circ \partial_1(\gamma) = f(\gamma(1) - \gamma(0)) = f(x - y) = 0$ so $f(x) = f(y)$. This shows that any two points, that lie in the same path-connected component have the same image under f. Hence this proves that f is constant on path-connected components. □

Proposition 2.27. *Let X be a topological space. Let A be an abelian group, and let $\pi_0(X)$ denote the set of path-connected components of X. Then $H^0(X, A) \cong A^{\pi_0(X)}$.*

Proof. Consider the cochain complex of X

$$0 \longrightarrow C^0(X, A) \xrightarrow{\delta_0} C^1(X, A) \longrightarrow \cdots$$

Now $H^0(X, A) \cong \mathrm{Ker}(\delta_0)$, which we can identify with set theoretic maps $f : X \to A$ that are constant on path-connected components using lemma 2.26. This means that such maps f are determined by their image in each path-connected component. That is, for each path-connected component f can send their elements to a unique element in A. Hence $H^0(X, A) \cong \mathrm{Ker}(\delta_0) \cong A^{\pi_0(X)}$. □

We will now calculate the singular cohomology groups of the spheres S^n and real projective n-space $\mathbb{P}^n(\mathbb{R})$.

Proposition 2.28. *Let A be an abelian group. The singular cohomology groups of the spheres S^n with coefficients in A are given as follows:*

$$H^p(S^n, A) \cong \begin{cases} A \oplus A, & n = 0, p = 0 \\ A, & n \geq 1, p \in \{0, n\} \\ 0, & \textit{otherwise} \end{cases}$$

Proof. We will treat S^0 first. By proposition 2.20 we know that $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_p(S^0) = 0$ for all $p \geq 1$. Now by proposition 2.27 we know that $H^0(S^0, A) \cong A \oplus A$ because S^0 consists of two path-connected components. Using the universal coefficient theorem 1.36 we obtain a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_0(S^0), A) \longrightarrow H^1(S^0, A) \longrightarrow \mathrm{Hom}(H_1(S^0), A) \longrightarrow 0$$

which simplifies to

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}, A) \longrightarrow H^1(S^0, A) \longrightarrow \mathrm{Hom}(0, A) \longrightarrow 0$$

We use proposition 1.31 and 1.28 to see that $\text{Ext}(\mathbb{Z}\oplus\mathbb{Z}, A) \cong \text{Ext}(\mathbb{Z}, A)\times\text{Ext}(\mathbb{Z}, A) = 0$. Obviously $\text{Hom}(0, A) = 0$ so $H^1(S^0, A) = 0$. For $p \geq 2$ we have $\text{Ext}(H_{p-1}(S^0), A) = \text{Ext}(0, A) = 0$ and $\text{Hom}(H_p(S^0), A) = \text{Hom}(0, A) = 0$ so $H^p(S^0, A) = 0$ for all $p \geq 1$.

Now let $n \geq 1$. Then $H_p(S^n) \cong \mathbb{Z}$ for $p = 0$ and $p = n$. Because S^n is path-connected we know by proposition 2.27 that $H^0(S^n, A) \cong A$. By the universal coefficient theorem we have a split exact sequence

$$0 \longrightarrow \text{Ext}(H_n(S^n), A) \longrightarrow H^{n+1}(S^n, A) \longrightarrow \text{Hom}(H_{n+1}(S^n), A) \longrightarrow 0$$

which simplifies to

$$0 \longrightarrow \text{Ext}(\mathbb{Z}, A) \longrightarrow H^{n+1}(S^n, A) \longrightarrow \text{Hom}(0, A) \longrightarrow 0$$

So again by proposition 1.28 we have $\text{Ext}(\mathbb{Z}, A) = \text{Hom}(0, A) = 0$ so $H^{n+1}(X, A) = 0$.

$$0 \longrightarrow \text{Ext}(H_{n-1}(S^n), A) \longrightarrow H^n(X, A) \longrightarrow \text{Hom}(H_n(S^n), A) \longrightarrow 0$$

simplifies to

$$0 \longrightarrow \text{Ext}(0, A) \longrightarrow H^n(S^n, A) \longrightarrow \text{Hom}(\mathbb{Z}, A) \longrightarrow 0$$

So $H^n(X, A) \cong \text{Hom}(\mathbb{Z}, A) \cong A$. For all $p \notin \{0, 1, n, n+1\}$ we have that $H_p(S^n) = H_{p-1}(S^n) = 0$ so $H^p(X, A) = 0$. This proves the claim. □

Theorem 2.29. *Let A be an abelian group. The singular cohomology groups of real projective n-space $\mathbb{P}^n(\mathbb{R})$ with coefficients in A are given as follows:*

$$H^p(\mathbb{P}^n(\mathbb{R}), A) \cong \begin{cases} A, & p = 0,\ p = n \textit{ if } n \textit{ is odd} \\ A/2A, & p \textit{ even } 0 < p \leq n \\ A[2], & p \textit{ odd } 0 < p < n \\ 0, & \textit{otherwise} \end{cases}$$

Proof. First of all note that $H^0(\mathbb{P}^n(\mathbb{R}), A) \cong A$ because $\mathbb{P}^n(\mathbb{R})$ is path-connected. Now let $0 < p < n$ odd, then by the universal coefficient theorem we have a split short exact sequence

$$0 \longrightarrow \text{Ext}(H_{p-1}(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow H^p(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow \text{Hom}(H_p(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow 0$$

we know that $p - 1$ is even so $H_{p-1}(\mathbb{P}^n(\mathbb{R})) = 0$ and $H_p(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ by proposition 2.23. Hence the short exact sequence is

$$0 \longrightarrow \text{Ext}(0, A) \longrightarrow H^p(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow \text{Hom}(\mathbb{Z}/2\mathbb{Z}, A) \longrightarrow 0$$

thus $H^p(\mathbb{P}^n(\mathbb{R})) \cong \text{Hom}(\mathbb{Z}/2\mathbb{Z}, A) \cong A[2]$. For $0 < p < n$ even we have that $H_p(\mathbb{P}^n(\mathbb{R}))) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_{p-1}(\mathbb{P}^n(\mathbb{R}))) = 0$ so the split exact sequence is

$$0 \longrightarrow \text{Ext}(\mathbb{Z}/2\mathbb{Z}, A) \longrightarrow H^p(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow \text{Hom}(0, A) \longrightarrow 0$$

And thus $H^p(\mathbb{P}^n(\mathbb{R})), A) \cong \text{Ext}(\mathbb{Z}/2\mathbb{Z}, A) \cong A/2A$ by proposition 1.30. Lastly, suppose n is even, then

$$0 \longrightarrow \text{Ext}(H_{n-1}(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow H^n(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow \text{Hom}(H_n(\mathbb{P}^n(\mathbb{R})), A) \longrightarrow 0$$

simplifies to

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/2\mathbb{Z},,A) \longrightarrow H^n(\mathbb{P}^n(\mathbb{R})),A) \longrightarrow \mathrm{Hom}(0,A) \longrightarrow 0$$

Thus $H^n(\mathbb{P}^n(\mathbb{R})),A) \cong A/2A$. If n is odd, then it simplifies to

$$0 \longrightarrow \mathrm{Ext}(0,,A) \longrightarrow H^n(\mathbb{P}^n(\mathbb{R})),A) \longrightarrow \mathrm{Hom}(\mathbb{Z},A) \longrightarrow 0$$

showing that $H^n(\mathbb{P}^n(\mathbb{R})),A) \cong \mathrm{Hom}(\mathbb{Z},A) \cong A$. For $p \geq n+1$ we have $\mathrm{Ext}(H_{p-1}(\mathbb{P}^n(\mathbb{R})),A) = \mathrm{Hom}(H_p(\mathbb{P}^n(\mathbb{R})),A)) = 0$ so $H^p(\mathbb{P}^n(\mathbb{R}),A) = 0$. □

Remark 2.30. Note that in both proposition 2.29 and theorem 2.28 we could calculate the singular cohomology groups from the singular homology groups. This is true in general. With sufficient knowledge about Ext-groups and Hom-groups one can always calculate the singular cohomology groups of a topological space X using the singular homology groups.

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