



Delft University of Technology

Propositional team logics

Yang, Fan; Väänänen, Jouko

DOI

[10.1016/j.apal.2017.01.007](https://doi.org/10.1016/j.apal.2017.01.007)

Publication date

2016

Document Version

Accepted author manuscript

Published in

Annals of Pure and Applied Logic

Citation (APA)

Yang, F., & Väänänen, J. (2016). Propositional team logics. *Annals of Pure and Applied Logic*.
<https://doi.org/10.1016/j.apal.2017.01.007>

Important note

To cite this publication, please use the final published version (if applicable).
Please check the document version above.

Copyright

Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

Takedown policy

Please contact us and provide details if you believe this document breaches copyrights.
We will remove access to the work immediately and investigate your claim.

Propositional Team Logics[☆]

Fan Yang^{a,1,*}, Jouko Väänänen^{b,2}

^a*Department of Values, Technology and Innovation, Delft University of Technology, Jaffalaan 5, 2628 BX Delft, The Netherlands*

^b*Department of Mathematics and Statistics, Gustaf Hållströmin katu 2b, PL 68, FIN-00014 University of Helsinki, Finland and University of Amsterdam, The Netherlands*

Abstract

We consider team semantics for propositional logic, continuing [34]. In team semantics the truth of a propositional formula is considered in a *set* of valuations, called a *team*, rather than in an individual valuation. This offers the possibility to give meaning to concepts such as dependence, independence and inclusion. We associate with every formula ϕ based on finitely many propositional variables the set $\llbracket \phi \rrbracket$ of *teams* that satisfy ϕ . We define a maximal propositional team logic in which every set of teams is definable as $\llbracket \phi \rrbracket$ for suitable ϕ . This requires going beyond the logical operations of classical propositional logic. We exhibit a hierarchy of logics between the smallest, viz. classical propositional logic, and the maximal propositional team logic. We characterize these different logics in several ways: first syntactically by their logical operations, and then semantically by the kind of sets of teams they are capable of defining. In several important cases we are able to find complete axiomatizations for these logics.

Keywords:

propositional team logics, team semantics, dependence logic, non-classical logic

2010 MSC: 03B60

1. Introduction

In classical propositional logic the propositional atoms, say $p_1, \dots, p_n$, are given a truth value 1 or 0 by what is called a valuation and then any propositional formula ϕ can be associated with the set $|\phi|$ of valuations giving ϕ the value 1. This constitutes a perfect analysis of the circumstances under which ϕ is true. The formula ϕ can be presented in so-called Disjunctive Normal Form based on taking the disjunction

[☆]Some results in this paper were included in the dissertation of the first author [32], which was supervised by the second author.

*Corresponding author

Email addresses: fan.yang.c@gmail.com (Fan Yang), jouko.vaananen@helsinki.fi (Jouko Väänänen)

¹This paper is partially based on research that was carried out in the Graduate School of Mathematics and Statistics of the University of Helsinki.

²The research was partially supported by grant 251557 of the Academy of Finland.

of descriptions of the valuations in $|\phi|$. Two fundamental results can be proved for classical propositional logic. The first says that *every* set of valuations of $p_1, \dots, p_n$ is equal to $|\phi|$ for some propositional formula ϕ . The second fundamental result says that there is a simple *complete* axiomatization of those ϕ that are valid in the sense that $|\phi|$ is the full set of all valuations on the propositional atoms occurring in ϕ .

In this paper, which continues [34], we consider a richer semantics called *team semantics* for propositional logic. In team semantics the truth of a propositional formula is evaluated in a *set* of valuations, called a *team*, rather than in an individual valuation. This offers the possibility of considering *probabilities* of formulas, as in [19], and the meaning of concepts such as *dependence*, *independence* and *inclusion*, as in [34]. It is the latter possibility that is our focus in this paper.

Team semantics was introduced by the second author in [27] on the basis of a new compositional semantics, due to Hodges [17, 18], for independence friendly logic [16, 25]. The monograph [27] was written in the context of predicate logic and team semantics was used to give meaning to a variable being totally determined by a sequence of other variables. In the context of propositional and modal logic team semantics was introduced in [28]. In propositional logic team semantics can be used to give meaning to a propositional variable being totally determined by a sequence of other variables. It took a few years before this idea was fully exploited in [31, 32]. Meanwhile modal dependence logic, i.e. team semantics for modal logic, was investigated e.g. in [7, 8, 9, 14, 15, 23, 26].

When propositional formulas are evaluated in a team—i.e. a set—of valuations, a whole new landscape opens in front of us. The first observation is a numerical explosion: If we have n propositional atoms, there are 2^n valuations, 2^{2^n} teams, and $2^{2^{2^n}}$ sets of teams. For $n = 3$ the third number is about 10^{77} . This emphasises the need for mathematical methods in team semantics. The truth table methods which list all possibilities is bad enough in ordinary propositional logic, but totally untenable in team semantics.

In classical propositional logic, we associate with every formula ϕ based on propositional atoms $p_1, \dots, p_n$ the set $|\phi|$ of *valuations* that satisfy ϕ . Similarly, in team semantics we associate with every formula ϕ based on propositional atoms $p_1, \dots, p_n$ the set $\llbracket \phi \rrbracket$ of *teams* that satisfy (in the sense defined below) ϕ . By choosing our formulas carefully we can express *every* set of teams in the form $\llbracket \phi \rrbracket$ for suitable ϕ , but this requires going beyond the logical operations of classical propositional logic. We can also axiomatize the propositional formulas that are *valid* i.e. satisfied by every team.

The rich structure of teams gives rise to a plethora of new propositional connectives. Most importantly, disjunction has several versions. To define when a team X satisfies $\phi \vee \psi$ we can say that this happens if X satisfies ϕ or it satisfies ψ , or we can say that this happens if X is the union of two sets Y and Z such that Y satisfies ϕ and Z satisfies ψ , or, finally, we can also say that this happens if, assuming $X \neq \emptyset$, the team X is the union of two sets $Y \neq \emptyset$ and $Z \neq \emptyset$ such that Y satisfies ϕ and Z satisfies ψ . If X is a singleton, which corresponds to the classical case, the first two disjunctions are equivalent, but the third is equivalent to $\phi \wedge \psi$. But for non-singleton teams there is a big difference in every respect. These distinctions, leading to different variants of familiar logical operations, reveal a hierarchy of logics between the smallest, viz.

classical propositional logic, and the maximal one capable of defining every set of teams. We characterize these different logics in several ways: first syntactically by their logical operations, and then semantically by the kind of sets of teams they are capable of defining. In several important cases we are able to find complete axiomatizations for these logic.

In our previous paper [34] we considered sets of teams that are *downward closed* in the sense that if a team is in the set, then every subteam is in the set, too. Respectively, the logics studied in [34] have the property that the sets of teams defined by their formulas are downward closed. We isolated five equivalent logics with this property, all based on some aspect of *dependence*. In these logics every downward closed set of teams is definable, and the logics have complete axiomatizations. The axiomatizations are by no means as simple as typical axiomatizations of classical propositional logic, but have still a certain degree of naturality.

In this paper we consider sets of teams, and related propositional logics, that are not downward closed. A property in a sense opposite to downward closure is closure under (set-theoretical) unions. In fact, a set of teams that is both closed downward and closed under unions is definable in classical propositional logic. So-called *inclusion logic*, to be defined below, is an example of a logic in which definable sets of teams are closed under unions. So-called *independence logic*, also to be defined below, is neither downward closed nor closed under unions. Our methods do not seem to apply to independence logic, we can merely approximate it from below and from above with logics that we understand better.

We do not rule out the possibility that a team is empty. Accordingly we distinguish whether a set of teams contains the empty team as an element or not. The basic dependence, independence and inclusion logics have the Empty Team Property i.e. every definable set of teams contains the empty team. However, many of our proofs depend on the ability to express the non-emptiness of a team. For this purpose we also consider a special atomic formula NE the only role of which is to say that the team is nonempty. This so-called non-emptiness NE was introduced in [32] and in [29]. We give examples which suggest that NE is not completely alien to common usage of language although it seems hopelessly abstract. The introduction of NE leads to two versions of each of our propositional logics: one without NE and one with NE.

This paper is structured as follows. In Section 2 we define the basic concepts and make some preliminary observations. We also define the propositional team logics we study in the paper, including *propositional dependence logic*, *propositional independence logic*, *propositional union closed logic*, *propositional inclusion logic* and *propositional team logic* as well as the strong version of each. In Section 3 we establish basic normal forms and use them to obtain semantic characterizations of our logics, whether strong or not. In Section 4 we prove some metalogical properties of our logics, including compactness and the closure under classical substitutions of the logics. In Section 5 we establish complete axiomatizations of the strong versions of our logics. Several open problems are listed in the concluding Section 6.

2. Preliminaries

Our propositional team logic follows the pattern set forth on first-order level by *dependence logic* [27], *independence logic* [13], as well as inclusion and exclusion logics [10]. The concepts of dependence and independence were earlier introduced in database theory, starting with [6]. However, in database theory the focus is on dependence and independence of attributes per se, while we take the dependence and independence as atomic formulas and use logical operations to build complex formulas. The benefit of considering complex formulas is that we can express very involved types of dependence and independence. A good example is the fact that first-order inclusion logic can express in finite models exactly all dependencies expressible in fixed point logic [11].

We follow here the reasoning of Wilfrid Hodges [17, 18] to the effect that a set of valuations, rather than a single valuation, permits the delineation of dependence and independence. We call such sets *teams*. Let us now give the formal definition of a team.

Definition 2.1. Throughout the paper we fix an infinite set $\text{Prop} = \{p_i \mid i \in \mathbb{N}\}$ of propositional variables. We sometimes use $\vec{x}, \vec{y}, \vec{z}, \dots$ to denote arbitrary sequences of propositional variables. A *valuation* s on a set N of indices (i.e. a set of natural numbers) is a function from N to the set $2 = \{0, 1\}$. A *team* X on N is a set of valuations on N . A *team* X on the set $\mathbb{N}$ of all natural numbers is called a *team*. If X is a team on N and $N' \subseteq N$, then we write $X \upharpoonright N'$ for the set $\{s \upharpoonright N' \mid s \in X\}$.

Table 1 shows an example of a team X consisting of six valuations. One possibility is to view a team as an information state as is done in *inquisitive logic* [5]. The idea is that there is one “true” valuation v and the valuations in the team are approximations of it as far as we know. The bigger the team the bigger is our uncertainty about v . On the other hand, if the team is as small as a singleton $\{v\}$, we know the valuation, and there is no uncertainty. This is just one intuition behind the team concept. A different intuition is that the valuations in a team arise from scientific observations. They may arise also from the organizational structure of a large company, etc.

We call propositional logics that have semantics based on teams *propositional team logics*. As the first step, let us examine the usual classical propositional logic in the setting of team semantics.

Definition 2.2. Well-formed formulas of *classical propositional logic* (**CPL**) are given by the following grammar

$$\phi ::= p_i \mid \neg p_i \mid \perp \mid (\phi \wedge \phi) \mid (\phi \otimes \phi).$$

Here we use the symbol $\otimes$ to denote the disjunction of **CPL**. A well-formed formula of **CPL** is said to be a *formula in the language of CPL* or a *classical formula*.

Definition 2.3. We define inductively the notion of a classical formula ϕ being *true* on a team X , denoted by $X \models \phi$, as follows:

- $X \models p_i$ iff for all $s \in X$, $s(i) = 1$

	p_0	p_1	p_2	p_3	$\dots$
s_1	1	1	1	1	
s_2	1	0	0	0	
s_3	0	1	1	1	
s_4	0	0	0	0	$\dots$
s_5	1	1	0	0	
s_6	0	1	0	1	

Table 1: A team $X = \{s_1, \dots, s_6\}$

- $X \models \neg p_i$ iff for all $s \in X$, $s(i) = 0$
- $X \models \perp$ iff $X = \emptyset$
- $X \models \phi \wedge \psi$ iff $X \models \phi$ and $X \models \psi$
- $X \models \phi \otimes \psi$ iff there exist two subteams $Y, Z \subseteq X$ with $X = Y \cup Z$ such that $Y \models \phi$ and $Z \models \psi$

We write $\phi(p_{i_1}, \dots, p_{i_n})$ if the propositional variables occurring in the formula ϕ are among $p_{i_1}, \dots, p_{i_n}$. The following lemma summarizes the main properties of classical formulas. The reader is referred to [34] for details on other properties of the team semantics of classical formulas.

Lemma 2.4. *Classical formulas have the Locality Property, the Flatness Property, the Downward Closure Property, the Union Closure Property, and the Empty Team Property defined as follows.*

(Locality Property) *Let X and Y be two teams, and $\phi(p_{i_1}, \dots, p_{i_n})$ a formula. If $X \upharpoonright \{i_1, \dots, i_n\} = Y \upharpoonright \{i_1, \dots, i_n\}$, then $X \models \phi \iff Y \models \phi$.*

(Flatness Property) $X \models \phi \iff \forall s \in X (\{s\} \models \phi)$

(Downward Closure Property) *If $X \models \phi$ and $Y \subseteq X$, then $Y \models \phi$*

(Union Closure Property) *If $X \models \phi$ for all $X \in \mathcal{X}$, then $\bigcup \mathcal{X} \models \phi$*

(Empty Team Property) $\emptyset \models \phi$ always holds

Under the usual single valuation semantics a classical formula $\phi(p_{i_1}, \dots, p_{i_n})$ defines a set $|\phi| = \{s \in 2^N : s \models \phi\}$ of valuations (a team!) on $N = \{i_1, \dots, i_n\}$; the same formula under the team semantics defines a set

$$\llbracket \phi \rrbracket := \{X \subseteq 2^N \mid X \models \phi\}$$

of teams on N . It is well-known that **CPL** is *expressively complete* under the usual single valuation semantics in the sense that every property $X \subseteq 2^N$ is definable by a classical formula ϕ , i.e., $X = |\phi|$. We now define a similar notion of *expressive completeness* for a set of team properties under the team semantics.

Definition 2.5. Let $\mathbb{P}$ be a set of team properties i.e. a set of sets of teams. We let $\mathbb{P}_N = \{P \upharpoonright N : P \in \mathbb{P}\}$, where each $P \upharpoonright N = \{X \upharpoonright N : X \in P\}$ is a team property on a finite set N of indices. We say that a propositional team logic L *characterizes* $\mathbb{P}$, if for each index set $N = \{i_1, \dots, i_n\}$,

$$\mathbb{P}_N = \{\llbracket \phi \rrbracket : \phi(p_{i_1}, \dots, p_{i_n}) \text{ is a formula in the language of } L\}.$$

If a logic characterizes a set $\mathbb{P}$ of team properties, then we also say that the logic is *expressively complete* for $\mathbb{P}$.

Below we define some interesting team properties, already inherent in Lemma 2.4:

Definition 2.6. A team property P , i.e., a set of teams, is called

- *flat* if $X \in P \iff \forall s \in X (\{s\} \in P)$;
- *downward closed* if $Y \subseteq X \in P \implies Y \in P$;
- *union closed* if $\mathcal{X} \subseteq P \implies \bigcup \mathcal{X} \in P$.

It follows from our previous paper [34] that several propositional logics of dependence (**PD**) (including propositional dependence logic and inquisitive logic) are expressively complete for the set of all nonempty downward closed team properties. In this paper we will study logics that are expressively complete for each of the team properties defined above. In particular, we will prove that **CPL** is expressively complete for the set of flat team properties and it is the biggest propositional team logic that defines both all downward closed team properties and all union closed team properties (Theorem 3.1).

The empty team is a member of any flat team property and of any nonempty downward closed team property. The familiar classical formulas and formulas in the language of **PD** that we studied in our previous paper [34] all have the empty team property. To define team properties that do not contain the empty team, we introduce a new atom NE, called *non-emptiness*, stating that the team in question is nonempty. To define also other interesting team properties, we now enrich the language of our logic.

Definition 2.7. Well-formed formulas of the *full propositional team logic* (**FPT**) are given by the following grammar

$$\begin{aligned} \phi ::= & p_i \mid \neg p_i \mid \text{NE} \mid \perp \mid p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m} \mid =(p_{i_1}, \dots, p_{i_k}, p_j) \\ & \mid p_{i_1}, \dots, p_{i_k} \subseteq p_{j_1} \dots p_{j_m} \mid (\phi \wedge \phi) \mid (\phi \otimes \phi) \mid (\phi \oplus \phi) \mid (\phi \vee \phi) \end{aligned}$$

The formulas $p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m}$, $=(p_{i_1}, \dots, p_{i_k}, p_j)$ and $p_{i_1}, \dots, p_{i_k} \subseteq p_{j_1} \dots p_{j_m}$ are called the *independence atom*, the *dependence atom* and the *inclusion atom*, respectively. The connectives $\otimes$, $\oplus$ and $\vee$ are called the *tensor (disjunction)*, the *nonempty disjunction* and the *Boolean disjunction*, respectively.

Definition 2.8. We define inductively the notion of a formula ϕ in the language of **FPT** being *true* on a team X , denoted by $X \models \phi$. All the cases are identical to those defined in Definition 2.3 and additionally:

- $X \models \text{NE}$ iff $X \neq \emptyset$
- $X \models p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m}$ iff for all $s, s' \in X$, there exists $s'' \in X$ such that

$$\langle s''(i_1), \dots, s''(i_k) \rangle = \langle s(i_1), \dots, s(i_k) \rangle$$

and

$$\langle s''(j_1), \dots, s''(j_m) \rangle = \langle s'(j_1), \dots, s'(j_m) \rangle$$

- $X \models (p_{i_1} \dots p_{i_k}, p_j)$ iff for all $s, s' \in X$,
if $\langle s(i_1), \dots, s(i_k) \rangle = \langle s'(i_1), \dots, s'(i_k) \rangle$, then $s(j) = s'(j)$
- $X \models p_{i_1} \dots p_{i_k} \subseteq p_{j_1} \dots p_{j_k}$ iff for all $s \in X$, there exists $s' \in X$ such that
 $\langle s(i_1), \dots, s(i_k) \rangle = \langle s'(j_1), \dots, s'(j_k) \rangle$
- $X \models \phi \otimes \psi$ iff $X = \emptyset$ or there are nonempty Y and Z such that $X = Y \cup Z$,
 $Y \models \phi$ and $Z \models \psi$
- $X \models \phi \vee \psi$ iff $X \models \phi$ or $X \models \psi$

We say that a formula ϕ is *valid*, denoted by $\models \phi$, if $X \models \phi$ holds for all teams X . We say that a formula ψ is a *logical consequence* of a set Γ of formulas, written $\Gamma \models \psi$, if for any team X such that $X \models \phi$ for all $\phi \in \Gamma$, we have $X \models \psi$. We also write $\phi \models \psi$ for $\{\phi\} \models \psi$. If $\phi \models \psi$ and $\psi \models \phi$, then we say that ϕ and ψ are *semantically equivalent*, in symbols $\phi \equiv \psi$.

Let L_1 and L_2 be two propositional team logics. We write $L_1 \leq L_2$ if every formula of L_1 is semantically equivalent to a formula of L_2 . If $L_1 \leq L_2$ and $L_2 \leq L_1$, then we write $L_1 \equiv L_2$ and say that L_1 and L_2 have the *same expressive power*.

Let us now spend a few moments with the atoms and connectives of **FPT**.

Independence atom

Let us first take a closer look at the independence atoms by considering the team X of Table 1. It can be verified that the independence atom $p_0 \perp p_3$ is satisfied by X . One may think of the team X as given data about $p_0, p_1, p_2, p_3, \dots$. For example, p_0, p_1, p_2 may be propositional variables which tell whether some valves V_0, V_1, V_2 controlling gas flow in an industrial process are open (1) or closed (0), and p_3 is a propositional variable indicating whether a warning lamp is on (1) or off (0). We can conclude on the basis of the team of Table 1 that the lamp is independent of the valve V_0 . However, the lamp is not completely independent of V_1 , because if V_1 is closed, the lamp is definitely off. Also, the lamp is not entirely independent of the valve V_2 , because if V_2 is closed, the lamp is again definitely off.

One way to describe the truth definition of $X \models p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m}$ is to compare it to Cartesian product: $X \models \{p_i : i \in I\} \perp \{p_j : j \in J\}$ if and only if

$$X \upharpoonright I \cup J = (X \upharpoonright I) \times (X \upharpoonright J).$$

This manifests the similarity between our concept of independence and the concept of independence of random variables in statistics.

The implication problem of independence atoms (i.e., the problem of asking whether an independence atom follows from a set of independence atoms) can be completely axiomatized by the axioms below, known in database theory as the Geiger-Paz-Pearl axioms ([12]):

- (i) If $\vec{x} \perp \vec{y}$, then $\vec{y} \perp \vec{x}$.
- (ii) If $\vec{x} \perp \vec{y}$, then $\vec{z} \perp \vec{y}$, where $\vec{z}$ is a subsequence of $\vec{x}$.
- (iii) If $\vec{x} \perp \vec{y}$, then $\vec{u} \perp \vec{v}$, where $\vec{u}$ and $\vec{v}$ are permutations of $\vec{x}$ and $\vec{y}$, respectively.
- (iv) If $\vec{x} \perp \vec{y}$ and $\vec{x}\vec{y} \perp \vec{z}$, then $\vec{x} \perp \vec{y}\vec{z}$.

While the downward closure property has a profound influence on properties of dependence logic as already mentioned, the independence atoms violate this property. For example, in Table 1 in the team X the attributes p_0 and p_3 are independent but in the subteam $Y = \{s_1, s_2, s_3\}$ they are not. We will see in the sequel that propositional independence logic and other propositional team logics have a completely different flavor than propositional logics of dependence.

Non-emptiness atom

Another formula that violates the downward closure property is the very simple atom NE that we call nonemptiness which states that the team is nonempty. An easy inductive proof shows that the NE-free fragment of **FPT** has the empty team property. But often when describing properties of teams, we do want to distinguish between the empty team and the nonempty teams. The atom NE is introduced exactly for this purpose.

The symbol NE is a logical symbol, on a par with $\perp$, with no internal structure and no proposition symbols occurring in it. While $\perp$ is generally conceived of as a symbol of contradiction, one may ask what is the intuitive meaning of NE? Does this symbol occur in natural language or in scientific discourse? Let us think of a natural language sentence that has the modality “might”:

I *might* come to the party.

Given a nonempty information state (i.e. a team) X , this sentence can be characterized as “there exists a nonempty substate Y in which I indeed come to the party”. This “might” modality (denoted by ∇) was considered by Hella and Stumpf in [15] and its team semantics is given by the clause

- $X \models \nabla \phi$ iff $X = \emptyset$ or there exists a nonempty team $Y \subseteq X$ such that $Y \models \phi$

The “might” modality can be expressed in terms of the more basic notion of non-emptiness of a team:

$$\nabla \phi \equiv \perp \vee ((\phi \wedge \text{NE}) \otimes \top).$$

Contradictions and linear implication

In the presence of the non-emptiness NE *contradiction* has two variants: the *weak contradiction* $\perp$ that is satisfied only by the empty team and the *strong contradiction* $\perp \wedge \text{NE}$ that is satisfied by no team at all.

A related logical constant is the *linear implication* $\multimap$ (introduced by Abramsky and Väänänen [1]) that has the semantics

- $X \models \phi \multimap \psi$ iff for any team Y , if $Y \models \phi$, then $X \cup Y \models \psi$

The strong contradiction is easily definable using the linear implication:

$$\perp \wedge \text{NE} \equiv \top \multimap \perp,$$

where $\top = p_{i_1} \otimes \neg p_{i_1}$. The reader is referred to [1] for details on linear implication. We only remark that in the presence of the downward closure property, we have

$$\phi \models \psi \iff \emptyset \models \phi \multimap \psi.$$

In other words, deciding whether ψ is a logical consequence of ϕ is reduced to deciding whether the linear implication $\phi \multimap \psi$ is satisfied by the empty team.

Disjunctions

Due to the way we define semantics there are more propositional operations than in the case of classical propositional logic. In particular, disjunction has three different incarnations, namely $\otimes$, $\oplus$ and $\vee$. These different forms arise from the difference between considering individual valuations and sets of valuations.

The tensor disjunction $\otimes$ generalizes the disjunction of classical propositional logic. The semantics of $\otimes$ and other connectives as defined in Definitions 2.3 and 2.8 is known in the literature (see e.g., [10]) as the *Lax Semantics* (in contrast to the *Strict Semantics*). An easy inductive proof shows that our logic **FPT** has the locality property. By contrast, if we replace the clause for tensor disjunction $\otimes$ in Definition 2.3 by the corresponding clause under strict semantics (denoted by $\models^s$)

- $X \models^s \phi \otimes \psi$ iff there exist two *disjoint* subteams $Y, Z \subseteq X$ with $X = Y \cup Z$ such that $Y \models^s \phi$ and $Z \models^s \psi$

the logic does not any more satisfy the local property. This is because, for instance, for the two valuations s_1 and s_2 defined in Table 1, we have $\{s_1, s_2\} \models^s (\text{NE} \wedge p_0) \otimes (\text{NE} \wedge p_0)$ while $\{s_1\} \not\models^s (\text{NE} \wedge p_0) \otimes (\text{NE} \wedge p_0)$, even though $\{s_1 \upharpoonright \{0\}, s_2 \upharpoonright \{0\}\} = \{(0, 1)\} = \{s_1 \upharpoonright \{0\}\}$. We refer the reader to [10] for further discussions on the difference between lax and strict semantics.

The nonempty disjunction $\oplus$ was introduced by Raine Rönholm (personal communication). One can easily verify that $\oplus$ can be defined in terms of the other disjunctions and the non-emptiness:

$$\phi \oplus \psi \equiv \perp \vee ((\phi \wedge \text{NE}) \otimes (\psi \wedge \text{NE})).$$

Moreover, it was observed in [15] that in any fragment of **FPT** that has the empty team property the might modality ∇ and the nonempty disjunction $\otimes$ as inter-definable:

$$\nabla\phi \equiv \phi \otimes \top \quad \text{and} \quad \phi \otimes \psi \equiv (\phi \otimes \psi) \wedge (\nabla\phi \wedge \nabla\psi) \quad (1)$$

To understand the meaning of $\otimes$ in natural language, let us think of the sentence (in the context of chess):

$$\text{Rook or queen was sacrificed in each play.} \quad (2)$$

It is clear what it means to say that a nonempty set X of plays satisfies this, and in each play in X either rook or queen was sacrificed. There is a slight difference in saying

$$\text{Rook or queen was sacrificed and both cases occurred in some plays.} \quad (3)$$

In our symbolic language, denoting “Rook was sacrificed” by ϕ and “queen was sacrificed” by ψ , (2) would be written

$$X \models \phi \otimes \psi$$

while (3) would be written

$$X \models (\phi \wedge \text{NE}) \otimes (\psi \wedge \text{NE}) \text{ or } X \models \phi \otimes \psi.$$

In a sense, $\phi \otimes \psi$ is an “honest” disjunction: if the team has anything in it at all, then it is divided between ϕ and ψ in the non-trivial way that both get a nonempty subteam. We can think that whoever says (3), means that if some plays were actually played, then in some of them a Rook was sacrificed and in some the Queen. To put it in a more general context, the formula $\phi \otimes \psi$ permits a type of “free choice” by having each disjunct nonvoid. This way the nonempty disjunction $\phi \otimes \psi$ provides more information than the tensor disjunction $\phi \otimes \psi$. In particular, uttering a disjunction with a void disjunct is actually less informative than simply stating one of the disjuncts.

The Boolean disjunction $\vee$ was called *intuitionistic disjunction* in our previous paper [34] in the context of propositional logics of dependence. In particular, in the presence of the downward closure property the intuitionistic disjunction has the *disjunction property*:

$$\models \phi \vee \psi \text{ implies } \models \phi \text{ or } \models \psi.$$

However, in the absence of the downward closure, this property, reminiscent of constructive logic, for $\vee$ fails, since, e.g., $\models \perp \vee \text{NE}$ and $\models (p \otimes \neg p) \vee (p \vee \neg p)$, whereas $\not\models \perp$, $\not\models \text{NE}$, $\not\models p \otimes \neg p$ and $\not\models p \vee \neg p$.

We define the empty disjunction for all three disjunctions as

$$\bigotimes \emptyset := \perp, \quad \bigotimes \emptyset := \perp \text{ and } \bigvee \emptyset := \perp \wedge \text{NE}.$$

We are interested in fragments of **FPT** that are expressively complete for some nice sets of team properties as defined in Definition 2.6. The languages of these fragments are determined in terms of which atoms and connective are allowed. Table 2 defines

Table 2: Propositional team logics

Logic	Atoms	Connectives
Classical propositional logic (CPL)	$p_i, \neg p_i, \perp$	$\wedge, \otimes$
Strong classical propositional logic (CPL ⁺)	$p_i, \neg p_i, \perp, \text{NE}$	$\wedge, \otimes$
Propositional independence logic (PI)	$p_i, \neg p_i, \perp,$ $p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m}$	$\wedge, \otimes$
Strong propositional independence logic (PI ⁺)	$p_i, \neg p_i, \perp, \text{NE},$ $p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m}$	$\wedge, \otimes$
Propositional team logic (PT)	$p_i, \neg p_i, \perp$	$\wedge, \otimes, \vee$
Strong propositional team logic (PT ⁺)	$p_i, \neg p_i, \perp, \text{NE}$	$\wedge, \otimes, \vee$
Propositional union closed logic (PU)	$p_i, \neg p_i, \perp$	$\wedge, \otimes, \oplus$
Strong propositional union closed logic (PU ⁺)	$p_i, \neg p_i, \perp, \text{NE}$	$\wedge, \otimes, \oplus$
Propositional inclusion logic (PInc)	$p_i, \neg p_i, \perp,$ $p_{i_1} \dots p_{i_k} \subseteq p_{j_1} \dots p_{j_k}$	$\wedge, \otimes$
Strong propositional inclusion logic (PInc ⁺)	$p_i, \neg p_i, \perp, \text{NE},$ $p_{i_1} \dots p_{i_k} \subseteq p_{j_1} \dots p_{j_k}$	$\wedge, \otimes$
Propositional dependence logic (PD)	$p_i, \neg p_i, \perp,$ $= (p_{i_1} \dots p_{i_k}, p_j)$	$\wedge, \otimes, \vee$
Strong propositional dependence logic (PD ⁺)	$p_i, \neg p_i, \perp,$ $= (p_{i_1} \dots p_{i_k}, p_j)$	$\wedge, \otimes, \vee, \multimap$
Full propositional team logic (FPT)	$p_i, \neg p_i, \perp, \text{NE},$ $p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m}$ $= (p_{i_1} \dots p_{i_k}, p_j)$ $p_{i_1} \dots p_{i_k} \subseteq p_{j_1} \dots p_{j_k}$	$\wedge, \otimes, \oplus, \vee$

the sets of atoms and connectives of the languages of these logics. Apart from **FPT**, we consider six other types of propositional logics, namely, classical logic, independence logic, team logic, union closed logic, inclusion logic and dependence logic, each of which has two variants, a weak version that has the empty team property, and a strong version that contains the NE atom or the linear implication $\multimap$ in its language. The propositional independence logic, propositional inclusion logic and propositional dependence logic we define here are propositional version of their first-order counterparts introduced in [10, 13, 27]. The propositional team logic defined as in Table 2 does not directly correspond to the propositional fragment of the first-order team logic studied in [21, 27], and the reader should not confuse the two logics. The propositional union closed logic is a new logic that was not previously considered in the literature.

3. Expressive Power and Normal Forms

In this section, we study the expressive power of the propositional team logics defined in Table 2 and their normal forms. We will also prove the interconnections illustrated in Figure 1.

Clearly, from the above theorem it follows that the standard disjunctive normal form of **CPL** under the usual single valuation semantics is also a normal form of **CPL** under the team semantics. Another immediate corollary of the expressive completeness of **CPL** for the set of all flat properties is that flatness is equivalent to being logically equivalent to a classical formula in propositional team logics, as stated in the theorem below. Flatness was originally introduced by Hodges [18] in the first-order context, and further studied in [27]. In the first-order dependence logic case all classical first-order formulas have the flatness property (see, e.g., [27]) but the converse is not true. For example, all first-order sentences (i.e., formulas without free variables) have the flatness property for the trivial reason that their truth in any model is decided by the truth in the singleton team $\{\emptyset\}$ of the empty assignment $\emptyset$ alone (see Lemma 1.1.14 in [32]).

Theorem 3.2. *Let ϕ be a formula in the language of **FPT**. The following are equivalent.*

- (i) ϕ has the flatness property.
- (ii) ϕ is semantically equivalent to a classical formula.
- (iii) ϕ has the downward closure property and the union closure property.

Proof. The equivalence of (i) and (ii) follows from Theorem 3.1 and the equivalence of (i) and (iii) is easy to verify. $\square$

Our characterization of classical propositional logic can be equivalently formulated as follows: The classical propositional logic cannot be extended in the context of team semantics to a propositional logic which satisfies both the downward closure and the union closure property. This is remotely reminiscent of the characterization of classical first-order logic, known as Lindström's Theorem [22], to the effect that classical first-order logic cannot be extended in the context of abstract logics to a logic which satisfies both the Downward Löwenheim-Skolem Theorem and the Compactness Theorem.

Recall from [34] that **PD** is expressively complete for the set of all downward closed team properties which contain the empty team. Having established also the expressive completeness of **CPL** for the set of all flat team properties, we now proceed towards the proof of the other expressive completeness results illustrated in Figure 1.

The set of subteams of a fixed team X on a finite set N is characterized by the classical formula Θ_X (in the sense of (4)). We now show that with the non-emptiness or the nonempty disjunction, a team can be characterized precisely.

Lemma 3.3. *Let X be a team on a finite set $N = \{i_1, \dots, i_n\}$ of indices. Define*

$$\Theta_X^* := \bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \text{NE}) \quad \text{and} \quad \Theta_X^{**} := \bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)}).$$

For any team Y on N , we have

- (i) $Y \models \Theta_X^* \iff Y = X$.
- (ii) $Y \models \Theta_X^{**} \iff Y = X \text{ or } Y = \emptyset$.

Proof. The direction “ $\Leftarrow$ ” for both items is obvious. For the direction “ $\Rightarrow$ ” of item (i), suppose $Y \models \Theta_X^*$. If $X = \emptyset$, then $\Theta_X^* = \perp$, hence $Y = \emptyset = X$. Otherwise, for each $s \in X$ there exists a set Y_s such that

$$Y = \bigcup_{s \in X} Y_s \text{ and } Y_s \models p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \text{NE}.$$

Clearly, $Y_s = \{s\}$ for each $s \in X$, implying $Y = X$.

For the direction “ $\Rightarrow$ ” of item (ii), suppose $Y \models \Theta_X^{**}$. If $Y = \emptyset$, then we are done. Otherwise, $Y = X$ is proved by a similar argument to that of item (i). $\square$

Next, we prove that many of the logics we defined in Section 2 (see Table 2) are expressively complete for certain sets of team properties (see Definitions 2.5 and 2.6 for definitions of the relevant notions).

Theorem 3.4. (i) Both $\mathbf{PT}^+$ and $\mathbf{FPT}$ characterize the set of all team properties, i.e., every team property is definable in the logics. In particular, $\mathbf{PT}^+ \equiv \mathbf{FPT}$.

(ii) $\mathbf{PT}$ characterizes the set of all team properties which contain the empty team.

(iii) $\mathbf{PU}$ characterizes the set of all union closed team properties which contain the empty team.

(iv) $\mathbf{PU}^+$ characterizes the set of all union closed team properties.

(v) $\mathbf{PD}$ characterizes the set of all downward closed team properties which contain the empty team.

(vi) $\mathbf{PD}^+$ characterizes the set of all downward closed team properties.

Proof. Let $N = \{i_1, \dots, i_n\}$ be an arbitrary finite set of indices.

(i) Obviously every formula $\phi(p_{i_1}, \dots, p_{i_n})$ in the language of $\mathbf{PT}^+$ or $\mathbf{FPT}$ defines a team property on N , i.e., $\llbracket \phi \rrbracket \subseteq \mathcal{P}(2^N)$. Conversely, for any team property $P \subseteq \mathcal{P}(2^N)$, we shall show $P = \llbracket \bigvee_{X \in P} \Theta_X^* \rrbracket$.

By Lemma 3.3(i), for any team Y on N ,

$$Y \models \bigvee_{X \in P} \Theta_X^* \iff \exists X \in P (Y = X) \iff Y \in P.$$

In particular, if $P = \emptyset$, then $\llbracket \bigvee_{X \in \emptyset} \Theta_X^* \rrbracket = \llbracket \perp \wedge \text{NE} \rrbracket = \emptyset$.

(ii) Since formulas $\phi(p_{i_1}, \dots, p_{i_n})$ in the language of $\mathbf{PT}$ have the empty team property, we have $\emptyset \in \llbracket \phi \rrbracket \subseteq \mathcal{P}(2^N)$. Conversely, for any team property $P \subseteq \mathcal{P}(2^N)$ with $\emptyset \in P$, we show $P = \llbracket \bigvee_{X \in P} \Theta_X^{**} \rrbracket$.

By Lemma 3.3(ii), for any team Y on N ,

$$Y \models \bigvee_{X \in P} \Theta_X^{**} \iff \exists X \in P (Y = X) \text{ or } Y = \emptyset \iff Y \in P.$$

(iii) It is easy to show by induction that every formula ϕ in the language of $\mathbf{PU}$ has the union closure property and the empty team property, which imply that $\llbracket \phi \rrbracket$ is

a union closed team property that contains the empty team. Conversely, for any union closed team property $P \subseteq \mathcal{P}(2^N)$ with $\emptyset \in P$, we show that $P = \llbracket \bigotimes_{X \in P} \Theta_X^{**} \rrbracket$.

If $Y \in P$, then, by Lemma 3.3(ii) we have $Y \models \Theta_Y^{**}$ and $Y \models \bigotimes_{X \in P} \Theta_X^{**}$. Conversely, if $Y \models \bigotimes_{X \in P} \Theta_X^{**}$, then for each $X \in P$ there is $Y_X \subseteq Y$ such that $Y = \bigcup_{X \in P} Y_X$ and $Y_X \models \Theta_X^{**}$. By Lemma 3.3(ii), we have $Y_X = \emptyset$ or $Y_X = X$ for each $X \in P$. Since $\emptyset \in P$ and P is a union closed team property, we conclude that $Y \in P$.

(iv) Obviously $\llbracket \text{NE} \rrbracket$ is a union closed team property. Thus, by item (iii), for every formula ϕ in the language of $\mathbf{PU}^+$, $\llbracket \phi \rrbracket$ is a union closed team property. Conversely, for any union closed team property $P \subseteq \mathcal{P}(2^N)$, we show that P is definable by some formula ϕ in the language of $\mathbf{PU}^+$.

If P contains the empty team, then $P = \llbracket \bigotimes_{X \in P} \Theta_X^{**} \rrbracket$ by item (iii). If $\emptyset \notin P$, then it is easy to verify that $P = \llbracket \text{NE} \wedge \bigotimes_{X \in P} \Theta_X^{**} \rrbracket$.

(v) This item is a consequence of results in [34]. Note that in [34], propositional dependence logic and some of its variants (including propositional inquisitive logic) are all shown to be expressively complete for the set of all downward closed team properties which contain the empty team. With a slight abuse of notation, we denote in this paper by $\mathbf{PD}$ any of these equivalent logics.

(vi) It is easy to show, by induction, that every formula ϕ in the language of $\mathbf{PD}^+$ has the downward closure property, which implies that $\llbracket \phi \rrbracket$ is a downward closed team property. Conversely, for any downward closed team property $P \subseteq \mathcal{P}(2^N)$, we show that P is definable by some formula in the language of $\mathbf{PU}^+$.

If P is a downward closed team property that contains the empty set, then by item (v), we know that P is definable by some formula in the language of $\mathbf{PD}$ (thus also in the language of $\mathbf{PD}^+$). If $\emptyset \notin P$, then since P is a downward closed team property, we must have $P = \emptyset$. Clearly, $P = \emptyset = \llbracket \top \multimap \perp \rrbracket$. $\square$

Results in Theorems 3.1 and 3.4 are illustrated in Figure 1, where all those expressively complete logics are represented in shaded rectangles labeled with their corresponding characteristic team properties. The logics $\mathbf{FPT}$, $\mathbf{PT}^+$ and $\mathbf{PT}$ do not have a label in Figure 1, as they characterize the set of arbitrary team properties (with or without the empty team). Except for the flat team property, each characteristic team property we study here has two variants. One with the empty team in the property and the other one without this constraint. We have an expressively complete logic for certain set of team properties that has the empty team property (indicated by a solid rectangle) and an expressively complete logic for the same set of team properties without the empty team property (indicated by a dashed rectangle).

We remarked in Section 2 that in any propositional team logic with the empty team property, the nonempty disjunction $\oplus$ and the might modality ∇ are inter-definable (Equation (1)). Consequently, classical propositional logic extended with the might modality ∇ has the same expressive power as $\mathbf{PU}$ and thereby is also expressively complete for the set of all union closed team properties which contain the empty team.

Propositional independence logic $\mathbf{PI}$ is expressively properly included in the expressively strongest logic $\mathbf{PT}^+$. In particular, the independence atoms are definable in $\mathbf{PT}^+$:

$$p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m} \equiv \bigvee_{X \in \mathcal{X}_{I,J}} \Theta_X^*, \quad (5)$$

Logic	Normal Form
PT ⁺ FPT	$\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \text{NE})$
PT	$\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$
PU ⁺	either $\text{NE} \wedge \bigotimes_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$ or $\bigotimes_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$
PU	$\bigotimes_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$
PD	$\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$
PD ⁺	either $\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$ or $\top \multimap \perp$
CPL	$\bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$

Table 3: Normal forms of propositional team logics

where $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_m\}$ and

$$\mathcal{X}_{I,J} = \{X \subseteq \{0,1\}^{I \cup J} \mid X = (X \upharpoonright I) \times (X \upharpoonright J)\}.$$

The expressive power of **PI** is an open problem.

Another immediate consequence of the expressive completeness of **PT**⁺ for the set of all team properties is that all the possible atoms and all the instances of all the possible connectives are expressible in the expressively strongest logic **PT**⁺. Consider the *Boolean negation* $\sim$ defined as

- $X \models \sim \phi$ iff $X \not\models \phi$

Clearly $\text{NE} \equiv \sim \perp$ and $\phi \vee \psi \equiv \sim (\sim \phi \wedge \sim \psi)$. Thus **CPL**⁺ extended with the Boolean negation $\sim$ has the same expressive power as **PT**⁺. See [24] for other properties of the Boolean negation $\sim$ and a complete axiomatization for propositional dependence logic extended with $\sim$. In this paper we will restrict our attention to the considerably simpler logical constant NE instead of $\sim$. Note that from the equivalence of the two logics, we only derive that *every instance* of $\sim \phi$ is expressible in **PT**⁺, but Boolean negation turns out to be not *uniformly definable* in **PT**⁺ (see [33]).

From the proofs of Theorem 3.1 and Theorem 3.4 we obtain interesting disjunctive normal forms for the logics, as listed in (the self-explanatory) Table 3. It is worth taking note of the many similarities and at the same time the subtle differences of these normal forms.

We now prove the results in Figure 1 concerning the comparison of the logics in terms of their expressive powers. In Figure 1, the logics placed in the same rectangle have the same expressive power. If a line connects two sets of logics, then the logics L_2 positioned above are expressively strictly stronger than the logics L_1 positioned

below, i.e., $L_1 < L_2$. If instead, L_1 and L_2 are connected by a double line, then only $L_1 \leq L_2$ is known. As discussed already, the logic $\mathbf{PT}^+$ and $\mathbf{FPT}$ are expressively complete for the set of all team properties, thus they are both the expressively strongest logics. The logic $\mathbf{PT}$ is expressively complete for the set of all team properties which contain the empty set, therefore it has stronger expressive power than the two logics $\mathbf{PU}$ and $\mathbf{PI}$ that characterize certain team properties which contain the empty set. For any other pair of logics that are linked by a solid line or a double line in Figure 1, if the one that is positioned above is an extension of the other, then it obviously has stronger expressive power. The logics $\mathbf{CPL}$, $\mathbf{PU}$, $\mathbf{PD}$, $\mathbf{PI}$ and $\mathbf{PT}$ that have the empty team property are strictly weaker than their corresponding logics $\mathbf{CPL}^+$, $\mathbf{PU}^+$, $\mathbf{PD}^+$, $\mathbf{PI}^+$ and $\mathbf{PT}^+$ that do not have the empty team property, respectively. $\mathbf{PD}$ is strictly stronger than $\mathbf{CPL}$ because classical formulas have the flatness property, while $\mathbf{PD}$ has formulas that lack the flatness property (see [34] for detail). The union closed logics $\mathbf{CPL}^+$, $\mathbf{PU}$ and $\mathbf{PU}^+$ are expressively different from the non-union closed logics $\mathbf{PI}^+$, $\mathbf{PT}$ and $\mathbf{PT}^+$, respectively. For instance, the formulas $p_0 \perp p_1$ and $p_0 \vee p_1$ are not closed under unions. Similarly, the downward closed logics $\mathbf{PD}$ and $\mathbf{PD}^+$ are expressively different from the non-downward closed logics $\mathbf{PI}$ and $\mathbf{PI}^+$, respectively. For instance, the formula $p_0 \perp p_1$ is not closed downward.

It now remains to prove that $\mathbf{PD}$ and $\mathbf{PD}^+$ are expressively weaker than $\mathbf{PI}$ and $\mathbf{PI}^+$, respectively. This reduces to showing that dependence atoms are expressible in $\mathbf{PI}$ and in $\mathbf{PI}^+$. First, note that the independence atoms $\vec{x} \perp \vec{y}$ are known as *unconditional* independence atoms in the literature of independence logic. A *conditional* independence atom [13] is written as $\vec{x} \perp_{\vec{z}} \vec{y}$ and its semantics is defined by the clause

- $X \models p_{j_1} \dots p_{j_b} \perp_{p_{i_1} \dots p_{i_a}} p_{k_1} \dots p_{k_c}$ iff for all $s, s' \in X$ with $s(\vec{i}) = s'(\vec{i})$, there exists $s'' \in X$ such that

$$s''(\vec{i}) = s(\vec{i}) = s'(\vec{i}), \quad s''(\vec{j}) = s(\vec{j}) \quad \text{and} \quad s''(\vec{k}) = s'(\vec{k}),$$

where $\vec{i} = i_1 \dots i_a$, $\vec{j} = j_1 \dots j_b$ and $\vec{k} = k_1 \dots k_c$.

In our setting conditional independence atoms are expressible in terms of unconditional ones:

$$\vec{x} \perp_{p_{i_1} \dots p_{i_a}} \vec{y} \equiv \bigotimes_{s \in 2^I} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_a}^{s(i_a)} \wedge (\vec{x} \perp \vec{y})),$$

where $I = \{i_1, \dots, i_a\}$. As observed already in [13] in the context of first-order logic, dependence atoms are definable in terms of conditional independence atoms:

$$=(\vec{x}, p_i) \equiv p_i \perp_{\vec{x}} p_i.$$

Putting these altogether, we conclude that dependence atoms are expressible in $\mathbf{PI}$ and in $\mathbf{PI}^+$. This completes the proof of the inter-relationships shown in Figure 1.

4. Metalogical properties

In this section, we study metalogical properties of propositional team logics. We will prove that propositional team logics are compact, and closed under classical substitutions.

4.1. Compactness

Propositional inquisitive logic was shown to be compact in [5]. As a consequence, propositional logics of dependence are all compact (see Theorem 3.3 in [34]). Moreover, from the same argument as given in [3] the compactness of all the other propositional team logics follows as well. Below we present a sketch of this proof (which makes essential use of König's Lemma) using the terminologies of this paper for the benefit of the reader.

Theorem 4.1 (Compactness Theorem). *For any set $\Gamma \cup \{\phi\}$ of formulas in the language of an arbitrary propositional team logic, if $\Gamma \models \phi$, then there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \phi$.*

Proof. Let $\Gamma = \{\theta_k \mid k \in \mathbb{N}\}$. For each $k \in \mathbb{N}$, define $\gamma_k = \theta_1 \wedge \dots \wedge \theta_k$. It is sufficient to show that $\gamma_k \models \phi$ for some $k \in \mathbb{N}$. Towards a contradiction, assume otherwise. Then for each $k \in \mathbb{N}$, there exists a team $X \subseteq 2^N$ such that $X \models \gamma_k$ and $X \not\models \phi$. Let N_k be the set of all indices of all propositional variables occurring in γ_k and ϕ . By the locality property of propositional team logics, the finite set

$$\mathcal{X}_k = \{X \subseteq 2^{N_k} : X \models \gamma_k \text{ and } X \not\models \phi\}$$

is nonempty. Put $T = \{\emptyset\} \cup \bigcup_{k \in \mathbb{N}} \mathcal{X}_k$. Define a relation $\leq$ on T by putting

- $\emptyset \leq X$ for all $X \in T$;
- $X \leq Y$ iff $Y \upharpoonright \text{dom}(X) = X$.

It is not hard to see that $(T, \leq)$ is a finitely branching infinite tree. By König's Lemma, the tree has an infinite branch $\langle X_k \mid k \in \mathbb{N} \rangle$, where $X_k \in \mathcal{X}_k$ for each $k \in \mathbb{N}$. Putting $N = \bigcup_{k \in \mathbb{N}} N_k$, this infinite branch determines a team $X \subseteq 2^N$ in such a way that $X \upharpoonright N_k = X_k$ for each $k \in \mathbb{N}$. Clearly, $X \not\models \phi$ and $X \models \gamma_k$ for each $k \in \mathbb{N}$. These contradict $\Gamma \models \phi$. $\square$

4.2. Closure under classical substitutions

A *substitution* of a propositional team logic L is a mapping σ from the set Form_L of all well-formed formulas of L into the set Form_L itself that commutes with the connectives and atoms of L . We say that L is *closed under* the substitution σ , if for any set $\Gamma \cup \{\phi\}$ of formulas of L ,

$$\Gamma \models \phi \implies \{\sigma(\gamma) \mid \gamma \in \Gamma\} \models \sigma(\phi).$$

If L is closed under all substitutions, then we say that L is *closed under uniform substitution*. The logics **FPT**, **PT**, **PT**⁺, **PD**, **PD**⁺, **PI** and **PI**⁺ are not closed under uniform substitution, because, for instance, we have

$$p_i \otimes p_i \models p_i \text{ and } p_i \circledast p_i \models p_i,$$

whereas

$$\models(p_i) \otimes \models(p_i) \not\models \models(p_i), \quad (p_i \perp p_i) \otimes (p_i \perp p_i) \not\models p_i \perp p_i,$$

$$(p_i \vee \neg p_i) \otimes (p_i \vee \neg p_i) \not\models (p_i \vee \neg p_i) \text{ and } (p_i \vee \neg p_i) \otimes (p_i \vee \neg p_i) \not\models (p_i \vee \neg p_i).$$

The logics $\mathbf{PU}^+$ and $\mathbf{PU}$ are not closed under uniform substitution either (for at least the trivial reason that strings of the form $\neg(\phi \otimes \psi)$ are not well-formed formulas), but nontrivial counter-examples of the above kind for the two logics are yet to be found. It was shown in [3] and [20] that propositional logics of dependence are, nevertheless, closed under *flat substitutions*, i.e., substitutions σ such that $\sigma(p)$ has the flatness property for any propositional variable p . Using the method in [20], we will prove in this section that propositional team logics are closed under *classical substitutions*, i.e., substitutions σ such that $\sigma(p)$ is a classical formula (i.e., a formula in the language of $\mathbf{CPL}$) for any propositional variable p .

Let us start by examining in detail the notion of substitution in our logics. The well-formed formulas of the propositional team logics we consider in this paper are assumed to be in *negation normal form* and we do not allow arbitrary formulas to occur in a dependence or independence atom. Strings of the form $\neg\phi$, $\vec{\phi} \perp \vec{\psi}$, $\vec{=}\langle\vec{\phi}, \vec{\psi}\rangle$ and $\vec{\phi} \subseteq \vec{\psi}$ are not necessarily well-formed formulas of our logics. As such, the notion of substitution is actually not well-defined in our logics. To derive our intended closure under substitution result we will then first need to seek for ways to make sense of the notion of substitution in our logics. In general, there are two possible solutions: either to expand the languages of the logics so as to allow more well-formed formulas, or to restrict the range of a substitution to a subset of the full set $\text{Form}_{\mathbf{L}}$ of well-formed formulas. We will take both approaches at the same time. We will confine ourselves to classical substitutions only and will also expand the language of our logics to include every substitution instance $\sigma(\phi)$ of a classical substitution σ to be a well-formed formula. Our reason for restricting attention to classical substitutions only is twofold. Conceptually, we do not have a good intuition of the intended semantics of the formulas $\neg(\phi \otimes \psi)$ and $\neg(\phi \vee \psi)$ or of the dependence and independence atoms $\vec{\phi} \perp \vec{\psi}$, $\vec{=}\langle\vec{\phi}, \vec{\psi}\rangle$ and $\vec{\phi} \subseteq \vec{\psi}$ with arbitrary arguments. Technically, arbitrary substitutions are not very interesting, as the logics are not closed under uniform substitution.

For simplicity, in what follows we will only work with $\mathbf{FPT}$ which has the maximal set of atoms and connectives among the propositional team logics we consider in this paper. Similar results for the other logics can be easily obtained as corollaries of those for $\mathbf{FPT}$. Let us now expand the language of $\mathbf{FPT}$ and include strings of the forms $\neg\alpha$, $\top$, $\vec{\alpha} \perp \vec{\beta}$, $\vec{=}\langle\vec{\alpha}, \vec{\beta}\rangle$ and $\vec{\alpha} \subseteq \vec{\beta}$ as well-formed formulas, where $\alpha, \beta, \vec{\alpha}$ and $\vec{\beta}$ are classical formulas or sequences of classical formulas. Denote the extended logic by $\mathbf{MPT}$. Observe that $\mathbf{MPT}$ will, clearly, have the same expressive power as $\mathbf{FPT}$, since the latter is already expressively complete for the set of all team properties (Theorem 3.4(i)).

We now define the semantics of $\mathbf{MPT}$. Given sequences $\vec{\alpha} = \alpha_1 \dots \alpha_k$ and $\vec{\beta} = \beta_1 \dots \beta_k$ of classical formulas, we define an equivalence relation $\sim_{(\vec{\alpha}, \vec{\beta})}$ on the set of all valuations as follows:

$$s \sim_{(\vec{\alpha}, \vec{\beta})} s' \quad \text{iff} \quad \forall i \in \{1, \dots, k\} (\{s\} \models \alpha_i \iff \{s'\} \models \beta_i).$$

We write $\sim_{\vec{\alpha}}$ for $\sim_{(\vec{\alpha}, \vec{\alpha})}$.

Definition 4.2. The team semantics of well-formed formulas of $\mathbf{MPT}$ is defined inductively in the same way as in Definitions 2.3 and 2.8 and additionally we have the following extra clauses:

- $X \models \neg\alpha$ iff $s \not\models \alpha$ in the usual sense for all $s \in X$.
- $X \models \top$ always holds.
- $X \models \vec{\alpha} \perp \vec{\beta}$ iff for all $s, s' \in X$, there exists $s'' \in X$ such that $s \sim_{\vec{\alpha}} s''$ and $s' \sim_{\vec{\beta}} s''$.
- $X \models =(\vec{\alpha}, \vec{\beta})$ iff for all $s, s' \in X$, if $s \sim_{\vec{\alpha}} s'$ then $s \sim_{\vec{\beta}} s'$.
- $X \models \vec{\alpha} \subseteq \vec{\beta}$ iff for all $s \in X$, there exists $s' \in X$ such that $s \sim_{(\vec{\alpha}, \vec{\beta})} s'$.

The above definition deserves some comments. In the literature of logics of dependence and independence, negation is usually treated only *syntactically*. That is, a negated classical formula $\neg\phi$ is defined to have the same semantics as the unique formula $\phi^\sim$ in *negation normal form* obtained by exhaustively applying the De Morgan's laws and the following syntactic rewrite rules:

$$\begin{array}{lll} p^\sim & \mapsto & \neg p \\ (\neg p)^\sim & \mapsto & p \end{array} \quad \begin{array}{lll} \top^\sim & \mapsto & \perp \\ \perp^\sim & \mapsto & \top \end{array} \quad \begin{array}{lll} (\phi \wedge \psi)^\sim & \mapsto & \phi^\sim \otimes \psi^\sim \\ (\phi \otimes \psi)^\sim & \mapsto & \phi^\sim \wedge \psi^\sim \end{array} \quad (6)$$

A routine inductive proof shows that $\neg\alpha \equiv \alpha^\sim$ for all classical formulas α (or see [20] for the proof), i.e., our negation as defined in Definition 4.2 coincides with the above syntactic negation when applied to classical formulas. It is also worth noting that our negation corresponds to the defined connective $\sim\downarrow$ in Hodges [17, 18].

The extended inclusion atom $\vec{\alpha} \subseteq \vec{\beta}$ is also studied in [15] in the context of modal inclusion logic. There it is shown that the usual modal logic extended with the extended inclusion atoms has the same expressive power as the usual modal logic extended with the might modality ∇ (see Section 2, under the heading “non-emptiness atom”). Given this result, it is natural to conjecture that classical propositional logic extended with the extended inclusion atoms has the same expressive power as classical propositional logic extended with the might modality ∇ , and it is also expressively complete for the set of union closed team properties which contain the empty team (see the remark after Theorem 3.4).

If α is a classical formula and σ is a classical substitution, then $\sigma(\alpha)$ is still a classical formula. In particular, given any classical substitution, the substitution instance of an extended dependence and independence atom is a well-formed formula of the extended logic $\mathbf{MPT}$. Having the notion of classical substitution well-defined in $\mathbf{MPT}$, we will now prove that $\mathbf{MPT}$ is closed under classical substitutions, namely the following theorem holds.

Theorem 4.3. *Let $\Gamma \cup \{\phi\}$ be a set of formulas in the language of $\mathbf{MPT}$, and σ a classical substitution. If $\Gamma \models \phi$, then $\{\sigma(\gamma) \mid \gamma \in \Gamma\} \models \sigma(\phi)$.*

Note that this theorem also implies that the original logic $\mathbf{FPT}$ is closed under those classical substitutions σ such that $\sigma(\phi)$ is a well-formed formula of $\mathbf{FPT}$, whenever ϕ

is. In this sense, Theorem 4.3 also characterizes the behavior of substitutions in the original logic **FPT**.

For the proof of Theorem 4.3, we first establish the following lemma, which generalizes Lemma 3.5 in [20] concerning flat substitutions (note that classical substitutions are also flat substitutions). For any valuation s and any substitution σ , define a valuation s_σ as

$$s_\sigma(i) = \begin{cases} 1 & \text{if } \{s\} \models \sigma(p_i); \\ 0 & \text{if } \{s\} \not\models \sigma(p_i). \end{cases}$$

For any team X , we write $X_\sigma = \{s_\sigma \mid s \in X\}$.

Lemma 4.4. *For any formula ϕ in the language of $\mathbb{MPT}$ and any classical substitution σ , we have*

$$X \models \sigma(\phi) \iff X_\sigma \models \phi.$$

Proof. We prove the lemma by induction on ϕ . The case $\phi = \{\text{NE}, \perp, \top\}$ is trivial. Case $\phi = p_i$. Since $\sigma(p_i)$ has the flatness property, we have

$$X \models \sigma(p_i) \iff \forall s \in X (\{s\} \models \sigma(p_i)) \iff \forall s_\sigma \in X_\sigma (\{s_\sigma\} \models p_i) \iff X_\sigma \models p_i.$$

Case $\phi = \vec{\alpha} \perp \vec{\beta}$, where $\vec{\alpha}$ and $\vec{\beta}$ are sequences of classical formulas. To show the direction “ $\implies$ ”, assume $X \models \sigma(\vec{\alpha}) \perp \sigma(\vec{\beta})$. For any $s_\sigma, s'_\sigma \in X_\sigma$, we have $s, s' \in X$ and there is $s'' \in X$ such that $s \sim_{\sigma(\vec{\alpha})} s''$ and $s' \sim_{\sigma(\vec{\beta})} s''$. By the induction hypothesis, we have $s_\sigma \sim_{\vec{\alpha}} s''_\sigma$ and $s'_\sigma \sim_{\vec{\beta}} s''_\sigma$, as required. The other direction “ $\impliedby$ ” is proved analogously.

The cases $\phi = \neg(\vec{\alpha}, \beta)$ and $\phi = \vec{\alpha} \subseteq \vec{\beta}$, where $\vec{\alpha}, \vec{\beta}, \beta$ are (sequences of) classical formulas, are proved analogously.

Case $\phi = \psi \otimes \chi$. For the direction “ $\implies$ ”, assuming $X \models \sigma(\psi) \otimes \sigma(\chi)$, if $X = \emptyset$, then $X_\sigma = \emptyset \models \psi \otimes \chi$. If $X \neq \emptyset$, then there are nonempty sets $Y, Z \subseteq X$ such that $X = Y \cup Z$ and $Y \models \sigma(\psi)$ and $Z \models \sigma(\chi)$. Since $Y_\sigma \cup Z_\sigma = X_\sigma$ and $Y_\sigma, Z_\sigma \neq \emptyset$, we obtain $X_\sigma \models \psi \otimes \chi$ by the induction hypothesis. The other direction “ $\impliedby$ ” is proved analogously.

The case $\phi = \psi \otimes \chi$ is proved analogously. The cases $\phi = \neg\alpha$ for α classical, $\phi = \psi \wedge \chi$ and $\phi = \psi \vee \chi$ follow readily from the induction hypothesis. $\square$

Finally, we give the proof of Theorem 4.3.

Proof of Theorem 4.3. If $\Gamma \models \phi$, then for any team X ,

$$\begin{aligned} X \models \sigma(\gamma) \text{ for all } \gamma \in \Gamma &\implies X_\sigma \models \gamma \text{ for all } \gamma \in \Gamma \quad (\text{by Lemma 4.4}) \\ &\implies X_\sigma \models \phi \quad (\text{by the assumption}) \\ &\implies X \models \sigma(\phi) \quad (\text{by Lemma 4.4}) \end{aligned}$$

Hence $\{\sigma(\gamma) \mid \gamma \in \Gamma\} \models \sigma(\phi)$. $\square$

5. Axiomatizations

In this section, we study the axiomatization problem of propositional team logics. For a set of n propositional variables, there are in total 2^{2^n} teams. Therefore propositional team logics are clearly *decidable*. Concrete axiomatizations for propositional logics of dependence can be found in [34]. In this section, we give natural deduction systems and prove the Completeness Theorem for the logics $\mathbf{PT}^+$, $\mathbf{CPL}^+$, $\mathbf{PI}^+$ and $\mathbf{PInc}^+$. Among the propositional team logics we have defined, these are the logics that have the non-emptiness NE but not the nonempty disjunction $\otimes$ in their languages. The problem of finding (nontrivial) axiomatizations for the other logics, especially for propositional independence logic $\mathbf{PI}$, is open.

5.1. $\mathbf{PT}^+$

In this subsection, we define a natural deduction system of strong propositional team logic ($\mathbf{PT}^+$), an expressively strongest logic, and prove the Soundness and Completeness Theorems for it.

We first present our natural deduction system. We adopt the standard conventions of natural deduction systems (readers who are not familiar with natural deduction systems are referred to, e.g., [2, 30]). For example, the letter D (with or without subscripts) in the following definition stand for an arbitrary derivation (with or without extra assumptions). As discussed in Section 4.2, $\mathbf{PT}^+$ (and also the other propositional team logics to be axiomatized) are *not* closed under uniform substitution. Therefore, the (sound and complete) deduction system given by the definition below will *not* admit the *Substitution Rule*

$$\frac{\phi(p_{i_1}, \dots, p_{i_n})}{\phi(\sigma(p_{i_1})/p_{i_1}, \dots, \sigma(p_{i_n})/p_{i_n})} \text{ Sub}$$

In particular, the axioms and rules of the system presented below using concrete formulas such as p_i should *not* be read as schemata. Only the metalanguage symbols ϕ , ψ and χ stand for arbitrary formulas.

Definition 5.1 (A natural deduction system of $\mathbf{PT}^+$).

AXIOMS

Atomic excluded middle	NE introduction
$\frac{}{p_i \otimes \neg p_i} \text{EM}_0$	$\frac{}{\perp \vee \text{NE}} \text{NEI}$

RULES

Conjunction introduction	Conjunction elimination
$\frac{D_1 \quad D_2}{\phi \wedge \psi} \wedge I$	$\frac{D}{\phi \wedge \psi} \wedge E$ $\frac{D}{\psi} \wedge E$

Boolean disjunction introduction $\frac{D}{\phi} \quad \frac{D}{\psi} \quad \frac{}{\phi \vee \psi} \vee I$	Boolean disjunction elimination $\frac{D_0 \quad \frac{[\phi]}{D_1} \quad \frac{[\psi]}{D_2}}{\phi \vee \psi \quad \chi \quad \chi} \vee E$
Tensor disjunction weak introduction $\frac{D}{\phi} \quad \frac{}{\phi \otimes \psi} (*) \otimes I^-$ (*) whenever ψ dose not contain NE	Tensor disjunction weakening $\frac{D}{\phi} \quad \frac{}{\phi \otimes \phi} \otimes W$
Tensor disjunction weak elimination $\frac{D_0 \quad \frac{[\phi]}{D_1} \quad \frac{[\psi]}{D_2}}{\phi \otimes \psi \quad \alpha \quad \alpha} (*) \otimes E^-$ (*) whenever α is a classical formula and the undischarged assumptions in the derivations D_1 and D_2 contain classical formulas only	Tensor disjunction weak substitution $\frac{D_1 \quad \frac{[\psi]}{D_2}}{\phi \otimes \psi \quad \chi} (*) \otimes Sub^-$ (*) whenever the undischarged assumptions in the derivation D_2 contain classical formulas only
Commutative and associative laws for tensor disjunction $\frac{D}{\phi \otimes \psi} \quad \frac{}{\psi \otimes \phi} Com \otimes$	Commutative and associative laws for tensor disjunction $\frac{D}{\phi \otimes (\psi \otimes \chi)} \quad \frac{}{(\phi \otimes \psi) \otimes \chi} Ass \otimes$
Contradiction introduction $\frac{D}{p_i \wedge \neg p_i} \perp I$	Contradiction elimination $\frac{D}{\phi \otimes \perp} \perp E$
Strong <i>ex falso</i> $\frac{D}{\perp \wedge NE} \quad \frac{}{\phi} ex\ falso^+$	Strong contradiction contraction $\frac{D}{\phi \otimes (\perp \wedge NE)} \quad \frac{}{\perp \wedge NE} \mathbf{0}Ctr$

Strong contradiction introduction $\frac{D \quad \left(\bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \text{NE}) \right) \wedge \left(\bigotimes_{s \in Y} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \text{NE}) \right)}{\perp \wedge \text{NE}} \quad (*) \text{ 0I}$ (*) whenever X and Y are distinct teams on $\{i_1, \dots, i_n\}$	
Distributive laws $\frac{D \quad \phi \otimes (\psi \vee \chi)}{(\phi \otimes \psi) \vee (\phi \otimes \chi)} \text{Dstr } \otimes \vee$ $\frac{D \quad \text{NE} \wedge \bigotimes_{i \in I} \phi_i}{\bigvee_{\emptyset \neq J \subseteq I} \bigotimes_{i \in J} (\text{NE} \wedge \phi_i)} \text{Dstr NE} \wedge \otimes$	

The above deduction system consists of two axioms and fifteen sets of rules. The atomic excluded middle axiom EM_0 is *not* an axiom schema. Especially, as discussed in the previous section, the substitution instances of $\neg p_i$ are not necessarily well-formed formulas of $\mathbf{PT}^+$. The conjunction $\wedge$ and the Boolean disjunction $\vee$ have the usual introduction and elimination rules. The usual commutative law, associative law and distributive laws for the two connectives can be derived easily in the system. Over classical formulas the tensor disjunction admits the usual Introduction Rule and Elimination Rule. Over non-classical formulas the rules for the tensor disjunction $\otimes$ are more subtle. The Introduction Rule $\otimes \text{I}^-$ is not in general sound in case ψ does not have the empty team property. For instance, we have $\perp \not\models \perp \otimes \text{NE}$. But the weakening rule $\otimes \text{W}$ is sound and we will apply this rule later on in our completeness proof. The elimination rule $\otimes \text{E}^-$ is not in general sound in case α is non-classical. For instance, we have $p_i \otimes \neg p_i \not\models p_i \vee \neg p_i$, even if $p_i \models p_i \vee \neg p_i$ and $\neg p_i \models p_i \vee \neg p_i$. The rules $\otimes \text{E}^-$ and $\otimes \text{Sub}^-$ have some side conditions concerning the undischarged assumptions in the sub-derivations. To see why these restrictions are necessary, note, for example, that for the rule $\otimes \text{E}^-$ we have

$$(\text{NE} \wedge p) \otimes (\text{NE} \wedge \neg p), p \otimes \neg p \not\models \perp,$$

$$\text{even if } (\text{NE} \wedge p) \otimes (\text{NE} \wedge \neg p), p \models \perp \text{ and } (\text{NE} \wedge p) \otimes (\text{NE} \wedge \neg p), \neg p \models \perp$$

and for the rule $\otimes \text{Sub}^-$ we have

$$\text{NE} \not\models \text{NE} \otimes (\text{NE} \wedge \perp), \text{ even if } \text{NE} \models \text{NE} \otimes \perp \text{ and } \text{NE}, \perp \models \text{NE} \wedge \perp.$$

In the absence of the usual (Strong) Introduction Rule and Elimination Rule for $\otimes$, we added the (weak) substitution, commutative and associative rules for $\otimes$ to the system. We also include some of the distributive laws that involve $\otimes$ in our system, and we derive the other sound distributive laws in the next proposition. The usual distributive laws that are not listed in Definition 5.1 are not sound in our system, see [34] for examples of the failure of these laws. The non-emptiness NE, the weak and strong negation ($\perp$ and $\perp \wedge \text{NE}$) have self-explanatory rules in our system.

Proposition 5.2. *The following are derivable in the natural deduction system of $\mathbf{PT}^+$:*

- (i) *Weak ex falso (ex falso⁻): If ϕ does not contain NE, then $\perp \vdash \phi$.*
- (ii) *The usual commutative law, associative law and distributive law for conjunction and Boolean disjunction.*
- (iii) *Distributive laws:*
 - (a) $\phi \otimes (\psi \wedge \chi) \vdash (\phi \otimes \psi) \wedge (\phi \otimes \chi)$ ($Dstr \otimes \wedge$)
 - (b) $\phi \vee (\psi \otimes \chi) \vdash (\phi \vee \psi) \otimes (\phi \vee \chi)$ ($Dstr \vee \otimes$)
 - (c) $(\phi \otimes \psi) \vee (\phi \otimes \chi) \vdash \phi \otimes (\psi \vee \chi)$ ($Dstr \otimes \vee$)
 - (d) *If α is a classical formula, then*
 - $(\alpha \wedge \psi) \otimes (\alpha \wedge \chi) \vdash \alpha \wedge (\psi \otimes \chi)$ ($Dstr^* \wedge \otimes$)
 - $\alpha \wedge (\psi \otimes \chi) \vdash (\alpha \wedge \psi) \otimes (\alpha \wedge \chi)$ ($Dstr^* \wedge \otimes$)

Proof. The rules in item (ii) are derived as usual. The items (a), (c) and (d) are not hard to derive (or see the proof of Proposition 4.6 in [34]). We will only give the derivation for the other rules.

For ex falso⁻, assuming that ϕ does not contain NE, we have the following derivation

$$\frac{\frac{\perp}{\perp \otimes \phi} \otimes I^-}{\phi} \perp E$$

For $Dstr \vee \otimes$, we have the following derivation

$$\frac{\phi \vee (\psi \otimes \chi) \quad \frac{\frac{[\phi]}{\phi \otimes \phi} \otimes W}{(\phi \vee \psi) \otimes (\phi \vee \chi)} \vee I, \otimes Sub^- \quad \frac{[\psi \otimes \chi]}{(\phi \vee \psi) \otimes (\phi \vee \chi)} \vee I, \otimes Sub^-}{(\phi \vee \psi) \otimes (\phi \vee \chi)} \vee E$$

□

Next, we prove the Soundness Theorem for our deduction system.

Theorem 5.3 (Soundness Theorem). *For any set $\Gamma \cup \{\phi\}$ of formulas in the language of $\mathbf{PT}^+$, we have $\Gamma \vdash \phi \implies \Gamma \models \phi$.*

Proof. We show that $\Gamma \models \phi$ holds for each derivation $D = \{\delta_1, \dots, \delta_k\}$ with the conclusion ϕ and the hypotheses in Γ .

If D is a singleton, then $\phi \in \Gamma$ or $\phi = p_i \otimes \neg p_i$ or $\phi = \perp \vee NE$. In the first case, obviously $\{\phi\} \models \phi$. The last two cases are also easy because the two formulas are clearly valid.

The induction steps for the rules 1-4, 6, 9-12, 14 and the distributive rules $Dstr \otimes \vee$ and $Dstr \otimes \vee \otimes$ are easy to verify. We only check the induction step for the other rules.

$\otimes I^-$: Assume that D is a derivation for $\Pi \vdash \phi$. We show that $\Pi \models \phi \otimes \psi$ whenever ψ does not contain NE. Suppose $X \models \theta$ for all $\theta \in \Pi$. By the induction hypothesis, we have $\Pi \models \phi$, thus $X \models \phi$ follows. Now, since ψ does not contain NE, an easy

inductive proof shows it has the empty team property, i.e., $\emptyset \models \psi$. Hence we obtain $\emptyset \cup X \models \phi \otimes \psi$ as required.

$\otimes E^-$: Assume that D_0 , D_1 and D_2 are derivations for $\Pi_0 \vdash \phi \otimes \psi$, $\Pi_1, \phi \vdash \alpha$ and $\Pi_2, \psi \vdash \alpha$, respectively. Assume that α is a classical formula and Π_1 and Π_2 contain classical formulas only. We show that $\Pi_0, \Pi_1, \Pi_2 \models \alpha$ follows from the induction hypothesis $\Pi_0 \models \phi \otimes \psi$, $\Pi_1, \phi \models \alpha$ and $\Pi_2, \psi \models \alpha$. Suppose $X \models \theta$ for all $\theta \in \Pi_0 \cup \Pi_1 \cup \Pi_2$. Then we have $X \models \phi \otimes \psi$, which means that there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$. For all $\theta \in \Pi_1 \cup \Pi_2$, since θ is a classical formula, by Lemma 2.4 we know that θ has the downward closure property. It then follows that $Y \models \theta_1$ for all $\theta_1 \in \Pi_1$ and $Z \models \theta_2$ for all $\theta_2 \in \Pi_2$. Thus $Y \models \alpha$ and $Z \models \alpha$. Since α is a classical formula, by Lemma 2.4 we know that α has the union closure property, which yields that $X \models \alpha$.

$\otimes Sub^-$: Assume that D_1 and D_2 are derivations for $\Pi_1 \vdash \phi \otimes \psi$ and $\Pi_2, \psi \vdash \chi$, respectively and Π_2 contains classical formulas only. We show that $\Pi_1, \Pi_2 \models \phi \otimes \chi$ follows from the induction hypothesis $\Pi_1 \models \phi \otimes \psi$ and $\Pi_2, \psi \models \chi$. Suppose $X \models \theta$ for all $\theta \in \Pi_1 \cup \Pi_2$. By the induction hypothesis, we have $X \models \phi \otimes \psi$, which means there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$. For all $\theta \in \Pi_2$, since θ is a classical formula, by Lemma 2.4 we know that θ has the downward closure property. Thus $Z \models \theta$. It then follows from the induction hypothesis that $Z \models \chi$. Hence we conclude $Y \cup Z = X \models \phi \otimes \chi$.

0I: It suffices to show that if X and Y are distinct teams on $\{i_1, \dots, i_n\}$, then $Z \not\models \Theta_X^* \wedge \Theta_Y^*$ for all teams Z . In view of the locality property, we may assume that Z is a team on $\{i_1, \dots, i_n\}$. By Lemma 3.3, if $Z \models \Theta_X^* \wedge \Theta_Y^*$, then $X = Z = Y$, which contradicts the assumption.

Dstr * $\wedge \otimes$: It suffices to show that $\phi \wedge (\psi \otimes \chi) \models (\phi \wedge \psi) \otimes (\phi \wedge \chi)$, whenever ϕ does not contain NE. Suppose $X \models \phi \wedge (\psi \otimes \chi)$. Then $X \models \phi$, $Y \models \psi$ and $Z \models \chi$ for some teams $Y, Z \subseteq X$ with $Y \cup Z = X$. Since ϕ does not contain NE, an easy inductive proof shows that ϕ has the downward closure property. Thus $Y \models \phi$ and $Z \models \phi$. Hence $Y \models \phi \wedge \psi$ and $Z \models \phi \wedge \chi$, implying $X \models (\phi \wedge \psi) \otimes (\phi \wedge \chi)$.

Dstr NE $\wedge \otimes$: It suffices to show that if $X \models NE \wedge \bigotimes_{i \in I} \phi_i$ for some team X , then $X \models \bigotimes_{i \in J} (NE \wedge \phi_i)$ for some nonempty $J \subseteq I$. The assumption implies that $X \neq \emptyset$ and there exist teams $X_i \subseteq X$ for each $i \in I$ such that $\bigcup_{i \in I} X_i = X$ and $X_i \models \phi_i$. Let $J \subseteq I$ be the set of indices $i \in I$ such that $X_i \neq \emptyset$. Since $X \neq \emptyset$, we must have that $J \neq \emptyset$. Hence $X_i \models NE \wedge \phi_i$ for each $i \in J$ and $X \models \bigotimes_{i \in J} (NE \wedge \phi_i)$. $\square$

We now proceed to prove the main result of this section, the Completeness Theorem for our system. Our argument is similar to that of the proof of the Completeness Theorem for propositional logics of dependence in [34]. The reader may compare what follows with Section 4.2 in [34]. To begin with, below we present a crucial lemma that is very similar to Lemma 3.2 in [34].

Lemma 5.4. *For any finite nonempty collections of teams $\{X_f \mid f \in F\}$, $\{Y_g \mid g \in G\}$ on some fixed domain, the following are equivalent:*

$$(a) \bigvee_{f \in F} \Theta_{X_f}^* \models \bigvee_{g \in G} \Theta_{Y_g}^*;$$

(b) *for each $f \in F$, we have $X_f = Y_{g_f}$ for some $g \in G$.*

Proof. Follows easily from Lemma 3.3. $\square$

Two formulas ϕ and ψ are said to be *provably equivalent* (written $\phi \dashv\vdash \psi$) if both $\phi \vdash \psi$ and $\psi \vdash \phi$ hold. Another crucial step for the completeness proof is to establish that every formula is provably equivalent in the deduction system to a formula in the disjunctive normal form shown in Table 3. Let us now state this as a lemma.

Lemma 5.5. *Every formula $\phi(p_{i_1}, \dots, p_{i_n})$ in the language of $\mathbf{PT}^+$ is provably equivalent to a formula in the normal form*

$$\bigvee_{f \in F} \Theta_{X_f}^*, \text{ where } \Theta_{X_f}^* = \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \text{NE}), \quad (7)$$

$\{X_f \mid f \in F\}$ a finite set of teams on $N = \{i_1, \dots, i_n\}$.

The detailed proof of the above lemma will be postponed. We now give the proof of the Completeness Theorem for our system.

Theorem 5.6 (Completeness Theorem). *For any formulas ϕ and ψ in the language of $\mathbf{PT}^+$, we have $\psi \models \phi \implies \psi \vdash \phi$. In particular, $\models \phi \implies \vdash \phi$.*

Proof. Suppose $\psi \models \phi$, where $\phi = \phi(p_{i_1}, \dots, p_{i_n})$ and $\psi = \psi(p_{i_1}, \dots, p_{i_n})$. By Lemma 5.5, we have

$$\phi \dashv\vdash \bigvee_{f \in F} \Theta_{X_f}^* \quad \text{and} \quad \psi \dashv\vdash \bigvee_{g \in G} \Theta_{Y_g}^* \quad (8)$$

for some finite sets $\{X_f \mid f \in F\}$ and $\{Y_g \mid g \in G\}$ of teams on $\{i_1, \dots, i_n\}$. The Soundness Theorem and (8) imply that

$$\bigvee_{f \in F} \Theta_{X_f}^* \models \bigvee_{g \in G} \Theta_{Y_g}^*. \quad (9)$$

If $F = \emptyset$, then $\phi \dashv\vdash \perp \wedge \text{NE}$. We obtain $\phi \vdash \psi$ by ex falso⁺. If $G = \emptyset$, then $\psi \dashv\vdash \perp \wedge \text{NE}$. In view of (9), we must have $F = \emptyset$ as well and $\phi \dashv\vdash \perp \wedge \text{NE}$. Hence $\phi \vdash \psi$.

If $F, G \neq \emptyset$, then by Lemma 5.4, for each $f \in F$ we have $X_f = Y_{g_f}$ and $\Theta_{X_f}^* = \Theta_{Y_{g_f}}^*$ for some $g_f \in G$, which implies that $\Theta_{X_f}^* \vdash \bigvee_{g \in G} \Theta_{Y_g}^*$ by $\vee\text{I}$. Hence, we obtain $\bigvee_{f \in F} \Theta_{X_f}^* \vdash \bigvee_{g \in G} \Theta_{Y_g}^*$ by applying $\vee\text{E}$. Finally, in view of (8) we conclude that $\phi \vdash \psi$. $\square$

Theorem 5.7 (Strong Completeness Theorem). *For any set $\Gamma \cup \{\phi\}$ of formulas in the language of $\mathbf{PT}^+$, we have $\Gamma \models \phi \implies \Gamma \vdash \phi$.*

Proof. By Theorem 5.6 and the Compactness Theorem (Theorem 4.1). $\square$

We end this section by supplying the proof of Lemma 5.5.

Proof of Lemma 5.5. Note that in the statement of the lemma we have fixed a set $\{p_{i_1}, \dots, p_{i_n}\}$ of variables. These variables all occur in the formula (7) in normal form, but not necessarily all of them actually occur in the formula $\phi(p_{i_1}, \dots, p_{i_n})$. In order to take care of this subtle point we first prove the following claim:

Claim. If $\{i_1, \dots, i_m\} \subset \{j_1, \dots, j_k\}$, then any formula $\psi(p_{i_1}, \dots, p_{i_m})$ in the normal form is provably equivalent to a formula $\theta(p_{j_1}, \dots, p_{j_k})$ in the normal form.

Proof of the claim. Without loss of generality we may assume that $K = \{j_1, \dots, j_k\} = \{i_1, \dots, i_m, i_{m+1}, \dots, i_k\}$ and $k > m$. By the assumption, we have

$$\psi(p_{i_1}, \dots, p_{i_m}) = \bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge \text{NE}),$$

where $\{X_f \mid f \in F\}$ is a finite set of teams on $M = \{i_1, \dots, i_m\}$. Let

$$\theta(p_{i_1}, \dots, p_{i_k}) = \bigvee_{f \in F} \bigvee_{\substack{Y \subseteq 2^K \\ Y \upharpoonright M = X_f}} \bigotimes_{s \in Y} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_k}^{s(i_k)} \wedge \text{NE}).$$

The following derivation proves $\theta \vdash \psi$:

$$\begin{aligned} (1) \quad & \bigvee_{f \in F} \bigvee_{\substack{Y \subseteq 2^K \\ Y \upharpoonright M = X_f}} \bigotimes_{s \in Y} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_k}^{s(i_k)} \wedge \text{NE}) \\ (2) \quad & \bigvee_{f \in F} \bigvee_{\substack{Y \subseteq 2^K \\ Y \upharpoonright M = X_f}} \bigotimes_{s \in Y} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge \text{NE}) \quad (\wedge E, \otimes \text{Sub}^-) \\ (3) \quad & \bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge \text{NE}) \quad (\vee E) \end{aligned}$$

Conversely, $\psi \vdash \theta$ is proved by the following derivation:

$$\begin{aligned} (1) \quad & \bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge \text{NE}) \\ (2) \quad & (p_{i_{m+1}} \otimes \neg p_{i_{m+1}}) \wedge \dots \wedge (p_{i_k} \otimes \neg p_{i_k}) \quad (\text{EM}_0, \wedge I) \\ (3) \quad & \bigotimes_{t \in 2^{K \setminus M}} (p_{i_{m+1}}^{t(i_{m+1})} \wedge \dots \wedge p_{i_k}^{t(i_k)}) \quad ((2), \text{Dstr}^* \wedge \otimes) \\ (4) \quad & \left(\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge \text{NE}) \right) \wedge \left(\bigotimes_{t \in 2^{K \setminus M}} (p_{i_{m+1}}^{t(i_{m+1})} \wedge \dots \wedge p_{i_k}^{t(i_k)}) \right) \\ & \quad ((1), (3), \wedge I) \\ (5) \quad & \bigvee_{f \in F} \bigotimes_{s \in X_f} \left(p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge (\text{NE} \wedge \bigotimes_{t \in 2^{K \setminus M}} (p_{i_{m+1}}^{t(i_{m+1})} \wedge \dots \wedge p_{i_k}^{t(i_k)})) \right) \\ & \quad (\text{Dstr}^* \wedge \otimes) \\ (6) \quad & \bigvee_{f \in F} \bigotimes_{s \in X_f} \left(p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge \bigvee_{\substack{\emptyset \neq Z \subseteq 2^{K \setminus M} \\ t \in Z}} \bigotimes_{t \in Z} (p_{i_{m+1}}^{t(i_{m+1})} \wedge \dots \wedge p_{i_k}^{t(i_k)} \wedge \text{NE}) \right) \\ & \quad (\text{Dstr NE} \wedge \otimes, \otimes \text{Sub}^-) \\ (7) \quad & \bigvee_{f \in F} \bigotimes_{s \in X_f} \bigvee_{\emptyset \neq Z \subseteq 2^{K \setminus M}} \left(p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge \bigotimes_{t \in Z} (p_{i_{m+1}}^{t(i_{m+1})} \wedge \dots \wedge p_{i_k}^{t(i_k)} \wedge \text{NE}) \right) \\ (8) \quad & \bigvee_{f \in F} \bigotimes_{s \in X_f} \bigvee_{\emptyset \neq Z \subseteq 2^{N \setminus M}} \bigotimes_{t \in Z} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge p_{i_{m+1}}^{t(i_{m+1})} \wedge \dots \wedge p_{i_k}^{t(i_k)} \wedge \text{NE}) \\ & \quad (\text{Dstr}^* \wedge \otimes) \end{aligned}$$

$$\begin{aligned}
(9) \quad & \bigvee_{f \in F} \bigvee_{G: X_f \rightarrow Z} \bigotimes_{s \in X_f} \bigotimes_{t \in G(s)} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_m}^{s(i_m)} \wedge p_{i_{m+1}}^{t(i_{m+1})} \wedge \dots \wedge p_{i_k}^{t(i_k)} \wedge \text{NE}) \\
& \text{where } Z = \{Z \subseteq 2^{K \setminus M} \mid Z \neq \emptyset\} \quad (\text{Dstr} \otimes \vee) \\
(10) \quad & \bigvee_{f \in F} \bigvee_{Y \subseteq 2^K} \bigotimes_{s \in Y} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_k}^{s(i_k)} \wedge \text{NE}) \quad (\text{since } \text{dom}(X_f) = M) \\
& Y \upharpoonright M = X_f
\end{aligned}$$

⊢

We now prove Lemma 5.5 by induction on $\phi(p_{i_1}, \dots, p_{i_n})$.

Case $\phi(p_{i_1}, \dots, p_{i_n}) = p_{i_k}$. We prove that $p_{i_k} \dashv\vdash \perp \vee (p_{i_k} \wedge \text{NE})$. For $p_{i_k} \vdash \perp \vee (p_{i_k} \wedge \text{NE})$, we have the following derivation:

$$\begin{aligned}
(1) \quad & p_{i_k} \\
(2) \quad & \perp \vee \text{NE} \quad (\text{NEI}) \\
(3) \quad & (p_{i_k} \wedge \perp) \vee (p_{i_k} \wedge \text{NE}) \quad ((1), (2), \wedge I, \text{Dstr}) \\
(4) \quad & \perp \vee (p_{i_k} \wedge \text{NE}) \quad (\wedge E)
\end{aligned}$$

Conversely, for $\perp \vee (p_{i_k} \wedge \text{NE}) \vdash p_{i_k}$, we have the following derivation

$$\begin{aligned}
(1) \quad & \perp \vee (p_{i_k} \wedge \text{NE}) \\
(2) \quad & p_{i_k} \vee p_{i_k} \quad (\text{ex falso}^-, \vee E) \\
(3) \quad & p_{i_k} \quad (\vee E)
\end{aligned}$$

By the Claim, the formula p_{i_k} is provably equivalent to a formula $\theta(p_{i_1}, \dots, p_{i_n})$ in the normal form.

Case $\phi(p_{i_1}, \dots, p_{i_n}) = \neg p_{i_k}$. Similar to the above case.

Case $\phi(p_{i_1}, \dots, p_{i_n}) = \text{NE}$. Note that NE is a formula with no propositional variable, but for the sake of the inductive proof, we need to prove the theorem for NE viewed as $\text{NE}(p_{i_1}, \dots, p_{i_n})$, a formula whose propositional variables are among $p_{i_1}, \dots, p_{i_n}$. By the claim, it suffices to derive the normal form for NE when it is viewed as $\text{NE}(p_{i_1})$. We prove that $\text{NE} \dashv\vdash \theta$, where

$$\theta := (p_{i_1} \wedge \text{NE}) \vee (\neg p_{i_1} \wedge \text{NE}) \vee ((p_{i_1} \wedge \text{NE}) \otimes (\neg p_{i_1} \wedge \text{NE})).$$

For $\text{NE} \vdash \theta$, we have the following derivation:

$$\begin{aligned}
(1) \quad & \text{NE} \\
(2) \quad & \text{NE} \wedge (p_{i_1} \otimes \neg p_{i_1}) \quad (\text{EM}_0, \wedge I) \\
(3) \quad & (p_{i_1} \wedge \text{NE}) \vee (\neg p_{i_1} \wedge \text{NE}) \vee ((p_{i_1} \wedge \text{NE}) \otimes (\neg p_{i_1} \wedge \text{NE})) \quad (\text{Dstr NE} \wedge \otimes)
\end{aligned}$$

For the other direction $\theta \vdash \text{NE}$, we have the following derivation:

$$\begin{aligned}
(1) \quad & (p_{i_1} \wedge \text{NE}) \vee (\neg p_{i_1} \wedge \text{NE}) \vee ((p_{i_1} \wedge \text{NE}) \otimes (\neg p_{i_1} \wedge \text{NE})) \\
(2) \quad & (p_{i_1} \wedge \text{NE}) \vee (\neg p_{i_1} \wedge \text{NE}) \vee ((p_{i_1} \otimes \neg p_{i_1}) \wedge \text{NE}) \quad (\text{Dstr}^* \wedge \otimes \wedge) \\
(3) \quad & \text{NE} \vee \text{NE} \vee \text{NE} \quad (\wedge E) \\
(4) \quad & \text{NE} \quad (\vee E)
\end{aligned}$$

Case $\phi(p_{i_1}, \dots, p_{i_n}) = \perp$. Trivially $\perp \Vdash \Theta_\emptyset^* = \perp$.

Case $\phi(p_{i_1}, \dots, p_{i_n}) = \psi(p_{i_1}, \dots, p_{i_n}) \vee \chi(p_{i_1}, \dots, p_{i_n})$. By the induction hypothesis, we have

$$\psi \Vdash \bigvee_{f \in F} \Theta_{X_f}^* \text{ and } \chi \Vdash \bigvee_{g \in G} \Theta_{X_g}^*, \quad (10)$$

where each $X_f, X_g \subseteq 2^N$. Then it follows from the rules $\vee E$ and $\vee I$ that

$$\psi \vee \chi \Vdash \bigvee_{f \in F} \Theta_{X_f}^* \vee \bigvee_{g \in G} \Theta_{X_g}^*.$$

If $\psi \Vdash \perp \wedge NE$ (i.e., $F = \emptyset$), then we obtain further by ex falso^+ , $\vee E$ and $\vee I$ that $\psi \vee \chi \Vdash \bigvee_{g \in G} \Theta_{X_g}^*$. Similarly for the case $\chi \Vdash \perp \wedge NE$.

Case $\phi(p_{i_1}, \dots, p_{i_n}) = \psi(p_{i_1}, \dots, p_{i_n}) \otimes \chi(p_{i_1}, \dots, p_{i_n})$. By the induction hypothesis, we have (10). If $\psi \Vdash \perp \wedge NE$ (i.e., $F = \emptyset$), then we derive $\psi \otimes \chi \Vdash \perp \wedge NE = \bigvee \emptyset$ by $(\mathbf{0Ctr})$ and (ex falso^+) . Similarly for the case $\chi \Vdash \perp \wedge NE$ (i.e., $G = \emptyset$).

Now, assume $F, G \neq \emptyset$. We show that $\psi \otimes \chi \Vdash \theta$, where

$$\theta := \bigvee_{f \in F} \bigvee_{g \in G} \Theta_{X_f \cup X_g}^*.$$

For the direction $\psi \otimes \chi \vdash \theta$, we have the following derivation:

$$\begin{aligned} (1) & \quad \psi \otimes \chi \\ (2) & \quad \left(\bigvee_{f \in F} \Theta_{X_f}^* \right) \otimes \left(\bigvee_{g \in G} \Theta_{X_g}^* \right) \\ (3) & \quad \bigvee_{f \in F} \bigvee_{g \in G} \left(\Theta_{X_f}^* \otimes \Theta_{X_g}^* \right) \quad (\text{Dstr } \otimes \vee) \\ (4) & \quad \bigvee_{f \in F} \bigvee_{g \in G} \Theta_{X_f \cup X_g}^* \quad (\otimes E^-, \otimes \text{Sub}^-) \end{aligned}$$

The other direction $\theta \vdash \psi \otimes \chi$ is proved similarly using $\otimes W$ and $\text{Dstr } \otimes \vee \otimes$.

Case $\phi(p_{i_1}, \dots, p_{i_n}) = \psi(p_{i_1}, \dots, p_{i_n}) \wedge \chi(p_{i_1}, \dots, p_{i_n})$. By the induction hypothesis, we have (10). If $\psi \Vdash \perp \wedge NE$ (i.e., $F = \emptyset$), then we derive $\psi \wedge \chi \Vdash \perp \wedge NE = \bigvee \emptyset$ by $(\wedge E)$ and (ex falso^+) . Similarly for the case $\chi \Vdash \perp \wedge NE$ (i.e., $G = \emptyset$).

Now, assume $F, G \neq \emptyset$. We show that $\psi \wedge \chi \Vdash \theta$, where

$$\theta := \bigvee_{h \in H} \Theta_{X_h}^* \text{ and } \{X_f \mid f \in F\} \cap \{X_g \mid g \in G\} = \{X_h \mid h \in H\}.$$

For $\psi \wedge \chi \vdash \theta$, we have the following derivation:

$$\begin{aligned} (1) & \quad \psi \wedge \chi \\ (2) & \quad \left(\bigvee_{f \in F} \Theta_{X_f}^* \right) \wedge \left(\bigvee_{g \in G} \Theta_{X_g}^* \right) \\ (3) & \quad \bigvee_{f \in F} \bigvee_{g \in G} \left(\Theta_{X_f}^* \wedge \Theta_{X_g}^* \right) \\ (4) & \quad \left(\bigvee_{\substack{(f,g) \in F \times G \\ X_f \neq X_g}} \left(\Theta_{X_f}^* \wedge \Theta_{X_g}^* \right) \right) \vee \left(\bigvee_{\substack{(f,g) \in F \times G \\ X_f = X_g}} \left(\Theta_{X_f}^* \wedge \Theta_{X_g}^* \right) \right) \end{aligned}$$

$$\begin{aligned}
(5) \quad & (\perp \wedge \text{NE}) \vee \bigvee_{\substack{(f,g) \in F \times G \\ X_f = X_g}} (\Theta_{X_f}^* \wedge \Theta_{X_g}^*) \quad (\mathbf{0I}) \\
(6) \quad & \bigvee_{\substack{(f,g) \in F \times G \\ X_f = X_g}} (\Theta_{X_f}^* \wedge \Theta_{X_g}^*) \quad (\mathbf{0E}) \\
(7) \quad & \bigvee_{h \in H} \Theta_{X_h}^* \quad (\wedge E, \vee E)
\end{aligned}$$

For the other direction $\theta \vdash \psi \wedge \chi$, we have the following derivation:

$$\begin{aligned}
(1) \quad & \bigvee_{h \in H} \Theta_{X_h}^* \\
(2) \quad & \left(\bigvee_{h \in H} \Theta_{X_h}^* \right) \wedge \left(\bigvee_{h \in H} \Theta_{X_h}^* \right) \quad (\wedge I) \\
(3) \quad & \left(\bigvee_{f \in F} \Theta_{X_f}^* \right) \wedge \left(\bigvee_{g \in G} \Theta_{X_g}^* \right) \quad (\vee I, H \subseteq F, G) \\
(4) \quad & \psi \wedge \chi
\end{aligned}$$

□

5.2. $\mathbf{CPL}^+$

We will give a complete axiomatization of $\mathbf{CPL}^+$ in the style of natural deduction. For this end, let us review our proof above of the Completeness Theorem for $\mathbf{PT}^+$. In a crucial step of the proof we transformed a formula into its disjunctive normal form $\bigvee_{f \in F} \Theta_{X_f}^*$. Each disjunct $\Theta_{X_f}^*$ is a formula in the language of $\mathbf{CPL}^+$, but $\mathbf{CPL}^+$ is a fragment of $\mathbf{PT}^+$ that does not have the Boolean disjunction $\vee$ in the language, so we seem to be in trouble. Our trick is that we view the set $\{\Theta_{X_f}^* \mid f \in F\}$ of formulas, rather than the disjunction of this set, as a *weak* normal form for formulas in the language of $\mathbf{CPL}^+$. On the basis of this plan, we can axiomatize $\mathbf{CPL}^+$ and prove the completeness theorem.

We will define a natural deduction system of $\mathbf{CPL}^+$ in which every formula ϕ is essentially provably equivalent to its disjunctive normal form $\bigvee_{f \in F} \Theta_{X_f}^*$. In particular, we will be able to essentially derive the provable equivalence between the non-emptiness NE and its disjunctive normal form $\bigvee_{\emptyset \neq Y \subseteq 2^N} \Theta_Y^*$. The behavior of the usual Introduction Rule and Elimination Rule of the Boolean disjunction $\vee$ ($\vee I$ and $\vee E$) will be simulated by two Strong Elimination Rules (SE_1 and SE_2) that do not involve $\vee$.

To define the Strong Elimination Rules, we will need to specify a particular occurrence of a subformula inside a formula. For this purpose, we identify a formula in the language of $\mathbf{CPL}^+$ with a finite string of symbols. A propositional variable p_i and the non-emptiness NE are symbols and the other symbols are $\wedge, \otimes, \neg$. Starting from the leftmost symbol, we number each symbol in a formula with a positive integer, as in the following example:

$$\begin{array}{cccccccc}
\text{NE} & \otimes & (& \neg & p_1 & \wedge & \text{NE} &) \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

If the m th symbol of a formula ϕ starts a string ψ which is a subformula of ϕ , we denote the subformula $[\psi, m]_\phi$, or simply $[\psi, m]$. We will sometimes refer to an occurrence of a formula χ inside a subformula ψ of ϕ . In this case we will use the same counting for the subformula ψ , rather than restart the counting from 1. We write $\phi(\beta/[\alpha, m])$ for the formula obtained from ϕ by replacing the occurrence of the subformula $[\alpha, m]$ by β . For example, for the formula $\phi = \text{NE} \otimes (\neg p_1 \wedge \text{NE})$, the second occurrence of the non-emptiness NE is denoted by $[\text{NE}, 7]$, and the same notation also designates the occurrence of NE inside the subformula $\neg p_1 \otimes \text{NE}$. The notation $\phi(\psi/[\text{NE}, 7])$ designates the formula $\text{NE} \otimes (\neg p_1 \wedge \psi)$.

Below we present the natural deduction system of **CPL**⁺.

Definition 5.8 (A natural deduction system of **CPL**⁺).

AXIOM

Atomic excluded middle

$$\frac{}{p_i \otimes \neg p_i} \text{EM}_0$$

RULES

All of the rules in Definition 5.1 that do not involve Boolean disjunction, i.e., the rules $\wedge I$, $\wedge E$, $\otimes I^-$, $\otimes W$, $\otimes E^-$, $\otimes \text{Sub}^-$, $\text{Com} \otimes$, $\text{Ass} \otimes$, $\perp I$, $\perp E$, ex falso^+ , **0I**, **0Ctr**, $\text{Dstr}^* \wedge \otimes$.

Strong elimination rules

$$\begin{array}{c} [\phi(\Theta_{Y_1}^*/[\text{NE}, m])] \quad [\phi(\Theta_{Y_k}^*/[\text{NE}, m])] \\ D_0 \quad D_1 \quad \dots \quad D_k \\ \phi \quad \theta \quad \dots \quad \theta \\ \hline \theta \end{array} \text{SE}_1$$

where $\{Y_1, \dots, Y_k\}$ is the set of all nonempty teams on a set N of indices

$$\begin{array}{c} [\phi(\psi \wedge \perp/[\psi, m])] \quad [\phi(\psi \wedge \text{NE}/[\psi, m])] \\ D_0 \quad D_1 \quad D_2 \\ \phi \quad \theta \quad \theta \\ \hline \theta \end{array} \text{SE}_2$$

All rules that do not involve the Boolean disjunction $\vee$ in the natural deduction system of **PT**⁺ (Definition 5.1) are included in the above system. Thus all clauses in Proposition 5.2 that do not involve $\vee$ are also derivable in the above system.

Let us ponder why we define the Strong Elimination Rules the way they are in our system in the absence of the Boolean disjunction $\vee$. The idea of the elimination rules

for the conjunction is simply that if we have inferred $\phi \wedge \psi$, we can infer both ϕ and ψ . The elimination rule for disjunction in classical logic is that if we have $\phi \vee \psi$ and we can derive θ separately from both ϕ and ψ then we have θ . In both cases the elimination rule builds into the syntax of the proof the semantics of the logical operation. This is roughly the general idea of natural deduction, due to Gentzen. We followed the same line of thinking when we introduce our rules for Boolean disjunction $\vee$ in the system for $\mathbf{PT}^+$. Now, we moved to the weak logic $\mathbf{CPL}^+$ which does not have Boolean disjunction in the language. But still, the non-emptiness NE is semantically equivalent to the formula $\bigvee_{i=1}^k \Theta_{Y_i}^*$ (in the language of $\mathbf{PT}^+$), where $\{Y_1, \dots, Y_k\}$ is the set of all nonempty teams on a set N of indices. To derive a formula θ from NE given some assumptions, in $\mathbf{PT}^+$ in the presence of the Boolean disjunction we could build up the following derivation:

$$\frac{\begin{array}{c} D_0 \\ \hline \text{NE} \\ \hline \bigvee_{i=1}^k \Theta_{Y_i}^* \end{array} \quad \begin{array}{c} [\Theta_{Y_1}^*] \\ D_1 \\ \hline \theta \end{array} \quad \dots \quad \begin{array}{c} [\Theta_{Y_k}^*] \\ D_k \\ \hline \theta \end{array}}{\theta}$$

Evidently such a derivation can be simulated in our deduction system of $\mathbf{CPL}^+$ using the rule SE_1 . More generally, a formula ϕ whose n th symbol is NE is semantically equivalent to the formula $\phi(\bigvee_{i=1}^k \Theta_{Y_i}^* / [\text{NE}, m])$ (in the language of $\mathbf{PT}^+$), which, as $\vee$ distributes over all connectives, is semantically equivalent to $\bigvee_{i=1}^k \phi(\Theta_{Y_i}^* / [\text{NE}, m])$. In the same way, to derive a formula θ from ϕ given some assumptions, it suffices to derive θ from each $\phi(\Theta_{Y_i}^* / [\text{NE}, m])$. This is exactly what the Strong Elimination Rule SE_1 characterizes. Analogously, the rule SE_2 characterizes the equivalence between a formula ϕ and $\phi(\psi \wedge (\perp \vee \text{NE}) / [\psi, m])$.

We now prove the Soundness Theorem for the deduction system of $\mathbf{CPL}^+$.

Theorem 5.9 (Soundness Theorem). *For any set $\Gamma \cup \{\phi\}$ of formulas in the language of $\mathbf{CPL}^+$, we have $\Gamma \vdash \phi \implies \Gamma \models \phi$.*

Proof. We show that for each derivation D with the conclusion ϕ and the hypotheses in Γ we have $\Gamma \models \phi$. We only verify the cases where the Strong Elimination Rules are applied. The other cases follow from the Soundness Theorem for $\mathbf{PT}^+$.

SE_1 : Put $\phi_i^* = \phi(\Theta_{Y_i}^* / [\text{NE}, m])$ for each $i \in \{1, \dots, k\}$. Assume that $D_0, D_1, \dots, D_k$ are derivations for $\Pi_0 \vdash \phi$, $\Pi_1, \phi_1^* \vdash \theta$, ..., $\Pi_k, \phi_k^* \vdash \theta$, respectively. We show that $\Pi_0, \Pi_1, \dots, \Pi_k \models \theta$ follows from the induction hypothesis $\Pi_0 \models \phi$, $\Pi_1, \phi_1^* \models \theta$, ..., $\Pi_k, \phi_k^* \models \theta$. This reduces to showing that $\phi \models \phi_1^* \vee \dots \vee \phi_k^*$ by induction on the subformulas ψ of ϕ .

Case $\psi = \text{NE}$. By the locality property, $X \models \text{NE} \iff X \upharpoonright N \models \text{NE}$ for any team X . Now, since $\{Y_1, \dots, Y_k\}$ is the set of all nonempty teams on N , we have $X \upharpoonright N \models \text{NE} \iff X \upharpoonright N \models \Theta_{Y_i}^*$ for some $i \in \{1, \dots, k\}$. Hence $\text{NE} \models \Theta_{Y_1}^* \vee \dots \vee \Theta_{Y_k}^*$.

If ψ is p_j or $\neg p_j$, then $\psi_i^* = \psi$ for each $i \in \{1, \dots, k\}$ and $\psi \models \psi_1^* \vee \dots \vee \psi_k^*$ holds trivially.

If $\psi = \theta \otimes \chi$ and without loss of generality we assume that the occurrence of the NE is in the subformula θ . Then by the induction hypothesis we have $\theta \models \theta_1^* \vee \dots \vee \theta_k^*$. Thus $\theta \otimes \chi \models (\theta_1^* \vee \dots \vee \theta_k^*) \otimes \chi \models (\theta_1^* \otimes \chi) \vee \dots \vee (\theta_k^* \otimes \chi)$.

The case $\psi = \theta \wedge \chi$ is proved similarly.

SE₂: Put $\phi_+^* = \phi(\psi \wedge \text{NE} / [\psi, m])$ and $\phi_-^* = \phi(\psi \wedge \perp / [\psi, m])$. Assume that D_0, D_1 and D_2 are derivations for $\Pi_0 \vdash \phi$, $\Pi_1, \phi_+^* \vdash \theta$ and $\Pi_2, \phi_-^* \vdash \theta$, respectively. We show that $\Pi_0, \Pi_1, \Pi_2 \models \theta$ follows from the induction hypothesis $\Pi_0 \models \phi$, $\Pi_1, \phi_+^* \models \theta$ and $\Pi_2, \phi_-^* \models \theta$. This is reduced to showing that $\phi \models \phi_+^* \vee \phi_-^*$ by induction on the subformulas δ of ϕ .

If δ is an atom and $\delta \neq [\psi, m]$, then $\delta_+^* = \delta = \delta_-^*$ and $\delta \models \delta_+^* \vee \delta_-^*$ holds trivially.

If $\delta = [\psi, m]$, then $\delta_+^* = \delta \wedge \text{NE}$ and $\delta_-^* = \delta \wedge \perp$. Since $\models \text{NE} \vee \perp$, we have $\delta \models \delta \wedge (\text{NE} \vee \perp) \models (\delta \wedge \text{NE}) \vee (\delta \wedge \perp)$.

The induction steps are proved analogously to the SE₁ case. $\square$

In the remainder of this section we prove the Completeness Theorem for our system. This proof is similar to the completeness proof for **PD** that we gave in [34]. The reader may compare this section with Section 4.3 in [34]. When proving the Completeness Theorem for the deduction system of **PT**⁺, we transformed a formula into its disjunctive normal form. Here in **CPL**⁺ we follow essentially the same idea. But in the absence of the Boolean disjunction we will not be able to express the relevant disjunctive normal form in the logic. Instead, we work with the weak normal form (i.e., the set of all disjuncts of a disjunctive normal form) and the behavior of the (strong) normal form can be simulated by using the Strong Elimination Rules SE₁ and SE₂. The disjuncts of the disjunctive normal of a formula (or elements in the weak normal form) can be obtained from what we call *strong realizations*. Our strong realizations are analogues of the “resolutions” in [4], and they are more complex than the “(weak) realizations” we defined in [34]. These strong realizations will play a crucial role in our argument. Let us now define this notion formally.

Let $\alpha \in \{p_i, \neg p_i, \text{NE}, \perp\}$ be an atom and $Y \subseteq 2^N$ a team on a set N of indices such that $Y \models \alpha$. A *strong realization* α_Y^* of α over Y is defined as

$$\alpha_Y^* := \Theta_Y^*.$$

Let $o = \langle [\alpha_1, m_1], \dots, [\alpha_c, m_c] \rangle$ be a sequence of some of the occurrences of atoms in a formula ϕ in the language of **CPL**⁺. A *strongly realizing sequence* of ϕ over o is a sequence $\Omega = \langle Y_1, \dots, Y_c \rangle$ such that $Y_i \models \alpha_i$ for each $i \in \{1, \dots, c\}$. We call the formula ϕ_Ω^* defined as follows a *strong realization* of ϕ over o :

$$\phi_{\langle Y_1, \dots, Y_c \rangle}^* := \phi((\alpha_1)_{Y_1}^* / [\alpha_1, m_1], \dots, (\alpha_c)_{Y_c}^* / [\alpha_c, m_c]).$$

Let O be the sequence of all occurrences of all atoms in ϕ . A strongly realizing sequence of ϕ over O is called a *maximal strongly realizing sequence*. A strong realization ϕ_Ω^* over O is called a *strong realization* of ϕ .

For example, consider the formula $\phi = \text{NE} \otimes (\neg p_1 \wedge \text{NE})$. Let Y_1 and Y_2 be two nonempty teams and $X = \{(1, 0)\}$ a team on $\{1\}$. Over $o = \langle [\text{NE}, 7] \rangle$ the sequence $\langle Y_1 \rangle$ is a strongly realizing sequence of ϕ and the formula $\text{NE} \otimes (\neg p_1 \wedge \text{NE}_{Y_1}^*)$ is a strong realization of ϕ . Both $\text{NE}_{Y_1}^* \otimes ((\neg p_1)_X^* \wedge \text{NE}_{Y_2}^*)$ and $\text{NE}_{Y_2}^* \otimes ((\neg p_1)_X^* \wedge \text{NE}_{Y_1}^*)$ are strong realizations of ϕ . Note that a formula always has at least one atom, so its maximal strongly realizing sequence is always a nonempty sequence.

In the next lemma we prove that every formula is semantically equivalent to the Boolean disjunction of all of its strong realizations over an arbitrary sequence of some occurrences of atoms, particularly of all its maximal strongly realizing sequences.

Lemma 5.10. *Let ϕ be a formula in the language of $\mathbf{CPL}^+$ and Λ the set of its strongly realizing sequences over a sequence o . Then $\phi \equiv \bigvee_{\Omega \in \Lambda} \phi_\Omega^*$.*

Proof. We prove the lemma by induction on the subformulas ψ of ϕ . Let N be the set of indices of propositional variables occurring in ϕ .

Base case: ψ is an atom. If the occurrence of ψ is not listed in o , then $\psi_\Omega^* = \psi$ for all $\Omega \in \Lambda$ and $\psi \equiv \bigvee_{\Omega \in \Lambda} \psi_\Omega^*$ holds trivially. Otherwise, the occurrence $\psi = [\psi, m_i]$ is in o and the set $\mathcal{Y} = \{Y_i \mid \langle Y_1, \dots, Y_c \rangle \in \Lambda\}$ consists of all teams on N that satisfy ψ . For any team X on N , by Lemma 5.4 we have $X \models \psi \iff X \in \mathcal{Y} \iff X \models \bigvee_{Y \in \mathcal{Y}} \Theta_Y^* \iff X \models \bigvee_{\Omega \in \Lambda} \psi_\Omega^*$.

The induction case $\psi = \theta \otimes \chi$ follows from the induction hypothesis and the fact that $[A \models A' \text{ and } B \models B'] \implies A \otimes B \models A' \otimes B'$ and that $A \otimes (B \vee C) \models (A \otimes B) \vee (A \otimes C)$ for all formulas A, B, C . Analogously for the case $\psi = \theta \wedge \chi$. $\square$

We will show that in our system one derives essentially the equivalence between a formula ϕ and the Boolean disjunction $\bigvee_{\Omega \in \Lambda} \phi_\Omega^*$ of its strong realizations over some sequence of occurrences of atoms. We first prove the direction that ϕ follows from $\bigvee_{\Omega \in \Lambda} \phi_\Omega^*$, which is simulated in our system by the derivation that each Boolean disjunct ϕ_Ω^* implies ϕ .

Lemma 5.11. *If Ω is a strongly realizing sequence of a formula ϕ in the language of $\mathbf{CPL}^+$ over a sequence of some occurrences of atoms in ϕ , then $\phi_\Omega^* \vdash \phi$.*

Proof. We derive the lemma by induction on the subformulas ψ of ϕ . Let $N = \{i_1, \dots, i_n\}$ be the set of indices of propositional variables occurring in ϕ .

The induction step is left to the reader. We only check the basic case when ψ is an atom. If the occurrence of ψ is not listed in o , then $\psi_\Omega^* = \psi$ and the statment holds trivially. Now, assume otherwise. Then $\psi_\Omega^* = \Theta_X^*$ and X is a team on N that satisfies ψ . If $\psi = \perp$, then $X = \emptyset$ and $\Theta_\emptyset^* = \perp$. Thus $\psi_\Omega^* \vdash \perp$ holds trivially. If $\psi = p_{i_k}$ and $X = \emptyset$, then $\Theta_\emptyset^* = \perp \vdash p_{i_k}$ follows from ex falso⁻. If $X \neq \emptyset$, then we have $s(i_k) = 1$ for all $s \in X$ and we derive $\Theta_X^* \vdash p_{i_k}$ as follows:

$$\begin{aligned}
(1) \quad & \bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \text{NE}) \\
(3) \quad & \bigotimes_{s \in X} p_{i_k} \quad (\wedge E, \otimes \text{Sub}^-) \\
(4) \quad & p_{i_k} \quad (\otimes E^-)
\end{aligned}$$

The case $\psi = \neg p_{i_k}$ is proved analogously. If $\psi = \text{NE}$, then $X \neq \emptyset$ and $\Theta_X^* \vdash \text{NE}$ is derived by a similar argument. $\square$

Next, we turn to prove that the Boolean disjunction $\bigvee_{\Omega \in \Lambda} \phi_\Omega^*$ of the strong realizations of some formula ϕ over some sequence o of occurrences of atoms follows

essentially from ϕ . Following the idea we used when we defined the Strong Elimination Rules, we simulate the derivation by proving in our system that a formula θ follows from ϕ , given that θ follows from each ϕ_Ω^* . We prove this in steps. First of all, if o is a sequence of one occurrence of the non-emptiness NE, then the statement follows by applying the SE_1 rule. We now generalize this result and show that the statement holds if o is a sequence of one occurrence of any atom.

Lemma 5.12. *If $\delta \vdash \theta$, then $\phi(\delta/[\psi, m]) \vdash \phi(\theta/[\psi, m])$.*

Proof. We prove the lemma by induction on the subformulas χ of ϕ .

If χ is an atom and $\chi \neq [\psi, m]$, then $\chi(\delta/[\psi, m]) = \chi = \chi(\theta/[\psi, m])$ and trivially $\chi(\delta/[\psi, m]) \vdash \chi(\theta/[\psi, m])$.

If $\chi = [\psi, m]$, then $\chi(\delta/[\psi, m]) = \delta$ and $\chi(\theta/[\psi, m]) = \theta$. Thus $\chi(\delta/[\psi, m]) \vdash \chi(\theta/[\psi, m])$ follows directly from the assumption.

Suppose $\chi = \chi_0 \otimes \chi_1$. Without loss of generality we may assume that the occurrence of the formula ψ is in the subformula χ_0 . By the induction hypothesis we have $\chi_0(\delta/[\psi, m]) \vdash \chi_0(\theta/[\psi, m])$. An application of the rule $\otimes Sub^-$ yields $\chi_0(\delta/[\psi, m]) \otimes \chi_1 \vdash \chi_0(\theta/[\psi, m]) \otimes \chi_1$.

The case $\chi = \chi_0 \wedge \chi_1$ is proved analogously by applying $\wedge E$ and $\wedge I$. $\square$

Lemma 5.13. *Let $[\alpha, m]$ be an occurrence of an atom in a formula ϕ and $\mathcal{Y}$ the set of all teams on $N = \{i_1, \dots, i_n\}$ that satisfy α . For any set $\Gamma \cup \{\theta\}$ of formulas in the language of $\mathbf{CPL}^+$, if $\Gamma, \phi_Y^* \vdash \theta$ for all $Y \in \mathcal{Y}$, then $\Gamma, \phi \vdash \theta$.*

Proof. If α is the non-emptiness NE, then the statement follows from SE_1 . If $\alpha = \perp$, then $\mathcal{Y} = \{\emptyset\}$, $\Theta_\emptyset^* = \perp$ and $\phi_\emptyset^* = \phi$. Thus the statement holds trivially. The nontrivial case is when α is p_{i_k} or $\neg p_{i_k}$. We only give the proof for the case $\alpha = p_{i_k}$. The case $\alpha = \neg p_{i_k}$ is proved similarly.

In view of SE_2 , to show $\Gamma, \phi \vdash \theta$ it suffices to show that $\Gamma, \phi(p_{i_k} \wedge NE/[p_{i_k}, m]) \vdash \theta$ and $\Gamma, \phi(p_{i_k} \wedge \perp/[p_{i_k}, m]) \vdash \theta$. To show the latter, first note that by the assumption we have $\Gamma, \phi(\Theta_\emptyset^*/[p_{i_k}, m]) \vdash \theta$, i.e., $\Gamma, \phi(\perp/[p_{i_k}, m]) \vdash \theta$. It then suffices to check that $\phi(p_{i_k} \wedge \perp/[p_{i_k}, m]) \vdash \phi(\perp/[p_{i_k}, m])$. But this follows from Lemma 5.12, as by $\wedge E$ we have $p_{i_k} \wedge \perp \vdash \perp$.

To show the former, in view of SE_1 it suffices to derive $\Gamma, \phi(p_{i_k} \wedge \Theta_X^*/[p_{i_k}, m]) \vdash \theta$ for all nonempty teams X on $N \setminus \{i_k\}$. By the rule $Dstr^* \wedge \otimes$, we have $p_{i_k} \wedge \Theta_X^* \vdash \Theta_Y^*$, where $Y \subseteq 2^N$ is defined as

$$Y = \{s : N \rightarrow 2 \mid s \upharpoonright N \setminus \{i_k\} \in X \text{ and } s(i_k) = 1\}.$$

It then follows from Lemma 5.12 that $\phi(p_{i_k} \wedge \Theta_X^*/[p_{i_k}, m]) \vdash \phi(\Theta_Y^*/[p_{i_k}, m])$. On the other hand, clearly $Y \in \mathcal{Y}$ and the assumptions implies that $\Gamma, \phi(\Theta_Y^*/[p_{i_k}, m]) \vdash \theta$. Hence we obtain $\Gamma, \phi(p_{i_k} \wedge \Theta_X^*/[p_{i_k}, m]) \vdash \theta$, as desired. $\square$

Now we are ready to prove the full statement for an arbitrary sequence o of occurrences of atoms.

Lemma 5.14. *Let Λ be the set of all strongly realizing sequences of ϕ over a sequence o of some occurrences of atoms in a formula ϕ . For any set $\Gamma \cup \{\theta\}$ of formulas in the language of $\mathbf{CPL}^+$, if $\Gamma, \phi_\Omega^* \vdash \theta$ for all $\Omega \in \Lambda$, then $\Gamma, \phi \vdash \theta$.*

Proof. Let $o = \langle [\alpha_1, m_1], \dots, [\alpha_c, m_c] \rangle$. By the assumption, for any Y_1 that satisfies α_1 we have

$$\Gamma, \phi((\alpha_1)_{Y_1}^* / [\alpha_1, m_1], (\alpha_2)_{X_2}^* / [\alpha_2, m_2], \dots, (\alpha_c)_{X_c}^* / [\alpha_c, m_c]) \vdash \theta$$

for all $\langle Y_1, X_2, \dots, X_c \rangle \in \Lambda$

Then we conclude by Lemma 5.13 that $\Gamma, \phi_\Omega^* \vdash \theta$ for all $\Omega \in \Lambda_1$, where Λ_1 is the set of all strongly realizing sequences of ϕ over a sequence $o_1 = \langle [\alpha_2, m_2], \dots, [\alpha_c, m_c] \rangle$.

By repeating this argument c times, we obtain $\Gamma, \phi \vdash \theta$ in the end. $\square$

For the sake of the proof of the Completeness Theorem, we need to further transform each strong realization ϕ_Ω^* into a formula $\Theta_{X_\Omega}^*$ in the normal form. To simplify notations, we write Θ_0^* for the formula $\perp \wedge \text{NE}$ and view $\mathbf{0}$ as a void team. Note that $\Theta_0^* \neq \Theta_\emptyset^* = \perp$.

Lemma 5.15. *Let Λ be the set of all maximal strongly realizing sequences of a formula $\phi(p_{i_1}, \dots, p_{i_n})$ in the language of CPL^+ .*

(i) *For each $\Omega \in \Lambda$, we have $\phi_\Omega^* \dashv\vdash \Theta_{X_\Omega}^*$ for some team X_Ω on $N = \{i_1, \dots, i_n\}$ or $X_\Omega = \mathbf{0}$.*

(ii) *Let $\Lambda_0 = \{\Omega \in \Lambda \mid X_\Omega \neq \mathbf{0}\}$. We have $\phi \equiv \bigvee_{\Omega \in \Lambda_0} \Theta_{X_\Omega}^*$.*

Proof. (i) We prove the lemma by induction on the subformulas ψ of ϕ . If ψ is an atom, then $\psi_\Omega^* = \Theta_{X_\Omega}^*$ for some team X_Ω on N that satisfies ψ and trivially $\psi_\Omega^* \dashv\vdash \Theta_{X_\Omega}^*$.

If $\psi = \delta \otimes \chi$, then by the induction hypothesis, we have

$$\delta_\Omega^* \dashv\vdash \Theta_{X_\Omega}^* \text{ and } \chi_\Omega^* \dashv\vdash \Theta_{Y_\Omega}^*, \quad (11)$$

By $\otimes \text{Sub}^-$ we have $\delta_\Omega^* \otimes \chi_\Omega^* \dashv\vdash \Theta_{X_\Omega}^* \otimes \Theta_{Y_\Omega}^*$. It then suffices to show that $\Theta_{X_\Omega}^* \otimes \Theta_{Y_\Omega}^* \dashv\vdash \Theta_Z^*$ for some team Z on N .

If $X_\Omega = \mathbf{0}$, then taking $Z = \mathbf{0}$, we derive $\Theta_0^* \dashv\vdash \Theta_0^* \otimes \Theta_{Y_\Omega}^*$ by ex falso^+ and $\mathbf{0}\text{Ctr}$. The case $Y_\Omega = \mathbf{0}$ is proved similarly. If $X_\Omega, Y_\Omega \neq \mathbf{0}$, then by $\otimes \text{W}$ and $\otimes \text{E}^-$, we derive $\Theta_{X_\Omega}^* \otimes \Theta_{Y_\Omega}^* \dashv\vdash \Theta_{X_\Omega \cup Y_\Omega}^*$.

If $\psi = \delta \wedge \chi$, then the induction hypothesis implies (11). By $\wedge \text{I}$ and $\wedge \text{E}$ we have $\delta_\Omega^* \wedge \chi_\Omega^* \dashv\vdash \Theta_{X_\Omega}^* \wedge \Theta_{Y_\Omega}^*$. It then suffices to show that $\Theta_{X_\Omega}^* \wedge \Theta_{Y_\Omega}^* \dashv\vdash \Theta_Z^*$ for some team Z on N .

If $X_\Omega = \mathbf{0}$, then taking $Z = \mathbf{0}$, we derive $\Theta_0^* \dashv\vdash \Theta_0^* \wedge \Theta_{Y_\Omega}^*$ by ex falso^+ and $\wedge \text{E}$. The case $Y_\Omega = \mathbf{0}$ is proved similarly. If $X_\Omega = Y_\Omega \neq \mathbf{0}$, then by $\wedge \text{E}$ and $\wedge \text{I}$, we derive $\Theta_{X_\Omega}^* \wedge \Theta_{Y_\Omega}^* \dashv\vdash \Theta_{X_\Omega}^*$. If $X_\Omega, Y_\Omega \neq \mathbf{0}$ and $X_\Omega \neq Y_\Omega$, then we derive $\Theta_0^* \dashv\vdash \Theta_{X_\Omega}^* \wedge \Theta_{Y_\Omega}^*$ by ex falso^+ and $\mathbf{0}\text{I}$.

(ii) It follows from the item (i), the Soundness Theorem and Lemma 5.10 that $\phi \equiv \bigvee_{\Omega \in \Lambda} \phi_\Omega^* \equiv \bigvee_{\Omega \in \Lambda} \Theta_{X_\Omega}^*$. If $\Lambda_0 \neq \emptyset$, then the statement clearly follows, as $\Theta_0^* \vee \psi = (\perp \wedge \text{NE}) \vee \psi \equiv \psi$ for all formulas ψ . If $\Lambda_0 = \emptyset$, then $\phi_\Omega^* \equiv \perp \wedge \text{NE}$ for each $\Omega \in \Lambda$. Thus $\phi \equiv \bigvee_{\Omega \in \Lambda} \phi_\Omega^* \equiv \bigvee_{\Omega \in \Lambda} (\perp \wedge \text{NE}) \equiv \perp \wedge \text{NE} \equiv \bigvee \emptyset$. $\square$

Finally, let us give the proof of the Completeness Theorem.

Theorem 5.16 (Completeness Theorem). *For any formulas ϕ and ψ in the language of $\mathbf{CPL}^+$, we have $\phi \models \psi \implies \phi \vdash \psi$.*

Proof. Suppose $\phi \models \psi$, where $\phi = \phi(p_{i_1}, \dots, p_{i_n})$ and $\psi = \psi(p_{i_1}, \dots, p_{i_n})$. By Lemma 5.15 and the Soundness Theorem, we have

$$\phi \equiv \bigvee_{\Omega \in \Lambda_0} \Theta_{X_\Omega}^* \models \bigvee_{\Upsilon \in \Lambda'_0} \Theta_{Y_\Upsilon}^* \equiv \psi. \quad (12)$$

where

- (i) Λ, Λ' are the sets of all strongly realizing sequences of ϕ and ψ , respectively and each X_Ω and Y_Υ are teams on $\{i_1, \dots, i_n\}$;
- (ii) $\phi_\Omega^* \dashv\vdash \Theta_{X_\Omega}^*$ and $\psi_\Upsilon^* \dashv\vdash \Theta_{Y_\Upsilon}^*$ for all $\Omega \in \Lambda$ and $\Upsilon \in \Lambda'$;
- (iii) $\Lambda_0 = \{\Omega \in \Lambda \mid X_\Omega \neq \mathbf{0}\}$ and $\Lambda'_0 = \{\Upsilon \in \Lambda' \mid Y_\Upsilon \neq \mathbf{0}\}$.

If $\Lambda_0, \Lambda'_0 \neq \emptyset$, then for any $\Omega \in \Lambda$, by Lemma 5.4 there exists $\Upsilon \in \Lambda'$ such that $X_\Omega = Y_\Upsilon$. We then have

$$\phi_\Omega^* \vdash \Theta_{X_\Omega}^* = \Theta_{X_\Upsilon}^* \vdash \psi_\Upsilon^* \vdash \psi$$

by (ii) and Lemma 5.11. Finally, we obtain $\phi \vdash \psi$ by Lemma 5.14.

If $\Lambda_0 = \emptyset$, then for each $\Omega \in \Lambda$ we have $\phi_\Omega^* \dashv\vdash \Theta_\emptyset^* = \perp \wedge \text{NE}$. Then by *ex falso*⁺ we derive $\phi_\Omega^* \vdash \psi$ and $\phi \vdash \psi$ follows from Lemma 5.14 again. If $\Lambda'_0 = \emptyset$, then $\psi \equiv \bigvee \emptyset = \perp \wedge \text{NE}$. But in view of (12), we must also have $\phi \equiv \perp \wedge \text{NE}$ and $\Lambda_0 = \emptyset$. This then reduces to the previous case. $\square$

5.3. $\mathbf{PI}^+$, $\mathbf{PInc}^+$ and other extensions of $\mathbf{CPL}^+$

The argument in the previous section can also be applied to axiomatize other propositional team logics obtained by adding new atoms with the empty team property to the language of $\mathbf{CPL}^+$, such as strong propositional independence logic ($\mathbf{PI}^+$) and strong propositional inclusion logic ($\mathbf{PInc}^+$). Throughout the section, we write $\mathbf{L}$ for an arbitrary such logic. In this section, we will show how to generalize the method in the previous section to axiomatize $\mathbf{L}$, and $\mathbf{PI}^+$ and $\mathbf{PInc}$ in particular.

What was crucial in the axiomatization of $\mathbf{CPL}^+$ was the notion of a *strong realization* of a formula. This notion can be generalized to richer languages, such as $\mathbf{PI}^+$ and $\mathbf{PInc}^+$. A *strong realization* α_Y^* of an atom α (such as independence atom and inclusion atom) over a team Y that satisfies α is defined as $\alpha_Y^* := \Theta_Y^*$. A *strongly realizing sequence of a formula ϕ over a sequence o* of some occurrences of atoms in ϕ and a *strong realization* (over o) are defined the same way as in the logic $\mathbf{CPL}^+$, except that a richer language $\mathbf{L}$ may contain more atoms.

Our natural deduction system of $\mathbf{L}$ consists of all of the axioms and the rules from the system of $\mathbf{CPL}^+$ (Definition 5.8), together with the Introduction Rule and the Elimination Rule ($\alpha\mathbf{I}$ and SE_α presented below) that characterize the equivalence between an arbitrary new atom α and the Boolean disjunction $\bigvee_Y \Theta_Y^*$ of its strong realizations.

Definition 5.17 (A natural deduction system of $\mathbf{L}$).

AXIOM

Atomic excluded middle $\frac{}{p_i \otimes \neg p_i} \text{EM}_0$

RULES

All of the rules from Definition 5.8, together with the following two rules for each new atom α in $\mathcal{L}$:
Atom α introduction $\frac{D \quad \Theta_Y^*}{\alpha} \alpha\text{I}$ where Y is a team on a set N of indices that satisfies α³
Strong elimination rule for α $\frac{\begin{array}{ccccccc} & [\phi(\Theta_{Y_1}^*/[\alpha, m])] & & [\phi(\Theta_{Y_k}^*/[\alpha, m])] & & & \\ D_0 & & D_1 & & D_k & & \\ \phi & & \theta & & \theta & & \end{array}}{\theta} \text{SE}_\alpha$ where $\{Y_1, \dots, Y_k\}$ is the set of all teams on a set N of indices that satisfy α

Next, we prove the Soundness Theorem and the (Strong) Completeness Theorem for the system.

Theorem 5.18 (Soundness Theorem). *For any set $\Gamma \cup \{\phi\}$ of formulas in the language of $\mathcal{L}$, we have $\Gamma \vdash \phi \implies \Gamma \models \phi$.*

Proof. We show that for each derivation D with the conclusion ϕ and the hypotheses in Γ we have $\Gamma \models \phi$. We only verify the cases when the rules αI and SE_α are applied.

αI : Assume that D is a derivation for $\Pi \vdash \Theta_Y^*$ where Y is a team on N that satisfies the atom α . We show that $\Pi \models \alpha$ follows from the induction hypothesis $\Pi \models \Theta_Y^*$. This is reduced to showing $\Theta_Y^* \models \alpha$. For any team X such that $X \models \Theta_Y^*$, by Lemma 5.4 we have $X = Y$. Thus $X \models \alpha$ follows from the assumption.

³On the surface this looks like a confusion between syntax and semantics. However, it is as in the Conjunction Introduction Rule “From ϕ and ψ we infer $\phi \wedge \psi$ ”. On the basic level the rules establish a connection between logical operations and their intended meaning. We could replace here the assumption “ Y satisfies the atom α ” by explicitly listing teams on N that satisfy α , but that would be more cumbersome.

SE_α : Put $\phi_i^* = \phi(\Theta_{Y_i}^*/[\alpha, m])$ for each $i \in \{1, \dots, k\}$. Assume that $D_0, D_1, \dots, D_k$ are derivations for $\Pi_0 \vdash \phi$, $\Pi_1, \phi_1^* \vdash \theta$, ..., $\Pi_k, \phi_k^* \vdash \theta$, respectively. We show that $\Pi_0, \Pi_1, \dots, \Pi_k \models \theta$ follows from the induction hypothesis $\Pi_0 \models \phi$, $\Pi_1, \phi_1^* \models \theta$, ..., $\Pi_k, \phi_k^* \models \theta$. This is reduced to showing that $\phi \models \phi_1^* \vee \dots \vee \phi_k^*$ by induction on ϕ . If ϕ is the atom α , then the statement follows from Lemma 5.4 and the choice of the Y_i 's. The other cases are left to the reader. $\square$

Theorem 5.19 (Completeness Theorem). *For any formulas ϕ and ψ in the language of $\mathbf{L}$, we have $\phi \models \psi \implies \phi \vdash \psi$.*

Proof. The theorem is proved by a similar argument to that of the proof of Theorem 5.16. Especially, all the lemmas leading to Theorem 5.16 can be easily generalized to the case of any extension $\mathbf{L}$ of $\mathbf{CPL}^+$ with new atoms having the empty team property. In particular, in the proof of the lemma that corresponds to Lemma 5.11, if ψ is a new atom α , then $\psi_\Omega \vdash \psi$ follows from α . When proving the lemma that corresponds to Lemma 5.13, one applies SE_α when α is a new atom. $\square$

Instantiating the new atoms α in the deduction system of Definition 5.17 with independence atoms or inclusion atoms, we obtain sound and (strongly) complete deduction systems of $\mathbf{PI}^+$ and $\mathbf{PInc}^+$. We end this section with a demonstration of the natural deduction system of $\mathbf{PI}^+$. Below we derive the Geiger-Paz-Pearl axioms [12]. To simplify notation, we write IndI for the Independence Atom Introduction Rule and SE_{Ind} for the Independence Atom Elimination Rule.

Example 5.20. Let $\vec{x} = p_{i_1} \dots p_{i_k}$, $\vec{y} = p_{j_1} \dots p_{j_m}$ and $\vec{z} = p_{l_1} \dots p_{l_n}$. The following Geiger-Paz-Pearl axioms are derivable in the natural deduction system of $\mathbf{PI}^+$:

- (i) $\vec{x} \perp \vec{y} \vdash \vec{y} \perp \vec{x}$
- (ii) $\vec{x} \perp \vec{y} \vdash \vec{z} \perp \vec{y}$, where $\vec{z}$ is a subsequence of $\vec{x}$.
- (iii) $\vec{x} \perp \vec{y} \vdash \vec{u} \perp \vec{v}$, where $\vec{u}$ is a permutation of $\vec{x}$ and $\vec{v}$ is a permutation of $\vec{y}$.
- (iv) $\vec{x} \perp \vec{y}$, $\vec{x}\vec{y} \perp \vec{z} \vdash \vec{x} \perp \vec{y}\vec{z}$.

Proof. We are going to use the Independence Atom Introduction Rule IndI a lot here. Therefore the proofs below seem entirely semantical. However, we have built the meaning of the independence atom into the rule IndI , so it is only natural that we refer to this meaning in the proofs. This is more complicated but, in principle, analogous to the way we often use in elementary logic the meaning of “and” and “or” when we prove e.g. distributivity laws using just the Elimination Rule and Introduction Rule for $\wedge$ and $\vee$.

Put $K = \{i_1, \dots, i_k\}$, $M = \{j_1, \dots, j_m\}$ and $N = \{l_1, \dots, l_n\}$.

(i) By SE_{Ind} , it suffices to show that for any team Y on $K \cup M$ such that $Y \models \vec{x} \perp \vec{y}$ we have $\Theta_Y^* \vdash \vec{y} \perp \vec{x}$. But this follows from IndI , as $Y \models \vec{y} \perp \vec{x}$ also holds.

(ii) By SE_{Ind} , it suffices to show that for any team Y on $K \cup M \cup N$ such that $Y \models \vec{x} \perp \vec{y}$ we have $\Theta_Y^* \vdash \vec{z} \perp \vec{y}$, where $\vec{z}$ is a subsequence of $\vec{x}$. In view of IndI , this is reduced to showing $Y \models \vec{z} \perp \vec{y}$. But this is obvious.

(iii) By SE_{Ind} , it suffices to show that for any team Y on $K \cup M$ such that $Y \models \vec{x} \perp \vec{y}$ we have $\Theta_Y^* \vdash \vec{u} \perp \vec{v}$, where $\vec{u} = p_{i_1} \cdots p_{i_a}$ is a permutation of $\vec{x}$ and $\vec{v} = p_{m_1} \cdots p_{m_b}$ is a permutation of $\vec{y}$. In view of IndI this is reduced to showing $Y \models \vec{u} \perp \vec{v}$, which follows from that $Y \models \vec{x} \perp \vec{y}$.

(iv) We will show that $(\vec{x} \perp \vec{y}) \wedge (\vec{x}\vec{y} \perp \vec{z}) \vdash \vec{x} \perp \vec{y}\vec{z}$. By SE_{Ind} , it suffices to show that for any team X on $K \cup M \cup N$ such that $X \models \vec{x} \perp \vec{y}$, we have $\Theta_X^* \wedge (\vec{x}\vec{y} \perp \vec{z}) \vdash \vec{x} \perp \vec{y}\vec{z}$. But this, by SE_{Ind} , is further reduced to showing that for any team Y on $K \cup M \cup N$ such that $Y \models \vec{x}\vec{y} \perp \vec{z}$, we have $\Theta_X^* \wedge \Theta_Y^* \vdash \vec{x} \perp \vec{y}\vec{z}$.

Now, if $X \neq Y$, then by **0I** and ex falso^+ , we derive $\Theta_X^* \wedge \Theta_Y^* \vdash \perp \wedge \text{NE} \vdash \vec{x} \perp \vec{y}\vec{z}$.

If $X = Y$, then $\Theta_X^* = \Theta_Y^*$. By IndI , it suffices to show $X \models \vec{x} \perp \vec{y}\vec{z}$. For any $s_1, s_2 \in X$, since $X \models \vec{x} \perp \vec{y}$, there exists $s_3 \in X$ such that $s_1(\vec{x}) = s_3(\vec{x})$ and $s_2(\vec{y}) = s_3(\vec{y})$. But as $X \models \vec{x}\vec{y} \perp \vec{z}$, there exists $s_4 \in X$ such that $s_4(\vec{x})s_4(\vec{y}) = s_3(\vec{x})s_3(\vec{y})$ and $s_4(\vec{z}) = s_2(\vec{z})$. We then conclude that $s_4(\vec{x}) = s_3(\vec{x}) = s_1(\vec{x})$ and $s_4(\vec{y})s_4(\vec{z}) = s_3(\vec{y})s_2(\vec{z}) = s_2(\vec{y})s_2(\vec{z})$. Hence $X \models \vec{x} \perp \vec{y}\vec{z}$ and this completes the proof. $\square$

6. Concluding remarks

In our previous work [34] we have investigated classical propositional logic, and several versions of propositional dependence logic. We proved their expressive completeness for downward closed team properties, and gave several complete axiomatizations of such logics. In this paper, we have studied propositional team logics more generally, recognizing that there is a whole hierarchy of them. We have established the results in Figure 1 concerning the expressive power of these logics. Several expressively complete logics are identified. For example, we have proved that $\mathbf{PU}^+$ is expressively complete for the set of union closed team properties. We also derived normal forms for many of the logics. Some of the logics we considered have the empty team property, and some do not. We axiomatized the logics ($\mathbf{PT}^+$, $\mathbf{CPL}^+$, $\mathbf{PI}^+$ and $\mathbf{PInc}^+$) without the empty team property, and we leave the concrete axiomatization of $\mathbf{PU}^+$ for future research. As is reflected by the sophisticated Strong Elimination Rules we gave in this paper, propositional logics of independence are more intricate than propositional logics of dependence. In particular, we feel that we have not yet fully understood the notion of independence, and neither a characterization of expressive power nor a complete axiomatization is given for $\mathbf{PI}$ here. Nevertheless, it is our hope that the results obtained concerning logics around $\mathbf{PI}$ will set the stage for further research in this field and lead to a better understanding of $\mathbf{PI}$, propositional independence logic, itself.

Acknowledgements

The authors would like to thank Maria Aloni, Pietro Galliani, Lauri Hella, Juha Kontinen, Vít Punčochář, Floris Roelfsen and Dag Westerståhl for useful conversations related to this paper. The authors also thank the anonymous referee for helpful comments concerning the presentation of this paper.

References

- [1] ABRAMSKY, S., AND VÄÄNÄNEN, J. From IF to BI. *Synthese* 167, 2 (2009), 207–230.
- [2] A.S.TROELSTRA, AND H.SCHWICHTENBERG. *Basic Proof Theory*. Cambridge University Press, 1996.
- [3] CIARDELLI, I. Inquisitive semantics and intermediate logics. Master’s thesis, University of Amsterdam, 2009.
- [4] CIARDELLI, I. *Questions in Logic*. PhD thesis, University of Amsterdam, 2016.
- [5] CIARDELLI, I., AND ROELOFSEN, F. Inquisitive logic. *Journal of Philosophical Logic* 40, 1 (2011), 55–94.
- [6] CODD, E. F. Further normalization of the data base relational model. *IBM Research Report, San Jose, California RJ909* (1971).
- [7] EBBING, J., HELLA, L., MEIER, A., MÜLLER, J.-S., VIRTEMA, J., AND VOLLMER, H. Extended modal dependence logic. In *20th International Workshop, WoLLIC 2013, Proceedings*, vol. 8071 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2013, pp. 126–137.
- [8] EBBING, J., AND LOHMANN, P. Complexity of model checking for modal dependence logic. In *SOFSEM, M. Bieliková, G. Friedrich, G. Gottlob, S. Katzenbeisser, and G. Turán, Eds.*, vol. 7147 of *Lecture Notes in Computer Science*. Springer, 2012, pp. 226–237.
- [9] EBBING, J., LOHMANN, P., AND YANG, F. Model checking for modal intuitionistic dependence logic. In *Post-proceedings of the Ninth Tbilisi Symposium on Language, Logic and Computation* (2013), vol. 7758 of *Lecture Notes in Computer Science*, pp. 231–256.
- [10] GALLIANI, P. Inclusion and exclusion in team semantics: On some logics of imperfect information. *Annals of Pure and Applied Logic* 163, 1 (January 2012), 68–84.
- [11] GALLIANI, P., AND HELLA, L. Inclusion logic and fixed point logic. In *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy* (2013), pp. 281–295.
- [12] GEIGER, D., PAZ, A., AND PEARL, J. Axioms and algorithms for inferences involving probabilistic independence. *Information and Computation* 91, 1 (March 1991), 128–141.
- [13] GRÄDEL, E., AND VÄÄNÄNEN, J. Dependence and independence. *Studia Logica* 101, 2 (April 2013), 399–410.

- [14] HELLA, L., LUOSTO, K., SANO, K., AND VIRTEMA, J. The expressive power of modal dependence logic. In *Advances in Modal Logic*, R. Goré, B. Kooi, and A. Kurucz, Eds., vol. 10. College Publications, 2014, pp. 294–312.
- [15] HELLA, L., AND STUMPF, J. The expressive power of modal logic with inclusion atoms. In *Proceedings Sixth International Symposium on Games, Automata, Logics and Formal Verification (2015)*, J. Esparza and E. Tronci, Eds., Electronic Proceedings in Theoretical Computer Science 193, pp. 129–143.
- [16] HINTIKKA, J., AND SANDU, G. Informational independence as a semantical phenomenon. In *Logic, Methodology and Philosophy of Science*, R. H. J. E. Fensstad, I. T. Frolov, Ed. Amsterdam: Elsevier, 1989, pp. 571–589.
- [17] HODGES, W. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL* 5 (1997), 539–563.
- [18] HODGES, W. Some strange quantifiers. In *Structures in Logic and Computer Science: A Selection of Essays in Honor of A. Ehrenfeucht*, J. Mycielski, G. Rozenberg, and A. Salomaa, Eds., vol. 1261 of *Lecture Notes in Computer Science*. London: Springer, 1997, pp. 51–65.
- [19] HYTTINEN, T., PAOLINI, G., AND VÄÄNÄNEN, J. Quantum team logic and Bell’s inequalities. *Rev. Symb. Log.* 8, 4 (2015), 722–742.
- [20] IEMHOFF, R., AND YANG, F. Structural completeness in propositional logics of dependence. *Archive for Mathematical Logic* 55, 7 (November 2016), 955–975.
- [21] KONTINEN, J., AND NURMI, V. Team logic and second-order logic. *Fundamenta Informaticae* 106 (2011), 259–272.
- [22] LINDSTRÖM, P. On extensions of elementary logic. *Theoria* 35 (1969), 1–11.
- [23] LOHMANN, P., AND VOLLMER, H. Complexity results for modal dependence logic. *Studia Logica* 101, 2 (2013), 343–366.
- [24] LÜCK, M. Axiomatizations for propositional and modal team logic. In *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)* (Dagstuhl, Germany, 2016), J.-M. Talbot and L. Regnier, Eds., vol. 62, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, pp. 33:1–33:18.
- [25] MANN, A. L., SANDU, G., AND SEVENSTER, M. *Independence-Friendly Logic: A Game-Theoretic Approach*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2011.
- [26] SEVENSTER, M. Model-theoretic and computational properties of modal dependence logic. *Journal of Logic and Computation* 19, 6 (2009), 1157–1173.
- [27] VÄÄNÄNEN, J. *Dependence Logic: A New Approach to Independence Friendly Logic*. Cambridge: Cambridge University Press, 2007.

- [28] VÄÄNÄNEN, J. Modal dependence logic. In *New Perspectives on Games and Interaction*, K. R. Apt and R. van Rooij, Eds., vol. 4 of *Texts in Logic and Games*. Amsterdam University Press, 2008, pp. 237–254.
- [29] VÄÄNÄNEN, J. Multiverse set theory and absolutely undecidable propositions. In *Interpreting Gödel*, J. Kennedy, Ed. Cambridge University Press, 2014, pp. 180–208.
- [30] VAN DALEN, D. *Logic and Structure*. Springer, 2012.
- [31] VIRTEMA, J. Complexity of validity for propositional dependence logics. In *Proceedings Fifth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2014* (2014), no. 161 in EPTCS, pp. 18–31.
- [32] YANG, F. *On Extensions and Variants of Dependence Logic*. PhD thesis, University of Helsinki, 2014.
- [33] YANG, F. *Uniform definability in team semantics setting*, manuscript, 2015.
- [34] YANG, F., AND VÄÄNÄNEN, J. Propositional logics of dependence. *Annals of Pure and Applied Logic* 167, 7 (July 2016), 557–589.