# Fourier Multiplier Methods in the Study of Growth and Decay of Semigroups

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# Fourier Multiplier Methods in the Study of Growth and Decay of Semigroups

# Fourier Multiplier Methods in the Study of Growth and Decay of Semigroups

## Dissertation

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at Delft University of Technology
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chair of the Board for Doctorates
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*Keywords:* Polynomial stability, strongly continuous semigroup, discrete semigroup, strongly Kreiss bounded operator, Fourier multiplier theory

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# SUMMARY


In this dissertation, we aim to apply Fourier multiplier theory as a unifying method to advance the study of semigroup theory and further develop the Fourier multiplier theory itself. These theories provide powerful tools for studying the behaviour of solutions to PDEs, including their existence, uniqueness, regularity, and stability.

Chapter 1 provides a brief introduction to the applications of semigroup theory and Fourier multiplier theory in modern PDE analysis. We outline the contributions of this dissertation and offer an overview for the subsequent chapters.

Chapter 2 introduces general notation and necessary preliminaries, including foundational concepts and theorems that will be used throughout this dissertation.

Chapter 3 studies polynomial decay rates of non-uniformly bounded $C_0$-semigroups in Banach spaces with Fourier type $p \in [1,2]$. We refine the earlier results of Rozendaal and Veraar by addressing the endpoint case of positive decay rates and improve those of Santana and Carvalho by removing the logarithmic correction in the decay rate. We also study the bounded case in weighted Lebesgue spaces, under certain a priori assumptions on the Fourier transform in such spaces.

Chapter 4 extends these ideas to the discrete setting, focusing on powers of bounded linear operators $T$ on Banach spaces with Fourier type $p \in [1,2]$. We analyse the decay of $T^n(I-T)^\tau, n \to \infty$ for some $\tau > 0$, where $I$ is the identity operator. We establish a connection between decay rates and the weak Ritt condition, a variational resolvent condition of the Ritt operator. Our results can be viewed as discrete counterparts to those of non-uniformly bounded $C_0$-semigroups.

Chapter 5 explores strongly Kreiss bounded operators, defined in close analogy to Ritt operators. We prove the growth bound of $T^n$ is strictly below $\frac{1}{2}$ in general UMD spaces. Our result extends the work of Arnold and Cuny, who considered the same problem in Lebesgue spaces.

Chapter 6 develops a Fourier multiplier theory that emphasizes the role of Banach space geometry. We establish a vector-valued Fourier multiplier theorem in weighted Lebesgue spaces, assuming the multiplier has $\mathscr{R}$-bounded range and satisfies an $\ell^r$-summability condition on its bounded $s$-variation seminorms over dyadic intervals, for some $s, r > 1$. The exponents $s$ and $r$ reflect the relationship between the geometric properties of Banach spaces and the boundedness of Fourier multiplier operators.

# SAMENVATTING

In dit proefschrift streven we ernaar om Fourier multipliertheorie toe te passen als een unificerende methode om de studie van halfgroepentheorie verder te brengen en Fourier multipliertheorie zelf verder te ontwikkelen. Deze vakgebieden bieden krachtige hulpmiddelen voor het bestuderen van het gedrag van oplossingen van partiële differentiaalvergelijkingen (PDVen), zoals hun bestaan, uniciteit, regulariteit en stabiliteit.

Hoofdstuk 1 biedt een korte inleiding tot de toepassingen van halfgroepentheorie en de Fourier multipliertheorie in de moderne PDV-analyse. We schetsen de bijdragen van dit proefschrift en geven een overzicht van de daaropvolgende hoofdstukken.

Hoofdstuk 2 introduceert algemene notatie en noodzakelijke preliminaries, inclusief de fundamentele concepten en stellingen die in dit proefschrift gebruikt zullen worden.

Hoofdstuk 3 bestudeert het polynomiale verval van niet-uniform begrensde $C_0$-halfgroepen in Banachruimten met Fourier type $p \in [1,2]$. We verfijnen de eerdere resultaten van Rozendaal en Veraar door het eindpuntgeval met positief verval te bestuderen en verbeteren resultaten van Santana en Carvalho door de logaritmische correctie in het verval te verwijderen. We onderzoeken ook het begrensde geval in gewogen Lebesgue ruimten, onder bepaalde a priori aannames over de Fourier transformatie in dergelijke ruimten.

Hoofdstuk 4 breidt deze ideeën uit naar het discrete geval, met een focus op machten van begrensde lineaire operatoren $T$ in Banachruimten met Fourier type $p \in [1,2]$. We analyseren het verval van $T^n(I-T)^\tau, n \to \infty$ voor sommige $\tau > 0$, waarbij $I$ de identiteitsoperator is. We leggen een verband tussen dit verval en de zwakke Ritt-voorwaarde, een variatie op de resolvente conditie in de Ritt-voorwaarde. Onze resultaten kunnen worden gezien als discrete tegenhangers van die van niet-uniform begrensde $C_0$-semigroepen.

Hoofdstuk 5 onderzoekt sterk Kreiss-begrensde operatoren, gedefinieerd in analogie met Ritt-operatoren. We bewijzen dat de groeibegrenzing van $T^n$ strikt onder $\frac{1}{2}$ ligt in algemene UMD-ruimten. Ons resultaat breidt het werk van Arnold en Cuny uit, die hetzelfde probleem in Lebesgue ruimten hebben beschouwd.

Hoofdstuk 6 ontwikkelt Fourier multipliertheorie die de rol van Banachruimte geometrie benadrukt. We bewijzen een vectorwaardige Fourier multiplierstelling in gewogen Lebesgue ruimten, onder de aanname dat de multiplier een $\mathscr{R}$-begrensd bereik heeft en voldoet aan een $\ell^r$-sommeerbaarheidsvoorwaarde op begrensde $s$-variatie halfnormen over dyadische intervallen, voor zekere $s, r > 1$. De exponenten $s$ en $r$ reflecteren de relatie tussen de geometrische eigenschappen van de Banachruimtes en de begrensdheid van Fourier multiplieroperatoren.

# PREFACE

During my PhD I was supervised by prof. dr. ir. M. C. Veraar (promotor) and dr. ir. E. Lorist (copromotor). I mainly finished the following papers with them:

[1] C. Deng, J. Rozendaal, and M. Veraar. "Improved polynomial decay for unbounded semigroups". In: *J. Evol. Equ.* 24.4 (2024), Paper No. 99.

[2] C. Deng, E. Lorist, and M. Veraar. "Strongly Kreiss bounded operators in UMD Banach spaces". In: *Semigroup Forum* 108.3 (2024), pp. 594–625.

[3] C. Deng. A weak Ritt condition on bounded linear operators. In preparation.

[4] C. Deng, E. Lorist, and M. Veraar. Multiplier theory in intermediate UMD Banach spaces. In preparation.

Among these, papers [1] and [2] have been published, while papers [3] and [4] are in preparation and will be submitted soon. Paper [1] corresponds to Chapter 3, paper [2] to Chapter 5, paper [3] to Chapter 4, and paper [4] to Chapter 6.

*Chenxi Deng*
*Delft, June 2025*

# 1

# INTRODUCTION

Partial differential equations (PDEs) are equations involving unknown functions of two or more variables along with their partial derivatives. PDEs are fundamental in many areas of science and engineering, as they provide a framework for modeling diverse phenomena. Examples include the heat equation (modeling temperature distribution), the wave equation (describing sound or light propagation), the Laplace and Poisson equations (governing steady-state and potential fields of electric systems), the Navier–Stokes equations (describing viscous fluid flow), the Euler equations (for inviscid fluids), the Schrödinger equation (modeling quantum wave functions) and so on.

In this chapter, we briefly introduce a modern analytical method for studying PDEs, present a brief review of the key results in later chapters, and provide an overview of the dissertation structure.

## 1.1. MODERN ANALYSIS OF PDES

The solvability of PDEs has always been one of the central challenges in mathematics. In the early stages, mathematicians primarily relied on classical analysis, such as separation of variables, integral transforms, and the method of characteristics to solve specific types of linear equations. Classical analysis provided the foundational mathematical framework for scientific research throughout the 18th and 19th centuries. However, as scientific problems became increasingly complex, the limitations of classical analysis gradually emerged. For instance, the fundamental solution method is a commonly used technique in classical PDE theory. However, the concept of a "fundamental solution" could not be precisely defined within the classical analysis. It was not until the establishment of distribution theory that this issue was resolved. Therefore, it is essential to develop modern analytical methods for the continued study of PDEs. For standard references on PDE theory, see [6, 51, 57, 81] and the references therein.

A modern theory widely used in initial value problems for partial differential equations is *operator semigroup theory* [50, 117]. By reformulating a partial differential equation as an abstract Cauchy problem, where the differential operator is redefined in a suitable functional framework, we can use the theory of operator semigroups to analyse the existence, uniqueness, and stability of solutions. A powerful approach to study semigroup theory is *Fourier multiplier theory*, which investigates the boundedness of Fourier multiplier operators on Lebesgue spaces, see [71–73, 147] and the references therein.

In this dissertation, we employ the Fourier multiplier method as a unifying framework to advance the study of semigroup theory; moreover, we will also develop the Fourier multiplier theory itself. The subsequent chapters will focus on the following questions.

1. The decay and growth of semigroups, in both continuous and discrete settings. This involves analysing the long-time behaviour of semigroups: Do they decay to zero? If so, at what rate and under what conditions? If the semigroup exhibits growth, can we characterize its power bound? How does the underlying space influence the results?

2. The development of Fourier multiplier theorems. We aim to establish a vector-valued Fourier multiplier theorem that uncovers the relation between the boundedness of Fourier multipliers and the geometric properties—such as type and cotype—of the underlying Banach spaces.

## 1.2. REVIEW OF THE MAIN RESULTS

In this section, we provide a concise overview of the main results of this dissertation. For a more detailed discussion of each topic, we refer readers to Chapters 3-6.

### 1.2.1. POLYNOMIAL STABILITY OF $C_0$-SEMIGROUPS

Semigroup theory arises when we study the abstract Cauchy problem on a Banach space $X$:

$$\begin{cases} u_t(t) &= Au(t), \quad t \geq 0, \\ u(0) &= u_0, \end{cases} \tag{1.2.1}$$

where $u_0 \in X$. Assume that equation (1.2.1) is well-posed, and the operator $A$ generates a $C_0$-semigroup $(T(t))_{t\geq 0}$ of bounded operators on $X$. Then the solution of (1.2.1) can be written as $u(t) := T(t)u_0$, for $u_0 \in X$. Note that a $C_0$-semigroup $(T(t))_{t\geq 0}$ is called *uniformly bounded* if there exists a constant $C \geq 0$ such that $\|T(t)\| \leq C$ for all $t \geq 0$; otherwise, it is *non-uniformly bounded* and satisfies $\lim_{t\to\infty} \|T(t)\| = \infty$.

We say the solution $u$ (or the semigroup) is *exponentially stable* if there exists a $\omega > 0$ such that

$$\|T(t)u_0\|_X \leq Ce^{-\omega t}\|u_0\|_X, \quad u_0 \in X, t \to \infty,$$

and *polynomially stable of order s* if there is a $s > 0$ such that

$$\|T(t)u_0\|_X \leq Ct^{-s}\|u_0\|_{D(A)}, \quad u_0 \in D(A), t \to \infty.$$

Direct calculation of $T(t)$ as $t \to \infty$ may be difficult; however, the resolvent of its generator (denoted by $R(\lambda, A)$ for $\lambda$ in the resolvent set $\rho(A)$) is often more accessible. Therefore, a classical approach to analysing the asymptotic behaviour of a semigroup is through studying the resolvent operator of its generator. This approach has been extensively investigated over the past decades, including the exponential stability and polynomial stability of the uniformly bounded and non-uniformly bounded $C_0$-semigroups.

Among them, the research on exponential stability has been relatively complete, see [56, 65, 107, 120, 145, 148] and the references therein. Therefore, the focus has shifted to the study of polynomial stability. We summarize some known results ([8, 12, 20, 130]) in the following table. For a detailed discussion of these results, see the introduction of Chapter 3. Note that in Table 1.1, $\mathbb{R}$ denotes the real numbers, $\mathbb{C}_+$ denotes the right half-plane,

Table 1.1: Comparison of uniformly bounded and non-uniformly bounded $C_0$-semigroups.

| | Resolvent condition | Hilbert space | Banach space |
|---|---|---|---|
| UB | $\|R(i\xi,A)\| \le C(1+\|\xi\|)^\beta, \quad \xi\in\mathbb{R}$ | $\|T(t)A^{-1}\| \le (C/t)^{1/\beta}$ | $\|T(t)A^{-1}\| \le (C\log(2+t)/t)^{1/\beta}$ |
| NUB | $\|R(\lambda,A)\| \le C(1+\|\lambda\|)^\beta, \quad \lambda\in\mathbb{C}_+$ | $\|T(t)(-A)^{-\tau}\| \le Ct^{-(\frac{\tau}{\beta}-1-\varepsilon)}$ | $\|T(t)(-A)^{-\tau}\| \le Ct^{-(\frac{\tau-1}{\beta}-1-\varepsilon)}$ |

$\beta > 0, \tau > \beta$, and $\varepsilon > 0$ is arbitrary. If $-A$ is an injective and sectorial operator, then the fractional power operator $(-A)^\tau$ is well-defined for all $\tau \in \mathbb{R}$. The polynomial stability results for non-uniformly bounded semigroups were unified in [130] using the theory of Fourier type $p \in [1,2]$, which means the Fourier transform $\mathscr{F} : L^p(\mathbb{R};X) \to L^{p'}(\mathbb{R};X)$ is bounded (see Section 2.3 for more details). To better understand our result, we first restate [130, Theorem 4.9] in a more concise form.

**Proposition 1.2.1.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\ge0}$ on a Banach space $X$ with Fourier type $p \in [1,2]$. Suppose that $\mathbb{C}_+ \subseteq \rho(A)$, and there exist $\beta > 0$ and $C \ge 0$ such that the resolvent of $A$ satisfies*

$$\|R(\lambda,A)\| \le C(1+|\lambda|)^\beta, \quad \lambda \in \mathbb{C}_+.$$

*Then for each $s \ge 0$ and all $\tau > (s+1)\beta + \frac{1}{p} - \frac{1}{p'}$, there exists a $C > 0$ such that*

$$\|T(t)x\|_X \le Ct^{-s}\|x\|_{D((-A)^\tau)}, \quad t \ge 1, x \in D((-A)^\tau).$$

Note that every Banach space has Fourier type 1 and every Hilbert space has Fourier type 2. It is clear that Proposition 1.2.1 generalizes the results in Table 1.1, which are originally from [8] in the Banach space setting. The Hilbert space case, on the other hand, follows from [130, Corollary 4.11]. However, in the latter work, the authors were only able to establish the endpoint case for Hilbert spaces, yielding $\tau = (s+1)\beta + \frac{1}{p} - \frac{1}{p'}$. For Banach spaces with Fourier type $p < 2$, the endpoint case remains an open problem. Recently, [134] achieved a result for $\tau = (s+1)\beta + \frac{1}{p} - \frac{1}{p'}$, incorporating a logarithmic correction in the decay rate. Our result shows that the logarithmic term can be removed.

**Theorem 1.2.2.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\ge0}$ on a Banach space $X$ with Fourier type $p \in [1,2]$. Suppose that $\mathbb{C}_+ \subseteq \rho(A)$, and there exist $\beta > 0$ and $C \ge 0$ such that*

$$\|R(\lambda,A)\| \le C(1+|\lambda|)^\beta, \quad \lambda \in \mathbb{C}_+.$$

*Let $s > 0$ and set $\tau := (s+1)\beta + \frac{1}{p} - \frac{1}{p'}$. Then there exists a $C \ge 0$ such that*

$$\|T(t)x\|_X \le Ct^{-s}\|x\|_{D((-A)^\tau)}, \quad t \ge 1, x \in D((-A)^\tau). \tag{1.2.2}$$

In Chapter 3, we will prove a more general version of the above theorem using Fourier multiplier theory, real interpolation, and properties of Besov spaces, see Theorem 3.1.1. Our result improves [134] by removing the logarithmic factor, and thereby entirely improves [130, Theorem 4.9]. However, this method fails when $s = 0$. To address this issue, additional assumptions on the underlying space $X$ are required (see Section 3.4 for details). Then, for $s = 0$, (1.2.2) also holds for the initial values in $D((-A)^\tau)$ for $\tau = \beta + \frac{1}{p} - \frac{1}{p'}$.

### 1.2.2. DISCRETE SEMIGROUPS: WEAK RITT OPERATORS AND KREISS OPERATORS

Inspired by the study of $C_0$-semigroups, we are also interested in the discrete case, which is widely used in the convergence of iterations of linear equations.

Let $T$ be a bounded linear operator on a Banach space $X$. One can view $(T^n)_{n\geq 0}$ as a discrete analogue of a $C_0$-semigroup. If there exists a constant $C \geq 0$ such that $\|T^n\| \leq C$ for all $n \geq 1$, then $T$ is called *power bounded*. The study of discrete semigroups can be divided into two aspects.

**(1) Decay rates of the difference norm $\|T^n(I-T)\|$.** Consider the following fixed point problem

$$x = Tx + g, \tag{1.2.3}$$

where $T$ is a bounded linear operator on a Banach space $X$ and $g$ is a given vector in $X$ such that this equation has at least a solution. To solve this problem, given an initial value $x_0 \in X$, consider successive approximations

$$x_k := Tx_{k-1} + g = T^k x_0 + \sum_{i=0}^{k-1} T^i g, \quad k \geq 1. \tag{1.2.4}$$

Moreover, notice that

$$x = Tx + g = T^k x + \sum_{i=0}^{k-1} T^i g, \quad k \geq 1.$$

If $x - x_0$ is in the range of $I - T$, then there exists a $y \in X$ such that

$$x - x_k = T^k(x - x_0) = T^k(I-T)y.$$

Therefore, $T^k(I-T)$ appears when investigating the convergence of iterations.

The breakthrough in this area is the Katznelson–Tzafriri theorem [75]: given a power bounded operator $T$, the difference norm $\|T^n(I-T)\| \to 0$ if and only if $\sigma(T) \cap \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \subseteq \{1\}$. Inspired by the theory of $C_0$-semigroups, studies of $T^n(I-T)$ shifted from whether it converges to zero to its rates of convergence. However, most studies focus on power bounded operators, see [9, 14, 34, 46, 47, 85, 109, 111, 112, 135] and the references therein. In particular, Seifert [135] proved that if $T$ is power bounded and $\sigma(T) \cap \{\lambda \mid |\lambda| = 1\} = \{1\}$ on a Banach space, then for $\beta \geq 1$, the resolvent condition

$$\|R(e^{i\theta}, T)\| \leq C|\theta|^{-\beta}, \quad 0 < |\theta| < \pi \tag{1.2.5}$$

implies $\|T^n(I-T)\| \le C\left(\frac{\log n}{n}\right)^{\frac{1}{\beta}}$. Ng and Seifert [112] investigated the result in Hilbert space, removing the log correction in the decay estimate.

By comparing the results of uniformly bounded $C_0$-semigroups (see Table 1.1) and power bounded operators, we conclude that power bounded operators $(T^n)_{n\ge 0}$ can be viewed as a discrete counterpart to uniformly bounded $C_0$-semigroups $(T(t))_{t\ge 0}$, while the operator $I-T$ plays a role analogous to the inverse of generators $A^{-1}$ in the context of uniformly bounded $C_0$-semigroups. One natural question is for general bounded operators, does an analogous result hold in between general bounded operators and non-uniformly bounded $C_0$-semigroups (Proposition 1.2.1)?

Note that an operator is called a Ritt operator if it satisfies the *Ritt condition*:

$$\|R(\lambda, T)\| \le \frac{C}{|\lambda - 1|}, \quad 1 < |\lambda| < 2.$$

By [98] and [106], Ritt operators are power bounded. By [31, Lemma 3.3], (1.2.5) can be viewed as a weak version of the Ritt condition. Inspired by this, we put forward the *weak Ritt condition*:

$$\|R(\lambda, T)\| \le \frac{C}{|\lambda - 1|^{\beta}}, \quad 1 < |\lambda| < 2, \tag{1.2.6}$$

for some $\beta > 1$. Since $\beta > 1$, the above condition does not necessarily imply power boundedness of $T$.

In Chapter 4, we explore the relation between the weak Ritt condition and the decay rates of bounded operators $(T^n)_{n\ge 0}$ when composed with operators of the form $(I-T)^{\tau}$ for some $\tau > 0$. A simple version of our main result reads as follows.

**Theorem 1.2.3.** *Let $X$ be a complex Banach space, $T$ be a bounded linear operator on $X$, and $\mathbb{D}$ be the unit disk. Assume that $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ and $\sigma(T) \cap \{\lambda \mid |\lambda| = 1\} = \{1\}$. Furthermore, suppose there exists a constant $\beta > 1$ such that the weak Ritt condition (1.2.6) holds. Then for each $s \in \mathbb{N}$ and all $\tau > (s+1)\beta - 1$, there exists a $C > 0$ such that*

$$\|T^n(I-T)^{\tau}\| \le Cn^{-s}, \quad n \ge 1.$$

From the above result, we conclude that achieving an improved decay rate inherently requires a trade-off in the form of increasing the exponent of the difference operator $I-T$. The result is established in the context of general Banach spaces. We further explore the decay rates on Banach spaces with Fourier type $p \in (1,2]$ (Theorem 4.3.4) and highlight the bounded case (Corollary 4.3.6). The primary approach involves adapting Fourier multiplier methods used in the continuous parameter setting, combined with fundamental resolvent calculations. A detailed comparison between our main results and other related findings is provided in Chapter 4.

**(2) Growth power bound of $\|T^n\|$.** Let us return to the problem (1.2.3). If $g = 0$, then (1.2.4) reduces to $x_k = T^k x$ for all $k \ge 1$. Such expressions commonly arise in numerical analysis as a result of time discretization. For the stability of a numerical scheme, it is desirable that $T^k$ does not grow rapidly when $T$ is not power bounded. In a $d$-dimensional space, the Kreiss matrix theorem (see [78, 86, 138]) asserts that if an operator $T$ is Kreiss bounded with constant $K$, then it is power bounded, and satisfies $\|T^n\| \le Ked$.

1

**Definition 1.2.4.** An operator $T$ is called *Kreiss bounded with constant $K$* if

$$\left\|(\lambda - T)^{-1}\right\| \le \frac{K}{|\lambda| - 1}, \quad |\lambda| > 1, \tag{1.2.7}$$

and $T$ is called *strongly Kreiss bounded with constant $K_s$* if

$$\left\|(\lambda - T)^{-n}\right\| \le \frac{K_s}{(|\lambda| - 1)^n}, \quad |\lambda| > 1, n \ge 1. \tag{1.2.8}$$

Clearly, by the Neumann series argument, power bounded operators are (strongly) Kreiss bounded. In finite dimensional spaces, Kreiss boundedness implies power boundedness as well by the Kreiss matrix theorem. However, in infinite dimensional spaces, Kreiss bounded operators are generally not power bounded. Counterexamples see, e.g. [7, 28, 43, 77, 96, 111]. In particular, we present [111, Example 4] in Section 4.4.2.

As mentioned above, in applications to numerical analysis, it is often desirable to ensure that $T^n$ does not grow very fast when $T$ is not power bounded. Therefore, investigating the power bound of $T^n$ becomes an essential subject. Let $T$ be a Kreiss bounded operator on a Banach space $X$ with constant $K$. By Cauchy's integral formula, we have

$$\left\|T^n\right\| \le Ke(n+1), \quad n \ge 1.$$

Besides, [96, Theorem 2.1] shows if $T$ is strongly Kreiss bounded with constant $K_s$, then

$$\left\|T^n\right\| \le K_s\sqrt{2\pi(n+1)}, \quad n \ge 1.$$

These results can be improved if further geometric assumptions are added to the space $X$, e.g. Hilbert spaces and $L^p$ spaces for $p \in (1,\infty)$. We refer to Chapter 5 for a thorough introduction to the previous works. In particular, if $X$ is a UMD Banach space, which means the Hilbert transform is bounded on $L^p(\mathbb{R};X)$, [37, Theorem 3.1] proved that the bound of Kreiss bounded operators can be improved to $\|T^n\| = O(n/(\log(n+2))^\beta)$ with $\beta = \frac{1}{\min\{q,q^*\}}$. Here $q$ and $q^*$ denote the (finite) cotypes of $X$ and the dual space $X^*$, respectively. However, the counterpart result for strongly Kreiss bounded operators in UMD Banach spaces was missing. In Chapter 5, we aim to solve this problem and mainly prove the following fact.

**Theorem 1.2.5.** *Let $X$ be a* UMD *Banach space and $T$ be a strongly Kreiss bounded operator on $X$, then there exist a $\alpha \in [0,1/2)$ depending on $X$ and a constant $C > 0$ depending on $X$ and $K_s$ such that*

$$\left\|T^n\right\| \le Cn^\alpha, \quad n \ge 1.$$

In Chapter 5, we derive a precise bound for strongly Kreiss bounded operators under additional geometric assumptions on the Banach space (see Theorem 5.3.1). Since UMD Banach spaces always satisfy these assumptions, Theorem 1.2.5 follows naturally as a corollary. The proof of Theorem 5.3.1 is inspired by the approach in [7], where the growth of strongly Kreiss bounded operators in $L^p$ spaces was studied. We extend their results to general UMD Banach spaces in Chapter 5, thanks to [21]. To achieve this, we introduce $\ell^q(L^p)$-Fourier decompositions, whose properties-such as duality, type, cotype, interpolation, and extrapolation—are thoroughly examined in Section 5.2. Furthermore, we also consider the growth of positive strongly Kreiss bounded operators on Banach function spaces, which is a direct extension of [7, Proposition 4.1].

### 1.2.3. FOURIER MULTIPLIER THEORY UNDER WEAK SMOOTH CONDITIONS

Let $X$ be a Banach space, $\mathscr{F}$ and $\mathscr{F}^{-1}$ be the Fourier transform and its inverse on the Schwartz function space $\mathscr{S}(\mathbb{R};X)$, respectively. Fourier multiplier theory studies sufficient conditions of a bounded scalar-valued or operator-valued function $m$ such that the Fourier multiplier operator $T_m := \mathscr{F}^{-1}(m\mathscr{F})$ can be extended to a bounded linear operator on $L^p(\mathbb{R};X)$ for some $p \in (1,\infty)$.

A foundational result in this field is the *Marcinkiewicz multiplier theorem*, which states that $T_m$ is bounded on $L^p(\mathbb{R})$ for all $p \in (1,\infty)$ if $m$ is of bounded (1-)variation over dyadic intervals. Denote $m|_J$ the restriction of $m$ on an interval $J \subseteq \mathbb{R}$. For $s \in [1,\infty)$, define the bounded $s$-variation space $\ell^\infty(V^s(\Delta))$ as the space of all $m \in L^\infty(\mathbb{R})$ such that

$$\|m\|_{\ell^\infty(V^s(\Delta))} := \sup_{J\in\Delta}(\|m|_J\|_{L^\infty(J)} + [m|_J]_{V^s(J)}) < \infty,$$

where $\Delta$ is the family of all dyadic intervals, and $[\cdot]_{V^s(J)}$ denotes the bounded $s$-variation seminorm on the interval $J \in \Delta$, i.e.

$$[m|_J]_{V^s(J)} := \sup_{\xi_0<\ldots<\xi_n:\xi_0,\ldots,\xi_n\in J}\Big(\sum_{k=1}^{n}|m(\xi_{k-1})-m(\xi_k)|^s\Big)^{\frac{1}{s}}.$$

In 1988, Coifman, Rubio de Francia, and Semmes [32] improved the Marcinkiewicz multiplier theorem on $L^p(\mathbb{R})$ for $p \in (1,\infty)$, assuming $m \in \ell^\infty(V^s(\Delta))$ where $s$ satisfies $1/s > |1/p - 1/2|$. We note that bounded $s$-variation is implied by $\frac{1}{s}$-Hölder smoothness, so larger $s$ corresponds to a weaker smoothness assumption. This approach was extended to the operator-valued setting by Hytönen and Potapov [68]. They assumed that the Banach space has the Littlewood–Paley–Rubio de Francia property (LPR$_p$ property), a condition known to hold only for a limited class of spaces, such as Hilbert spaces and usual Lebesgue spaces. Later, [1] generalized [32] to Banach function spaces.

Motivated by the $\ell^q(L^p)$-type Fourier decompositions discussed in Chapter 5, and building on the strategies developed in [1, 68], we investigate the variational Carleson operator on $L^p$ spaces and apply these insights to the study of Fourier multiplier theory. We derive sufficient conditions in terms of the $\ell^r$-summability of bounded $s$-variation seminorms of the multiplier, for some $s > 1$, under the assumption that the multiplier has $\mathscr{R}$-bounded range. A simplified formulation of our main result is stated below.

**Theorem 1.2.6.** *Let $X_0$ and $Y$ be* UMD *Banach spaces, $H$ a Hilbert space and define $X = [X_0, H]_\theta$ for some $\theta \in (0,1]$. Suppose that $X$ has cotype $q$ and $Y$ has type $t$ and set $\frac{1}{r} := \frac{1}{t} - \frac{1}{q}$. Let $s \in [1, \frac{2}{2-\theta})$ and $m\colon \mathbb{R} \to \mathscr{L}(X,Y)$ have $\mathscr{R}$-bounded range and*

$$\|m\|_{\ell^r(\dot{V}^s(\Delta;\mathscr{L}(X,Y)))} := \Big(\sum_{J\in\Delta}[m|_J]^r_{V^s(J;\mathscr{L}(X,Y))}\Big)^{\frac{1}{r}} < \infty,$$

*where $\mathscr{L}(X,Y))$ denotes the bounded linear operators from $X$ to $Y$. Then $T_m$ is bounded from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$ for all $p \in (s,\infty)$.*

Theorem 1.2.6 is proved in Chapter 6 as Theorem 6.5.4, where we also extend the result to weighted Lebesgue spaces. Our proof strategy is inspired by techniques from [1,

1

32, 68]. We begin by establishing a weighted, vector-valued estimate for the variational Carleson operator, which serves as a key ingredient for developing a Fourier multiplier theory in atomic $R$-spaces (see Section 6.2.3).

The main idea is to decompose the multiplier $m$ into two parts: $m-N$ and $N$, where $N$ lies in the closure of the convex hull of the range of $m$. For the difference $m-N$, we exploit the embedding of bounded $s$-variation spaces into $R$-spaces to derive bounds on $T_{m-N}$ in terms of the $\ell^r$-summability of the bounded $s$-variation seminorms of $m$. For the component $N$, we apply the Littlewood-Paley inequality together with the $\mathscr{R}$-boundedness of the multiplier range to control $T_N$. Combining these two estimates yields the desired operator norm bound for $T_m$.

In contrast to the approach in [1, 68], our method avoids the need for the $\mathrm{LPR}_p$ property and the stronger $\ell^2(\ell^r)$-boundedness condition on the multiplier range. Instead, we require only $\mathscr{R}$-boundedness of the range, along with an $\ell^r$-summability condition on the bounded $s$-variation seminorms of $m$, where the exponent $r$ is determined by the type and cotype of the underlying Banach spaces. Furthermore, our framework allows for operator-valued multipliers $m \in \mathscr{L}(X,Y)$ with $X$ and $Y$ being two (possibly distinct) UMD Banach spaces, and only $X$ is assumed to be a complex interpolation space. As a result, our theorem encompasses new classes of multipliers that lie beyond the scope of current operator-valued Fourier multiplier theorems.

## 1.3. OVERVIEW

The dissertation is organized as follows. In Chapter 2, we introduce general notation and provide necessary preliminaries that will be frequently used throughout this work, including important properties of spectrum and resolvents, Fourier multiplier theorems, type and cotype properties of Banach spaces, etc.

In Chapters 3–5, we focus on the application of Fourier multiplier theory to the decay and growth of semigroups. In Chapter 3, we investigate the decay rates of non-uniformly bounded $C_0$-semigroups, improving the results in [130, 134]. In Chapter 4, we extend these ideas to the discrete case of bounded linear operators, establishing a connection between decay rates and the weak Ritt condition. Chapter 5 explores strongly Kreiss bounded operators, defined in a manner analogous to weak Ritt operators. We show that the growth is below $\frac{1}{2}$ in general UMD spaces. Inspired by the Littlewood-Paley-Rubio de Francia decomposition, we further develop the so-called $\ell^q(L^p)$-Fourier decompositions and apply them in the proof of our main results.

In Chapter 6, we are dedicated to establishing new multiplier theorems assuming the type and cotype properties of the underlying Banach spaces. Motivated by the form of $\ell^q(L^p)$-Fourier decompositions in Chapter 5, we derive sufficient conditions for the Fourier multiplier in terms of its $\ell^r$-summability of bounded $s$-variation seminorms, for some $s,r>1$ decided by the geometry of the Banach spaces. Our results encompass a broader class of multipliers that fall outside the scope of current operator-valued Fourier multiplier theorems.

# 2

# PRELIMINARIES

## 2.1. GENERAL NOTATION

In this dissertation, we adopt the following notation and conventions.

We denote the set of integers by $\mathbb{Z}$, the positive integers by $\mathbb{N} := \{1, 2, \ldots\}$, and the natural numbers (including zero) by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The scalar field is represented by $\mathbb{K}$, which encompasses both the complex plane $\mathbb{C}$ and the real numbers $\mathbb{R}$. We define the right half of the complex plane as $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$, where $\operatorname{Re}\lambda$ denotes the real part of $\lambda$, and the left half as $\mathbb{C}_- := -\mathbb{C}_+$. The torus is denoted by $\mathbb{T} := [0, 1]$, and for $n \in \mathbb{Z}$, we write $e_n(\cdot) := e^{2\pi i n \cdot}$.

Banach spaces are represented by $X$, $Y$, and $Z$. The dual space of $X$ is denoted by $X^*$, and the norm on $X$ is written as $\|\cdot\|_X$, with the subscript omitted when no ambiguity arises. The space of bounded linear operators between complex Banach spaces $X$ and $Y$ is denoted by $\mathscr{L}(X, Y)$, and $\mathscr{L}(X) := \mathscr{L}(X, X)$. We write $X \subseteq Y$ if $X$ embeds in $Y$ continuously. The space of $X$-valued Schwartz functions on $\mathbb{R}$ is denoted by $\mathscr{S}(\mathbb{R}; X)$, and the space of $X$-valued tempered distributions by $\mathscr{S}'(\mathbb{R}; X)$. For $p \in [1, \infty]$ and a measurable weight function $w : \mathbb{R} \to [0, \infty)$, the Bochner space of equivalence classes of strongly measurable, $p$-integrable, $X$-valued functions on $\mathbb{R}$ with respect to $w$ is denoted by $L^p(\mathbb{R}, w; X)$. When $w \equiv 1$, this space is simply written as $L^p(\mathbb{R}; X)$.

Let $\mathbb{P}$ denote a probability measure, and $(\Omega, \mathscr{A}, \mathbb{P})$ represent a probability space. The expectation and variance of a random variable $\varepsilon$ are denoted by $\mathbb{E}[\varepsilon]$ and $\operatorname{Var}(\varepsilon)$, respectively.

For an operator $A$ on $X$, its domain and range are denoted by $D(A)$ and $\operatorname{Ran}(A)$, respectively. The adjoint operator of $A$ is written as $A^*$, and the identity operator is denoted by $I$. The indicator function of a set $S$ is represented by $\mathbf{1}_S$.

Constants depending on parameters $a, b, \ldots$ are denoted by $C_{a,b,\ldots}$, and their values may vary from line to line. We use $f(s) \lesssim_{a,b,\ldots} g(s)$ to mean $f(s) \leq C_{a,b,\ldots} g(s)$ for all $s$ and some constant $C_{a,b,\ldots} \geq 0$ independent of $s$, with analogous definitions for $f(s) \gtrsim_{a,b,\ldots} g(s)$ and $f(s) \eqsim_{a,b,\ldots} g(s)$. The restriction of a function $f : \mathbb{R} \to X$ on an interval $J \subseteq \mathbb{R}$ is defined by $f|_J$. The notation $c \uparrow c_0$ and $c \downarrow c_0$ signifies that $c$ monotonically increases or decreases, respectively, to a constant $c_0$. The floor function $\lfloor a \rfloor$ denotes the largest integer less than or equal to $a$, while the ceiling function $\lceil a \rceil$ denotes the smallest integer greater than or equal to $a$. The complement of a set $S$ is written as $S^c$. The maximum and minimum of $p$ and $q$ are denoted by $p \vee q$ and $p \wedge q$, respectively. The Hölder conjugate $p' \in [1,\infty]$ of $p \in [1,\infty]$ is defined by the relation $1 = \frac{1}{p} + \frac{1}{p'}$.

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## 2.2. SOME BASIC FUNCTIONAL ANALYSIS

We first recall some basic knowledge of resolvent and spectrum that will be used several times in the later chapters. For further details, we refer the reader to monographs on functional analysis, such as [108, 150].

Let $A : D(A) \subseteq X \to X$ be a closed operator. The *resolvent set* of $A$ is defined as

$$\rho(A) := \{\lambda \in \mathbb{C} \,|\, \lambda - A : D(A) \to X \text{ is invertible and } (\lambda - A)^{-1} \in \mathscr{L}(X)\}.$$

The *resolvent* $(\lambda - A)^{-1}$ is denoted by $R(\lambda, A)$ and the *spectrum* is the complement of $\rho(A)$ in $\mathbb{C}$, denoted by $\sigma(A)$. Note that if $\rho(A)$ is not empty, then $A$ is closed. If $A$ is a densely defined operator with $\rho(A) \neq \emptyset$, then for every $\lambda \in \rho(A)$, the space $D(A)$ with the norm $\|z\|_{D(A)} := \|(\lambda - A)z\|$ for all $z \in D(A)$ is a Banach space. The norms defined above for different $\lambda \in \rho(A)$ are equivalent to the graph norm. Moreover, the embedding $D(A) \subseteq X$ is continuous. The *resolvent identity* is the following identity.

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A), \quad \lambda, \mu \in \rho(A).$$

Let $A$ be a linear operator on a Banach space $X$. For $\omega \in (0, \pi)$, set

$$S_\omega := \{z \in \mathbb{C} \setminus \{0\} \,|\, |\arg(z)| < \omega\}.$$

Then $-A$ is a *sectorial operator* if there exists an $\omega \in (0, \pi)$ such that $\sigma(-A) \subseteq \overline{S_\omega}$, and

$$\sup\left\{\|\lambda(\lambda + A)^{-1}\|_{\mathscr{L}(X)} \,|\, \lambda \in \mathbb{C} \setminus \overline{S_{\omega_1}}\right\} < \infty \tag{2.2.1}$$

for each $\omega_1 \in (\omega, \pi)$. If $-A$ is a sectorial operator, then the fractional power $(-A)^\alpha$ is well-defined for each $\alpha \in \mathbb{C}_+$, see [63, Chapter 3]. If, additionally, $A$ is injective, then $(-A)^\alpha$ is well-defined for all $\alpha \in \mathbb{C}$. Note that $D((-A)^\beta) \subseteq D((-A)^\alpha)$ whenever $\beta \in \mathbb{C}$ satisfies $\operatorname{Re}\beta > \operatorname{Re}\alpha$.

**Lemma 2.2.1.** *If $A \in \mathscr{L}(X)$ is such that $\|A\| < 1$, then $I - A$ is invertible and*

$$R(1, A) = \sum_{j=0}^{\infty} A^j, \text{ and } \|R(1, A)\| \leq \frac{1}{1 - \|A\|}.$$

**Lemma 2.2.2.** *For $\lambda \in \rho(A)$, $R(\lambda, A)$ is an analytic $\mathscr{L}(X)$-valued function. If $\mu \in \rho(A)$, $|\mu - \lambda| < \frac{1}{\|R(\mu, A)\|}$, then*

$$R(\lambda, A) = \sum_{j=0}^{\infty} R(\mu, A)^{j+1} (\mu - \lambda)^j.$$

*In particular,*

$$R(\lambda, A)^{(k)} = (-1)^k k! R(\lambda, A)^{(k+1)}, \quad k \in \mathbb{N}.$$

*Remark* 2.2.3. By Lemma 2.2.2, the set $\rho(A)$ is open, thereby, $\sigma(A)$ is closed. Moreover, for all $\lambda \in \rho(A)$,

$$\|R(\lambda, T)\| \geq \frac{1}{\inf_{s\in\sigma(A)} |s-\lambda|}.$$

Finally, we collect some theorems throughout this dissertation. The following density argument can be found in [108, Proposition 1.18]. With this, we only need to prove that a linear operator is well-defined and bounded on a dense subspace in many cases.

**Lemma 2.2.4** (Density argument)**.** *Let $X$ and $Y$ be Banach spaces, and let $X_0$ be a dense subspace of $X$. Suppose that the linear operator $T : X_0 \to Y$ satisfies $\|Tx\|_Y \lesssim \|x\|_X$ for all $x \in X_0$. Then $T$ extends uniquely to a bounded linear operator on $X$.*

The Minkowski's inequality can be found in [71, Proposition 1.2.22].

**Lemma 2.2.5** (Minkowski's inequality)**.** *Let $(S_1, \mathscr{A}_1, \mu_1)$ and $(S_2, \mathscr{A}_2, \mu_2)$ be measure spaces and let $X$ be a Banach space. For all $1 \leq p \leq q < \infty$, we have the contractive embedding*

$$L^p(S_1; L^q(S_2; X)) \subseteq L^q(S_2; L^p(S_1; X)).$$

With the following result from [71, Theorem 2.1.9], we only need to consider scalar spaces in many cases when proving results in Hilbert spaces.

**Proposition 2.2.6** (Paley–Marcinkiewicz–Zygmund)**.** *Let $(S_1, \mathscr{A}_1, \mu_1)$ and $(S_2, \mathscr{A}_2, \mu_2)$ be measure spaces and let $p_1, p_2 \in [1, \infty)$. Consider a bounded linear operator $T$ from $L^{p_1}(S_1)$ to $L^{p_2}(S_2)$. Let $H$ be a Hilbert space. Then $T \otimes I_H$ uniquely extends to a bounded operator from $L^{p_1}(S_1; H)$ to $L^{p_2}(S_2; H)$ and its norm satisfies*

$$\|T\| \leq \|T \otimes I_H\| \lesssim_{p_1,p_2} \|T\|.$$

## 2.3. FOURIER MULTIPLIER THEORY

This section focuses on the vector-valued Fourier multiplier theory, which forms the cornerstone of the methods used in later chapters. For further details on scalar-valued Fourier multiplier theory, see [61, Chapter 6]. For a comprehensive treatment of operator-valued Fourier multiplier theory, we refer the reader to [71–73].

### 2.3.1. FOURIER TYPE

**Definition 2.3.1.** Let $p \in [1,2]$, $q \in [2,\infty]$, $X$ is said to have *Fourier type $p$* if the Fourier transform $\mathscr{F} : L^p(\mathbb{R}; X) \to L^{p'}(\mathbb{R}; X)$ is bounded. Moreover, $X$ is said to have *Fourier cotype $q$* if $X$ has Fourier type $q'$.

Every Banach space $X$ has Fourier type 1. Moreover, Example 2.3.6 below indicates that the Fourier type necessarily requires $p \in [1,2]$. Hausdorff–Young theorem states the Fourier transform is bounded, with the norm at most one, as an operator from $L^p(\mathbb{R}; H)$ to $L^{p'}(\mathbb{R}; H)$ for any $p \in [1,2]$ (see [71, Corollary 2.4.10]). Furthermore, by [71, Propositions 2.4.16 and 2.4.20], $X$ has Fourier type $p$ if and only if $X^*$ has Fourier type $p$.

The following statement is known as the Plancherel theorem, which implies that every Hilbert space $H$ has Fourier type 2. Conversely, any Banach space with Fourier type 2 is isomorphic to a Hilbert space by Kwapien's theorem (cf. [83] and [71, Theorem 2.1.18]).

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**Proposition 2.3.2** (Plancherel theorem)**.** *Let $H$ be a Hilbert space. If $f \in L^2(\mathbb{R};H) \cap L^1(\mathbb{R};H)$, then $\widehat{f} \in L^2(\mathbb{R};H)$ and $\|\widehat{f}\|_{L^2(\mathbb{R};H)} = \|f\|_{L^2(\mathbb{R};H)}$. In particular, the Fourier transform (more precisely, its restriction to $L^2(\mathbb{R};H) \cap L^1(\mathbb{R};H)$) extends to an isometry on $L^2(\mathbb{R};H)$.*

The following lemma shows that the notion of Fourier type could equivalently be defined in terms of the Fourier transform on the torus, see [71, Proposition 2.4.20].

**Lemma 2.3.3.** *Let $X$ be a Banach space, $p \in [1,2]$. Then the following are equivalent:*

*(i) $\mathscr{F}$ extends to a bounded operator from $L^p(\mathbb{R};X)$ to $L^{p'}(\mathbb{R};X)$;*

*(ii) $\mathscr{F}$ extends to a bounded operator from $L^p(\mathbb{T};X)$ to $\ell^{p'}(\mathbb{Z};X)$;*

*(iii) $\mathscr{F}$ extends to a bounded operator from $\ell^p(\mathbb{Z};X)$ to $L^{p'}(\mathbb{T};X)$;*

### 2.3.2. FOURIER MULTIPLIERS

Let $X$ and $Y$ be Banach spaces and $p \in (1,\infty)$. For a symbol $m \in L^\infty(\mathbb{R};\mathscr{L}(X,Y))$ we define the *Fourier multiplier operator* $T_m : \mathscr{S}(\mathbb{R};X) \to \mathscr{S}'(\mathbb{R};Y)$ as

$$T_m f = \mathscr{F}^{-1}(m\widehat{f}), \quad f \in \mathscr{S}(\mathbb{R};X).$$

Then $m \in L^\infty(\mathbb{R};\mathscr{L}(X,Y))$ is said to be an *$L^p$-multiplier* (on $\mathbb{R}$) if there exists a constant $C < \infty$ such that

$$\|T_m f\|_{L^p(\mathbb{R};Y)} \le C\|f\|_{L^p(\mathbb{R};X)}, \quad f \in \mathscr{S}(\mathbb{R};X).$$

We also have analogous formulations for the torus. A bounded sequence $(m_n)_{n\in\mathbb{Z}}$ is called an *$L^p$-Fourier multiplier* (on $\mathbb{T}$) if there exists a constant $C < \infty$ such that for all $f \in \mathscr{P}(\mathbb{T};X)$

$$\Big\| \sum_{n\in\mathbb{Z}} m_n \widehat{f}(n) e_n \Big\|_{L^p(\mathbb{T};Y)} \le C\|f\|_{L^p(\mathbb{T};X)}.$$

By density the mappings $f \mapsto \mathscr{F}^{-1}(m\widehat{f})$ and $f \mapsto \sum_{n\in\mathbb{Z}} m_n \widehat{f}(n) e_n$ can be uniquely extended to bounded linear operators which will be denoted by $T_m \in \mathscr{L}(L^p(\mathbb{R};X); L^p(\mathbb{R};Y))$ and $T_m \in \mathscr{L}(L^p(\mathbb{T};X), L^p(\mathbb{R};Y))$, respectively.

Using transference it is possible to transform multipliers for the real line to the periodic case and vice versa (see [71, Section 5.7]):

**Lemma 2.3.4.** *Let $X$ and $Y$ be Banach spaces and $p \in (1,\infty)$. Let $m \in L^\infty(\mathbb{R};\mathscr{L}(X,Y))$ be a Fourier multiplier from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$. Suppose that for all $x \in X$, the point $k \in \mathbb{Z}$ is a Lebesgue point of $m(\cdot)x$ and set $m_k x := m(k)x$. Then $\{m_k\}_{k\in\mathbb{Z}}$ is a Fourier multiplier from $L^p(\mathbb{T};X)$ to $L^p(\mathbb{T};Y)$, and in fact,*

$$\|T_{\{m_k\}_{k\in\mathbb{Z}}}\|_{\mathscr{L}(L^p(\mathbb{T};X),L^p(\mathbb{T};Y))} \le \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X),L^p(\mathbb{R};Y))}.$$

**Lemma 2.3.5.** *Suppose $m \in L^\infty(\mathbb{R};\mathscr{L}(X,Y))$ is Riemann integrable over any bounded interval. Suppose that $\{m(\varepsilon k)\}_{k\in\mathbb{Z}\setminus\{0\}}$ are Fourier multipliers from $L^p(\mathbb{T};X)$ to $L^p(\mathbb{T};Y)$ having uniformly bounded multiplier norm for some sequence of numbers $\varepsilon \downarrow 0$. Then $m$ is a Fourier multiplier from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$, and*

$$\|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X),L^p(\mathbb{R};Y))} \le \liminf_{\varepsilon\downarrow 0} \|T_{\{m(\varepsilon k)\}_{k\in\mathbb{Z}\setminus\{0\}}}\|_{\mathscr{L}(L^p(\mathbb{T};X),L^p(\mathbb{T};Y))}.$$

The example below implies that the Fourier type necessarily requires $p \in [1,2]$.

*Example* 2.3.6. Define $m : \mathbb{R} \to \mathbb{C}$ by $m := e^{i\pi|\xi|^2}$ for $\xi \in \mathbb{R}$ and let $p \in [1,\infty)$. Define the Fourier multiplier operator $T_m : \mathscr{S}(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ by

$$T_m f := \mathscr{F}^{-1}(m\widehat{f}), \quad f \in \mathscr{S}(\mathbb{R}).$$

By calculation we have $\mathscr{F}^{-1}(m) = e^{i\frac{\pi}{4}} e^{-i\pi|x|^2}$. Then by Young's inequality,

$$\|T_m f\|_{L^\infty(\mathbb{R})} = \|\mathscr{F}^{-1}(m) * f\|_{L^\infty(\mathbb{R})} \le \|\mathscr{F}^{-1}(m)\|_{L^\infty(\mathbb{R})} \|f\|_{L^1(\mathbb{R})},$$

implying that $T_m \in \mathscr{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$. On the other hand, by Plancherel theorem (Proposition 2.3.2), $T_m \in \mathscr{L}(L^2(\mathbb{R}))$ since

$$\|T_m f\|_{L^2(\mathbb{R})} = \|m\widehat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}. \tag{2.3.1}$$

Thus, by Riesz–Thorin Theorem (cf. [124] and [71, Theorem 2.2.1]), $T_m \in \mathscr{L}(L^p(\mathbb{R}), L^{p'}(\mathbb{R}))$ for all $p \in [1,2]$. If $T_m \in \mathscr{L}(L^p(\mathbb{R}))$ for some $p < 2$, then using Riesz–Thorin Theorem again yields $T_m \in \mathscr{L}(L^p(\mathbb{R}), L^2(\mathbb{R}))$. By (2.3.1),

$$\|f\|_{L^2(\mathbb{R})} = \|T_m f\|_{L^2(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})},$$

which derives a contradiction. If $T_m \in \mathscr{L}(L^p(\mathbb{R}))$ for some $p > 2$, by a similar analysis we yield $T_m \in \mathscr{L}(L^2(\mathbb{R}), L^p(\mathbb{R}))$, implying the same contradiction. Therefore, $T_m \in \mathscr{L}(L^p(\mathbb{R}))$ if and only if $p = 2$.

However, by Hausdorff–Young theorem, for any $q \in (2,\infty)$, $\mathscr{F} \in \mathscr{L}(L^{q'}(\mathbb{R}), L^q(\mathbb{R}))$. If $\mathbb{C}$ has Fourier type $q$, i.e. $\mathscr{F}$ is a bounded operator from $L^q(\mathbb{R})$ to $L^{q'}(\mathbb{R})$, then for every $m \in L^\infty(\mathbb{R})$ we get

$$\begin{aligned}\|T_m f\|_{L^{q'}(\mathbb{R})} &\le \|\mathscr{F}\|_{\mathscr{L}(L^q(\mathbb{R}),L^{q'}(\mathbb{R}))} \|m\widehat{f}\|_{L^q(\mathbb{R})} \\ &\le \|\mathscr{F}\|_{\mathscr{L}(L^q(\mathbb{R}),L^{q'}(\mathbb{R}))} \|m\|_{L^\infty(\mathbb{R})} \|\widehat{f}\|_{L^q(\mathbb{R})} \\ &\le \|\mathscr{F}\|_{\mathscr{L}(L^q(\mathbb{R}),L^{q'}(\mathbb{R}))} \|m\|_{L^\infty(\mathbb{R})} \|\mathscr{F}\|_{\mathscr{L}(L^{q'}(\mathbb{R}),L^q(\mathbb{R}))} \|f\|_{L^{q'}(\mathbb{R})}.\end{aligned}$$

This contradicts the above analysis when $m = e^{i\pi|\cdot|^2}$. Therefore, even for the scalar case, the definition of Fourier type cannot be extended to $(2,\infty)$.

## 2.4. UMD BANACH SPACES

In this section, we give a brief introduction to the so-called class of UMD Banach spaces, which in many ways provides the correct setting for vector-valued analysis. For an introduction to UMD Banach spaces we refer the reader to [71, Chapters 4 and 5] and [118].

Let $p \in (1,\infty)$, $N \in \mathbb{N}$, $(S,\mathscr{A},\mu)$ be a $\sigma$-finite measure space, $(\mathscr{F}_n)_{n=0}^N$ be a $\sigma$-finite filtration, and let $(f_n)_{n=0}^N$ be a finite martingale in $L^p(S;X)$. Define its martingale differences by $df_n := f_n - f_{n-1}$ for $n \ge 1$.

2

**Definition 2.4.1.** A Banach space is said to have the property of *unconditional martingale differences (*UMD *property)* if for all $p \in (1,\infty)$ and $N \in \mathbb{N}$, there exists a finite constant $C \geq 0$ (depending on $p$ and $X$) such that for all finite martingales $f_n \in L^p(S;X)$ and all scalars $|\varepsilon_n| = 1, n = 1, \cdots, N$, we have

$$\left\| \sum_{n=1}^{N} \varepsilon_n d f_n \right\|_{L^p(S;X)} \leq C \left\| \sum_{n=1}^{N} d f_n \right\|_{L^p(S;X)}.$$

If this condition holds, then $X$ is said to be a UMD *Banach space*.

Although the UMD property is defined probabilistically, it turns out to be equivalent to a purely analytic statement–the boundedness of Hilbert transform $H$. For $f \in \mathscr{S}(\mathbb{R};X)$, the *Hilbert transform* is the principal value integral

$$Hf(x) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y, \quad x \in \mathbb{R}. \tag{2.4.1}$$

By [71, Proposition 5.2.2], we have for $f \in \mathscr{S}(\mathbb{R};X)$,

$$\widehat{Hf}(\xi) = -i\,\text{sgn}(\xi)\widehat{f}(\xi).$$

In particular, we have

$$H = -iS_{[0,\infty)} + iS_{(-\infty,0]}, \quad iH + I = 2S_{[0,\infty)}, \quad -iH + I = 2S_{(-\infty,0]} \tag{2.4.2}$$

where $S_{[0,\infty)}$ is called the *Riesz projection*. By the seminal works of Burkholder [24] and Bourgain [22], this characterisation is equivalent to the probabilistic definition of the UMD property in terms of the Unconditionality of Martingale Differences, see the theorem below.

**Theorem 2.4.2.** *Let $X$ be a Banach space and let $p \in (1,\infty)$ be fixed. The following assertions are equivalent:*

1. *$X$ is a* UMD *space;*
2. *for every $f \in L^p(\mathbb{R};X)$, the Hilbert transform extends to a bounded operator on $L^p(\mathbb{R};X)$.*
3. *for every $f \in L^p(\mathbb{R};X)$, the Riesz projection extends to a bounded operator on $L^p(\mathbb{R};X)$.*

Next, we state some properties that will be used throughout this dissertation. We refer the readers to [71, Chapter 4].

- Let $p \in (1,\infty)$, $L^p(S;X)$ is a UMD Banach space whenever $X$ is a UMD Banach space.
- UMD spaces are reflexive and super-reflexive.
- If $X$ is a UMD Banach space, then $X^*$ is a UMD Banach space.

Hilbert spaces (in particular $\mathbb{R}$ and $\mathbb{C}$), as well as all closed subspaces and quotient spaces of $L^p(S)$ for $p \in (1,\infty)$ are UMD spaces (see [24]). Note that $L^1(S)$ and $L^\infty(S)$ do not satisfy the UMD property, as they are not reflexive.

The following lemma provides a sufficient condition for the duality of $L^p(S;X)$ when $X$ is a UMD Banach space.

**Lemma 2.4.3.** *If $X$ is a* UMD *Banach space, then for any $p \in [1,\infty)$,*

$$L^{p'}(S;X^*) \eqsim (L^p(S;X))^*.$$

*Proof.* If $X$ is a UMD Banach space, then $X^*$ is also a UMD Banach space and therefore reflexive. By [71, Theorem 1.3.21], $X^*$ possesses the Radon-Nikodým property (see [71, Definition 1.3.9]) with respect to the measure space $(S,\mathscr{A},\mu)$. The conclusion then follows from [71, Theorem 1.3.10]. □

2

## 2.5. TYPE, COTYPE, AND RELATED PROPERTIES

In this section, we introduce some notions related to the geometry of the Banach space such as type and cotype, see [72] for more details.

A *Rademacher variable* is a random variable $\varepsilon : \Omega \to \mathbb{K}$ defined on some probability space $(\Omega,\mathscr{A},\mathbb{P})$, which is uniformly distributed over $S_{\mathbb{K}} := \{z \in \mathbb{K} \,|\, |z| = 1\}$. A Rademacher sequence is a sequence $(\varepsilon_n)_{n\geq 1}$ of independent Rademacher variables. The following lemma from [72, Theorem 6.2.4] will be used several times in later chapters, we present it here for the reader's convenience.

**Lemma 2.5.1** (Kahane–Khintchine inequality)**.** *Let $(\varepsilon_n)_{n\geq1}$ be a Rademacher sequence. Then for all $0 < p < q < \infty$ and all sequences $(x_n)_{n=1}^N$ in any Banach space $X$, we have*

$$\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^q(\Omega;X)} \leq C\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^p(\Omega;X)}.$$

The space $X$ is said to have *type* $p \in [1,\infty]$ if there exists a constant $\tau_{p,X} > 0$ such that for all $x_1,\cdots,x_n \in X$, we have

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{L^2(\Omega;X)} \leq \tau_{p,X}\Big(\sum_{k=1}^n \|x_k\|_X^p\Big)^{\frac{1}{p}}.$$

The space $X$ is said to have *cotype* $q \in [1,\infty]$ if there exists a constant $c_{q,X} > 0$ such that for all $x_1,\cdots,x_n \in X$, we have

$$\Big(\sum_{k=1}^n \|x_k\|_X^q\Big)^{\frac{1}{q}} \leq c_{q,X}\Big\|\sum_{k=1}^n \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.$$

In the above, the complex Rademacher sequence can be replaced by a real Rademacher sequence (see [72, Proposition 6.1.19]). By Kahane–Khintchine inequality, the space $L^2(\Omega;X)$ in the left hand side of the inequality can be changed to $L^p(\Omega;X)$ for any $p \in [1,\infty)$. By the triangle inequality, every Banach space has type 1. Note that Rademacher sequences are real-symmetric, then by [72, Proposition 6.1.5], we have

$$\|x_1\|_X = \|\varepsilon_1 x_1\|_{L^1(\Omega;X)} \leq C\Big\|\sum_{k=1}^n \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.$$

2

Hence, every Banach space has cotype $\infty$. The space $X$ has type 2 and cotype 2 if and only if $X$ is isomorphic to a Hilbert space ([83, Proposition 3.1]). Moreover, by considering the scalar case, it follows that necessarily $p \in [1,2]$ and $q \in [2,\infty]$ in the above definitions. Indeed, recalling that a Rademacher sequence is a sequence $(\varepsilon_n)_{n=1}^N$ of independent Rademacher random variables, then

$$\mathbb{E}\left\|\sum_{n=1}^N \varepsilon_n x_n\right\|^2 = \operatorname{Var}\left(\sum_{n=1}^N \varepsilon_n x_n\right) = \sum_{n=1}^N \operatorname{Var}(\varepsilon_n x_n) = \sum_{n=1}^N \mathbb{E}\|\varepsilon_n x_n\|^2 = \sum_{n=1}^N \|x_n\|^2.$$

If $\mathbb{C}$ has type $p > 2$, then

$$\left(\sum_{n=1}^N |x_n|^2\right)^{\frac{1}{2}} = \left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^2(\Omega)} \le \tau_p \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}},$$

which leads to a contradiction since $\ell^p \not\subseteq \ell^2$ for any $p > 2$. A similar argument shows that $q \in [2,\infty]$. We say that $X$ has *non-trivial type* if X has type $p \in (1,2]$, and *finite cotype* if it has cotype $q$ for some $q \in [2,\infty)$. If $X$ has type $p$ (cotype $q$), it also has type $u \in [1,p]$ (cotype $v \in [q,\infty]$). If $X$ has type $p$, then $X^*$ has cotype $p'$ (see [72, Proposition 7.1.13]). If $X$ has non-trivial type and finite cotype $q$, then $X^*$ has type $q'$ (see [72, Proposition 7.4.10 and Theorem 7.4.23]). This is the case when $X$ is a UMD or super-reflexive space. Finally, note that Fourier type $p$ implies type $p$ and cotype $p'$ (see [72, Proposition 7.3.6]).

## 2.6. FUNCTION SPACES

In this section, we present two classes of function spaces: variation spaces and Banach function spaces. For comprehensive treatments, we refer the reader to [54] for variation spaces and to [90, 151] and the recent survey [95] for Banach function spaces.

### 2.6.1. BOUNDED VARIATION

We begin by introducing the bounded variation space $V^1$ and its applications in multiplier theory. We will delve deeper into bounded $s$-variation spaces and develop a variational multiplier theory based on the Marcinkiewicz multiplier theorem in Chapter 6.

We say that a sequence $(a_n)_{n\in\mathbb{Z}} \subseteq \ell^\infty(\mathbb{Z};X)$ is said to be of *bounded variation* if

$$[a]_{V^1(\mathbb{Z};X)} := \sum_{n\in\mathbb{Z}} |a_{n+1} - a_n| < \infty,$$

and we denote by $V^1$ the space of all such sequences equipped with the norm

$$\|a\|_{V^1(\mathbb{Z};X)} := \|a\|_{\ell^\infty(\mathbb{Z};X)} + [a]_{V^1(\mathbb{Z};X)}.$$

The following vector-valued Marcinkiewicz multiplier theorem is due to [23]. A corresponding version on $\mathbb{R}$ with operator-valued $m$ can be found in [72, Theorem 8.3.9]. Via transference, the periodic case can also be derived from [72].

**Lemma 2.6.1** (Marcinkiewicz multiplier theorem)**.** *Let $X$ be a* UMD *space and* $1 < p < \infty$*. If $m \in \ell^\infty \cap V^1$, then $m$ is an $L^p$-Fourier multiplier on $\mathbb{T}$, and there exists a constant $M_{p,X} \ge 1$ such that*

$$\|T_m\| \le M_{p,X}\|m\|_{V^1(\mathbb{Z};X)}.$$

### 2.6.2. BANACH FUNCTION SPACES

We briefly introduce Banach function spaces and the associate space, see [90, 151] for more details.

**Definition 2.6.2.** Let $(S, \mathscr{A}, \mu)$ be a $\sigma$-finite measure space and denote the space of measurable functions $f: S \to \mathbb{C}$ by $L^0(S)$. A vector space, $X \subseteq L^0(S)$ equipped with a norm $\|\cdot\|$, is called a *Banach function space over S* if it satisfies the following properties:

- *Ideal property:* If $f \in X$ and $g \in L^0(S)$ with $|g| \le |f|$, then $g \in X$ with $\|g\| \le \|f\|$.
- *Fatou property:* If $0 \le f_n \uparrow f$ for $(f_n)_{n\ge1}$ in $X$ and $\sup_{n\ge1}\|f_n\| < \infty$, then $f \in X$ and $\|f\| = \sup_{n\ge1}\|f_n\|$.
- *Saturation property:* For every measurable $E \subseteq S$ of positive measure, there exists a measurable $F \subseteq E$ of positive measure with $\mathbf{1}_F \in X$.

2

We note that the saturation property is equivalent to the assumption that there is an $f \in X$ such that $f > 0$ almost everywhere. Moreover, the Fatou property ensures that $X$ is complete.

We define the associate space $X'$ of a Banach function space $X$ as the space of all $g \in L^0(S)$ such that

$$\|g\|_{X'} := \sup_{\|f\|\le1} \int_S |fg| \, \mathrm{d}\mu < \infty,$$

which is again a Banach function space. For $g \in X'$, define $\varphi_g : X \to \mathbb{C}$ by

$$\varphi_g(f) := \int_S fg \, \mathrm{d}\mu,$$

which is a bounded linear functional on $X$, i.e. $\varphi_g \in X^*$. Hence, by identifying $g$ and $\varphi_g$, one can regard $X'$ as a closed subspace of $X^*$. Moreover, if $X$ is reflexive (or, more generally, *order-continuous*), then $X' = X^*$.

The following notions, closely connected to type and cotype, will play an important role in Section 5.4 (see [90, Section 1.d] for details).

**Definition 2.6.3.** Let $1 \le p \le q \le \infty$. We call $X$ *p-convex* if for all finite sequences $(x_n)_{n=1}^N \subseteq X$,

$$\left\| \left( \sum_{n=1}^N |x_n|^p \right)^{\frac{1}{p}} \right\| \le \left( \sum_{n=1}^N \|x_n\|^p \right)^{\frac{1}{p}},$$

and we call $X$ *q-concave* if for all finite sequences $(x_n)_{n=1}^N \subseteq X$,

$$\left( \sum_{n=1}^N \|x_n\|^q \right)^{\frac{1}{q}} \le \left\| \left( \sum_{n=1}^N |x_n|^q \right)^{\frac{1}{q}} \right\|.$$

Note that any Banach function space is 1-convex by the triangle inequality, and any Banach function space is $\infty$-concave. Indeed, for any $x_n, x_m \in X$, since $|x_n| \le ||x_n| \vee |x_m||$ and $|x_m| \le ||x_n| \vee |x_m||$, then by the ideal property,

$$\sup_{1\le n\le N} \|x_n\| \le \| \sup_{1\le n\le N} |x_n| \|, \quad x_1, \cdots, x_N \in X.$$

2

One often defines $p$-convexity and $q$-concavity using finite sums of elements from $X$ and a constant in the defining inequalities. However, by [90, Proposition 1.d.8], one can always renorm $X$ such that these constants are equal to one, yielding our definition. Moreover, $X$ is $p$-convex ($p$-concave) if and only if $X^*$ or $X'$ is $p'$-concave ($p'$-convex) by [90, Proposition 1.d.4], since $X'$ is a close subspace of $X^*$. As a simple example, we note that $L^r(S)$ is $p$-convex for all $p \in [1,r]$ and $q$-concave for all $q \in [r,\infty]$. Indeed,

$$\left\|\Big(\sum_{n=1}^N |f_n|^r\Big)^{\frac{1}{r}}\right\|_{L^r(S)}^r = \int_S \sum_{n=1}^N |f_n|^r \,\mathrm{d}\mu = \sum_{n=1}^N \|f_n\|_{L^r(S)}^r,$$

and the other inequalities follow from Minkowski's inequality (see Lemma 2.2.5).

Let $(S,\mathscr{A},\mu)$ be a $\sigma$-finite measure space and denote the space of measurable functions $f\colon S\to\mathbb{C}$ by $L^0(S)$. For $s\in(0,\infty)$ and a Banach function space $X$, define the $s$-*concavification*

$$X^s := \left\{f \in L^0(S) : |f|^{\frac{1}{s}} \in X\right\},$$

equipped with the quasi-norm

$$\|f\|_{X^s} := \left\||f|^{\frac{1}{s}}\right\|_X^s. \tag{2.6.1}$$

If $s\le 1$, $X^s$ is always a Banach function space. For $s>1$, $X^s$ is a Banach function space if and only if $X$ is $s$-convex. Note that for $0<s,p<\infty$, $f\in(L^p(S))^s$,

$$\|f\|_{(L^p(S))^s} = \left(\int_S |f|^{\frac{p}{s}}\,\mathrm{d}\mu\right)^{\frac{s}{p}} < \infty,$$

then we have $(L^p(S))^s = L^{\frac{p}{s}}(S)$. For more details we refer to [2].

## 2.7. Interpolation spaces

In this section, we recall some basic properties of complex and real interpolation spaces. See [97, 142] for more details.

### 2.7.1. Complex interpolation

**Definition 2.7.1.** A Banach space $X$ is called a $\theta$-*intermediate* UMD *Banach space* if it is the complex interpolation space between a UMD Banach space $Z$ and a Hilbert space $H$, i.e. $X=[Z,H]_\theta$ for some $\theta\in(0,1]$.

Note that $X=H$ if $\theta=1$. We say $X$ is an intermediate UMD Banach space if it is $\theta$-intermediate UMD for some $\theta\in(0,1]$. Note that reflexive Lebesgue spaces, Sobolev spaces, Triebel–Lizorkin spaces, and Besov spaces are all intermediate UMD Banach spaces. Since the UMD property is stable under complex interpolation, any intermediate UMD Banach space is a UMD Banach space. Conversely, by a result of Rubio de Francia [133], all UMD Banach *function* spaces are intermediate UMD Banach function spaces. For general Banach spaces, this is an open problem.

Let $0 < \theta_0 < \theta_1 < 1, \lambda := (1-\theta)\theta_0 + \theta\theta_1$, we have the following reiteration identity ([87, Example 6.6]):

$$\big[[Z,H]_{\theta_0},[Z,H]_{\theta_1}\big]_\theta = [Z,H]_\lambda. \tag{2.7.1}$$

The following result on complex interpolation spaces of $L^p$-spaces will be used several times later, see [71, Theorem 2.2.6].

**Proposition 2.7.2.** *Let $1 \le p_0 \le p_1 < \infty$ or $1 \le p_0 < p_1 = \infty$ and let $0 < \theta < 1$. For any interpolation couple $(Z,H)$ of complex Banach spaces and any measure space $(S,\mathscr{A},\mu)$ we have*

$$\big[L^{p_0}(S;Z),L^{p_1}(S;H)\big]_\theta = L^p(S;[Z,H]_\theta),$$

*isometrically, with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.*

### 2.7.2. REAL INTERPOLATION

Let $-A$ be a sectorial operator on a Banach space $X$, and let $\tau \in (0,\infty)$ and $q \in [1,\infty]$. Then the real interpolation space associated with $A$, $\tau$ and $q$ is

$$D_A(\tau,q) := (X,D((-A)^\alpha))_{\tau/\alpha,q}, \tag{2.7.2}$$

where $\alpha \in (\tau,\infty)$ is arbitrary. It follows from reiteration that $D_A(\tau,q)$ is independent of the choice of $\alpha$. In particular, one has

$$D_A(\tau,q) = (X,D((-A)^m))_{\tau/m,q} = (X,D(A^m))_{\tau/m,q}$$

whenever $m \in \mathbb{N}$ satisfies $m > \tau$. By basic properties of interpolation spaces [142, Theorem 1.1.3],

$$D_A(\tau_0,q_0) \subseteq D_A(\tau_1,q_1)$$

if $\tau_1 < \tau_0$, or if $\tau_1 = \tau_0$ and $q_0 \le q_1$. By [63, Corollary 6.6.3],

$$D_A(\tau,1) \subseteq D((-A)^\tau) \subseteq D_A(\tau,\infty) \tag{2.7.3}$$

for all $\tau > 0$. Also, $D((-A)^\alpha)$ is a dense subset of $D_A(\tau,q)$ for all $\alpha > \tau$ and $q < \infty$, by [63, Theorem 6.6.1]. Similarly, we also have reiteration identity ([142, Theorem 1.10.2]):

$$\big(D_A(\tau_0,p),D_A(\tau_1,p)\big)_{\theta,q} = D_A(\tau,q), \tag{2.7.4}$$

where $\tau = (1-\theta)\tau_0 + \theta\tau_1, p,q \in [1,\infty]$. If $X$ has Fourier type $p \in [1,2]$, $-A$ is a sectorial operator and $0 \in \rho(-A)$. Then both $D((-A)^\tau)$ and $D_A(\tau,p)$ also have Fourier type $p$, for all $\tau > 0$.

# 3

# Improved polynomial decay for non-uniformly bounded semigroups

## 3.1. Introduction

### 3.1.1. Setting

We study the asymptotic behaviour of solutions to the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= Au(t), \quad t \geq 0, \\ u(0) &= x, \end{cases} \tag{3.1.1}$$

on a Banach space $X$. We assume that (3.1.1) is well posed, so that the solution operators form a $C_0$-semigroup $(T(t))_{t\geq 0} \subseteq \mathscr{L}(X)$ of bounded operators, with generator $A$. Throughout, we will consider $A$ satisfying $\overline{\mathbb{C}_+} \subseteq \rho(A)$, where $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Under these assumptions, there are two well-known flavors of results that relate information about the resolvent operators to asymptotic behaviour of the semigroup orbits.

Firstly, the classical Gearhart–Huang–Prüss–Greiner theorem [56, 65, 120] says that, if $X$ is a Hilbert space, then the semigroup $(T(t))_{t\geq 0}$ is uniformly stable ($\|T(t)\| \to 0$ as $t \to \infty$), and all orbits decay exponentially to zero, if and only if

$$\sup_{\lambda \in \mathbb{C}_+} \|R(\lambda, A)\|_{\mathscr{L}(X)} < \infty. \tag{3.1.2}$$

Versions of this theorem on non-Hilbertian Banach spaces were discovered later [107, 145, 148]. Here an assumption such as (3.1.2) typically guarantees exponential decay only for sufficiently smooth initial data, with the degree of smoothness depending on

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the geometry of the underlying Banach space. It is relevant to note that all these results make no a priori assumptions on the growth of the semigroup; only spectral information is required.

On the other hand, a more recent line of research considers the setting where the resolvent is not bounded on the right half-plane, but instead blows up along the imaginary axis at a specified rate. In this case the semigroup is not uniformly stable, and one can at best hope to obtain uniform decay rates for sufficiently smooth initial data. Semigroups with these properties arise naturally in the study of the damped wave equation

$$\partial_t^2 u(t,x) = \Delta_g u(t,x) - a(x)\partial_t u(t,x), \quad (t,x) \in \mathbb{R} \times M, \tag{3.1.3}$$

on a Riemannian manifold $(M,g)$, where $a \in C(M)$ [4, 25, 27, 84, 121, 122]. A succession of results in semigroup theory [8, 11, 12, 20, 128] has elucidated the relationship between the rate of resolvent blowup and the rate of decay of classical solutions to (3.1.1), in the case where the semigroup is a priori assumed to be *uniformly bounded* ($\|T(t)\| \le C$ for some $C > 1$). The latter assumption is in turn satisfied if the damping function $a$ in (3.1.3) is non-negative.

However, when considering functions $a$ in (3.1.3) that change sign, the associated semigroup need not be uniformly bounded and one may encounter unexpected spectral behaviour (see e.g. [123, 129]). Moreover, polynomially growing semigroups appear naturally in the analysis of Schrödinger operators with unbounded potentials [38, 62], perturbed wave equations [59, 115], delay differential equations [137] and hyperbolic equations on non-Hilbertian Banach spaces [33, 127].

Hence it is natural to wonder what can be said when one combines some of the difficulties of both the lines of research mentioned above, that is, if the semigroup $(T(t))_{t\ge0}$ is *non-uniformly bounded* ($\|T(t)\| \lesssim e^{\omega t}$ for some $\omega > 0$) and the resolvent is not uniformly bounded on the right half-plane. This is the setting that will be considered in this chapter.

### 3.1.2. PREVIOUS WORK

Throughout this chapter, we will consider $C_0$-semigroups $(T(t))_{t\ge0}$ with generator $A$ such that $\mathbb{C}_+ \subseteq \rho(A)$ and

$$\|R(\lambda,A)\|_{\mathscr{L}(X)} \le C(1+|\lambda|)^\beta, \quad \lambda \in \mathbb{C}_+, \tag{3.1.4}$$

where $\beta, C \ge 0$ are independent of $\lambda$. Under these assumptions, $-A$ is a sectorial operator of angle $\pi/2$. Indeed, the semigroup generation property implies that $\|R(\lambda,A)\|_{\mathscr{L}(X)} \lesssim 1/\operatorname{Re}\lambda$ for $\operatorname{Re}\lambda$ large, which gives a uniform bound in (2.2.1) for $|\lambda|$ large if $\omega' > \pi/2$. On the other hand, (3.1.4) implies that $\overline{\mathbb{C}_+} \subseteq \rho(A)$, which in turn yields the required bound in (2.2.1) for $|\lambda|$ small.

Unless $\beta = 0$, i.e. unless (3.1.2) holds, the resolvent might blow up along the imaginary axis, with polynomial rate at most $O(|\lambda|^\beta)$. As in the work for uniformly bounded semigroups mentioned above, one hopes to derive polynomial rates of decay for semigroup orbits with sufficiently smooth initial data.

In this regard, it was first shown in [8] that, on general Banach spaces, for each $\rho \ge 0$ and $\tau > (\rho+1)\beta + 1$ one has

$$\|T(t)x\|_X \lesssim t^{-\rho}\|x\|_{D((-A)^\tau)} \tag{3.1.5}$$

for all $t \geq 1$ and $x \in D((-A)^{\tau})$. Later, [130] improved this estimate under additional geometric assumptions on the underlying Banach space. Namely, if $X$ has Fourier type $p \in [1,2]$ (see Section 2.3.2), then (3.1.5) holds for each $\tau > (\rho+1)\beta + \frac{1}{p} - \frac{1}{p'}$. Moreover, if $p = 2$, i.e. if $X$ is a Hilbert space, then one may let $\tau = (\rho+1)\beta$. However, it was left as an open question whether one may also let $\tau = (\rho+1)\beta + \frac{1}{p} - \frac{1}{p'}$ for Banach spaces with Fourier type $p \in [1,2)$ (see also [126, Appendix B]).

Recently, it was shown in [134] that the results from [130] regarding (3.1.5) can in fact be improved. More precisely, for each $\rho > 0$ and $\sigma > \frac{1}{p} - \frac{1}{p'}$ one has



$$\|T(t)x\|_X \lesssim t^{-\rho} \log(t)^{\sigma} \|x\|_{D((-A)^{\tau})} \tag{3.1.6}$$

for $t \geq 2$ and $x \in D((-A)^{\tau})$, where $\tau = (\rho+1)\beta + \frac{1}{p} - \frac{1}{p'}$. That is, for $\rho > 0$ and $p \in [1,2)$, (3.1.6) attains the missing endpoint exponent from [130], up to a logarithmic loss. In fact, [134] combined methods from [130] with ones from the theory for bounded semigroups in [11] and considered resolvents with more general growth behaviour, but specializing to polynomially growing resolvents leads to (3.1.6).

Finally, it is important to emphasize that the results from [130] and [134] are far from optimal if the semigroup $(T(t))_{t\geq 0}$ is uniformly bounded. Indeed, in this case [12] yields, on general Banach spaces and for all $\rho \geq 0$,

$$\|T(t)x\|_X \lesssim t^{-\rho} \log(t)^{\rho} \|x\|_{D((-A)^{\tau})} \tag{3.1.7}$$

for $t \geq 2$ and $x \in D((-A)^{\tau})$, where $\tau = \rho\beta$. Moreover, by [20], if $X$ is a Hilbert space then the logarithmic factor in (3.1.7) can be removed, yielding (3.1.5) for $\tau = \rho\beta$. On the other hand, for non-uniformly bounded semigroups on Hilbert spaces and for $\rho = 0$ one cannot in general expect to obtain (3.1.5) for $\tau < (\rho+1)\beta$, as follows from an example of Wrobel (see [149, Example 4.1] and [130, Example 4.20]). We also refer to [130, Section 4.7.1] for an application to polynomially growing semigroups of the combination of (3.1.7) and a rescaling argument.

### 3.1.3. MAIN RESULT

For $\tau > 0$ and $q \in [1,\infty]$, we will work with the real interpolation space

$$D_A(\tau,q) := (X, D(A^m))_{\tau/m,q},$$

where $m \in \mathbb{N}$ with $m > \tau$ arbitrary (see also (2.7.2)). Moreover, we refer to (3.4.2) and (3.4.3) for the definitions of Hardy–Littlewood type and Hardy–Littlewood cotype, respectively. The following is our main result.

**Theorem 3.1.1.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ with Fourier type $p \in [1,2]$. Suppose that $\mathbb{C}_+ \subseteq \rho(A)$, and that there exist $\beta > 0$ and $C \geq 0$ such that (3.1.4) holds. Let $\rho \geq 0$ and set $\tau := (\rho+1)\beta + \frac{1}{p} - \frac{1}{p'}$. Then there exists a $C_\rho \geq 0$ such that*

$$\|T(t)x\|_X \leq C_\rho t^{-\rho} \|x\|_{D_A(\tau,p)} \tag{3.1.8}$$

*for all $t \geq 1$ and $x \in D_A(\tau,p)$. If $\rho > 0$, then (3.1.8) also holds with $D_A(\tau,p)$ replaced by $D_A(\tau,q)$ for any $q \in [1,\infty]$, or by $D((-A)^{\tau})$.*

*Suppose, additionally, that $p > 1$ and that $X$ has Hardy–Littlewood type $p$ or Hardy–Littlewood cotype $p'$. Then, for $\rho = 0$, (3.1.8) also holds with $D_A(\tau, p)$ replaced by $D((-A)^\tau)$.*

The first two statements of Theorem 3.1.1 are contained in the main text as Theorem 3.3.5, while the last statement is Theorem 3.4.3.

Given that any Banach space has Fourier type $p = 1$, the first part of Theorem 3.1.1 applies to general Banach spaces. For $p \in (1,2]$, the assumptions on $X$ in Theorem 3.1.1 are satisfied in particular if $X$ is isomorphic to a closed subspace of $L^r(S)$, for $(S, \mathscr{A}, \mu)$ a measure space and $r = p$ or $r = p'$ (see Section 3.4).

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For $p \in [1,2)$, the first part of Theorem 3.1.1 improves (3.1.6) by removing the logarithmic factor for $\rho > 0$, and it yields an endpoint result for $\rho = 0$. The second part of Theorem 3.1.1 in turn fully extends (3.1.5) to $\rho = 0$ and $\tau = \beta + \frac{1}{p} - \frac{1}{p'}$, under additional geometric assumptions. Also note that, for all $p \in [1,2]$ and $\rho > 0$, (3.1.8) involves a larger space of initial data than considered in [130] and [134], since $D((-A)^\tau) \subseteq D_A(\tau, \infty)$. On the other hand, for $\rho = 0$, (3.1.8) complements the main result of [130] on Hilbert spaces, since in general one neither has $D((-A)^\tau) \subseteq D_A(\tau, 2)$ nor $D_A(\tau, 2) \subseteq D((-A)^\tau)$.

The exponent $\tau$ in Theorem 3.1.1 is sharp for $p = 2$ and $\rho = 0$, as noted above, and for general $p \in [1,2]$ as $\beta \to 0$, as follows from a modification of an example of Arendt concerning exponential stability (see [5, Example 5.1.11] and [148, Section 4]). We do not know whether, for a general Banach space with Fourier type $p \in [1,2)$ and for $\rho = 0$, (3.1.8) also holds with $D_A(\tau, p)$ replaced by $D((-A)^\tau)$.

For any $C_0$-semigroup $(T(t))_{t\geq 0}$ there exists an $\omega \in \mathbb{R}$ such that

$$\|T(t)\|_{\mathscr{L}(X)} \lesssim e^{\omega t}, \quad t \geq 0. \tag{3.1.9}$$

It follows from (3.1.9) that (3.1.4) holds whenever $\operatorname{Re}\lambda \geq \omega + 1$. This is because by [50, Theorem I.1.10],

$$\|R(\lambda, A)\|_{\mathscr{L}(X)} \lesssim \frac{1}{\operatorname{Re}\lambda - \omega} \lesssim (1 + |\lambda|)^\beta.$$

Moreover, (3.1.4) directly extends to $\lambda \in i\mathbb{R}$ as well. Hence (3.1.4) is in fact an assumption on the growth of the resolvent as $\lambda$ tends to infinity in the strip $\{\lambda \in \mathbb{C} \mid 0 \leq \operatorname{Re}\lambda \leq \omega + 1\}$.

One may weaken assumption (3.1.4) somewhat, by requiring instead that

$$\|R(\lambda, A)\|_{\mathscr{L}(X)} \lesssim (1 + |\lambda|)^{\beta_0}, \quad \lambda \in \mathbb{C}_+ \tag{3.1.10}$$

for *some* $\beta_0 > 0$, and that

$$\|R(i\xi, A)\|_{\mathscr{L}(X)} \lesssim (1 + |\xi|)^\beta, \quad \xi \in \mathbb{R}.$$

Then the conclusion of Theorem 3.1.1 still holds, and the specific value of $\beta_0$ in (3.1.10) plays no role. Indeed, the place in the proof of Theorem 3.1.1 where one genuinely uses polynomial resolvent bounds for $\lambda \in \mathbb{C}_+$ is in the proofs of Theorems 3.3.4 and 3.4.3, to obtain a dense subset of initial values for which the semigroup orbits are integrable, and there the value of $\beta_0$ is irrelevant. Instead, as in the theory for uniformly bounded semigroups, to obtain concrete rates of decay we work with the behaviour of the resolvent on the imaginary axis.

As in [130, 134], our techniques in principle also allow for $A$ to have a singularity at zero. More precisely, one could suppose that (3.1.4) holds for $|\lambda| \geq 1$, and that there exists an $\alpha > 0$ such that $\|R(\lambda, A)\|_{\mathscr{L}(X)} \lesssim |\lambda|^{-\alpha}$ for $|\lambda| < 1$. In this case one has to assume additionally that $-A$ is an injective sectorial operator, and the initial values have to be restricted to the range of a suitable fractional power of $-A$. For simplicity, we will not consider such a setting in this article.

### 3.1.4. THE STRATEGY OF THE PROOF AND ORGANIZATION

Our approach is similar to that in [130] (see also [126]), applying Fourier multiplier theory to the resolvent on the imaginary axis. However, whereas [130] mostly involved Fourier multipliers from $L^p(\mathbb{R}; Y)$ to $L^q(\mathbb{R}; X)$ for suitable $1 \le p \le q \le \infty$ and $Y \subseteq X$, in the present chapter we proceed differently.

Namely, the first part of Theorem 3.1.1 is proved using Proposition 3.2.2, which considers multipliers between the Besov space $B^s_{p,p}(\mathbb{R}; Y)$ and $L^{p'}(\mathbb{R}; X)$, for suitable values of $p$ and $s$. Working with such multipliers allows us to obtain endpoint estimates. In turn, Besov spaces are intimately connected to the real interpolation method, and in Proposition 3.3.2 we show that real interpolation spaces can also be used effectively to cancel out resolvent growth, as is required to satisfy the conditions of our Fourier multiplier theorems. This somewhat different approach also necessitates other changes to the setup from [130].

On the other hand, for the second part of Theorem 3.1.1 we consider Fourier multipliers between weighted spaces $L^p(\mathbb{R}, w; Y)$ and $L^q(\mathbb{R}, v; X)$, for suitable weights $w$ and $v$. This setting allows us to obtain endpoint results involving fractional domains, at the cost of having to make a priori assumptions about the mapping properties of the Fourier transform between such weighted spaces.

This chapter is arranged as follows. In Section 3.3 we then prove the first part of Theorem 3.1.1, and in Section 3.4 we prove the final statement in Theorem 3.1.1.

## 3.2. PRELIMINARIES

### 3.2.1. NOTATION IN THIS CHAPTER

In this chapter, we define the Fourier transform $\mathscr{F} : L^1(\mathbb{R}; X) \to L^\infty(\mathbb{R}; X)$ by

$$\mathscr{F} f(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}} e^{-i\xi t} f(t)\,\mathrm{d}t, \;\; \xi \in \mathbb{R},$$

and the inverse Fourier transform $\mathscr{F}^{-1} : L^1(\mathbb{R}; X) \to L^\infty(\mathbb{R}; X)$ is defined by

$$\mathscr{F}^{-1} f(t) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} f(\xi)\,\mathrm{d}\xi, \quad t \in \mathbb{R}.$$

We write $f'$ for the first-order derivative of a function $f : \mathbb{R} \to X$.

Let $Y$ be a Banach space and $m : \mathbb{R} \to \mathscr{L}(X, Y)$. We say that $m$ is $X$-*strongly measurable* if $\xi \mapsto m(\xi)y$ is a strongly measurable $X$-valued map for every $y \in Y$. Throughout this chapter, we will consider $m$ that has the additional property that there exist $\beta, C_\beta \ge 0$ such that $\|m(\xi)\|_{\mathscr{L}(X,Y)} \le C_\beta (1 + |\xi|)^\beta$ for all $\xi \in \mathbb{R}$. In this case, we may set

$$T_m f := \mathscr{F}^{-1}(m\widehat{f}) \quad f \in \mathscr{S}(\mathbb{R}; X).$$

Then $T_m : \mathscr{S}(\mathbb{R}; X) \to \mathscr{S}'(\mathbb{R}; Y)$ is the *Fourier multiplier operator* with *symbol* $m$.

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### 3.2.2. BESOV SPACES

Next we introduce the space we mainly considered in this chapter: the Besov space. We refer the readers to [73] for more details. Throughout this chapter, fix an inhomogeneous Littlewood–Paley sequence $(\phi_k)_{k\geq 0} \subseteq C_c^\infty(\mathbb{R})$. That is, one has $\phi_1(\xi) = 0$ if $|\xi| \notin [1/2, 2]$, $\phi_k(\xi) = \phi_1(2^{-k+1}\xi)$ for each $k > 1$ and $\xi \in \mathbb{R}$, and

$$\sum_{k=0}^{\infty} \phi_k(\xi) = 1, \quad \xi \in \mathbb{R}.$$

Let $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Then the Besov space $B^s_{p,q}(\mathbb{R}; X)$ consists of all $f \in \mathscr{S}'(\mathbb{R}; X)$ such that $\mathscr{F}^{-1}(\phi_k) * f \in L^p(\mathbb{R}; X)$ for each $k \geq 0$, and such that

$$\|f\|_{B^s_{p,q}(\mathbb{R};X)} := \|(2^{ks}\mathscr{F}^{-1}(\phi_k) * f)_{k\geq 0}\|_{\ell^q(L^p(\mathbb{R};X))} < \infty.$$

Then $\mathscr{S}(\mathbb{R}; X) \subseteq B^s_{p,q}(\mathbb{R}; X)$ and the embedding is dense if $p, q < \infty$ ([73, Proposition 14.4.3]). Finally, we will use the simple observation that

$$B^{s_1}_{p,q_1}(\mathbb{R}; X) \subseteq B^{s_2}_{p,q_2}(\mathbb{R}; X) \tag{3.2.1}$$

for all $p, q_1, q_2 \in [1, \infty]$, $s_1 > s_2$, or $q_1 \leq q_2, s_1 = s_2$. Furthermore, by [73, Proposition 14.4.18],

$$B^0_{p,1}(\mathbb{R}; X) \subseteq L^p(\mathbb{R}; X) \subseteq B^0_{p,\infty}(\mathbb{R}; X) \tag{3.2.2}$$

for all $p \in [1, \infty]$.

The following lemma will be used in the proof of Proposition 3.3.1.

**Lemma 3.2.1.** *Let $p \in [1, \infty)$, $q \in [1, \infty]$ and $s \in (0, 1/p)$. Then there exists a $C \geq 0$ such that $\mathbf{1}_{(0,\infty)} f \in B^s_{p,q}(\mathbb{R}; X)$ for all $f \in B^s_{p,q}(\mathbb{R}; X)$, and*

$$\|\mathbf{1}_{(0,\infty)} f\|_{B^s_{p,q}(\mathbb{R};X)} \leq C\|f\|_{B^s_{p,q}(\mathbb{R};X)}.$$

*Proof.* For $p > 1$, the statement in fact holds for $s \in (-1/p, 1/p)$, as is shown in [73, Corollary 14.6.35]. In the proof of the latter result, one can see that for $s \in (0, 1/p)$ one may also allow $p = 1$. □

Finally, the following Fourier multiplier result, [73, Proposition 14.5.7], is one of the key ingredients in the proof of the first half of Theorem 3.1.1.

**Proposition 3.2.2.** *Let $X$ and $Y$ be Banach spaces with Fourier type $p \in [1, 2]$, and let $m : \mathbb{R} \to \mathscr{L}(Y, X)$ be $X$-strongly measurable, with $\sup_{\xi\in\mathbb{R}} \|m(\xi)\|_{\mathscr{L}(Y,X)} < \infty$. Then*

$$T_m : B^{1/p-1/p'}_{p,p}(\mathbb{R}; Y) \to L^{p'}(\mathbb{R}; X)$$

*is bounded.*

## 3.3. POLYNOMIAL STABILITY ON REAL INTERPOLATION SPACES

This section is devoted to the proof of the first half of Theorem 3.1.1. To this end, we need some preliminary results.

The following proposition, connecting interpolation spaces to the Besov spaces from Chapter 2, will play a key role in the proof of part of Theorem 3.3.4.

**Proposition 3.3.1.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$, and suppose that $-A$ is a sectorial operator. Let $M \geq 0$, $\omega \in \mathbb{R}$ be such that $\|T(t)\|_{\mathscr{L}(X)} \leq Me^{(\omega-1)t}$ for all $t \geq 0$, and let $p \in [1,\infty)$, $q \in [1,\infty]$ and $s \in (0,1/p)$. Then there exists a constant $C \geq 0$ such that $[t \mapsto \mathbf{1}_{(0,\infty)}(t)e^{-\omega t}T(t)x] \in B^s_{p,q}(\mathbb{R};X)$ for all $x \in D_A(s,q)$, with*

$$\|[t \mapsto \mathbf{1}_{(0,\infty)}(t)e^{-\omega t}T(t)x]\|_{B^s_{p,q}(\mathbb{R};X)} \leq C\|x\|_{D_A(s,q)}.$$

*Proof.* Let $J : X \to L^p(\mathbb{R};X)$ be the bounded linear operator given by $Jx(t) := e^{-\omega|t|}T(|t|)x$, for $x \in X$ and $t \in \mathbb{R}$. Since $(Jx)'(t) = -\mathrm{sign}(t)e^{-\omega|t|}T(|t|)(\omega - A)x$ for $x \in D(A)$ and $t \neq 0$, then

$$\|Jx\|_{W^{1,p}(\mathbb{R};X)} \lesssim \|Jx\|_{L^p(\mathbb{R};X)} + \|(Jx)'\|_{L^p(\mathbb{R};X)} \lesssim \|x\|_X + \|(\omega - A)x\|_X \lesssim \|x\|_{D(A)}.$$

This proves the restricted operator $J : D(A) \to W^{1,p}(\mathbb{R};X)$ is bounded. Note that by [73, Theorem 14.4.31],

$$(X, D(A))_{s,q} = D_A(s,q), \quad (L^p(\mathbb{R};X), W^{1,p}(\mathbb{R};X))_{s,q} = B^s_{p,q}(\mathbb{R};X).$$

Real interpolation shows that $J : D_A(s,q) \to B^s_{p,q}(\mathbb{R};X)$ is bounded as well. Now the proof is concluded by applying Lemma 3.2.1. □

In turn, the following proposition will be crucial for the proof of Theorem 3.1.1.

**Proposition 3.3.2.** *Let $A$ be be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. Suppose that $\mathbb{C}_+ \subseteq \rho(A)$, and that there exist $\beta > 0$ and $C \geq 0$ such that*

$$\|R(\lambda, A)\|_{\mathscr{L}(X)} \leq C(1 + |\lambda|)^\beta, \quad \lambda \in \mathbb{C}_+. \tag{3.3.1}$$

*Then $i\mathbb{R} \subseteq \rho(A)$, and*

$$\sup\{\|R(i\xi, A)^k\|_{\mathscr{L}(D_A((n+1)\beta,q),X)} \mid \xi \in \mathbb{R}, k \in \{0,\ldots,n+1\}\} < \infty$$

*for all $n \in \mathbb{N}_0$ and $q \in [1,\infty]$.*

*Proof.* The required statement is trivial for $k = 0$. If $k > 0$ and $|\xi| < 1$, for all $n \in \mathbb{N}_0$ and $q \in [1,\infty]$, (3.3.1) yields

$$\|R(i\xi, A)^k x\|_X \leq C2^{(n+1)\beta}\|x\|_X \leq C2^{(n+1)\beta}\|x\|_{D_A((n+1)\beta,q)}, \quad x \in D_A((n+1)\beta, q).$$

Henceforth we will consider $k > 0$ and $\xi \in \mathbb{R}$ with $|\xi| \geq 1$.

By basic properties of resolvents, it follows from (3.3.1) that $i\mathbb{R} \subseteq \rho(A)$ and

$$\|R(i\xi, A)\|_{\mathscr{L}(X)} \leq C(1 + |\xi|)^\beta, \quad \xi \in \mathbb{R}. \tag{3.3.2}$$

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Moreover, we conclude from the assumptions that $D(-A)\cap \operatorname{Ran}(-A)$ is dense in $X$, and $-A$ is a sectorial operator of angle $\pi/2$. Then by [73, Proposition 15.2.12], $D((-A)^{\beta}) = D((1-A)^{\beta})$ for any $\beta>0$ with equivalent graph norms. Note that

$$\|R(i\xi,A)(-A)^{-\beta}\|_{\mathscr{L}(X)} \eqsim \|R(i\xi,A)\|_{\mathscr{L}(D((-A)^{\beta}),X)},$$

(3.3.2) and [130, Proposition 3.4] yield

$$\begin{aligned}\sup_{|\xi|\geq 1}\|R(i\xi,A)(-A)^{-\beta}\|_{\mathscr{L}(X)} &\eqsim \sup_{|\xi|\geq 1}\|R(i\xi,A)\|_{\mathscr{L}(D((-A)^{\beta}),X)}\\ &\eqsim \sup_{|\xi|\geq 1}\|R(i\xi,A)\|_{\mathscr{L}(D((1-A)^{\beta}),X)}<\infty.\end{aligned} \tag{3.3.3}$$

Then, for $k\in\{1,\dots,n\}$ and $|\xi|\geq 1$,

$$\begin{aligned}\|R(i\xi,A)^k\|_{\mathscr{L}(D((-A)^{n\beta}),X)} &\eqsim \|R(i\xi,A)^k(-A)^{-n\beta}\|_{\mathscr{L}(X)}\\ &\leq \|R(i\xi,A)^k(-A)^{-k\beta}\|_{\mathscr{L}(X)}\|(-A)^{-(n-k)\beta}\|_{\mathscr{L}(X)} \lesssim 1.\end{aligned}$$

Together with (3.3.2), this implies

$$\|R(i\xi,A)^k\|_{\mathscr{L}(D((-A)^{n\beta}),X)} \lesssim (1+|\xi|)^{\beta}, \quad k\in\{1,\dots,n+1\}. \tag{3.3.4}$$

On the other hand, another application of [130, Proposition 3.4] shows that

$$\|R(i\xi,A)(-A)^{-\beta-1}\|_{\mathscr{L}(X)} \lesssim (1+|\xi|)^{-1}.$$

This, combined with (3.3.3), yields

$$\begin{aligned}&\|R(i\xi,A)^k\|_{\mathscr{L}(D((-A)^{(n+1)\beta+1}),X)} \eqsim \|R(i\xi,A)^k(-A)^{-(n+1)\beta-1}\|_{\mathscr{L}(X)}\\ &\leq \|R(i\xi,A)(-A)^{-\beta}\|^{k-1}_{\mathscr{L}(X)}\|R(i\xi,A)(-A)^{-\beta-1}\|_{\mathscr{L}(X)}\|(-A)^{-\beta}\|^{n-k+1}_{\mathscr{L}(X)}\\ &\lesssim (1+|\xi|)^{-1},\end{aligned} \tag{3.3.5}$$

for all $k\in\{1,\dots,n+1\}$ and $|\xi|\geq 1$.

Now, by (2.7.3), (3.3.4) and (3.3.5), we have

$$\begin{aligned}\|R(i\xi,A)^k\|_{\mathscr{L}(D_A(n\beta,1),X)} &\lesssim (1+|\xi|)^{\beta},\\ \|R(i\xi,A)^k\|_{\mathscr{L}(D_A((n+1)\beta+1,1),X)} &\lesssim (1+|\xi|)^{-1},\end{aligned}$$

for $|\xi|\geq 1$ and $k\in\{1,\dots,n+1\}$. Finally, by (2.7.4)

$$D_A((n+1)\beta,q) = (D_A(n\beta,1),D_A((n+1)\beta+1,1))_{\frac{\beta}{1+\beta},q},$$

interpolating these estimates yields $\sup_{|\xi|\geq 1}\|R(i\xi,A)^k\|_{\mathscr{L}(D_A((n+1)\beta,q),X)}<\infty$. □

We next give the following extension of [130, Proposition 3.2] to the mixed Besov-Lebesgue setting.

**Proposition 3.3.3.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$, and let $Y$ be a Banach space that is continuously embedded in $X$. Suppose that $i\mathbb{R}\subseteq\rho(A)$, and that there exist $\beta>0$ and $C\geq 0$ such that*

$$\|R(i\xi,A)\|_{\mathscr{L}(Y,X)}\leq C(1+|\xi|)^{\beta},\quad \xi\in\mathbb{R}. \tag{3.3.6}$$

*Let $p\in[1,\infty)$ and $s\in[0,\infty)$ be such that either $s>0$, or $s=0$ and $p=1$, and suppose that there exist $q\in[1,\infty]$ and $n\in\mathbb{N}$ such that*

$$T_{R(i\cdot,A)^j}: B^s_{p,p}(\mathbb{R};Y)\to L^q(\mathbb{R};X) \tag{3.3.7}$$

*is bounded for each $j\in\{n-1,n\}\cap\mathbb{N}$. Then*

$$T_{R(i\cdot,A)^n}: B^s_{p,p}(\mathbb{R};Y)\to L^\infty(\mathbb{R};X)$$

*is bounded.*

We only assume (3.3.6) to guarantee that the Fourier multiplier operator in (3.3.7) is well-defined; the specific value of $\beta$ plays no role here.

*Proof.* The proof is analogous to that of [130, Proposition 3.2]. For the convenience of the reader, we provide the argument. By Lemma 2.2.4, it suffices to show that there exists a $C\geq 0$ such that

$$\sup_{k\leq\gamma\leq k+1}\|T_{R(i\cdot,A)^n}f(\gamma)\|_X\leq C\|f\|_{B^s_{p,p}(\mathbb{R};Y)} \tag{3.3.8}$$

for every $f\in\mathscr{S}(\mathbb{R};Y)$ and $k\in\mathbb{Z}$, since $\mathscr{S}(\mathbb{R};Y)$ is a dense subset of $B^s_{p,p}(\mathbb{R};Y)$.

By the assumption, for each $j\in\{n-1,n\}\cap\mathbb{N}$, there exists a $K_j\geq 0$ independent of $f$ such that

$$\|T_{R(i\cdot,A)^j}f\|_{L^q(\mathbb{R};X)}\leq K_j\|f\|_{B^s_{p,p}(\mathbb{R};Y)}. \tag{3.3.9}$$

Hence for every $k\in\mathbb{Z}$, there exists a $t\in[k-1,k]$ such that

$$\|T_{R(i\cdot,A)^j}f(t)\|_X\leq K_j\|f\|_{B^s_{p,p}(\mathbb{R};Y)}. \tag{3.3.10}$$

Otherwise, there exists some $k_0$ such that for all $t\in[k_0-1,k_0]$,

$$\|T_{R(i\cdot,A)^j}f(t)\|_X> K_j\|f\|_{B^s_{p,p}(\mathbb{R};Y)}.$$

Then

$$\|T_{R(i\cdot,A)^j}f\|_{L^q(\mathbb{R};X)}\geq\|T_{R(i\cdot,A)^j}f\|_{L^q([k_0-1,k_0];X)}> K_j\|f\|_{B^s_{p,p}(\mathbb{R};Y)},$$

which is contradict to (3.3.9).

Now let $\tau\in[0,2]$. By direct calculation,

$$\begin{aligned} T(\tau)T_{R(i\cdot,A)^n}f(t) &= \tfrac{1}{2\pi}\int_{\mathbb{R}}e^{i\xi(t+\tau)}e^{-i\xi\tau}T(\tau)R(i\xi,A)R(i\xi,A)^{n-1}\widehat{f}(\xi)\,\mathrm{d}\xi\\ &= \tfrac{1}{2\pi}\int_{\mathbb{R}}e^{i\xi(t+\tau)}R(i\xi,A)^n\widehat{f}(\xi)\,\mathrm{d}\xi\\ &\quad -\tfrac{1}{2\pi}\int_{\mathbb{R}}\int_0^\tau e^{i\xi(t+\tau-r)}T(r)R(i\xi,A)^{n-1}\widehat{f}(\xi)\,\mathrm{d}r\,\mathrm{d}\xi\\ &= T_{R(i\cdot,A)^n}f(t+\tau)-\int_0^\tau T(r)T_{R(i\cdot,A)^{n-1}}f(t+\tau-r)\,\mathrm{d}r. \end{aligned} \tag{3.3.11}$$

Hence, by (3.3.10), Hölder's inequality and (3.3.9), for $n>1$ one has

$$\begin{aligned}\|T_{R(i\cdot,A)^n}f(t+\tau)\|_X &\lesssim \|T_{R(i\cdot,A)^n}f(t)\|_X + \int_0^\tau \|T_{R(i\cdot,A)^{n-1}}f(t+\tau-r)\|_X\,\mathrm{d}r\\ &\leq K_n\|f\|_{B^s_{p,p}(\mathbb{R};Y)} + \tau^{1/q'}\|T_{R(i\cdot,A)^{n-1}}f\|_{L^q(\mathbb{R};X)}\\ &\lesssim \|f\|_{B^s_{p,p}(\mathbb{R};Y)}.\end{aligned}$$

This implies (3.3.8) for $n>1$.

Finally, for $n=1$, by the assumptions on $p$ and $s$ as well as (3.2.1) and (3.2.2), one has $B^s_{p,p}(\mathbb{R};Y)\subseteq B^0_{p,1}(\mathbb{R};Y)\subseteq L^p(\mathbb{R};Y)$. Hence Hölder's inequality gives

$$\int_0^\tau \|f(t+\tau-r)\|_X\,\mathrm{d}r \lesssim \int_0^\tau \|f(t+\tau-r)\|_Y\,\mathrm{d}r \leq \tau^{1/p'}\|f\|_{L^p(\mathbb{R};Y)} \lesssim \|f\|_{B^s_{p,p}(\mathbb{R};Y)}.$$

Combining this with (3.3.11) gives

$$\|T_{R(i\cdot,A)}f(t+\tau)\|_X \lesssim \|T_{R(i\cdot,A)}f(t)\|_X + \int_0^\tau \|f(t+\tau-r)\|_X\,\mathrm{d}r \lesssim \|f\|_{B^s_{p,p}(\mathbb{R};Y)},$$

finishing the proof. □

We will also rely on the following version of [130, Theorem 4.6] in the mixed Besov-Lebesgue setting.

**Theorem 3.3.4.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. Suppose that $\mathbb{C}_+\subseteq\rho(A)$, and that there exist $\beta>0$ and $C\geq 0$ such that*

$$\|R(\lambda,A)\|_{\mathscr{L}(X)} \leq C(1+|\lambda|)^\beta, \quad \lambda\in\mathbb{C}_+. \tag{3.3.12}$$

*Let $\gamma>0$, $p\in[1,\infty)$ and $s\in(0,1/p)$, and suppose that there exist $n\in\mathbb{N}_0$ and $q\in[1,\infty]$ such that*

$$T_{R(i\cdot,A)^k}: B^s_{p,p}(\mathbb{R};D_A(\gamma,p)) \to L^q(\mathbb{R};X) \tag{3.3.13}$$

*is bounded for each $k\in\{n-1,n,n+1\}\cap\mathbb{N}$. Then there exists a $C_n\geq 0$ such that*

$$\|T(t)x\|_X \leq C_n t^{-n}\|x\|_{D_A(\gamma+s,p)} \tag{3.3.14}$$

*for all $t\geq 1$ and $x\in D_A(\gamma+s,p)$.*

Note that (3.3.12) automatically extends to all $\lambda\in\overline{\mathbb{C}_+}$, so (3.3.13) is well-defined. Also, as in Proposition 3.3.3, the specific value of $\beta$ in (3.3.12) plays no role.

*Proof.* We want to show that $\|T(t)x\|_X\leq C_n t^{-n}\|x\|_{D_A(\gamma+s,p)}$ for all $t\geq 1$ and $x\in D_A(\gamma+s,p)$. As stated in Section 2.7, $D(A^l)$ is dense in $D_A(\gamma+s,p)$ whenever $l\in\mathbb{N}$ satisfies $l>\gamma+s$, we then may suppose throughout that $x\in D(A^l)$ for some large $l$ due to Lemma 2.2.4. Hence, setting $g(t):=t^n\mathbf{1}_{(0,\infty)}(t)T(|t|)x$ for $t\in\mathbb{R}$, by [130, Lemma 4.3], one can find a $\rho<\frac{\tau-1}{\beta}-1$ large enough such that $n-\rho<-1$, and $\|T(t)\|\leq C_\rho t^{-\rho}$ for $t\geq 1$. Thus,

$$\int_{\mathbb{R}} t^n\mathbf{1}_{\mathbb{R}_+}(t)T(|t|)x\,dt = \int_1^\infty t^nT(t)x\,dt + \int_0^1 t^nT(t)x\,dt$$

$$\leq C_\rho \int_1^\infty t^n t^{-\rho} \|x\| dt + \int_0^1 t^n \|T(t)\| \|x\| dt < \infty.$$

Then $g \in L^1(\mathbb{R}; X)$. In turn, [130, Lemma 3.1] then implies that $\widehat{g}(\xi) = n! R(i\xi, A)^{n+1} x$ for all $\xi \in \mathbb{R}$.

Next, let $(T(t))_{t\geq 0}$ restrict to a $C_0$-semigroup on $D_A(\gamma, p)$, we may fix $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\|_{\mathcal{L}(D_A(\gamma,p))} \leq Me^{(\omega-1)t}$ for all $t \geq 0$. Set $f(t) := \mathbf{1}_{(0,\infty)}(t) e^{-\omega t} T(|t|)x$. Recalling that $D_A(\gamma + s, p) \subseteq D_A(\gamma, p)$, then

$$\|f\|_{L^\infty(\mathbb{R}; D_A(\gamma,p))} \leq M\|x\|_{D_A(\gamma,p)} \lesssim \|x\|_{D_A(\gamma+s,p)}. \tag{3.3.15}$$

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Moreover, the generator of $(T(t))_{t\geq 0}$ is the part of $A$ in $D_A(\gamma, p)$, which has domain $D_A(\gamma + 1, p)$. Then

$$\|f\|_{B^s_{p,p}(\mathbb{R}; D_A(\gamma,p))} \lesssim \|x\|_{(D_A(\gamma,p), D_A(\gamma+1,p))_{s,p}} \lesssim \|x\|_{D_A(\gamma+s,p)}, \tag{3.3.16}$$

as follows from Proposition 3.3.1 and [142, Theorem 1.10.2]. By [143, Proposition 2.3.1], we have $\omega + i\xi \in \rho(A)$ and for all $\xi \in \mathbb{R}$,

$$R(\omega + i\xi, A)x = \int_0^\infty e^{-(\omega+i\xi)t} T(t)x \, \mathrm{d}t = \widehat{f}(\xi).$$

In particular, if we set

$$m(\xi) := n!(R(i\xi, A)^n + \omega R(i\xi, A)^{n+1}),$$

then $m(\xi)\widehat{f}(\xi) = \widehat{g}(\xi)$.

By combining all this, we see that

$$\begin{aligned} \sup_{t\geq 0} \|t^n T(t)x\|_X &= \|T_m f\|_{L^\infty(\mathbb{R};X)} \\ &\leq n!\big(\|T_{R(i\cdot,A)^n} f\|_{L^\infty(\mathbb{R};X)} + \omega \|T_{R(i\cdot,A)^{n+1}} f\|_{L^\infty(\mathbb{R};X)}\big). \end{aligned} \tag{3.3.17}$$

For $n > 0$, one can apply (3.3.13) and Proposition 3.3.3 to the final line, and then use (3.3.16) as well, to obtain

$$\sup_{t\geq 0} \|t^n T(t)x\|_X \lesssim \|f\|_{B^s_{p,p}(\mathbb{R}; D_A(\gamma,p))} \lesssim \|x\|_{D_A(\gamma+s,p)}.$$

For $n = 0$, the same reasoning can be used for the second term in brackets in (3.3.17), while for the first term one can directly rely on (3.3.15), since $T_{R(i\cdot,A)^0} f = f$. □

We are now ready to prove the first part of Theorem 3.1.1.

**Theorem 3.3.5.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ with Fourier type $p \in [1,2]$. Suppose that $\mathbb{C}_+ \subseteq \rho(A)$, and that there exist $\beta > 0$ and $C \geq 0$ such that*

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq C(1 + |\lambda|)^\beta, \quad \lambda \in \mathbb{C}_+.$$

*Let $\rho \geq 0$ and set $\tau := (\rho + 1)\beta + \frac{1}{p} - \frac{1}{p'}$. Then there exists a $C_\rho \geq 0$ such that*

$$\|T(t)x\|_X \leq C_\rho t^{-\rho} \|x\|_{D_A(\tau,p)} \tag{3.3.18}$$

*for all $t \geq 1$ and $x \in D_A(\tau, p)$. Moreover, if $\rho > 0$, then (3.3.18) also holds with $D_A(\tau, p)$ replaced by $D_A(\tau, q)$ for any $q \in [1,\infty]$, or by $D((-A)^\tau)$.*

*Proof.* We first consider the case where $\rho \in \mathbb{N}_0$. Note that $X$ having Fourier type $p$ implies $D_A((\rho+1)\beta, p)$ also have Fourier type $p$. Moreover, by Proposition 3.3.2,

$$\sup_{\xi\in\mathbb{R}} \|R(i\xi, A)^k\|_{\mathcal{L}(D_A((\rho+1)\beta,p),X)} < \infty$$

for all $k \in \{0,\dots,\rho+1\}$. Hence Proposition 3.2.2 implies that

$$T_{R(i\cdot,A)^k} : B^{1/p-1/p'}_{p,p}(\mathbb{R}; D_A((\rho+1)\beta, p)) \to L^{p'}(\mathbb{R}; X)$$

is bounded for every $k \in \{0,\dots,\rho+1\}$ since $R(i\cdot, A)^k$ is $X$-strongly measurable. Finally, Theorem 3.3.4 yields (3.3.18).

To extend (3.3.18) to general $\rho \geq 0$ we proceed as follows. Fix $t \geq 1$ and $\rho > 0$. Let $\rho_0, \rho_1 \in \mathbb{N}_0$ be such that $\rho_0 < \rho < \rho_1$, and let $\theta \in (0,1)$ be such that $\rho = (1-\theta)\rho_0 + \theta\rho_1$. Set $\tau_i := (\rho_i + 1)\beta + \frac{1}{p} - \frac{1}{p'}$ for $i \in \{0,1\}$. Then, by what we have already shown,

$$\|T(t)\|_{\mathcal{L}(D_A(\tau_i,p),X)} \leq C_{\rho_i} t^{-\rho_i}$$

for some constant $C_{\rho_i} \geq 0$ independent of $t$. Now, due to reiteration, real interpolation with parameters $\theta$ and $q \in [1,\infty]$ gives

$$\|T(t)\|_{\mathcal{L}(D_A(\tau,q),X)} \leq C_\rho t^{-\rho}$$

for some $C_\rho \geq 0$ independent of $t$. This proves both (3.3.18) and the final statement of the theorem, since $D((-A)^\tau) \subseteq D_A(\tau,\infty)$. □

## 3.4. Polynomial Stability on Fractional Domains

This section is devoted to the proof of the final statement in Theorem 3.1.1. For $\gamma \in \mathbb{R}$, denote the weight $w_\gamma : \mathbb{R} \to [0,\infty)$ by

$$w_\gamma(x) := |x|^\gamma, \quad x \in \mathbb{R}. \tag{3.4.1}$$

We say that $X$ has *Hardy–Littlewood type* $p \in (1,2]$ if

$$\mathcal{F} : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}, w_{p-2}; X) \tag{3.4.2}$$

is bounded, where $w_{p-2}$ is defined in (3.4.1). Moreover, $X$ has *Hardy–Littlewood cotype* $q \in [2,\infty)$ if

$$\mathcal{F} : L^q(\mathbb{R}, w_{q-2}; X) \to L^q(\mathbb{R}; X) \tag{3.4.3}$$

is bounded. Note that, if $X = \mathbb{C}$, then (3.4.2) is the Hardy–Littlewood inequality. In the latter case, and in fact for any Hilbert space $X$, (3.4.2) holds for all $p \in (1,2]$, and (3.4.3) for all $q \in [2,\infty)$ by [71, Theorem 2.1.9].

If $X$ has Fourier type $p_0 \in (1,2]$, then $X$ has Hardy–Littlewood type $p$ for all $p \in (1, p_0)$, and Hardy–Littlewood cotype for all $q \in (p_0',\infty)$ (see [42, Proposition 3.5]). Also, if $X$ is a Banach lattice which is $p$-convex and $p$-concave with $p \in (1,\infty)$, then $X$ has Fourier type $p$ and Hardy–Littlewood type $p$ if $p \leq 2$, and Fourier type $p'$ and Hardy–Littlewood cotype $p$ if $p \geq 2$ (see [55, Proposition 2.2] and [42, Proposition 6.9]). This holds in particular if $X$ is isomorphic to a closed subspace of $L^p(S)$, for any measure space $(S, \mathscr{A}, \mu)$. For more on the relation between the notions of Fourier type, Hardy–Littlewood (co)type, and convexity and concavity in Banach lattices, we refer to [42].

We first show the weighted space $L^r(\mathbb{R}, w_{\delta r}; X)$ is a subspace of $\mathscr{S}'(\mathbb{R}; X)$, which will be used in Proposition 3.4.2.

**Lemma 3.4.1.** *Let $r \in [1, \infty)$, $\delta \in (-\infty, 1/r')$. Then $L^r(\mathbb{R}, w_{\delta r}; X) \subseteq \mathscr{S}'(\mathbb{R}; X)$.*



*Proof.* Let $f \in \mathscr{S}(\mathbb{R})$, $g \in L^r(\mathbb{R}, w_{\delta r}; X)$. Define $L_g : \mathscr{S}(\mathbb{R}) \to X$ by

$$L_g(f) := \int_{\mathbb{R}} f(x) g(x) \, dx.$$

Then

$$\begin{aligned} \|L_g(f)\| &\leq \textstyle\int_{|x|\leq 1} \|f(x)g(x)\| \, dx + \int_{|x|>1} \|f(x)g(x)\| \, dx \\ &\leq \|g\|_{L^1([-1,1];X)} \|f\|_{L^\infty([0,1])} + \textstyle\int_{|x|>1} \|f(x)g(x)\| \, dx. \end{aligned} \tag{3.4.4}$$

By Hölder's inequality,

$$\begin{aligned} \int_{|x|\leq 1} \|g(x)\| \, dx &= \int_{|x|\leq 1} \|g(x)\| |x|^{\delta} |x|^{-\delta} \, dx \\ &\leq \left( \int_{|x|\leq 1} \|g(x)\|^r |x|^{\delta r} \, dx \right)^{1/r} \left( \int_{|x|\leq 1} |x|^{-\delta r'} \, dx \right)^{1/r'} \\ &\leq \|g\|_{L^r(\mathbb{R}, w_{\delta r}; X)} \left( \int_{|x|\leq 1} |x|^{-\delta r'} \, dx \right)^{1/r'} < \infty. \end{aligned}$$

Thus, the first term in the right hand side of (3.4.4) is finite since $f$ is continuous.

For the second term in the right hand side of (3.4.4), note that $1 + |x| \leq 2|x|$ for $|x| > 1$, choose a number $a$ large enough such that $a + \delta r' > 1$. Then

$$\begin{aligned} \int_{|x|>1} \|f(x)g(x)\| \, dx &= \int_{|x|>1} |f(x)||x|^{-\delta} |x|^{\delta} \|g(x)\| \, dx \\ &\leq \left( \int_{|x|>1} \|g(x)\|^r |x|^{\delta r} \, dx \right)^{1/r} \left( \int_{|x|>1} |x|^{-\delta r'} |f(x)|^{r'} \, dx \right)^{1/r'} \\ &\leq \|g\|_{L^r(\mathbb{R}, w_{\delta r}; X)} \left( \int_{|x|>1} (1+|x|)^a (1+|x|)^{-a} |f(x)|^{r'} |x|^{-\delta r'} \, dx \right)^{1/r'} \\ &\leq \|g\|_{L^r(\mathbb{R}, w_{\delta r}; X)} \sup_{x\in\mathbb{R}} (1+|x|)^{a/r'} |f(x)| \left( \int_{|x|>1} (1+|x|)^{-a} |x|^{-\delta r'} \, dx \right)^{1/r'} \\ &\lesssim \|g\|_{L^r(\mathbb{R}, w_{\delta r}; X)} \sup_{x\in\mathbb{R}} (1+|x|)^{a/r'} |f(x)| \left( \int_{|x|>1} (1+|x|)^{-a-\delta r'} \, dx \right)^{1/r'} \\ &\lesssim \|g\|_{L^r(\mathbb{R}, w_{\delta r}; X)} \sup_{x\in\mathbb{R}} (1+|x|)^{a/r'} |f(x)|. \end{aligned}$$

By the above analysis, we have

$$\|L_g(f)\| \lesssim \|g\|_{L^r(\mathbb{R}, w_{\delta r}; X)} \sup_{x\in\mathbb{R}} (1+|x|)^{\lceil a/r' \rceil} |f(x)|,$$

finishing the proof. □

The following proposition will play the same role in this section that Proposition 3.2.2 did in the previous section. For $\gamma \in \mathbb{R}$, recall the definition of the weight $w_\gamma : \mathbb{R} \to [0,\infty)$ from (3.4.1).

**Proposition 3.4.2.** *Let $p,q \in [1,\infty]$, $r \in [1,\infty)$, $\delta_1 \in \mathbb{R}$ and $\delta_2 \in (-\infty, 1/r')$. Let $Y$ be a Banach space such that*

$$\mathscr{F} : L^p(\mathbb{R};Y) \to L^r(\mathbb{R}, w_{\delta_1 r}; Y) \tag{3.4.5}$$

*is bounded, and let $X$ be a Banach space such that*

$$\mathscr{F} : L^r(\mathbb{R}, w_{\delta_2 r}; X) \to L^q(\mathbb{R};X) \tag{3.4.6}$$

*is bounded. Let $m : \mathbb{R}\setminus\{0\} \to \mathscr{L}(Y,X)$ be an $X$-strongly measurable map for which there exists a $C \geq 0$ such that $\|m(\xi)\|_{\mathscr{L}(Y,X)} \leq C|\xi|^{\delta_1-\delta_2}$ for all $\xi \in \mathbb{R}\setminus\{0\}$. Then $T_m : L^p(\mathbb{R};Y) \to L^q(\mathbb{R};X)$ is bounded.*

*Proof.* Note that since $\delta_2 < 1/r'$, $L^r(\mathbb{R}, w_{\delta_2 r}; X) \subseteq \mathscr{S}'(\mathbb{R};X)$ by Lemma 3.4.1; Thus, the inverse Fourier transform is well-defined. Simply combining the assumptions on $X$, $m$ and $Y$, we get

$$\|T_m f\|_{L^q(\mathbb{R};X)} \lesssim \|m\widehat{f}\|_{L^r(\mathbb{R},w_{\delta_2 r};X)} \lesssim \|w_{\delta_2} w_{\delta_1-\delta_2}\widehat{f}\|_{L^r(\mathbb{R};Y)} \lesssim \|f\|_{L^p(\mathbb{R};Y)}$$

for all $f \in L^p(\mathbb{R};Y)$. □

We are now ready to prove the last statement in Theorem 3.1.1, as a special case of the following result.

**Theorem 3.4.3.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ with Fourier type $p \in (1,2]$, and suppose that $X$ has Hardy–Littlewood type $p$ or Hardy–Littlewood cotype $p'$. Suppose that $\mathbb{C}_+ \subseteq \rho(A)$, and that there exist $\beta > 0$ and $C \geq 0$ such that*

$$\|R(\lambda, A)\|_{\mathscr{L}(X)} \leq C(1+|\lambda|)^\beta, \quad \lambda \in \mathbb{C}_+.$$

*Let $\rho \geq 0$ and set $\tau := (\rho+1)\beta + \frac{1}{p} - \frac{1}{p'}$. Then there exists a $C_\rho \geq 0$ such that*

$$\|T(t)x\|_X \leq C_\rho t^{-\rho} \|x\|_{D((-A)^\tau)}$$

*for all $t \geq 1$ and $x \in D((-A)^\tau)$.*

Note that Theorem 3.4.3 is independent of Theorem 3.3.5 in the special case where $\rho = 0$. For $\rho > 0$ the conclusion of Theorem 3.4.3 already follows from Theorem 3.3.5.

*Proof.* The proof is analogous to that of [130, Theorem 4.9], and as such also similar to the proof of Theorem 3.3.5. We will indicate the key steps.

By interpolation, cf. [130, Lemma 4.2], it suffices to consider $\rho \in \mathbb{N}_0$. Note that the resolvent assumption extends to $i\mathbb{R}$ as before. We first show

$$\|R(i\xi, A)^k\|_{\mathscr{L}(D((-A)^\tau),X)} \lesssim (1+|\xi|)^{-(\frac{1}{p}-\frac{1}{p'})} \tag{3.4.7}$$

for all $k \in \{1,\dots,\rho+1\}$ and an implicit constant independent of $\xi \in \mathbb{R}$. If $|\xi| \le 1$,

$$\|R(i\xi,A)^k x\|_X \le \frac{C(1+|\xi|)^\beta (1+|\xi|)^{\frac{1}{p}-\frac{1}{p'}}}{(1+|\xi|)^{\frac{1}{p}-\frac{1}{p'}}}\|x\|_X \lesssim (1+|\xi|)^{-(\frac{1}{p}-\frac{1}{p'})}\|x\|_{D((-A)^\tau)}, \quad x \in D((-A)^\tau).$$

If $|\xi|>1$, using the assumption and [130, Proposition 3.4], one gets

$$\begin{aligned}\|R(i\xi,A)(-A)^{-\beta-(\frac{1}{p}-\frac{1}{p'})}\|_{\mathscr{L}(X)} &\eqsim \|R(i\xi,A)^k\|_{\mathscr{L}(D((-A)^{\beta+\frac{1}{p}-\frac{1}{p'}}),X)} \\ &\lesssim |\xi|^{-(\frac{1}{p}-\frac{1}{p'})} \lesssim (1+|\xi|)^{-(\frac{1}{p}-\frac{1}{p'})},\end{aligned}$$

where we use $|\xi| > \frac{1+|\xi|}{2}$. Note that we still have (3.3.3) here, then for all $k \in \{1,\dots,\rho+1\}$,

$$\begin{aligned}&\|R(i\xi,A)^k\|_{\mathscr{L}(D((-A)^\tau),X)} \eqsim \|R(i\xi,A)^k(-A)^{-(\rho+1)\beta-(\frac{1}{p}-\frac{1}{p'})}\|_{\mathscr{L}(X)} \\ &\le \|R(i\xi,A)(-A)^{-\beta}\|_{\mathscr{L}(X)}^{k-1}\|R(i\xi,A)(-A)^{-\beta-(\frac{1}{p}-\frac{1}{p'})}\|_{\mathscr{L}(X)}\|(-A)^{-\beta}\|_{\mathscr{L}(X)}^{\rho-k+1} \\ &\lesssim (1+|\xi|)^{-(\frac{1}{p}-\frac{1}{p'})}.\end{aligned}$$

Next, since $D((-A)^\tau)$ has the same Fourier type and Hardy–Littlewood type and cotype as $X$, because $X$ and $D((-A)^\tau)$ are isomorphic. In particular, if $X$ has Hardy–Littlewood type $p$, then one may apply Proposition 3.4.2 with $r=p$, $q=p'$, $\delta_1 = \frac{1}{p'}-\frac{1}{p}$ and $\delta_2=0$. On the other hand, if $X$ has Hardy–Littlewood cotype $p'$, then one can apply Proposition 3.4.2 with $q=r=p'$, $\delta_1=0$ and $\delta_2=\frac{1}{p}-\frac{1}{p'}$. In both cases, it follows from (3.4.7) that

$$T_{R(i\cdot,A)^k} : L^p(\mathbb{R};D((-A)^\tau)) \to L^{p'}(\mathbb{R};X)$$

is bounded for all $k \in \{1,\dots,\rho+1\}$, which satisfies [130, Theorem 4.6] with $\psi \equiv 0$. The result follows from [130, Theorem 4.6]. $\square$

# 4

# A WEAK RITT CONDITION ON BOUNDED LINEAR OPERATORS

## 4.1. INTRODUCTION

In the last chapter, we studied the decay rates of non-uniformly bounded $C_0$-semigroups, which plays an important role in the analysis of stability of damped wave equations (see [11, 20, 91–93] and the references therein). In this chapter, we focus on the decay of the discrete analogues and their applications in the stability of non-autonomous Cauchy problems and the convergence rates of numerical schemes.

Let $T$ be a bounded linear operator on a Banach space $X$. One can view $(T^n)_{n\geq 0}$ as a discrete analogue of a $C_0$-semigroup. A foundational study along this direction is the Katznelson–Tzafriri theorem [75], saying if $T$ is a power bounded operator on a Banach space, then $\|T^n(I-T)\| \to 0$ if and only if the $\sigma(T)\cap\{\lambda \,|\, |\lambda|=1\} \subseteq \{1\}$. This can be regarded as a characterization of the decay of discrete semigroups. Many improvements and extensions have been made in this direction, ranging from the question of whether $\|T^n(I-T)\|$ converges to 0 to quantitative analysis of convergence rates, see [9, 14, 34, 46, 47, 85, 109, 111] and the references therein.

A classical technique in studying the stability of evolution equations is to study the relation between the decay rates of $C_0$-semigroups and the resolvent condition of the generator of the semigroups, see [8, 12, 20, 65, 128, 148] and the references therein. By adapting the methods used for the $C_0$-semigroups, similar results can be obtained for discrete semigroups. Seifert [135] proved that if $T$ is power bounded and $\sigma(T)\cap\{\lambda \,|\, |\lambda| = 1\} = \{1\}$, then for $\beta \geq 1$, the resolvent condition

$$\|R(e^{i\theta}, T)\| \leq C|\theta|^{-\beta}, \quad \theta \to 0 \tag{4.1.1}$$

implies the following decay estimate

$$\|T^n(I-T)\| \leq C\left(\frac{\log n}{n}\right)^{\frac{1}{\beta}} \tag{4.1.2}$$

on Banach spaces for all sufficiently large $n \geq 0$. Later, given the same resolvent condition, Ng and Seifert [112] gave a similar result to (4.1.2) in Hilbert spaces, removing the log correction on the right hand side of (4.1.2). The above works are discrete time analogues of uniformly bounded $C_0$-semigroup ($\|T(t)\| \le C$ for some $C \ge 1$ and all $t \ge 0$) results of Batty and Duyckaerts [12] on Banach spaces, and those of Borichev and Tomilov [20] on Hilbert spaces, respectively. Let $\beta \ge 1$. These results are summarised in the following table. By Table 4.1, power bounded operator $(T^n)_{n\in\mathbb{N}}$ can be regarded

Table 4.1: Comparison of continuous and discrete time semigroups.

| | Resolvent condition | Banach space | Hilbert space |
|---|---|---|---|
| Continuous | $\|R(i\xi,A)\| \lesssim (1+\|\xi\|)^\beta, \xi \in \mathbb{R}$ | $\|T(t)A^{-1}\| \lesssim \left(\frac{\log t}{t}\right)^{\frac{1}{\beta}}$ | $\|T(t)A^{-1}\| \lesssim t^{-1/\beta}$ |
| Discrete | $\|R(e^{i\theta},T)\| \lesssim \|\theta\|^{-\beta}, \theta \in (0,2\pi)$ | $\|T^n(I-T)\| \lesssim \left(\frac{\log n}{n}\right)^{\frac{1}{\beta}}$ | $\|T^n(I-T)\| \lesssim n^{-1/\beta}$ |

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as discrete-time analogues of uniformly bounded $C_0$-semigroups $(T(t))_{t\ge0}$, while $(I-T)$ corresponds to the inverse of the generator of a bounded $C_0$-semigroup $A^{-1}$. Recall that Rozendaal and Veraar [130] provided a result for non-uniformly bounded $C_0$-semigroup on Banach spaces with Fourier type $p \in [1,2]$ (see Proposition 1.2.1). A natural question is whether analysing the resolvent condition of $T$ allows us to obtain a similar result to Proposition 1.2.1 for not power bounded operators $(T^n)_{n\ge0}$?

Note that by [31, Lemma 3.3], the resolvent condition (4.1.1) for power bounded operators is equivalent to the *weak Ritt condition*:

$$\|R(\lambda,T)\| \le \frac{C}{|\lambda-1|^\beta}, \quad 1<|\lambda|<2,$$

for some $\beta > 1$. If $\beta = 1$, the above estimate is the well-known Ritt condition, implying power boundedness of $T$. Since $\beta > 1$, the weak Ritt condition does not necessarily imply power boundedness of $T$. To the best of the authors' knowledge, there is little research studying the relation between the weak Ritt condition and the discrete semigroup $(T^n)_{n\ge0}$ for $T$ a bounded linear operator that is not power bounded. In the early 2000s, Nevanlinna [111, Theorem 9] proved that under the weak Ritt condition, a Kreiss bounded operator $T$ (see (1.2.7)) has a decay rate $\|T^n(I-T)^\tau\| \lesssim n^{-s}$ for $s \in \mathbb{N}$ and $\tau = \beta(s+1)-1$.

In this chapter, we are interested in the relation between the weak Ritt condition and the decay rates of bounded operators $(T^n)_{n\ge0}$ under composition with operators of the form $(I-T)^\tau$ for some $\tau > 0$. A simple version of our main result is as follows (also see Theorem 4.3.2):

**Theorem 4.1.1.** *Let $X$ be a complex Banach space, $T$ be a bounded linear operator on $X$, and $\mathbb{D}$ be the unit disk. Assume that $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ and $\sigma(T) \cap \{\lambda \,|\, |\lambda| = 1\} = \{1\}$. Furthermore, suppose there exists a constant $\beta > 1$ such that*

$$\|R(\lambda,T)\| \le \frac{C}{|\lambda-1|^\beta}, \quad 1<|\lambda|<2$$

*holds. Then for each $s \in \mathbb{N}$ and all $\tau > (s+1)\beta - 1$, there exists a $C > 0$ such that*

$$\|T^n(I-T)^\tau\| \le Cn^{-s}, \quad n \ge 1.$$

It is easy to see our result may be viewed as a discrete analogue of the results in [130] for non-uniformly bounded $C_0$-semigroups on general Banach spaces. We also investigate the relation between the decay rates and the geometry of the underlying Banach spaces (i.e. the Fourier type), see Theorem 4.3.4. Theorem 4.1.1 is obtained by a direct calculation, whereas Theorem 4.3.4 is obtained by adapting the Fourier multiplier theory, which is similar to the techniques used in [130]. Furthermore, see Proposition 4.3.1, where we consider a sector condition (4.2.5) and obtain a better decay for the discrete semigroups; this result directly extends [109, Theorem 4.9.3]. We will give a comprehensive comparison of these results in Section 4.4.

The chapter is organized as follows. In Section 4.2, we collect some preliminaries on the weak Ritt condition and other related concepts. In Section 4.3, we prove our main results, Theorems 4.3.2 and 4.3.4, and in Section 4.4, we provide a detailed comparison between our main results and related results. Finally, in Section 4.5, we present two applications of our results: the asymptotic behaviour of solutions to non-autonomous Cauchy problems and the convergence rates of polynomial acceleration methods for fixed point problems.

4

## 4.2. PRELIMINARIES

### 4.2.1. NOTATION IN THIS CHAPTER

Let $\mathbb{D} := \{\lambda \in \mathbb{C} \,|\, |\lambda| < 1\}$ be the unit disc, $\partial\mathbb{D} := \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$ be the boundary of $\mathbb{D}$, and $\overline{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}$ be the closure of $\mathbb{D}$. We denote the null space of $T \in \mathscr{L}(X)$ by $\operatorname{Ker}(T) := \{x \in X \,|\, Tx = 0\}$. We denote by $C^k(\mathbb{R})$ with $k \in \mathbb{N}$ the space of $k$-th order continuously differentiable functions in $\mathbb{R}$. For a function $f \in C^k(\mathbb{R})$, we denote by $f^{(k)}$ the $k$-th order derivative of $f$.

In this chapter, we define the *Fourier transform* $\mathscr{F}: \ell^1(\mathbb{Z}; X) \to L^\infty(\mathbb{T}; X)$, $f := (f_n)_{n\in\mathbb{Z}} \mapsto \widehat{f}$ as

$$\mathscr{F}(f)(\xi) := \widehat{f}(\xi) := \sum_{n\in\mathbb{Z}} f_n e_{-n}(\xi), \quad \xi \in \mathbb{T},$$

and the *inverse Fourier transform* $\mathscr{F}^{-1}: L^1(\mathbb{T}; X) \to \ell^\infty(\mathbb{Z}; X), g \mapsto \check{g}$ as

$$\mathscr{F}^{-1}(g)(n) := \check{g}(n) := \int_{\mathbb{T}} g(t) e_n(t)\,\mathrm{d}t, \quad n \in \mathbb{Z},$$

where $e_n(t) := e^{2\pi i n t}$ for $t \in \mathbb{T}$ and $n \in \mathbb{Z}$.

### 4.2.2. $\ell^p_q(\mathbb{Z}; \mathscr{L}(X, Y))$-FOURIER MULTIPLIER THEORY

Define the space

$$c_{00}(\mathbb{Z}; X) := \{a := (a_n)_{n\in\mathbb{Z}} \subseteq X \,|\, \text{there exists an } N > 0 \text{ such that } a_n = 0 \text{ for all } |n| \geq N\}.$$

Clearly, $c_{00}(\mathbb{Z}; X)$ is dense in $\ell^p(\mathbb{Z}; X)$ for all $p \in [1, \infty)$. Let $m \in L^\infty(\mathbb{T}; \mathscr{L}(X, Y))$. Define the *Fourier multiplier operator* $T_m : c_{00}(\mathbb{Z}; X) \to c_{00}(\mathbb{Z}; Y)$ by

$$T_m f(n) := \mathscr{F}^{-1}(m\widehat{f}) = \int_{\mathbb{T}} m(\xi)\widehat{f}(\xi) e_n(\xi)\,\mathrm{d}\xi, \quad f \in c_{00}(\mathbb{Z}; X),\ n \in \mathbb{Z}.$$

If there exists a constant $C>0$ such that for all $f \in c_{00}(\mathbb{Z};X)$,

$$\|T_m f\|_{\ell^q(\mathbb{Z};Y)} \leq C\|f\|_{\ell^p(\mathbb{Z};X)},$$

then we say $m$ is an $\ell^p_q(\mathbb{Z};\mathscr{L}(X,Y))$-*Fourier multiplier*. By density, $T_m$ can be uniquely extended to a bounded linear operator from $\ell^p(\mathbb{Z};X)$ to $\ell^q(\mathbb{Z};X)$.

The following result is a discrete analogue of [131, Proposition 3.9].

**Proposition 4.2.1.** *Let $X$ be a Banach space with Fourier type $p \in [1,2]$, $Y$ be a Banach space with Fourier cotype $q \in [2,\infty]$. Let $r \in [1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Let $m : \mathbb{T} \to \mathscr{L}(X,Y)$ be an $X$-strongly measurable map such that $m \in L^r(\mathbb{T};\mathscr{L}(X,Y))$. Then $m$ is a $\ell^p_q(\mathbb{Z};\mathscr{L}(X,Y))$-Fourier multiplier with*



$$\|T_m\| \leq C_{q',Y}(\mathbb{T})C_{p,X}(\mathbb{Z})\,\|m\|_{L^r(\mathbb{T};\mathscr{L}(X,Y))}. \tag{4.2.1}$$

*Proof.* By a density argument, it suffices to consider $f \in c_{00}(\mathbb{Z};X)$. Since $X$ has Fourier type $p$ and $Y$ has Fourier cotype $q$, let $C_{p,X}(\mathbb{Z})$ and $C_{q',Y}(\mathbb{T})$ denote the operator norms of $\mathscr{F} : \ell^p(\mathbb{Z};X) \to L^{p'}(\mathbb{T};X)$ and $\mathscr{F}^{-1} : L^{q'}(\mathbb{T};Y) \to \ell^q(\mathbb{Z};Y)$, respectively. By Hölder's inequality,

$$\|m\widehat{f}\|_{L^{q'}(\mathbb{T};Y)} \leq \|m\|_{L^r(\mathbb{T};\mathscr{L}(X,Y))}\|\widehat{f}\|_{L^{p'}(\mathbb{T};X)} \leq C_{p,X}(\mathbb{Z})\|m\|_{L^r(\mathbb{T};\mathscr{L}(X,Y))}\|f\|_{\ell^p(\mathbb{Z};X)}.$$

Hence,

$$\begin{aligned}\|T_m f\|_{\ell^q(\mathbb{Z};Y)} = \|\mathscr{F}^{-1}(m\widehat{f})\|_{\ell^q(\mathbb{Z};Y)} &\leq C_{q',Y}(\mathbb{T})\|m\widehat{f}\|_{L^{q'}(\mathbb{T};Y)}\\ &\leq C_{q',Y}(\mathbb{T})C_{p,X}(\mathbb{Z})\|m\|_{L^r(\mathbb{T};\mathscr{L}(X,Y))}\|f\|_{\ell^p(\mathbb{Z};X)}.\end{aligned}$$

□

### 4.2.3. FRACTIONAL POWER OPERATORS

In this section, we define fractional power operators under the weak Ritt condition (4.2.9) and the sector condition (4.2.5) below.

Let $\lambda \in \rho(T)$, $\tau \in \mathbb{R}$. Fix a branch cut from 1 to $\infty$ along the real axis, so that $(1-\lambda)^\tau := e^{\tau \log(1-\lambda)}$ is single-valued and analytic on $\mathbb{C}\backslash[1,\infty)$. Let $T$ be a bounded linear operator on a Banach space $X$ such that $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Define the *fractional power operator* $(I-T)^\tau$ by

$$(I-T)^\tau := \frac{1}{2\pi i}\int_\Gamma (1-\lambda)^\tau R(\lambda,T)\,\mathrm{d}\lambda. \tag{4.2.2}$$

where $\Gamma$ is a contour surrounding $\sigma(T)$, avoiding the cut by starting and ending at 1 along rays(see Figure 4.1). The following proposition shows that the definition is well-defined.

**Proposition 4.2.2.** *Let $\beta > 1$ and $\tau > \beta - 1$. Let $T$ be a bounded linear operator on a Banach space $X$ such that $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Assume that*

$$\|R(\lambda,T)\| \leq \frac{C}{|\lambda-1|^\beta}, \quad 1 < |\lambda| < 2.$$

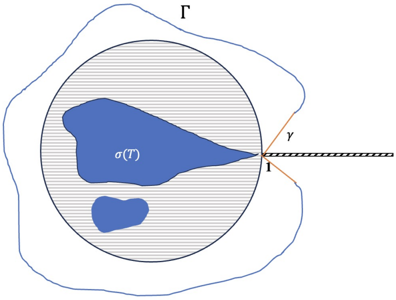


Figure 4.1: A contour Γ as in (4.2.2).



*Then the fractional power operator defined by* (4.2.2) *is well defined and satisfies the semigroup property: for* $\tau_1, \tau_2 > \beta - 1$,

$$(I-T)^{\tau_1}(I-T)^{\tau_2} = (I-T)^{\tau_1+\tau_2}.$$

*Proof.* Let Γ be a contour surrounding $\sigma(T)$, starting from and ending at 1, and assume that near 1 it coincides with a ray

$$\gamma := \{1 + te^{i\theta_0} \mid t \in (0,\varepsilon)\} \subseteq \Gamma$$

for some $\varepsilon > 0$ and $|\theta_0| \in (0, \frac{\pi}{2})$ (see Figure 4.1). Since the only singularity of the integrand in (4.2.2) lies at $\lambda = 1$, it suffices to analyse the behaviour as $\lambda \in \gamma$. A direct computation gives

$$\begin{aligned}\left\| \frac{1}{2\pi i} \int_\gamma (1-\lambda)^\tau R(\lambda, T)\, d\lambda \right\| &\lesssim \int_0^\varepsilon |te^{i\theta_0}|^\tau \frac{1}{|te^{i\theta_0}|^\beta}\, dt \\ &\lesssim \int_0^\varepsilon t^{\tau-\beta}\, dt < \infty,\end{aligned}$$

due to $\tau > \beta - 1$. Hence $(I-T)^\tau$ is a bounded operator on $X$, and the definition is well posed.

To prove the semigroup property, let $\Gamma_1$ and $\Gamma_2$ be two admissible contours starting and ending at 1, with $\Gamma_2$ strictly inside $\Gamma_1$ and intersecting only at 1. Using the resolvent identity and Fubini's theorem, one obtains

$$\begin{aligned}(I-T)^{\tau_1}(I-T)^{\tau_2} &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} (1-\lambda)^{\tau_1} \Big( \int_{\Gamma_2} (1-\mu)^{\tau_2} R(\mu, T)\, \mathrm{d}\mu \Big) R(\lambda, T)\, \mathrm{d}\lambda \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{(1-\lambda)^{\tau_1}(1-\mu)^{\tau_2}}{\lambda - \mu} \big( R(\mu, T) - R(\lambda, T) \big)\, \mathrm{d}\mu\, \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_2} \left( \frac{1}{2\pi i} \int_{\Gamma_1} \frac{(1-\lambda)^{\tau_1}}{\lambda - \mu}\, \mathrm{d}\lambda \right) (1-\mu)^{\tau_2} R(\mu, T)\, \mathrm{d}\mu \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma_1} \left( \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(1-\mu)^{\tau_2}}{\lambda - \mu}\, \mathrm{d}\mu \right) (1-\lambda)^{\tau_1} R(\lambda, T)\, \mathrm{d}\lambda.\end{aligned} \tag{4.2.3}$$

We now evaluate the two terms in the right hand side of (4.2.3) separately. For the first term, note that $(1-\lambda)^{\tau_1}$ is continuous on the closure of the region bounded by $\Gamma_1$ and analytic in its interior. Since $\Gamma_2$ intersects $\Gamma_1$ only at the point 1, and this single point has measure zero for the outer integral, we may ignore it. Hence, by the Cauchy integral formula,

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{(1-\lambda)^{\tau_1}}{\lambda-\mu}\,\mathrm{d}\lambda = (1-\mu)^{\tau_1}, \quad \mu\in\Gamma_2\setminus\{1\}.$$

For the second term, the same reasoning allows us to disregard the case $\lambda = 1$. For $\lambda\in\Gamma_1\setminus\{1\}$, fix $1<t<2$ and choose the branch cut along $[t,\infty)$. By the dominated convergence theorem,

$$\frac{1}{2\pi i}\int_{\Gamma_2}\frac{(1-\mu)^{\tau_2}}{\lambda-\mu}\,d\mu = \lim_{t\downarrow 1}\frac{1}{2\pi i}\int_{\Gamma_2}\frac{(t-\mu)^{\tau_2}}{\lambda-\mu}\,d\mu = 0.$$

Therefore, the second term vanishes. Substituting back into (6.4.1) yields

$$(I-T)^{\tau_1}(I-T)^{\tau_2} = \frac{1}{2\pi i}\int_{\Gamma_2}(1-\mu)^{\tau_1+\tau_2}R(\mu,T)\,\mathrm{d}\mu = (I-T)^{\tau_1+\tau_2}.$$

This proves the semigroup property. □

Next, we introduce the sector condition. For details, we refer to [109, Chapter 4]. Let $\delta>0$ and

$$K_\delta := \left\{\lambda = 1+te^{i\theta}\,\middle|\, t>0, |\theta|<\frac{\pi}{2}+\delta\right\}. \tag{4.2.4}$$

We say the resolvent of $T$ satisfies the *sector condition* if

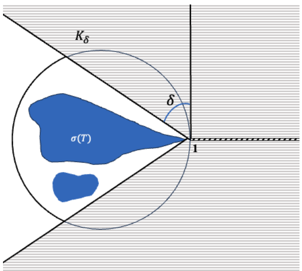


Figure 4.2: the shape of $K_\delta$.

$$\|R(\lambda,T)\| \le \frac{C}{|\lambda-1|^{\beta}}, \quad \lambda\in\{\lambda\in K_\delta \,|\, |\lambda|<2\}, \tag{4.2.5}$$

for some $\beta\ge 1$. A similar proof of Proposition 4.2.2 shows the definition in (4.2.2) for $\tau>\beta-1$ is well-defined and also satisfies the semigroup property if the sector condition holds. We will explore this further in Section 4.3.1.

### 4.2.4. WEAK RITT CONDITION

Let $T$ be a bounded linear operator on a Banach space $X$ satisfying $\sigma(T) \subseteq \{\lambda \,|\, |\lambda| < 1\} \cup \{1\}$ and *the Ritt condition*

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|}, \quad |\lambda| > 1. \tag{4.2.6}$$

In 1953, Ritt [125] proved that (4.2.6) implies $\|T^n\| = o(n)$, which is an important condition in ergodic theory [44, 99]. Tamdor [141] improved the result to $\|T^n\| = O(\log n)$. Lyubich [98] and Nagy and Zemánek [106] finally proved independently that the Ritt condition (4.2.6) holds if and only if $T$ is *power bounded* and $\|T^n(I - T)\| = O(n^{-1})$.

The Ritt condition has many equivalent characterizations. We summarize them as follows.

**Lemma 4.2.3.** *Let $T$ be a bounded linear operator on a Banach space $X$ such that $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Then the following are equivalent:*

*(1) There exists a constant $C > 1$ such that (4.2.6) holds.*

*(2) There exists a constant $C > 1$ such that*

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|}, \quad 1 < |\lambda| < 2. \tag{4.2.7}$$

*(3) There exists a constant $C > 1$ such that for some $\delta > 0$,*

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|}, \quad \lambda \in K_\delta, \tag{4.2.8}$$

*where $K_\delta$ is defined in (4.2.4).*

*Proof.* The fact that (1) implies (3) is a brilliant result proven independently by Lyubich [98] and Nagy and Zemánek [106] independently. Moreover, it is clear (3) implies (2). Therefore, it suffices to show (2) implies (1). Let $f(\lambda) := |\lambda - 1| \|R(\lambda, T)\|$ for $\lambda \in \{\lambda \in \mathbb{C} \,|\, |\lambda| > \|T\|\}$, then

$$f(\lambda) \leq |1 - \lambda^{-1}| \sum_{k=0}^{\infty} \|T/\lambda\|^k \to 1, \quad |\lambda| \to \infty.$$

Moreover, define $S := \{\lambda \in \mathbb{C} \,|\, 2 \leq |\lambda| \leq \|T\|\}$. Since $S$ is compact and $f(\lambda) := |\lambda - 1| \|R(\lambda, T)\|$ is a continuous function on $S$, $f$ is uniformly bounded on $S$. Therefore, there exists a constant $C > 1$ such that for $|\lambda| \geq 2$, $f(\lambda) \leq C$.

□

Motivated by (4.2.7), we give the following definition.

**Definition 4.2.4.** Let $T$ be a bounded linear operator on a Banach space $X$. We say $T$ satisfies the *weak Ritt condition* if there exists a constant $C \geq 1$ such that

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|^\beta}, \quad 1 < |\lambda| < 2, \tag{4.2.9}$$

for some $\beta > 1$.

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*Remark* 4.2.5. The case that $\beta = 1$ is the Ritt condition which implies $T$ is power bounded; The case that $\beta < 1$ is not of interest in this context, because then $1 \in \rho(T)$ by $\|R(\lambda, T)\| \geq \frac{1}{\text{dist}(\lambda, \sigma(T))}$. Thus the spectral radius is smaller than 1 and thereby $\|T^n\|$ decays exponentially.

Next, we recall the definition of polynomial bounded operators and its resolvent estimate.

**Definition 4.2.6.** Let $X$ be a Banach space and $T$ be a bounded linear operator on $X$. We say $T$ is *polynomial bounded* if $\|T^n\| \leq p(n)$ for some polynomial $p$ and all $n \in \mathbb{N}$.

Obviously, a polynomial bounded operator may not be a power bounded operator. Without loss of generality we assume the polynomial to be of the form $p(t) = Ct^\beta$ for some $\beta \in \mathbb{N}$ and all $t \in \mathbb{R}$. The following lemma gives a characterization of the relation between polynomial bounded operators and its resolvent in a neighbourhood of 1 (see [49, Theorem 2.4] and [48, Theorem II.1.17]).

4

**Lemma 4.2.7.** *Let $T$ be a bounded linear operator on a Banach space $X$ with the spectral radius $r(T) \leq 1$. If for some $\beta \geq 0$,*

$$\limsup_{|\lambda| \to 1_+} (|\lambda| - 1)^\beta \|R(\lambda, T)\| < \infty, \tag{4.2.10}$$

*then*

$$\|T^n\| \leq Cn^\beta, \quad n \in \mathbb{N}. \tag{4.2.11}$$

*Moreover, if* (4.2.11) *holds for some $\beta = d$, then* (4.2.10) *holds with $\beta = d + 1$.*

We end this section with two examples of operators satisfying the weak Ritt condition (4.2.9).

*Example* 4.2.8. Let $X := \mathbb{R}^2$, $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. By calculation,

$$\left\| T^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} n \\ 1 \end{pmatrix} \right\| = \sqrt{n^2 + 1} \to \infty, \quad n \to \infty.$$

Thus, $T$ is not power bounded. Furthermore, for any $\lambda \in \mathbb{R}$, $\lambda - T = \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix}$, which shows $\sigma(T) = \{1\}$. By a direct calculation,

$$(\lambda - T)^{-1} = \frac{1}{|\lambda - 1|^2} \begin{pmatrix} \lambda - 1 & 1 \\ 0 & \lambda - 1 \end{pmatrix}.$$

Define $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A := \begin{pmatrix} \lambda - 1 & 1 \\ 0 & \lambda - 1 \end{pmatrix}$, then by a direct calculation, for $1 < |\lambda| < 2$, there exists a constant $C > 0$ such that

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \sqrt{|(\lambda - 1)x_1 + x_2|^2 + |(\lambda - 1)x_2|^2} \leq C,$$

so $\|(\lambda - T)^{-1}\| \le \frac{C}{|\lambda-1|^2}$. On the other hand, we have $\|A\| \ge \left\| A\begin{pmatrix}0\\1\end{pmatrix}\right\| \ge 1$. Therefore, there exists a constant $C > 0$ such that

$$\frac{1}{|\lambda-1|^2} \le \|(\lambda - T)^{-1}\| \le \frac{C}{|\lambda-1|^2}.$$

*Example* 4.2.9. Let $A$ be a linear operator on a Banach space $X$. Assume that there exists $\omega_1 \in (0, \frac{\pi}{2})$ such that $-A$ generates an analytic semigroup $(e^{-zA})_{z\in\Sigma_{\omega_1}}$ on $\Sigma_{\omega_1} := \{z \in \mathbb{C} \,|\, |\arg z| < \omega_1\}$ and

$$\|e^{-zA}\| \lesssim |z| + 1, \quad z \in \Sigma_{\omega_1}.$$

Let $T := (1 + A)^{-1}$, then $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ and

$$\|R(\lambda, T)\| \le \frac{C}{|\lambda-1|^2}, \quad 1 < |\lambda| < 2. \tag{4.2.12}$$

Indeed, by [88, Lemma 2.3], there exists $\omega_2 \in (0, \frac{\pi}{2})$, such that $\sigma(A) \subseteq \overline{\Sigma_{\omega_2}}$ and

$$\|(z + A)^{-1}\| \lesssim \frac{1}{|z|^2} + \frac{1}{|z|}, \quad z \in \Sigma_{\pi-\omega_2}.$$

Thus, $\mathbb{C}_- \subseteq \rho(A)$ and $T$ is well-defined. We now show that $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$. Since

$$\lambda - T = \lambda - (1 + A)^{-1} = \lambda(1 - \lambda^{-1} + A)(1 + A)^{-1}. \tag{4.2.13}$$

Let $z := \lambda^{-1} - 1$, then $\lambda = \frac{1}{z+1}$. Note that $\lambda \in \sigma(T)$ if and only if $z \in \sigma(A)$. It is easy to see that for $z \in \sigma(A) \subseteq \mathbb{C}_+$, one has $|z + 1| \ge 1$, and the equal sign holds if and only if $z = 0$. Therefore, $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$.

Next, we show the resolvent estimate (4.2.12) holds. Since $z \in \rho(A)$ is equivalent to $\lambda \in \rho(T)$. By (4.2.13), one has

$$R(\lambda, T) = \lambda^{-1}(1 + A)(1 - \lambda^{-1} + A)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda^2}(1 - \lambda^{-1} + A)^{-1}.$$

For $1 < |\lambda| < 2$, we have $1 - \lambda^{-1} \in \Sigma_{\pi-\omega_2}$. If $|\lambda - 1| < 1$, then $\frac{1}{|\lambda-1|} < \frac{1}{|\lambda-1|^2}$; Otherwise, we have $1 \le |\lambda - 1| < 3$ and then $\frac{1}{|\lambda-1|} \le 1 = \frac{|\lambda-1|^2}{|\lambda-1|^2} \le \frac{9}{|\lambda-1|^2}$. The same calculation yields $\frac{1}{|\lambda|} \le \frac{9}{|\lambda-1|^2}$. Therefore,

$$\begin{aligned}\|R(\lambda, T)\| &\lesssim \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\left(\frac{1}{|1 - \frac{1}{\lambda}|^2} + \frac{1}{|1 - \frac{1}{\lambda}|}\right) \\ &\lesssim \frac{1}{|\lambda|} + \frac{1}{|\lambda-1|^2} + \frac{1}{|\lambda-1|} \\ &\lesssim \frac{1}{|\lambda-1|^2}, \quad 1 < |\lambda| < 2.\end{aligned}$$

Furthermore, by Lemma 4.2.7, (4.2.12) implies $\|T^n\| \le Cn^2$. In other words, the operator $T$ is polynomially bounded, but not power bounded.

*Remark* 4.2.10. An operator $A$ satisfying the assumptions in Example 4.2.9 can be found in [88].

4

### 4.2.5. ERGODIC THEORY

We now introduce some basic concepts from ergodic theory in order to present the splitting theorem for the space $X$. For more details, we refer the readers to [45] and [79].

Let $A_n(T) := \frac{1}{n}\sum_{i=0}^{n-1} T^i$, where $T$ is a bounded linear operator on a Banach space $X$.

**Definition 4.2.11.** An operator $T$ is said to be *Cesàro bounded* if there exists a constant $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^i x \right\| \le C\|x\|, \quad x \in X.$$

Moreover, $T$ is said to be *absolutely Cesàro bounded* if there exists a constant $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=0}^{n-1} \|T^i x\| \le C\|x\|, \quad x \in X.$$



Obviously, power bounded operators are (absolutely) Cesàro bounded, but the converse is not true. A counterexample is the following (also see [17, Corollary 2.3]).

*Example* 4.2.12. Let $1 \le p < \infty$. Denote the standard canonical basis $d_k$ in $\ell^p$, where the $k$th coordinate is 1 and the other coordinates are 0. Let $T$ be the unilateral weighted backward shift on $\ell^p$ defined by $Td_k = 0, k \le 1$ and $Td_k = w_k d_{k-1}, k \ge 2$, where $w_k := \left(\frac{k}{k-1}\right)^\alpha$ for $k \ge 2$, with $0 < \alpha < \frac{1}{p}$. Then by [17, Theorem 2.1], $T$ is absolutely Cesáro bounded on $\ell^p$. Moreover, we have $\|T^n d_k\| = 0$ for $k \le n$ and $\|T^n d_k\| = \left(\frac{k}{k-n}\right)^\alpha$ for $k > n$. Then,

$$\|T^n\| \ge \|T^n d_{n+1}\| = (n+1)^\alpha,$$

so $T$ is not power bounded.

On the other hand, let $x := \sum_{k=-\infty}^{\infty} c_k d_k$, where $c_k \in \mathbb{R}, k \in \mathbb{Z}$ satisfies that $\sum_{k>n} |c_k|^p = 1$, we obtain

$$\begin{aligned} \|T^n\| = \sup_{\|x\|_{\ell^p}=1} \|T^n x\| &= \left(\sum_{k>n} \|c_k T^n d_k\|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{k>n} |c_k|^p\right)^{\frac{1}{p}} \sup_{k>n} \|T^n d_k\| \\ &\le (n+1)^\alpha, \quad n \in \mathbb{N}. \end{aligned}$$

Therefore, $\|T^n\| = (n+1)^\alpha$ for $n \in \mathbb{N}$.

We next recall a stronger property.

**Definition 4.2.13.** We say $T$ is *mean ergodic* if $A_n(T)$ converges in the strong operator topology.

Define

$$X_{me} := \left\{x \in X \middle| \lim_{n\to\infty} A_n(T)x \text{ exists} \right\}.$$

If $T$ is mean ergodic, then $X_{me} = X$. Furthermore, since $\frac{T^{n-1}}{n} = A_n(T) - \frac{n-1}{n} A_{n-1}(T)$, we have

$$\lim_{n\to\infty} \frac{T^{n-1}x}{n} = 0, \quad x \in X. \tag{4.2.14}$$

It is clear mean ergodic implies Cesàro bounded and (4.2.14). Conversely, we have the following result, see [45, Corollary VIII · 5 · 4].

**Lemma 4.2.14.** *If $X$ is a reflexive Banach space, then $T$ is mean ergodic if and only if $T$ is Cesàro bounded and* (4.2.14) *holds for all $x \in X$.*

Recall related splitting conclusions in $X$, see [79, Theorem 2.1.3].

**Theorem 4.2.15.** *Let $T$ be a Cesàro bounded operator on a Banach space $X$ and* (4.2.14) *holds for all $x \in X$. Then*

*(1) The space $X_{me} = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}(I-T)}$.*

*(2) The operator $P : X_{me} \to \mathrm{Ker}(I-T)$ defining by $Px = \lim_{n\to\infty} A_n(T)x$ is a projection of $X_{me}$ on to $\mathrm{Ker}(I-T)$. Moreover, $P = P^2 = PT = TP$.*

*(3) For any $z \in X$, $\lim_{n\to\infty} A_n(T)x = 0$ if and only if $z \in \overline{\mathrm{Ran}(I-T)}$.*

We immediately have

**Corollary 4.2.16.** *If $T$ is mean ergodic, then*

$$X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}(I-T)}.$$

Now we claim that if $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}(I-T)}$, certain subspaces are always dense in $X$.

**Lemma 4.2.17.** *Let $T$ be a bounded linear operator on $X$. If $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}(I-T)}$, then for every $\tau \in \mathbb{N}$, $X_\tau := \mathrm{Ker}(I-T) \oplus \mathrm{Ran}(I-T)^\tau$ is dense in $X$. In particular, if $T$ is a contraction on $X$, i.e. $\|T\| \le 1$, then the conclusion holds for every $\tau > 0$.*

*Proof.* We first prove the first conclusion. For any $\varepsilon > 0$ and $x \in \overline{\mathrm{Ran}(I-T)}$, there exists $x^1 \in X$, such that $x^1 = \tilde{x}_1 + x_1$, where $\tilde{x}_1 \in \mathrm{Ker}(I-T)$, $x_1 \in \overline{\mathrm{Ran}(I-T)}$ and

$$\|x - (I-T)x_1\| = \|x - (I-T)x^1\| \le \frac{\varepsilon}{\tau}.$$

For such $x_1 \in \overline{\mathrm{Ran}(I-T)}$, there exists $x^2 \in X$, such that $x^2 = \tilde{x}_2 + x_2$, where $\tilde{x}_2 \in \mathrm{Ker}(I-T)$, $x_2 \in \overline{\mathrm{Ran}(I-T)}$ and

$$\|x_1 - (I-T)x_2\| = \|x_1 - (I-T)x^2\| \le \frac{\varepsilon}{\tau\|I-T\|}.$$

Similarly, for $x_{k-1} \in \overline{\mathrm{Ran}(I-T)}, 3 \le k \le \tau$, there exists $x^k \in X$, such that $x^k = \tilde{x}_k + x_k$, where $\tilde{x}_k \in \mathrm{Ker}(I-T)$, $x_k \in \overline{\mathrm{Ran}(I-T)}$ and

$$\|x_{k-1} - (I-T)x_k\| = \|x_{k-1} - (I-T)x^k\| \le \frac{\varepsilon}{\tau\|I-T\|^{k-1}}.$$

Then, by the triangle inequality,

$$\begin{aligned}\|x-(I-T)^\tau x_k\| &= \|x-(I-T)x_1+(I-T)x_1-(I-T)^2x_2+\cdots+(I-T)^{\tau-1}x_{k-1}-(I-T)^\tau x_k\| \\ &\le \|x-(I-T)x_1\|+\sum_{k=2}^{\tau}\|I-T\|^{k-1}\|x_{k-1}-(I-T)x_k\| \le \varepsilon,\end{aligned}$$

which means $\mathrm{Ran}(I-T)^\tau$ is dense in $\overline{\mathrm{Ran}(I-T)}$. Therefore, $X_\tau = \mathrm{Ker}(I-T)\oplus\mathrm{Ran}(I-T)^\tau$ is dense in $X$.

Note that a contraction is mean ergodic, $X=\mathrm{Ker}(I-T)\oplus\overline{\mathrm{Ran}(I-T)}$. If $\tau\in(0,1)$, the result follows directly from [41, Proposition 2.1]. For any $\tau\ge 1$, $\varepsilon>0$ and $y\in\overline{\mathrm{Ran}(I-T)}$, there exists $z_1\in\mathrm{Ker}(I-T)$ and $z_2\in\overline{\mathrm{Ran}(I-T)}$, such that $z=z_1+z_2\in X$ and $\|y-(I-T)^{\lfloor\tau\rfloor}z\|\le\frac{\varepsilon}{2}$. Moreover, by [41, Proposition 2.1], $\mathrm{Ran}(I-T)^{\tau-\lfloor\tau\rfloor}$ is dense in $\overline{\mathrm{Ran}(I-T)}$. Let $x\in X$ be such that $\|z_2-(I-T)^{\tau-\lfloor\tau\rfloor}x\|\le\frac{\varepsilon}{2\|(I-T)\|^{\lfloor\tau\rfloor}}$. Then

$$\begin{aligned}\|y-(I-T)^\tau x\| &\le \|y-(I-T)^{\lfloor\tau\rfloor}z_2\|+\|(I-T)^{\lfloor\tau\rfloor}z_2-(I-T)^\tau x\| \\ &\le \|y-(I-T)^{\lfloor\tau\rfloor}z\|+\|(I-T)\|^{\lfloor\tau\rfloor}\|z_2-(I-T)^{\tau-\lfloor\tau\rfloor}x\|\le\varepsilon.\end{aligned}$$

□

*Remark* 4.2.18. Recall the unilateral weighted backward shift operator $T$ on $\ell^p, 1<p<\infty$ in Example 4.2.12. We have that $X_\tau=\mathrm{Ker}(I-T)\oplus\mathrm{Ran}(I-T)^\tau$ is a dense subspace in $\ell^p$ for $\tau\in\mathbb{N}$. Indeed, $T$ is absolutely Cesàro bounded on $\ell^p$ , then by [17, Corollary 2.7], $T$ is mean ergodic. Therefore, the result follows from Corollary 4.2.16 and Lemma 4.2.17.

## 4.3. MAIN RESULT

In this section, we study the polynomial decay rates of the discrete semigroup $(T^n)_{n\ge0}$ when composed with the operator $(I-T)^\tau$ for some $\tau>0$, in general Banach spaces in Section 4.3.1 and in Banach spaces with Fourier type in Section 4.3.2. Our main results are Theorems 4.3.2 and 4.3.4.

### 4.3.1. RESULTS IN GENERAL BANACH SPACES

We first investigate the polynomial decay of $T^n(I-T)^\tau$ in general Banach spaces as $n\to\infty$. Two cases are considered: when the resolvent of $T$ satisfies the sector condition (4.2.5) and when it satisfies the weak Ritt condition (4.2.9).

Recall the definition of $K_\delta$ in (4.2.4). The following result directly extends [109, Theorem 4.9.3].

**Proposition 4.3.1.** *Let $X$ be a complex Banach space, $T\in\mathcal{L}(X)$, $\sigma(T)\subseteq\overline{\mathbb{D}}$ and $\sigma(T)\cap\partial\mathbb{D}=\{1\}$. Suppose that $K_\delta\subseteq\rho(T)$ for some $\delta>0$, and there exists a constant $C\ge1$ such that*

$$\|R(\lambda,T)\|\le\frac{C}{|\lambda-1|^\beta},\qquad \lambda\in\{u\in K_\delta\,|\,|u|<2\},$$

*holds for some $\beta>1$. Then for every $s>0$, there exists a constant $C_{\delta,\tau,\beta}$ such that*

$$\|T^n(I-T)^\tau\|\le C_{\delta,s,\beta}n^{-s},\qquad n\to\infty,$$

*where $\tau=s+\beta-1$.*

*Proof.* By (4.2.2),

$$T^n(I-T)^\tau := \frac{1}{2\pi i}\int_\gamma \lambda^n(1-\lambda)^\tau R(\lambda,T)\,\mathrm{d}\lambda$$

holds for $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3$ where $\gamma_1$ is a line segment of the form $\{\lambda = 1 + te^{i(\frac{\pi}{2}+\delta)} \mid t \geq 0\}$, $\gamma_2$ is a circular arc of the form $\{\lambda = re^{i\theta} \mid r < 1 \text{ fixed and } \theta \text{ varies}\}$, and $\gamma_3$ is symmetric with $\gamma_1$ and joints it at 1. Since $\sigma(T) \cap \partial\mathbb{D} = \{1\}$, we may choose $r$ in such a way such that $\|(1-\lambda)^\tau R(\lambda,T)\|$ is uniformly bounded on $\gamma_2$. Then the contour $\gamma$ is inside the unit circle and surrounds $\sigma(T)$ (see Figure 4.3).

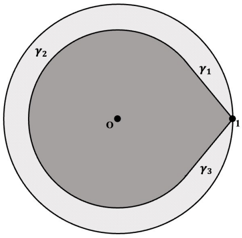


Figure 4.3: a contour $\gamma$.

On $\gamma_2$, we have

$$\left\|\frac{1}{2\pi i}\int_{\gamma_2} \lambda^n(1-\lambda)^\tau R(\lambda,T)\,\mathrm{d}\lambda\right\| \leq C_\tau r^n.$$

By an elementary calculation, there exists a constant $c_\delta > 0$ such that $c_\delta \leq -\frac{\log(1+t^2-2t\sin\delta)}{2t}$ on $\gamma_1$, implying $|\lambda| \leq e^{-c_\delta t}$ while $c_\delta \to 0$ when $\delta \to 0$. Note that on $\gamma_1$ we also have (4.2.5) holds, then

$$\begin{aligned}
\left\|\frac{1}{2\pi i}\int_{\gamma_1} \lambda^n(1-\lambda)^\tau R(\lambda,T)\,\mathrm{d}\lambda\right\| &\leq \frac{1}{2\pi}\int_{\gamma_1} |\lambda^n||\lambda-1|^\tau \|R(\lambda,T)\||\mathrm{d}\lambda| \\
&\leq C\int_{\gamma_1} |\lambda^n||\lambda-1|^{\tau-\beta}|\mathrm{d}\lambda| \\
&\leq C\int_0^\infty e^{-c_\delta n t} t^{\tau-\beta+1-1}\,\mathrm{d}t \\
&\leq C(c_\delta n)^{-(\tau-\beta+1)}\int_0^\infty e^{-h} h^{\tau-\beta+1-1}\,\mathrm{d}h \\
&= C_{\delta,\tau,\beta}\Gamma(\tau-\beta+1)n^{-(\tau-\beta+1)}.
\end{aligned}$$

The treatment of $\gamma_3$ is identical. Then

$$\|T^n(I-T)^\tau\| \leq C_{\delta,\tau,\beta}\Gamma(\tau-\beta+1)n^{-(\tau-\beta+1)} + C2^s r^n, \quad n \in \mathbb{N}.$$

Note that $r < 1$, the conclusion follows since $r^n$ decays faster than $n^{-(\tau-\beta+1)}$ as $n \to \infty$. □

A decay result for $(T^n(I-T)^\tau)_{n\geq 1}$ under the weak Ritt condition is the following.

**Theorem 4.3.2.** *Let $X$ be a complex Banach space, and $T$ be a bounded linear operator on $X$. Assume that $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Furthermore, suppose there exists a constant $\beta > 1$ such that the weak Ritt condition (4.2.9) holds. Then for each $s \in \mathbb{N}$,*

- *for all $\tau > \beta(s+1) - 1$, there exists a constant $C_\tau > 0$ such that*
$$\|T^n(I-T)^\tau\| \le C_\tau n^{-s}, \quad n \in \mathbb{N};$$
- *for $\tau = \beta(s+1) - 1$, there exists a constant $C > 0$ such that*
$$\|T^n(I-T)^\tau\| \le C n^{-s} \log n, \quad n \in \mathbb{N}.$$

*Proof.* Let $s \in \mathbb{N}, \tau \ge \beta(s+1)-1$. Fix $\delta \in (0, \frac{\pi}{2})$ and choose $\varepsilon \in (0,\delta)$. Set $r := \frac{\sin\delta}{\sin(\delta-\varepsilon)}$, which is motivated by the geometric configuration in Figure 4.4. For sufficiently small $\varepsilon > 0$ we have $r \in (1, \frac{3}{2})$ and $r \to 1$ as $\varepsilon \to 0$. Define the contour $\Gamma := \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{re^{i\theta} \mid \varepsilon \le \theta \le 2\pi - \varepsilon\}, \qquad \Gamma_2 = \{1 + \mu e^{i\delta} \mid 0 < \mu \le \tfrac{\sin\varepsilon}{\sin(\delta-\varepsilon)}\}.$$

By (4.2.2),

$$\begin{aligned} T^n(I-T)^\tau x &= \tfrac{1}{2\pi i}\int_\Gamma \lambda^n(1-\lambda)^\tau R(\lambda,T)x\,\mathrm{d}\lambda, \\ &= \tfrac{1}{2\pi i}\left(\int_{\Gamma_1} \lambda^n(1-\lambda)^\tau R(\lambda,T)x\,\mathrm{d}\lambda + \int_{\Gamma_2} \lambda^n(1-\lambda)^\tau R(\lambda,T)x\,\mathrm{d}\lambda\right), \\ &= \tfrac{1}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} r^{n+1}e^{i(n+1)\theta}(1-re^{i\theta})^\tau R(re^{i\theta},T)x\,\mathrm{d}\theta \\ &\quad + \tfrac{1}{2\pi i}\int_{\Gamma_2} \lambda^n(1-\lambda)^\tau R(\lambda,T)x\,\mathrm{d}\lambda. \end{aligned}$$

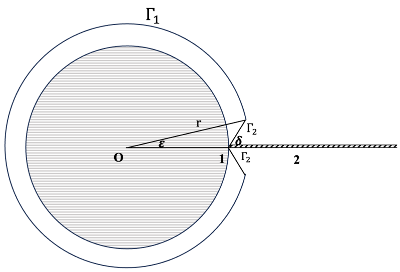


Figure 4.4: the integral path Γ.

Therefore,

$$\begin{aligned} \left\|(n+1)^s T^n(I-T)^\tau x\right\| \le &\left\|\frac{(n+1)^s}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} r^{n+1}e^{i(n+1)\theta}(1-re^{i\theta})^\tau R(re^{i\theta},T)x\,\mathrm{d}\theta\right\| \\ &+ \left\|\frac{(n+1)^s}{2\pi}\int_{\Gamma_2} \lambda^n(1-\lambda)^\tau R(\lambda,T)x\,\mathrm{d}\lambda\right\|. \end{aligned} \tag{4.3.1}$$

For the first term in the right hand side of (4.3.1), we have

$$\begin{aligned} &\left\|\tfrac{(n+1)^s}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} r^{n+1}e^{i(n+1)\theta}(1-re^{i\theta})^\tau R(re^{i\theta},T)x\,\mathrm{d}\theta\right\| \\ &= \left\|\tfrac{(-i)^s r^{n+1}}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} [e^{i(n+1)\theta}]^{(s)}(1-re^{i\theta})^\tau R(re^{i\theta},T)x\,\mathrm{d}\theta\right\| \\ &\le \tfrac{r^{n+1}}{2\pi}\left\|\textstyle\sum_{j=0}^{s-1}[e^{i(n+1)\theta}]^{(j)}[(1-re^{i\theta})^\tau R(re^{i\theta},T)x]^{(s-1-j)}\Big|_\varepsilon^{2\pi-\varepsilon}\right\| \\ &\quad + \tfrac{r^{n+1}}{2\pi}\left\|\int_\varepsilon^{2\pi-\varepsilon} e^{i(n+1)\theta}P(re^{i\theta},T)x\,\mathrm{d}\theta\right\|, \end{aligned} \tag{4.3.2}$$

where $P(re^{i\theta},T) := [(1-re^{i\theta})^\tau R(re^{i\theta},T)]^{(s)} = \sum_{k=0}^{s}\binom{s}{k}[(1-re^{i\theta})^\tau]^{(k)}R(re^{i\theta},T)^{(s-k)}$, which is independent of $n$ and $\varepsilon$. By an elementary calculation, for $k=1,\cdots,s$,

$$R(re^{i\theta},T)^{(k)} = i^k\sum_{j=1}^{k} a_j R(re^{i\theta},T)^{j+1}r^j e^{ij\theta},$$
$$[(1-re^{i\theta})^\tau]^{(k)} = i^k\sum_{j=1}^{k} b_j(1-re^{i\theta})^{\tau-j}r^j e^{ij\theta}.$$

where $a_j, b_j, j=1,2,\cdots,k$ are constants. For writing convenience, we omit the constants from now. By Leibniz's law,

$$\begin{aligned}
P(re^{i\theta},T) &= \sum_{k=0}^{s}[(1-re^{i\theta})^\tau]^{(k)}R(re^{i\theta},T)^{(s-k)}\\
&= \sum_{k=1}^{s-1}\Big(\sum_{j=1}^{k} r^j e^{ij\theta}(1-re^{i\theta})^{\tau-j}\Big)\Big(\sum_{l=1}^{s-k}R(re^{i\theta},T)^{l+1}r^l e^{il\theta}\Big)\\
&\quad + (1-re^{i\theta})^\tau\sum_{l=1}^{s}R(re^{i\theta},T)^{l+1}r^l e^{il\theta} + R(re^{i\theta},T)\sum_{j=1}^{s}r^j e^{ij\theta}(1-re^{i\theta})^{\tau-j}.
\end{aligned}$$

The assumption (4.2.9) further implies

$$\begin{aligned}
\|P(re^{i\theta},T)\| &\le r^s\sum_{k=1}^{s-1}\Big(\sum_{j=1}^{k}|1-re^{i\theta}|^{\tau-j}\Big)\Big(\sum_{l=1}^{s-k}\frac{1}{|1-re^{i\theta}|^{\beta(l+1)}}\Big)\\
&\quad + r^s|1-re^{i\theta}|^\tau\sum_{l=1}^{s}\frac{1}{|1-re^{i\theta}|^{\beta(l+1)}} + \frac{r^s}{|1-re^{i\theta}|^\beta}\sum_{j=1}^{s}|1-re^{i\theta}|^{\tau-j}\\
&= r^s\Big(\sum_{k=1}^{s-1}\Big(\sum_{j=1}^{k}\sum_{l=1}^{s-k}|1-re^{i\theta}|^{\tau-j-\beta(l+1)}\Big) + \sum_{l=1}^{s}|1-re^{i\theta}|^{\tau-\beta(l+1)}\\
&\quad + \sum_{j=1}^{s}|1-re^{i\theta}|^{\tau-j-\beta}\Big).
\end{aligned} \tag{4.3.3}$$

Since $\beta>1$, if $\tau>\beta(s+1)-1$, one has $\tau-k-\beta(s-k+1)>-1$ for $k=0,1,\cdots,s$, and

$$\begin{cases}
\tau-j-\beta(l+1)>-1, & 1\le j\le k, 1\le l\le s-k, 1\le k\le s-1;\\
\tau-\beta(l+1)>-1, & 1\le l\le s;\\
\tau-j-\beta>-1, & 1\le j\le s.
\end{cases}$$

which implies $P(re^{i\theta},T)\in L^1(0,2\pi;\mathscr{L}(X))$ by [100, Lemma 3.1 (1-a)]. On the other hand, if $\tau=\beta(s+1)-1$, by [100, Lemma 3.1 (1-b)],

$$\int_0^{2\pi}\|P(re^{i\theta},T)\|_{\mathscr{L}(X)}\,\mathrm{d}\theta \le Cr^s\log\frac{1}{r-1}. \tag{4.3.4}$$

Moreover, for any $\theta\in(0,2\pi)$ and $j=0,\cdots,s-1$,

$$\begin{aligned}
&\Big\|[e^{i(n+1)\theta}]^{(j)}[(1-re^{i\theta})^\tau R(re^{i\theta},T)x]^{(s-1-j)}\Big\|\\
&= \Big\|i^j(n+1)^j e^{i(n+1)\theta}\sum_{k=0}^{s-1-j}[(1-re^{i\theta})^\tau]^{(k)}R(re^{i\theta},T)^{(s-1-j-k)}x\Big\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| i^{s-1}(n+1)^j e^{i(n+1)\theta}\Big[\sum_{k=1}^{s-2-j}\Big(\sum_{n=1}^{k} r^n e^{in\theta}(1-re^{i\theta})^{\tau-n}\Big)\Big(\sum_{l=1}^{s-1-j-k} R(re^{i\theta},T)^{l+1} r^l e^{il\theta}\Big)\\
&\quad + (1-re^{i\theta})^{\tau}\sum_{l=1}^{s-1-j} R(re^{i\theta},T)^{l+1} r^l e^{il\theta} + R(re^{i\theta},T)\sum_{n=1}^{s-1-j} r^n e^{in\theta}(1-re^{i\theta})^{\tau-n}\Big]\Big\|\\
&\le (n+1)^j r^{s-1-j}\Big[\sum_{k=1}^{s-2-j}\Big(\sum_{n=1}^{k}\sum_{l=1}^{s-1-j-k} |1-re^{i\theta}|^{\tau-n-\beta(l+1)}\Big)\\
&\quad + r^{s-1-j}\Big(\sum_{l=1}^{s-1-j} |1-re^{i\theta}|^{\tau-\beta(l+1)} + \sum_{n=1}^{s-1-j} |1-re^{i\theta}|^{\tau-n-\beta}\Big).
\end{aligned}
$$

Using a similar analysis as above with $s$ replaced by $s-1-j$. Since $\tau \ge \beta(s+1)-1$, one gets

$$\tau - k - (s-j-k)\beta > 0, \quad j = 0,1,\dots,s-1,\ k = 0,1,\dots,s-1-j.$$

Take $\theta = \pm\varepsilon$. By the Sine rule,

$$|1-re^{\pm i\varepsilon}| = \frac{|\sin\varepsilon|}{|\sin(\delta-\varepsilon)|}.$$

Therefore, the boundary terms in (4.3.2) vanish as $\varepsilon \to 0_+$.

For the second term of the right hand side of (4.3.1), we have

$$
\begin{aligned}
&\Big\| \tfrac{(n+1)^s}{2\pi}\int_{\Gamma_2} \lambda^n (1-\lambda)^\tau R(\lambda,T)x\,\mathrm{d}\lambda\Big\|\\
&\lesssim \tfrac{(n+1)^s}{2\pi}\int_0^{\frac{\sin\varepsilon}{\sin(\delta-\varepsilon)}} |1-\mu e^{i\delta}|^n |\mu e^{i\delta}|^\tau \|R(1+\mu e^{i\delta},T)x\|\,|\mathrm{d}\mu|\\
&\lesssim (n+1)^s \tfrac{\sin\varepsilon}{\sin(\delta-\varepsilon)}(1+\tfrac{\sin\varepsilon}{\sin(\delta-\varepsilon)})^{n+\tau-\beta}\|x\|,
\end{aligned}
\tag{4.3.5}
$$

where we use the weak Ritt condition again.

Let $\varepsilon \to 0_+$ in (4.3.1), note that we also have $r \to 1$. If $\tau > \beta(s+1)-1$, by (4.3.1)-(4.3.5) we obtain

$$\|(1+n)^s T^n (I-T)^\tau x\| \le \frac{1}{2\pi}\int_0^{2\pi} \|P(e^{i\theta};T)\|_{\mathscr{L}(X)}\|x\|\,\mathrm{d}\theta \le C_\tau\|x\|.$$

If $\tau = \beta(s+1)-1$, let $r = 1+\frac{1}{n}$, then by (4.3.4),

$$
\begin{aligned}
\|(1+n)^s T^n (I-T)^\tau x\| &\le \tfrac{r^{n+1}}{2\pi}\int_0^{2\pi} \|P(e^{i\theta};T)\|_{\mathscr{L}(X)}\|x\|\,\mathrm{d}\theta\\
&\le C(1+\tfrac{1}{n})^{n+1+s}\log n\|x\|.
\end{aligned}
\tag{4.3.6}
$$

Since $(1+\frac{1}{n})^{n+1+s} \to e$ as $n \to \infty$, the right hand side of (4.3.6) is uniformly bounded in $n$. □

### 4.3.2. Results in Banach space with Fourier type $p \in (1,2]$

In this section, we add more assumptions on the geometry of the underlying Banach space $X$ and get an $\ell^p$-norm decay of discrete semigroups, see Theorem 4.3.4.

We first introduce a lemma which will be used in the main result.

**Lemma 4.3.3.** *Let $T$ be a bounded operator on a complex Banach space $X$, $S \subseteq \rho(T)$ be a bounded set, and let $\beta \in \mathbb{N}$, $s \in [1,\infty]$. Then $R(\cdot,T)(I-T)^\beta \in L^s(S;\mathcal{L}(X))$ if and only if $R(\cdot,T)(I-\cdot)^\beta \in L^s(S;\mathcal{L}(X))$, with constants depending on $\beta$, $\|T\|$, and $\sup_{\lambda\in S}|\lambda|$.*

*Proof.* The case $\beta = 1$ is clear. If $\beta > 1$, let $\lambda \in S$, then there exists a constant $C$ such that $|\lambda| \le C$. By binomial theorem,

$$(I-\lambda)^\beta = (I-T+T-\lambda)^\beta = \sum_{j=0}^{\beta}\binom{\beta}{j}(I-T)^j(T-\lambda)^{\beta-j}.$$

Therefore,

$$R(\lambda,T)(I-\lambda)^\beta = R(\lambda,T)(I-T)^\beta - \sum_{j=1}^{\beta}\binom{\beta}{j}(I-T)^{\beta-j}(T-\lambda)^{j-1}. \tag{4.3.7}$$

Since $|\lambda| \le C$, the last term of (4.3.7) is bounded, finishing the proof. □

Our main result on a Banach space with Fourier type $p$ is stated as follows.

**Theorem 4.3.4.** *Let $X$ be a complex Banach space with Fourier type $p \in [1,2]$ and $\omega \in [1,\infty]$ be such that $\frac{1}{\omega} = \frac{1}{p} - \frac{1}{p'}$. Assume that $T \in \mathcal{L}(X)$ with $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Let $s \in \mathbb{N}_0$ and $\beta \ge 1$ be such that*

- *$\beta - \lfloor\beta\rfloor \in [0, \frac{1}{\omega(s+1)})$ if $p < 2$;*
- *$\beta \in \mathbb{N}$ if $p = 2$.*

*Suppose there exists a constant $C \ge 1$ such that the weak Ritt condition*

$$\|R(\lambda,T)\| \le \frac{C}{|\lambda-1|^\beta}, \quad 1 < |\lambda| < 2,$$

*holds for such $\beta$. Let $\tau = (s+1)\lfloor\beta\rfloor$. Then*

$$\|n \mapsto n^s T^n (I-T)^\tau x\|_{\ell^{p'}} \le C_\tau \|x\|, \quad x \in X.$$

*Remark* 4.3.5. In the case $\beta = 1$, we already saw that $T$ is power bounded. Consequently, we have $\|T^n(I-T)\| \le C\left(\frac{\log n}{n}\right)^{1/\beta}$ in Banach spaces, as established in [135]. Furthermore, the logarithmic correction can be removed in the Hilbert space setting, as shown in [112].

*Proof of Theorem 4.3.4.* Without loss of generality, fix $s \in \mathbb{N}$, the case $s = 0$ can be proved similarly. Let $T^n = 0$ if $n < 0$. By Neumann series expansion, we have for $\theta \in \mathbb{T}$,

$$\sum_{n\ge 0} e_{-n}(\theta) T^n r^{-n} = re^{2\pi i\theta} R(re^{2\pi i\theta}, T), \quad 1 < r < 2. \tag{4.3.8}$$

and for $k \ge 1$,

$$R(re^{2\pi i\theta}, T)^{(k)} = \sum_{j=1}^{k} a_j R(re^{2\pi i\theta}, T)^{j+1} r^j e^{2\pi i j\theta}, \quad 1 < r < 2, \tag{4.3.9}$$

where $a_j, j=1,\cdots,k$ are constants. By Lemma 4.2.7,

$$\|e_{-n}(\theta)n^s r^{-n}T^n(I-T)^\tau x\| \lesssim \frac{n^{s+\beta}}{r^n}\|(I-T)^\tau x\|, \quad n\in\mathbb{N}.$$

Since the right hand side of this inequality is summable, we get the sum

$$\sum_{n\geq 0} e_{-n}(\theta)n^s r^{-n}T^n(I-T)^\tau x$$

is uniformly convergent. Define

$$A_{r,\theta} := 1+\sum_{k=1}^{s}\binom{s}{k}\sum_{j=1}^{k} a_j r^j e^{2\pi i j\theta}R(re^{2\pi i\theta},T)^j.$$

4

By (4.3.8) and (4.3.9), we have

$$\begin{aligned}
&\sum_{n\geq 0} e_{-n}(\theta)n^s r^{-n}T^n(I-T)^\tau x = \left(\frac{i}{2\pi}\right)^s \sum_{n\geq 0}(e^{-2\pi i n\theta})^{(s)} r^{-n}T^n(I-T)^\tau x\\
&= \left(\frac{i}{2\pi}\right)^s \left(\sum_{n\geq 0} e^{-2\pi i n\theta} r^{-n}T^n(I-T)^\tau x\right)^{(s)} = \left(\frac{i}{2\pi}\right)^s \left(re^{2\pi i\theta}R(re^{2\pi i\theta},T)\right)^{(s)}(I-T)^\tau x\\
&= \left(\frac{i}{2\pi}\right)^s \left(\sum_{k=1}^{s}\binom{s}{k}R(re^{2\pi i\theta},T)^{(k)}(re^{2\pi i\theta})^{(s-k)}(I-T)^\tau x\right.\\
&\qquad \left. +(2\pi i)^s re^{2\pi i\theta}R(re^{2\pi i\theta},T)(I-T)^\tau x\right)\\
&= \left(\frac{i}{2\pi}\right)^s \left((2\pi i)^{s-k} re^{2\pi i\theta}\sum_{k=1}^{s}\binom{s}{k}\left((2\pi i)^k\sum_{j=1}^{k} a_j R(re^{2\pi i\theta},T)^{j+1} r^j e^{2\pi i j\theta}\right)(I-T)^\tau x\right.\\
&\qquad \left. +(2\pi i)^s re^{2\pi i\theta}R(re^{2\pi i\theta},T)(I-T)^\tau x\right)\\
&= (-1)^s re^{2\pi i\theta}R(re^{2\pi i\theta},T)\left(1+\sum_{k=1}^{s}\binom{s}{k}\sum_{j=1}^{k} a_j r^j e^{2\pi i j\theta}R(re^{2\pi i\theta},T)^j\right)(I-T)^\tau x\\
&= (-1)^s re^{2\pi i\theta}R(re^{2\pi i\theta},T)A_{r,\theta}(I-T)^\tau x.
\end{aligned} \tag{4.3.10}$$

Let $f_r(n) := e^{-\nu n}r^{-n}T^n x$, where $\nu>0$ such that

$$\log n < \frac{\nu}{\beta+\frac{2}{p}}n, \quad n\in\mathbb{N}.$$

We deduce from $\|T^n\|\lesssim n^\beta$ that

$$\begin{aligned}
\|f_r\|_{\ell^p} &= \Big(\sum_{n=0}^{\infty}\|e^{-\nu n}r^{-n}T^n x\|^p\Big)^{\frac{1}{p}} \lesssim \Big(\sum_{n=0}^{\infty}(e^{-\nu n}n^\beta)^p\Big)^{\frac{1}{p}}\|x\|\\
&\lesssim \Big(\sum_{n=0}^{\infty}(n^{-(\beta+\frac{2}{p})}n^\beta)^p\Big)^{\frac{1}{p}}\|x\| = \Big(\sum_{n=0}^{\infty}\frac{1}{n^2}\Big)^{\frac{1}{p}}\|x\| \leq C\|x\|.
\end{aligned} \tag{4.3.11}$$

Therefore, by (4.3.8),

$$\mathscr{F}f_r(\theta) = \sum_{n=0}^{\infty} e^{-2\pi i n\theta}e^{-\nu n}r^{-n}T^n x = re^{\nu+2\pi i\theta}R(re^{\nu+2\pi i\theta},T)x, \quad \theta\in\mathbb{T}. \tag{4.3.12}$$

Let

$$m_r(\theta) := (-1)^s A_{r,\theta}(I-T)^\tau \left(re^{2\pi i\theta}R(re^{2\pi i\theta},T)(1-e^{-\nu})+e^{-\nu}\right),$$

and let $g(n) := n^s r^{-n}T^n(I-T)^\tau x, n \in \mathbb{N}$. By (4.3.10) and (4.3.12),

$$\begin{aligned} m_r\mathscr{F}f_r &= (-1)^s A_{r,\theta}(I-T)^\tau \left(re^{2\pi i\theta}R(re^{2\pi i\theta},T)(1-e^{-\nu})+e^{-\nu}\right)re^{\nu+2\pi i\theta}R(re^{\nu+2\pi i\theta},T)x \\ &= (-1)^s re^{2\pi i\theta}A_{r,\theta}(I-T)^\tau R(re^{\nu+2\pi i\theta},T)\left(re^{2\pi i\theta}R(re^{2\pi i\theta},T)(e^\nu-I)+I\right)x \\ &= (-1)^s re^{2\pi i\theta}A_{r,\theta}(I-T)^\tau R(re^{\nu+2\pi i\theta},T)R(re^{2\pi i\theta},T)\left(re^{\nu+2\pi i\theta}-T\right)x \\ &= (-1)^s re^{2\pi i\theta}R(re^{2\pi i\theta},T)A_{r,\theta}(I-T)^\tau x \\ &= \mathscr{F}g. \end{aligned}$$

Hence,

$$n^s r^{-n}T^n(I-T)^\tau x = \mathscr{F}^{-1}(m_r\mathscr{F}f_r)(n) = T_{m_r}f_r(n).$$

Assume for a while that $T_{m_r}: \ell^p \to \ell^{p'}$ is uniformly bounded with respect to $1<r<2$, recalling (4.3.11), one has

$$\|n \mapsto n^s r^{-n}T^n(I-T)^\tau x\|_{\ell^{p'}} \le C_\tau\|x\|, \quad x \in X.$$

For any $N \in \mathbb{N}$,

$$\begin{aligned} \left(\sum_{n=1}^N \|n^sT^n(I-T)^\tau x\|^{p'}\right)^{\frac{1}{p'}} &= \left(\sum_{n=1}^N \lim_{r\to 1_+}\|n^s r^{-n}T^n(I-T)^\tau x\|^{p'}\right)^{\frac{1}{p'}} \\ &= \lim_{r\to 1_+}\left(\sum_{n=1}^N \|n^s r^{-n}T^n(I-T)^\tau x\|^{p'}\right)^{\frac{1}{p'}} \\ &\le C_\tau\|x\|, \quad x\in X. \end{aligned}$$

By the monotone convergence theorem, this yields

$$\|n \mapsto n^sT^n(I-T)^\tau x\|_{\ell^{p'}} \le C_\tau\|x\|, \quad x \in X.$$

Finally, we show that $T_{m_r}: \ell^p \to \ell^{p'}$ is uniformly bounded with respect to $1<r<2$. We only need to prove that $R(re^{2\pi i\theta},T)^j(I-T)^\tau$ is a $\ell^p_{p'}(\mathbb{Z};\mathscr{L}(X))$-Fourier multiplier for $1\le j\le s+1$ and $1<r<2$. By Proposition 4.2.1, it suffices to show that $R(re^{2\pi i\theta},T)^j(I-T)^\tau \in L^\omega(\mathbb{T};\mathscr{L}(X))$, $1\le j\le s+1$ with uniform estimate in $r$.

For each $1<r<2$, let $S_r := \{\lambda \,|\, |\lambda| = r\}$. If $\omega=\infty$ (i.e. $p=2$), we immediately conclude from (4.2.9) that $R(re^{2\pi i\theta},T)(I-re^{2\pi i\theta})^{\lfloor\beta\rfloor} \in L^\infty(S_r;\mathscr{L}(X))$; if $\omega<\infty$, recall that $0\le\beta-\lfloor\beta\rfloor<\frac{1}{\omega(s+1)}$, and hence

$$\int_{S_r}\|R(\lambda,T)(I-\lambda)^{\lfloor\beta\rfloor}\|^{j\omega}d\lambda = \int_{\mathbb{T}}\|R(re^{2\pi i\theta},T)(I-re^{2\pi i\theta})^{\lfloor\beta\rfloor}\|^{j\omega}d\theta$$

$$\leq \int_{\mathbb{T}} \frac{C}{|I - re^{2\pi i\theta}|^{(\beta - \lfloor\beta\rfloor)\omega j}} d\theta$$
$$\leq \int_{\mathbb{T}} \frac{C}{|I - e^{2\pi i\theta}|^{(\beta - \lfloor\beta\rfloor)\omega j}} d\theta < \infty.$$

Therefore, $R(re^{2\pi i\theta}, T)(I - re^{2\pi i\theta})^{\lfloor\beta\rfloor} \in L^{j\omega}(S_r; \mathscr{L}(X))$ with uniform estimate in $r$. Recall that $r < 2$, by Lemma 4.3.3, we have $R(re^{2\pi i\theta}, T)(I - T)^{\lfloor\beta\rfloor} \in L^{j\omega}(S_r; \mathscr{L}(X))$ with uniform estimate in $r$. Note that for $\tau \in \mathbb{N}$ satisfying $\tau \geq (s+1)\lfloor\beta\rfloor$, we have

$$\begin{aligned} \int_{\mathbb{T}} \|R(re^{2\pi i\theta}, T)^j (I-T)^\tau\|^\omega d\theta &\leq \int_{\mathbb{T}} \|R(re^{2\pi i\theta}, T)^j (I-T)^{j\lfloor\beta\rfloor}\|^\omega \|(I-T)^{\tau - j\lfloor\beta\rfloor}\|^\omega d\theta \\ &\leq C_\tau \int_{\mathbb{T}} \|R(re^{2\pi i\theta}, T)(I-T)^{\lfloor\beta\rfloor}\|^{j\omega} d\theta < \infty, \end{aligned}$$

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finishing the proof. □

In particular, we get the following boundedness result, which shows $\|T^n(I-T)x\|$ can be bounded without assuming that $T$ is power bounded. This result coincides with [100, Theorem 3.4] if $p = 1$.

**Corollary 4.3.6.** *Let $X$ be a complex Banach space with Fourier type $p \in [1,2]$, Let $0 \leq \alpha < \frac{1}{p} - \frac{1}{p'}$ if $p \neq 2$ and $\alpha = 0$ if $p = 2$. Assume that $T \in \mathscr{L}(X)$ with $\sigma(T) \subseteq \overline{\mathbb{D}}$, $\sigma(T) \cap \partial\mathbb{D} = \{1\}$ and*

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|^{1+\alpha}}, \quad 1 < |\lambda| < 2.$$

*Then*
$$\|n \mapsto T^n(I-T)x\|_{\ell^{p'}} \leq C\|x\|, \quad x \in X.$$

## 4.4. COMPARISON

In this section, we present a comparison of our findings alongside relevant results from prior studies.

### 4.4.1. COMPARISON BETWEEN MAIN RESULTS

In this chapter, we studied the decay rates of the discrete semigroup $(T^n)_{n\geq 0}$ when composed with operators of the form $(I-T)^\tau$ for some $\tau > 0$. We assumed throughout that the spectrum of $T$ is contained in the closed unit disk while only point 1 lies on the boundary of the spectrum. Regarding the resolvent assumption, we examined two cases.

First, we considered the sector condition: for some $\beta > 1$ and $\delta > 0$,

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|^\beta}, \quad \lambda \in \{\lambda \in K_\delta \,|\, |\lambda| < 2\}.$$

Under this condition, we established Proposition 4.3.1 on general Banach spaces, which shows for every $s_1 > 0$, there exists $\tau_1 = s_1 + \beta_1 - 1$ such that

$$\|T^n(I-T)^{\tau_1}\| \leq Cn^{-s_1}, \quad n \to \infty.$$

This is a direct extension of [109, Theorem 4.9.3].

Second, we considered the weak Ritt condition: for $\beta > 1$,

$$\|R(\lambda, T)\| \le \frac{C}{|\lambda - 1|^{\beta}}, \quad 1 < |\lambda| < 2.$$

Note that if $\beta = 1$, i.e. $T$ is a Ritt operator, the above resolvent conditions are equivalent by Lemma 4.2.3. However, when $\beta > 1$, clearly the sector condition implies the weak Ritt condition but the converse is unknown. Our main interest was the weak Ritt condition.

Under the weak Ritt condition, we focused on the effect of the underlying space $X$ on the decay rate, and we mainly considered two types of spaces:

- General Banach spaces;
- Banach spaces with Fourier type $p \in (1,2]$.

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Using the Cauchy formula and differential calculations, we established Theorem 4.3.2, which states that for every $s_2 \in \mathbb{N}$, and $\tau_2 > \beta_2(s_2+1) - 1$, one has

$$\|T^n (I-T)^{\tau_2}\| \lesssim n^{-s_2}.$$

Comparing Theorem 4.3.2 with Proposition 4.3.1, one may conclude that the decay rate $s_1$ is better than $s_2$. Indeed, given $\tau_1 = \tau_2, \beta_1 = \beta_2 > 1$, Proposition 4.3.1 yields the decay rate $s_1 = \tau_1 + 1 - \beta_1$ while Theorem 4.3.2 shows $s_2 \le \frac{\tau_2 + 1 - \beta_2}{\beta_2} < s_1$. This reflects the fact that the sector condition is stronger than the weak Ritt condition.

If $X$ is a Banach space with Fourier type $p \in [1,2)$, then for every $s_3 \in \mathbb{N}$, there exists $\tau_3 = (s_3+1)\lfloor \beta_3 \rfloor$ where $\beta_3 - \lfloor \beta_3 \rfloor < \frac{1}{s_3+1}(\frac{1}{p} - \frac{1}{p'})$ and

$$\|n \mapsto n^{s_3} T^n (I-T)^{\tau_3} x\|_{\ell^{p'}} \le C\|x\|, \quad x \in X.$$

Clearly, this implies

$$\|T^n (I-T)^{\tau_3}\| \le C n^{-s_3}.$$

Let $p = 1, \beta_2 = \beta_3 = \beta, s_2 = s_3 = s$. If $\beta \in \mathbb{N}$, then Theorem 4.3.2 is stronger than Theorem 4.3.4. However, if $0 < \beta - \lfloor \beta \rfloor < \frac{1}{s+1}$, Theorem 4.3.2 yields

$$\tau_2 > \beta(s+1) - 1 = (s+1)\lfloor \beta \rfloor + (s+1)(\beta - \lfloor \beta \rfloor) - 1 \ge \tau_3.$$

Hence, Theorem 4.3.4 is stronger than Theorem 4.3.2.

There are a few minor differences between the above results. Theorem 4.3.2 imposes no restriction on $\beta_2$, whereas Theorem 4.3.4 requires the additional assumption: $\beta_3 - \lfloor \beta_3 \rfloor < \frac{1}{s_3+1}(\frac{1}{p} - \frac{1}{p'})$. However, Theorem 4.3.2 does not cover the boundedness case while we get the boundedness of $n \mapsto T^n(I-T)$ in $\ell^{p'}$ as a corollary of Theorem 4.3.4 (see Corollary 4.3.6). Moreover, in the above three results, $s_1$ need not be an integer, whereas $s_2, s_3$ have to be integers due to the techniques used in the proof.

The convergence in $\ell^{p'}$ provides stronger information and, in certain cases, leads to improved decay estimates. To illustrate this point, we first introduce Lemma 4.4.1 below, which was shown to us by Mario Ullrich.

**Lemma 4.4.1.** *Let $(a_n)_{n\geq 1} \in \ell^1$ be a positive decreasing sequence. Assume that $(a_n n^s)_{n\geq 1} \in \ell^1$. Then*

$$a_n \lesssim n^{-s-1}, \quad n \in \mathbb{N}.$$

*Proof.* Since $(a_n)_{n\geq 1}$ is a decreasing sequence, then

$$a_n \leq \min\{a_k : n/2 < k \leq n\} \leq \frac{1}{\lceil \frac{n}{2} \rceil} \sum_{k>\frac{n}{2}} a_k \lesssim \frac{1}{n} \sum_{k>\frac{n}{2}} a_k, \quad \text{if n is an odd number,}$$

and

$$a_n \leq \min\{a_k : n/2 < k \leq n\} \lesssim \frac{1}{n} \sum_{k>\frac{n}{2}} a_k, \quad \text{if n is a even number.}$$

Here we use the fact that the minimum is smaller than the average. Moreover, since $\sum_{k\geq 1} a_k k^s < \infty$, then

$$\sum_{k>\frac{n}{2}} a_k \lesssim n^{-s} \sum_{k>\frac{n}{2}} a_k k^s \lesssim n^{-s}, \quad n \in \mathbb{N}.$$

Thus,

$$a_n \lesssim \frac{1}{n} \sum_{k>\frac{n}{2}} a_k \lesssim n^{-s-1}, \quad n \in \mathbb{N}.$$

□

*Remark* 4.4.2. Suppose that the assumptions in Theorem 4.3.4 hold. Given $x \in X$, let $a_n := \|T^n(I-T)^{\tau_3} x\|^{p'}$, then by Theorem 4.3.4, $\sum_{n\geq 1} a_n n^{p's_3} \lesssim \|x\|^{p'}$. If $(a_n)_{n\geq 0}$ is a decreasing sequence, Lemma 4.4.1 yields

$$\|T^n(I-T)^{\tau_3} x\| \lesssim n^{-s_3-\frac{1}{p'}} \|x\|,$$

which improves the original result $\|T^n(I-T)^{\tau_3} x\| \lesssim n^{-s_3} \|x\|$ in Theorem 4.3.4.

### 4.4.2. COMPARISON WITH OTHER KNOWN RESULTS

In this chapter, we assume $T$ is a bounded linear operator on $X$. While our general framework assumes only boundedness, stronger decay estimates can be obtained under additional operator-theoretic assumptions.

Recall the following result of Nevanlinna in [111, Example 4 and Theorem 9]. Let $X$ denote the space of analytic functions in $\mathbb{D}$ such that $f^{(1)}$ has boundary values in the Hardy space $H^1$, equipped with the norm

$$\|f\| := |f|_\infty + |f'|_1 = \sup_{|z|\leq 1} |f(z)| + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f^{(1)}(e^{i\theta}) \right| \mathrm{d}\theta.$$

Let $T := \frac{1}{2}(1+M_z)$ where $M_z$ is the multiplication operator with the independent variable $z$. Then $T$ is Kreiss bounded (see (1.2.7)) and $\sigma(T) = \{\lambda \,|\, |\lambda - 1/2| \leq 1/2\}$. In particular, $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Moreover, we have

(1) There exists a constant $C_1 > 0$ such that

$$C_1^{-1}\sqrt{n+1} \leq \|T^n\| \leq C_1\sqrt{n+1}, \quad n \in \mathbb{N};$$

(2) There exists a constant $C_2 > 0$ such that

$$\|R(\lambda, T)\| \leq \frac{C_2}{|\lambda - 1|^2}, \quad 1 < |\lambda| < 2;$$

(3) There exists a constant $C_3 > 0$ such that

$$C_3^{-1} \leq \|T^n(I-T)\| \leq C_3;$$

(4) Then for every integer $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that

$$\|T^n(I-T)^{2k+1}\| \leq \frac{C_k}{(n+1)^k}, \quad n \in \mathbb{N}.$$



However, if we apply Theorem 4.3.2 with exponent $\beta = 2$ to achieve the same decay rate of order $k$, we obtain the estimate $\|T^n(I-T)^\tau\| \leq \frac{C_k}{n^k}$ for any $\tau > 2k+1$. This is slightly weaker than the [111, Example 4], which further requires the assumption of Kreiss boundedness of $T$.

Recently, [100] considered the so-called Ritt–Kreiss condition, or the $(\beta,\gamma)$-RK condition: for $\beta,\gamma \geq 0$, there exists a constant $C > 0$ such that

$$\|R(\lambda, T)\| \leq C\frac{|\lambda|^{\beta+\gamma-1}}{|\lambda-1|^\beta(|\lambda|-1)^\gamma}, \quad |\lambda| > 1.$$

Clearly, the $(\beta,\gamma)$-RK condition is similar to the weak Ritt condition discussed in this chapter. The difference is that the $(\beta,\gamma)$-RK condition will blow up both at 1 and $\infty$, while the weak Ritt condition only blows up at 1. In [100], the authors discussed the power difference of $T^n$ under the $(\beta,\gamma)$-RK condition, achieving several bounded and growth results with respect to the choice of $\beta$ and $\gamma$. In particular, under the $(\beta,0)$-RK condition, [100, Theorem 3.4] gave the boundedness of $\|T^{n+1} - T^n\|$ for $1 < \beta < 2$, which is consistent with Corollary 4.3.6 in this chapter.

While [100] investigates the upper bound of power difference $T^n(I-T)$ under the $(\beta,\gamma)$-RK condition, we focus on establishing decay rates under the weak Ritt condition. To achieve decay rates, we introduce a trade-off in the composed operator: instead of considering $T^n(I-T)$, we study $T^n(I-T)^\tau$ for some $\tau > 1$. This approach parallels the relationship between uniformly bounded strongly continuous semigroups and power bounded operators (see Table 4.1). Additionally, we explore how the geometry of the underlying Banach space $X$ influences the achievable decay rates, as made precise in Theorem 4.3.4.

## 4.5. APPLICATION

In this section, we illustrate the application of our main results to two significant areas: the asymptotic behaviour of solutions to non-autonomous Cauchy problems and the convergence rates of iterative and polynomial acceleration methods for fixed point problems.

### 4.5.1. POLYNOMIAL STABILITY OF DAMPED WAVE EQUATIONS

The study of the asymptotic behaviour of orbits of evolution families is inspired by the work of [116]. However, our approach differs from that of [116] in a key aspect: while [116] focuses primarily on uniformly bounded evolution families, we extend the analysis to include evolution families that are not necessarily uniformly bounded.

Let $X$ be a reflexive Banach space. Recall the definition of evolution family $(U(t,s))_{t\geq s\geq 0}$ as follows.

**Definition 4.5.1.** A two parameter family of bounded linear operators $(U(t,s))_{t\geq s\geq 0}$ on $X$ is called an *evolution family* if the following two conditions are satisfied:

(1) $U(s,s)=I$, $U(t,r)U(r,s)=U(t,s)$ for all $0\leq s\leq r\leq t\leq T$;



(2) $(t,s)\to U(t,s)$ is strongly continuous for all $0\leq s\leq t\leq T$.

The evolution family $(U(t,s))_{t\geq s\geq 0}$ is usually used in non-autonomous Cauchy problems, we consider the homogeneous case:

$$\begin{cases} z_t(t) &= A(t)z(t), \quad t\geq 0,\\ z(0) &= x, \end{cases} \tag{4.5.1}$$

where $A(t), t\geq 0$ are closed, densely defined linear operators and $x\in D(A(0))$. The evolution family $(U(t,s))_{t\geq s\geq 0}$ is said to be *$k$-periodic* for some $k\in\mathbb{N}$, if $U(t+k,s+k)=U(t,s)$, for all $t\geq s\geq 0$.

Let $k,\tau\in\mathbb{N}$. Define the operator $T:=U(k,0)$ and the space $X_\tau:=\mathrm{Ker}(I-T)\oplus\mathrm{Ran}(I-T)^\tau$. If $T$ is mean ergodic, then $X_\tau$ is dense in $X$ by Corollary 4.2.16 and Lemma 4.2.17. We further impose the following assumptions:

$(H_1)$ The family $(A(t))_{t\geq 0}$ is sufficiently well behaved so that there exists an associated evolution family $(U(t,s))_{t\geq s\geq 0}$, and the solution of (4.5.1) can be represented as

$$z(t)=U(t,0)x,\quad t\geq 0.$$

$(H_2)$ The evolution family $(U(t,s))_{t\geq s\geq 0}$ is *$k$-periodic* for some $k\in\mathbb{N}$.

$(H_3)$ The operator $T:=U(k,0)$ is mean ergodic.

$(H_4)$ $X_\tau\subseteq D(A(t))$ for every $t>0$, and there exists a projection operator $P$ from $X$ to $\mathrm{Ker}(I-T)$.

We refer the readers to [117, Chapter 5] and [50] for a detailed exposition of the relation between the family $(A(t))_{t\geq 0}$, the evolution family $(U(t,s))_{t\geq s\geq 0}$, and the solutions of non-autonomous Cauchy problems (4.5.1). Moreover, if $A(t)=A$ is independent of $t$, then $U(t,s)=U(t-s)$ and the two parameter family of operators reduces to the semigroup $(U(t))_{t\geq 0}$ generated by $A$. One says the evolution family $(U(t,s))_{t\geq s\geq 0}$ associated with $(A(t))_{t\geq 0}$ is *uniformly bounded* if there exists a constant $C\geq 1$ such that $\sup_{0\leq s\leq t}\|U(t,s)\|\leq C$. Our focus is on the asymptotic behaviour of the solution to equation

(4.5.1) when the initial value $x \in X_\tau$, and $(U(t,s))_{t\geq s\geq 0}$ is non-uniformly bounded, further implying $T$ is not power bounded. For previous results concerning the stability of periodic evolution families, see [10, 13, 144] and the references therein.

Let $z^*$ be the periodic solution of (4.5.1) with the initial value $z^*(0) = Px$, i.e. $z^*(t) = z^*(0)$ if $t = nk$ for $n \in \mathbb{N}$. The conclusion is as follows.

**Theorem 4.5.2.** *Let $X$ be a complex Banach space and assumptions* $(\mathrm{H}_1)-(\mathrm{H}_4)$ *hold. Assume that $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T)\cap\partial\mathbb{D} = \{1\}$. Suppose that there exists a $\beta > 1$ such that the weak Ritt condition (4.2.9) holds. Then for $x \in X_{\lceil 2\beta-1\rceil}$, the solution of equation (4.5.1) satisfies*

$$\|z(t) - z^*(t)\| \to 0, \quad t\to\infty.$$

*Proof.* Since $T$ is mean ergodic, then $X_{\lceil 2\beta-1\rceil}$ is dense in $X$ by Corollary 4.2.16 and Lemma 4.2.17. Therefore, for any $x \in X_{\lceil 2\beta-1\rceil}$, $x = Px + z$ where $z \in \mathrm{Ran}(I-T)^{\lceil 2\beta-1\rceil}$. Since $(I-P)x \in \mathrm{Ran}(I-T)^{\lceil 2\beta-1\rceil}$, there exists $y \in X$ such that $(I-P)x = (I-T)^{\lceil 2\beta-1\rceil}y$. By Theorem 4.3.2,

$$\|T^n x - Px\| = \|T^n(I-P)x\| = \|T^n(I-T)^{\lceil 2\beta-1\rceil}y\| \lesssim n^{-1}\|y\| \to 0, \quad n\to\infty. \tag{4.5.2}$$

Given $t \geq 0$, choose a proper $n$ such that $t-nk \in [0,1)$, then by periodicity of $(U(t,s))_{t\geq s\geq 0}$ and the boundedness of $U(t-nk,0)$, we have $T^n = U(k,0)^n = U(nk,(n-1)k)U((n-1)k,(n-2)k)\cdots U(2k,k)U(k,0) = U(nk,0)$ and

$$\begin{aligned}\|z(t)-z^*(t)\| &= \|U(t,0)(x-Px)\| = \|U(t-nk+nk,nk)U(nk,0)(x-Px)\| \\ &= \|U(t-nk,0)(T^n x - Px)\| \leq C\|T^n x - Px\|.\end{aligned} \tag{4.5.3}$$

The proof is completed by (4.5.2) and (4.5.3). □

*Remark* 4.5.3. Unfortunately, we can not extend this result to $x \in X$ since $X_{\lceil 2\beta-1\rceil}$ is not closed by [109, Theorem 4.4.5].

### 4.5.2. POLYNOMIAL ACCELERATION OF FIXED POINT PROBLEMS

In this section, we study the application of our main results in polynomial acceleration of fixed point problems; for details, we refer to [109].

Let $T$ be a bounded linear operator on a Banach space $X$, and $g \in (I-T)^\tau X$ for some $\tau \in \mathbb{R}$ be a given vector such that the following fixed point problem

$$x = Tx + g \tag{4.5.4}$$

admits a solution on $X$. One simple method to solve equation (4.5.4) is *successive approximation.* We construct a sequence of approximations $(x_k)_{k\geq 0}$ defined by

$$x_{k+1} := Tx_k + g,$$

with a given starting vector $x_0 \in X$. Define

$$d_k := x_{k+1} - x_k = Tx_k - x_k + g$$

as the associated *residual*. We seek an approximation to the solution $x$ in the subspace $x_0 + \operatorname{span}\{d_0, Td_0, \cdots, T^{k-1}d_0\}$. In other words, we look for coefficients $a_{jk}$ such that

$$x_k = x_0 + \sum_{i=0}^{k-1} a_{ik} T^i d_0.$$

By a direct calculation,

$$d_k = Tx_k - x_k + g = \left(1 - (I-T)\sum_{i=0}^{k-1} a_{ik} T^i\right) d_0.$$

This motivates the definition of the polynomials



$$q_{k-1}(\lambda) = \sum_{i=0}^{k-1} a_{ik}\lambda^i, \quad p_k(\lambda) = 1 - (1-\lambda)q_{k-1}(\lambda).$$

Recall the following lemma from [109, Proposition 1.4.2].

**Lemma 4.5.4.** *Let $q$ be an arbitrary polynomial and set*

$$p(\lambda) := 1 - (1-\lambda)q(\lambda).$$

*If $x$ solves the fixed point problem (4.5.4), then it solves the problem*

$$x = p(T)x + q(T)g. \tag{4.5.5}$$

*Conversely, if additionally* $\operatorname{Ker} q(T) = \{0\}$, *then (4.5.5) implies (4.5.4) as well.*

For $y \in \operatorname{Ran}(I-T)^\tau$, set

$$\|y\|_\tau := \inf\{\|a\| \mid y = (I-T)^\tau a\}.$$

Let $p_k(\lambda) := \lambda^k, q_{k-1}(\lambda) = \sum_{i=0}^{k-1}\lambda^i$. Then $p_k(\lambda) = 1-(1-\lambda)q_{k-1}(\lambda)$. By a direct calculation,

$$x_k := p_k(T)x_0 + q_{k-1}(T)g, \quad k \in \mathbb{N}. \tag{4.5.6}$$

We are interested in studying the iteration speed of $x_k$ for the fixed point problem (4.5.4) when $g \in \operatorname{Ran}(I-T)^{\tau+1}$, which implies $x$ stays in $\operatorname{Ran}(I-T)^\tau$. Here, the parameter $\tau$ quantifies the degree of "amenability" of the data $g$, which is intrinsically linked to the decay rate $s$ of the iteration process. Specifically, a larger value of $\tau$ typically corresponds to faster decay, reflecting a more regular $g$ by Theorem 4.3.4. The conclusion is as follows, which is similar to [109, Theorem 4.9.1].

**Theorem 4.5.5.** *Let $X$ be a complex Banach space with Fourier type $p \in [1,2]$ and $\omega \in [1,\infty]$ be such that $\frac{1}{\omega} = \frac{1}{p} - \frac{1}{p'}$. Assume that $T \in \mathscr{L}(X)$ with $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Let $s \in \mathbb{N}_0$. If $p < 2$, let $\beta \geq 1$ be such that $\beta - \lfloor\beta\rfloor \in [0, 1/\omega(s+1))$; If $p = 2$, let $\beta \in \mathbb{N}$. Suppose the weak Ritt condition (4.2.9) holds for such $\beta$. Assume that (4.5.4) has a solution $x \in \operatorname{Ran}(I-T)^\tau$ for $\tau = (s+1)\lfloor\beta\rfloor$. Then*

$$\|k \mapsto k^s T^k (I-T)^\tau x\|_{\ell^{p'}} \leq C_\tau \|x\|_\tau.$$

*Proof.* By Lemma 4.5.4, if $x$ solves the fixed point problem (4.5.4), then it also solves the problem

$$x = p_k(T)x + q_{k-1}(T)g.$$

Let $a_{\inf}$ be a vector satisfying $x - x_0 = (I-T)^\tau a_{\inf}$ with $\|a_{\inf}\| = \inf\{\|a\| \mid x - x_0 = (I-T)^\tau a\}$. Then by Theorem 4.3.2,

$$\begin{aligned} \|k \mapsto k^s(x - x_k)\|_{\ell^{p'}} &= \|k \mapsto k^s p_k(T)(x - x_0)\|_{\ell^{p'}} \\ &\lesssim \|k \mapsto k^s T^k (I-T) a_{\inf}\|_{\ell^{p'}} \\ &\lesssim \|a_{\inf}\| = \|x - x_0\|_\tau, \end{aligned}$$

which completes the proof. □

A similar argument, replacing Theorem 4.3.4 with Theorem 4.3.2, yields the following decay rate result.

**Proposition 4.5.6.** *Let $T$ be a bounded linear operator on a Banach space $X$ with $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$. Suppose that there exists a $\beta > 1$ such that the weak Ritt condition (4.2.9) holds. Let $s \in \mathbb{N}$ and $\tau > (s+1)\beta - 1$. Assume that (4.5.4) has a solution $x \in \operatorname{Ran}(I - T)^\tau$. Then for any $x_0 \in \operatorname{Ran}(I-T)^\tau$, we have*

$$\|x - x_k\| \lesssim k^{-s}\|x - x_0\|_\tau, \quad k \to \infty,$$

*where $x_k$ is defined in* (4.5.6).

# 5

# Strongly Kreiss Bounded Operators in UMD Banach Spaces

## 5.1. Introduction

In the last chapter, we studied the decay rates of discrete semigroups $(T^n)_{n\geq 0}$ whose generator $T$ is a weak Ritt operator $T$. In this chapter, we study the growth bounds for $(T^n)_{n\geq 0}$ where $T$ satisfies a slightly weaker resolvent condition than the Ritt condition.

Let $X$ be a Banach space. If $T$ is power bounded, then by the Neumann series

$$(\lambda - T)^{-1} = \sum_{n\geq 0} \frac{T^n}{\lambda^{n+1}}, \quad |\lambda| > 1,$$

one obtains

$$\left\|(\lambda - T)^{-1}\right\| \leq \sum_{n\geq 0} \frac{\|T^n\|}{|\lambda|^{n+1}} \leq \frac{C}{|\lambda| - 1}, \quad |\lambda| > 1.$$

One can repeat the above calculation after differentiation, to see that

$$\left\|(\lambda - T)^{-k}\right\| \leq \frac{C}{(|\lambda| - 1)^k}, \quad |\lambda| > 1, k \in \mathbb{N}.$$

These observations motivate the definitions of (strongly) Kreiss bounded operators, which we already mentioned in Definition 1.2.4. By the above observations, we see any power bounded operator (Ritt operator) is (strongly) Kreiss bounded. By letting $\lambda \to \infty$ one sees that $K_s, K \geq 1$.

This chapter is based on the article [39]: C. Deng, E. Lorist, and M. Veraar. "Strongly Kreiss bounded operators in UMD Banach spaces". In: Semigroup Forum 108.3 (2024), pp. 594–625.

In applications to numerics and ergodic theory, one often needs power boundedness of $T$ or sharp estimates for $\|T^n\|$ as $n \to \infty$, which can be difficult to obtain directly. However, it is often possible to check (strong) Kreiss boundedness. Therefore, it is useful to investigate the converse to the above observations:

(i) Does (strong) Kreiss boundedness imply power boundedness?

(ii) If this is not the case, which growth of $\|T^n\|$ can one obtain from the (strong) Kreiss boundedness of $T$?

In the continuous time setting, the Hille-Yosida theorem provides a result of this form. It gives the equivalent characterization between bounded $C_0$-semigroups and the powers of the resolvent of its generator. Moreover, the Hille-Yosida theorem yields that $T$ is strongly Kreiss bounded with constant $K_s$ if and only if

$$\left\| e^{\xi T} \right\| \le K_s e^{|\xi|}, \quad \xi \in \mathbb{C}, \tag{5.1.1}$$

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see [110, Proposition 1.1].

There is a gap between (5.1.1) and power boundedness of $T$, stemming from the gap between the growth of an entire function and the decay of its Taylor coefficients (see [110]). Therefore, the answer to Question (i) is unfortunately negative: not every (strongly) Kreiss bounded operator is power bounded. Counterexamples to this and related questions can be found in [7, 28, 43, 77, 96].

Question (ii) has been extensively studied. For instance, using Cauchy's integral formula, one can check that if $T$ is Kreiss bounded with constant $K$, then (see [139, p.9]) we have

$$\left\| T^n \right\| \le Ke(n+1), \quad n \in \mathbb{N}, \tag{5.1.2}$$

and, if $T$ is strongly Kreiss bounded with constant $K_s$, then (see [96, Theorem 2.1]) we have

$$\left\| T^n \right\| \le K_s \sqrt{2\pi(n+1)}, \quad n \in \mathbb{N}. \tag{5.1.3}$$

Moreover, these growth rates in $n$ are known to be optimal in general Banach spaces, see [136, 139] for Kreiss bounded operators, and [96, Example 2.2] for strongly Kreiss bounded operators.

Under geometric assumptions on $X$ one can improve the above bounds. In the special case that $X$ is $d$-dimensional, the "Kreiss matrix theorem" (see [78, 86, 138]) asserts that Kreiss boundedness with constant $K$ implies $T$ is power bounded with $\|T^n\| \le Ked$. In applications, the dimension may be very large (see [43]), so it is of interest to study the sharpness with respect to $d$, which was established in [77] up to multiplicative constants. In the finite dimensional setting, this seemed the end of the story. However, 20 years later in [114], it was shown that the bound can be improved to sublinear growth in $d$ under further conditions.

In the infinite dimensional setting several results are known which improve the estimate (5.1.2) for Kreiss bounded operators and the estimate (5.1.3) for strongly Kreiss bounded operators:

- If $X$ is a Hilbert space:
  - (5.1.2) can be improved to $\|T^n\| = O\big(n/\sqrt{\log(n+2)}\big)$ (see [19, Theorem 5] and [30, Theorem 4.1]).
  - (5.1.3) can be improved to $\|T^n\| = O((\log(n+2))^\beta)$ for some $\beta > 0$ depending on $T$ (see [30, Theorem 4.5]). Moreover it is also shown in [30, Proposition 4.9] that $\beta$ can be arbitrary large.
- If $X = L^p$ with $p \in (1,\infty)\setminus\{2\}$:
  - (5.1.2) can be improved to $\|T^n\| = O\big(n/\sqrt{\log(n+2)}\big)$ as well (see [37, Corollary 3.2]).
  - (5.1.3) can be improved to $\|T^n\| = O(n^{\left|\frac{1}{2}-\frac{1}{p}\right|}(\log(n+2))^\beta)$ for some $\beta > 1$, where the number $\left|\frac{1}{2}-\frac{1}{p}\right|$ is optimal (see [7, Theorem 1.1]).
- If $X$ is a UMD space, $q$ and $q^*$ denote the (finite) cotypes of $X$ and $X^*$, respectively:
  - (5.1.2) can be improved to $\|T^n\| = O(n/(\log(n+2))^\beta)$ with $\beta = \frac{1}{q\wedge q^*}$ (see [37, Theorem 3.1]).

Table 5.1: Growth rates for (strongly) Kreiss bounded operators in various spaces.

| | Banach | Hilbert | $L^p$ | UMD |
|---|---|---|---|---|
| KB | $O(n)$ | $O\big(n/\sqrt{\log(n+2)}\big)$ | $O(n/\sqrt{\log(n+2)})$ | $O(n/(\log(n+2))^\beta)$ |
| SKB | $O(\sqrt{n})$ | $O((\log(n+2)))^\beta)$ | $O(n^{\left\|\frac{1}{2}-\frac{1}{p}\right\|}(\log(n+2))^\beta)$ | This chapter |

See Table 5.1 for an overview of these results. An improvement of (5.1.3) for general UMD spaces seems to be missing. The main results of this paper give such improvements. Moreover, we recover the results for strongly Kreiss bounded operators from [30, Theorem 4.5] and [7, Theorem 1.1] in the Hilbert and $L^p$-cases, respectively. The following two results (see Corollaries 5.3.2 and 5.3.3) are special cases of our main result:

- If $X$ is a UMD space, there exists an $\alpha \in (0,1/2)$ depending only on $X$ such that $\|T^n\| = O(n^\alpha)$;
- If $X = [Y,H]_\theta$ (complex interpolation), where $Y$ is a UMD space and $H$ is a Hilbert space with $\theta \in (0,1)$, then there exists an $\alpha \in (0,(1-\theta)/2)$ depending only on $X$ such that $\|T^n\| = O(n^\alpha)$.

For instance, the above conclusions can be applied to $L^p$-spaces both in the commutative setting and non-commutative setting. Improvements of (5.1.3) for Banach function spaces are discussed in Theorems 5.4.1 and 5.4.4.

The previously mentioned improvements of (5.1.3) follow from one single theorem, in which the main condition on $X$ is formulated in terms of upper and lower estimates for decompositions in the Fourier domain, which we introduce and study in detail. The definitions and properties of these decompositions will be given in Section 5.2.

**Theorem 5.1.1.** *Let $X$ be a Banach space which has upper $\ell^{q_0}(L^p)$-decompositions and lower $\ell^{q_1}(L^p)$-decompositions, where $p \in (1,\infty)$ and $1 < q_0 \le q_1 < \infty$. If $T$ is a strongly Kreiss bounded operator on $X$, then there exist constants $C, \beta > 0$ depending on $X$ and $T$ such that*

$$\|T^n\| \le C n^{\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q_1}\right)}(\log(n+2))^{\beta}, \quad n \ge 1.$$

One can see that $q_0 = q_1$ would lead to logarithmic growth. However, this equality can only occur if $X$ is isomorphic to a Hilbert space. This follows from Propositions 5.2.10 and 5.2.11 and Kwapień's theorem (see [83]).

The structure of this chapter is as follows. We explain our main tool: Fourier decomposition properties in Section 5.2, including duality, interpolation, extrapolation, (Fourier) type, and cotype properties. With the help of Fourier decompositions, we can prove our main results in Section 5.3 for general UMD space and in Section 5.4 for UMD Banach function spaces. In Section 5.5 we collect some open problems related to the results of paper.



## 5.2. FOURIER DECOMPOSITIONS

In Theorem 6.1.2, we used an assumption in terms of decompositions in the Fourier domain. In this section, we will introduce these concepts.

### 5.2.1. $L^p(\mathbb{T};\mathscr{L}(X,Y))$-FOURIER MULTIPLIER THEORY

**Definition 5.2.1.** Define the *Fourier transform* $\mathscr{F}: L^1(\mathbb{T};X) \to \ell^\infty(\mathbb{Z};X)$, $f \mapsto \widehat{f}$ as

$$\mathscr{F}(f)(n) := \widehat{f}(n) := \int_{\mathbb{T}} f(t)\overline{e_n(t)}\,\mathrm{d}t, \quad n \in \mathbb{Z}.$$

We call $\widehat{f}(n)$ the $n$-th Fourier coefficient of $f$. For a sequence $g = (g_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z};X)$, define the *inverse Fourier transform* $\mathscr{F}^{-1}: \ell^1(\mathbb{Z};X) \to L^\infty(\mathbb{T};X), g \mapsto \check{g}$ as

$$\mathscr{F}^{-1}(g)(t) := \check{g}(t) := \sum_{n\in\mathbb{Z}} g_n e_n(t), \quad t \in \mathbb{T}.$$

In this chapter, we mainly consider the Fourier transform of trigonometric polynomials, the definition is as follows.

**Definition 5.2.2.** A *trigonometric polynomial* $f : \mathbb{T} \to X$ is a function of the form

$$f(x) := \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} a_n e_n,$$

where $(a_n)_{n\in\mathbb{Z}}$ is a finitely supported sequence in $\mathbb{Z}$.

Let $\mathscr{P}(\mathbb{T};X)$ denote the set of all trigonometric polynomial $f : \mathbb{T} \to X$, then $\mathscr{P}(\mathbb{T};X)$ is dense in $L^p(\mathbb{T};X)$ for all $p \in [1,\infty)$. Observe that in view of the orthonormality of the exponentials we have for all $n \in \mathbb{Z}$ and $f \in \mathscr{P}(\mathbb{T};X)$, $\widehat{f}(n) = a_n$.

Let $Y$ be another Banach space, for a bounded sequence $m = (m_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z};\mathscr{L}(X,Y))$ and $f \in \mathscr{P}(\mathbb{T};X)$, define the Fourier multiplier operator

$$T_m f := \mathscr{F}^{-1}(m\widehat{f}) = \sum_{n\in\mathbb{Z}} m_n \widehat{f}(n) e_n,$$

which is well-defined since $\widehat{f}$ is a finitely nonzero sequence. Let $p \in [1,\infty)$. If there exists a constant $C > 0$ such that for all $f \in \mathscr{P}(\mathbb{T};X)$,

$$\|T_m f\|_{L^p(\mathbb{T};Y)} \le C\|f\|_{L^p(\mathbb{T};X)},$$

we call $m$ a $L^p(\mathbb{T};\mathscr{L}(X,Y))$-*Fourier multiplier*. In this case, $T_m$ can be uniquely extended to a bounded linear operator on $L^p(\mathbb{T};X)$ by Lemma 2.2.4.

### 5.2.2. $\ell^q(L^p)$-FOURIER DECOMPOSITIONS

After the above preparation we can now introduce the $\ell^q(L^p)$-Fourier decompositions. A family $\mathscr{I}$ of subsets of $\mathbb{Z}$ is called an *interval partition* if it is a partition of $\mathbb{Z}$ and each $I \in \mathscr{I}$ is an interval. For an interval $I \subseteq \mathbb{Z}$ and $f \in \mathscr{P}(\mathbb{T};X)$, define

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$$S_I f := T_{\mathbf{1}_I} f = \mathscr{F}^{-1}(\mathbf{1}_I \widehat{f}) = \sum_{n\in I} \widehat{f}(n) e_n.$$

Note that $S_I$ is called the Riesz projection if $I := \mathbb{Z}_+$. For $p \in (1,\infty)$, we will write $R_{X,p} := \|T_{\mathbf{1}_{\mathbb{Z}_+}}\|_{\mathscr{L}(L^p(\mathbb{T};X))}$. By Theorem 2.4.2, $X$ is a UMD Banach space if and only if the Riesz projection is bounded. For any $I := [a,b] \subseteq \mathbb{Z}$, denote $M_a f(x) := e^{2\pi i a x} f(x)$. For $x \in \mathbb{T}$, we have

$$\begin{aligned}\|S_I f\|_{L^p(\mathbb{T};X)} &= \|S_{[a,\infty)} f - S_{[b,\infty)} f\|_{L^p(\mathbb{T};X)} \\ &= \|M_a S_{\mathbf{1}_{\mathbb{Z}_+}} M_{-a} f - M_b S_{\mathbf{1}_{\mathbb{Z}_+}} M_{-b} f\|_{L^p(\mathbb{T};X)} \\ &\le 2R_{X,p}\|f\|_{L^p(\mathbb{T};X)}.\end{aligned} \tag{5.2.1}$$

**Definition 5.2.3.** Let $X$ be a Banach space and $p, q \in [1,\infty]$.

(i) The space $X$ is said to have *upper* $\ell^q(L^p)$*-decompositions* if there exists a constant $U > 0$ such that for each interval partition $\mathscr{I}$ and all $f \in \mathscr{P}(\mathbb{T};X)$,

$$\|f\|_{L^p(\mathbb{T};X)} \le U\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^q_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q}}.$$

(ii) The space $X$ is said to have *lower* $\ell^q(L^p)$*-decompositions* if there exists a constant $L > 0$ such that for each interval partition $\mathscr{I}$ and all $f \in \mathscr{P}(\mathbb{T};X)$,

$$\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^q_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q}} \le L\|f\|_{L^p(\mathbb{T};X)}.$$

By the triangle inequality,

$$\|f\|_{L^p(\mathbb{T};X)} = \Big\|\sum_{I\in\mathscr{I}} S_I f\Big\|_{L^p(\mathbb{T};X)} \le \sum_{I\in\mathscr{I}} \|S_I f\|_{L^p(\mathbb{T};X)}.$$

Hence every Banach space has upper $\ell^1(L^p)$-decompositions for any $p \in [1,\infty]$. Moreover, if $X$ is nonzero, then $q < \infty$ and in fact $q \le 2 \wedge p'$ for the upper decompositions (see Proposition 5.2.10).

Any UMD Banach space has lower $\ell^\infty(L^p)$-decompositions for $p \in (1,\infty)$, which follows from (5.2.1). In Theorem 5.2.9 we shall see that this can be improved. Moreover, the UMD property and $p \in (1,\infty)$ cannot be avoided for the lower decompositions. In other words, if there exist $p, q \in [1,\infty]$ such that (a nonzero) $X$ has lower $\ell^q(L^p)$-decompositions, then $p \in (1,\infty)$, and $X$ is a UMD space. To see this, take $I_1 = \mathbb{Z}_+$ and $I_2 = \mathbb{Z}_- \setminus \{0\}$. Definition 5.2.3(ii) immediately implies that for all $f \in \mathscr{P}(\mathbb{T};X)$,

$$\|S_{I_1} f\|_{L^p(\mathbb{T};X)} \le L\|f\|_{L^p(\mathbb{T};X)},$$

which gives the boundedness of the Riesz projection. Thus, $X$ is a UMD space and $p \in (1,\infty)$ due to the reflexivity of $X$.

### 5.2.3. BASIC PROPERTIES



Let us discuss some basic properties of the upper and lower decompositions. We start with a simple duality result.

**Proposition 5.2.4** (Duality)**.** *Let $X$ be a Banach space, $p \in (1,\infty)$ and $q \in [1,\infty]$. The following are equivalent:*

*(1) $X$ is a UMD space which has upper $\ell^q(L^p)$-decompositions.*

*(2) $X^*$ has lower $\ell^{q'}(L^{p'})$-decompositions.*

*Proof.* If $q = 1$, by the analysis in Section 5.2.2, $X^*$ has lower $\ell^\infty(L^{p'})$-decompositions implies $X^*$ has UMD and so does $X$. It is trivial that $X$ has upper $\ell^1(L^p)$-decompositions for any $p \in [1,\infty]$. On the other hand, if $X$ is a UMD space, then $X^*$ also has UMD. Hence $X^*$ has lower $\ell^\infty(L^p)$-decompositions for all $p \in (1,\infty)$. By Propositions 5.2.10 and 5.2.11 below, $q$ cannot be infinity. We next consider the case $q \in (1,\infty)$.

(2)⇒(1): We already noted that $X^*$ is a UMD space in Section 5.2.2. Then $X^{**}$ is also a UMD space, thereby reflexive. Thus $X = X^{**}$. To show the upper estimate, let $\mathscr{I}$ be an interval partition, let $f \in \mathscr{P}(\mathbb{T};X)$ and $g \in \mathscr{P}(\mathbb{T};X^*)$. If $q < \infty$, by Hölder's inequality and the assumption, we have

$$\begin{aligned}
|\langle f,g\rangle_{L^p(\mathbb{T};X),L^{p'}(\mathbb{T};X^*)}| &= \Big|\int_{\mathbb{T}} \langle f,g\rangle_{X,X^*}\,\mathrm{d}t\Big| = \Big|\int_{\mathbb{T}} \sum_{I\in\mathscr{I}} \langle S_I f, S_I g\rangle_{X,X^*}\,\mathrm{d}t\Big| \\
&\le \sum_{I\in\mathscr{I}} \|S_I f\|_{L^p(\mathbb{T};X)} \|S_I g\|_{L^{p'}(\mathbb{T};X^*)} \\
&\le \Big(\sum_{I\in\mathscr{I}} \|S_I f\|^q_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q}} \Big(\sum_{I\in\mathscr{I}} \|S_I g\|^{q'}_{L^{p'}(\mathbb{T};X^*)}\Big)^{\frac{1}{q'}} \\
&\le L\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^q_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q}} \|g\|_{L^{p'}(\mathbb{T};X^*)}.
\end{aligned}$$

Taking the supremum over all $g$ which satisfy $\|g\|_{L^{p'}(\mathbb{T};X^*)} \le 1$, it follows from Lemma 2.4.3 that $X$ has upper $\ell^q(L^p)$-decompositions.

(1)⇒(2): Let $g \in \mathscr{P}(\mathbb{T}; X^*)$. Let $f_I \in \mathscr{P}(\mathbb{T}; X)$ for $I \in \mathscr{I}$, where we suppose that only finitely many $f_I$ are nonzero. Let $f := \sum_{I\in\mathscr{I}} S_I f_I$. Hölder's inequality and the assumption give that

$$\begin{aligned}\Big|\sum_{I\in\mathscr{I}} \int_{\mathbb{T}} \langle f_I, S_I g\rangle_{X,X^*}\,\mathrm{d}t\Big| = \Big|\int_{\mathbb{T}} \langle f, g\rangle_{X,X^*}\,\mathrm{d}t\Big| &\le \|f\|_{L^p(\mathbb{T};X)}\|g\|_{L^{p'}(\mathbb{T};X^*)}\\ &\le U\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^q_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q}}\|g\|_{L^{p'}(\mathbb{T};X^*)}\\ &\le 2UR_{X,p}\Big(\sum_{I\in\mathscr{I}} \|f_I\|^q_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q}}\|g\|_{L^{p'}(\mathbb{T};X^*)}.\end{aligned}$$

where, in the last step, we applied the boundedness of the Riesz projection. Taking the supremum over all $(f_I)_{I\in\mathscr{I}} \in \ell^q(L^p(\mathbb{T};X))$ with $\sum_{I\in\mathscr{I}} \|f_I\|^q_{L^p(\mathbb{T};X)} \le 1$, it follows from Lemma 2.4.3 that $X^*$ has lower $\ell^{q'}(L^{p'})$-decompositions. □

In the following proposition, we show that one can trade $\ell^q$-summability for polynomial growth in the number of intervals in the decomposition properties, which seems like a natural way to prove upper decompositions. A similar result holds for the lower decompositions case.

**Proposition 5.2.5** ($\ell^q$-summability versus growth $\alpha$)**.** *Let $X$ be a Banach space and let $p, q \in [1,\infty]$.*

*(1) If $X$ has upper $\ell^q(L^p)$-decompositions, then there exists a constant $U > 0$ such that for $r \in [q,\infty]$, all finite families of disjoint intervals $\mathscr{I}$ and $f \in \mathscr{P}(\mathbb{T}; X)$ with support in $\cup\{I \in \mathscr{I}\}$,*

$$\|f\|_{L^p(\mathbb{T};X)} \le U(\#\mathscr{I})^{\frac{1}{q}-\frac{1}{r}}\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^r_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{r}},$$

*where $\#\mathscr{I}$ is the number of intervals in $\mathscr{I}$.*

*(2) Conversely, if there exists an $r \in (q,\infty]$ and a constant $U > 0$ such that for all finite families of disjoint intervals $\mathscr{I}$ and $f \in \mathscr{P}(\mathbb{T}; X)$ with support in $\cup\{I \in \mathscr{I}\}$,*

$$\|f\|_{L^p(\mathbb{T};X)} \le U(\#\mathscr{I})^{\frac{1}{q}-\frac{1}{r}}\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^r_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{r}},$$

*then $X$ has upper $\ell^s(L^p)$-decompositions for $1 \le s < q$.*

*Proof.* Note that (1) follows directly from Hölder's inequality:

$$\begin{aligned}\|f\|_{L^p(\mathbb{T};X)} &\le U\Big(\sum_{I\in\mathscr{I}} 1\cdot\|S_I f\|^q_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q}} \le U\left(\sum_{I\in\mathscr{I}} 1^{(r/q)'}\right)^{\frac{1}{q}-\frac{1}{r}}\left(\sum_{I\in\mathscr{I}} \|S_I f\|^r_{L^p(\mathbb{T};X)}\right)^{\frac{1}{r}}\\ &\le U(\#\mathscr{I})^{\frac{1}{q}-\frac{1}{r}}\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^r_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{r}}.\end{aligned}$$

For (2), let $\mathscr{I} = \{I_k : k \geq 1\}$ be an interval partition of $\mathbb{Z}$. For a trigonometric polynomial $f : \mathbb{T} \to X$, set $f_k := S_{I_k} f$. Without loss of generality, we may assume that

$$\|f_k\|_{L^p(\mathbb{T};X)} \geq \|f_{k+1}\|_{L^p(\mathbb{T};X)}, \quad k \geq 1.$$

Otherwise, one may relabel the indices of the intervals in the partition. By the triangle inequality and the assumption, we get

$$\begin{aligned}
\|f\|_{L^p(\mathbb{T};X)} = \Big\| \sum_{k=1}^{\infty} f_k \Big\|_{L^p(\mathbb{T};X)} &\leq \sum_{j=1}^{\infty} \Big\| \sum_{k=2^{j-1}}^{2^j-1} f_k \Big\|_{L^p(\mathbb{T};X)} \\
&\leq U \sum_{j=1}^{\infty} 2^{(j-1)(\frac{1}{q}-\frac{1}{r})} \Big( \sum_{k=2^{j-1}}^{2^j-1} \|f_k\|^r_{L^p(\mathbb{T};X)} \Big)^{\frac{1}{r}} \\
&\leq U \sum_{j=1}^{\infty} 2^{\frac{j-1}{q}} \|f_{2^{j-1}}\|_{L^p(\mathbb{T};X)} \\
&\leq U \sum_{j=1}^{\infty} 2^{\frac{j-1}{q}} \left( \frac{\|f_1\|^s_{L^p(\mathbb{T};X)} + \cdots + \|f_{2^{j-1}}\|^s_{L^p(\mathbb{T};X)}}{2^{j-1}} \right)^{\frac{1}{s}} \\
&\leq U \sum_{j=1}^{\infty} 2^{(j-1)(\frac{1}{q}-\frac{1}{s})} \cdot \Big( \sum_{k=1}^{\infty} \|f_k\|^s_{L^p(\mathbb{T};X)} \Big)^{\frac{1}{s}}.
\end{aligned}$$

Since $\frac{1}{q} - \frac{1}{s} < 0$, assertion (2) follows. ☐

In the next proposition, we discuss a complex interpolation result for the decomposition properties.

**Proposition 5.2.6** (Interpolation)**.** *Let $(X_0, X_1)$ be an interpolation couple of* UMD *spaces. Let $p_0, p_1 \in (1,\infty)$ and $q_0, q_1 \in [1,\infty]$. Let $\theta \in (0,1)$, set $X_\theta = [X_0, X_1]_\theta$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

*If $X_i$ has upper (lower) $\ell^{q_i}(L^{p_i})$-decompositions for $i = 0,1$, then $X_\theta$ has upper (lower) $\ell^q(L^p)$-decompositions.*

*Proof.* We start with the proof of the lower case. Since $X_\theta$ is a UMD space, it has lower $\ell^\infty(L^p)$-decompositions. Thus we may assume without loss of generality that $q < \infty$ and thus $\min\{q_0, q_1\} < \infty$. Let $\mathscr{I}$ be an interval partition of $\mathbb{Z}$. Let

$$T : L^{p_i}(\mathbb{T}; X_i) \to \ell^{q_i}(\mathscr{I}; L^{p_i}(\mathbb{T}; X_i))$$

be given by $Tf = (S_I f)_{I \in \mathscr{I}}$ for $i = 0,1$. From the assumption we see that $T$ is bounded of norm $L_i$ for $i = 0,1$. Therefore, by Proposition 2.7.2, we obtain that $T : L^p(\mathbb{T}; X_\theta) \to \ell^q(L^p(\mathbb{T}; X_\theta))$ is bounded and

$$\|T\|_{\mathscr{L}(L^p(\mathbb{T};X_\theta),\ell^q(L^p(\mathbb{T};X_\theta)))} \leq L_0^{1-\theta} L_1^\theta.$$

This gives the required result.

For the upper case, in Proposition 5.2.10 below, we will show that $q_0, q_1 < \infty$. Define an operator $T\colon \ell^{q_i}(\mathscr{I}; L^{p_i}(\mathbb{T}; X_i)) \to L^{p_i}(\mathbb{T}; X_i)$ by

$$T((f_I)_{I\in\mathscr{I}}) := \sum_{I\in\mathscr{I}} S_I f_I$$

for $i = 0, 1$. Note that

$$\begin{aligned}\|T((f_I)_{I\in\mathscr{I}})\|_{L^{p_i}(\mathbb{T};X_i)} &= \Big\|\sum_{I\in\mathscr{I}} S_I f_I\Big\|_{L^{p_i}(\mathbb{T};X_i)} \\ &\le U_i\Big(\sum_{J\in\mathscr{I}}\Big\|S_J\sum_{I\in\mathscr{I}} S_I f_I\Big\|^{q_i}_{L^{p_i}(\mathbb{T};X_i)}\Big)^{\frac{1}{q_i}} \\ &\le U_i\Big(\sum_{I\in\mathscr{I}}\Big\|S_I f_I\Big\|^{q_i}_{L^{p_i}(\mathbb{T};X_i)}\Big)^{\frac{1}{q_i}} \\ &\le 2U_i R_{X_i,p_i}\|(f_I)_{I\in\mathscr{I}}\|_{\ell^{q_i}(\mathscr{I};L^{p_i}(\mathbb{T};X_i))},\end{aligned}$$

where, in the last step, we applied (5.2.1) again. Therefore, by Proposition 2.7.2, we obtain that $T\colon \ell^q(\mathscr{I}; L^p(\mathbb{T}; X_\theta)) \to L^p(\mathbb{T}; X_\theta)$ is bounded. Applying this to $(f_I)_{I\in\mathscr{I}} = (S_I f)_{I\in\mathscr{I}}$ for $f \in \mathscr{P}(\mathbb{T}; X_\theta)$, then $T((f_I)_{I\in\mathscr{I}}) = \sum_{I\in\mathscr{I}} S_I S_I f = \sum_{I\in\mathscr{I}} S_I f$ and

$$\|f\|_{L^p(\mathbb{T};X_\theta)} = \Big\|\sum_{I\in\mathscr{I}} S_I f\Big\|_{L^p(\mathbb{T};X_\theta)} \le 2(U_0 R_{X_0,p_0})^{1-\theta}(U_1 R_{X_1,p_1})^{\theta}\|(S_I f)_{I\in\mathscr{I}}\|_{\ell^q(\mathscr{I};L^p(\mathbb{T};X_\theta))},$$

completing the proof. □

With a similar method we obtain the following "extrapolation result".

**Proposition 5.2.7** (Extrapolation)**.** *Let $X$ be a* UMD *space, $p \in (1,\infty)$ and $q \in (1,\infty)$.*

*(1) If $X$ has upper $\ell^q(L^p)$-decompositions, then $X$ has upper $\ell^s(L^r)$-decompositions for every $s \in [1, q)$ and $r \in (1, ps'/q') \cup ((p's'/q')', \infty)$.*

*(2) If $X$ has lower $\ell^q(L^p)$-decompositions, then $X$ has lower $\ell^s(L^r)$-decompositions for every $s \in (q, \infty]$ and $r \in (1, ps/q) \cup ((p's/q)', \infty)$.*

*Proof.* We first prove (2). For every $s \in (q, \infty]$, let $\theta = \frac{q}{s}$, if $r < \frac{ps}{q}$, then there exists $t \in (1,\infty)$ such that

$$\frac{1}{r} = \frac{1-\theta}{t} + \frac{\theta}{p}. \tag{5.2.2}$$

If $r > (p's/q)'$, then there exists $t \in (1,\infty)$ such that $\frac{1}{r'} = \frac{1-\theta}{t'} + \frac{\theta}{p'}$, which implies (5.2.2) as well. Moreover, $X$ has lower $\ell^\infty(L^t)$-estimates for all $t \in (1,\infty)$ by the boundedness of the Riesz projection. It therefore follows from Proposition 5.2.6 that $X$ has lower $\ell^s(L^r)$-estimates by (5.2.2) and $\frac{1}{s} = \frac{1-\theta}{\infty} + \frac{\theta}{q}$.

By the duality result in Proposition 5.2.4 and ((2)), we have if $X$ has upper $\ell^q(L^p)$-decompositions, then $X$ has lower $\ell^{q'}(L^{p'})$-decompositions, and then $X$ has lower $\ell^{s'}(L^{r'})$-decompositions for every $s' \in (q', \infty]$ and $r' \in (1, p's'/q') \cup (ps'/q', \infty)$. Using Proposition 5.2.4 again conclude the result. □

The decomposition properties also behave well in the following sense, where we note that extrapolation to other exponents can be deduced from Proposition 5.2.5 and Corollary 5.2.7.

**Proposition 5.2.8.** *Let $(S, \mathscr{A}, \mu)$ be a $\sigma$-finite measure space. Let $X$ be a Banach space and let $p, q \in (1, \infty)$.*

*(1) If $X$ has upper $\ell^q(L^p)$-decompositions, then $L^p(S;X)$ has upper $\ell^{p\wedge q}(L^p)$-decompositions.*

*(2) If $X$ has lower $\ell^q(L^p)$-decompositions, then $L^p(S;X)$ has lower $\ell^{p\vee q}(L^p)$-decompositions.*

*Proof.* (1): By Fubini's theorem, the assumption, the contractive embedding $\ell^{p\wedge q} \subseteq \ell^q$, and Minkowski's inequality (see Lemma 2.2.5), we obtain that for $f \in \mathscr{P}(\mathbb{T}; L^p(S;X))$,

$$\begin{aligned}
\|f\|_{L^p(\mathbb{T};L^p(S;X))} &= \|f\|_{L^p(S;L^p(\mathbb{T};X))} \\
&\le U\|(S_I f)_{I\in\mathscr{I}}\|_{L^p(S;\ell^q(\mathscr{I};L^p(\mathbb{T};X)))} \\
&\le U\|(S_I f)_{I\in\mathscr{I}}\|_{L^p(S;\ell^{p\wedge q}(\mathscr{I};L^p(\mathbb{T};X)))} \\
&\le U\|(S_I f)_{I\in\mathscr{I}}\|_{\ell^{p\wedge q}(\mathscr{I};L^p(S;L^p(\mathbb{T};X)))} \\
&= U\|(S_I f)_{I\in\mathscr{I}}\|_{\ell^{p\wedge q}(\mathscr{I};L^p(\mathbb{T};L^p(S;X)))}.
\end{aligned}$$

(2): This can be proved in the same way:

$$\begin{aligned}
\|(S_I f)_{I\in\mathscr{I}}\|_{\ell^{p\vee q}(\mathscr{I};L^p(\mathbb{T};L^p(S;X)))} &= \|(S_I f)_{I\in\mathscr{I}}\|_{\ell^{p\vee q}(\mathscr{I};L^p(S;L^p(\mathbb{T};X)))} \\
&\le \|(S_I f)_{I\in\mathscr{I}}\|_{L^p(S;\ell^{p\vee q}(\mathscr{I};L^p(\mathbb{T};X)))} \\
&\le \|(S_I f)_{I\in\mathscr{I}}\|_{L^p(S;\ell^q(\mathscr{I};L^p(\mathbb{T};X)))} \\
&\le L\|f\|_{L^p(S;L^p(\mathbb{T};X))} \\
&= \|f\|_{L^p(\mathbb{T};L^p(S;X))},
\end{aligned}$$

where we use the contractive embedding $\ell^q \subseteq \ell^{p\vee q}$. □

A Banach space $X$ is *super-reflexive* if and only if for any $C < \infty$, there exists a constant $M$ and two numbers $p > 1$ and $q < \infty$ such that

$$\frac{1}{M}\Big(\sum_{n\in\mathbb{Z}} \|x_n\|^q\Big)^{1/q} \le \Big\|\sum_{n\in\mathbb{Z}} x_n\Big\| \le M\Big(\sum_{n\in\mathbb{Z}} \|x_n\|^p\Big)^{1/p},$$

for any basic sequence $(x_n)_{n\in\mathbb{Z}}$ in $X$ of constant $\le C$ (see [74]). The following result is much deeper and follows from [21] and [52]. It will play a role in some of the results below.

**Theorem 5.2.9.** *Let $X$ be a Banach space and $p \in (1, \infty)$.*

*(1) $X$ is super-reflexive if and only if there exists a $q \in (1, \infty)$ such that $X$ has upper $\ell^q(L^p)$-decompositions.*

*(2) $X$ is a UMD space if and only if there exists a $q \in (1, \infty)$ such that $X$ has lower $\ell^q(L^p)$-decompositions.*

*Proof.* Note that an $X$-valued block sequence on the characters of $\mathbb{T}$ is $(S_{I_k} f)_{k\geq 1}$ for $f \in L^p(\mathbb{T};X)$ and $(I_k)_{k\in\mathbb{N}}$ an interval partition of $\mathbb{Z}$. Then (1) is immediate from [21, Theorem 10] and [52, Theorem 9.25]. For (2), note that if $X$ is a UMD space, then $X^*$ is a UMD space as well, and thus super-reflexive by [71, Corollary 4.3.8]. By (1) for each $p' \in (1,\infty)$, there exists a $q' \in (1,\infty)$ such that $X^*$ has upper $\ell^{q'}(L^{p'})$-decompositions. Applying Proposition 5.2.4, $X$ has lower $\ell^q(L^p)$-decompositions. The converse implication has already been observed below Definition 5.2.3. □

### 5.2.4. NECESSITY OF TYPE AND COTYPE PROPERTIES

We have already seen that super-reflexivity and UMD are necessary for upper and lower decompositions, respectively. Our next aim is to show that the decomposition properties also imply (Fourier) type and cotype. Recall that the space $X$ has *Fourier type* $p \in [1,2]$ if there exists a constant $\varphi_{X,p} > 0$ such that for all finitely nonzero $(x_n)_{n\in\mathbb{Z}}$ in $X$, we have

$$\Big\| \sum_{n\in\mathbb{Z}} e_n x_n \Big\|_{L^{p'}(\mathbb{T};X)} \leq \varphi_{X,p} \, \|(x_n)_{n\geq 1}\|_{\ell^p(\mathbb{Z};X)}.$$

To deduce type and cotype properties, we will present the details in the case of upper decompositions. The lower case will be derived by duality.

**Proposition 5.2.10** (Upper decompositions implies type and cotype)**.** *Let $X$ be a Banach space and $p,q \in [1,\infty]$. If $X$ has upper $\ell^q(L^p)$-decompositions, then $q \in [1, p' \wedge 2]$ and*

*(1) $X$ has type $q$;*

*(2) $X$ has Fourier type $r'$ and cotype $r$ for any $r \in \left(\frac{2q'}{p\wedge 2}, \infty\right)$.*

*Proof.* By the assumption applied to the trigonometric polynomial $f = \sum_{k=1}^n e_k x_k$, and $I_k = \{k\}$ for $k \in \mathbb{Z}$, we obtain

$$\Big\| \sum_{n\in\mathbb{Z}} e_n x_n \Big\|_{L^p(\mathbb{T};X)} \leq U \, \|(x_n)_{n\geq 1}\|_{\ell^q(\mathbb{Z};X)}. \tag{5.2.3}$$

This implies $q \in [1,p']$. Indeed, if $q > p'$, (5.2.3) yields

$$\Big\| \sum_{n\in\mathbb{Z}} e_n x_n \Big\|_{L^p(\mathbb{T};X)} \leq \|(x_n)_{n\geq 1}\|_{\ell^{p'}(\mathbb{Z};X)}.$$

This leads to an improvement of the classical Hausdorff–Young inequalities, which is known to be false for $\mathbb{C}$ (see Example 2.3.6 and Lemma 2.3.3) and thus for one-dimensional subspaces of $X$. We also refer the readers to another proof in [42, below (4.6) with $a \in (0,1)$].

(1): For fixed $t \in \mathbb{T}$, $(\varepsilon_n)_{n=1}^N$ and $(\varepsilon_n e_n(t))_{n=1}^N$ are identically distributed Rademacher sequences. For any $N \in \mathbb{N}$ and $(x_n)_{n=1}^N$, Fubini's theorem and (5.2.3) yield

$$\Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^p(\Omega;X)}^p = \int_{\mathbb{T}} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^p(\Omega;X)}^p \, \mathrm{d}t$$

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$$= \int_{\mathbb{T}} \Big\| \sum_{n=1}^{N} \varepsilon_n e_n(t) x_n \Big\|_{L^p(\Omega;X)}^p \, \mathrm{d}t$$
$$= \int_{\Omega} \Big\| \sum_{n=1}^{N} \varepsilon_n e_n(t) x_n \Big\|_{L^p(\mathbb{T};X)}^p \, \mathrm{d}\mathbb{P}$$
$$\le \int_{\Omega} \| (\varepsilon_n x_n)_{n=1}^N \|_{\ell^q(\mathbb{Z};X)}^p \, \mathrm{d}\mathbb{P} \le \Big( \sum_{n=1}^{N} \|x_n\|^q \Big)^{\frac{p}{q}}.$$

By Kahane–Khintchine inequality, $X$ has type $q$, and thus in particular $q \le 2$.

(2): If $p \le 2$, interpolating (5.2.3) with the trivial bound

$$\Big\| \sum_{k=1}^{n} e_k x_k \Big\|_{L^\infty(\mathbb{T};X)} \le \sum_{k=1}^{n} \|x_k\|,$$

and setting $\theta = \frac{p}{2}$ and $\frac{1}{s} = \frac{1-\theta}{1} + \frac{\theta}{q} = 1 - \frac{p}{2q'}$, we deduce from the Riesz–Thorin Theorem that

$$\Big\| \sum_{k=1}^{n} e_k x_k \Big\|_{L^2(\mathbb{T};X)} \le U^{p/2} \Big( \sum_{k=1}^{n} \|x_k\|^s \Big)^{1/s}.$$

Note that $s \in [1,2]$ as a consequence of $q \le p'$. Therefore, Hölder's inequality implies that

$$\Big\| \sum_{k=1}^{n} e_k x_k \Big\|_{L^2(\mathbb{T};X)} \le U^{p/2} n^{\frac{1}{s}-\frac{1}{2}} \Big( \sum_{k=1}^{n} \|x_k\|^2 \Big)^{1/2}. \tag{5.2.4}$$

On the other hand, if $p > 2$, (5.2.3) and Hölder's inequality yield

$$\Big\| \sum_{n\in\mathbb{Z}} e_n x_n \Big\|_{L^2(\mathbb{T};X)} \le U \Big( \sum_{k=1}^{n} \|x_k\|^q \Big)^{1/q} \le U n^{\frac{1}{q}-\frac{1}{2}} \Big( \sum_{k=1}^{n} \|x_k\|^2 \Big)^{1/2}. \tag{5.2.5}$$

By [73, Lemma 13.1.32], (5.2.4) and (5.2.5) imply Fourier type $r'$ if $\frac{1}{r'} > \frac{1}{s} = 1 - \frac{p\wedge 2}{2q'}$, which is the required result. Since Fourier type $r'$ implies cotype $r$ by [72, Proposition 7.3.6], this completes the proof. □

**Proposition 5.2.11** (Lower decompositions implies type and cotype)**.** *Let $X$ be a Banach space, $p \in (1,\infty)$ and $q \in [1,\infty]$. If $X$ has lower $\ell^q(L^p)$-decompositions, then $q \in [p' \vee 2, \infty]$ and*

*(1) $X$ has cotype $q$;*

*(2) $X$ has Fourier type $r'$ and type $r'$ for any $r \in \left(\frac{2q}{p'\wedge 2}, \infty\right)$.*

*Proof.* Note that the assumption implies that $X$ is a UMD Banach space. By Proposition 5.2.4, we know that $X^*$ has upper $\ell^{q'}(L^{p'})$-decompositions. Thus Proposition 5.2.10 gives that $q' \in [1, p \wedge 2]$, and $X^*$ has type $q'$. Therefore, $X$ has cotype $q$ and this proves (1). Similarly, $X^*$ has Fourier type $r'$ for any $r \in \left(\frac{2q}{p'\wedge 2}, \infty\right)$. This implies that $X$ has Fourier type $r'$, and thus also type $r'$ by [72, Proposition 7.3.6]. □

### 5.2.5. EXAMPLES

We have already seen that every UMD space admits nontrivial upper and lower $\ell^q(L^p)$-decompositions. In this section, we give some concrete spaces and indicate what the admissible $p$ and $q$ are on these spaces.

*Example* 5.2.12. Let $H$ be a Hilbert space. Then

- $H$ has upper $\ell^p(L^p)$-decompositions for $p \in (1,2]$.
- $H$ has upper $\ell^q(L^p)$-decompositions for $p \in [2,\infty)$ and $q \in [1, p')$.
- $H$ has lower $\ell^p(L^p)$-decompositions for $p \in [2,\infty)$.
- $H$ has lower $\ell^q(L^p)$-decompositions for $p \in (1,2]$ and $q \in (p',\infty]$.

Indeed, it suffices to prove the the last two statements because the first two claims follow by the duality (Proposition 5.2.4). Moreover, for the last two statements it suffices to consider $H = \mathbb{C}$ by Proposition 2.2.6. By Rubio de Francia's Littlewood–Paley inequality for arbitrary intervals ([132, Theorem 1.2]) and [71, Proposition 5.7.1], there is a $C > 0$ such that for $p \in [2,\infty)$, each interval partition $\mathscr{I}$ and all $f \in \mathscr{P}(\mathbb{T})$, we have

$$\Big(\sum_{I\in\mathscr{I}} \|S_I f\|_{L^p(\mathbb{T})}^p\Big)^{\frac{1}{p}} \leq \Big\|\Big(\sum_{I\in\mathscr{I}} |S_I f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})}.$$

It is trivial that $\mathbb{C}$ has lower $\ell^q(L^p)$-decompositions for $p = 2$ and $q \in (2,\infty]$. For $p \in (1,2)$, $q \in (p',\infty)$, by Minkowski's inequality (Lemma 2.2.5) with $q > p' > p$, and Rubio de Francia's inequality ([132, Section 7]) and transference, there is a $C > 0$ such that

$$\Big(\sum_{I\in\mathscr{I}} \|S_I f\|_{L^p(\mathbb{T})}^q\Big)^{\frac{1}{q}} \leq \Big\|\Big(\sum_{I\in\mathscr{I}} |S_I f|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})}.$$

The endpoint $q = p'$ is missing when $p \neq 2$. We leave it as an open problem whether the endpoint holds, see Problem 5.5.3.

*Example* 5.2.13. Let $(S,\mathscr{A},\mu)$ be a $\sigma$-finite measure space and $p \in (1,\infty)$. From Example 5.2.12 and Proposition 5.2.8 we immediately obtain

- $L^p(S)$ has upper $\ell^p(L^p)$-decompositions for $p \in (1,2]$ and upper $\ell^q(L^p)$-decompositions for $p \in [2,\infty)$ and $q \in [1, p')$.
- $L^p(S)$ has lower $\ell^p(L^p)$-decompositions for $p \in [2,\infty)$ and lower $\ell^q(L^p)$-decompositions for $p \in (1,2]$ and $q \in (p',\infty]$.

The claims about $\ell^p(L^p)$-decompositions are optimal, which follows from the optimality of Corollary 5.3.5 below. Whether the endpoints $q = p'$ hold is even unclear in the case $S$ is a singleton thus $L^p(S) = \mathbb{C}$, see Problem 5.5.3. The spaces $X = L^1(S)$ and $X = L^\infty(S)$ are not reflexive in general and thus not UMD. By Theorem 5.2.9, they do not have nontrivial upper and lower estimates.

An efficient method to create many examples is through interpolation. It is actually an open problem if all UMD spaces can be written as an interpolation space as below. For UMD lattices this is indeed the case (see [133]).

*Example* 5.2.14. Let $X := [Y, H]_\theta$, where $Y$ is a UMD Banach space and $H$ is a Hilbert space such that $(Y, H)$ is an interpolation couple, and $\theta \in (0,1)$. Let $p \in ((1-\theta/2)^{-1}, 2/\theta)$. Then there exists a $\theta_0 > \theta$ depending on $\theta, p$ and $Y$ such that

- $X$ has lower $\ell^{\frac{2}{\theta_0}}(L^p)$-decompositions.
- $X$ has upper $\ell^{\frac{2}{2-\theta_0}}(L^p)$-decompositions.

As a trivial consequence, the same holds with $\theta_0 = \theta$. To derive the above, we first explain the lower case. By the assumption on $p$ we can find $p_0 \in (1,\infty)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}. \tag{5.2.6}$$

By Theorem 5.2.9 there exists an $s \in (1,\infty)$ such that $Y$ has lower $\ell^s(L^{p_0})$-decompositions. Note that $s \geq p_0' \vee 2$ by Proposition 5.2.11. Since $H$ has lower $\ell^2(L^2)$-decompositions, Proposition 5.2.6 gives that $X$ has lower $\ell^r(L^p)$-decompositions where $r \in [2,s]$ satisfies $\frac{1}{r} = \frac{1-\theta}{s} + \frac{\theta}{2} > \frac{\theta}{2}$. Then there exists a $\theta_0 > \theta$ such that $r = \frac{2}{\theta_0}$. This gives the result in the lower case.

The upper case can be proved similarly. We can still find a $p_0 \in (1,\infty)$ such that (5.2.6) holds. By Theorem 5.2.9 and Proposition 5.2.10, there exists an $s \in [1, p_0' \wedge 2]$ such that $Y$ has upper $\ell^s(L^{p_0})$-decompositions. Since $H$ has upper $\ell^2(L^2)$-decompositions, we get $X$ has upper $\ell^r(L^p)$-decompositions where $r \in [s,2]$ satisfies $\frac{1}{r} = \frac{1-\theta}{s} + \frac{\theta}{2} < 1 - \frac{\theta}{2}$. Then there exists a $\theta_0 > \theta$ such that $r = \frac{2}{2-\theta_0}$. This completes the proof.

## 5.3. MAIN RESULTS ON UMD BANACH SPACES

### 5.3.1. STATEMENT OF THE RESULTS

In this section, we prove Theorem 6.1.2 and discuss several consequences. We use a slightly more general formulation below, as this is required to obtain sharp estimates in Corollary 5.3.5. The main extra ingredient is to allow growth in the upper and lower decompositions.

**Theorem 5.3.1.** *Let $X$ be a Banach space, let $p, q_0, q_1 \in (1,\infty)$ and let $\gamma_0 \in [0, 1/q_0')$, $\gamma_1 \in [0, 1/q_1)$. Suppose that the following conditions hold:*

*(1) There exists a constant $U > 0$ such that for all finite families of disjoint intervals $\mathscr{I}$ and all $f \in \mathscr{P}(\mathbb{T}; X)$ with support in $\cup\{I \in \mathscr{I}\}$,*

$$\|f\|_{L^p(\mathbb{T};X)} \leq U(\#\mathscr{I})^{\gamma_0}\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^{q_0}_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q_0}},$$

*where $\#\mathscr{I}$ is the number of intervals in $\mathscr{I}$.*

*(2) There exists a constant $L > 0$ such that for all finite families of disjoint intervals $\mathscr{I}$ and all $f \in \mathscr{P}(\mathbb{T}; X)$ with support in $\cup\{I \in \mathscr{I}\}$,*

$$\Big(\sum_{I\in\mathscr{I}} \|S_I f\|^{q_1}_{L^p(\mathbb{T};X)}\Big)^{\frac{1}{q_1}} \leq L(\#\mathscr{I})^{\gamma_1}\|f\|_{L^p(\mathbb{T};X)}.$$

*Suppose that $T \in \mathscr{L}(X)$ is strongly Kreiss bounded with constant $K_s$. Then there exist constants $C, \beta > 0$ depending on $X$ and $K_s$ such that*

$$\|T^n\| \le C n^{\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q_1}+\gamma_0+\gamma_1\right)}(\log(n+2))^{\beta}, \quad n \ge 1.$$

For $\gamma_0 = \gamma_1 = 0$, the conditions (1) and (2) in Theorem 5.3.1 are equivalent to the upper $\ell^{q_0}(L^p)$-decompositions and lower $\ell^{q_1}(L^p)$-decompositions of $X$, respectively. Moreover, this implies $q_0 \le 2, q_1 \ge 2$ by Proposition 5.2.10. In many cases it is sufficient to consider $\gamma_0 = \gamma_1 = 0$. Furthermore, note that by Proposition 5.2.5, the estimate in (1) implies that $X$ has upper $\ell^s(L^p)$-decompositions for all $s$ satisfying $\frac{1}{s} > \frac{1}{q_0} + \gamma_0 \ge \frac{1}{2}$. A similar implication holds from (2) to lower decompositions of $X$, and one has $\frac{1}{q_1'} + \gamma_1 \ge \frac{1}{2}$. Finally, note that it is not useful to consider $\gamma_0 \ge 1/q_0'$ or $\gamma_1 \ge 1/q_1$, because the obtained bound in the theorem

$$\frac{1}{2}\left(\frac{1}{q_0} - \frac{1}{q_1} + \gamma_0 + \gamma_1\right) \ge \frac{1}{2}\left(\frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_0'} + \frac{1}{q_1'} - 1\right) \ge \frac{1}{2},$$

which is worse than the known result, see (5.1.3).

Before we turn to the proof, we derive several immediate consequences. Using Theorem 5.2.9, we obtain:

**Corollary 5.3.2** (General UMD case)**.** *Let $X$ be a* UMD *Banach space. Suppose that $T \in \mathscr{L}(X)$ is strongly Kreiss bounded with constant $K_s$. Then there exists an $\alpha \in [0, \frac{1}{2})$ depending on $X$, and a constant $C$ depending on $X$ and $K_s$ such that*

$$\|T^n\| \le C n^{\alpha}, \quad n \ge 1.$$

For interpolation spaces we can also provide explicit growth rates.

**Corollary 5.3.3** (Intermediate UMD)**.** *Let $X := [Y, H]_\theta$, where $Y$ is a* UMD *Banach space and $H$ is a Hilbert space such that $(Y, H)$ is an interpolation couple, and $\theta \in (0,1)$. Suppose that $T \in \mathscr{L}(X)$ is strongly Kreiss bounded with constant $K_s$. Then there exists an $\alpha \in [0, (1-\theta)/2)$ depending on $X$, and a constant $C > 0$ depending on $X$ and $K_s$ such that*

$$\|T^n\| \le C n^{\alpha}(\log(n+2))^{\beta}, \quad n \ge 1.$$

In particular, one can also take $\alpha = (1-\theta)/2$ in the above.

*Proof.* By Example 5.2.14 we know that $X$ has lower $\ell^{\frac{2}{\theta_0}}(L^2)$-decompositions for some $\theta_0 > \theta$. Then by duality, $X$ has upper $\ell^{\frac{2}{2-\theta_0}}(L^2)$-decompositions. Thus it remains to observe that $\alpha := \frac{2-\theta_0}{4} - \frac{\theta_0}{4} = \frac{1-\theta_0}{2} < \frac{1-\theta}{2}$. □

Similarly, the results of [30, Theorem 4.5] follow from Example 5.2.12.

**Corollary 5.3.4** (Hilbert spaces)**.** *Let $X$ be a Hilbert space. Suppose that $T \in \mathscr{L}(X)$ is strongly Kreiss bounded with constant $K_s$. Then there exist constants $C, \beta > 0$ depending $K_s$ such that*

$$\|T^n\| \le C(\log(n+2))^{\beta}, \quad n \ge 1.$$

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We can also recover [7, Theorem 1.1], for which we will need the parameters $\gamma_0, \gamma_1$ in Theorem 5.3.1. Note that in [7, Proposition 1.2], the authors showed for every $1 \le p \le \infty$, there exists a strongly Kreiss bounded operator on $\ell^p(\mathbb{Z})$ and some constant $C_p \ge 1$ such that for every $n \in \mathbb{N}$,

$$n^{\left|\frac{1}{2}-\frac{1}{p}\right|}/C_p \le \|T^n\| \le C_p n^{\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

Hence, the exponent $\left|\frac{1}{2}-\frac{1}{p}\right|$ cannot be improved.

**Corollary 5.3.5** ($L^p$-spaces)**.** *Let $(S, \mathscr{A}, \mu)$ be a $\sigma$-finite measure space and let $X = L^p(S)$ with $p \in (1,\infty)$. Suppose that $T \in \mathscr{L}(X)$ is strongly Kreiss bounded with constant $K_s$. Then there exist constants $C, \beta > 0$ depending on $p$ and $K_s$ such that*

$$\|T^n\| \le C n^{\left|\frac{1}{2}-\frac{1}{p}\right|}(\log(n+2))^{\beta}, \quad n \ge 1.$$

*Proof.* Due to the missing endpoint, using Example 5.2.13 would yield the asymptotic $n^\alpha$ for $\alpha > \left|\frac{1}{2}-\frac{1}{p}\right|$. We therefore argue differently, using the growth parameters $\gamma_0, \gamma_1$ in Theorem 5.3.1.

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By duality, it suffices to consider $p \in (1,2]$. By Example 5.2.13 we know that assumption (1) in Theorem 5.3.1 holds with $q_0 = p$ and $\gamma_0 = 0$. Next we claim that assumption (2) in Theorem 5.3.1 is satisfied with $q_1 = 2$ and $\gamma_1 = \frac{1}{p} - \frac{1}{2}$. This readily follows from [7]. Here we include the details for convenience.

Since $\mathscr{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is an invertible isometry, then by Fubini's theorem, for finite families of disjoint intervals $\mathscr{I}$ and all $f \in \mathscr{P}(\mathbb{T})$ with support in $\cup\{I \in \mathscr{I}\}$,

$$\begin{aligned}\Big\|\Big(\sum_{I\in\mathscr{I}} |S_I f|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{T})} &= \Big(\sum_{I\in\mathscr{I}} \|S_I f\|_{L^2(\mathbb{T})}^2\Big)^{\frac{1}{2}} = \Bigg(\sum_{I\in\mathscr{I}} \Big(\sum_{n\in I} |\widehat{f}(n)|^2\Big)^{\frac{1}{2}\cdot 2}\Bigg)^{\frac{1}{2}} \\ &= \Big(\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2\Big)^{\frac{1}{2}} = \|f\|_{L^2(\mathbb{T})}.\end{aligned}$$

By the boundedness of the Riesz projection from $L^1(\ell^2)$ into $L^{1,\infty}(\ell^2)$ (see [76]), we get

$$\Big\|\Big(\sum_{I\in\mathscr{I}} |S_I f|^2\Big)^{\frac{1}{2}}\Big\|_{L^{1,\infty}(\mathbb{T})} \le C(\#\mathscr{I})^{\frac{1}{2}} \|f\|_{L^1(\mathbb{T})}.$$

Applying the Minkowski's inequality and Marcinkiewicz interpolation theorem (see [71, Theorem 2.2.3]) yields

$$\Big(\sum_{I\in\mathscr{I}} \|S_I f\|_{L^p(\mathbb{T})}^2\Big)^{\frac{1}{2}} \le \Big\|\Big(\sum_{I\in\mathscr{I}} |S_I f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{T})} \le c_p C^{\frac{2}{p}-1} (\#\mathscr{I})^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^p(\mathbb{T})}.$$

By Minkowski's inequality and Fubini's theorem ($L^p(S;\ell^2(L^p(\mathbb{T}))) \subseteq \ell^2(L^p(\mathbb{T};L^p(S)))$), we obtain

$$\Big(\sum_{I\in\mathscr{I}} \|S_I f\|_{L^p(\mathbb{T};L^p(S))}^2\Big)^{\frac{1}{2}} \le c_p C^{\frac{2}{p}-1} (\#\mathscr{I})^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^p(\mathbb{T};L^p(S))},$$

which implies the claim.

From the above and Theorem 5.3.1 we see that

$$\|T^n\| \le C n^\alpha (\log(n+2))^\beta, \quad n \ge 1,$$

with

$$\alpha = \frac{1}{2}\Big(\frac{1}{q_0} - \frac{1}{q_1} + \gamma_0 + \gamma_1\Big) = \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{2} + 0 + \frac{1}{p} - \frac{1}{2}\Big) = \frac{1}{p} - \frac{1}{2}.$$

□

A further application for Banach function spaces will be presented in Theorems 5.4.1 and 5.4.4.

### 5.3.2. PREPARATORY LEMMAS

Before we prove Theorem 5.3.1, we need several preparatory lemmas. We start by noting the key property that we will use of strongly Kreiss bounded operators, which follows from [60] and [105, Corollary 3.2].

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**Lemma 5.3.6.** *If $T$ is a strongly Kreiss bounded operator on a Banach space $X$ with constant $K_s$, then we have*

$$\Big\|\sum_{k=0}^{n} \lambda^k T^k\Big\| \le 20 K_s (n+1), \quad |\lambda| = 1, n \in \mathbb{N}. \tag{5.3.1}$$

*Proof.* It was shown in [60] that if $T$ is a strongly Kreiss bounded operator with constant $K_s$, then we have

$$\sup_{n\ge 0}\Big\|\sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}}\Big\| \le \frac{4K_s}{|\lambda|-1}, \quad |\lambda| > 1.$$

By [105, Corollary 3.2], this is equivalent to

$$\Big\|\sum_{k=0}^{n} \lambda^k T^k\Big\| \le 20 K_s (n+1), \quad |\lambda| = 1, n \ge 5.$$

If $n \le 4$, (5.3.1) holds because of (5.1.3). The proof is complete. □

Next, we present some technical estimates based on the standard Stirling formula. These are quantified and optimized versions of results from [7].

**Lemma 5.3.7.** *Let $n \ge 2$. Then for all integers $k \in [0, 2\sqrt{n}]$,*

$$\frac{e^n}{28\sqrt{n}} \le \frac{n^{n-k}}{(n-k)!} \le \frac{e^n}{\sqrt{\frac{8\pi}{5} n}}. \tag{5.3.2}$$

*Proof.* The case $n \in [2,99]$ can be checked by hand. In the following we assume $n \geq 100$. It is elementary to check that $\log(1-x) \leq -\frac{2x}{2-x}$ for $x \in [0,1)$. Therefore, setting $g(x) = \frac{x^2}{2n-x}$, we find

$$\begin{aligned}\log\left(e^x\left(1-\tfrac{x}{n}\right)^{n-x}\right) &= x + (n-x)\log\left(1-\tfrac{x}{n}\right) \\ &\leq x - (n-x)\frac{2x}{2n-x} = g(x).\end{aligned}$$

The function $g\colon [0,2\sqrt{n}] \to [0,\infty)$ is increasing. It follows that for all $x \in [0,2\sqrt{n}]$,

$$g(x) \leq g(2\sqrt{n}) = \frac{4n}{2n-2\sqrt{n}} = \frac{2}{1-\frac{1}{\sqrt{n}}} \leq \frac{20}{9},$$

where in the last step we used $n \geq 100$. Therefore, we can conclude

$$e^k\left(1-\tfrac{k}{n}\right)^{n-k} \leq e^{\frac{20}{9}}. \tag{5.3.3}$$

Next, we show that (5.3.2) holds via the standard Stirling formula

$$e^{\frac{1}{12n+1}}\sqrt{2\pi n}\left(\tfrac{n}{e}\right)^n < n! < e^{\frac{1}{12n}}\sqrt{2\pi n}\left(\tfrac{n}{e}\right)^n, \quad n \geq 1. \tag{5.3.4}$$

Let $n \geq 100$ and $k \in [0,2\sqrt{n}]$. It follows that $n-k \geq n-2\sqrt{n} \geq 80$. Thus, by the upper estimate of (5.3.4) and (5.3.3), we have

$$\begin{aligned}(n-k)! &\leq \sqrt{2\pi(n-k)}\left(\tfrac{n-k}{e}\right)^{n-k} e^{\frac{1}{12(n-k)}} \\ &\leq \sqrt{2\pi n}\left(\tfrac{n}{e}\right)^{n-k}\left(1-\tfrac{k}{n}\right)^{n-k} e^{\frac{1}{12\cdot 80}} \\ &\leq \sqrt{2\pi}\sqrt{n}\left(\tfrac{n}{e}\right)^{n-k} e^{-k+\frac{20}{9}} e^{\frac{1}{960}} \\ &= e^{\frac{20}{9}} e^{\frac{1}{960}}\sqrt{2\pi}\sqrt{n}n^{n-k}e^{-n} \\ &\leq 28\sqrt{n}n^{n-k}e^{-n}.\end{aligned}$$

The first estimate in (5.3.2) is proved.

On the other hand, since $f(x) := e^x\left(1-\frac{x}{n}\right)^{n-x}$ is increasing, we have $f(x) \geq f(0) = 1$ for $x \in [0,2\sqrt{n}]$. Note that $n-k \geq n-2\sqrt{n} \geq \frac{4}{5}n$ due to $n \geq 100$. Thus the lower estimate of (5.3.4) gives

$$\begin{aligned}(n-k)! &\geq \sqrt{2\pi(n-k)}\left(\tfrac{n-k}{e}\right)^{n-k} e^{\frac{1}{12(n-k)+1}} \\ &\geq \sqrt{\frac{8\pi}{5}}\sqrt{n}n^{n-k}e^{-n}e^k\left(1-\tfrac{k}{n}\right)^{n-k} \\ &\geq \sqrt{\frac{8\pi}{5}}\sqrt{n}n^{n-k}e^{-n},\end{aligned}$$

finishing the proof. □

The following lemma will be used in the calculation of a self-improvement result of strongly Kreiss bounded operators. Note that we use the notation $\sum_{a\le m\le b}$ for $a, b \in \mathbb{R}$ to denote the sum over all integers $m \in \mathbb{Z}$ such that $a \le m \le b$.

**Lemma 5.3.8.** *For $n \ge 2$ and $m \in [n-\sqrt{n}, n]$, define*

$$b_{n,m} := \sum_{m-\sqrt{n}\le k\le m-1} \frac{n^k}{k!},$$
$$a_{n,m} := e^n b_{n,m}^{-1},$$

*and $a_{n,m} = b_{n,m} = 0$ otherwise. Then we have $\|(a_{n,m})_{m\in\mathbb{Z}}\|_{\ell^\infty} \le 32$ and $[(a_{n,m})_{m\in\mathbb{Z}}]_{V^1} \le 978$.*

*Proof.* The case $n \in [2,99]$ can be checked by hand. In the following we assume $n \ge 100$. We start with the boundedness of $a_{n,m}$ for $m \in [n-\sqrt{n}, n]$. From (5.3.2) it is almost immediate that

$$|a_{n,m}| = \frac{e^n}{b_{n,m}} \le \frac{28e^n}{\sum_{m-\sqrt{n}\le k\le m-1} \frac{e^n}{\sqrt{n}}} \le 28\big(1+\frac{1}{\sqrt{n}-1}\big) \le 32.$$

Next, we show that $(a_{n,m})_{m\in\mathbb{Z}}$ has bounded variation. First we fix $m \in [n-\sqrt{n}, n-1]$ and let $L := \lceil m-\sqrt{n}\rceil$. By (5.3.2),

$$\begin{aligned} |a_{n,m+1}-a_{n,m}| &= e^n\big|b_{n,m+1}^{-1} - b_{n,m}^{-1}\big| = e^n \frac{\left|\frac{n^m}{m!} - \frac{n^L}{L!}\right|}{b_{n,m}b_{n,m+1}} \\ &\le e^{-n}\left(\frac{n^m}{m!} + \frac{n^L}{L!}\right)\cdot a_{n,m}a_{n,m+1} \\ &\le e^{-n}\cdot 2\frac{e^n}{\sqrt{\frac{8\pi}{5}n}}\cdot 32^2 \le \frac{914}{\sqrt{n}}. \end{aligned}$$

Therefore, we can conclude

$$[(a_{n,m})_{m\in\mathbb{Z}}]_{V^1} \le 2\sup_{m\ge 1}|a_{n,m}| + \sum_{n-\sqrt{n}\le m\le n-1} |a_{n,m+1}-a_{n,m}| \le 978,$$

finishing the proof. □

The following key lemma will provide a way to obtain a special self-improvement of bounds for strongly Kreiss bounded operators. The proof is a straightforward extension of [7], where $X = L^p$ was considered. In order to obtain not too large explicit constant, some adjustment and optimization seemed necessary. Moreover, it can be helpful to see where the geometry of the space $X$ enters.

**Lemma 5.3.9.** *Let $X$ be a* UMD *space and $p \in (1,\infty)$. Let $T \in \mathcal{L}(X)$ be strongly Kreiss bounded with constant $K_s$. Suppose that there exists an increasing function $h : \mathbb{R}_+ \to [1,\infty)$ such that for all $x \in X$ and $n \ge 2$,*

$$\Big\|\sum_{1\le m\le n} e_m T^m x\Big\|_{L^p(\mathbb{T};X)} \le h(n)\|x\|. \tag{5.3.5}$$

*Then there exists a constant $C_{X,p} > 0$ such that for all $j \geq 0$, $n \geq 1$ and $x \in X$,*

$$\Big\| \sum_{n-\sqrt{n}+j \leq m \leq n} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \leq C_{X,p} K_s h(\sqrt{n}) \|x\|. \tag{5.3.6}$$

*Proof.* Define $S_n := \sum_{1 \leq m \leq \sqrt{n}} e_m T^m$. Then

$$\begin{aligned} e^{e_1 nT} S_n &= \sum_{k \geq 0} \frac{e_k (nT)^k}{k!} \sum_{1 \leq m \leq \sqrt{n}} e_m T^m \\ &= \sum_{k \geq 0} \sum_{k+1 \leq m \leq k+\sqrt{n}} \frac{n^k}{k!} e_m T^m \\ &= \sum_{1 \leq m \leq \sqrt{n}} \widetilde{b}_{n,m} e_m T^m + \sum_{m \geq \lfloor \sqrt{n} \rfloor + 1} b_{n,m} e_m T^m, \end{aligned}$$

where $\widetilde{b}_{n,m} := \sum_{0 \leq k \leq m-1} \frac{n^k}{k!}$ and $b_{n,m} := \sum_{m-\sqrt{n} \leq k \leq m-1} \frac{n^k}{k!}$.

We first consider the case $n \geq 6$ and thus $\sqrt{n} \geq 2$. Fix $j \geq 0$ and let $I_n = [n-\sqrt{n}+j, n] \cap \mathbb{N}$. Note that $f(x) := x^2 - 2x - 1$ is increasing in $[\sqrt{6}, \infty)$, then

$$n - \sqrt{n} \geq \lfloor \sqrt{n} \rfloor + 1.$$

By the boundedness of the Riesz projection with constant $R_{X,p}$, (5.1.1) (which uses the strong Kreiss boundedness), and (5.3.5) we obtain

$$\begin{aligned} \Big\| \sum_{m \in I_n} b_{n,m} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} &\leq 2R_{X,p} \| e^{e_1 nT} S_n x \|_{L^p(\mathbb{T};X)} \\ &\leq 2K_s R_{X,p} e^n \|S_n x\|_{L^p(\mathbb{T};X)} \\ &\leq 2K_s R_{X,p} e^n h(\sqrt{n}) \|x\|. \end{aligned} \tag{5.3.7}$$

Let $a_{n,m} := e^n b_{n,m}^{-1}$ for $m \in I_n$ and zero otherwise. Then by Lemma 5.3.8, $\|(a_{n,m})_{m \in \mathbb{Z}}\|_{\ell^\infty} \leq 32$ and $[(a_{n,m})_{m \in \mathbb{Z}}]_{V^1} \leq 978$. Therefore, the Fourier multiplier Lemma 2.6.1 and (5.3.7) imply that

$$\begin{aligned} \Big\| \sum_{m \in I_n} e^n e_m T^m x \Big\|_{L^p(\mathbb{T};X)} &= \Big\| \sum_{m \in I_n} a_{n,m} b_{n,m} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \\ &\leq 1010 M_{X,p} \Big\| \sum_{m \in I_n} b_{n,m} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \\ &\leq 2020 M_{X,p} K_s R_{X,p} e^n h(\sqrt{n}) \|x\|. \end{aligned}$$

Dividing by $e^n$ gives (5.3.6) with

$$C_{X,p} := 2020 M_{X,p} R_{X,p}.$$

To prove the estimate for $n \leq 5$, note that by Lemma 5.3.6 we can write

$$\begin{aligned} \Big\| \sum_{m \in I_n} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} &\leq 2R_{X,p} \Big\| \sum_{m=0}^{n} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \\ &\leq 40(n+1) K_s R_{X,p} \|x\| \\ &\leq C_{X,p} K_s h(\sqrt{n}) \|x\|. \end{aligned}$$

□

Combining Lemma 5.3.9 with the upper $\ell^q(L^p)$-decompositions, we obtain the following self-improvement result.

**Proposition 5.3.10.** *Let* $1 < p, q < \infty$, $\gamma \in [0, 1/q')$, *and suppose that* $X$ *is a* UMD *which satisfies Theorem 5.3.1(1) with* $(q_0, \gamma_0)$ *replaced by* $(q, \gamma)$. *Let* $T \in \mathcal{L}(X)$ *be strongly Kreiss bounded with constant* $K_s$. *Suppose that there exist constants* $d \in [0,1]$ *and* $P \geq 1$ *such that for all* $x \in X$ *and* $n \geq 1$,

$$\Big\| \sum_{m=1}^{n} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \leq P n^d \|x\|. \tag{5.3.8}$$

*Then there is a constant* $\widetilde{C}_{X,p} > 0$ *such that for all* $x \in X$, *and* $n \geq 1$,

$$\Big\| \sum_{m=1}^{n} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \leq PU\widetilde{C}_{X,p} K_s n^{\frac{1}{2}(d+\frac{1}{q}+\gamma)} \|x\|.$$

*Proof.* Let $N \in \mathbb{N}$ be such that $(N-1)^2 < n \leq N^2$ and $\mathscr{I} := \{[k^2+1, k^2+k] \cup [k^2+k+1, (k+1)^2] \mid k = 0, \cdots, N-1\}$. Letting $n_1 = k^2+k$, then $k \leq \sqrt{n_1} \leq k+1$ and $n_1 \geq k^2+1 = n_1 - k + 1 \geq n_1 - \sqrt{n_1} + 1$. Hence by Lemma 5.3.9 with $h(n_1) = P n_1^d$, we obtain

$$\Big\| \sum_{m=k^2+1}^{k^2+k} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \leq K_s C_{X,p} P (k+1)^d.$$

By similar discussion, we also have

$$\Big\| \sum_{m=k^2+k+1}^{(k+1)^2} e_m T^m x \Big\|_{L^p(\mathbb{T};X)} \leq K_s C_{X,p} P (k+1)^d.$$

Then by the boundedness of the Riesz projection with constant $R_{X,p}$, the upper decompositions with constant $U$, we find

$$\begin{aligned}
\Big\| \sum_{m=1}^{n} e_m T^m x \Big\|_{L^p(\mathbb{T};X)}^q &\leq (2R_{X,p})^q \Big\| \sum_{m=1}^{N^2} e_m T^m x \Big\|_{L^p(\mathbb{T};X)}^q \\
&\leq U^q (2R_{X,p})^q (2N)^{q\gamma} \sum_{k=0}^{N-1} \Big( \Big\| \sum_{m=k^2+1}^{k^2+k} e_m T^m x \Big\|_{L^p(\mathbb{T};X)}^q \\
&\qquad + \Big\| \sum_{m=k^2+k+1}^{(k+1)^2} e_m T^m x \Big\|_{L^p(\mathbb{T};X)}^q \Big) \\
&\leq P^q U^q (2R_{X,p})^q C_{X,p}^q K_s^q (2N)^{q\gamma} \sum_{k=0}^{N-1} 2(k+1)^{dq} \|x\|^q \\
&\leq 2^{1+q\gamma+q} P^q R_{X,p}^q U^q C_{X,p}^q K_s^q N^{(d+\gamma)q+1} \|x\|^q,
\end{aligned}$$

where $C_{X,p}$ is the constant defined in the proof of Lemma 5.3.9. If $N = 1$, then $n = N = 1$; If $N \geq 2$, then by assumption $N \leq 2N - 2 < 2\sqrt{n}$. In both cases we get $N \leq 2\sqrt{n}$, this gives the result with constant (use $d + \frac{2}{q} + 2\gamma < d + 2 \leq 3$)

$$2^{\frac{1}{q}+\gamma+1} R_{X,p} C_{X,p} 2^{d+\frac{1}{q}+\gamma} \leq 16 R_{X,p} C_{X,p} := \widetilde{C}_{X,p}. \qquad \square$$

### 5.3.3. PROOF OF THEOREM 5.3.1

We can finally turn to the proof of the main result, which is an extension of the argument in [7].

*Proof of Theorem 5.3.1.* Since $T$ and $T^*$ are both strongly Kreiss bounded, it follows from Lemma 5.3.6 and (5.1.3) that for $S \in \{T, T^*\}$ and $n \geq 1$ we have

$$\Big\| \sum_{k=1}^{n} e_k S^k x \Big\|_{L^p(\mathbb{T};X)} \leq \min\{20K_s(n+1)+1, K_s n\sqrt{2\pi(n+1)}\}\|x\| \leq 21K_s n\|x\|,$$

using the first term in the minimum for $n > 64$ and the second term for $n \leq 64$. Therefore, (5.3.8) holds for $T$ and $T^*$ with $d = c_0 = d_0 := 1$ and $P := 21K_s$.

Using the assumption Theorem 5.3.1(2), by a similar duality argument as in Proposition 5.2.4, one can check that the estimate Theorem 5.3.1(1) holds with $(X, p, q_0, \gamma_0, U)$ replaced by $(X^*, p', q_1', \gamma_1, L)$. Define $c_N$ and $d_N$ for $N \in \mathbb{N}$ by

$$c_N = \frac{1}{2^N q_0'} - \frac{\gamma_0}{2^N}, \quad \text{and} \quad d_N = \frac{1}{2^N q_1} - \frac{\gamma_1}{2^N}.$$

Let $F_{X,p} := U\widetilde{C}_{X,p}K_s$ and $F_{X^*,p'} = L\widetilde{C}_{X^*,p'}K_s$, where $\widetilde{C}_{X,p}$ is the constant defined in the proof of Proposition 5.3.10. By Proposition 5.3.10 and an induction argument one sees that for every $N \geq 1$,

$$\Big\| \sum_{k=1}^{n} e_k T^k x \Big\|_{L^p(\mathbb{T};X)} \leq P(F_{X,p})^N n^{c_N + \frac{1}{q_0} + \gamma_0} \|x\|, \quad n \geq 1, x \in X,$$

$$\Big\| \sum_{k=1}^{n} e_k T^{*k} x^* \Big\|_{L^{p'}(\mathbb{T};X^*)} \leq P(F_{X^*,p'})^N n^{d_N + \frac{1}{q_1'} + \gamma_1} \|x^*\|, \quad n \geq 1, x^* \in X^*.$$

Let $n \geq 14$ and thus $n+2 \geq e^e$. We claim that there exist $N \in \mathbb{N}$ and $w_0, w_1 > 0$ such that

$$(F_{X,p})^N n^{c_N} \leq (\log(n+2))^{w_0}, \tag{5.3.9}$$

$$(F_{X^*,p'})^N n^{d_N} \leq (\log(n+2))^{w_1}. \tag{5.3.10}$$

Indeed, let $N \in \mathbb{N}$ be such that $2^N < \frac{\log(n+2)}{\log(\log(n+2))} \leq 2^{N+1}$. Then

$$\begin{aligned} n^{c_N} &\leq (n+2)^{c_N} = e^{c_N \log(n+2)} = e^{\log(\log(n+2)) \frac{\log(n+2)}{\log(\log(n+2))} c_N} \\ &= (\log(n+2))^{\frac{\log(n+2)}{\log(\log(n+2))} c_N} \leq (\log(n+2))^{2/q_0' - 2\gamma_0}, \\ n^{d_N} &\leq (\log(n+2))^{2/q_1 - 2\gamma_1}. \end{aligned}$$

Moreover, from $2^N \leq \frac{\log(n+2)}{\log(\log(n+2))}$ and $\log(\log(n+2)) \geq 1$, we obtain that $2^N \leq \log(n+2)$, then $N \leq \frac{\log(\log(n+2))}{\log 2}$. Therefore, since $(F_{X,p})^N \geq 1, (F_{X^*,p'})^N \geq 1$,

$$(F_{X,p})^N = e^{N \log F_{X,p}} \leq e^{\frac{\log(\log(n+2))}{\log 2} \log F_{X,p}} = (\log(n+2))^{\frac{\log F_{X,p}}{\log 2}}$$

$$(F_{X^*,p'})^N = (\log(n+2))^{\frac{\log F_{X^*,p'}}{\log 2}}.$$

This gives (5.3.9) with

$$w_0 = \frac{2}{q_0'} - 2\gamma_0 + \frac{\log F_{X,p}}{\log 2}, \quad w_1 = \frac{2}{q_1} - 2\gamma_1 + \frac{\log F_{X^*,p'}}{\log 2}.$$

From (5.3.9) and (5.3.10) we can conclude that for all $n \geq 14$, and $x \in X, x^* \in X^*$,

$$\Big\| \sum_{k=1}^{n} e_k T^k x \Big\|_{L^p(\mathbb{T};X)} \leq P(\log(n+2))^{w_0} n^{\frac{1}{q_0}+\gamma_0} \|x\|, \tag{5.3.11}$$

$$\Big\| \sum_{k=1}^{n} e_k T^{*k} x^* \Big\|_{L^{p'}(\mathbb{T};X^*)} \leq P(\log(n+2))^{w_1} n^{\frac{1}{q_1'}+\gamma_1} \|x^*\|. \tag{5.3.12}$$

Since $\frac{1}{q_0} + \gamma_0 \geq \frac{1}{2}$ and $w_0 \geq \frac{\log F_{X,p}}{\log 2} \geq \frac{\log(8080)}{\log 2}$, one can readily check that (5.3.11) extends to $n \leq 13$ with a larger $P$:

$$\Big\| \sum_{k=1}^{n} e_k T^k x \Big\|_{L^p(\mathbb{T};X)} \leq 13 K_s \sqrt{2\pi(n+1)} \|x\| \leq P(\log(n+2))^{w_0} n^{\frac{1}{q_0}+\gamma_0} \|x\|,$$

where we used the bound (5.1.3) once more. The same holds for (5.3.12).

If $n \geq 3$, note that $1 + \sqrt{n} \leq 2\sqrt{n}, \log(\sqrt{n}+3) \leq \log(n+2)$, applying (5.3.12) we find

$$\Big\| \sum_{1 \leq k \leq 1+\sqrt{n}} e_k T^{*k} x^* \Big\|_{L^{p'}(\mathbb{T};X^*)} \leq 2P(\log(n+2))^{w_1} n^{\frac{1}{2q_1'}+\frac{\gamma_1}{2}} \|x^*\|. \tag{5.3.13}$$

If $n \leq 2$, by (5.1.3) we get

$$\Big\| \sum_{1 \leq k \leq 1+\sqrt{n}} e_k T^{*k} x^* \Big\|_{L^{p'}(\mathbb{T};X^*)} \leq \sum_{1 \leq k \leq 2} K_s \sqrt{2\pi(k+1)} \leq P(\log(n+2))^{w_1} n^{\frac{1}{2q_1'}+\frac{\gamma_1}{2}} \|x^*\|.$$

Hence, (5.3.13) holds for $n \geq 1$.

By Lemma 5.3.9 and (5.3.11), with $h(n) := P(\log(n+2))^{w_0} n^{\frac{1}{q_0}+\gamma_0}$, we obtain

$$\Big\| \sum_{n-\sqrt{n} \leq k \leq n} e_k T^k x \Big\|_{L^p(\mathbb{T};X)} \leq P K_s C_{X,p} (\log(n+2))^{w_0} n^{\frac{1}{2q_0}+\frac{\gamma_0}{2}} \|x\|. \tag{5.3.14}$$

It follows that for all $n \geq 1$ and $x \in X, x^* \in X^*$,

$$\begin{aligned}
&(1 + \lfloor\sqrt{n}\rfloor) \big|\langle x^*, T^{n+1} x\rangle_{X^*,X}\big| \\
&= \Big| \sum_{1 \leq k \leq 1+\sqrt{n}} \langle T^{*k} x^*, T^{n+1-k} x\rangle_{X^*,X} \Big| \\
&= \Big| \Big\langle \sum_{1 \leq k \leq 1+\sqrt{n}} T^{*k} x^*, \sum_{1 \leq m \leq 1+\sqrt{n}} T^{n+1-m} x \Big\rangle_{X^*,X} \int_{\mathbb{T}} e_{k-m} \,\mathrm{d}t \Big| \\
&= \Big| \int_{\mathbb{T}} \Big\langle \sum_{1 \leq k \leq 1+\sqrt{n}} e_k T^{*k} x^*, \sum_{1 \leq m \leq 1+\sqrt{n}} \bar{e}_m T^{n+1-m} x \Big\rangle_{X^*,X} \mathrm{d}t \Big|
\end{aligned}$$

$$\leq \Big\| \sum_{1\leq k\leq 1+\sqrt{n}} e_k T^{*k} x^* \Big\|_{L^{p'}(\mathbb{T};X^*)} \Big\| \sum_{n-\sqrt{n}\leq k\leq n} e_k T^k x \Big\|_{L^p(\mathbb{T};X)}$$
$$\leq 2P^2 K_s C_{X,p} (\log(n+2))^{w_0+w_1} n^{\frac{1}{2}(\frac{1}{q_0}+\gamma_0+\frac{1}{q_1'}+\gamma_1)} \|x\| \|x^*\|,$$

where in the last step we used (5.3.13) and (5.3.14). Taking the supremum over $\|x\| \leq 1$ and $\|x^*\| \leq 1$, we obtain for all $n \geq 1$,

$$\big\| T^n \big\| \leq C(\log(n+2))^{\beta} n^{\frac{1}{2}(\frac{1}{q_0}+\gamma_0-\frac{1}{q_1}+\gamma_1)},$$

where $C := 2P^2 K_s C_{X,p}$ and $\beta = w_0 + w_1$. □

## 5.4. RESULTS IN BANACH FUNCTION SPACES

In this section we will discuss the growth of strongly Kreiss bounded operators in the particular case when $X$ is a Banach *function* space. Recall the terminologies convexity, concavity and $s$-concavification $X^s$ of a Banach function space $X$ in Section 2.6.2.

Our main result of Banach function spaces reads as follows.

**Theorem 5.4.1.** *Let $X$ be a Banach function space over $S$ and $s \in (1,2)$. Suppose $X$ is $s$-convex and $s'$-concave, and*

$$X_s := \big((X^s)'\big)^{\frac{1}{2-s}}$$

*is a* UMD *Banach function space. Suppose that $T \in \mathcal{L}(X)$ is strongly Kreiss bounded with constant $K_s$. Then there exist constants $C, \beta > 0$ depending on $X$ and $K_s$ such that*

$$\big\| T^n \big\| \leq C n^{\frac{1}{2}-\frac{1}{s'}} (\log(n+2))^{\beta}, \quad n \geq 1.$$

*Proof.* Let $f \in X^s$, then $|f|^{\frac{1}{s}} \in X$. Since $X$ is $s'$-concave, then for any finite sequences $(f_n)_{n=1}^N \subseteq X$,

$$\Big(\sum_{n=1}^N \|f_n\|_{X^s}^{\frac{s'}{s}}\Big)^{\frac{s}{s'}} = \Big(\sum_{n=1}^N \Big\||f_n|^{\frac{1}{s}}\Big\|_X^{s'}\Big)^{\frac{s}{s'}} \leq \Big\|\Big(\sum_{n=1}^N |f_n|^{\frac{s'}{s}}\Big)^{\frac{1}{s'}}\Big\|_X^s = \Big\|\Big(\sum_{n=1}^N |f_n|^{\frac{s'}{s}}\Big)^{\frac{s}{s'}}\Big\|_{X^s}.$$

Hence, $X^s$ is $\frac{s'}{s}$-concave, so $(X^s)'$ is $\left(\frac{s'}{s}\right)'$-convex, i.e. $\frac{1}{2-s}$-convex, then we can conclude that $X_s$ is a well-defined Banach function space because of the line below (2.6.1). By [113, Corollary 2.12] and [26] we have

$$X = ((X_s)')^{1-\frac{2}{s'}} \cdot L^{s'}(S) = \big[(X_s)', L^2(S)\big]_{\frac{2}{s'}}.$$

Since $\frac{1-2/s'}{2} = \frac{1}{2} - \frac{1}{s'}$, Corollary 5.3.3 yields the result. □

Let us illustrate Theorem 5.4.1 and the space $X_s$ with some examples. We start by calculating the space $X_s$ for $X = L^p(S)$.

*Example* 5.4.2. Let $(S,\mathscr{A},\mu)$ be a $\sigma$-finite measure space and let $X = L^p(S)$ with $p \in (1,\infty)$. Let $1 \le s < p \wedge p'$. Note that $L^p(S)$ is $s$-convex and $s'$-concave. Moreover, we have

$$X_s = \left(\left(L^{\frac{p}{s}}(S)\right)'\right)^{\frac{1}{2-s}} = \left(L^{\frac{p}{p-s}}(S)\right)^{\frac{1}{2-s}} = L^{\frac{(2-s)p}{p-s}}(S) =: L^q(S).$$

Since

$$q = \frac{(2-s)p}{p-s} < \infty, \quad \text{and} \quad q' = \frac{(2-s)p}{p+s-sp} = \frac{(2-s)p'}{p'-s} < \infty,$$

we observe that $q \in (1,\infty)$ and thus that $X_s$ is a UMD Banach function space. Therefore, Theorem 5.4.1 yields that for any strongly Kreiss bounded operator $T \in \mathscr{L}(X)$, there are $C, \beta > 0$ depending on $p$ and $K_s$ such that

$$\|T^n\| \le C n^{\frac{1}{2}-\frac{1}{s'}} (\log(n+2))^\beta, \quad n \ge 1.$$

The above method can also be extended to non-commutative $L^p$-spaces. Note that the result in Example 5.4.2 is almost as sharp as Corollary 5.3.5. So, in this particular case, the general result in Theorem 5.4.1 almost recovers the specialized result in Corollary 5.3.5. Of course, the advantage of Theorem 5.4.1 is that it is applicable to many other Banach function spaces, such as Lorentz, Orlicz and variable Lebesgue spaces. Let us illustrate the result for variable Lebesgue spaces:

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*Example* 5.4.3. Let $(S,\mathscr{A},\mu)$ be a $\sigma$-finite measure space, fix $p_0, p_1 \in (1,\infty)$ and assume $p\colon S \to [p_0,p_1]$ is measurable. Let $X = L^{p(\cdot)}(S)$ be the space of all $f \in L^0(S)$ such that

$$\int_S |f(x)|^{p(x)} \,\mathrm{d}\mu(x) < \infty,$$

which, equipped with the corresponding Luxemburg norm, is a Banach function space. Let $1 \le s < p_0 \wedge p_1'$ and note that $L^{p(\cdot)}(S)$ is $s$-convex and $s'$-concave. Moreover, by the same computation as in Example 5.4.2, we have $X_s = L^{q(\cdot)}(S)$, where $q\colon S \to (1,\infty)$ satisfies

$$q(x) = \frac{(2-s)p(x)}{p(x)-s} < \frac{(2-s)p_1}{p_0-s} < \infty, \quad x \in S,$$
$$q(x)' = \frac{(2-s)p(x)'}{p(x)'-s} < \frac{(2-s)p_0'}{p_1'-s} < \infty, \quad x \in S.$$

So, by [89, Corollary 1.2] we know that $X_s$ is a UMD Banach function space. Therefore, Theorem 5.4.1 yields that for any strongly Kreiss bounded operator $T \in \mathscr{L}(X)$, there are $C, \beta > 0$ depending on $p$ and $K_s$ such that

$$\|T^n\| \le C n^{\frac{1}{2}-\frac{1}{s'}} (\log(n+2))^\beta, \quad n \ge 1.$$

### 5.4.1. POSITIVE STRONGLY KREISS BOUNDED OPERATORS

We end this chapter by considering positive strongly Kreiss bounded operators $T$ (i.e. $Tf \ge 0$ for all $f \ge 0$) on a Banach function space. The main result is the following extension of [7], where the case $X = L^p$ was considered.

**Theorem 5.4.4.** *Let $X$ be a Banach function space. Suppose that $T \in \mathcal{L}(X)$ is a positive operator which is strongly Kreiss bounded with constant $K_s$.*

*(1) If $X$ is $p$-convex with $p \in (2, \infty]$, then there exist constants $C, \beta \geq 0$ depending on $X$ and $K_s$ such that*

$$\|T^n\| \leq C n^{\frac{1}{p}} (\log(n+2))^{\beta}, \quad n \geq 1.$$

*(2) If $X$ is $q$-concave with $q \in [1,2)$, then there exist constants $C, \beta \geq 0$ depending on $X$ and $K_s$ such that*

$$\|T^n\| \leq C n^{\frac{1}{q'}} (\log(n+2))^{\beta}, \quad n \geq 1.$$

*Proof.* The case $n = 1$ is clear. In the following, we assume $n \geq 2$.

(2): Let $x \geq 0$. By Lemma 5.3.7, we have $\frac{e^n}{28\sqrt{n}} \leq \frac{n^k}{k!}$, for $k \in [n - 2\sqrt{n}, n]$. Using the positivity of $T$ and $\ell^1 \hookrightarrow \ell^q$, we obtain

$$\begin{aligned} \frac{e^n}{28\sqrt{n}} \Big( \textstyle\sum_{n-\sqrt{n} \leq k \leq n} (T^k x)^q \Big)^{\frac{1}{q}} &\leq \Big( \textstyle\sum_{n-\sqrt{n} \leq k \leq n} \Big( \frac{n^k T^k x}{k!} \Big)^q \Big)^{\frac{1}{q}} \\ &\leq \Big( \textstyle\sum_{k \geq 0} \Big( \frac{n^k T^k x}{k!} \Big)^q \Big)^{\frac{1}{q}} \\ &\leq \textstyle\sum_{k \geq 0} \frac{n^k T^k x}{k!}. \end{aligned} \tag{5.4.1}$$

Since $X$ is $q$-concave, it follows from (5.1.1) that

$$\begin{aligned} \frac{e^n}{28\sqrt{n}} \Big( \sum_{n-\sqrt{n} \leq k \leq n} \|T^k\|^q \Big)^{\frac{1}{q}} &\leq \frac{e^n}{28\sqrt{n}} \Big\| \Big( \sum_{n-\sqrt{n} \leq k \leq n} (T^k x)^q \Big)^{\frac{1}{q}} \Big\| \\ &\leq \Big\| \sum_{k \geq 0} \frac{n^k T^k x}{k!} \Big\| = \|e^{nT} x\| \leq K_s e^n \|x\|, \end{aligned}$$

Therefore,

$$\Big( \sum_{n-\sqrt{n} \leq k \leq n} \|T^k x\|^q \Big)^{\frac{1}{q}} \leq 28 K_s \sqrt{n} \|x\|. \tag{5.4.2}$$

For all $x \in X, x^* \in X^*$, from (5.4.2) we can estimate

$$\begin{aligned} (1 + \lfloor \sqrt{n} \rfloor) |\langle T^{n+1} x, x^* \rangle|^q &= \sum_{1 \leq k \leq 1+\sqrt{n}} |\langle T^{n+1-k} x, T^{*k} x^* \rangle|^q \\ &\leq \sum_{1 \leq k \leq 1+\sqrt{n}} \|T^{n+1-k} x\|^q \|T^{*k} x^*\|^q \\ &\leq \sum_{n-\sqrt{n} \leq k \leq n} \|T^k x\|^q \sup_{1 \leq k \leq 1+\sqrt{n}} \|T^{*k}\|^q \|x^*\|^q \\ &\leq (28 K_s)^q n^{\frac{q}{2}} \|x\|^q \sup_{1 \leq k \leq 2\sqrt{n}} \|T^k\| \|x^*\|^q. \end{aligned}$$

Therefore, since $\sqrt{n} \leq 1 + \lfloor \sqrt{n} \rfloor$, taking the supremum over $\|x\|, \|x^*\| \leq 1$, and using $\|T^n\| = \|T^{*n}\|$, we find that

$$\|T^{n+1}\| \leq 28 K_s n^{\frac{1}{2q'}} \sup_{1 \leq k \leq 2\sqrt{n}} \|T^k\| \leq 28 K_s (n+1)^{\frac{1}{2q'}} \sup_{1 \leq k \leq 2\sqrt{n+1}} \|T^k\|.$$

Changing $n+1$ to $n$ yields

$$\|T^n\| \le 28K_s n^{\frac{1}{2q'}} \sup_{1\le k\le 2\sqrt{n}} \|T^k\|. \tag{5.4.3}$$

Since $\|T^n\| \le K_s\sqrt{2\pi(n+1)}$ in any Banach space by (5.1.3), from (5.4.3) we obtain

$$\|T^n\| \le 28K_s \cdot 2\sqrt{\pi}K_s \cdot 2^{\frac{1}{2}} \cdot n^{\frac{1}{2q'}+\frac{1}{4}}.$$

Since $q' > 2$, combining the above and (5.4.3) we get

$$\begin{aligned}\|T^n\| &\le (28K_s)^2 \cdot 2\sqrt{\pi}K_s \cdot 2^{\frac{1}{2}+\frac{1}{2q'}+\frac{1}{4}} \cdot n^{\frac{1}{2q'}+\frac{1}{2}(\frac{1}{2q'}+\frac{1}{4})}\\ &\le (28K_s)^2 \cdot 2\sqrt{\pi}K_s \cdot 2^{\frac{1}{2}\cdot 2} \cdot n^{\frac{1}{q'}+\frac{1}{2^2}(\frac{1}{2}-\frac{1}{q'})}.\end{aligned}$$

By induction that for any $N \ge 0$,

$$\|T^n\| \le 2\sqrt{\pi}K_s \cdot Q^N n^{\frac{1}{q'}+(\frac{1}{2}-\frac{1}{q'})2^{-N}},$$

where $Q = 28\sqrt{2}K_s$.

Now we proceed as in the proof of Theorem 5.3.1 below (5.3.9), with $P := 2\sqrt{\pi}K_s$, $F_{X,p} := Q = 28\sqrt{2}K_s \ge 1$, $c_N := \left(\frac{1}{2} - \frac{1}{q'}\right)2^{-N}$ for $N \ge 0$. If $n \ge 14$, let $N \in \mathbb{N}$ be such that $2^N < \frac{\log(n+2)}{\log(\log(n+2))} \le 2^{N+1}$. Then

$$n^{c_N} \le (n+2)^{c_N} \le (\log(n+2))^{1-\frac{2}{q'}}.$$

From $2^N \le \frac{\log(n+2)}{\log(\log(n+2))}$ and $\log(\log(n+2)) \ge 1$, we obtain $N \le \frac{\log(\log(n+2))}{\log 2}$. Since $Q \ge 1$, $Q^N = e^{N\log Q} \le (\log(n+2))^{\frac{\log Q}{\log 2}}$. This gives

$$Q^N n^{c_N} \le (\log(n+2))^{\frac{\log Q}{\log 2}+1-\frac{2}{q'}}.$$

Letting $\beta := \frac{\log Q}{\log 2} + 1 - \frac{2}{q'}$, we obtain

$$\|T^n\| \le P n^{\frac{1}{q'}} (\log(n+2))^{\beta}, \quad n \ge 14. \tag{5.4.4}$$

If $n \le 13$, by (5.1.3), $\|T^n\| \le K_s\sqrt{2\pi(13+1)} \le 10K_s$. Substituting $P$ with $C := 10K_s$ in (5.4.4) yields the conclusion for $n \ge 2$.

(1): If $X$ is $p$-convex with $p \in (2,\infty]$, then by duality $X^*$ is $p'$-concave. Applying (2) on $X^*$ gives

$$\|T^n\| = \|T^{*n}\| \le C n^{\frac{1}{p}} (\log(n+2))^{\beta}, \quad n \ge 1.$$

□

## 5.5. Open Problems

In this section we collect some open problems related to the results of the chapter.

The upper and lower decompositions imply (Fourier) type and cotype properties of $X$ as we have seen in Propositions 5.2.10 and 5.2.11. It would be interesting to know if a converse result holds.

*Problem* 5.5.1. Let $X$ be a UMD space and $p, q \in (1, \infty)$. Find a sufficient condition for $\ell^q(L^p)$-upper or lower decompositions in terms of (Fourier) type and cotype of the space $X$.

Our decomposition properties are Fourier decomposition properties on $\mathbb{T}$. One may similarly define Fourier decomposition properties on $\mathbb{R}$, in which case it is natural to wonder if these properties would be equivalent. Note that transference methods are not directly applicable.

*Problem* 5.5.2. Are the decomposition properties equivalent to their counterparts on $\mathbb{R}$?

Even for the scalar field, we do not know for which $p$ and $q$ the upper and lower $\ell^q(L^p)$-decompositions hold. The following problem concerns the missing sharp endpoints.

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*Problem* 5.5.3. Does the scalar field $\mathbb{C}$ have lower $\ell^{p'}(L^p)$-decompositions for $p \in (1,2)$?

If one reverses the roles of $\ell^{p'}$ and $L^p$, then the above estimate fails as was observed in [35], which answered a problem left open in [132]. In particular, a positive answer to Problem 5.5.3 would be a special case of the following:

*Problem* 5.5.4. Does Proposition 5.2.7 hold in the sharp case $r = \frac{ps}{q}$ or $r = \left(\frac{p's}{q}\right)'$?

In Corollary 5.3.5 we have seen a sharp result for $X = L^p(S)$ for strongly Kreiss bounded operators. It is natural to ask if this result can be extended to non-commutative $L^p$-spaces.

*Problem* 5.5.5. Does Corollary 5.3.5 hold for non-commutative $L^p$-spaces?

It seems that the bounds for positive operators obtained in Section 5.4.1 are non-optimal. Especially for $L^p(S)$-spaces we expect that there is an improvement. The bounds obtained from Corollary 5.3.5 and Theorem 5.4.4 are different. Moreover, as observed in [7], the bound of Theorem 5.4.4 is worse than the one in Corollary 5.3.5 if $p \in (4/3, 4)$. It is unclear to us if and how positivity can help in the case $p \in (4/3, 4)$. Given the results for $L^1(S)$, $L^2(S)$ and $L^\infty(S)$ (see [7]), one could even hope that $\theta = 0$ in the case of positive operators.

*Problem* 5.5.6. Let $T$ be a positive operator on $L^p(S)$ with $p \in (1, \infty) \setminus \{2\}$ which is strongly Kreiss bounded. What is the infimum of all $\theta \in [0, 1/2)$ for which there exists a $C$ such that $\|T^n\| \le C n^\theta$ for all $n \ge 1$.

There has been a lot of interest in Kreiss bounded operators in finite dimensions (see [78, 86, 138]). However, it seems to be unknown whether the obtained bounds in terms of the dimension can be improved for strongly Kreiss bounded operators.

*Problem* 5.5.7. Let $X$ be $d$-dimensional. Let $T$ be strongly Kreiss bounded. Determine the best $\theta \in (0, 1]$ for which there exists a $C$ such that $\|T^n\| \le C d^\theta$ for all $n \ge 1$.

# 6

# Multiplier theory in intermediate UMD Banach spaces

## 6.1. Introduction

In Chapter 5, we demonstrated that $\ell^q(L^p)$-Fourier decompositions improve the power bounds of strongly Kreiss bounded operators in Banach spaces. Motivated by this result, we will study the variational Carleson operator and explore its potential applications in multiplier theory.

Let $X$ and $Y$ be Banach spaces. Recall that a vector-valued Fourier multiplier operator is an operator $T_m$ of the form

$$T_m f = \mathscr{F}^{-1}(m\mathscr{F}(f)), \quad f \in \mathscr{S}(\mathbb{R}; X),$$

where $m\colon \mathbb{R} \to \mathscr{L}(X, Y)$, $\mathscr{S}(\mathbb{R}; X)$ is the class of $X$-valued Schwartz functions and $\mathscr{F}$ denotes the Fourier transform. In the scalar-valued case $X = Y = \mathbb{C}$, sufficient conditions for the boundedness of $T_m$ on Lebesgue spaces are provided by the classical Fourier multiplier theorems due to Marcinkiewicz, Mihlin and Hörmander [64, 101, 103, 104], etc. The classical multiplier theorems were extended to UMD spaces in two stages. The case $X = Y$ with scalar-valued multipliers was first addressed in the 1980s by McConnell [102] and Bourgain [23]. Subsequently, in the early 2000s, the more general setting with $X \neq Y$ and operator-valued multipliers was developed. The operator-valued analogue of the Mihlin multiplier theorem was proven by Weis [146], and shortly after an analogue of the Marcinkiewicz multiplier theorem by Štrkalj and Weis [140].

Recall that the classical scalar-valued Marcinkiewicz multiplier theorem states as follows.

**Theorem 6.1.1.** *Let $1 < p < \infty$ and $\Delta$ be the dyadic interval partition of $\mathbb{R}$. Let $m \in L^\infty(\mathbb{R})$ such that for $J \in \Delta$, $m|_J \in V^1(J)$ and*

$$\|m\|_{\ell^\infty(V^1(\Delta))} := \sup_{J\in\Delta} \|m|_J\|_{V^1(J)} < \infty, \tag{6.1.1}$$

*where $V^1(J)$ is the bounded variation space. Then $T_m$ is bounded on $L^p(\mathbb{R})$.*

In 1988, Coifman, Rubio de Francia and Semmes [32] extended the assumption of bounded variation in Theorem 6.1.1 to bounded $s$-variation (see Section 6.2.3 below) for some $s > 1$. Note that bounded $s$-variation is implied by $\frac{1}{s}$-Hölder smoothness, so larger $s$ corresponds to a weaker smoothness assumption. Coifman et al. proved the result in three steps. First, they showed boundedness of $T_m$ on $L^p(\mathbb{R})$ with $p \in [2,\infty)$ under the assumption that $m \in \ell^\infty(V^s(\Delta))$ with $s \in [1,2)$, using the Littlewood–Paley–Rubio de Francia estimate [132]. Then by duality, the boundedness of $T_m$ extends to $p \in (1,2)$. Finally, by Plancherel theorem, the map $(m, f) \mapsto T_m f$ is bounded from $L^\infty(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ for all $p \in (1,\infty)$. Since $\ell^\infty(V^\infty(\Delta)) = L^\infty(\mathbb{R})$, by bilinear interpolation this yields that $T_m$ is bounded on $L^p(\mathbb{R})$ if $m \in \ell^\infty(V^s(\Delta))$ with $\frac{1}{s} > |\frac{1}{p} - \frac{1}{2}|$.

Later, this approach was extended to the operator-valued setting with $X = Y$ by Hytönen and Potapov [68]. To this end, they assume that $X$ is a complex interpolation space between another Banach space $X_0$ and a Hilbert space $H$, thereby enabling the use of interpolation techniques. Furthermore, they assume $X$ has the LPR$_p$-property, a vector-valued version of the Littlewood–Paley–Rubio de Francia estimate introduced by Berkson, Gillespie, and Torrea [15] and further studied in [2, 58, 70, 119]. However, the known examples of Banach spaces with the LPR$_p$-property are limited to Banach *function* spaces satisfying specific geometric conditions [119]. For this reason, Amenta, Lorist, and Veraar [1] extended the results of [68] to Banach function spaces $X$ and $Y$. Besides the LPR$_p$-property, their theory required a strengthening of the $\mathscr{R}$-boundedness condition on the range of $m$, called $\ell^2(\ell^r)$-boundedness. Notably, $\mathscr{R}$-boundedness itself, which strengthens uniform boundedness, was shown to be necessary by Clément and Prüss [29].

Motivated by the $\ell^q(L^p)$ structure of Fourier decompositions and the strategy of [1], we investigate a weighted operator-valued Fourier multiplier theorem under the $\ell^r$ sum of the bounded $s$-variation seminorm of the multiplier on dyadic intervals for some $s, r > 1$. This reveals the relation between geometric (type and cotype) conditions on the underlying Banach spaces and the boundedness of Fourier multiplier operators. A simplified version of our main result is as follows.

**Theorem 6.1.2.** *Let $X$ be a $\theta$-intermediate* UMD *Banach space and $Y$ be a* UMD *Banach space for some $\theta \in (0,1]$. Suppose that $X$ has cotype $q$ and $Y$ has type $t$ and set $\frac{1}{r} := \frac{1}{t} - \frac{1}{q}$. Let $s \in [1, \frac{2}{2-\theta})$ and $m\colon \mathbb{R} \to \mathscr{L}(X,Y)$ has $\mathscr{R}$-bounded range and*

$$\|m\|_{\ell^r(\dot{V}^s(\Delta;\mathscr{L}(X,Y)))} := \Big(\sum_{J\in\Delta} [m|_J]^r_{V^s(J;\mathscr{L}(X,Y))}\Big)^{\frac{1}{r}} < \infty,$$

*Then $T_m$ is bounded from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$ for all $p \in (s,\infty)$.*

Theorem 6.1.2 is established in the main text as Theorem 6.5.4, where Muckenhoupt weights are also incorporated. Weighted extensions of such multiplier results have been previously studied in the scalar-valued setting [16, 80, 82], and in the operator-valued setting in [1]. Our approach begins by proving a weighted vector-valued estimate for a variational Carleson operator, which serves as the foundation for developing a multiplier theory in atomic $R$-spaces (see Section 6.2.3). The central idea is to decompose the multiplier operator into two components: $T_{m-N}$ and $T_N$, where $N$ lies in the closure of the convex hull of the range of $m$. For the operator $T_{m-N}$, we utilize the embedding of bounded $s$-variation spaces into $R$-spaces to establish boundedness in terms of the $\ell^r$-summability of the bounded $s$-variation seminorms of $m$ over dyadic intervals. For the operator $T_N$, we apply the Littlewood–Paley inequality along with the $\mathscr{R}$-boundedness of the range of $m$ to obtain the desired norm estimate. Combining the bounds for both components yields the full conclusion.

The conceptual framework of our proof is inspired by [1]. However, we replace the $\mathrm{LPR}_p$-property with a vector-valued variational Carleson estimate, which is verifiable using an interpolation assumption [3]. This assumption is known to hold not only for Banach function spaces but also for, e.g. the Schatten classes. Moreover, in contrast to the $\ell^2(\ell^r)$-boundedness condition required in [1], we only assume $\mathscr{R}$-boundedness of the multiplier range, along with mild decay conditions on $m(\xi)$ as $|\xi|\to\infty$. More specifically, we replace (6.1.1) with an $\ell^r$-summability condition on the bounded $s$-variation seminorms of $m$, where $r$ is determined by the type and cotype properties of the underlying spaces.

This chapter is organized as follows. In Section 6.2 we discuss some preliminaries on weighted $L^p$-spaces and bounded $s$-variation spaces. Then in Section 6.3 we briefly introduce $\mathscr{R}$-boundedness which is one of the key assumptions in our main result. In Section 6.4 we prepare for our main result by establishing a weighted version of the vector-valued variational Carleson estimate. Afterwards, we establish our main result in Section 6.5. Finally, we compare the main result to the literature and give an example in Section 6.6.



## 6.2. PRELIMINARIES

### 6.2.1. NOTATION IN THIS CHAPTER

Let $\Delta$ denote the dyadic partition of $\mathbb{R}$ (i.e. $\Delta := \bigcup_{k\in\mathbb{Z}} \pm[2^k, 2^{k+1})$), and let $\overline{\mathrm{conv}}(S)$ denote the closure of the convex hull of a set $S$. The Fourier transform of $f \in \mathscr{S}'(\mathbb{R};X)$ is denoted by $\mathscr{F}f$ or $\widehat{f}$. If $f\in L^1(\mathbb{R};X)$, then

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i\xi t} f(t)\,\mathrm{d}t, \quad \xi\in\mathbb{R}.$$

Define the space of $X$-valued smooth compactly supported function on $\mathbb{R}\setminus\{0\}$ as $\mathscr{D}(\mathbb{R}\setminus\{0\};X)$, and define

$$\check{\mathscr{D}}(\mathbb{R}\setminus\{0\};X) := \left\{ g\in\mathscr{S}(\mathbb{R};X) \,\middle|\, g = \check{f} \text{ for some } f\in\mathscr{D}(\mathbb{R}\setminus\{0\};X)\right\},$$

where $\check{f}$ denotes the inverse Fourier transform of $f$.

### 6.2.2. FOURIER MULTIPLIERS IN WEIGHTED $L^p$-SPACES

For $p\in(1,\infty)$, we define the *Muckenhoupt $A_p$-class* as the class of all locally integrable weights $w\colon\mathbb{R}\to(0,\infty)$ such that

$$[w]_{A_p} := \sup_J \Big(\frac{1}{|J|}\int_J w(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|J|}\int_J w(x)^{1-p'}\,\mathrm{d}x\Big)^{p-1} < \infty,$$

where the supremum is taken over all intervals $J \subseteq \mathbb{R}$. By Hölder's inequality, we have $[w]_{A_p} \geq 1$ for $p \in [1,\infty)$:

$$1 = \frac{1}{|J|}\int_J \mathrm{d}x = \frac{1}{|J|}\int_J w(x)^{\frac{1}{p}} w(x)^{-\frac{1}{p}}\,\mathrm{d}x \leq [w]_{A_p}^{\frac{1}{p}}.$$

Moreover, these classes are increasing in $p$, i.e. if $p_0 < p_1$, then $A_{p_0} \subseteq A_{p_1}$ with $[w]_{A_{p_1}} \leq [w]_{A_{p_0}}$. We call $w' := w^{1-p'} = w^{-\frac{1}{p-1}}$ the *dual weight* of $w \in A_p$. By [61, Proposition 7.1.5], $w \in A_p$ if and only if $w' \in A_{p'}$ and

$$[w']_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}. \tag{6.2.1}$$

Then $(w')' = (w^{-\frac{1}{p-1}})^{-\frac{1}{p'-1}} = w$.

We have the following self-improvement lemma.

**Lemma 6.2.1.** *Let $p \in (1,\infty)$, then for any $w \in A_p$ we have*

$$[w]_{A_{p-\varepsilon}} \lesssim_p [w]_{A_p}, \quad 0 \leq \varepsilon \leq \frac{p-1}{1 + C_d [w]_{A_p}^{(p-1)^{-1}}}.$$

*Proof.* Recall the definitions of $[w]_{A_\infty}$ and $[w]_{A_\infty}^{\exp}$ in [69]:



$$[w]_{A_\infty} := \sup_J \Big(\frac{1}{|J|}\int_J M(w(x)\mathbf{1}_J)\,\mathrm{d}x\Big),$$

$$[w]_{A_\infty}^{\exp} := \sup_J \Big(\frac{1}{|J|}\int_J w(x)\,\mathrm{d}x\Big)\exp\Big(\frac{1}{|J|}\int_J \log w(x)^{-1}\,\mathrm{d}x\Big),$$

where $M$ stands for the usual uncentered Hardy–Littlewood maximal operator. Then by Jensen's inequality ([61, Exercise 1.1.3(b)]),

$$\exp\Big(\frac{1}{|J|}\int_J \log w'(x)^{-1}\,\mathrm{d}x\Big) \leq \Big(\frac{1}{|J|}\int_J w(x)\,\mathrm{d}x\Big)^{p'-1}.$$

Combining this with [67, Proposition 2.2] and (6.2.1), we get

$$[w']_{A_\infty} \lesssim [w']_{A_\infty}^{\exp} \leq [w']_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$

Let $\varepsilon_0 := \frac{p-1}{1+C[w']_{A_\infty}}$. By [69, Theorem 1.2], for every $w \in A_p$, we have $w \in A_{p-\varepsilon_0} \subseteq A_{p-\varepsilon}$ with

$$[w]_{A_{p-\varepsilon}} \leq [w]_{A_{p-\varepsilon_0}} \lesssim_p [w]_{A_p}.$$

$\square$

Let $X$ and $Y$ be Banach spaces. For $p \in (1,\infty)$ and $w \in A_p$ we define $L^p(\mathbb{R}, w; X)$ as the space of all strongly measurable $f : \mathbb{R} \to X$ such that

$$\|f\|_{L^p(\mathbb{R},w;X)} := \Big(\int_{\mathbb{R}} \|f(x)\|_X^p w(x)\,\mathrm{d}x\Big)^{1/p} < \infty.$$

The following lemma follows immediately from [53, Lemma 3.3] since $\check{\mathscr{D}}(\mathbb{R}\backslash\{0\}; X) \subseteq \mathscr{S}(\mathbb{R}; X)$.

**Lemma 6.2.2.** *Let $X$ be a Banach space, $p \in (1,\infty)$ and $w \in A_p$. Then $\mathscr{S}(\mathbb{R};X)$ is dense in $L^p(\mathbb{R},w;X)$.*

Let $m \in L^\infty(\mathbb{R};\mathscr{L}(X,Y))$, define the *Fourier multiplier operator* $T_m : \mathscr{S}(\mathbb{R};X) \to \mathscr{S}'(\mathbb{R};Y)$ as

$$T_m f = \mathscr{F}^{-1}(m\widehat{f}), \quad f \in \mathscr{S}(\mathbb{R};X).$$

If, for some $p \in (1,\infty)$ and $w \in A_p$, we have

$$\|T_m f\|_{L^p(\mathbb{R},w;Y)} \lesssim \|f\|_{L^p(\mathbb{R},w;X)}$$

for all $f \in \mathscr{S}(\mathbb{R};X)$, then $T_m$ can be extended to a bounded operator from $L^p(\mathbb{R},w;X)$ to $L^p(\mathbb{R},w;Y)$ by density with the operator norm $\|m\|_{L^p(\mathbb{R},w;\mathscr{L}(X,Y))}$, and $m$ is called a $L^p(\mathbb{R},w;\mathscr{L}(X,Y))$*-Fourier multiplier.*

For an interval $I \subseteq \mathbb{R}$, we define $S_I$ as the Fourier multiplier operator with symbol $m = \mathbf{1}_I$. We have the following weighted, vector-valued Littlewood–Paley inequality, see [53, Theorem 3.4]. Note that the unweighted case, i.e. $w \equiv 1$, was first proved in [23].

**Proposition 6.2.3.** *Let $X$ be a* UMD *Banach space, $(\varepsilon_J)_{J\in\Delta}$ be a Rademacher sequence, $p \in (1,\infty)$ and $w \in A_p$. Then we have for all $f \in L^p(\mathbb{R},w;X)$,*

$$\mathbb{E}\Big\|\sum_{J\in\Delta}\varepsilon_J S_J f\Big\|_{L^p(\mathbb{R},w;X)} \lesssim_{X,p} [w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(\mathbb{R},w;X)},$$

$$\|f\|_{L^p(\mathbb{R},w;X)} \lesssim_{X,p} [w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}\mathbb{E}\Big\|\sum_{J\in\Delta}\varepsilon_J S_J f\Big\|_{L^p(\mathbb{R},w;X)}.$$

### 6.2.3. THE FUNCTION SPACES $R^s$, $V^s$ AND $C^\alpha$

In Section 2.6.1, we introduced the bounded variation space $V^1$. We now extend this framework to the bounded $s$-variation spaces $V^s$ for $s \in [1,\infty)$, which will play a central role in the main result of this chapter.

Let $X$ be a Banach space and $J \subseteq \mathbb{R}$ be a bounded interval, $s \in [1,\infty)$. Define

$$[f]_{V^s(J;X)} := \sup_P \left(\sum_{i=0}^{N-1}\|f(t_{i+1})-f(t_i)\|_X^s\right)^{1/s}, \tag{6.2.2}$$

where $P := \{(t_0,\cdots,t_N) \mid N \in \mathbb{N}, t_i < t_{i+1}, i = 0,\cdots,N-1\}$ is a finite partition of $J$. Denote $f \in \dot{V}^s(J;X)$ if (6.2.2) is finite. We say that $f$ has *bounded $s$-variation for $s \in [1,\infty)$*, denoted by $f \in V^s(J;X)$, if

$$\|f\|_{V^s(J;X)} := \|f\|_{L^\infty(J;X)} + [f]_{V^s(J;X)} < \infty. \tag{6.2.3}$$

Furthermore we define $V^\infty(J;X) := L^\infty(J;X)$ by convention.

Given a collection of bounded disjoint intervals $\mathscr{J}$ in $\mathbb{R}$ and $r \in [1,\infty]$. We define the spaces $\ell^r(V^s(\mathscr{J};X))$ and $\ell^r(\dot{V}^s(\mathscr{J};X))$ as

$$\|f\|_{\ell^r(V^s(\mathscr{J};X))} := \Big(\sum_{J\in\mathscr{J}}\|f|_J\|_{V^s(J;X)}^r\Big)^{1/r} < \infty, \quad f \in L^\infty(\mathbb{R};X),$$

and

$$\|f\|_{\ell^r(\dot{V}^s(\mathscr{J};X))} := \Big(\sum_{J\in\mathscr{J}} [f|_J]^r_{V^s(J;X)}\Big)^{1/r} < \infty, \quad f \in L^\infty(\mathbb{R};X),$$

with the usual modifications for $r=\infty$.

We say that a function $a: J\to X$ is an $R^s(J;X)$*-atom*, written as $a\in R^s_{\mathrm{at}}(J;X)$, if there exists a set $\mathscr{I}$ of mutually disjoint subintervals of $J$ and a set of vectors $(c_I)_{I\in\mathscr{I}}\subseteq X$ such that

$$a = \sum_{I\in\mathscr{I}} c_I \mathbf{1}_I \quad\text{and}\quad \Big(\sum_{I\in\mathscr{I}} \|c_I\|^s\Big)^{1/s} \le 1.$$

Define $R^s(J;X)\subseteq L^\infty(J;X)$ by

$$R^s(J;X) := \Big\{f : f = \sum_{k=1}^\infty \lambda_k a_k, (\lambda_k)_{k\ge1}\in\ell^1, (a_k)_{k\ge1}\subseteq R^s_{\mathrm{at}}(J;X)\Big\},$$

with norm

$$\|f\|_{R^s(J;X)} := \inf\Big\{\|(\lambda_k)_{k\ge1}\|_{\ell^1} : f = \sum_{k=1}^\infty \lambda_k a_k \text{ as above}\Big\}.$$

For a collection of bounded disjoint intervals $\mathscr{J}$ in $\mathbb{R}$ and $r\in[1,\infty]$, the space $\ell^r(R^s(\mathscr{J};X))$ consists of all $f\in L^\infty(\mathbb{R};X)$ such that

$$\|f\|_{\ell^r(R^s(\mathscr{J};X))} := \Big(\sum_{J\in\mathscr{J}} \|f|_J\|^r_{R^s(J;X)}\Big)^{1/r} < \infty,$$

with the usual modification for $r=\infty$.

For $\alpha\in(0,1]$ we define the space of $\alpha$*-Hölder continuous functions* $C^\alpha(J;X)$ as the space of all $f: J\to X$ with

$$\|m\|_{C^\alpha(J;X)} := \|m\|_{L^\infty(J;X)} + [m]_{C^\alpha(J;X)},$$

where

$$[m]_{C^\alpha(J;X)} := \sup_{t_1,t_2\in J} \frac{\|m(t_1)-m(t_2)\|}{|t_1-t_2|^\alpha}.$$

The next lemma gives the interpolation inclusions of $V^s$ spaces.

**Lemma 6.2.4.** *Let $X_0, X_1$ be Banach spaces and $J\subseteq\mathbb{R}$ be a bounded interval. For $s_0, s_1\in[1,\infty], \theta\in(0,1]$ we have the continuous embedding*

$$[V^{s_0}(J;X_0), V^{s_1}(J;X_1)]_\theta \subseteq V^{s_\theta}(J;[X_0,X_1]_\theta),$$

*where $\frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$.*

*Proof.* Let $X$ be a Banach space and $s\in[1,\infty]$. Note that the norm (6.2.3) of $V^s(J;X)$ equivalent to

$$\|f\|_{V^s(J;X)} \eqsim \|f(t_0)\|_X + \sup_P \Bigg(\sum_{i=0}^{N-1} \|f(t_{i+1})-f(t_i)\|_X^s\Bigg)^{\frac1s} < \infty, \tag{6.2.4}$$

where $P := \{(t_0, \cdots, t_N) \mid N \in \mathbb{N}, t_i < t_{i+1}, i = 0, \cdots, N-1\}$ is a finite partition of $J$. Define an operator $T_P : V^s(J; X) \to \ell^s(X)$ as

$$T_P f := \big(f(t_0), f(t_1) - f(t_0), \cdots, f(t_N) - f(t_{N-1})\big).$$

Then

$$\|T_P f\|_{\ell^s(X)} = \left(\|f(t_0)\|_X^s + \sum_{i=0}^{N-1} \|f(t_{i+1}) - f(t_i)\|_X^s\right)^{\frac{1}{s}}.$$

By (6.2.4), it is clear

$$\|f\|_{V^s(J;X)} \le \sup_P \|T_P f\|_{\ell^s(X)} \le \|f\|_{V^s(J;X)}, \tag{6.2.5}$$

where the supremum is taken over all finite partitions. Then for any fixed partition $P$, we have $T_p$ is a bounded linear operator from $V^{s_0}(J; X_0)$ to $\ell^{s_0}(X_0)$ and from $V^{s_1}(J; X_1)$ to $\ell^{s_1}(X_1)$. Therefore, by complex interpolation and [71, Theorem 2.2.6], $T_p$ is a bounded linear operator from $[V^{s_0}(J; X_0), V^{s_1}(J; X_1)]_\theta$ to $[\ell^{s_0}(J; X_0), \ell^{s_1}(J; X_1)]_\theta = \ell^{s_\theta}([X_0, X_1]_\theta)$. This combined with (6.2.5) yields

$$\|f\|_{V^{s_\theta}(J;[X_0,X_1]_\theta)} \lesssim \sup_P \|T_P f\|_{\ell^{s_\theta}([X_0,X_1]_\theta)} \lesssim \|f\|_{[V^{s_0}(J;X_0),V^{s_1}(J;X_1)]_\theta},$$

finishing the proof. □

We recall the definition of dyadic martingale differences. Let $k \in \mathbb{Z}$. We denote the set of all dyadic intervals with the length of $2^{-k}$ by

$$\mathscr{D}_k := \left\{\left[n2^{-k}, (n+1)2^{-k}\right), n \in \mathbb{Z}\right\}$$

and the set of all dyadic intervals by $\mathscr{D} := \cup_{k\in\mathbb{Z}} \mathscr{D}_k$.

**Definition 6.2.5.** Let $f : [0,1] \to X$ be a integrable function and $I \in \mathscr{D}_k$ be a dyadic interval. The *conditional expectation of f* with respect to the increasing family of $\sigma$-algebras $\sigma(\mathscr{D}_k)$ generated by $\mathscr{D}_k$ is defined as

$$E_k(f)(t) := \sum_{I\in\mathscr{D}_k} \mathrm{Avg}_I(f)\mathbf{1}_I(t), \quad k \in \mathbb{N}_0.$$

Here, $\mathrm{Avg}_I(f)$ denotes the average value of $f$ over the interval $I$. The *dyadic martingale difference* of $f$ is then defined as

$$D_k(f) := E_k(f) - E_{k-1}(f), \quad k \in \mathbb{N},$$

and we denote $D_k$ as the *dyadic martingale difference operator.*

Next, we recall the result of Coifman, Rubio de Francia, and Semmes, originally established concisely in [32, Lemma 2]. A detailed proof was later provided by Manuel Bohnert (Karlsruhe Institute of Technology) in his Master's thesis [18]. Using Bohnert's argument, we extend the result in a different direction: given $s_1 < s_2 \in [1, \infty)$, for every function $m \in V^{s_1}$, there exists an operator $N \in \overline{\mathrm{conv}}(\mathrm{Ran}(m))$ such that the $R^{s_2}$-norm of $m - N$ can be controlled by the $V^{s_1}$-seminorm of $m$ in the vector-valued setting.

**Lemma 6.2.6.** *Let $X$ be a Banach space, $J \subseteq \mathbb{R}$ be an interval, and $s_1, s_2 \in [1,\infty)$ satisfying $s_1 < s_2$. Then for every $m \in \dot{V}^{s_1}(J;X)$, there exists an operator $N_J \in \overline{\operatorname{conv}}(\operatorname{Ran}(m))$ such that*

$$\|m - N_J\|_{R^{s_2}(J;X)} \lesssim [m]_{V^{s_1}(J;X)},$$

*Proof.* If $m(t) \equiv x \in X$ for all $t \in J$, then $m = \|x\| \cdot \frac{x}{\|x\|}\mathbf{1}_J \in R^{s_2}(J;X)$ where $\frac{x}{\|x\|}\mathbf{1}_J \in R^{s_2}_{\mathrm{at}}(J;X)$ and the conclusion holds clearly with $N_J := x$. From now on we assume $m \in \dot{V}^{s_1}(J;X)$ with $[m]_{V^{s_1}(J;X)} = 1$. Without loss of generality, let $J := [a,b] \subseteq \mathbb{R}$, $n \in \mathbb{N}$. Define $g : J \to [0,1]$ as

$$g(t) := \sup_{P_t} \sum_{i\geq 0} \|m(t_{i+1}) - m(t_i)\|^{s_1}, \tag{6.2.6}$$

where $P_t$ is a partition of $[a,t]$. Define $\mathbb{M} : [0,1] \to X$ by

$$\mathbb{M}(y) := m(t), \quad t \in J,\ y := g(t). \tag{6.2.7}$$

Note that $\mathbb{M}$ is well-defined. Indeed, if there exist $t, \tilde{t} \in J$ such that $t < \tilde{t}$ and $g(t) = g(\tilde{t}) = y$, then by the definition of $g$, we have $m(t) \equiv m(\tilde{t})$. Finally, define

$$N_J := \int_0^1 \mathbb{M}(y)\,\mathrm{d}y. \tag{6.2.8}$$

Then by [71, Proposition 1.2.12], $N_J \in \overline{\operatorname{conv}}(\operatorname{Ran}(\mathbb{M})) = \overline{\operatorname{conv}}(\operatorname{Ran}(m))$.

Moreover, let $\tilde{t} > t$ and $P_{\tilde{t}}$ be an arbitrary partition of $[a,\tilde{t}]$, then

$$g(\tilde{t}) = \sup_{P_{\tilde{t}}} \left(\sum_{i\geq 0} \|m(t_{i+1}) - m(t_i)\|^{s_1}\right) \geq g(t) + \|m(\tilde{t}) - m(t)\|^{s_1}.$$

Thus, we conclude from the above and $\tilde{y} := g(\tilde{t})$ that

$$\|\mathbb{M}(\tilde{y}) - \mathbb{M}(y)\| = \|m(\tilde{t}) - m(t)\| \leq |g(\tilde{t}) - g(t)|^{\frac{1}{s_1}} = |\tilde{y} - y|^{\frac{1}{s_1}},$$

which means $\mathbb{M}$ is $\frac{1}{s_1}$-Hölder continuous.

Now we seek a representation of $\mathbb{M}$ using dyadic martingale differences. For each $y \in [0,1)$, there exists a unique sequence $(I_{i,y})_{i\in\mathbb{N}}$ with $I_{i,y} \in \mathscr{D}_i$ including $y$ and $\bigcap_{i=1}^{\infty} \overline{I_{i,y}} = \{y\}$. Then

$$\lim_{i\to\infty} E_i(\mathbb{M})(y) = \lim_{i\to\infty} \sum_{I\in\mathscr{D}_i} \operatorname{Avg}_I(\mathbb{M})\mathbf{1}_I(y) = \lim_{i\to\infty} \operatorname{Avg}_{I_{i,y}}(\mathbb{M}) = \mathbb{M}(y).$$

Here we use $\mathbb{M}$ is Hölder continuous and thereby continuous. Recalling that $D_k(\mathbb{M})(y) = E_k(\mathbb{M})(y) - E_{k-1}(\mathbb{M})(y)$, $k \geq 1$, we have

$$\sum_{k=1}^{\infty} D_k(\mathbb{M})(y) = \lim_{n\to\infty} \sum_{k=1}^{n} D_k(\mathbb{M})(y) = \lim_{n\to\infty} E_n(\mathbb{M})(y) - E_0(\mathbb{M})(y) = \mathbb{M}(y) - E_0(\mathbb{M})(y).$$

Therefore, we obtain

$$\mathbb{M}(y) = E_0(\mathbb{M})(y) + \sum_{k=1}^{\infty} D_k(\mathbb{M})(y), \quad y \in [0,1]. \tag{6.2.9}$$

Next, we show $D_k(\mathbb{M})$ is pointwise bounded for every $k \in \mathbb{N}$. For any $y \in [0,1)$, there exists $n \in \mathbb{Z}$ such that $y \in I := [n2^{-(k-1)}, (n+1)2^{-(k-1)})$. Define $I_l := [2n2^{-k}, (2n+1)2^{-k})$ and $I_r := [(2n+1)2^{-k}, 2(n+1)2^{-k})$, it is obvious that $I \in \mathscr{D}_{k-1}, I_l, I_r \in \mathscr{D}_k, I_l \cup I_r = I, I_l \cap I_r = \emptyset$, and $y$ lies in either $I_l$ or $I_r$. We assume that $y \in I_l$. The calculation for $y \in I_r$ is similar. Note that $\mathbb{M}$ is $\frac{1}{s_1}$-Hölder continuous, let $\tilde{t} := t + 2^{-k}$, then

$$\begin{aligned}
\left\|D_k(\mathbb{M})(y)\right\| &= \left\|E_k(\mathbb{M})(y) - E_{k-1}(\mathbb{M})(y)\right\| \\
&= \left\|\sum_{I\in\mathscr{D}_k} \mathrm{Avg}_I(\mathbb{M})\mathbf{1}_I(y) - \sum_{I\in\mathscr{D}_{k-1}} \mathrm{Avg}_I(\mathbb{M})\mathbf{1}_I(y)\right\| \\
&= \left\|\mathrm{Avg}_{I_l}(\mathbb{M}) - \mathrm{Avg}_I(\mathbb{M})\right\| = \left\|\frac{1}{|I_l|}\int_{I_l} \mathbb{M}(t)\,\mathrm{d}t - \frac{1}{|I|}\int_I \mathbb{M}(t)\,\mathrm{d}t\right\| \\
&= \left\|2^k \int_{2n2^{-k}}^{(2n+1)2^{-k}} \mathbb{M}(t)\,\mathrm{d}t - 2^{k-1}\int_{n2^{-(k-1)}}^{(n+1)2^{-(k-1)}} \mathbb{M}(t)\,\mathrm{d}t\right\| \\
&= \left\|2^{k-1} \int_{2n2^{-k}}^{(2n+1)2^{-k}} 2\mathbb{M}(t)\,\mathrm{d}t - 2^{k-1}\int_{2n2^{-k}}^{(2n+1)2^{-k}} \mathbb{M}(t)\,\mathrm{d}t - 2^{k-1}\int_{(2n+1)2^{-k}}^{2(n+1)2^{-k}} \mathbb{M}(t)\,\mathrm{d}t\right\| \\
&= \left\|2^{k-1} \int_{2n2^{-k}}^{(2n+1)2^{-k}} \mathbb{M}(t)\,\mathrm{d}t - 2^{k-1}\int_{(2n+1)2^{-k}}^{2(n+1)2^{-k}} \mathbb{M}(t)\,\mathrm{d}t\right\| \\
&= \left\|2^{k-1} \int_{(2n+1)2^{-k}}^{2(n+1)2^{-k}} \mathbb{M}(\tilde{t})\,\mathrm{d}\tilde{t} - 2^{k-1}\int_{(2n+1)2^{-k}}^{2(n+1)2^{-k}} \mathbb{M}(t)\,\mathrm{d}t\right\| \\
&= 2^{k-1} \int_{(2n+1)2^{-k}}^{2(n+1)2^{-k}} \left\|\mathbb{M}(\tilde{t}) - \mathbb{M}(t)\right\|\,\mathrm{d}t \\
&\le 2^{k-1} \int_{(2n+1)2^{-k}}^{2(n+1)2^{-k}} |t-\tilde{t}|^{\frac{1}{s_1}}\,\mathrm{d}t \le 2^{-\frac{k}{s_1}-1},
\end{aligned}$$

and then

$$\|D_k(\mathbb{M})\|_{L^\infty(0,1;X)} \le 2^{-\frac{k}{s_1}-1}.$$

By definition, it is clear that the dyadic martingale difference is constant on each dyadic interval in $\mathscr{D}_k$. In particular, $D_k(\mathbb{M})$ is constant on $[n2^{-k}, (n+1)2^{-k})$ for $0 \le n < 2^k$. Thus, we have the representation $D_k(\mathbb{M}) = \sum_{n=0}^{2^k-1} \mu_n \mathbf{1}_{[n2^{-k},(n+1)2^{-k})}$, where $\mu_n$ satisfies

$$\left\|\mu_n\right\| \le \|D_k(\mathbb{M})\|_{L^\infty(0,1;X)} \le 2^{-\frac{k}{s_1}-1}, \quad n = 0, \cdots, 2^{k-1}.$$

We conclude that

$$\left(\sum_{n=0}^{2^k-1} \left\|\mu_n\right\|^{s_2}\right)^{\frac{1}{s_2}} \le \left(\sum_{n=0}^{2^k-1} \left(2^{-\frac{k}{s_1}-1}\right)^{s_2}\right)^{\frac{1}{s_2}} \le 2^{\frac{k}{s_2}} 2^{-\frac{k}{s_1}-1} \le 2^{-k\left(\frac{1}{s_1}-\frac{1}{s_2}\right)} := U_k.$$

Thus, $D_k(\mathbb{M}) = U_k \sum_{n=0}^{2^k-1} \frac{\mu_n}{U_k} \mathbf{1}_{[n2^{-k},(n+1)2^{-k})}$ with $\sum_{n=0}^{2^k-1} \left(\frac{\|\mu_n\|}{U_k}\right)^{s_2} \le 1$, implying

$$\|D_k(\mathbb{M})\|_{R^{s_2}([0,1);X)} \le U_k = 2^{-k\left(\frac{1}{s_1}-\frac{1}{s_2}\right)}.$$

Now we conclude from (6.2.9) that

$$\begin{aligned} \|\mathbb{M} - E_0(\mathbb{M})\|_{R^{s_2}([0,1);X)} &\le \textstyle\sum_{k=1}^{\infty} \|D_k(\mathbb{M})\|_{R^{s_2}([0,1)};X) \le \frac{4}{2^{\left(\frac{1}{s_1}-\frac{1}{s_2}\right)}-1} \\ &\lesssim_{s_1,s_2} [m]_{V^{s_1}(J;X)}. \end{aligned} \tag{6.2.10}$$

Hence, $\mathbb{M} - E_0(\mathbb{M}) \in R^{s_2}([0,1);X)$. Then by definition, there exist sequences $(\lambda_k)_{k\ge1} \in \ell^1$ and $(a_k)_{k\ge1} \subseteq R^{s_2}_{\text{at}}([0,1);X)$ such that $\mathbb{M} - E_0(\mathbb{M}) = \sum_{k\ge1} \lambda_k a_k$, where $a_k = \sum_{I\in\mathscr{I}_k} c_{I,k}\mathbf{1}_I$ with $\mathscr{I}_k$ a family of mutually disjoint subintervals of $[0,1)$ and $(c_{I,k})_{k\ge1} \subseteq \mathscr{L}(X,Y)$ satisfying $\sum_{I\in\mathscr{I}_k} \|c_{I,k}\|^{s_2} \le 1$.

For $k \ge 1$, define $\widetilde{I} := \{t \in J \mid y \in I \in \mathscr{I}_k\}$ and $\widetilde{\mathscr{I}}_k := \cup \widetilde{I}$. Then, for each $t \in \tilde{I} \in \widetilde{\mathscr{I}}_k$, there exists $y \in I \in \mathscr{I}_k$ such that

$$m(t) - N_J = \mathbb{M}(y) - E_0(\mathbb{M}) = \sum_{k\ge1} \lambda_k c_{I,k}.$$

Define $c_{\tilde{I},k} = c_{I,k}$ and $\tilde{a}_k := \sum_{\tilde{I}\in\widetilde{\mathscr{I}}_k} c_{\tilde{I},k}\mathbf{1}_I$. Next, we define $I_0 := \{t \in J \mid y = 1\}$, $\tilde{a}_0 := \frac{\mathbb{M}(1)-N_J}{\|\mathbb{M}(1)-N_J\|}\mathbf{1}_{I_0}$, $\lambda_0 := \|\mathbb{M}(1) - N_J\|$. Note that

$$|\lambda_0| = \|\mathbb{M}(1) - N_J\| \le \int_0^1 \|\mathbb{M}(1) - \mathbb{M}(y)\|\,\mathrm{d}y \le [m]_{V^{s_1}(J;X)}. \tag{6.2.11}$$

Thus, we can construct

$$m(t) - N_J = \sum_{k\ge0} \lambda_k \tilde{a}_k, \quad t \in J.$$

Therefore, by (6.2.10) and (6.2.11), we obtain

$$\left\|m - N_J\right\|_{R^{s_2}(J;X)} \le \sum_{k\ge0} |\lambda_k| \lesssim_{s_1,s_2} [m]_{V^{s_1}(J;X)}.$$

Moreover, we also conclude from (6.2.10) that $E_0(\mathbb{M}) \in R^{s_2}([0,1];X)$ and

$$\|m\|_{R^{s_2}(J;X)} = \|\mathbb{M}\|_{R^{s_2}([0,1];X)} \lesssim_{s_1,s_2} \|m\|_{V^{s_1}(J;X)}.$$

□

Note that the spaces $V^s(J;X)$, $R^s(J;X)$ and $C^\alpha(J;X)$ are Banach spaces, and satisfy the following embedding results by [1, Lemma 4.3].

**Lemma 6.2.7.** *Let $X$ be a Banach space, let $J \subseteq \mathbb{R}$ be a closed, bounded interval, and $s \in [1,\infty)$.*

(i) *We have $R^s(J;X) \subseteq V^s(J;X)$ and for all $f \in R^s(J;X)$,*

$$\left\|f\right\|_{V^s(J;X)} \lesssim \left\|f\right\|_{R^s(J;X)}.$$

(ii) *We have $C^{\frac{1}{s}}(J;X) \subseteq V^s(J;X)$ and for all $f \in C^{\frac{1}{s}}(J;X)$ we have*

$$\left\|f\right\|_{V^s(J;X)} \le \left\|f\right\|_{L^\infty(J;X)} + |J|^{\frac{1}{s}}\,[f]_{C^{1/s}(J;X)}.$$

## 6.3. RELATION TO $\mathscr{R}$-BOUNDEDNESS

In this section, we briefly introduce the basic properties $\mathscr{R}$-boundedness which plays a key role in our main result. For details we refer to [72, Chapter 8].

Let $X$ and $Y$ be Banach spaces and $(\varepsilon_n)_{n\geq 1}$ be a Rademacher sequence on a fixed probability space $(\Omega, \mathbb{P})$. Consider a family of operators $\mathscr{T} \subseteq \mathscr{L}(X,Y)$.

**Definition 6.3.1.** We say $\mathscr{T}$ is *$\mathscr{R}$-bounded* if there exists a constant $C \geq 0$ such that for all finite sequences $(T_n)_{n=1}^N$ in $\mathscr{T}$ and $(x_n)_{n=1}^N$ in $X$,

$$\Big\|\sum_{n=1}^N \varepsilon_n T_n x_n\Big\|_{L^2(\Omega;Y)} \leq C\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.$$

The least admissible constant in the inequality is called the $\mathscr{R}$-bound, denoted by $\mathscr{R}(\mathscr{T})$.

Note that using Kahane-Khintchine inequality one can replace $L^2(\Omega;X)$ with $L^p(\Omega;X)$ for $1 < p < \infty$ in Definition 6.3.1. By [72, Propositions 8.1.21 and 8.1.22], if $\mathscr{T}$ is $\mathscr{R}$-bounded in $\mathscr{L}(X,Y)$, then $\mathrm{conv}(\mathscr{T})$ and $\overline{\mathrm{conv}}(\mathscr{T})$ are also $\mathscr{R}$-bounded with

$$\mathscr{R}(\mathscr{T}) = \mathscr{R}(\mathrm{conv}(\mathscr{T})) = \mathscr{R}(\overline{\mathrm{conv}}(\mathscr{T})). \tag{6.3.1}$$

In the next result, we show a function in $R^r(J;\mathscr{L}(X,Y))$ for $r > 0$, $J \subseteq \mathbb{R}$ is $\mathscr{R}$-bounded under cotype and type assumptions on $X$ and $Y$.

**Proposition 6.3.2.** *Let $X$, $Y$ be Banach spaces and $\mathscr{J}$ be a collection of disjoint intervals $J \subseteq \mathbb{R}$. Suppose that $X$ has cotype $q \in [2,\infty]$ and $Y$ has type $t \in [1,2]$. Let $\frac{1}{r} := \frac{1}{t} - \frac{1}{q}$. Then $f \in \ell^r(R^r(\mathscr{J};\mathscr{L}(X,Y)))$ has $\mathscr{R}$-bounded range.*

*Proof.* Let $J \in \mathscr{J}$ and $\mathscr{I}$ be a family of mutually disjoint subintervals of $J$. For each $a \in R^r_{\mathrm{at}}(J;\mathscr{L}(X,Y))$, there exists a sequence $\{c_I\}_{I\in\mathscr{I}} \subseteq \mathscr{L}(X,Y)$ such that $a = \sum_{I\in\mathscr{I}} c_I \mathbf{1}_I$. Then by [72, Proposition 8.1.20],

$$\mathscr{R}(\mathrm{Ran}(a)) = \mathscr{R}(\cup_{I\in\mathscr{I}}\{c_I\}) \lesssim \Big(\sum_{I\in\mathscr{I}} \|c_I\|^r\Big)^{\frac{1}{r}} \leq 1.$$

Thus, for each $f|_J \in R^r(J;\mathscr{L}(X,Y))$, there exist sequences $(\lambda_k)_{k\geq 1} \subseteq \mathbb{C}$ and $(a_k)_{k\geq 1} \subseteq R^r_{\mathrm{at}}(J;\mathscr{L}(X,Y))$ such that $f|_J = \sum_{k\geq 1} \lambda_k a_k$. Then

$$\mathscr{R}(\mathrm{Ran}(f|_J)) \leq \sum_{k\geq 1} |\lambda_k| \mathscr{R}(\mathrm{Ran}(a_k)) \leq \|f|_J\|_{R^r(J;\mathscr{L}(X,Y))}.$$

Let $f \in \ell^r(R^r(\mathscr{J};\mathscr{L}(X,Y)))$, then $\mathrm{Ran}(f) = \cup_{J\in\mathscr{J}} \mathrm{Ran}(f|_J)\mathbf{1}_J$ where $f|_J \in R^r(J;\mathscr{L}(X,Y))$. Therefore, again by [72, Proposition 8.1.20],

$$\mathscr{R}(\mathrm{Ran}(f)) \lesssim \sum_{J\in\mathscr{J}} (\|\mathscr{R}(\mathrm{Ran}(f|_J))\|^r)^{\frac{1}{r}} \lesssim \|f\|_{\ell^r(R^r(\mathscr{J};\mathscr{L}(X,Y)))}.$$

□

We immediately conclude the following corollary from Lemmas 6.2.6 and 6.2.7 and Proposition 6.3.2 for Hölder continuous functions and bounded $s$-variation functions.

**Corollary 6.3.3.** *Let $X$, $Y$ be Banach spaces and $\mathscr{J}$ be a collection of disjoint intervals $J \subseteq \mathbb{R}$. Suppose that $X$ has cotype $q \in [2,\infty]$ and $Y$ has type $t \in [1,2]$. Let $\frac{1}{r} := \frac{1}{t} - \frac{1}{q}$ and $s \in (1,r)$. Then any function in $\ell^r(C^{\frac{1}{s}}(\mathscr{J};\mathscr{L}(X,Y)))$ or $\ell^r(V^s(\mathscr{J};\mathscr{L}(X,Y)))$ has $\mathscr{R}$-bounded range with*

$$\mathscr{R}\left(\mathrm{Ran}(f)\right) \lesssim \|f\|_{\ell^r(C^{\frac{1}{s}}(\mathscr{J};\mathscr{L}(X,Y)))}, \quad f \in \ell^r(C^{\frac{1}{s}}(\mathscr{J};\mathscr{L}(X,Y))),$$
$$\mathscr{R}\left(\mathrm{Ran}(g)\right) \lesssim \|g\|_{\ell^r(V^s(\mathscr{J};\mathscr{L}(X,Y)))}, \quad g \in \ell^r(V^s(\mathscr{J};\mathscr{L}(X,Y))),$$

*where the implied constant depends on the various parameters in the statement, but is independent of the function.*

## 6.4. WEIGHTED, VECTOR-VALUED VARIATIONAL CARLESON ESTIMATES

In this section, we prepare for our main result by establishing a weighted version of the vector-valued variational Carleson estimate. We start by extending the boundedness of the vector-valued Carleson operator from [66] to a weighted setting. For $f \in \mathscr{S}(\mathbb{R};X)$ and $a \in \mathbb{R}$ define

$$\mathscr{C}_a f(x) := \int_{-\infty}^{a} \widehat{f}(\xi)\mathrm{e}^{2\pi i\xi x}\,\mathrm{d}\xi, \quad x \in \mathbb{R}.$$

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The *Carleson maximal operator* is defined as

$$\mathscr{C}_* f(x) := \sup_{a\in\mathbb{R}} \|\mathscr{C}_a f(x)\|_X = \|a \mapsto \mathscr{C}_a f(x)\|_{L^\infty(\mathbb{R};X)}, \quad x \in \mathbb{R}.$$

**Proposition 6.4.1.** *Let $X$ be an intermediate* UMD *Banach space and let $p \in (1,\infty)$. There exists a non-decreasing function $\phi_{X,p} : [1,\infty) \to [1,\infty)$ such that for all $w \in A_p$ and $f \in L^p(\mathbb{R},w;X)$,*

$$\|\mathscr{C}_* f\|_{L^p(\mathbb{R},w)} \le \phi_{X,p}([w]_{A_p}) \|f\|_{L^p(\mathbb{R},w;X)}.$$

*Proof.* Since $\mathscr{C}_a f = f - S_{[a,\infty)}f$, by Lemma 2.2.4 it suffices to prove that there exists a non-decreasing function $\phi_{X,p} : [1,\infty) \to [1,\infty)$ such that for all $w \in A_p$ and $f \in \mathscr{S}(\mathbb{R}^d;X)$,

$$\left\|\sup_{a\in\mathbb{R}} \|S_{[a,\infty)}f\|_X\right\|_{L^p(\mathbb{R},w)} \le \phi_{X,p}([w]_{A_p}) \|f\|_{L^p(\mathbb{R},w;X)}. \tag{6.4.1}$$

Define $\mathscr{M}^a f := x \mapsto \mathrm{e}^{-2\pi i x a} f(x)$ and note that

$$\|S_{[0,\infty)}\mathscr{M}^a f\|_X = \left\|e^{-2\pi i x a} S_{[a,\infty)} f\right\|_X = \|S_{[a,\infty)}f\|_X.$$

Let $H$ denote Hilbert transform. By (2.4.2), it is equivalent to show that there exists a non-decreasing function $\phi_{X,p} : [1,\infty) \to [1,\infty)$ such that for all $w \in A_p$ and $f \in \mathscr{S}(\mathbb{R};X)$,

$$\left\|\sup_{a\in\mathbb{R}} \|H\mathscr{M}^a f\|_X\right\|_{L^p(\mathbb{R},w)} \le \phi_{X,p}([w]_{A_p}) \|f\|_{L^p(\mathbb{R},w;X)}. \tag{6.4.2}$$

To prove (6.4.2), we want to apply the sparse domination result in [94]. To do so, let $p_0 \in (1,\infty)$ and define $T : L^{p_0}(\mathbb{R};X) \to L^{p_0}(\mathbb{R};L^\infty(\mathbb{R};X))$ by

$$Tf(x,a) := H\mathcal{M}^a f(x), \quad x,a \in \mathbb{R},\ f \in L^{p_0}(\mathbb{R};X).$$

If $w \equiv 1$, [66, Theorem 1.1] shows that (6.4.1) holds and therefore (6.4.2) also holds, which implies that $T$ is well-defined and bounded by density. For $f \in L^{p_0}(\mathbb{R};X)$, furthermore define

$$M_T^\# f(x) := \sup_{I \ni x} \operatorname*{ess\,sup}_{x_1,x_2 \in I} \left\| T(f\mathbf{1}_{\mathbb{R}\setminus 5I})(x_1) - T(f\mathbf{1}_{\mathbb{R}\setminus 5I})(x_2) \right\|_{L^\infty(\mathbb{R};X)}, \quad x \in \mathbb{R},$$

where the supremum is taken over all intervals $I$ containing $x$.

Fix $x \in \mathbb{R}$ and an interval $I$ containing $x$. Take $x_1, x_2 \in I$ and denote the length of $I$ by $\varepsilon$. Note that for $y \in \mathbb{R}\setminus 5I$,

$$|x-y| \eqsim |x_1-y| \eqsim |x_2-y| \gtrsim 2\varepsilon.$$

Then by (2.4.1),

$$\begin{aligned}
&\left\| T(f\mathbf{1}_{\mathbb{R}\setminus 5I})(x_1) - T(f\mathbf{1}_{\mathbb{R}\setminus 5I})(x_2) \right\|_{L^\infty(\mathbb{R};X)} \\
&= \sup_{a\in\mathbb{R}} \frac{1}{\pi} \left\| \int_{\mathbb{R}\setminus 5I} \left( \frac{1}{x_1-y} - \frac{1}{x_2-y} \right) e^{-2\pi i y a} f(y)\,\mathrm{d}y \right\|_X \\
&\lesssim \int_{\mathbb{R}\setminus 5I} \frac{|x_1-x_2|}{|x_1-y||x_2-y|} \|f(y)\|_X \,\mathrm{d}y \\
&\lesssim \varepsilon \int_{\mathbb{R}\setminus 5I} \frac{1}{|x-y|^2} \|f(y)\|_X \,\mathrm{d}y \\
&\lesssim \varepsilon \sum_{k=1}^\infty \int_{2^k\varepsilon < |x-y| \le 2^{k+1}\varepsilon} \frac{1}{|x-y|^2} \|f(y)\|_X \,\mathrm{d}y \\
&\lesssim \sum_{k=1}^\infty \frac{1}{2^k} \frac{2}{2^{k+1}\varepsilon} \int_{|x-y| \le 2^{j+1}\varepsilon} \|f(y)\|_X \,\mathrm{d}y \\
&\lesssim \sum_{k=1}^\infty \frac{1}{2^k} \cdot M\Big(\|f\|_X\Big)(x) \lesssim M\Big(\|f\|_X\Big)(x),
\end{aligned}$$

where $M$ is the Hardy-Littlewood maximal operator. Therefore we obtain

$$M_T^\# f(x) \lesssim M\Big(\|f\|_X\Big)(x), \quad x \in \mathbb{R}.$$

Since $M$ is weak $L^1$-bounded by [71, Theorem 2.3.2], $M_T^\#$ extends to a weak $L^1$-bounded operator as well. We have now checked all assumptions of [94, Theorem 1.1] with $p_1 = p_0$, $p_2 = 1$, $r = 1$ and $(S,d,\mu)$ is $\mathbb{R}$ with the Euclidean metric and the Lebesgue measure. By [94, Corollary 1.2], we conclude for all $p \in (p_0,\infty)$, $w \in A_{p/p_0}$ and $f \in L^p(\mathbb{R},w;X)$,

$$\left\| \sup_{a\in\mathbb{R}} \left\| H\mathcal{M}^a f \right\|_X \right\|_{L^p(\mathbb{R},w)} \le \frac{p^2}{p-p_0} \cdot C_{X,p_0} \cdot [w]_{A_{p/p_0}}^{\max\{\frac{1}{p-p_0},1\}} \|f\|_{L^p(\mathbb{R},w;X)}, \tag{6.4.3}$$

where the factor $\frac{p^2}{p-p_0}$ arises from the proof of [94, Proposition 4.1] and $C_{X,p_0}$ is comparable to the constant in (6.4.2) for the case $w \equiv 1$, i.e. the constant from [66, Theorem 1.1]. By inspection of the proof, it is clear that $C_{X,p_0} \to \infty$ as $p_0 \downarrow 1$.

Now, for fixed $w \in A_p$, let $p_0 \in (1,p)$ such that $\frac{p}{p_0} = p - \varepsilon$ with $\varepsilon > 0$ as in Lemma 6.2.1. Combined with (6.4.3), we conclude that there is a non-decreasing function $\phi_{X,p} : [1,\infty) \to [1,\infty)$ such that (6.4.2) holds, finishing the proof. □

We will use Proposition 6.4.1 to extend the boundedness of the vector-valued variational Carleson operator from [3] to the weighted setting as well. Note that we only rely on Proposition 6.4.1 (and thus the main result of [66]) in the proof. Hence, we also provide an alternative way to prove the unweighted boundedness of the vector-valued variational Carleson operator, originally proven in [3].

Recalling that a family $\mathscr{I}$ of subsets of $\mathbb{R}$ is called an *interval partition* if it is a partition of $\mathbb{R}$ and each $I \in \mathscr{I}$ is an interval. For $s_0 \in [1,\infty)$ and $f \in \mathscr{S}(\mathbb{R};X)$ we define the *variational Carleson operator*

$$\mathscr{C}_*^{s_0} f(x) := \big[a \mapsto \mathscr{C}_a f(x)\big]_{V^{s_0}(\mathbb{R};X)} = \sup_{\mathscr{I}} \Big(\sum_{I\in\mathscr{I}} \|S_I f\|^{s_0}\Big)^{\frac{1}{s_0}}, \quad x \in \mathbb{R},$$

where the supremum is taken over all finite interval partitions $\mathscr{I}$ of $\mathbb{R}$.

**Theorem 6.4.2.** *Let $X$ be a $\theta$-intermediate* UMD *Banach space for some $\theta \in (0,1]$. Let $s_0 \in (2/\theta,\infty)$ and $p \in (s_0',\infty)$. There exists a non-decreasing function $\phi_{X,p,s_0} : [1,\infty) \to [1,\infty)$ such that for all $w \in A_{p/s_0'}$ and $f \in L^p(\mathbb{R},w;X)$,*

$$\big\|\mathscr{C}_*^{s_0} f\big\|_{L^p(\mathbb{R},w)} \le \phi_{X,p,s_0}([w]_{A_{p/s_0'}}) \big\|f\big\|_{L^p(\mathbb{R},w;X)}. \tag{6.4.4}$$

*Proof.* Let $n \in \mathbb{N}$, define $J_n := [-n,n]$ and

$$\mathscr{C}_{*,n}^{s_0} f(x) := \big[a \mapsto \mathscr{C}_a f(x)\big]_{V^{s_0}(J_n;X)}, \quad x \in \mathbb{R}.$$

By monotone convergence theorem, it suffices to prove the statement for $\mathscr{C}_{*,n}^{s_0} f$.

Fix $u > 2$. We start by proving an estimate on a Hilbert space $H$. Note that $H$ is isomorphic to $L^2$, then the $u'$-concavification $H^{u'}$ is a UMD Banach function space since $L^{2/u'}$ is a UMD Banach function space, see Section 2.6.2 for more details. For any $p_0 > u'$, by [2, Theorem 5.2] and Proposition 6.4.1, there exists a non-decreasing function $\phi_{H,p_0,u} : [1,\infty) \to [1,\infty)$ such that for all $v \in A_{\frac{p_0}{u'}}$ and $f \in L^{p_0}(\mathbb{R},v;H)$,

$$\big\|\mathscr{C}_{*,n}^{u} f\big\|_{L^{p_0}(\mathbb{R},v)} \le \phi_{H,p_0,u}([v]_{A_{p_0/u'}}) \big\|f\big\|_{L^{p_0}(\mathbb{R},v;H)},$$

and

$$\begin{aligned} \big\|(x,a) \mapsto \mathscr{C}_a f(x)\big\|_{L^{p_0}(\mathbb{R},v;V^u(J_n;H))} &\le \Big\|\big\|a \mapsto \mathscr{C}_a f(\cdot)\big\|_{L^\infty(J_n;H)}\Big\|_{L^{p_0}(\mathbb{R},v)} \\ &\quad + \Big\|\big[a \mapsto \mathscr{C}_a f(\cdot)\big]_{V^u(J_n;H)}\Big\|_{L^{p_0}(\mathbb{R},v)} \\ &\le \phi_{H,p_0,u}([v]_{A_{p_0/u'}}) \big\|f\big\|_{L^{p_0}(\mathbb{R},v;H)}. \end{aligned} \tag{6.4.5}$$

Let $X_0$ be a UMD Banach space, $H$ be a Hilbert space such that $X := [X_0,H]_\theta$. Set $\theta_0 := \frac{u}{s_0}$. Then by reiteration identity (2.7.1), there exists an $\eta \in (0,1)$ small enough such that $X_1 := [X_0,H]_\eta$ and

$$X = [X_0,H]_\theta = [X_1,H]_{\theta_0}.$$

Proposition 6.4.1 yields for all $p_1 \in (1,\infty)$, there exists a non-decreasing function $\phi_{X_1,p_1} : [1,\infty) \to [1,\infty)$ such that for $v \in A_{p_1}$, $f \in L^{p_1}(\mathbb{R}, v; X_1)$,

$$\big\| (x,a) \mapsto \mathscr{C}_a f(x) \big\|_{L^{p_1}(\mathbb{R},v;L^\infty(J_n;X_1))} \leq \phi_{X_1,p_1}([v]_{A_{p_1}}) \, \|f\|_{L^{p_1}(\mathbb{R},v;X_1)}. \tag{6.4.6}$$

Fix $w \in A_{p/s_0'}$, choose

$$p_0 := \tfrac{p}{s_0'} u' > u', \quad p_1 := \tfrac{p}{s_0'} > 1.$$

Then

$$\tfrac{1-\theta_0}{p_1} + \tfrac{\theta_0}{p_0} = \tfrac{1}{p}, \quad \text{and} \quad \tfrac{\theta_0}{u} = \tfrac{1}{s_0},$$

so by Lemma 6.2.4 and [71, Theorem 2.2.6] we have

$$[L^{p_1}(\mathbb{R}, w; L^\infty(J_n; X_1)), L^{p_0}(\mathbb{R}, w; V^u(J_n; H))]_{\theta_0} \subseteq L^p(\mathbb{R}, w; V^{s_0}(J_n; X)).$$

Hence, using complex interpolation between (6.4.5) and (6.4.6), we conclude that there is a non-decreasing function $\phi_{X,p,s_0} : [1,\infty) \to [1,\infty)$ such that for all $f \in L^p(\mathbb{R}, w; X)$,

$$\big\| (x,a) \mapsto \mathscr{C}_a f(x) \big\|_{L^p(\mathbb{R},w;V^{s_0}(J_n;X))} \leq \phi_{X,p,s_0}([w]_{A_{p/s_0'}}) \, \|f\|_{L^p(\mathbb{R},w;X)}.$$

6

□

## 6.5. MAIN RESULT

Having shown the required weighted, vector-valued variational Carleson estimate, we now turn to the proof of our main theorem, Theorem 6.5.4. We start with a corollary of the variational Carleson estimate in Theorem 6.4.2 and the Littlewood–Paley inequality in Proposition 6.2.3.

**Proposition 6.5.1.** *Let $X$ be a $\theta$-intermediate* UMD *Banach space for some $\theta \in (0,1]$. Assume that $X$ has cotype $q \in [2,\infty)$. Let $s \in (1, \frac{2}{2-\theta})$ and $p \in [s,\infty)$. Then there exists a non-decreasing function $\phi_{X,p,s} : [1,\infty) \to [1,\infty)$ such that for all $w \in A_{\frac{p}{s}}$ and $f \in L^p(\mathbb{R}, w; X)$,*

$$\Big\| \Big( \sum_{J\in\Delta} \sup_{\mathscr{I}_J} \Big( \sum_{I\in\mathscr{I}_J} \|S_I f\|_X^{s'} \Big)^{\frac{q}{s'}} \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R},w)} \leq \phi_{X,p,s}([w]_{A_{\frac{p}{s}}}) \, \|f\|_{L^p(\mathbb{R},w;X)}, \tag{6.5.1}$$

*where the supremum is taken over all finite interval partitions $\mathscr{I}_J$ of $J$.*

*Proof.* By Lemma 6.2.2, it suffices to prove the conclusion for $f \in \check{\mathscr{D}}(\mathbb{R}\setminus\{0\}; X)$, for which there are only finite non-zero terms in its Littlewood–Paley decomposition. Furthermore, by rescaled Rubio de Francia extrapolation (see [36, Corollary 3.14]), it suffices to prove the proposition for $p = q$ and $w \in A_{q/s}$. Indeed, by Fubini's theorem, Theorem

6.4.2 ($s < \frac{2}{2-\theta}$, then $s' > \frac{2}{\theta}$), cotype $q$ of $X$, Kahane-Khintchine inequality, and Proposition 6.2.3, there exists a non-decreasing function $\phi_{X,q,s} : [1,\infty) \to [1,\infty)$ such that

$$
\begin{aligned}
\Big\| \Big( \textstyle\sum_{J\in\Delta} \sup_{\mathcal{I}_J} & \big( \textstyle\sum_{I\in\mathcal{I}_J} \|S_I f\|_X^{s'} \big)^{\frac{q}{s'}} \Big)^{\frac{1}{q}} \Big\|_{L^q(\mathbb{R},w)} \\
&= \Big( \textstyle\sum_{J\in\Delta} \Big\| \sup_{\mathcal{I}_J} \big( \textstyle\sum_{I\in\mathcal{I}_J} \|S_I S_J f\|_X^{s'} \big)^{\frac{1}{s'}} \Big\|_{L^q(\mathbb{R},w)}^q \Big)^{\frac{1}{q}} \\
&\lesssim \Big( \textstyle\sum_{J\in\Delta} \Big\| \mathscr{C}_*^{s'} S_J f \Big\|_{L^q(\mathbb{R},w)}^q \Big)^{\frac{1}{q}} \\
&\lesssim \phi_{X,q,s}([w]_{A_{\frac{q}{s}}}) \Big( \textstyle\sum_{J\in\Delta} \|S_J f\|_{L^q(\mathbb{R},w;X)}^q \Big)^{\frac{1}{q}} \\
&\lesssim \phi_{X,q,s}([w]_{A_{\frac{q}{s}}}) \Big\| \big( \textstyle\sum_{J\in\Delta} \|S_J f\|_X^q \big)^{\frac{1}{q}} \Big\|_{L^q(\mathbb{R},w)} \\
&\lesssim \phi_{X,q,s}([w]_{A_{\frac{q}{s}}}) \Big\| \big\| \textstyle\sum_{J\in\Delta} \varepsilon_J S_J f \big\|_{L^q(\Omega;X)} \Big\|_{L^q(\mathbb{R},w)} \\
&\lesssim \phi_{X,q,s}([w]_{A_{\frac{q}{s}}}) \mathbb{E} \big\| \textstyle\sum_{J\in\Delta} \varepsilon_J S_J f \big\|_{L^q(\mathbb{R},w;X)} \\
&\lesssim \phi_{X,q,s}([w]_{A_{\frac{q}{s}}}) \|f\|_{L^q(\mathbb{R},w;X)},
\end{aligned}
$$

where $(\varepsilon_J)_{J\in\Delta}$ denotes a Rademacher sequence. □

We are now ready to prove an analogue of our main result for $R^s$-spaces, which will imply the corresponding result for $V^s$-spaces.

**Proposition 6.5.2.** *Let $X$ be a $\theta$-intermediate* UMD *Banach space and $Y$ be a* UMD *Banach space for some $\theta \in (0,1]$. Suppose that $X$ has cotype $q$ and $Y$ has type $t$ and set $\frac{1}{r} := \frac{1}{t} - \frac{1}{q}$. Let $s \in [1, \frac{2}{2-\theta})$ and $m \in \ell^r(R^s(\Delta;\mathscr{L}(X,Y)))$. Then for all $p \in (s,\infty)$, there exists a non-decreasing function $\phi_{X,Y,p,s}: [1,\infty) \to [1,\infty)$ such that for all $w \in A_{p/s}$,*

$$\|T_m\|_{\mathscr{L}(L^p(\mathbb{R},w;X),L^p(\mathbb{R},w;Y))} \le \phi_{X,Y,p,s}([w]_{A_{p/s}})\|m\|_{\ell^r(R^s(\Delta;\mathscr{L}(X,Y)))}.$$

*Remark* 6.5.3. Note that our assumptions in Proposition 6.5.2 imply $m$ has $\mathscr{R}$-bounded range by Proposition 6.3.2. Indeed, by interpolation, we see $\frac{1}{q} > \frac{\theta}{2}$, then $\frac{1}{r} < 1 - \frac{\theta}{2} < \frac{1}{s}$. Thus, we obtain $\ell^r(R^s(\Delta;\mathscr{L}(X,Y))) \subseteq \ell^r(R^r(\Delta;\mathscr{L}(X,Y)))$.

*Proof of Proposition 6.5.2.* By Lemma 6.2.2, it suffices to prove the conclusion for $f \in \check{\mathscr{D}}(\mathbb{R}\backslash\{0\};X)$, for which there are only finite non-zero terms in its Littlewood–Paley decomposition. By Fatou's lemma we may further assume for any $J \in \Delta$, $m|_J$ admits finite non-zero functions $a_{k,J} \in R^s(J;\mathscr{L}(X,Y)), 1 \le k \le n_J$ such that

$$m|_J = \sum_{k=1}^{n_J} \lambda_{k,J} a_{k,J}, \quad a_{k,J} = \sum_{I\in B_{k,J}} c_I \mathbf{1}_I,$$

where $\|m|_J\|_{R^s(J;\mathscr{L}(X,Y))} = \sum_{k=1}^{n_J} |\lambda_{k,J}| < \infty$, $c_I \in \mathscr{L}(X,Y)$, and $B_{k,J}$ is a family of mutually disjoint subintervals $I \subseteq J$.

Applying Proposition 6.2.3 and the type $t$ of $Y$ and $A_{p/s} \subseteq A_p$, we know there exists a non-decreasing function $\phi_{Y,p} : [1,\infty) \to [1,\infty)$ such that

$$\begin{aligned}
\|T_m f\|_{L^p(\mathbb{R},w;Y)} &= \|\mathscr{F}^{-1}\big(\textstyle\sum_{J\in\Delta} m|_J \mathbf{1}_J \widehat{f}\big)\|_{L^p(\mathbb{R},w;Y)} \\
&= \Big\|\textstyle\sum_{J\in\Delta} S_J\big(\sum_{k=1}^{n_J} \lambda_{k,J} \sum_{I\in B_{k,J}} c_I S_I f\big)\Big\|_{L^p(\mathbb{R},w;Y)} \\
&\lesssim \phi_{Y,p}([w]_{A_p})\mathbb{E}\Big\|\textstyle\sum_{J\in\Delta} \varepsilon_J S_J\big(\sum_{k=1}^{n_J} \lambda_{k,J} \sum_{I\in B_{k,J}} c_I S_I f\big)\Big\|_{L^p(\mathbb{R},w;Y)} \\
&\lesssim \phi_{Y,p}([w]_{A_{\frac{p}{s}}})\Big\|\Big(\textstyle\sum_{J\in\Delta} \big\|\sum_{k=1}^{n_J} \lambda_{k,J} \sum_{I\in B_{k,J}} c_I S_I f\big\|_Y^t\Big)^{\frac{1}{t}}\Big\|_{L^p(\mathbb{R},w)},
\end{aligned}$$

where $(\varepsilon_J)_{J\in\Delta}$ denotes a Rademacher sequence.

Further estimating by Hölder's inequality and Proposition 6.5.1, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{J\in\Delta}\Big\|\sum_{k=1}^{n_J}\lambda_{k,J}\sum_{I\in B_{k,J}} c_I S_I f\Big\|_Y^t\Big)^{\frac{1}{t}}\Big\|_{L^p(\mathbb{R},w)} \\
&\le \Big\|\Big(\sum_{J\in\Delta}\Big(\sum_{k=1}^{n_J}|\lambda_{k,J}|\sum_{I\in B_{k,J}}\|c_I\|_{\mathscr{L}(X,Y)}\|S_I f\|_X\Big)^t\Big)^{\frac{1}{t}}\Big\|_{L^p(\mathbb{R},w)} \\
&\le \Big\|\Big(\sum_{J\in\Delta}\Big(\sum_{k=1}^{n_J}|\lambda_{k,J}|\Big[\Big(\sum_{I\in B_{k,J}}\|c_I\|^s_{\mathscr{L}(X,Y)}\Big)^{\frac{1}{s}}\Big(\sum_{I\in B_{k,J}}\|S_I f\|_X^{s'}\Big)^{\frac{1}{s'}}\Big]\Big)^t\Big)^{\frac{1}{t}}\Big\|_{L^p(\mathbb{R},w)} \\
&\le \Big\|\Big(\sum_{J\in\Delta}\Big(\sum_{k=1}^{n_J}|\lambda_{k,J}|\Big(\sum_{I\in B_{k,J}}\|S_I f\|_X^{s'}\Big)^{\frac{1}{s'}}\Big)^t\Big)^{\frac{1}{t}}\Big\|_{L^p(\mathbb{R},w)} \\
&\le \Big\|\Big(\sum_{J\in\Delta}\Big(\Big(\sum_{k=1}^{n_J}|\lambda_{k,J}|\Big)\cdot\Big(\sup_{1\le k\le n_J}\sum_{I\in B_{k,J}}\|S_I f\|_X^{s'}\Big)^{\frac{1}{s'}}\Big)^t\Big)^{\frac{1}{t}}\Big\|_{L^p(\mathbb{R},w)} \\
&\le \Big\|\Big(\sum_{J\in\Delta}\|m|_J\|^t_{R^s(J;\mathscr{L}(X,Y))}\cdot\sup_{1\le k\le n_J}\Big(\sum_{I\in B_{k,J}}\|S_I f\|_X^{s'}\Big)^{\frac{t}{s'}}\Big)^{\frac{1}{t}}\Big\|_{L^p(\mathbb{R},w)} \\
&\le \|m\|_{\ell^r(R^s(\Delta;\mathscr{L}(X,Y)))}\Big\|\Big(\sum_{J\in\Delta}\sup_{1\le k\le n_J}\Big(\sum_{I\in B_{k,J}}\|S_I f\|_X^{s'}\Big)^{\frac{q}{s'}}\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R},w)} \\
&\le \phi_{X,p,s}([w]_{A_{\frac{p}{s}}})\|m\|_{\ell^r(R^s(\Delta;\mathscr{L}(X,Y)))}\big\|f\big\|_{L^p(\mathbb{R},w;X)},
\end{aligned}$$

with the usual modification for $r=\infty$. Combined with the previous estimates, this finishes the proof. □

As announced, our main result is now a consequence of Proposition 6.5.2 and Lemma 6.2.6.

**Theorem 6.5.4.** *Let $X$ be a $\theta$-intermediate* UMD *Banach space and $Y$ be a* UMD *Banach space for some $\theta\in(0,1]$. Suppose that $X$ has cotype $q\in[2,\infty]$ and $Y$ has type $t\in[1,2]$ and set $\frac{1}{r} := \frac{1}{t}-\frac{1}{q}$. Let $s\in[1,\frac{2}{2-\theta})$. Assume that $m\in\ell^r(\dot{V}^s(\Delta;\mathscr{L}(X,Y)))$ has $\mathscr{R}$-bounded range. Then for all $p\in(s,\infty)$ there exists a non-decreasing function $\phi_{X,Y,p,s}\colon[1,\infty)\to[1,\infty)$ such that for all $w\in A_{p/s}$,*

$$\|T_m\|_{\mathscr{L}(L^p(\mathbb{R},w;X),L^p(\mathbb{R},w;Y))} \le \phi_{X,Y,p,s}([w]_{A_{p/s}})\big(\|m\|_{\ell^r(\dot{V}^s(\Delta;\mathscr{L}(X,Y)))}+\mathscr{R}(\mathrm{Ran}(m))\big).$$

*Proof.* Denote the restriction of $m$ on $J$ by $m_J$. By Fatou's lemma, it suffices to consider finite non-zero terms of $m_J$ for $J \in \Delta$. Then by Lemma 6.2.6, for every $m_J$, there exists an operator $N_J \in \overline{\operatorname{conv}}(\operatorname{Ran}(m|_J)) \subseteq \overline{\operatorname{conv}}(\operatorname{Ran}(m))$ such that

$$\|m_J - N_J\|_{R^{s_1}(J;\mathcal{L}(X,Y))} \lesssim [m_J]_{V^s(J;\mathcal{L}(X,Y))}.$$

Define $N(\cdot) := \sum_{J\in\Delta} N_J \mathbf{1}_J(\cdot)$. By Lemma 6.2.1, for every $w \in A_{\frac{p}{s}}$, there exists $s_1 \in (s, \frac{2}{2-\theta} \wedge p)$ depending on $p, s, [w]_{A_{\frac{p}{s}}}$ such that $w \in A_{\frac{p}{s_1}}$ and $[w]_{A_{\frac{p}{s_1}}} \lesssim_{p,s} [w]_{A_{\frac{p}{s}}}$. Then by Proposition 6.5.2, there exists a non-decreasing function $\phi_{X,Y,p,s} : [1,\infty) \to [1,\infty)$ such that

$$\begin{aligned}
\|T_{m-N}\| &\leq \phi_{X,Y,p,s}([w]_{A_{p/s_1}})\Big(\sum_{J\in\Delta} \|m_J - N_J\|^r_{R^{s_1}(J;\mathcal{L}(X,Y))}\Big)^{\frac{1}{r}} \\
&\leq \phi_{X,Y,p,s}([w]_{A_{p/s}})\Big(\sum_{J\in\Delta} [m_J]^r_{V^s(J;\mathcal{L}(X,Y))}\Big)^{\frac{1}{r}} \\
&= \phi_{X,Y,p,s}([w]_{A_{p/s}})\|m\|_{\ell^r(\dot{V}^s(\Delta;\mathcal{L}(X,Y)))}.
\end{aligned}$$

On the other hand, since $\operatorname{Ran}(m)$ is $\mathscr{R}$-bounded, by (6.3.1), $\overline{\operatorname{conv}}(\operatorname{Ran}(m))$ is also $\mathscr{R}$-bounded with $\mathscr{R}(\overline{\operatorname{conv}}(\operatorname{Ran}(m))) = \mathscr{R}(\operatorname{Ran}(m))$. Then Proposition 6.2.3 yields

$$\begin{aligned}
\|T_N f\|_{L^p(\mathbb{R},w;Y)} &\lesssim_{Y,p,s} \phi_{X,Y,p,s}([w]_{A_{p/s}})\mathbb{E}\Big\|\sum_{J\in\Delta} \varepsilon_J \mathscr{F}^{-1}\mathbf{1}_J\Big(\sum_{I\in\Delta} N_I \mathbf{1}_I \widehat{f}\Big)\Big\|_{L^p(\mathbb{R},w;Y)} \\
&\lesssim_{Y,p,s} \phi_{X,Y,p,s}([w]_{A_{p/s}})\mathbb{E}\Big\|\sum_{J\in\Delta} \varepsilon_J N_J S_J f\Big\|_{L^p(\mathbb{R},w;Y)} \\
&\lesssim_{X,Y,p,s} \phi_{X,Y,p,s}([w]_{A_{p/s}})\mathscr{R}(\operatorname{Ran}(m))\|f\|_{L^p(\mathbb{R},w;X)}.
\end{aligned}$$

Combining the above estimates yields the conclusion. □

Note that Fourier type $t$ implies type $t$ and cotype $t'$, we then have the following corollary when $Y = X$.

**Corollary 6.5.5.** *Let $X$ be a $\theta$-intermediate* UMD *Banach space for some $\theta \in (0,1]$. Assume that $X$ has Fourier type $t \in [1,2]$. Let $\frac{1}{r} := \frac{1}{t} - \frac{1}{t'}$ and $s \in [1, \frac{2}{2-\theta})$. Assume that $m \in \ell^r(\dot{V}^s(\Delta;\mathcal{L}(X)))$ has $\mathscr{R}$-bounded range. Then for all $p \in (s,\infty)$, there exists a non-decreasing function $\phi_{X,p,s} : [1,\infty) \to [1,\infty)$ such that for all $w \in A_{p/s}$,*

$$\|T_m\|_{\mathcal{L}(L^p(\mathbb{R},w;X))} \leq \phi_{X,p,s}([w]_{A_{p/s}})\left(\|m\|_{\ell^r(\dot{V}^s(\Delta;\mathcal{L}(X)))} + \mathscr{R}(\operatorname{Ran}(m))\right).$$

## 6.6. COMPARISON AND EXAMPLE

As mentioned in the introduction, this chapter is motivated by the strategy of [1]. In this section, we compare our result to [1] and show an application to Theorem 6.5.4.

### 6.6.1. COMPARISON

We first recall the statement of [1, Theorem 5.18].

**Theorem 6.6.1.** *Let $p \in (1,\infty)$ and $\theta \in (0,1)$. Let $Y$ and $H$ be an interpolation couple, with $Y$ a* UMD *Banach space, $H$ a Hilbert space, and $Y \cap H$ dense in both $Y$ and $H$. Let $X = [Y,H]_\theta$. Suppose $\mathscr{T} \subseteq \mathscr{L}(Y \cap H)$ is a Banach disc which is $\mathscr{R}$-bounded on $Y$ and uniformly bounded on $H$. Let $s \in (1,\infty)$ satisfy $\frac{1}{s} > \max\{\frac{1}{p}, 1-\frac{\theta}{2}\}$ and $m \in \ell^\infty(V^s(\Delta,\mathscr{T}))$. Then for all $w \in A_{p/s}$, there exists a non-decreasing function $\phi_{Y,p,s,\theta} : [1,\infty) \to [1,\infty)$ such that*

$$\|T_m\|_{\mathscr{L}(L^p(w,X))} \le \phi_{Y,p,s,\theta}([w]_{A_{p/s}})\|m\|_{\ell^\infty(V^s(\Delta,\mathscr{T}))}[\mathscr{T}]_{\mathscr{R}}.$$

Theorem 6.5.4 and Theorem 6.6.1 both study the boundedness of the multiplier operator $T_m$ on weighted vector-valued $L^p$-spaces, under the assumptions that the multiplier $m$ lies in a bounded $s$-variation space and has $\mathscr{R}$-bounded range. Their key differences are summarized as follows:

- Space assumptions. In Theorem 6.5.4, we allow $m$ to be an operator-valued multiplier in $\mathscr{L}(X,Y)$ with $X, Y$ being two (possibly distinct) UMD Banach spaces and only $X$ needs to be an interpolation space. In contrast, Theorem 6.6.1 restricts to the case $X = Y$, where $X$ is explicitly assumed to be a complex interpolation space between a UMD space and a Hilbert space.

- Variation and summability. Both theorems assume a bounded $s$-variation condition on the multiplier, with $s > 1$ satisfying $\frac{1}{s} > \max\{\frac{1}{p}, 1-\frac{\theta}{2}\}$. However, Theorem 6.5.4 highlights the role of Banach space geometry by requiring $m \in \ell^r(\dot{V}^s(\Delta;\mathscr{L}(X,Y)))$, where $r$ is determined by the cotype and type properties of $X$ and $Y$. In contrast, Theorem 6.6.1 assumes $r = \infty$, which is a weaker assumption.

- Range assumptions. Theorem 6.5.4 assumes that the range of $m$ is $\mathscr{R}$-bounded. In Theorem 6.6.1, the range of $m$ lies in a Banach disc that is $\mathscr{R}$-bounded on $Y$ and uniformly bounded on the associated Hilbert space $H$.

We next consider a special case of Theorem 6.5.4, i.e. $X$ is a Lebesgue space.

**Proposition 6.6.2.** *Let $u \in (1,\infty)$, $r = \frac{2u}{|2-u|}$ and $r = \infty$ if $u = 2$. Let $s \in (1, u \wedge u')$, $p \in (s,\infty)$, $w \in A_{\frac{p}{s}}$. Assume that $m \in \ell^r(\dot{V}^s(\Delta;\mathscr{L}(L^u(S))))$ has $\mathscr{R}$-bounded range. Then $m$ is a $L^p(\mathbb{R},w;\mathscr{L}(L^u(S)))$-Fourier multiplier.*

*Proof.* Note that by [72, Proposition 7.1.4], $L^u(S)$ has type $u \wedge 2$ and cotype $u \vee 2$. For any $u > 2$, there exists $2 < u < v < \infty, \theta \in (0,1)$ such that $L^u(S) = [L^v(S), L^2(S)]_\theta$ where $\frac{1}{u} = \frac{1-\theta}{v} + \frac{\theta}{2} > \frac{\theta}{2}$. By Theorem 6.5.4 we see $r = \frac{2u}{u-2}$ and $s < \frac{2}{2-\theta} < \frac{2u}{2u-2} = u'$. The case $u = 2$ is directly from Theorem 6.5.4 with $\theta = 1$. Finally, if $u < 2$, there exists $1 < v < u < 2, \theta \in (0,1)$ such that $L^u(S) = [L^v(S), L^2(S)]_\theta$ where $\frac{1}{u} = \frac{1-\theta}{v} + \frac{\theta}{2} < 1 - \frac{\theta}{2}$. By Theorem 6.5.4 we see $r = \frac{2u}{2-u}$ and $s < \frac{2}{2-\theta} < \frac{2u}{2u-2u+2} = u$. □

*Remark* 6.6.3. Comparing Proposition 6.6.2 with [1, Example 5.12], we observe the following:

- When $u = 2$, our result closely parallels [1, Example 5.12(i-a)], with the exception that the endpoint case $p = s$ is not included in our setting.

- When $u > 2$, our assumptions on the $p$ and $w$ are weaker than those in [1, Example 5.12(ii-a)]. However, we further assume $m \in \ell^{\frac{2u}{u-2}}(\dot{V}^s(\Delta;\mathscr{L}(L^u(S))))$ for $s \in [1,u')$ and $m$ has $\mathscr{R}$-bounded range. Regarding the range condition, [1] assumes $\ell^2$-boundedness of the multiplier range, which is equivalent to the $\mathscr{R}$-boundedness.
- When $u < 2$, our assumptions on $p$, $s$, and $w$ coincide with those in [1, Example 5.12(iii-b)]. The key difference lies in the range condition: we assume that the multiplier $m \in \ell^{\frac{2u}{2-u}}(\dot{V}^s(J;\mathscr{L}(L^u(S))))$ has $\mathscr{R}$-bounded range, whereas [1] imposes the stronger $\ell^2(\ell^{s'})$-boundedness condition.

### 6.6.2. EXAMPLE

At the end of this chapter, we show that for $s<2$, a $\frac{1}{s}$-Hölder continuous function $m$ is a Fourier multiplier if it has $\mathscr{R}$-bounded range and its restriction on each $J \in \Delta$ satisfies a decay condition in terms of the length of $J$.

**Proposition 6.6.4.** *Let $X$ be a $\theta$-intermediate* UMD *Banach space and $Y$ be a* UMD *Banach space for some $\theta \in (0,1]$. Suppose that $X$ has cotype $q \in [2,\infty]$ and $Y$ has type $t \in [1,2]$ and set $\beta > \frac{1}{r} := \frac{1}{t} - \frac{1}{q}$. Let $s \in [1, \frac{2}{2-\theta})$, $p \in (s,\infty)$ and $w \in A_{\frac{p}{s}}$. Assume that $m : \mathbb{R} \to \mathscr{L}(X,Y)$ has $\mathscr{R}$-bounded range and for each $J \in \Delta$, $m|_J$ is in $\in C^{\frac{1}{s}}(J;\mathscr{L}(X,Y))$ and*

$$\|m|_J\|_{C^{\frac{1}{s}}(J;\mathscr{L}(X,Y))} \le \frac{1}{1+|\log_2|J||^{\beta}}.$$

*Then $m$ is a $L^p(\mathbb{R},w;\mathscr{L}(X,Y))$-Fourier multiplier.*

*Proof.* It suffices to prove that $m \in \ell^r(\dot{V}^s(\Delta;\mathscr{L}(X,Y)))$, then the conclusion follows from Theorem 6.5.4. Let $J_n^{\pm} := \pm[2^n, 2^{n+1})$, $m_{1,n}(\cdot) := m|_{J_n^+}(\cdot)$ and $m_{2,n}(\cdot) := m|_{J_n^-}(\cdot)$. By assumption we can write $m = \sum_{n\in\mathbb{Z}}(\mathbf{1}_{J_n^+}m_{1,n} + \mathbf{1}_{J_n^-}m_{2,n})$. Define $\phi_n, \psi_n : [1,2] \to \mathscr{L}(X,Y)$ as

$$\phi_n(y) := m_{1,n}(t), \quad y := 2^{-n}t,\ t \in J_n^+,$$

and

$$\psi_n(y) := m_{2,n}(t), \quad y := -2^{-n}t,\ t \in J_n^-.$$

Then $\phi_n, \psi_n \in C^{\frac{1}{s}}([1,2];\mathscr{L}(X,Y))$ and

$$\|\phi_n\|_{C^{\frac{1}{s}}([1,2];\mathscr{L}(X,Y))}, \|\psi_n\|_{C^{\frac{1}{s}}([1,2];\mathscr{L}(X,Y))} \le \frac{1}{1+|n|^{\beta}}.$$

By direct calculation,

$$\begin{aligned}
[m]_{C^{\frac{1}{s}}(J_n^+;\mathscr{L}(X,Y))} = [m_{1,n}]_{C^{\frac{1}{s}}(J_n^+;\mathscr{L}(X,Y))} &= \sup_{t_1,t_2\in J_n^+} \frac{\|m_{1,n}(t_1)-m_{1,n}(t_2)\|}{|t_1-t_2|^{\frac{1}{s}}} \\
&= \sup_{t_1,t_2\in J_n^+} \frac{\|\phi_n(2^{-n}t_1)-\phi_n(2^{-n}t_2)\|}{|t_1-t_2|^{\frac{1}{s}}} \\
&\le \sup_{y_1,y_2\in[1,2]} \frac{\|\phi_n(y_1)-\phi_n(y_2)\|}{2^{\frac{n}{s}}|y_1-y_2|^{\frac{1}{s}}} = 2^{-\frac{n}{s}}[\phi_n]_{C^{\frac{1}{s}}([1,2];\mathscr{L}(X,Y))}.
\end{aligned}$$

On the other hand, let $P_n = \{(t_0, \cdots, t_N) \mid N \in \mathbb{N}, t_{j-1} < t_j, j = 1, \cdots, N\}$ is a finite partition of $J_n^+$. Since $C^{\frac{1}{s}}(J_n^+; \mathscr{L}(X,Y)) \subseteq V^s(J_n^+; \mathscr{L}(X,Y))$, we get

$$[m]_{V^s(J_n^+;\mathscr{L}(X,Y))} \leq [m]_{C^{\frac{1}{s}}(J_n^+;\mathscr{L}(X,Y))} |J_n^+|^{\frac{1}{s}} \leq [\phi_n]_{C^{\frac{1}{s}}([1,2];\mathscr{L}(X,Y))} \lesssim \frac{1}{1+|n|^\beta}.$$

By a similar calculation, we also have

$$[m]_{V^s(J_n^-;\mathscr{L}(X,Y))} \leq [\psi_n]_{C^{\frac{1}{s}}([1,2];\mathscr{L}(X,Y))} \lesssim \frac{1}{1+|n|^\beta}.$$

Therefore,

$$\|m\|^r_{\ell^r(\dot{V}^s(\Delta;\mathscr{L}(X,Y)))} = \sum_{n\in\mathbb{Z}} [m|_{J_n^\pm}]^r_{V^s(J_n^\pm;\mathscr{L}(X,Y))} \lesssim \sum_{n\in\mathbb{Z}} \left(\frac{1}{1+|n|^\beta}\right)^r < \infty.$$

□

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# Curriculum Vitæ

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