Evolution of the opinion dynamics of N populations

by

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Lay Summary

In many situations, when we have a group of people, they all form opinions on a subject. Everyone in a population influences each others opinion. Naturally, people how the opinion of children are influenced by other children differs from how adults influence their opinions. Between different age groups there are all kind of different interactions.

This thesis aims to model how such opinions evolve and influence each other over time. First We assume an individual can either have a negative or positive opinion on a subject. To model this, we use the Ising model, originally developed for the description of magnetism in metals. We model opinion changes as random processes influenced by the opinion of other individuals in the population. We divide the population into subgroups of people who interact similarly. In this thesis, we prove that if we make the total group of people larger and larger, this random process becomes a deterministic process. Just like when you flip a coin infinitely many times, you end up with heads 50% of the time. We then determine how the different populations influence each other's opinions. Understanding group opinion dynamics can help explain the spread of misinformation on social media, the emergence and disappearance of political parties, or how companies can predict or start trends.

Summary

In this thesis we show we can model population opinion dynamics, where we split the population into a general number of N groups. For this we first used a variant of the Ising model, used to describe magnetism in metals, for the opinions. We found that we could describe the opinion model using the Hamiltonian given by:

$$H(\sigma) = \sum_{j=1}^{N} \sum_{\sigma_i | i \in I_j} \sigma_i [\beta_{1j} m_n(\sigma)_1 \rho_1 + \cdots + \beta_{Nj} m_n(\sigma)_N \rho_N]$$

We used this Hamiltonian to determine the flip rate of the opinions given by:

$$q_{ji} = e^{-\beta \Delta H}.$$

.

We used this flip rate to describe the opinion model as a Markov process. We proved that this Markov process converges to the deterministic solution of the N-dimensional system of differential equations given by:

$$\begin{cases} \frac{d\vec{x}}{dt} \\ x(0) = x_0, \end{cases}$$

where $F(\vec{x})$ is the N-dimensional vector given by:

$$F(\vec{x}) = \begin{pmatrix} (1-x_1)e^{\Gamma_N} - (1+x_N)e^{-\Gamma_1} \\ \vdots \\ (1-x_N)e^{\Gamma_N} - (1+x_N)e^{-\Gamma_N} \end{pmatrix}$$

Depending on the strength of $F(\vec{x})$ we see different behavior in the limiting ordinary differential equation. We did local bifurcation analysis on the two-dimensional system around the origin. For the two-dimensional model we found that the following conditions hold the system undergoes an Hopf bifurcation at $(0,0)$

$$\begin{aligned} &\alpha_{11} + \alpha_{22} = 2, \\ &(\alpha_{11} - \alpha_{22})^2 + \alpha_{12}\alpha_{21} < 0. \end{aligned}$$

Finally we showed that the findings of the bifurcation analysis were correct by performing numerical analysis for a three different systems.

Contents

1 Introduction

The analysis of opinion dynamics is a field of study which gets increasingly more important in the modern era. With the internet people get connected and influence one another more than ever before. Understanding the evolution of the opinion of different groups of people in society helps us predict the next social media hit or fashion trend. Naturally this is very useful information for companies. But, it is also important for governments to understand population dynamics. It can be used to predict and stop the spread of fake news and propaganda.

To help develop a model for opinion dynamics we take a look at a model from statistical physics. Ferromagnets are a class of metals, with the special property that they can become magnetic, even in the absence of an external magnetic field. This phenomenon can be explained using the quantum mechanical descriptions of atoms. Each electron in each atom of the metal has a small magnetic field within it. This magnetic field can be in two directions: up or down. In the initial state of the metal, all the electrons within it are pointing in random directions, giving a mean zero magnetic field(Because opposite spins cancel each other out). When the metal is put into an external magnetic field, all the electrons will align with this magnetic field. After turning off the external magnetic field the electrons will stay in this aligned position, causing them to create their own magnetic field and therefore causing the metal to become magnetic.

In the twenties of the twentieth century a mathematical model to describe the behavior of the interacting spins in the ferromagnets was developed. It was developed by Ersnt Ising under the supervision of Wilhelm Lenz. The former being the inspiration of the name of the so-called Ising model. In the Ising model, each electron is assigned one of the values -1 or +1, representing the up or down state respectively. These spins(or electrons) are positioned on a lattice, with each spin only interacting with its nearest neighbors. Neighboring spins with the same direction have a low energy and neighboring spins with opposite directions have a high energy. The total energy of the system is given by the following Hamiltonian:

$$H = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j - \sum_j h_j\sigma_j \tag{1.1}$$

Here the sums are taken over all pairs of neighboring electrons in the model, σ_j represents the state of the spin, the electron interaction coefficient is given by J_{ij} and the effect of an external magnetic field is given by h_j. In this thesis we will only look at situations without an external magnetic field. Thus, from now on we will set $h_j = 0$. This version of the model is shown in Figure 1.1 In this version of the model particles are only influenced by adjacent particles. The next assumption we make is that the model is location independent, i.e., each particles influences every other particle.

Later on in the twentieth century Pierre Curie and Pierre Weiss developed a simplification of the Ising model, called the Curie-Weiss model. This mathematical model is a mean-field approximation, which assumes that the interaction between the spins are symmetric and that the behavior of each spin is influenced by the mean behavior of all the other spins in the system. This system is shown in Figure 1.2 and has the following Hamiltonian:

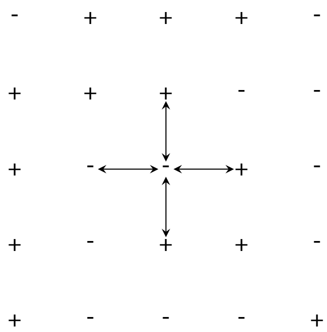


Figure 1.1: Schematic view of the Ising Model, where each particle only interacts with its direct neighbour.

$$H(\sigma) = -\frac{\beta}{N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - \sum_{j=1}^{N}h_j\sigma_j \tag{1.2}$$

The Curie-Weiss model assumes that all electrons in the system interact with each other in the same manner.

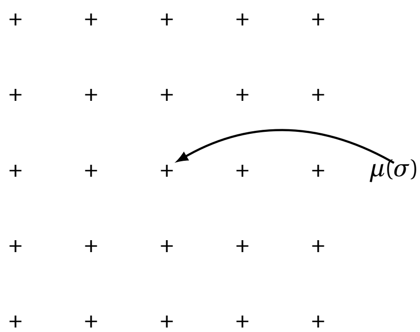


Figure 1.2: Schematic view of the Ising model, where each particle is only influenced by the mean field of the entire population.

This is of course not always applicable. For this we introduce our model which is the main subject matter of this thesis: The N-population Curie-Weiss model. This model assumes that all the electrons in the system are dividable in N populations. Each population i has a set of interaction factors $\{\beta_{ij}\}_{j=1,..n}$ for its interaction with all the populations. This model is shown in Figure 1.3. The Hamiltonian of one of these populations is given by:

$$H(I_j) = -\sum_{i=1}^{n_j}\sigma_{ji}\sum_{k=1}^{N}\beta_{kj}\mu(\sigma_k)\rho_k \tag{1.3}$$

In this equation, σ_{ji} represents the spin of the i-th electron in population I_j, n_j represents the size of population I_j, β_{kj} represents the influence of population I_k on populations I_j, $\mu(x)$ is the mean function and ρ_k is the proportion of population I_k of the total population.
You can find that model lies somewhere in between the Curie-Weiss and Ising model. When we have $N = 1$ populations we just get the Curie-Weiss model and when we take that there are only populations of size 1, we get a Ising model if we say that only neighboring populations have a $\beta_{ij} \neq 0$.

Now that we have defined the model we can go back to the subject of population opinion dynamics with an example. Suppose we have research the opinion of a population of 100.000 people on a social media trend over time. Each person in the population has either a positive or a negative opinion on the social media trend. Let's divide this total population in to three groups. The first group consists only of children, the second group are the young adults, and the final group contains the people over 30 years old.
Now suppose there is a new social media app on the market. Initially a few people in the young adult group

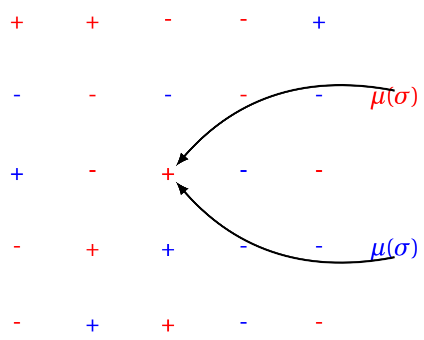


Figure 1.3: Schematic view of the 2-dimensional Ising model where each particle is only influenced by the mean field of each of the two populations.

use the app. So, they have a positive opinion about the app. The other groups don't know about the app yet, so their opinion will be negative. The new app starts spreading within the young adult group and peoples opinion start changing to positive. Then the young brothers and sisters of the young adult group start using the app, so the app spreads to the children group. Now also the opinion in the children group starts flipping to positive. Some of the parents of these children see their children using the app and start using it themselves. But it doesn't really spread within in the 30+ group, because the 30+'ers don't have time to talk about the app with other people.
Finally the young adults start noticing the 30+'ers using the app. The young adults stop using the app, because it's not cool anymore since there are parents on it. This starts the rapid flip to a negative opinion in the young adult group. For social media companies it is really important to know what causes the flips of the opinions of the groups. If they can predict why and when the happen they can act on them.

Let's continue back to the subject of this paper. The opinions of the populations are the spins, the effect between and within the populations are the interaction coefficients, the flips of the populations are phase transitions. Our model can be used to predict those big changes of opinions of the groups. This example shows the importance of studying the phase transitions. It also show that expanding our model to arbitrary N group, is important. The more groups the better you can describe the real situation. For example we could expand the model to separate between sociological generations, like: generation X, the babyboomers, the millennials etc.

In this thesis we will derive a time evolution of the N-populations Curie-Weiss model. We start this in Chapter 2, by introducing some necessarily foundational mathematics. In Chapter 3 we develop the 2-dimensional mean-field Ising model and show its convergence to a two-dimensional system of ordinary differential equations. In Chapter 4 we build upon this and expand the model and the convergence to a general number N. In Chapter 5 we will analyze the local bifurcations at the equilibrium point at the origin of the two- and three-dimensional model. Then in Chapter 6, we will discuss the results. Finally in Chapter 7 we will show the conclusion.

2

Markov processes and generators

In this chapter we will introduce the reader to some relevant background theory, which is necessary to understand the results shown in the report. In the first section we will discuss Markov processes, generators and other related concepts. In the second section of this chapter we will give two theorems used in this thesis.

2.1. Markov processes and Generators

Let's look at a discrete time process $\{X_n\}_{n=1}^{\infty}$. Now let us assume that each state X_m depends solely on its previous state X_{m-1}. This kind of process is called a Markov process. A formal definition is given by[5]:

Definition 2.1 (Markov Process). *Let $X = \{X_n\}$ be a stochastic process for $n = 1,2,3,...,$ with state space S. We say that X is a discrete-time Markov process if for all n:*

$$\mathbb{P}(X_n|X_{n-1}, X_{n-2}, ..., X_1) = \mathbb{P}(X_n|X_{n-1}) \tag{2.1}$$

We want to extend the discrete time Markov process to a continuous time Markov process. For this we introduce the concept called the holding time. This is the time it takes for a system in state A to jump to any other state. The holding time is exponentially distributed with rate λ_A. This rate is given by:

$$\lambda_A = \sum_{B \in S \setminus \{A\}} w(A \to B) \tag{2.2}$$

Where the sum goes over all possible states. Here $w(i \to j)$ is the flip rate from state i to state j, given by[5]:

$$w(A \to B) = \frac{\partial \mathbb{P}(X_t = B|X_0 = A)}{\partial t} \tag{2.3}$$

This allows us toe define the continuous time Markov Process.

Definition 2.2 (Continuous-Time Markov Process). *Let $x(t)$ be a stochastic process with state space S. This process is called Markov if for all for all $0 < s < t$, we have:*

$$\mathbb{P}(x(t+s)|x(r)\ \forall r \leq t) = \mathbb{P}(x(t+s)|x(t)) \tag{2.4}$$

We say that a Markov Process is time-homogeneous if:

$$\mathbb{P}(x(t+s) = B|x(t) = A) = \mathbb{P}(x(t) = B|x(0) = A) \tag{2.5}$$

We then denote this probability by $p_{AB}(t)$.
Next up we will introduce the master equations. This theorem allows us to say something about the time-dynamics of the system.

Theorem 2.1 (Master Equation). *Suppose we have a continuous time Markov process X with state space S. Let $P_A(t)$ the probability that the system is in state A at time t, then the following holds:*

$$\frac{dP_A(t)}{dt} = \sum_{B \in S \setminus \{A\}} w(B \to A) P_B(t) - w(A \to B) P_A(t).$$

Each Markov process has a corresponding semigroup. We say that a Markov process $X = \{X(t)\}$ corresponds to a semigroup $\{T(t)\}$ on the function space $C_b(x)$ if for all $f \in L$, $s, t \geq 0$ and all x in the state space of X, we have: [2]

$$\mathbb{E}[f(X(t+s))|X(t) = x] = T(s)f(x) \tag{2.6}$$

The generator of a semigroup $\{T(t)\}$ on L is the operator $\mathscr{L}$ with domain $\mathscr{D}(\mathscr{L})$ for which this limit exists is defined by: [2]

$$\mathscr{L}f = \lim_{t\to 0}(T(t)f - f). \tag{2.7}$$

Using the definitions of the generator and semigroup we can use the following lemma to find the generator of the Markov Process:

Lemma 2.2. *For the generator $\mathscr{L}$ corresponding to the time-homogeneous Markov process X is given by:*

$$\mathscr{L}f(A) = \sum_{B\in S\setminus\{A\}} w(A\to B)(f(B) - f(A))$$

Proof of Lemma 2.2. Let $\{T(t)\}_{t\leq 0}$ be the semigroup corresponding to the markov process X with state space S, then:

$$T(t)f(A) := \mathbb{E}_i(f(X_t)) = \mathbb{E}(f(X_t)|X_0 = A), \tag{2.8}$$

Using the definition of the generator we can find the statement we set out to prove:

$$\begin{aligned}\mathscr{L}f(A) &= \lim_{t\to 0}\frac{T(t)f(A) - f(A)}{t}\\ &= \lim_{t\to 0}\frac{\mathbb{E}(f(X_t)|X_0 = A) - f(A)}{t}\\ &= \lim_{t\to 0}\frac{\sum_{B\in S}\mathbb{P}(X_t = B|X_0 = A)f(B) - f(A)}{t}\end{aligned}$$

Substituting $\sum_{B\in S}\mathbb{P}(X_t = B|X_0 = A) = 1$ in this equation gives:

$$\mathscr{L}f(A) = \lim_{t\to 0}\frac{\sum_{B\in S}\mathbb{P}(X_t = B|X_0 = A)}{t}(f(B) - f(A)).$$

Now, using that $f(B) - f(A) = 0$ if $B = A$, we get:

$$\mathscr{L}f(A) = \lim_{t\to 0}\frac{\sum_{B\in S\setminus\{A\}}\mathbb{P}(X_t = B|X_0 = A)}{t}(f(B) - f(A))$$

We know that $\mathbb{P}(X_0 = B|X_0 = A) = 0$ if $B \neq A$. Adding this into the fraction gives:

$$\mathscr{L}f(A) = \lim_{t\to 0}\frac{\sum_{B\in S\setminus\{A\}}\mathbb{P}(X_t = B|X_0 = A) - P(X_0 = B|X_0 = A)}{t}(f(B) - f(A))$$

Now when we move the limit inside of the sum, we get a partial derivative:

$$\mathscr{L}f(A) = \sum_{B\in S\setminus\{A\}}\left.\frac{\partial\mathbb{P}(X_t = B|X_0 = A)}{\partial t}\right|_{t=0}(f(B) - f(A))$$

Finally using Equation 2.3, this partial derivative is the flip rate and we get:

$$\mathscr{L}f(A) = \sum_{B\in S\setminus\{A\}} w(A\to B)(f(B) - f(A))$$

Finally, if $B = A$, then $f(B) - f(A) = 0$, so we get:

$$\mathscr{L}f(A) = \sum_{B\in S\setminus A} w(A\to B)(f(B) - f(A)).$$

□

Now we have defined all the terms associated with the Markov Process, we would like the introduce a final theorem. This theorem shows that we can use the convergence of generators to prove the convergence of the corresponding Markov process .

Theorem 2.3. *Let E_n be a discrete spaces and define the map $\eta_n : E_n \to \mathbb{R}^N$. Let X_n be a Markov process on E_n with generator $(A_n, \mathscr{D}(A_n))$ on $C_b(E_n)$. Let Y be a Markov process on $\mathbb{R}^N$ with generator $(A, \mathscr{D}(A))$ on $C_b(\mathbb{R}^N)$. Set $Y_n = \eta_n(X_n)$ and suppose:*

$$a)\ Y_n(0) \to Y(0) \tag{2.9}$$

$$b)\ \forall f \in \mathscr{D}(A) : \lim_n \sup_{x \in E_n} \left| A_n(f \circ \eta_n)(x) - Af(\eta_n(x)) \right| = 0 \tag{2.10}$$

Then for all t $Y_n(t)$ converges to $Y(t)$ in distribution. [2]

2.2. Other Theorems

Theorem 2.4 (Taylor's Theorem[1])**.** *If $f : \mathbb{R}^N \to \mathbb{R}$ is a real-valued function of class C^2 on the open set containing the line segment L from $\vec{x}$ to $\vec{x} + \vec{h}$, then there exists a point $\theta \in L$ such that:*

$$f\left(\vec{x} + \vec{h}\right) = f(\vec{x}) + \sum_{i=1}^{N} \frac{\partial}{\partial i} f(\vec{x}) h_i + \frac{1}{2} \vec{h}^T \vec{H}_f(\theta) \vec{h},$$

where $\vec{H}_f$ is the Hessian matrix given by :

$$\vec{H}_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_N^2} \end{pmatrix}.$$

[4]

Definition 2.3 (Bifurcation value)**.** *Suppose we have system of ordinary differential equations given by:*

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \mu), \tag{2.11}$$

depending on a parameter $\mu \in \mathbb{R}$. A value of $\mu_0 \in \mathbb{R}$ is called a bifurcation value if the vector field $\vec{f}(\vec{x}, \mu)$ is structurally unstable at $\mu = \mu_0$. [7]

Theorem 2.5 (Saddle-Node Bifurcation)**.** *Suppose we have the system given by Equation 2.11. Suppose $f(x_0, \mu_0) = 0$ and that the $N \times N$ matrix $A \equiv Df(x_0, \mu_0)$ has a simple eigenvalue $\lambda = 0$ with eigenvector v and that A^T has an eigenvector w corresponding to the eigenvalue $\lambda = 0$. If the following conditions are satisfied:*

$$\vec{w}^T \vec{f}_\mu(x_0, \mu_0) \neq 0, \qquad \vec{w}^T D^2 \vec{f}(x_0, \mu_0)(\vec{v}, \vec{v}) \neq 0. \tag{2.12}$$

Then depending on the signs in Equation 2.12 there are no equilibrium points of the system near x_0 when $\mu < \mu_0$, there are two equilibrium points of the system near x_0 when $\mu > \mu_0$ and the system experiences a saddle-node bifurcation as μ passes through the bifurcation value $\mu = \mu_0$. [7]

Theorem 2.6 (Transcritical Bifurcation)**.** *Suppose we have the system given by Equation 2.11. Suppose $f(x_0, \mu_0) = 0$ and that the $N \times N$ matrix $A \equiv Df(x_0, \mu_0)$ has a simple eigenvalue $\lambda = 0$ with eigenvector v and that A^T has an eigenvector w corresponding to the eigenvalue $\lambda = 0$. If the following conditions are satisfied:*

$$\vec{w}^T \vec{f}_\mu(x_0, \mu_0) = 0, \tag{2.13}$$

$$\vec{w}^T D\vec{f}_\mu(x_0, \mu_0)\vec{v} \neq 0, \tag{2.14}$$

$$\vec{w}^T D^2 \vec{f}(x_0, \mu_0)(\vec{v}, \vec{v}) \neq 0, \tag{2.15}$$

then the system experiences a transcritical bifurcation at x_0 as the parameter μ passes through the bifurcation value $\mu = \mu_0$. At the transcritical bifurcation two branches of equilibria intersect and exchange stability. So for $\mu < \mu_0$ we could say equilibrium one is stable and equilibrium 2 is unstable, then for $\mu > \mu_0$, we have equilibrium one is unstable and equilibrium 2 is stable. [7]

Theorem 2.7 (Hopf-Bifurcation). *Suppose we have the system given by Equation 2.11. Suppose* $f(x_0, \mu_0) = 0$ *and that* $Df(x_0, \mu_0)$ *has a pair of pure imaginary eigenvalues and no other eigenvalues with zero real part. Suppose the following condition holds:*

$$\frac{d}{d\mu}\left[\Re(\lambda_\mu)\right]_{\mu=\mu_0} \neq 0. \tag{2.16}$$

Then the system undergoes a Hopf bifurcation at $x = x_0$ *as the parameter* μ *passes through* μ_0*. [7]*

3

The 2-dimensional mean-field Ising model

In this chapter we analyze the mean-field Ising model with two populations ($N = 2$). This will serve as the foundation for studying the general N population model. The goal of this chapter is to show the convergence of the two-dimensional mean-field Ising model to a 2-dimensional system of ordinary differential equations. In Section 3.1 we will introduce the model and its mathematical description using an example from population opinion dynamics. In the subsequent section, we we prove the convergence of its generator to the generator of a two-dimensional system of ordinary differential equations.

3.1. Description of the Model

To introduce the model, we will use an example of opinion dynamics. Suppose we have a group of n individuals, who all have an opinion on a subject. That opinion can be either positive or negative. Our goal is to model how these individuals influence each other's opinions. Figure 3.1 gives a schematic view of a group of $n = 121$ individuals with either a positive or negative opinion, denoted with a plus or minus respectively. Each individual in the system influence each other. We begin by introducing a few simplifications to get to the desired model.

+	-	-	-	+	-	-	+	+	+	+
-	-	-	-	+	+	+	-	+	-	-
-	+	-	+	+	+	-	-	+	-	-
+	-	+	-	-	+	+	-	+	+	+
-	+	+	+	+	+	-	+	+	+	+
+	+	+	+	-	-	-	+	-	-	-
-	-	+	-	+	+	-	-	-	+	-
+	-	-	+	-	-	+	+	+	+	+
-	-	-	+	+	+	+	-	-	-	-
+	-	+	-	-	-	-	-	-	-	+
+	-	-	+	+	-	-	-	+	-	+

Figure 3.1: Schematic overview of a model with 121 individuals. A plus represents and individual with a positive opinion and a minus represents an individual with a negative opinion.

The first simplification we want to make is to assume that the location of the individuals does not matter. Two individuals on opposite sides of the grid have the same influence on one another as two adjacent individuals. Secondly we divide the individuals in two groups. Let's say we have a group of young adults and a group of children. We assume the interactions in the model are characterized by four coefficients. The intra- and inter population coefficients, i.e: The effects of the group of children on the group of young adults, the effect of the group of children on other children, the effect of the group of young adults on the group of children and the effect of the group of young adults on other young adults. A schematic view of this model is shown in Figure 3.2.

Finally we introduce the mean-field approximation. This states that instead of each individual influencing

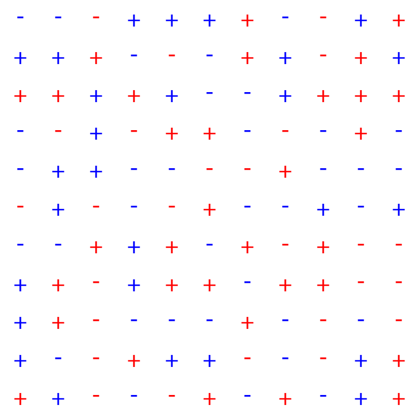


Figure 3.2: Schematic view of the two-population model. A plus represents and individual with a positive opinion and a minus represents an individual with a negative opinion. The model is divided in a group of children and a group of young adults in red and blue respectively.

each and every other individual, each individual is only influenced by the mean of each of the two populations. This is the two-population mean-field Ising model. Figure 3.3 gives a schematic view of the final version of this model.

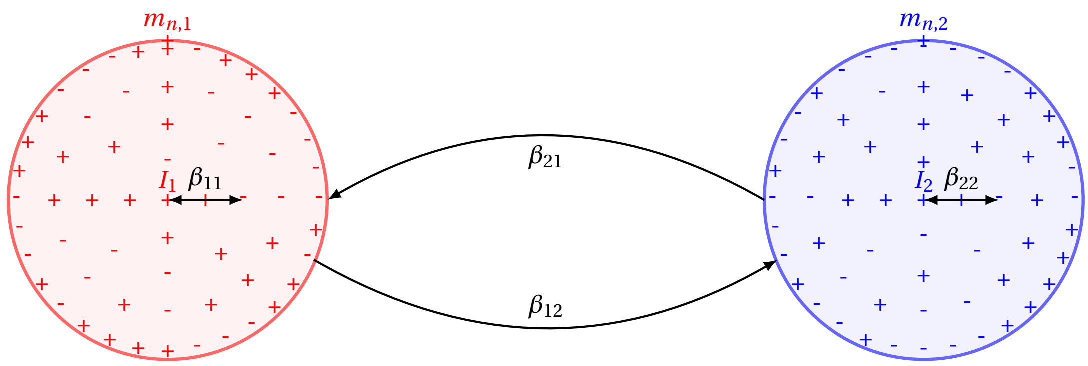


Figure 3.3: Schematic view of the two-population Mean-Field Ising model. A plus represents and individual with a positive opinion and a minus represents an individual with a negative opinion. The model is divided in a group of children and a group of young adults in red and blue respectively. The interaction coefficients are shown by the arrows.

We will now introduce the mathematical notation corresponding to the model. To describe the opinion of our n individuals over time we introduce a sequence of random variables $\sigma = \{\sigma_1\sigma_2\dots,\sigma_n\}$. As the individuals can either have a positive or negative opinion, each random variable takes values in the state space of $E = \{-1,+1\}$. The total state space of the population σ is therefore given by $\Omega = E^n = \{-1,+1\}^n$. To split the individuals into two populations we will introduce two index sets I_1 and I_2. If the index of an individual is part of I_1, that individual is part of population 1, conversely if the index of an individual is part of I_2 that individual is part of population 2. Obviously we then have that $I_1 \cap I_2 = \varnothing$ and $I_1 \cup I_2 = \{1,\dots,n\}$, since an individual is part of exactly one population. The sizes of the populations are given by $n_1 = \text{card}(I_1)$ and $n_2 = \text{card}(I_2)$. Subsequently the size of the total population is given by $n = \text{card}(\sigma) = n_1 + n_2$. We assume $\rho_1, \rho_2 > 0$, meaning both populations are non-empty. This gives that the proportion of the populations in respect to the total population are given by $\rho_1 = \frac{n_1}{n}$ and $\rho_2 = \frac{n_2}{n}$. We denote the total population with individual j's opinion flipped by σ^j.

Because we are using a mean-field model, we need to define the mean. For this we introduce the mean function $\mu(\sigma)$:

Definition 3.1 (Mean Function)**.** *Let I_j be the index set of j-th sub population of total population σ, then mean function $\mu : E^{n_j} \to [-1,1]$ is defined by:*

$$\mu(\sigma_i, i \in I_j) = \frac{1}{n_j}\sum_{i \in I_j} \sigma_i. \tag{3.1}$$

The 2-dimensional mean vector of the system m_n is given by: $m_n(\sigma_0) = \begin{pmatrix}\mu(\sigma_i, i \in I_1)\\ \mu(\sigma_i, i \in I_2)\end{pmatrix} = \begin{pmatrix} m_n(\sigma)_1 \\ m_n(\sigma)_2\end{pmatrix}$.

Next up we look at the Hamiltonian of the system. The Hamiltonian for the Mean-Field Ising model is given by[3]:

$$H(\sigma) = -\beta \sum_{i=1}^{n} \sigma_i \mu(\sigma). \tag{3.2}$$

Here β is the interaction coefficient. We adjust this Hamiltonian, by splitting it up to two populations. This gives us the Hamiltonian for the two-population mean-field Ising model, given by:

$$H(\sigma) = -\left[\sum_{\sigma_i | i \in I_1} \sigma_i \left[\beta_{11} m_n(\sigma)_1 \rho_1 + \beta_{21} m_n(\sigma)_2 \rho_2\right] + \sum_{\sigma_i | i \in I_2} \sigma_i \left[\beta_{12} m_n(\sigma)_1 \rho_1 + \beta_{22} m_n(\sigma)_2 \rho_2\right]\right]. \tag{3.3}$$

Here β_{11}, β_{12}, β_{21} and β_{22} denote the effect of population 1 on itself, population 1 on population 2, population 2 on population 1 and population 2 on itself, as shown in in Figure 3.3.

We can use the Hamiltonian of the system the determine the energy change when the opinion of an individual in the model flips. This change of energy can be used to find the flip rate of an individual in the model. The energy change when individual σ_i flips is given by:

$$\begin{aligned}\Delta E_i &= H(\sigma^i) - H(\sigma),\\ &= \begin{cases} 2\sigma_i \left[\beta_{11} m_n(\sigma)_1 \rho_1 + \beta_{21} m_n(\sigma)_2 \rho_2\right] & \text{if } i \in I_1,\\ 2\sigma_i \left[\beta_{12} m_n(\sigma)_1 \rho_1 + \beta_{22} m_n(\sigma)_2 \rho_2\right] & \text{if } i \in I_2.\end{cases}\end{aligned}$$

Before we can go any further, we define the flip rate of a single spin in the model. For this we apply the Master equation from Theorem 2.1 on the model:

$$\frac{dP(\sigma,t)}{dt} = \sum_{i}^{n} (w(\sigma^i \to \sigma) P(\sigma^i, t) - w(\sigma \to \sigma^i) P(\sigma, t)), \tag{3.4}$$

$$= \sum_{j=1}^{N} \sum_{\sigma_i | i \in I_j} (w(\sigma^i \to \sigma) P(\sigma^i, t) - w(\sigma \to \sigma^i) P(\sigma, t)). \tag{3.5}$$

We know that in equilibrium the probability of the state is proportional to the Boltzmann distribution[6], i.e., $P_{eq}(\sigma, t) \propto e^{-H(\sigma)}$. Here we can use the energy difference described in the Equation 3.4. Plugging the probability in equilibrium in the master equation gives us the following condition:

$$\frac{w(\sigma \to \sigma^i)}{w(\sigma^i \to \sigma)} = \frac{P_{eq}(\sigma^i)}{P_{eq}(\sigma)} = e^{-\beta(H(\sigma^i) - H(\sigma))} = e^{-\beta \Delta E_i}. \tag{3.6}$$

We make the following choice of definition for the flip rate, which satisfies this condition:

$$w(\sigma \to \sigma^i) = e^{-\frac{\beta \Delta E_i}{2}} \tag{3.7}$$

$$= \begin{cases} e^{-\sigma_i \{\beta_{11} m_n(\sigma)_1 \rho_1 + \beta_{21} m_n(\sigma)_2 \rho_2\}} & \text{if } i \in I_1,\\ e^{-\sigma_i \{\beta_{12} m_n(\sigma)_1 \rho_1 + \beta_{22} m_n(\sigma)_2 \rho_2\}} & \text{if } i \in I_2.\end{cases} \tag{3.8}$$

Here we have absorbed the Boltzmann coefficient β into the interaction coefficient. We would like to split this into a flip rate for each population. This results in:

$$q_{1i} = e^{-\sigma_i \{\beta_{11} m_n(\sigma)_1 \rho_1 + \beta_{21} m_n(\sigma)_2 \rho_2\}} = e^{-\sigma_i \Gamma_1},$$

and

$$q_{2i} = e^{-\sigma_i \{\beta_{12} m_n(\sigma)_1 \rho_1 + \beta_{22} m_n(\sigma)_2 \rho_2\}} = e^{-\sigma_i \Gamma_2}.$$

Here q_{1i} and q_{2i} are the flip rates for particles in populations 1 and 2 respectively. We introduced the terms Γ_1 and Γ_2 for convenience. Those are given by:

$$\Gamma_1 = \beta_{11} m_n(\sigma)_1 \rho_1 + \beta_{21} m_n(\sigma)_2 \rho_2,$$
and
$$\Gamma_2 = \beta_{12} m_n(\sigma)_1 \rho_1 + \beta_{22} m_n(\sigma)_2 \rho_2.$$

Now that we have defined the flip rate, we can find the generator of σ using Lemma 2.2:

$$\mathscr{L}_n f(\sigma) = \sum_{j \neq i} w(S_i \to S_j)(f(S_j) - f(S_i)). \tag{3.9}$$

In our model $w(S_i \to S_j) = 0$, if $S_j \neq S_i^k$ for some $k = 1, \dots, n$, where S_i^k denotes population S_i with the k-th individual flipped. So we get:

$$\mathscr{L}_n f(\sigma) = \sum_{i=1}^{n} w(\sigma \to \sigma^i)(f(\sigma^i) - f(\sigma)). \tag{3.10}$$

We can split this sum up in a sum for each populations, giving us:

$$\mathscr{L}_n f(\sigma) = \sum_{i \in I_1} w(\sigma \to \sigma^i)(f(\sigma^i) - f(\sigma)), \tag{3.11}$$
$$+ \sum_{i \in I_2} w(\sigma \to \sigma^i)(f(\sigma^i) - f(\sigma)). \tag{3.12}$$

Finally we can insert the definition for the flip rate for each population giving us:

$$\mathscr{L}_n f(\sigma) = \sum_{i \in I_1} q_{1i}(f(\sigma^i) - f(\sigma)), \tag{3.13}$$
$$+ \sum_{i \in I_2} q_{2i}(f(\sigma^i) - f(\sigma)). \tag{3.14}$$

3.2. Convergence of the generators

Now we have found the generator of our system. We want to show that when taking the limit the system converges in probability to a two-dimensional system of ordinary differential equations. First we will give this system with its generator. Then we will show that the generator of the two-dimensional mean-field Ising model converges to the generator of the two-dimensional system of ordinary differential equations.
We start by showing the two-dimensional ordinary differential equation with its generator. Let $x(t)$ be the solution to the two-dimensional ordinary differential equation given by:

$$\begin{cases} \frac{d\vec{x}}{dt} = F(x), \\ x(0) = x_0. \end{cases} \tag{3.15}$$

where $F(\vec{x})$ is the vector given by:

$$F(\vec{x}) = \begin{pmatrix} (1-x_1)e^{\Gamma_1} - (1+x_1)e^{-\Gamma_1}, \\ (1-x_2)e^{\Gamma_2} - (1+x_2)e^{-\Gamma_2}. \end{pmatrix} \tag{3.16}$$

The generator $\mathscr{A}$ working on $f(x(t))$ is can be found using the definition of the infinitesimal generator and corresponds to the semigroup $\{T(t)\}$ with $T(t)f(s) = f(s+t)$. This gives us:

$$\mathscr{A} f(s) = \lim_{t \to 0} \frac{T(t)f(s) - f(s)}{t}, \tag{3.17}$$
$$= \lim_{t \to 0} \frac{f(s+t) - f(s)}{t}, \tag{3.18}$$
$$= \frac{df}{dt}, \tag{3.19}$$
$$= \langle \nabla f, \frac{dx}{dt} \rangle = \langle \nabla f, F(x) \rangle. \tag{3.20}$$

Where in the last step we used the chain rule.

Proposition 3.1. *The stochastic process given by $t \mapsto m_n(t)$ is a process with generator $\mathscr{A}_n$ given by:*

$$\begin{aligned}\mathscr{A}_n(f\circ m_n(\sigma)) = &\frac{\rho_1 n(1+m_n(\sigma)_1)}{2}e^{-\Gamma_1}\left(f(m_n(\sigma)-\frac{2}{\rho_1 n}\hat{e}_1)-f(m_n(\sigma))\right),\\ &+\frac{\rho_1 n(1-m_n(\sigma)_1)}{2}e^{\Gamma_1}\left(f(m_n(\sigma)+\frac{2}{\rho_1 n}\hat{e}_1)-f(m_n(\sigma))\right),\\ &+\frac{\rho_2 n(1+m_n(\sigma)_2)}{2}e^{-\Gamma_2}\left(f(m_n(\sigma))-\frac{2}{\rho_2 n}\hat{e}_2)-f(m_n(\sigma))\right),\\ &+\frac{\rho_2 n(1-m_n(\sigma)_2)}{2}e^{\Gamma_2}\left(f(m_n(\sigma)+\frac{2}{\rho_2 n}\hat{e}_2)-f(m_n(\sigma)))\right).\end{aligned} \tag{3.21}$$

Proof. Let the generator $\mathscr{A}_n$ work on the function $f\circ m_n(\sigma)$:

$$\begin{aligned}\mathscr{A}_n(f\circ m_n(\sigma)) = &\sum_{i\in I_1} q_{2i}\left(f(m_n(\sigma^i))-f(m_n(\sigma))\right),\\ &+\sum_{i\in I_2} q_{1i}\left(f(m_n(\sigma^i))-f(m_n(\sigma))\right).\end{aligned}$$

Next up we want to rewrite the term $m_n(\sigma^i)$. For this we first introduces the unit vectors $\hat{e}_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $\hat{e}_2=\begin{pmatrix}0\\1\end{pmatrix}$. We now look at the definition of m_n and plug in σ^i. This gives us:

$$m_n(\sigma^i)=\begin{pmatrix}\mu(\sigma^i_j, j\in I_2)\\ \mu(\sigma^i_j, j\in I_1)\end{pmatrix}.$$

If $i\in I_1$, then $\mu(\sigma^i_j, j\in I_2)=\mu(\sigma_j, j\in I_2)$. Similarly if $i\in I_2$, then $\mu(\sigma^i_j, j\in I_1)=\mu(\sigma_j, j\in I_1)$. Now let's have a look what happens to $\mu(\sigma^i_j, j\in I_1)$ if $i\in I_1$. We get:

$$\mu(\sigma^i_j, j\in I_1)=\frac{1}{n_1}\left[\sum_{j\in I_1|j\neq i}\sigma_j-\sigma_i\right], \tag{3.22}$$

$$=\frac{1}{n_1}\left[\sum_{j\in I_1}\sigma_j-2\sigma_i\right], \tag{3.23}$$

$$=\mu(\sigma_j, j\in I_1)-2\frac{\sigma_i}{n_1}. \tag{3.24}$$

We get a similar result for the situation the flipped individual is in population 2. This allows us to write:

$$m_n(\sigma^i)=\begin{cases}m_n(\sigma)-2\frac{\sigma_i}{n_1}e_1, & \text{if } i\in I_1,\\ m_n(\sigma)-2\frac{\sigma_1}{n_2}e_2, & \text{if } i\in I_2.\end{cases}$$

Substituting this result into our expression for the generator gives us:

$$\begin{aligned}\mathscr{A}_n(f\circ m_n(\sigma)) = &\sum_{i\in I_1} e^{-\sigma_i\Gamma_1}\left(f(m_n(\sigma)-\frac{2\sigma_i}{n_1}e_1)-f(m_n(\sigma))\right),\\ &+\sum_{i\in I_2} e^{-\sigma_i\Gamma_2}\left(f(m_n(\sigma)-\frac{2\sigma_i}{n_2}e_2)-f(m_n(\sigma))\right).\end{aligned}$$

Next, we split the sums over the populations into two parts, a sum over all individuals with a positive opinion and a sum over all individuals with a negative opinion. This allows to substitute all value of -1 or $+1$ for all the σ_1 terms. This gives us:

$$\begin{aligned}\mathscr{A}_n(f\circ m_n(\sigma)) &= \sum_{i\in I_1|\sigma_i=1} e^{-\Gamma_1}\left(f(m_n(\sigma)-\frac{2}{n_1}\hat{e}_1)-f(m_n(\sigma))\right),\\ &+\sum_{i\in I_1|\sigma_i=-1} e^{\Gamma_1}\left(f(m_n(\sigma)+\frac{2}{n_1}\hat{e}_1)-f(m_n(\sigma))\right),\\ &+\sum_{i\in I_2|\sigma_i=1} e^{-\Gamma_2}\left(f(m_n(\sigma)-\frac{2}{n_2}\hat{e}_2)-f(m_n(\sigma))\right),\\ &+\sum_{i\in I_2|\sigma_i=-1} e^{\Gamma_2}\left(f(m_n(\sigma)+\frac{2}{n_2}\hat{e}_2)-f(m_n(\sigma))\right).\end{aligned}$$

Now all the terms over which we sum have vanished and we only sum over one. We can rewrite this sum as an expression including the m_n. We know that each sum is the number of positive or negative particles in each populations. We will look at the definition of the mean function μ to find this expression:

$$\begin{aligned}\mu(\sigma) &= \frac{1}{n}\sum_{i}^{n} = \sigma_i,\\ &= \frac{1}{n}\left\{\sum_{i|\sigma_i=-1}^{n} -1 \sum_{i|\sigma_i=1}^{n} 1\right\},\\ &= \frac{1}{n}\left\{n^+ - n^-\right\}.\end{aligned}$$

Here n_+ and n_- denote the number of positive and negative individuals in the model in population respectively. Using $n_+ = n - n^-$ we can get the following expressing expressions for n^+ and n^-:

$$n^+ = \sum_{i|\sigma_i=1} 1 = \frac{n\left\{1+\mu(\sigma)\right\}}{2}, \tag{3.25}$$

$$n^- = \sum_{i|\sigma_i=-1} 1 = \frac{n\left\{1-\mu(\sigma)\right\}}{2}. \tag{3.26}$$

Substituting this into the expression for the generator gives:

$$\begin{aligned}\mathscr{A}_n(f\circ m_n(\sigma)) &= \frac{\rho_1 n(1+m_n(\sigma)_1)}{2} e^{-\Gamma_1}\left(f(m_n(\sigma)-\frac{2}{\rho_1 n}\hat{e}_1)-f(m_n(\sigma))\right),\\ &+\frac{\rho_1 n(1-m_n(\sigma)_1)}{2} e^{\Gamma_1}\left(f(m_n(\sigma)+\frac{2}{\rho_1 n}\hat{e}_1)-f(m_n(\sigma))\right),\\ &+\frac{\rho_2 n(1+m_n(\sigma)_2)}{2} e^{-\Gamma_2}\left(f(m_n(\sigma))-\frac{2}{\rho_2 n}\hat{e}_2)-f(m_n(\sigma))\right),\\ &+\frac{\rho_2 n(1-m_n(\sigma)_2)}{2} e^{\Gamma_2}\left(f(m_n(\sigma)+\frac{2}{\rho_2 n}\hat{e}_2)-f(m_n(\sigma)))\right).\end{aligned}$$

□

Now we have found the generator of the model, we can introduce the final proposition which shows the convergence of the models.

Proposition 3.2. *The Markov process given by $t\mapsto m_n(\sigma(t))$ with $m_n(\sigma(0))\to x_0$ converges to $x(t)$ the solution to the two-dimensional differential equation given by Equation 3.15*

Proof. Using the generator we found in Proposition 3.1, We can apply theorem 2.3, to show the convergence of the generators. Suppose $m_n(\sigma(t)) \to \vec{x}(t)$ as n goes to infinity we then have:

$$\begin{aligned}\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) &= \lim_{n\to\infty} \frac{\rho_1 n(1+x_1)}{2} e^{-\Gamma_1}\left(f(\vec{x}-\frac{2}{\rho_1 n}\hat{x}_1) - f(\vec{x})\right),\\ &+ \lim_{n\to\infty} \frac{\rho_1 n(1-x_1)}{2} e^{\Gamma_1}\left(f(\vec{x}+\frac{2}{\rho_1 n}\hat{x}_1) - f(\vec{x})\right),\\ &+ \lim_{n\to\infty} \frac{\rho_2 n(1+x_2)}{2} e^{-\Gamma_2}\left(f(\vec{x}-\frac{2}{\rho_2 n}\hat{x}_2) - f(\vec{x})\right),\\ &+ \lim_{n\to\infty} \frac{\rho_2 n(1-x_2)}{2} e^{\Gamma_2}\left(f(\vec{x}+\frac{2}{\rho_2 n}\hat{x}_2) - f(\vec{x})\right).\end{aligned}$$

Now since $f \in C^2$ we can apply the Theorem 2.4. For this we will first only look at the part $f(\vec{x}-\frac{2}{\rho_1 n}\hat{x}_1)$. We know that there exists a $\theta_1 \in L_1$, where L_1 is the path between $\vec{x}$ and $\vec{x}-\frac{2}{\rho_1 n}$, such that:

$$f(\vec{x}-\frac{2}{\rho_1 n}\hat{x}_1) = f(\vec{x}) - \frac{2}{\rho_1 n}\frac{\partial}{\partial x_1} f(\vec{x}) + \frac{1}{2}\frac{4}{\rho_1^2 n^2}\frac{\partial^2}{\partial x_1^2} f(\theta).$$

We can get similar results for $f(\vec{x}+\frac{2}{\rho_1 n}\hat{x}_1)$, $f(\vec{x}-\frac{2}{\rho_2 n}\hat{x}_2)$ and $f(\vec{x}+\frac{2}{\rho_2 n}\hat{x}_2)$, which corresponding θ_2, θ_3 and θ_4 in paths L_2, L_3 and L_4 respectively. Substituting these results into our limit gives:

$$\begin{aligned}\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) &= \lim_{n\to\infty} \frac{\rho_1 n(1+x_1)}{2} e^{-\Gamma_1}\left(f(\vec{x}) - \frac{2}{\rho_1 n}\frac{\partial}{\partial x_1} f(\vec{x}) + \frac{2}{\rho_1^2 n^2}\frac{\partial^2}{\partial x_1^2} f(\theta_1) - f(\vec{x})\right),\\ &+ \lim_{n\to\infty} \frac{\rho_1 n(1-x_1)}{2} e^{\Gamma_1}\left(f(\vec{x}) + \frac{2}{\rho_1 n}\frac{\partial}{\partial x_1} f(\vec{x}) + \frac{2}{\rho_1^2 n^2}\frac{\partial^2}{\partial x_1^2} f(\theta_2) - f(\vec{x})\right),\\ &+ \lim_{n\to\infty} \frac{\rho_2 n(1+x_2)}{2} e^{-\Gamma_2}\left(f(\vec{x}) - \frac{2}{\rho_2 n}\frac{\partial}{\partial x_2} f(\vec{x}) + \frac{2}{\rho_2^2 n^2}\frac{\partial^2}{\partial x_2^2} f(\theta_3) - f(\vec{x})\right),\\ &+ \lim_{n\to\infty} \frac{\rho_2 n(1-x_2)}{2} e^{\Gamma_2}\left(f(\vec{x}) + \frac{2}{\rho_2 n}\frac{\partial}{\partial x_2} f(\vec{x}) + \frac{2}{\rho_2^2 n^2}\frac{\partial^2}{\partial x_2^2} f(\theta_4) - f(\vec{x})\right).\end{aligned}$$

If we now, cancel out the $f(\vec{x})$ parts and move the limits and the components $\frac{n\rho_1}{2}$ and $\frac{n\rho_2}{2}$ to inside the brackets we get:

$$\begin{aligned}\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) &= (1+x_1)\, e^{-\Gamma_1}\left(-\frac{\partial}{\partial x_1} f(\vec{x}) + \lim_{n\to\infty}\frac{1}{\rho_1 n}\frac{\partial f(\theta_1)}{\partial x_1^2}\right),\\ &+ (1-x_1)\, e^{\Gamma_1}\left(\frac{\partial}{\partial x_1} f(\vec{x}) + \lim_{n\to\infty}\frac{1}{\rho_1 n}\frac{\partial f(\theta_2)}{\partial x_1^2}\right),\\ &+ (1+x_2)\, e^{-\Gamma_2}\left(-\frac{\partial}{\partial x_2} f(\vec{x}) + \lim_{n\to\infty}\frac{1}{\rho_2 n}\frac{\partial f(\theta_3)}{\partial x_2^2}\right),\\ &+ (1-x_2)\, e^{\Gamma_2}\left(\frac{\partial}{\partial x_2} f(\vec{x}) + \lim_{n\to\infty}\frac{1}{\rho_2 n}\frac{\partial f(\theta_4)}{\partial x_2^2}\right).\end{aligned}$$

Now since $\frac{1}{n}$ goes to zero as n goes to infinity we get:

$$\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) = -(1+x_1)e^{-\Gamma_1}\frac{\partial f}{\partial x_1} + (1-x_1)e^{\Gamma_1}\frac{\partial f}{\partial x_1} - (1+x_2)e^{-\Gamma_2}\frac{\partial f}{\partial x_2} + (1-x_2)e^{\Gamma_2}\frac{\partial f}{\partial x_2}.$$

Rewriting this gives us what we wanted to show:

$$\begin{aligned}\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) &= < \begin{bmatrix}\frac{\partial f}{\partial x_1}\\ \frac{\partial f}{\partial x_2}\end{bmatrix}, \begin{bmatrix}(1-x_1)e^{\Gamma_1} - (1+x_1)e^{-\Gamma_1}\\ (1-x_2)e^{\Gamma_2} - (1+x_2)e^{-\Gamma_2}\end{bmatrix} >,\\ &= < \nabla f, F > = \mathscr{A} f.\end{aligned}$$

Using this we clearly see that:

$$\lim_{n} \sup_{x \in E_n} \left| \mathscr{A}_n (f \circ m_n)(x) - \mathscr{A} f(m_n(x)) \right| = 0 \tag{3.27}$$

We have now shown that as n goes to infinity and $m_n(t)$ goes to x_t we have that the generator $\mathscr{A}_n$ converges to the generator $\mathscr{A}$. Therefore by Theorem 2.3, the two-population Mean-Field Ising model converges to two-dimensional system of Ordinary Differential equations described by Equation 3.15

□

4

The N-dimensional mean-field Ising model

In this chapter we will expand the framework, we created in Chapter 2 to discuss the two-population Mean-Field Ising model, to create an analysis of the "general N"-population Mean-Field Ising model. In the first section of this chapter we will give a quick introduction to the model and will make some alterations and additions to the mathematical notation introduced in Chapter 2. In the second section of this chapter we will proof the convergence of the model to a N-dimensional system of Ordinary Differential Equations.

4.1. Description of the Model

When describing the N-Population Mean-Field Ising model, we will make similar steps as before in the $N = 2$ case. We again have a group of n individuals who can have either a negative or a positive opinion on a subject. We again make the same simplifications as before. We have a Mean-Field and location independent model. Where in the $N = 2$ case, we split the populations into two groups, now we split the population into a general number N groups. Figure 4.1 gives an schematic overview for a three-population Mean-Field Ising model. We expanded our model to a general N, so we can split the total population on different criteria. We could for example split the total population into three ages groups: children, young adults and 30+'ers.

To achieve our goal, we need to make some alterations to the mathematical notation introduced with the two-dimensional model. We again introduce a sequence of n random variables $\sigma = \{\sigma_1, \dots, \sigma_n\}$ representing the individuals in the total population. Each individual has a state space $E = \{-1, +1\}$ and the entire population has total state space $\Omega = \{-1, +1\}^n$. Now we diverge from the two-population model. We introduce N index sets $I_1, \dots, I_N$, to split the total population. All the indices for individuals in Population j are stored in index set I_j for al $j = 1, \dots, N$. Again we have chosen the populations such that each individual is in exactly one population. This gives the two requirements for the index sets: $\prod_{j=1}^{N} i_j = \emptyset$ and $\bigcup_{j=1}^{N} I_j = \{1, \dots, n\}$. For each population j the the population size n_j is given by $n_j = \text{card}(I_j)$ and the proportion out of the total population ρ_j is given by $\rho_j = \frac{n_j}{n}$. Here we have chosen ρ_j such that $\lim_{n\to\infty} \frac{n_j}{n} = \rho_j > 0$, meaning that the populations are non-empty. Obviously this gives $\sum_{j=1}^{N} n_j = n$ and $\sum_{j=1}^{N} \rho_j = 1$. As before we denote the k-th individual in the total population being flipped by σ^k.

The mean function defined for the two-population model can also be applied to the N-population model, allowing us to define the N-dimensional mean vector $m_n(\sigma)$ of the model by:

$$m_n(\sigma) = \begin{pmatrix} \mu(\sigma_i, i \in I_1) \\ \vdots \\ \mu(\sigma_i, i \in I_N) \end{pmatrix} = \begin{pmatrix} m_n(\sigma)_1 \\ \vdots \\ m_n(\sigma)_N \end{pmatrix}. \tag{4.1}$$

For the N-dimensional model we also store all the population fractions in a N-dimensional vector given by:

$$\rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}. \tag{4.2}$$

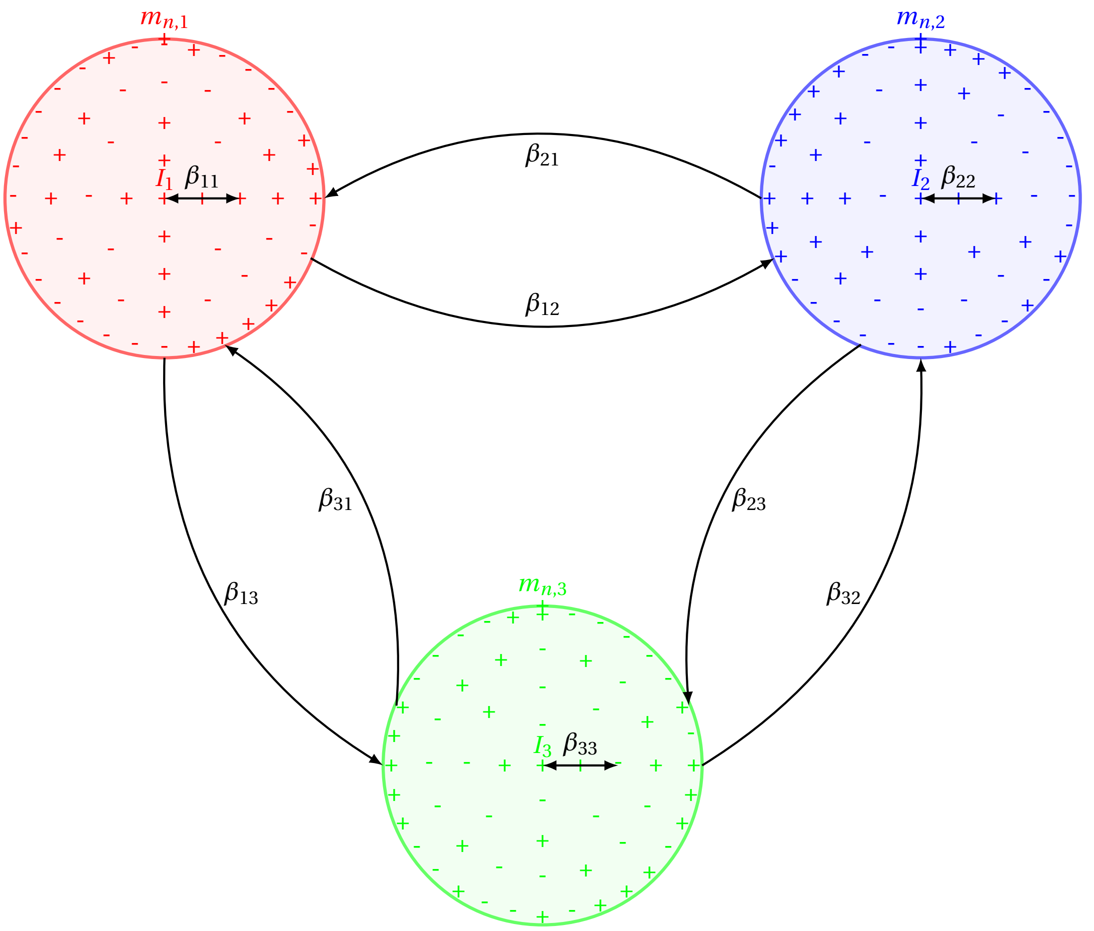


Figure 4.1: Schematic view of the 2-population Mean-Field Ising model. A plus represents and individual with a positive opinion and a minus represents an individual with a negative opinion. The model is divided in population 1 and population 2 in red and blue respectively. The interaction coefficients are shown by the arrows.

Now to find the flipping rate of the system we again look at the Hamiltonian. We can continue from Equation 3.2, and split it up for N populations giving:

$$H(\sigma) = -\sum_{j=1}^{N} \sum_{\sigma_i | i \in I_j} \sigma_i [\beta_{1j} m_n(\sigma)_1 \rho_1 + \cdots + \beta_{Nj} m_n(\sigma)_N \rho_N]. \tag{4.3}$$

Here β_{jk} is the interaction coefficient denoting the effect of population j on population k. All the interaction coefficients are stored in the interaction matrix C, which is a $N \times N$ matrix given by:

$$C = \begin{pmatrix} \beta_{11} & \dots & \beta_{1N} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \dots & \beta_{NN} \end{pmatrix}. \tag{4.4}$$

We now again look at the energy difference when a individual flips and we see:

$$\Delta E_i = H(\sigma^i) - H(\sigma) \tag{4.5}$$

$$= \begin{cases} 2\sigma_i [\beta_{11} m_n(\sigma)_1 \rho_1 + \cdots + \beta_{N1} m_n(\sigma)_N \rho_N], \text{ if } i \in I_1, \\ \vdots \\ 2\sigma_i [\beta_{1N} m_n(\sigma)_1 \rho_1 + \cdots + \beta_{NN} m_n(\sigma)_N \rho_N], \text{ if } i \in I_N. \end{cases} \tag{4.6}$$

Similarly to the two-population mean-field Ising model we find the flip rate for the i-th individual in population j q_{ij} to be given by:

$$q_{ji} = e^{-\sigma_i \Gamma_j}. \tag{4.7}$$

Where Γ_j is given by:

$$\Gamma_j = \sum_{i=1}^{N} \beta_{ij} m_n(\sigma)_i \rho_i. \tag{4.8}$$

No we have found the flip rate of the N-dimensional model, we can define its generator:

$$\mathscr{L}_n f(\sigma) = \sum_{j=1}^{N} \sum_{i \in I_j} q_{ji} \Big(f(\sigma^i) - f(\sigma) \Big). \tag{4.9}$$

4.2. Convergence of the generators

Now we have found the generator of our system. We want to show that when taking the limit the system converges in probability to a N-dimensional system of ordinary differential equations. First we will give this system with its generator. Then we will show that the generator of the N-dimensional mean-field Ising model converges to the generator of the N-dimensional system of ordinary differential equations.

We start by showing the N-dimensional ordinary differential equation with its generator. Let $x(t)$ be the solution to the N-dimensional ordinary differential equation given by:

$$\begin{cases} \frac{d\vec{x}}{dt} = F(x), \\ x(0) = x_0. \end{cases} \tag{4.10}$$

where $F(\vec{x})$ is the vector given by:

$$F(\vec{x}) = \begin{pmatrix} (1-x_1)e^{\Gamma_1} - (1+x_1)e^{-\Gamma_1} \\ \vdots \\ (1-x_N)e^{\Gamma_N} - (1+x_N)e^{-\Gamma_N} \end{pmatrix}. \tag{4.11}$$

The generator $\mathscr{A}$ working on $f(x(t))$ is can be found by using the definition of the infinitesimal generator and corresponds to the semigroup $\{T(t)\}$ with $T(t)f(s) = f(s+t)$. This gives us:

$$\mathscr{A} f(s) = \lim_{t\to 0} \frac{T(t)f(s) - f(s)}{t}, \tag{4.12}$$

$$= \lim_{t\to 0} \frac{f(s+t) - f(s)}{t}, \tag{4.13}$$

$$= \frac{df}{dt}, \tag{4.14}$$

$$= \langle \nabla f, \frac{dx}{dt} \rangle = \langle \nabla f, F(x) \rangle. \tag{4.15}$$

Where in the last step we used the chain rule.

Proposition 4.1. *The stochastic process given by $t \mapsto m_n(t)$ is a process with generator $\mathscr{A}_n$ given by:*

$$\mathscr{A}_n(f \circ m_n(\sigma)) = \sum_{j=1}^{N} \frac{\rho_j n \big(1 + m_n(\sigma)_j\big)}{2} e^{-\Gamma_j} \left(f(m_n(\sigma) - \frac{2}{\rho_j n} \hat{e}_j) - f(m_n(\sigma)) \right), \tag{4.16}$$

$$+ \sum_{j=1}^{N} \frac{\rho_j n \big(1 - m_n(\sigma)_j\big)}{2} e^{\Gamma_j} \left(f(m_n(\sigma) + \frac{2}{\rho_j n} \hat{e}_j) - f(m_n(\sigma)) \right).$$

Proof. Let the generator $\mathscr{A}_n$ work on $f \circ m_n(\sigma)$:

$$\mathscr{A}_n(f \circ m_n(\sigma)) = \sum_{j=1}^{N} \sum_{i \in I_j} q_{ji} \Big(f(m_n(\sigma^i)) - f(m_n(\sigma)) \Big).$$

Next up we want to rewrite the term $m_n(\sigma^i)$. For this we first introduce the N-dimensional unit vector $\hat{e}_j$, which is zero except for its j-th index, where it has value one. Now we look at the definition of m_n and plug in σ^i. This gives us:

$$m_n(\sigma^i) = \begin{pmatrix} \mu(\sigma^i_j, j \in I_1) \\ \vdots \\ \mu(\sigma^i_j, j \in I_N) \end{pmatrix}.$$

If $i \in I_k$, then $\mu(\sigma^i_j, j \in I_l) = \mu(\sigma_j, j \in I_l)$ for $l \neq k$. Now let's have a look what happens to $\mu(\sigma^i_j, j \in I_k)$ if $i \in I_k$. We get:

$$\mu(\sigma^i_j, j \in I_k) = \frac{1}{n_k}\left[\sum_{j \in I_k | j \neq i} \sigma_j - \sigma_i\right], \tag{4.17}$$

$$= \frac{1}{n_k}\left[\sum_{j \in I_k} \sigma_j - 2\sigma_i\right], \tag{4.18}$$

$$= \mu(\sigma_j, j \in I_k) - 2\frac{\sigma_i}{n_k}. \tag{4.19}$$

This allows us to write:

$$m_n(\sigma^i) = \begin{cases} m_n(\sigma) - 2\frac{\sigma_i}{n_1} e_1 \text{if } i \in I_1, \\ \vdots \\ m_n(\sigma) - 2\frac{\sigma_i}{n_N} e_N \text{if } i \in I_N. \end{cases}$$

Substituting this result into our expression for the generator gives us:

$$\mathscr{A}_n(f \circ m_n(\sigma)) = \sum_{j=1}^{N} \sum_{i \in I_j} e^{-\sigma_i \Gamma_j} \left(f(m_n(\sigma) - \frac{2\sigma_i}{n_j} e_j) - f(m_n(\sigma)) \right).$$

Next up we split the sums over the populations into two parts, a sum over all individuals with a positive opinion and a sum over all individuals with a negative opinion. This allows to substitute all value of -1 or $+1$ for all te σ_1 terms. This gives us:

$$\mathscr{A}_n(f \circ m_n(\sigma)) = \sum_{j=1}^{N} \sum_{i \in I_j | \sigma_i = 1} e^{-\Gamma_j} \left(f(m_n(\sigma) - \frac{2}{n_j}\hat{e}_j) - f(m_n(\sigma)) \right),$$
$$+ \sum_{j=1}^{N} \sum_{i \in I_j | \sigma_i = -1} e^{\Gamma_j} \left(f(m_n(\sigma) + \frac{2}{n_j}\hat{e}_j) - f(m_n(\sigma)) \right).$$

Now all the term over which we sum have vanished and we only sum over one. We can now substitute Equation 3.25:

$$\mathscr{A}_n(f \circ m_n(\sigma)) = \sum_{j=1}^{N} \frac{\rho_j n \left(1 + m_n(\sigma)_j\right)}{2} e^{-\Gamma_j} \left(f(m_n(\sigma) - \frac{2}{\rho_j n}\hat{e}_j) - f(m_n(\sigma)) \right),$$
$$+ \sum_{j=1}^{N} \frac{\rho_j n \left(1 - m_n(\sigma)_j\right)}{2} e^{\Gamma_j} \left(f(m_n(\sigma) + \frac{2}{\rho_j n}\hat{e}_j) - f(m_n(\sigma)) \right).$$

We have now find our desired expression for the generator. We can now apply theorem 2.3, to show the convergence of the generators.

□

Proposition 4.2. *The Markov process given by* $t \mapsto m_n(\sigma(t))$ *with* $m_n(\sigma(0) \to x_0)$ *converges to* $x(t)$ *the solution of the N-dimensional differential equation given by Equation 4.10.*

Proof. Suppose $m_n(\sigma(t)) \to \vec{x}(t)$ as n goes to infinity we then have:

$$\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) = \sum_{j=1}^{N} \lim_{n\to\infty} \frac{\rho_j n\left(1+x_j\right)}{2} e^{-\Gamma_j}\left(f(\vec{x}-\frac{2}{\rho_j n}\hat{x_j}) - f(\vec{x})\right),$$
$$+\sum_{j=1}^{N} \lim_{n\to\infty} \frac{\rho_j n\left(1-x_j\right)}{2} e^{\Gamma_j}\left(f(\vec{x}+\frac{2}{\rho_j n}\hat{x_j}) - f(\vec{x})\right).$$

Now we want to apply the Theorem 2.4. For this we will first only look at the part $f(\vec{x}-\frac{2}{\rho_j n}\hat{x}_1)$. We know that there exists a $\theta_1 \in L_1$, where L_1 is the path between $\vec{x}$ and $\vec{x}-\frac{2}{\rho_j n}$, such that:

$$f(\vec{x}-\frac{2}{\rho_j n}\hat{x}_j) = f(\vec{x}) - \frac{2}{\rho_j n}\frac{\partial}{\partial x_j} f(\vec{x}) + \frac{1}{2}\frac{4}{\rho_j^2 n^2}\frac{\partial^2}{\partial x_j^2} f(\theta).$$

We can get similar results for $f(\vec{x}+\frac{2}{\rho_j n}\hat{x}_1)$ with corresponding θ_2 in path L_2. Substituting these results into our limit gives:

$$\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) = \sum_{j=1}^{N} \lim_{n\to\infty} \frac{\rho_j n\left(1+x_j\right)}{2} e^{-\Gamma_j}\left(f(\vec{x}) - \frac{2}{\rho_j n}\frac{\partial}{\partial x_j} f(\vec{x}) + \frac{2}{\rho_j^2 n^2}\frac{\partial^2}{\partial x_j^2} f(\theta_1) - f(\vec{x})\right),$$
$$+\sum_{j=1}^{N} \lim_{n\to\infty} \frac{\rho_j n\left(1-x_j\right)}{2} e^{\Gamma_j}\left(f(\vec{x}) + \frac{2}{\rho_j n}\frac{\partial}{\partial x_j} f(\vec{x}) + \frac{2}{\rho_j^2 n^2}\frac{\partial^2}{\partial x_j^2} f(\theta_2) - f(\vec{x})\right).$$

If we now, cancel out the $f(\vec{x})$ parts and move the limits and the component $\frac{n\rho_{1j}}{2}$ to inside the brackets we get:

$$\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) = \sum_{j=1}^{N} \left(1+x_j\right) e^{-\Gamma_j}\left(-\frac{\partial}{\partial x_j} f(\vec{x}) + \lim_{n\to\infty}\frac{1}{\rho_j n}\frac{\partial f(\theta_1)}{\partial x_j^2}\right),$$
$$+\sum_{j=1}^{N} \left(1-x_j\right) e^{\Gamma_j}\left(\frac{\partial}{\partial x_j} f(\vec{x}) + \lim_{n\to\infty}\frac{1}{\rho_j n}\frac{\partial f(\theta_2)}{\partial x_j^2}\right).$$

Now since $\frac{1}{n}$ goes to zero as n goes to infinity we get:

$$\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) = \sum_{j=1}^{N} -(1+x_j)e^{-\Gamma_j}\frac{\partial f}{\partial x_j} + (1-x_j)e^{\Gamma_j}\frac{\partial f}{\partial x_j}.$$

Rewriting this gives us what we wanted to show:

$$\lim_{n\to\infty} \mathscr{A}_n f(\vec{x}) =< \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}, \begin{bmatrix} (1-x_1)e^{\Gamma_1} - (1+x_1)e^{-\Gamma_1} \\ (1-x_2)e^{\Gamma_2} - (1+x_2)e^{-\Gamma_2} \end{bmatrix} >,$$
$$=< \nabla f, F >= \mathscr{A} f.$$

Using this we clearly see that:

$$\lim_n \sup_{x\in E_n} \left|\mathscr{A}_n(f\circ m_n)(x) - \mathscr{A} f(m_n(x))\right| = 0 \tag{4.20}$$

We have now shown that as n goes to infinity and $m_n(t)$ goes to x_t we have that the generator $\mathscr{A}_n$ converges to the generator $\mathscr{A}$. Therefore by Theorem 2.3, the N-population Mean-Field Ising model converges in probability to the N-dimensional system of Ordinary Differential equations described by Equation 4.10. □

5
Local bifurcations of the model for two and three populations

In this chapter we will be characterizing the two- and three-dimensional system of ordinary differential equations for when local bifurcations around the origin occur. In the first section we will be looking at the vector F obtained in Chapter 4 and determining when bifurcations occur in the two-dimensional model. In the second section of this chapter we will be doing analysis of the two-population model for a Hopf bifurcation. In the third section of this chapter we will focus on the analysis of a Hopf bifurcation in the three-dimensional model and in the fourth and final section we will do two numerical simulations of Hopf bifurcations in the three-dimensional model. This chapter will solely focus on local bifurcations. Global bifurcations are outside the scope of this thesis. We will first make some notation changes to the N-dimensional ordinary differential equation given by:

$$\begin{cases} \frac{d\vec{x}}{dt} = F(\vec{x}) \\ x(0) = x_0, \end{cases} \tag{5.1}$$

where $F(\vec{x})$ is the N-dimensional vector given by:

$$F(\vec{x}) = \begin{pmatrix} (1-x_1)e^{\Gamma_N} - (1+x_N)e^{-\Gamma_1} \\ \vdots \\ (1-x_N)e^{\Gamma_N} - (1+x_N)e^{-\Gamma_N} \end{pmatrix} \tag{5.2}$$

The change in notation we make, has to do with the interaction matrix β. For this we first diagonalize the fraction vector ρ giving:

$$\rho_d = \text{diag}(\rho_1, \rho_2, \dots, \rho_N) = \begin{pmatrix} \rho_1 & 0 & \dots & 0 \\ 0 & \rho_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_N \end{pmatrix} \tag{5.3}$$

We now define a new interaction matrix α, which includes the populations fractions ρ. α is defined by:

$$\alpha = \rho_d \beta = \begin{pmatrix} \rho_1\beta_{11} & \dots & \rho_1\beta_{1N} \\ \vdots & \ddots & \vdots \\ \rho_N\beta_{N1} & \dots & \rho_N\beta_{NN} \end{pmatrix} \tag{5.4}$$

This allows us to write Γ_j as:

$$\Gamma_j(\vec{x}) = \vec{x}^T \alpha e_j \tag{5.5}$$

The second change in notation we introduce is rewriting the vector F using the hyperbolic sine and cosine given by:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{5.6}$$

$$\text{and} \tag{5.7}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{5.8}$$

Inserting these two equations into the definition of the vector $F(\vec{x})$ gives:

$$F(\vec{x}) = \begin{pmatrix} 2\sinh(\Gamma_1) - 2x_1\cosh(\Gamma_1) \\ \vdots \\ 2\sinh(\Gamma_N) - 2x_N\cosh(\Gamma_N) \end{pmatrix} \tag{5.9}$$

5.1. Bifurcations in the 2-dimensional Mean-Field Ising model

For the analysis of the bifurcation we will be looking at an example of a group of men population 1 and a group of women population 2. We will be looking at the equilibrium point $x_0 = (0,0)$. This means that the women and the men both have an average of 0 opinion on the subject. Meaning half of each group has an positive opinion and half of each has an negative opinion on the subject. The two-population system of ordinary differential equations discussed in this section is given by:

$$\begin{cases} \frac{d\vec{x}}{dt} = F(\vec{x}) \\ x(0) = x_0 \end{cases} \tag{5.10}$$

Where $F(\vec{x})$ is the two-dimensional vector given by:

$$F(\vec{x}) = \begin{pmatrix} 2\sinh(\Gamma_1) - 2x_1\cosh(\Gamma_1) \\ 2\sinh(\Gamma_2) - 2x_2\cosh(\Gamma_2) \end{pmatrix} \tag{5.11}$$

We start this chapter by showing that two types of bifurcations are not possible at $x = (0,0)$.

Lemma 5.1. *The two-dimensional system of differential equations given by Equation 5.10 cannot have the following bifurcation types at* $(0,0)$*: (a) A saddle-point bifurcation and (b) a transcritical bifurcation.*

Proof. Suppose there is either a saddle-point of transcritical bifurcation at $(0,0)$. Then by Theorem 2.5 and Theorem 2.6, we have $w^t D^2 F|_{x=(0,0)}(v,v) \neq \vec{0}$. However if we look at D^2F we get:

$$D^2F = \begin{pmatrix} \left([2\alpha_{11}^2 - 4\alpha_{11}]\, v_1^2 + [2\alpha_{21}^2 - 2\alpha_{21}]\, v_2^2 + [4\alpha_{11}\alpha_{21} - 4\alpha_{21}]\, v_1 v_2\right)\sinh(\Gamma_1) - \left(2\alpha_{11}^2 v_1^2 + 2\alpha_{21}^2 v_2^2 + 4\alpha_{11}\alpha_{21} v_1 v_2\right) x_1\cosh(\Gamma_1) \\ \left([2\alpha_{22}^2 - 4\alpha_{22}]\, v_1^2 + [2\alpha_{12}^2 - 2\alpha_{12}]\, v_2^2 + [4\alpha_{22}\alpha_{12} - 4\alpha_{12}]\, v_1 v_2\right)\sinh(\Gamma_2) - \left(2\alpha_{22}^2 v_1^2 + 2\alpha_{12}^2 v_2^2 + 4\alpha_{22}\alpha_{12} v_1 v_2\right) x_2\cosh(\Gamma_2) \end{pmatrix}.$$

Now substituting for $x = (0,0)$ gives:

$$D^2F|_{x=(0,0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.12}$$

This is leads to a contradiction. So by contradiction we have shown there cannot be a saddle-point or transcritical bifurcation at $(0,0)$. □

Now we are going to take a look at the Hopf bifurcation, which indicates that limit cycles occur.

Lemma 5.2. *The two-dimensional ordinary differential equation given by Equation 5.10 has a Hopf bifurcation at* $(0,0)$ *with bifurcation parameter* $\mu = \alpha_{11}$ *if the following are true:*

$$\alpha_{11} + \alpha_{22} = 2, \tag{5.13}$$

$$(\alpha_{11} - \alpha_{22})^2 + \alpha_{12}\alpha_{21} < 0. \tag{5.14}$$

Proof. For this we will be using Theorem 2.7. The Jacobian of vector F at $(0,0)$ is given by:

$$J(F(\vec{x}))|_{\vec{x}=(0,0)} = \begin{pmatrix} 2\alpha_{11}-2 & 2\alpha_{21} \\ 2\alpha_{12} & 2\alpha_{22}-2 \end{pmatrix} \tag{5.15}$$

Now let us determine the eigenvalues of this matrix. Using $\det\left(J(F)|_{x=(0,0)} - \lambda I\right) = 0$ we get the equality:

$$\lambda^2 + (4-2\alpha_{22}-2\alpha_{11})\lambda - 4(\alpha_{11}+\alpha_{22}-\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}-1) \tag{5.16}$$

Solving for λ using the quadratic formula gives:

$$\lambda_{\pm} = \alpha_{11}+\alpha_{22}-2 \pm \frac{1}{2}\sqrt{(4-\alpha_{22}2-2\alpha_{11})^2 - 16(\alpha_{11}+\alpha_{22}-\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}-1)} \tag{5.17}$$

We can simplify this equation to:

$$\lambda_{\pm} = \alpha_{11}+\alpha_{22}-2 \pm \sqrt{(\alpha_{11}-\alpha_{22})^2 + 4\alpha_{12}\alpha_{21}} \tag{5.18}$$

Now suppose $\alpha_{11}+\alpha_{22} = 2$ and $(\alpha_{11}-\alpha_{22})^2 + \alpha_{12}\alpha_{21} < 0$ hold. Substituting the former in Equation 5.18, gives:

$$\lambda_{\pm} = (\alpha_{11}-\alpha_{22})^2 + \alpha_{12}\alpha_{21} < 0 \tag{5.19}$$

Now we get $\Im(\lambda) \neq 0$ and $\Re(\lambda) = 0$. If we now take the derivative of the real part of the eigenvalues w.r.t. μ we get:

$$\frac{d}{d\mu}[\mu + \alpha_{22}]_{\mu=\mu_0} = 1 \tag{5.20}$$

This is not equal to zero, so by Theorem 2.7 there is a Hopf bifurcation at $(0,0)$ as μ passes through μ_0. □

5.2. Numerical methods on the 2-dimensional Mean-Field Ising Model

To perform the numerical analysis of the 2-dimensional Mean-Field Ising model we want to construct an interaction matrix which satisfies the conditions of Lemma 5.2.For this we start with a matrix A, which has a conjugate pair of purely imaginary eigenvalue $\lambda_{\pm} = \pm\omega i$, with $\omega \in \mathbb{R}$. This matrix is given by:

$$A = \begin{pmatrix} \omega i & 0 \\ 0 & -\omega i \end{pmatrix} \tag{5.21}$$

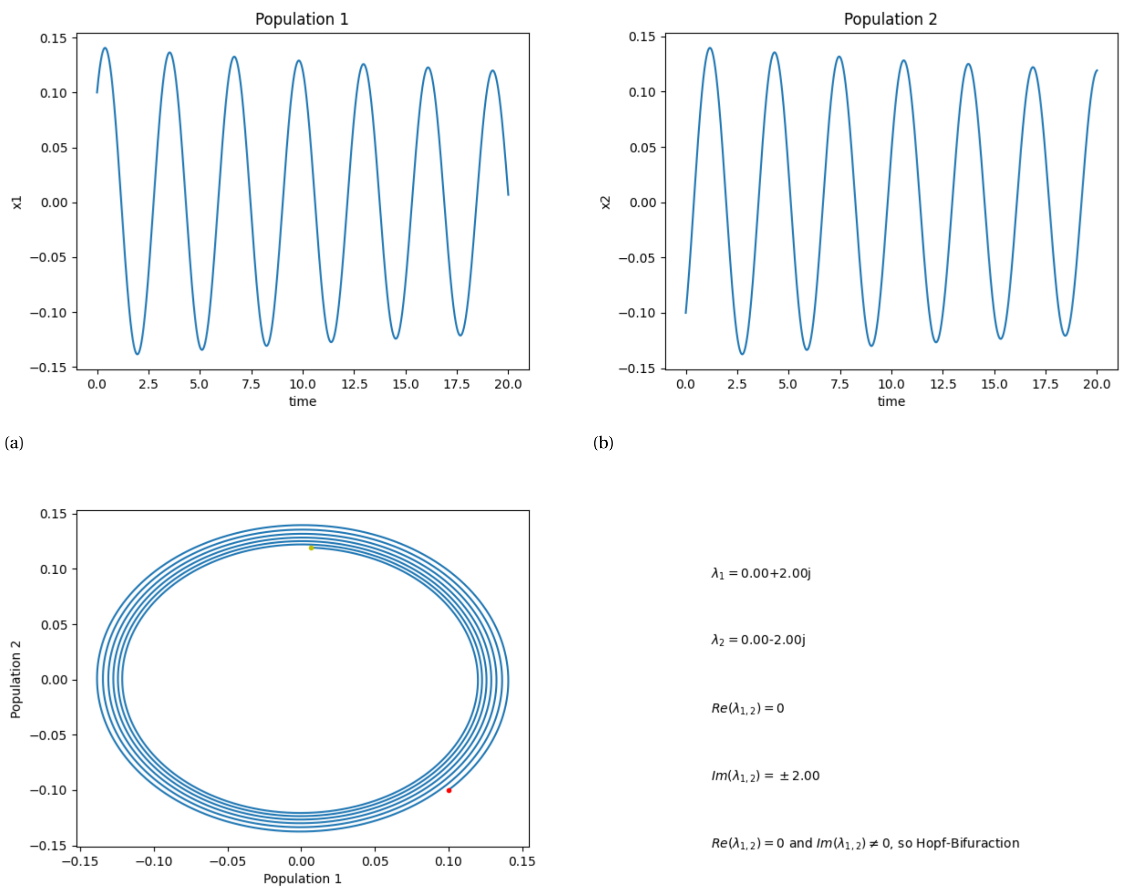
Next up we alter the matrix such that it only contains real numbers. This gives us:

$$A_{real} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{5.22}$$

Next we want to find the interaction matrix C, which Jacobian is equal to matrix A_{real}. This matrix is given by:

$$C = \begin{pmatrix} 1 & -\frac{\omega}{2} \\ \frac{\omega}{2} & 1 \end{pmatrix} \tag{5.23}$$

This matrix satisfies the conditions of Lemma 5.2. We set α_{11} to be the bifurcation parameter, then the bifurcation value $\mu_0 = 1$. We see that for $\mu < \mu_0$, $\Re_(\lambda_{\pm}) < 0$, for $\mu > \mu_0$, $\Re_(\lambda_{\pm}) > 0$ and for $\mu = \mu_0$, $\Re(\lambda_{\pm}) = 0$. Next up we performed a numerical simulation on the two-dimensional system of ordinary differential equations given by Equation 5.10, with interaction matrix C. For this we have chosen $\omega = 2$. We used the forward difference method, with time-step $dt = 10^{-5}$, for a total of two million steps. The simulation started close to $(0,0)$ at the point $x = (0.1, -0.1)$ The results of the simulation are shown in figure 5.1. The simulation shows the occurrence of a limit cycle, due to the Hopf bifurcation.



(a) (b) (c) (d)

Figure 5.1: (a) shows the evolution of the average opinion of population 1, (b) shows the evolution of the average opinion of population 2, (c) plots the average opinion of population 1 vs the average opinion of population 2. The start point of the simulation is shown in red en the end point in green. (d) shows the check if a Hopf bifurcation is possible.

5.3. Bifurcations of the 3-dimensional Mean-Field Ising model

In this section we will be discussing the three-dimensional Mean-field Ising, as derived in Chapter **??**. This system is given by:

$$\begin{cases} \frac{d\vec{x}}{dt} = F(\vec{x}) \\ x(0) = x_0 \end{cases} \tag{5.24}$$

Where $F(\vec{x})$ is the three-dimensional vector given by:

$$F(\vec{x}) = \begin{pmatrix} 2\sinh(\Gamma_1) - 2x_1\cosh(\Gamma_1) \\ 2\sinh(\Gamma_2) - 2x_2\cosh(\Gamma_2) \\ 2\sinh(\Gamma_3) - 2x_3\cosh(\Gamma_3) \end{pmatrix} \tag{5.25}$$

We will start by stating the conditions on when a Hopf bifurcation occurs.

Lemma 5.3. *Let $x(t)$ be the solution of the three-dimensional system of ordinary differential equations given by Equation 5.24. Let 3×3-matrix D be given by:*

$$D = \begin{pmatrix} 2\alpha_{11} - 2 & 2\alpha_{21} & 2\alpha_{31} \\ 2\alpha_{12} & 2\alpha_{22} - 2 & 2\alpha_{32} \\ 2\alpha_{13} & 2\alpha_{23} & 2\alpha_{33} - 2 \end{pmatrix}. \tag{5.26}$$

Suppose there is a choice of parameter μ and bifurcation value μ_0 such that:

$$\frac{d}{d\mu}[\Re(\lambda_\mu)]|_{\mu=\mu_0} \neq 0. \tag{5.27}$$

The system undergoes a Hopf-bifurcation at $x = (0,0)$ *as* μ *goes through* μ_0 *if* D *has to eigenvalues of the form* $\lambda = \pm\omega i$ *and one eigenvalue of the form* $\lambda = d$*, with* $w, d \in \mathbb{R}$

Proof. Suppose D has eigenvalues and a bifurcation value as described in the lemma. By theorem 2.7 we know that the Jacobian of vector F at $x = (0,0)$ has to have a conjugate pair of pure imaginary eigenvalues and a real eigenvalue. After long and cumbersome computations we have found the Jacobian of F at to be given by:

$$J(F)|_{x_0=(0,0)} = \begin{pmatrix} 2\alpha_{11}-2 & 2\alpha_{21} & 2\alpha_{31} \\ 2\alpha_{12} & 2\alpha_{22}-2 & 2\alpha_{32} \\ 2\alpha_{13} & 2\alpha_{23} & 2\alpha_{33}-2 \end{pmatrix}. \tag{5.28}$$

. Clearly this is equal to D. So $J(F)|_{x=(0,0)}$ has the same eigenvalues of D. Therefore by Theorem 2.7, there is an Hopf bifurcation at $x = (0,0)$. □

5.4. Numerical methods on the 3-dimensional Mean-Field Ising model

We want to construct a matrix which satisfies the conditions laid out in Lemma 5.3. For this we start with a matrix A, which has a conjugate pair of pure imaginary eigenvalues $\lambda_\pm = \pm\omega i$ and a real eigenvalue $\lambda_{real} = a$, with $a, \omega \in \mathbb{R}$. We construct this by making A a diagonal matrix with the eigenvalues on the diagonal:

$$A = \begin{pmatrix} \omega i & 0 & 0 \\ 0 & -\omega i & 0 \\ 0 & 0 & a \end{pmatrix} \tag{5.29}$$

Next up we want this matrix to only contain real numbers, so we create the matrix A_{real}, with the same eigenvalues given by:

$$A_{real} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \tag{5.30}$$

Clearly this matrix has the same eigenvalues. Next we want to find a interaction matrix C such that its Jacobian at zero is equal to matrix A. This interaction matrix is given by:

$$C = \begin{pmatrix} 1 & \frac{\omega}{2} & 0 \\ -\frac{\omega}{2} & 1 & \\ 0 & 0 & \frac{a}{2}+1 \end{pmatrix} \tag{5.31}$$

This matrix satisfies the conditions of Lemma 5.3. Next up we did a numerical analysis on the three-dimensional system of ordinary differentials equations given by Equation 5.24 with the interaction matrix being C, where we have chosen $\omega = 2$ and $a = -2$. For this we used the forward difference methodwith a timestep $dt = 10^{-5}$ for a total of two million steps. The simulation started at starting point $x = (0.1, -0.2, 0.07)$. The results of this analysis are shown in Figure 5.2.

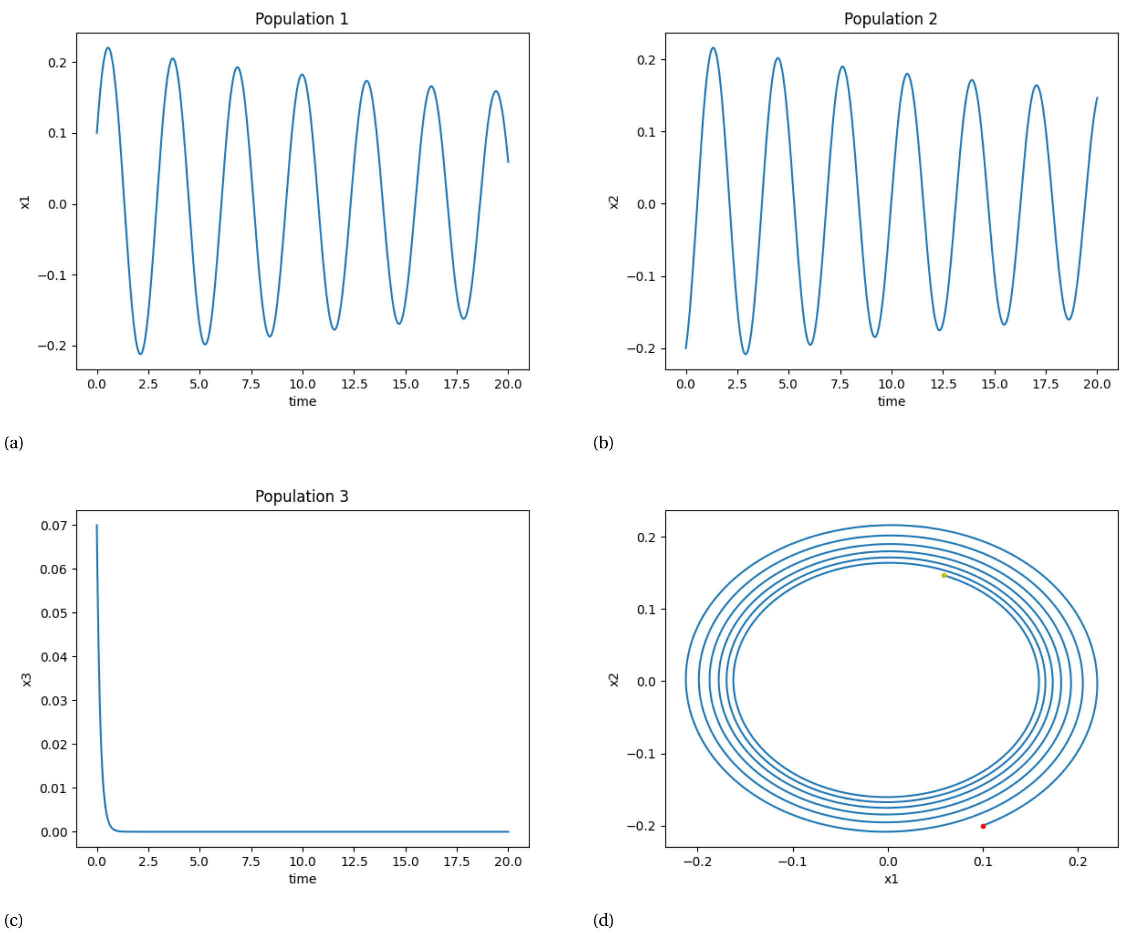


Figure 5.2: (a) shows the evolution of the average opinion of population 1, (b) shows the evolution of the average opinion of population 2, (c) shows the evolution of the average opinion of population 3, (d) plots the average opinion of population 1 vs the average opinion of population 2. The start point of the simulation is shown in red en the end point in green.

The figures show the occurrence of two periodic populations and one stable population. The Hopf bifurcation caused the occurrence of a limit cycle between two of the three populations, as expected.

Lastly we construct a third numerical simulation. We wanted to have a simulation where all three populations where oscillating. What we did was to let the third population act as a "carrier" population. We started with an interaction matrix where two populations oscillate given by:

$$C_{original} = \begin{pmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.32}$$

Then we switch the interaction factors α_{12} and α_{13}. We also set the interaction factor $\alpha_{32} = 3$. This caused population 1 to influence population 2 through population 3. This gave us the interaction matrix C given by:

$$C = \begin{pmatrix} 2 & 0 & -3 \\ 3 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix}. \tag{5.33}$$

ext up we did a numerical analysis on the three-dimensional system of ordinary differentials equations given by Equation 5.24 with the interaction matrix being C. For this we used the forward difference method with a timestep $dt = 10^{-5}$ for a total of two million steps. The simulation started at starting point $x = (0.1, -0.2, 0.07)$. The results are shown in Figure 5.3. It shows that we succeeded in our goal to make all three populations oscillate.

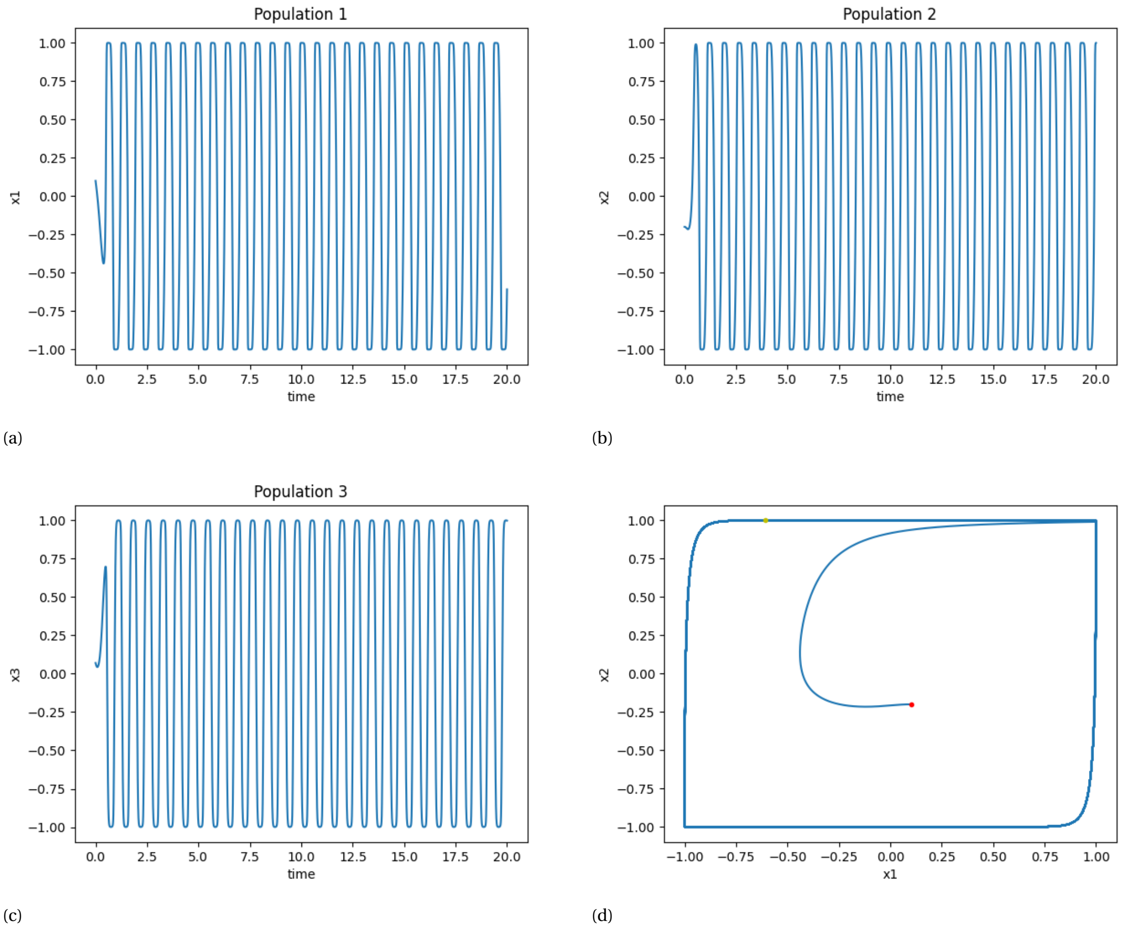


Figure 5.3: (a) shows the evolution of the average opinion of population 1, (b) shows the evolution of the average opinion of population 2, (c) shows the evolution of the average opinion of population 3, (d) plots the average opinion of population 1 vs the average opinion of population 2. The start point of the simulation is shown in red en the end point in green.

6
Discussion

The objective of the study in this paper was to extend the two-population mean-field Ising model to a general N-population mean-field Ising model, with the aim to use it to model opinion dynamics. We proved that the two- and N-population mean-field Ising model converge to systems of ordinary differential equations. This result bridges the gap between a statistical process and a deterministic process.

We found that by created a new interaction matrix by taking the production of the old interaction matrix and the population fractions. This shows that a big population can be modeled as a small population with a big interaction coefficient and vice versa.

We characterized when Hopf bifurcations occur for the two- and three-dimensional models. This result is of good use for the modeling of opinion dynamics. A Hopf bifurcation coincides with a big change in system behavior. This could be used to model big changes in opinions to changes in society.

In this study we only focused on local bifurcation at $x_0 = (0,0)$. Further research should be done on other equilibrium points and global bifurcations. This would allow us the predict the population dynamics even better. The bifurcation analysis in this study was solely focused on the two- and three-dimensional models. Further research should be done on bifurcations in the general N model.

When we defined the system, we excluded the external field. The effect of an external field should be researched and whether the convergence from statistic model to deterministic model still happens.

One of the limitations of the model is the binary behavior of the opinions. Each individual could either have a positive or negative opinion. In reality opinions are a lot more nuanced and a person could for example also have a neutral opinion on a subject. Further research should be done on adding more opinion possibilities to the model.

In the final section of Chapter 5 we showed that it is possible to have three populations oscillating at the same time. The conjugate complex pair of eigenvalue of the Jacobian of this interaction matrix, has a non-zero eigenvalue. Further research should be done what the conditions are for three populations to oscillate.

7
Conclusion

The objective of the study in this paper was to analyze population dynamics, by expanding the two-population mean-field Ising model to a general N-population mean-field Ising model and showing its convergence to a system of ordinary differential equations. In Chapter 3, we defined the notation of the two-dimensional mean-field Ising and showed its convergence to the two-dimensional system of ordinary differential equations given by Equation 3.15.
In Chapter 4 we expanded the notation of the two-dimension mean-field Ising model to the general N situation. Subsequently we showed the convergence of model the N-dimensional system of ordinary differential equations given by Equation 4.10 using convergence of generators. In chapter 5 we showed that for the two-dimensional model transcritical and sandle-node bifurcations are not possible at $x_0 = 0,0$. We found that a Hopf-bifurcation is possible for the three-dimensional model if the following requirements for the interactions coefficients are met:

$$\alpha_{11} + \alpha_{22} = 2, \tag{7.1}$$

$$(\alpha_{11} - \alpha_{22})^2 + \alpha_{12}\alpha_{21} < 0. \tag{7.2}$$

With a numerical analysis using the forward difference method we showed that when these conditions are met a limit cycle occurs close to x_0.
For the three-dimensional model we gave the requirements needed for a Hopf bifurcation in Lemma 5.3. Here we again performed a numerical analysis and again found limit cycles close to x_0. We were also able to construct a model were three population oscillate.

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