Technische Universiteit Delft
Faculteit Elektrotechniek, Wiskunde en Informatica
Delft Institute of Applied Mathematics

# Numerieke prijsbepaling van verschillende soorten Aziatische opties

# (Engelse titel: Numerical pricing of several types of Asian options)

Verslag ten behoeve van het
Delft Institute of Applied Mathematics
als onderdeel ter verkrijging

van de graad van

**BACHELOR OF SCIENCE**
**in**
**TECHNISCHE WISKUNDE**

**door**

**IRIS KOOIJMAN**

**Delft, Nederland**
**December 2018**

**BSc verslag TECHNISCHE WISKUNDE**

# "Numerieke prijsbepaling van verschillende soorten Aziatische opties"

**(Engelse titel: "Numerical pricing of several types of Asian options")**


IRIS KOOIJMAN

**Technische Universiteit Delft**


**Begeleider**

Prof.dr.ir. C.W. Oosterlee

**Overige commissieleden**

Dr. J.G. Spandaw

Dr. J.L.A Dubbeldam

December, 2018

Delft

# Abstract

Option valuation is one of the more applied areas of mathematics. Options are financial derivatives whose value depends on the value of an underlying asset. They are frequently used in hedging to minimize the risk when trading in the underlying stock and therefore require accurate pricing. Though quite some research has been conducted on standard option types such as European options and American options, more exotic options such as conditional Asian options or Asian tail options are less known and therefore enjoyed far less attention from the research community.

In this bachelor thesis we will focus on computing the correct price for the Asian option and various types of Asian options. A short introduction on these types of options will be given, after which mathematical models for pricing these types of options will be derived. Several numerical methods such as the Monte Carlo method and the finite difference method (applied to the Black-Scholes partial differential equation) will be used to approximate the true value of these types of options.

# Contents

# Frequently used notation

| | |
|---|---|
| $(A-B)_+$ | $\max(A-B,0)$ |
| $K$ | Strike price |
| $r$ | Interest rate |
| $t$ | Time variable |
| $\sigma$ | Volatility of the underlying stock |
| $V_C$ | Value of a call option |
| $V_P$ | Value of a put option |
| $T$ | Maturity time of the option |
| $S$ | Price of the underlying stock |
| $A$ | Average price of the underlying stock |
| $\mu$ | Drift of the stock |
| $\Lambda(x)$ | Payoff function of the option |
| $\mathbb{Q}$ | Risk neutral probability measure |
| $W_t$ | Brownian motion under $\mathbb{Q}$ |
| $Z_t$ | Running sum of stock price |
| $b$ | Threshold value for the conditional Asian option |
| $M$ | Number of computed asset price paths in the Monte Carlo method |
| $\Delta t$ | Size of the time step |
| $N_t$ | Number of time steps |
| $V^a$ | Asian option with arithmetic average |
| $V^g$ | Asian option with geometric average |
| $F_t$ | Filtration generated by discrete option value |

# Chapter 1

# Introduction

The financial market is a concept that has generated a lot of interest throughout history. It is a place where one does not only trade physical products, but also immaterial products such as stocks, futures, bonds and options. An option is a type of *derivative*, which means that its value is derived from another asset, in this case from the value of the underlying stock. There is an enormous amount of variety in types of options, the most standard being the *European call and put options*.

**Definition 1.0.1.** a) A **European call option** gives its holder the right (but not the obligation) to purchase from the writer a prescribed asset for a prescribed price at a prescribed time in the future [11].

b) A **European put option** gives its holder the right (but not the obligation) to sell to the writer a prescribed asset for a prescribed price at a prescribed time in the future [11].

The holder is the person who owns the option, the writer is the person who sold the option to the holder. Whereas the holder has the option, the writer has the obligation to buy/sell. The prescribed time in the future is commonly referred to as the *expiry date* or *maturity time* and the prescribed price is called the *strike price*.

Suppose the holder owns a European call option on the stock of company $A$ with strike price $K$. At the maturity time $T$, the holder of the option is given the choice to either exercise or not to exercise the option. Exercising the option means: to buy an asset of company $A$ for price $K$. Whether or not the holder will exercise the option, depends on the stock price $S$ of company $A$ at time $T$. If the stock price is larger than the strike price, the holder can simply exercise the option to buy the assets for price $K$ and immediately sell them on the market for price $S$. This will give the holder an immediate, risk free profit of $S - K > 0$. However, if the stock price is smaller than the strike price, exercising the option will result in a loss and the holder will choose not to exercise. We therefore say that the **value of the call option at time** $T$ or **payoff** is equal to $\max(S - K, 0)$.
We denote this by:

$$V_C(S,T) = (S_T - K)_+$$

Here $V_C(S,T)$ denotes the value of the call option for stock price $S$ at maturity time $T$.

The same reasoning applies for the European put option, with the value at time $t = T$ denoted by:

$$V_P(S,T) = (K - S_T)_+$$

The values of the European options are known at $t = T$, but the question for financial engineers is to evaluate the value of the option at $t = 0$. In this way, the stock traders know the (mathematically) correct price they have to pay at $t = 0$.

European options are often called *vanilla options*, since they are the most basic options available on the market. In addition to European options, there is a extensive amount of *exotic options* available, such as American options, Bermudan options, barrier options etc. each with distinct properties that make them useful for traders in different sectors of the financial market.

## 1.1 Problem description

In this thesis, we pay special attention to valuating one of the exotic options: **Asian options**. Whereas the value of the European option at final time $T$ depends on the stock price at time $T$, the value of the Asian option depends on the average of the stock price during the lifetime of the option. It is therefore often called a *path-dependant option.*

Options are often used for *hedging.* This is a type of strategy that traders use to cover for the risk they are exposed to when trading in stocks. It is not necessarily a method to make a lot of money, but rather a method that protects the trader. Since traders are sometimes investing large sums of money in the market, they heavily rely on options to be correctly priced. This makes the research done in the field of financial engineering (and quantitative finance in particular) fascinating and essential.

A lot of scientific literature has been written about option valuation. In an initial effort to value options, Myron Scholes and Fisher Black introduced the Black-Scholes partial differential equation (see chapter 3) which described the price of the European option and gave an exact solution [1]. A while later, Kemna and Vorst were one of the first to introduce a pricing method for the Asian option, based on the Monte Carlo method [12].
After that, many papers were written proposing different methods to value the Asian option. Some papers described a method that uses a dimension reduction of the Black-Scholes PDE for continuously sampled Asian options in order to try and find the mathematically fair price of the option (e.g. "Robust Numerical Methods for PDE Models of Asian Options" by Zvan et al. [18]). Other papers (e.g. "Convergence of numerical methods for valuing path-dependent options using interpolation" by Forsyth et al.[9]) introduced the "jump condition" in order to use the Black-Scholes PDE for European options for discretely sampled Asian options. This idea will be explained in chapter 2 by using the finite difference method.

Though the European and Asian options are often discussed in papers such as those mentioned above, several variations on Asian options are not. In this thesis we will use the methods described by those papers to focus on valuing several types of Asian options and look at numerical methods to price those types of options. In chapter 2 an outline of the type of financial products will be given and the mathematical model of the stock price will be described. In chapter 3 the numerical methods like the Monte Carlo method and the finite difference method and their relation to option valuation will be discussed, after which in chapter 4 the results of those methods will be presented. Finally, we will derive a conclusion from the presented material, which will lead to our recommendations for future work on the matter.

## 1.2 Assumptions

In order to mathematically model the stock price and evaluate the value of the option, some assumptions about the financial market have to be made.

1. The asset price may take any non-negative value.
2. Buying and selling an asset may take place at any time $0 \leq t \leq T$.
3. It is possible to buy and sell any amount of the asset.
4. The bid-ask spread (difference in the bid-price and the ask-price) is zero - the price for buying equals the price for selling.
5. There are no transaction costs.
6. There are no dividends or stock splits.
7. Short selling (selling assets though one does not (yet) possesses them) is allowed - it is possible to hold a negative amount of the asset.
8. There is a single, constant, risk-free interest rate that applies to any amount of money borrowed from or deposited in the bank.

[11]

One of the essential principles for pricing options is the principle of *no arbitrage*. This principle states that "there is never an opportunity to make a risk-free profit that gives a greater return than that provided by the interest from a bank deposit" [11]. This is a quite evident principle, since if this did not apply, people would simply borrow money from the bank for the given interest rate and invest it in the project, securing an immediate, risk-free profit.
In the real financial world arbitrage opportunities do occur. However, as soon as these opportunities are discovered, the market tries to exploit those and thereby making sure the opportunity ceases to exist.

# Chapter 2

# Mathematical model

In this chapter we begin by describing several types of options for which we later propose a way to value them, after which we describe two mathematical models which can be used to describe and find the value of the option. Finally, we introduce and use the Feynman-Kac formula.

## 2.1 European options

As mentioned in chapter 1, the European options are the most common options traded on the market. Let $K$ denote the strike price and $S_T$ the price of the stock at time $T$. The value of the European option depends on the price of the stock at final time $T$ and its payoff at time $T$ is given by:

$$
\begin{aligned}
V_P(S,T) &= (K - S_T)_+ \\
V_C(S,T) &= (S_T - K)_+
\end{aligned}
$$

Here $V_C$ denotes the value of the call option and $V_P$ denotes the value of the put option.

## 2.2 Asian option

Asian options are a type of exotic options that are fairly popular on the financial market. They were developed in the 1980's by Mark Standish and David Spaughton, both working for the British company Bankers Trust at the time. During a business trip to Tokyo, the men came up with the idea to create an new type of option and, since they were in Asia at the time, they decided to call it the Asian option [7].

The Asian option differs from the European option in the sense that its value does not just depend on the value of the stock at the expiry date, but on all previous values of the stock from the moment the option was sold to the holder. This makes the Asian option a so called *path dependent option*, since it depends on the average of the stock price.

Since the value of the Asian option depends on the average stock price over a period of time instead of the final price at the expiry date, the option has a lower volatility compared to the European or American options. They are interesting when the underlying stock is highly volatile, or for example when a person has to deal with the exchange rate of a currency over time [3].

Suppose we have an Asian option with expiry time $T$ and strike price $K$. Let $A_t$ be some type of average of the underlying stock price up to time $t$. There are two main types of Asian options:

- the **average price Asian option**: for this option the final stock price $S_T$ in the European option is replaced by $A_T$.
  - average price Asian call option: $V(S,T) = (A_T - K)_+$
  - average price Asian put option: $V(S,T) = (K - A_T)_+$
- the **average strike Asian option**: for this option the strike price $K$ in the European option is replaced by the average price $A_T$.
  - average strike Asian call option: $V(S,T) = (S_T - A_T)_+$
  - average strike Asian put option: $V(S,T) = (A_T - S_T)_+$ [11]

For this project, the definition of the **average price Asian option** will be used in the computation of pricing the option.
Though the described two types of options are the main two types, there is a lot of variation possible through the definition of the average. The average can be affected by several factors, such as:

- Period of averaging
- Arithmetic or geometric averaging
- Weighted or unweighted averaging
- Discrete or continuous sampling of the asset price [15]

In this thesis, we choose to first look at an Asian option with maturity time $T$ where the average is taken over the entire interval $[0,T]$ (thus without excluding sub-intervals), later also including options where the average is taken over a part of the interval $[0,T]$ (Asian tail options). The method of averaging is unweighted. Furthermore, we will focus first on the arithmetic averaging, but will later compare the results of the pricing with the geometric average. Additionally, we will shift our focus to the discrete sampling.
Let $S_{t_i}$ be the price of the stock at time $t_i$, $K$ the strike price of the option and $T$ its maturity time. Split the time line $T$ in $n$ equal parts. The value of the discrete average price Asian put option with arithmetic averaging can then be written as:

$$(K - \frac{1}{n}\sum_{i=1}^{n} S_{t_i})_+ \tag{2.1}$$

Similarly, the value of the average price Asian call option can be written as:

$$(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K)_+ \tag{2.2}$$

At each time $t_i$ (with $0 \leq i \leq n$), the value of the Asian option will be updated according to the stock price at time $t_i$. Therefore, the option does not just depend on the stock price $S$ and time $t$, it now also depends on the average stock price $A(S,t)$. This gives an extra dimension to take into consideration when computing the fair price of $V(S,A,t)$.

## 2.3 Conditional Asian options

Within the spectrum of Asian options there exist many slightly different types of Asian options. They can differ in e.g. time span, imposed conditions or type of average. One instantiation is the conditional Asian option. As the name suggests, this type of Asian option is *conditioned* on a certain property. Like the Asian option, it takes the average of the stock value over a certain period, but the payoff is determined only by the average of the stock values which are larger than a certain threshold.
Let $b$ denote the threshold. The average of the stock value can than be written as:

$$A = \frac{\sum_{i=1}^{n} S_{t_i} I_{\{S_{t_i} > b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}} \tag{2.3}$$

Here $I$ is the indicator function.
The pricing of this option is usually done through simulations (see chapter 4), although a Fourier based, semi-analytic pricing method was proposed by Feng and Volkmer in 2015 [8]. Since the conditional Asian option adheres to a certain condition, its value is naturally different from the regular Asian option. This difference is different for the put and the call option.

### 2.3.1 Conditional Asian put option

Observe that the value of the discrete conditional Asian put option can be written as [8]:

$$\Big(K - \frac{\sum_{i=1}^{n} S_{t_i} I_{\{S_{t_i} > b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}}\Big)_+$$

Notice that for $b = 0$ the conditional Asian option is actually a regular Asian option.

We now prove that conditional Asian puts are cheaper than regular Asian puts. Observe that:

$$\begin{aligned}
\frac{\sum_{i=1}^{n} S_{t_i} I_{\{S_{t_i} > b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}} &= \sum_{j=1}^{n} \frac{S_{t_j} I_{\{S_{t_j} > b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}} \\
&= \frac{1}{n} \cdot n \sum_{j=1}^{n} \frac{S_{t_j} I_{\{S_{t_j} > b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}} \\
&= \frac{1}{n} \sum_{j=1}^{n} n \frac{S_{t_j} I_{\{S_{t_j} > b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}} \\
&= \frac{1}{n} \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} I_{\{S_{t_i} \leq b\}} + I_{\{S_{t_i} > b\}}\Big) \frac{S_{t_j} I_{\{S_{t_j} > b\}}}{\sum_{i=1}^{n} I_{\{S_{t_j} > b\}}} \\
&= \frac{1}{n} \sum_{j=1}^{n} \frac{\sum_{i=1}^{n} I_{\{S_{t_i} \leq b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}} S_{t_j} I_{\{S_{t_j} > b\}} + \frac{1}{n} \sum_{j=1}^{n} S_{t_j} I_{\{S_{t_j} > b\}} \\
&= \frac{1}{n} \frac{\sum_{i=1}^{n} I_{\{S_{t_i} \leq b\}}}{\sum_{i=1}^{n} I_{\{S_{t_i} > b\}}} \sum_{j=1}^{n} S_{t_j} I_{\{S_{t_j} > b\}} + \frac{1}{n} \sum_{j=1}^{n} S_{t_j} I_{\{S_{t_j} > b\}}
\end{aligned}$$

Notice that $\sum_{j=1}^n S_{t_j} I_{\{S_{t_j}>b\}} \geq \sum_{j=1}^n b I_{\{S_{t_j}>b\}}$ because of the indicator function $I_{\{S_{t_j}>b\}}$. This implies:

$$\begin{aligned}
\frac{1}{n}\frac{\sum_{i=1}^n I_{\{S_{t_i}\leq b\}}}{\sum_{i=1}^n I_{\{S_{t_i}>b\}}}\sum_{j=1}^n S_{t_j} I_{\{S_{t_j}>b\}} &\geq \frac{1}{n}\frac{\sum_{i=1}^n I_{\{S_{t_i}\leq b\}}}{\sum_{i=1}^n I_{\{S_{t_i}>b\}}}\sum_{j=1}^n b I_{\{S_{t_j}>b\}} \\
&= \frac{b}{n}\frac{\sum_{i=1}^n I_{\{S_{t_i}\leq b\}}}{\sum_{i=1}^n I_{\{S_{t_i}>b\}}}\sum_{j=1}^n I_{\{S_{t_j}>b\}} \\
&= \frac{b}{n}\sum_{i=1}^n I_{\{S_{t_i}\leq b\}}
\end{aligned}$$

Therefore we can conclude that:

$$\begin{aligned}
\frac{\sum_{i=1}^n S_i I_{\{S_{t_i}>b\}}}{\sum_{i=1}^n I_{\{S_{t_i}>b\}}} &\geq \frac{b}{n}\sum_{i=1}^n I_{\{S_{t_i}\leq b\}} + \frac{1}{n}\sum_{j=1}^n S_{t_j} I_{\{S_{t_j}>b\}} \\
&= \frac{1}{n}\sum_{j=1}^n b I_{\{S_{t_j}\leq b\}} + \frac{1}{n}\sum_{j=1}^n S_{t_j} I_{\{S_{t_j}>b\}} \\
&\geq \frac{1}{n}\sum_{j=1}^n S_{t_j} I_{\{S_{t_j}\leq b\}} + \frac{1}{n}\sum_{j=1}^n S_{t_j} I_{\{S_{t_j}>b\}} \\
&= \frac{1}{n}\sum_{j=1}^n S_{t_j}
\end{aligned}$$

Therefore $\left(K - \frac{\sum_{i=1}^n S_{t_i} I_{\{S_{t_i}>b\}}}{\sum_{i=1}^n I_{\{S_{t_i}>b\}}}\right)_+ \leq (K - \frac{1}{n}\sum_{i=1}^n S_{t_i})_+$.
In other words: the value of the conditional Asian put option is less than the value of the regular Asian put option.

### 2.3.2 Conditional Asian call option

For the conditional Asian call option, the opposite applies: it is worth *more* than the regular Asian call option. Notice that the value of the discrete conditional Asian call option can be written as:

$$\Big(\frac{\sum_{i=1}^n S_{t_i} I_{\{S_{t_i}>b\}}}{\sum_{i=1}^n I_{\{S_{t_i}>b\}}} - K\Big)_+$$

It can be proven that:

$$\Big(\frac{\sum_{i=1}^n S_{t_i} I_{\{S_{t_i}>b\}}}{\sum_{i=1}^n I_{\{S_{t_i}>b\}}} - K\Big)_+ \geq (K - \frac{1}{n}\sum_{i=1}^n S_{t_i})_+$$

The proof is similar to the proof in section 2.3.1.

### 2.3.3 Financial interpretation

Now that we mathematically understand the price difference between conditional and regular Asian options, we look at the financial interpretation.

This type of option is relatively new and not very well known nor well traded in the financial market. Those who trade this type of option view it as a viable hedging and risk management instrument [8]. The advantage of the conditional Asian option w.r.t the regular Asian option is that the conditional Asian option has lower volatility. This is because of the constrained composition of the average, resulting in a less volatile average and thereby resulting in a less volatile payoff. Additionally, the average of the conditional Asian option is usually higher than that of the regular Asian option, which makes the conditional Asian put option cheaper and the Asian call more expensive.

## 2.4 Asian tail options

The Asian tail option is a regular European option where the averaging "Asian" feature is only active in the final part of the life of the option. This means that the payoff for the option is determined by the average of not the entire stock price path, but only the last part. This type of option is popular by traders in options with a long life time, e.g. employee options, since it protects the owner against last-minute price changes. Usually the tail kicks in ten to twenty days before the maturity time. [5]

Moreover, it reduces price manipulations. Suppose a writer sold a European call option to a holder. At maturity time $T$, the payoff is equal to: $V(S,T) = (S_T - K)_+$. The writer would like to have a low stock price, since that would reduce the value of the option, so he/she might be tempted to (when possible) manipulate the stock price right before the maturity time, for example by releasing negative information about the company. The Asian tail feature reduces the occurrence of this happening.

## 2.5 Asset price model

In order to value the option, a price path of the asset is needed. The obvious problem is that at $t = 0$, the price path of the asset is still unknown. Therefore, an *asset path model* is needed, which is typically the solution of a stochastic differential equation (SDE).
Let $\mathbb{Q}$ be the risk neutral probability measure.

**Definition 2.5.1.** The **risk neutral probability measure** is a probability measure such that each stock price is exactly equal to the discounted expectation of the share price under this measure.

[4]

**Definition 2.5.2.** Let $S_0, S_1, ..., S_{n-1}, S_n$ be a sequence of random variables, with $S_0$ constant. This process is called a **martingale** if:

1. $\mathbb{E}(S_i) < \infty \quad \forall i \in [1, n]$
2. $S_i = \mathbb{E}(S_{i+1}|S_1, ..., S_i) \quad \forall i \in [1, n]$

[14]

Observe the SDE below:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.4}$$

This SDE describes the movement of the stock price. In this equation $\mu$ is a constant parameter and represents the *drift*: the general movement of the stock. $\sigma$ represents the volatility of the stock, $S_t$ the price of the asset at time $t$ and $W_t$ is a Brownian motion under $\mathbb{Q}$. The derivation of the solution can be found through Itô Calculus in [16] and is given by:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}X_i} \text{ with } X_i \sim N(0,1) \tag{2.5}$$

$S_t$ follows a Geometric Brownian Motion (GBM).
Divide the time line [0,T] in $N_t$ equidistant intervals of length $\Delta t$. Our discrete-time model can be described by: [11]

$$S_{t_{i+1}} \quad = \quad S_{t_i} e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}X_i} \text{ with } X_i \sim N(0,1) \tag{2.6}$$

This is a discrete version of equation (2.5) where $S_{t_i}$ is the price of the asset at time $t_i$, $\Delta t = t_{i+1} - t_i$ and $X_i$ is a random variable from the standard normal distribution. Since $X_i$ is a random variable, this makes $S_{t_{i+1}}$ a random variable as well.

It is important to keep in mind that $\mu$ needs to be equal to $r$ (interest provided by the bank) to avoid *arbitrage* (see section 1.2), which is the risk neutral price process.

An important detail to keep in mind is that "the current asset price reflects all past information" [11]. This is known as the *efficient market hypothesis*. This means that all information about the company (such as profits, returns, recent developments) have already been incorporated in the price of the asset and are therefore no longer relevant. The only element that can influence the price is new information. This, in combination with definitions 2.5.1 and 2.5.2, show that the discounted stock price $\tilde{S}_t := e^{-rt}S_t$ is a $\mathbb{Q}$-martingale.

Let $\Lambda(x) = (K - x)_+$ denote the payoff function of an arbitrary put option at final time $T$. The expected payoff of the option is then denoted by $\mathbb{E}(\Lambda(x))$. The (reasonable) assumption is made that $|\Lambda(x)| = \Lambda(x) < \infty$ and therefore $\mathbb{E}(\Lambda(x)) < \infty$. The value of the put option at $t = 0$ is then equal to:

$$\begin{aligned} V(S,0) &= e^{-rT}V(S,T) \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}(\Lambda(x)) \quad [11] \end{aligned} \tag{2.7}$$

For the call option the situation is similar, with a payoff function of the form $\Lambda(x) = (x - K)_+$. The value $e^{-rT}$ is called the *discounting factor*. The assumption is that the interest is continuously compounding, the discounting factor compensates for the built up interest.

## 2.6 Black-Scholes model

Section 2.5 shows how to value the option at $t = 0$ by simulating an asset path and computing its expected value. However, there is also another method to value the option: the Black-Scholes model. The Black-Scholes model is a solution to the Black-Scholes equation, which differs for the European and Asian options.

### 2.6.1 European options

The option value for European options has been described in a Nobel prize winning equation called: the *Black-Scholes equation*, formulated by Fisher Black and Myron Scholes. The Black-Scholes equation is a partial differential equation and for the European option it is given by: [15]

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{2.8}$$

In this equation $V(S,t)$ represents the option value, $t$ the time, $r$ the interest rate, $S$ the value of the stock and $\sigma$ the volatility. This partial differential equation has a closed-form solution.

This is a second-order differential equation in terms of the stock price, and first-order in time. We therefore need two boundary conditions in terms of stock price and one initial condition.

The time condition is not difficult to find, it was already given in chapter 1:

$$\begin{aligned} V_P(S,T) &= (K - S_T)_+ \\ V_C(S,T) &= (S_T - K)_+ \end{aligned}$$

We now look at what happens to the value of the option when $S = 0$. If $S$ equals zero at some point in the asset path, it will stay zero for every time step after that. The reason for this is that if a company goes bankrupt ($S = 0$), it will stay bankrupt after that time. This results in the following boundary conditions: [11]

$$\begin{aligned} V_P(0,t) &= Ke^{-r(T-t)}, \text{ for all } 0 \leq t \leq T. \\ V_C(0,t) &= 0, \text{ for all } 0 \leq t \leq T. \end{aligned}$$

If $S$ takes on a very large value, i.e. $S \to \infty$, it is likely to stay very large until final time $T$. This results in the boundary conditions: [11]

$$\begin{aligned} V_P(S,t) &\approx 0, \text{ for } S \to \infty \\ V_C(S,t) &\approx S, \text{ for } S \to \infty \end{aligned}$$

### 2.6.2 Asian options

For the Asian option, the partial differential equation is given by: [15]

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial Z} - rV = 0 \tag{2.9}$$

An extra variable $Z$ is added in order to take the average value of the option into account. It is the running sum of the stock price: $Z_t = \int_0^t S_t dt$ such that $A(T) = \frac{Z_T}{T}$
Unfortunately, equation (2.9) does not have a closed, analytic solution for the Asian option with arithmetic average. Other valuation techniques are therefore necessary in order to estimate the value of this type of option, see chapter 3.

### Boundary conditions

This section expands upon the work of Lee Tse Yueng [16]. Equation (2.9) is a second-order derivative in terms of the stock price, first-order in time and first-order in $Z$, which means that two boundary conditions in terms of stock price, one initial condition and one condition for $Z$ are needed.
The initial conditions for the Asian put ($V_P$) and call ($V_C$) are given at final time $t = T$ and are quite straightforward:

$$V_P(Z_T, S_T, T) = (K - \frac{Z_T}{T})_+ \tag{2.10}$$

$$V_C(Z_T, S_T, T) = (\frac{Z_T}{T} - K)_+ \tag{2.11}$$

The initial conditions are simply equal to the payoff of the Asian option at final time $T$.

The computation of the boundary conditions require a bit more effort.

Here we examine the left boundary condition: $V(Z, 0, t)$. This boundary condition can be derived from the initial conditions (2.10) and (2.11). An important property of stocks is that if $S_t = 0$, then $S_\tau = 0 \; \forall \tau > t$. I.e. if a company goes bankrupt ($S = 0$) it will not have value after that time. This implies that, at $S = 0$:

$$\begin{aligned} Z_T &= \int_0^T S_\tau d\tau \\ &= \int_0^t S_\tau d\tau + \int_t^T S_\tau d\tau \\ &= \int_0^t S_\tau d\tau + 0 \\ &= Z_t \end{aligned}$$

The left boundary condition is therefore given by:

$$\begin{aligned} V_P(Z_t, 0, t) &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}(K - \frac{Z_T}{T})_+ | S_t = 0\Big] \\ &= e^{-r(T-t)}(K - \frac{Z_t}{T})_+ \end{aligned} \tag{2.12}$$

$$V_C(Z_t, 0, t) = e^{-r(T-t)}(\frac{Z_t}{T} - K)_+ \quad [16] \tag{2.13}$$

Additionally, a boundary condition for the $Z$-direction needs to be constructed. Though a boundary for $Z = 0$ seems the most logical choice, it is not the most efficient one. Notice that in the payoff of the option the value $Z_t = KT$ marks the border between the option being in the money (having a positive value) or out of the money (having value zero).
First, the boundary for the call option is analyzed. Let $Z_t > KT$ (the option is in the money). The

value of the option is then equal to the discounted expected value of the payoff.

$$\begin{aligned}
V_C(Z_t, S_t, t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[(\frac{Z_T}{T} - K)_+)|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{T}\int_0^T S_\tau d\tau - K|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{T}\int_0^t S_\tau d\tau + \frac{1}{T}\int_t^T S_\tau d\tau - K|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{T}\int_0^t S_\tau d\tau|\mathcal{F}_t\Big] + e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{T}\int_t^T S_\tau d\tau - K|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{Z_t}{T} - K|\mathcal{F}_t\Big] + \frac{1}{T}e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T S_\tau d\tau|\mathcal{F}_t\Big] \\
&= e^{-r(T-t)}\Big[\frac{Z_t}{T} - K\Big] + \frac{1}{T}e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T S_\tau d\tau|\mathcal{F}_t\Big]
\end{aligned}$$

Here $\mathcal{F}_t$ is an element of the filtration generated by $V(Z_T, S_T, T)$.

**Definition 2.6.1.** Let $\mathbb{F} = \mathcal{F}_t, t \in I)$ with $I \subset \mathbb{R}$ be a family of $\sigma$-algebras with $\mathcal{F}_t \subset \mathcal{F}$ for all $t \in I$. $\mathbb{F}$ is called a **filtration** if $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s, t \in I$ with $s \leq t$. [13]

The filtration $\mathcal{F}_t$ represents the information known at time $t$.

The value $\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T S_\tau d\tau|\mathcal{F}_t\Big]$ needs to be determined. In order to find that value $d(e^{rt}\tilde{S}_t)$ will be observed. Notice that:

$$d(e^{rt}\tilde{S}_t) = re^{rt}\tilde{S}_t dt + e^{rt}d\tilde{S}_t \tag{2.14}$$

Recall that $\tilde{S}_t = e^{-rt}S_t$. The value $d\tilde{S}_t$ can be calculated as follows:

$$\begin{aligned}
d\tilde{S}_t &= d(e^{-rt}S_t) \\
&= -re^{-rt}S_t dt + e^{-rt}dS_t \\
&= -r\tilde{S}_t dt + e^{-rt}dS_t \\
&= -r\tilde{S}_t dt + e^{-rt}[\mu dt + \sigma dW_t]S_t \quad \text{(recall equation (2.4))} \\
&= -r\tilde{S}_t dt + [\mu dt + \sigma dW_t]\tilde{S}_t \\
&= [(\mu - r)dt + \sigma dW_t]\tilde{S}_t \\
&= \sigma\tilde{S}_t dW_t \quad \text{(due to the risk neutral assumption } r = \mu\text{, see chapter 2.5)}
\end{aligned}$$

Integrating equation (2.14) gives:

$$\begin{aligned}
\int_t^T d(e^{r\tau}\tilde{S}_\tau)d\tau &= \int_t^T re^{r\tau}\tilde{S}_\tau d\tau + \int_t^T e^{r\tau}\sigma\tilde{S}_\tau dW_\tau \iff \\
e^{rT}\tilde{S}_T - e^{rt}\tilde{S}_t &= \int_t^T re^{r\tau}\tilde{S}_\tau d\tau + \int_t^T e^{r\tau}\sigma\tilde{S}_\tau dW_\tau
\end{aligned}$$

Taking conditional expectation results in:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[e^{rT}\tilde{S}_T - e^{rt}\tilde{S}_t|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T re^{r\tau}\tilde{S}_\tau d\tau|\mathcal{F}_t\Big] + \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{r\tau}\sigma\tilde{S}_\tau dW_\tau|\mathcal{F}_t\Big] \iff \\
e^{rT}\tilde{S}_t - e^{rt}\tilde{S}_t &= \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T re^{r\tau}\tilde{S}_\tau d\tau|\mathcal{F}_t\Big] + 0
\end{aligned}$$

The value $\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{r\tau}\sigma\tilde{S}_\tau dW_\tau|\mathcal{F}_t\Big] = 0$ because the mean is zero under the risk neutral probability measure $\mathbb{Q}$. Additionally $\mathbb{E}^{\mathbb{Q}}[e^{rT}\tilde{S}_T|\mathcal{F}_t] = e^{rT}\tilde{S}_t$ since $\tilde{S}_t$ is a martingale.
The above results therefore in the requested value:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{r\tau}\tilde{S}_\tau dt|\mathcal{F}_t\Big] &= \frac{1}{r}(e^{rT}\tilde{S}_t - e^{rt}\tilde{S}_t) \\ &= \frac{1}{r}(e^{r(T-t)}S_t - S_t) \\ &= \frac{S_t}{r}(e^{r(T-t)} - 1) \qquad (2.15)\end{aligned}$$

This concludes to:

$$V_C(Z_t, S_t, t) = e^{-r(T-t)}\Big[\frac{Z_t}{T} - K\Big] + \frac{S_t}{rT}(1 - e^{-r(T-t)}) \text{ for } Z_t \geq KT \quad [16] \qquad (2.16)$$

Next, the value of the Asian put needs to be calculated for $Z_t \geq KT$.
Assume that $Z_t \geq KT$ and consider the difference between the value of the put and call option:

$$\begin{aligned}V_C(Z_t, S_t, t) - V_P(Z_t, S_t, t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[(\frac{Z_T}{T} - K)_+ - (K - \frac{Z_T}{T})_+|\mathcal{F}_t\Big] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{Z_T}{T} - K|\mathcal{F}_t\Big] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\frac{Z_T}{T}|\mathcal{F}_t] + e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[-K]|\mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{T}\int_0^t S_\tau d\tau + \frac{1}{T}\int_t^T S_\tau d\tau|\mathcal{F}_t\Big] - e^{-r(T-t)}K \\ &= \frac{e^{-r(T-t)}}{T}Z_t + e^{-r(T-t)}(\frac{S_t}{rT}[e^{r(T-t)} - 1]) - e^{-r(T-t)}K \\ &= e^{-r(T-t)}[\frac{Z_t}{T} - K] + e^{-r(T-t)}(\frac{S_t}{rT}[e^{r(T-t)} - 1]) \\ &= e^{-r(T-t)}[\frac{Z_t}{T} - K] + \frac{S_t}{rT}[1 - e^{-r(T-t)}] \\ &= V_C(Z_t, S_t, t)\end{aligned}$$

The above proofs that:

$$V_P(Z_t, S_t, t) = 0 \text{ for } Z_t \geq KT \quad [16] \qquad (2.17)$$

The final boundary condition that needs to be derived is the condition for $S \to \infty$.
If $S \to \infty$, this automatically results in $Z \to \infty$, since $Z_t = \int_0^t S_t$. Therefore the boundary condition for $S \to \infty$ is the same as the boundary for $Z_t \geq KT$.

To summarize, the boundary conditions of the Black-Scholes partial differential equation (2.9) are given by:

- Time condition at final time $T$:
  $V_P(Z_T, S_T, T) = (K - \frac{Z_T}{T})_+$
  $V_C(Z_T, S_T, T) = (\frac{Z_T}{T} - K)_+$

- Boundary condition for $S = 0$:
  $V_P(Z_t, 0, t) = e^{-r(T-t)}(K - \frac{Z_t}{T})_+$
  $V_C(Z_t, 0, t) = e^{-r(T-t)}(\frac{Z_t}{T} - K)_+$
- Boundary condition for $S \to \infty$
  $V_P(Z_t, S_t, t) = 0$
  $V_C(Z_t, S_t, t) = e^{-r(T-t)}[\frac{Z_t}{T} - K] + \frac{S_t}{rT}[1 - e^{-r(T-t)}]$
- Boundary condition for $Z_t \geq KT$:
  $V_P(Z_t, S_t, t) = 0$
  $V_C(Z_t, S_t, t) = e^{-r(T-t)}[\frac{Z_t}{T} - K] + \frac{S_t}{rT}[1 - e^{-r(T-t)}]$

## 2.7 Feynman-Kac formula

If we compare the asset price model in section 2.5 to the Black-Scholes model in sections 2.6.1 and 2.6.2 we notice that there are two ways of modeling the value of an option: one using simulations and computing the expected value and the other using partial differential equations to describe the movement of the option value.
The reason for using two models is that simulating the asset price is an ideal model for the Monte Carlo method, whereas describing the option value with a partial differential equation is ideal for the finite difference method.

An important detail that needs to be checked is whether the two different approaches result in the same value for the options. Fortunately, the Feynman-Kac formula proofs this (here for the European options):

**Proposition 1.** (**Feynman-Kac formula**)
Consider the partial differential equation

$$\frac{\partial V}{\partial t}(S,t) + k(S,t)\frac{\partial V}{\partial S}(S,t) + \frac{1}{2}m^2(S,t)\frac{\partial^2 V}{\partial S^2}(S,t) - P(S,t)V(S,t) + f(S,t) = 0$$

defined for all $S \in \mathbb{R}$ and $t \in [0,T]$, subject to the terminal condition

$$V(S,T) = \phi(x),$$

where $k,\ m,\ \phi,\ P,\ f$ are known functions, $T$ is a parameter and $V : \mathbb{R} \times [0,T] \to \mathbb{R}$ is the unknown. Then the solution can be written as a conditional expectation:

$$V(S,t) = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^\tau P(S_\tau,\tau)d\tau} f(S_\tau,\tau)d\tau + e^{-\int_t^T P(S_\tau,\tau)d\tau}\phi(S_T)|S_t = S\Big]$$

under the probability measure $\mathbb{Q}$ such that $S$ follows the Geometric Brownian Motion. [6]

In the case of the Black-Scholes PDE (2.8) $f(S,t) \equiv 0$, $m^2(S,t) = \sigma^2 S^2$, $P(S,t)$ equals the parameter $r$, $\phi(x) = \Lambda(x)$ and $k(S,t) = rS$. Using the Feynman-Kac formula results in:

$$\begin{aligned} V(S,0) &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T 0 + e^{-\int_0^T r d\tau}\Lambda(S_T)|S_t = S\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-rT}\Lambda(S_T)|S_t = S\Big] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[\Lambda(S_T)] \end{aligned} \tag{2.18}$$

This shows that equation (2.18) is equal to equation (2.7) and both approaches therefore give the same result. A similar result can be found for Asian options.

# Chapter 3

# Numerical methods

The goal of option valuation in general is to find the value of the option at the start, i.e. $t = 0$. This is the main value where traders are interested in, since it determines the price of the option at the moment it is sold.

Since there is no closed, analytic solution for the Black-Scholes PDE for the Asian option with arithmetic average, numerical methods are needed to approach its value. In this chapter we look at two of those methods: the Monte Carlo method and the finite difference method.

## 3.1 The Monte Carlo method

The Monte Carlo method was developed during the second world war by American scientists Stanislaw Ulam and John Von Neumann in the context of the Manhattan project. This project was a research and development operation in the second World War that produced the first nuclear weapons. Since their research was very secretive, the scientists needed a code name for this method and came up with the name "Monte Carlo method", named after the Monte Carlo casino in Monaco which the uncle of Ulam frequented [10].

As mentioned before, the Monte Carlo method uses the asset path simulation in section 2.5. In order to price the option, one computes a large number of asset paths, say $M$ paths. For each path the value of the option at $t = T$ can be calculated, simply by looking at the price path.

Computing $M$ asset paths results in $M$ simulated values $V_1, ..., V_M$ of the European option at final time $T$. With these values the sample mean $a_M$ and sample standard deviation $b_M$ can be computed.

$$\begin{aligned} a_M &= \frac{1}{M}\sum_{i=1}^{M} V_i(S,T) \\ b_M^2 &= \frac{1}{M-1}\sum_{i=1}^{M}(V_i(S,T) - a_M)^2 \quad [11] \end{aligned}$$

The strong law of large numbers implies:

$$a_M \rightarrow \mathbb{E}^{\mathbb{Q}}(V(S,T)) \text{ when } M \rightarrow \infty \quad [11]$$

The advantage of this method is that it can relatively easy be adapted to evaluate various types of options, even exotic options, when compared to other numerical methods (such as the finite difference method, see section 3.2).

### 3.1.1 Issues with the Monte Carlo method

Although the Monte Carlo method is very easy to implement for a large variety of exotic options, there are certain known issues.

Let $\mathbb{E}^{\mathbb{Q}}[V(S,0)] = a$ and $\text{var}[V(S,0)] = b^2$. Notice that $\sum_{i=1}^{M} V_i$ behaves like an $\mathrm{N}\big(Ma, Mb^2\big)$ random variable. This means that $a_M - a$ is approximately $\mathrm{N}\big(0, \frac{b^2}{M}\big)$. So the normal distribution is scaled by a factor $\frac{b}{\sqrt{M}}$. What is interesting about this is that when the number of samples $M$ is multiplied by a thousand, this only results in a reduction of factor 10 of the standard deviation. This means that the Monte Carlo can be very *computationally intensive*. Quite some research has been conducted in order to make the Monte Carlo method more efficient, e.g. by introducing antithetic variates and control variates [2] [11].

The above shows that the number of asset paths influences the standard error and thereby the width of the confidence interval. Though a small confidence interval is preferable, it will not be of any use if the estimate of the value is completely wrong. The accuracy of the estimate is determined by the step size in time and the number of paths. A smaller step size results in a more accurate estimate, since the discrete asset path better resembles the real life continuous asset path as the number of time steps increase.

However, as it turns out in chapter 4 the Monte Carlo method works quite well for several types of Asian options and the computation time proves not to be a problem. One of the reasons that may have contributed to this is that the value of the type of Asian options all rely on the average stock price instead of individual stock prices. In the average stock price, large ”deviations” of the stock price are evened out by taking the average, and thereby the standard deviation of the value of the Asian option is reduced.

## 3.2 Finite difference method

The finite difference method is a method which uses Taylor approximations to approximate the derivatives in a (partial) differential equation. It is a discrete method, in the sense that the domain of the equation is discretized with the use of a (sometimes equidistant) grid, where the value of the equation is approximated at every grid point.

### 3.2.1 European options

In this section, the finite difference method is applied to the Black-Scholes partial differential equation in order to numerically price European options. The section expends upon the work by Higham [11].

As mentioned before, the Black-Scholes partial differential equation is the equation which describes the price of the European call or put option under the Black-Scholes model. The Black-Scholes equation is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.1}$$

The boundary conditions have been described in section 2.6.1:

$$\begin{aligned}
V_P(S,T) &= (K - S_T)_+ && (3.2)\\
V_P(0,t) &= Ke^{-r(T-t)} && \text{for all } 0 \le t \le T \;(3.3)\\
V_P(S,t) &\approx 0 && \text{for } S \to \infty \;(3.4)\\
V_C(S,T) &= (S_T - K)_+ && (3.5)\\
V_C(0,t) &= 0 && \text{for all } 0 \le t \le T \;(3.6)\\
V_C(S,t) &\approx S && \text{for } S \to \infty \;(3.7)
\end{aligned}$$

Notice that the third and last condition make it difficult to apply the finite difference method, since $S \in [0, \infty]$, which is not an interval that can be divided in a finite number of bounded sub-intervals. Therefore, we assume $S \in [0, S_{\max}]$ for $S_{\max}$ large enough and replace the boundary condition for the put option by $V_P(S_{\max}, t) = 0$ and for the call option by $V_C(S_{\max}, t) = S_{\max}$.

We are now ready to apply the method of finite differences. In order to apply this method, we create a grid over the $(S, t)$ surface. In the $S$ direction, we create $N_x$ points spaced at equal distance $l = \frac{S_{\max}}{N_x}$ from each other. In the $t$ direction, we create $N_t$ points spaced at equal distance $k = \frac{T}{N_t}$ from each other. We have now created a grid $\{jl, hk\}_{j=0,i=0}^{N_x,\ N_t}$ on which we can apply the method of finite differences.

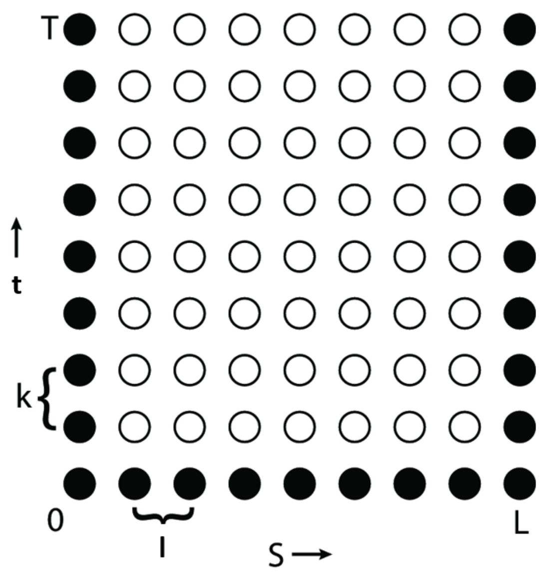



Figure 3.1: Grid for the method of finite differences.

$\mathbf{V}^h$ is the numerical solution at time step $h$, denoted by

$$\mathbf{V}^h := \begin{bmatrix} V_1^h \\ V_2^h \\ \vdots \\ \vdots \\ V_{N_x-1}^i \end{bmatrix} \in \mathbb{R}^{N_x-1}$$

The vector $\mathbf{V}^0$ is specified for the put and call option by the initial conditions (3.2) and (3.5) respectively. The boundary values $V_0^h$ and $V_{N_x}^h$ are specified by (3.3) and (3.4) for the put option and by (3.6) and (3.7) for the call option.

The finite difference method uses finite difference operators to approximate (partial) derivatives. In this paragraph, we use these operators to approximate the partial derivatives in the Black-Scholes equation.

The first partial derivative in the Black-Scholes equation is the first order time derivative $\frac{\partial V}{\partial t}$. We use two different difference operators: the forward operator and the backward operator.

- Forward operator: $\frac{\partial V}{\partial t} = \frac{V^{h+1}-V^h}{k} + O(k)$.
- Backward operator: $\frac{\partial V}{\partial t} = \frac{V^h - V^{h-1}}{k} + O(k)$

These operators are later combined for the Crank-Nicholson operator. $O(k)$ represents the order of the error made by the approximation of the operator.

For the stock partial derivatives central difference operators are used. These operators have a higher order of accuracy since the order of the error is larger.

- Central difference operator first order derivative: $\frac{\partial V}{\partial S} = \frac{V_{j+1}-V_{j-1}}{2l} + O(l^2)$.
- Central difference operator second order derivative: $\frac{\partial^2 V}{\partial S^2} = \frac{V_{j+1}-2V_j+V_{j-1}}{l^2} + O(l^2)$

One might ask the question why central difference operators are not used for the time derivative, since they have a higher order of accuracy than both the forward and the backward operator. The reason for this is that the central difference operator for first order partial derivative at time $t_i$ does not use the value $V^h$, but the values $V^{h-1}$ and $V^{h+1}$. This creates a jump in time. The second order central difference operator uses the value $V_j$ at stock price $S_{t_j}$, which is why this is not a problem for the stock price derivatives.
Using the forward in time operator the Black-Scholes PDE can be rewritten like this:

$$\begin{aligned}
\frac{V_j^{h+1} - V_j^h}{k} + O(k) + \frac{1}{2}\sigma^2 (jl)^2 \frac{V_{j+1}^h - 2V_j^h + O(l^2) + V_{j-1}^h}{l^2} + rjl\frac{V_{j+1}^h - V_{j-1}^h}{2l} + O(l^2) - rV_j^h = 0 \iff \\
\frac{V_j^{h+1}}{k} = \frac{V_j^h}{k} - \frac{1}{2}\sigma^2 j^2 l^2 \frac{V_{j+1}^h - 2V_j^h + V_{j-1}^h}{l^2} - rjl\frac{V_{j+1}^h - V_{j-1}^h}{2l} + rV_j^h + O(k) + O(l^2) \iff \\
\frac{V_j^{h+1}}{k} = \frac{V_j^h}{k} - \frac{1}{2}\sigma^2 j^2 (V_{j+1}^h - 2V_j^h + V_{j-1}^h) - \frac{1}{2}rj(V_{j+1}^h - V_{j-1}^h) + rV_j^h + O(k) + O(l^2) \iff \\
V_j^{h+1} = V_j^h - \frac{1}{2}\sigma^2 kj^2 (V_{j+1}^h - 2V_j^h + V_{j-1}^h) - \frac{1}{2}rkj(V_{j+1}^h - V_{j-1}^h) + rkV_j^h + O(k) + O(l^2) \iff \\
V_j^{h+1} = (1+rk)V_j^h - \frac{1}{2}\sigma^2 kj^2 (V_{j+1}^h - 2V_j^h + V_{j-1}^h) - \frac{1}{2}rkj(V_{j+1}^h - V_{j-1}^h) + O(k) + O(l^2)
\end{aligned}$$

We can now write the approximation of $V$ as

$$\mathbf{V^{h+1}} = F\mathbf{V^h} + \mathbf{p^h} \tag{3.8}$$

with $F = (1 + rk)I - \frac{1}{2}k\sigma^2 D_2T_2 - \frac{1}{2}rkD_1T_1 \in \mathbb{R}^{(N_x-1)\times(N_x-1)}$ and: [11]

$$D_1 = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & N_x - 1 \end{bmatrix} \in \mathbb{R}^{(N_x-1)\times(N_x-1)},$$

$$D_2 = \begin{bmatrix} 1^2 & & & & & \\ & 2^2 & & & & \\ & & 3^2 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & (N_x - 1)^2 \end{bmatrix} \in \mathbb{R}^{(N_x-1)\times(N_x-1)}$$

$$T_1 = \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & -1 & 0 \end{bmatrix} \in \mathbb{R}^{(N_x-1)\times(N_x-1)}, \ T_2 = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{(N_x-1)\times(N_x-1)}$$

$$\mathbf{p}^h = \begin{bmatrix} (\frac{1}{2}k\sigma^2 - \frac{1}{2}rk)V_0^h \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x - 1)(\sigma^2(N_x - 1) + r)V_{N_x}^h \end{bmatrix} \in \mathbb{R}^{N_x-1}$$

Consider the vector $\mathbf{p}^h$. For the put option, (3.4) gives that the last term of the vector will be equal to zero. In the same way for the call option, (3.6) gives that the first term of the vector will be equal to zero.

Now take the backward in time operator for the time derivative. The Black-Scholes PDE can be rewritten in a similar way to the form of:

$$G\mathbf{V}_{h+1} = \mathbf{V^h} + \mathbf{q}^h \tag{3.9}$$

with $G = (1+rk)I - \frac{1}{2}k\sigma^2 D_2 T_2 - \frac{1}{2}krD_1T_1 \in \mathbb{R}^{(N_x-1)\times(N_x-1)}$ and

$$\mathbf{q}^h = \begin{bmatrix} \frac{1}{2}k(\sigma^2 - r)V_0^{h+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x-1)(\sigma^2(N_x-1)+r)V_{N_x}^{h+1} \end{bmatrix} \in \mathbb{R}^{N_x-1}.$$

The Crank-Nicolson method is a combination of the forward in time operator and the backward in time operator by computing "the average" of both methods. It can be written as follows:

$$\frac{1}{2}(I+B)\mathbf{V}^{h+1} = \frac{1}{2}(I+F)V^h + \frac{1}{2}(\mathbf{p}^h + \mathbf{q}^h) \quad [11] \tag{3.10}$$

Observe that this is an *implicit* method. However, the advantage is that, next to the fact that it is *unconditionally stable*, it has a numerical error of $O(k^2) + O(l^2)$.

In figure 3.2 below, the Crank-Nicolson method is applied to a European put option.

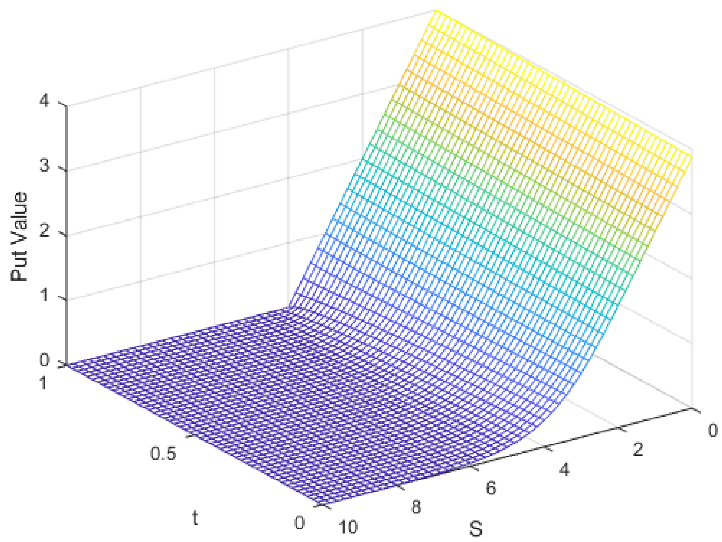



Figure 3.2: Value European put with strike price $K = 4$, volatility $\sigma = 0.3$, interest $r = 0.03$ and maturity time $T = 1$.

Beforehand, one does not know which path the stock price will follow. Therefore the value of the put option is calculated for every possible discrete combination of $S$ and $t$. Figure 3.2 shows that at $t = T$ the value of the put option equals zero for stock prices higher than 4. This makes sense since the figure displays a put option with strike price of $K = 4$ and a payoff of $(K - S_T)_+$.

### 3.2.2 Asian options

This section expands upon the work of Lee Tse Yueng [16].
In the same way the finite difference method can be applied to the Black-Scholes equation for Asian options. Recall that the Black-Scholes PDE is given by

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial Z} - rV = 0 \tag{3.11}$$

and the boundary conditions are given by

- Time condition at final time $T$:
  $V_P(Z_T, S_T, T) = (K - \frac{Z_T}{T})_+$
  $V_C(Z_T, S_T, T) = (\frac{Z_T}{T} - K)_+$
- Boundary condition for $S = 0$:
  $V_P(Z_t, 0, t) = e^{-r(T-t)}(K - \frac{Z_t}{T})_+$
  $V_C(Z_t, 0, t) = e^{-r(T-t)}(\frac{Z_t}{T} - K)_+$
- Boundary condition for $S \to \infty$
  $V_P(Z_t, S_t, t) = 0$
  $V_C(Z_t, S_t, t) = e^{-r(T-t)}[\frac{Z_t}{T} - K] + \frac{S_t}{rT}[1 - e^{-r(T-t)}]$
- Boundary condition for $Z_t = KT$:
  $V_P(Z_t, S_t, t) = 0$
  $V_C(Z_t, S_t, t) = e^{-r(T-t)}[\frac{Z_t}{T} - K] + \frac{S_t}{rT}[1 - e^{-r(T-t)}]$

For $Z_t > KT$ the option value can be described analytically:

$$\begin{aligned} V_P(Z_t, S_t, t) &= 0 \\ V_C(Z_t, S_t, t) &= e^{-r(T-t)}[\frac{Z_t}{T} - K] + \frac{S_t}{rT}[1 - e^{-r(T-t)}] \end{aligned}$$

For $Z < KT$ the finite difference method will be used to approximate the value of the option.

The discretization is similar to the discretization of the Black-Scholes PDE for the European option. Let $(Z_i, S_j, t_h)$ form a equidistant grid in $[0, Z_{\max}] \times [0, S_{\max}] \times [0, T]$. In order to create this grid, the assumption needs to be made that $Z_t$ and $S_t$ have maximum values $Z_{\max}$ and $S_{\max}$ respectively. Suppose the $Z$ axis is divided in $N_Z$ equidistant points, the $S$ axis in $N_S$ equidistant points and the $t$ axis in $N_t$ equidistant points. Then $\Delta Z = \frac{Z_{\max}}{N_Z}$, $\Delta S = \frac{S_{\max}}{N_S}$ and $\Delta t = \frac{T}{N_t}$. Let $Z_i = i\Delta Z$, $S_j = j\Delta S$, $t_h = h\Delta t$. Notice that $V^h_{i,j} = V(Z_i, S_j, t_h)$.
Let $L^h_{i,j}$ be the numerical discretization of $rS\frac{\partial V}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial Z} - rV$. Discretization gives:

- $$\begin{aligned} rS\frac{\partial V}{\partial S} &= rS_j \frac{V^h_{i,j+1} - V^h_{i,j-1}}{2\Delta S} + O(\Delta S^2) \\ &= rj\Delta S \frac{V^h_{i,j+1} - V^h_{i,j-1}}{2\Delta S} + O(\Delta S^2) \\ &= \frac{1}{2} rj(V^h_{i,j+1} - V^h_{i,j-1}) + O(\Delta S^2) \end{aligned}$$

- $$\begin{aligned}\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} &= \frac{1}{2}\sigma^2 S_j^2 \frac{V_{i,j+1}^h - 2V_{i,j}^h + V_{i,j-1}^h}{\Delta S^2} + O(\Delta S^2)\\ &= \frac{1}{2}\sigma^2 (j\Delta S)^2 \frac{V_{i,j+1}^h - 2V_{i,j}^h + V_{i,j-1}^h}{\Delta S^2} + O(\Delta S^2)\\ &= \frac{1}{2}\sigma^2 j^2 (V_{i,j+1}^h - 2V_{i,j}^h + V_{i,j-1}^h) + O(\Delta S^2)\end{aligned}$$

- $$\begin{aligned}S\frac{\partial V}{\partial Z} &= S_j \frac{V_{i+1,j}^h - V_{i,j}^h}{\Delta Z} + O(\Delta Z)\\ &= j\Delta S \frac{V_{i+1,j}^h - V_{i,j}^h}{\Delta Z} + O(\Delta Z)\\ &= \frac{j\Delta S}{\Delta Z}(V_{i+1,j}^h - V_{i,j}^h) + O(\Delta Z)\end{aligned}$$

- $$rV = rV_{i,j}^h$$

Therefore:

$$\begin{aligned}L_{i,j}^h &= \frac{1}{2}rj(V_{i,j+1}^h - V_{i,j-1}^h) + \frac{1}{2}\sigma^2 j^2 (V_{i,j+1}^h - 2V_{i,j}^h + V_{i,j-1}^h)\\ &+ \frac{j\Delta S}{\Delta Z}(V_{i+1,j}^h - V_{i,j}^h) - rV_{i,j}^h + O(\Delta S^2) + O(\Delta Z)\\ &= (\frac{1}{2}\sigma^2 j^2 + \frac{1}{2}rj)V_{i,j+1}^h + (-r - \sigma^2 j^2 - \frac{j\Delta S}{\Delta Z})V_{i,j}^h\\ &+ (\frac{1}{2}\sigma^2 j^2 - \frac{1}{2}rj)V_{i,j-1}^h + \frac{j\Delta S}{\Delta Z}V_{i+1,j}^h \quad [16]\end{aligned}$$

Next the time derivative will be discretized. Forward in time discretization at (i,j,h) results in:

$$\frac{V_{i,j}^{h+1} - V_{i,j}^h}{\Delta t} + L_{i,j}^h = 0$$

Discretization backward in time at (i,j,h+1) results in:

$$\frac{V_{i,j}^{h+1} - V_{i,j}^h}{\Delta t} + L_{i,j}^{h+1} = 0$$

Combining both provides the Crank-Nicolson scheme of $O(\Delta t^2, \Delta S^2, \Delta Z)$:

$$\begin{gathered}\frac{V_{i,j}^{h+1} - V_{i,j}^h}{\Delta t} + \frac{1}{2}(L_{i,j}^h + L_{i,j}^{h+1}) = 0\\ V_{i,j}^{h+1} + \frac{\Delta t}{2}L_{i,j}^{h+1} = V_{i,j}^h - \frac{\Delta t}{2}L_{i,j}^h \quad [16]\end{gathered}$$

Filling in the expression for $L^h_{i,j}$ and $L^{h+1}_{i,j}$ results in:

$$\begin{aligned}\frac{\Delta t}{2}(\frac{1}{2}\sigma^2 j^2 + \frac{1}{2}rj)V^{h+1}_{i,j+1} + (\frac{\Delta t}{2}(-r - \sigma^2 j^2 - \frac{j\Delta S}{\Delta Z}) + 1)V^{h+1}_{i,j} \\ + \frac{\Delta t}{2}(\frac{1}{2}\sigma^2 j^2 - \frac{1}{2}rj)V^{h+1}_{i,j-1} + \frac{j\Delta S\Delta t}{2\Delta Z}V^{h+1}_{i+1,j} \\ = \frac{-\Delta t}{2}(\frac{1}{2}\sigma^2 j^2 + \frac{1}{2}rj)V^{h}_{i,j+1} + (\frac{-\Delta t}{2}(-r - \sigma^2 j^2 - \frac{j\Delta S}{\Delta Z}) + 1)V^{h}_{i,j} \\ + \frac{-\Delta t}{2}(\frac{1}{2}\sigma^2 j^2 - \frac{1}{2}rj)V^{h}_{i,j-1} + \frac{-j\Delta S\Delta t}{2\Delta Z}V^{h}_{i+1,j}\end{aligned}$$

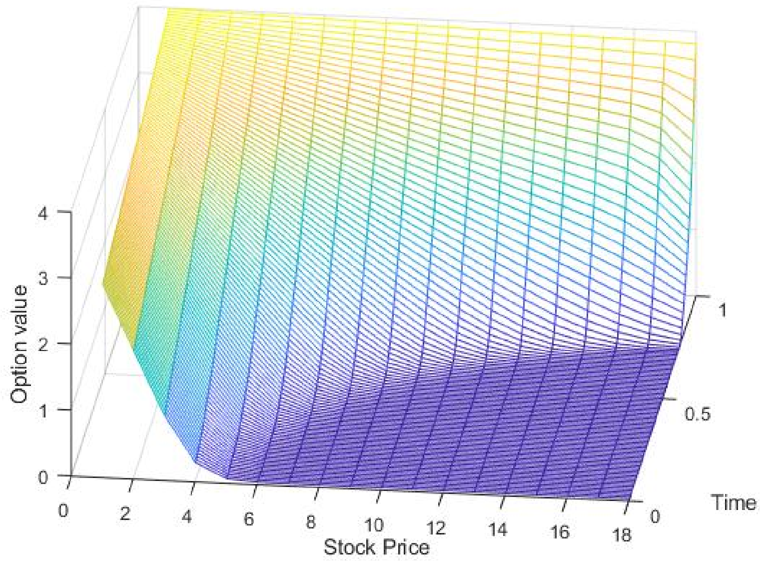
This scheme can be represented in matrix-vector form as follows:

$$AV^{h+1} = BV^h + p_a$$

$V^h$ is the vector $V^h = (V^h_{i,1}, V^h_{i,2}, ..., V^h_{i,N-1})^T$. Vector $p_a$ contains the boundary values and the values of the term $\frac{j\Delta S\Delta t}{2\Delta Z}V^h_{i+1,j}$. Matrices $A$ and $B$ can be found in appendix A.

An important note must be made about the domain used in this discretization: not all combinations of $(Z_i, S_j, t_h)$ are possible in the real financial market. For example: if $S_j = 0\ \forall j$, then in the real financial market $Z_i = 0\ \forall i$. Additionally, $Z_i \geq S_j\ \forall i, j$ since $Z$ is the running sum of $S$. Though it might look like this causes problems when using the option values generated by the finite difference method in the real financial market, this is actually not the case. The finite difference method gives a solution in the form of a 3-dimensional grid. Financial engineers that want to know the option value can simply look it up in the grid for their combination of $(Z_i, S_j, t_h)$ of which they know it exists.

Figure 3.3 shows the value of the Asian put option for fixed $Z < KT$:




option.jpg

Figure 3.3: Value Asian put with strike price $K = 4$, volatility $\sigma = 0.3$, interest $r = 0.03$ and maturity time $T = 1$.

The figure shows that at maturity time $T$ the value of the option is constant. This agrees with the time condition $V(Z, S, T) = (K - \frac{Z_T}{T})_+$. As the algorithm progresses to $t = 0$, the value of the option decreases faster for higher values of the stock price than for lower values of the stock price. This makes sense, since a higher value for the stock price will likely result in a higher value of the average.

### 3.2.3 Conditional Asian option

Although the finite difference method is applicable for European and Asian options, we run into problems when applying it to the conditional Asian option. The reason for this is that applying the finite difference method to the Black-Scholes equation for Asian options results in a 3-dimensional solution: every combination of $(Z, S, t)$ that is on the grid gives a value for $V$. The issue lies with the variable for the average. For the regular Asian option $Z_t = \int_0^T S_t dt$ and $A_T = \frac{Z_T}{T}$. All values of $S_\tau$ with $\tau < t$ are taken into account in this integral. However, this no longer holds for the conditional Asian option, where for some time intervals $S$ might be smaller or equal to $b$ and will not be taken into account. The issue is that it is impossible to determine the length of the time intervals that $S \leq b$ since the finite difference method does not generate or simulate an asset path. This causes problems when computing the average $A_T = \frac{Z_T}{\text{time}}$, since we do not know the length of the time which we should divide the sum of the stock prices by.

Though the conditional option is difficult to price with the finite difference method, we can use this method to price the conditional Asian option with the adaption that the average of the regular option is used: $A_T = \frac{Z_T}{T}$. The idea of the conditional option is that only values of $S$ larger than a certain threshold $b$ are taken into account when computing the average. This implies that $S > b$, but also that $A > b$. The logical way to adapt the discretization in section 3.2.2 is therefore to reduce the grid on which the finite difference method is applied to the grid: $[b, S_{\max}] \times [b, Z_{\max}] \times [0, T]$. Suppose the $Z$ axis is divided in $M$ equidistant points, the $S$ axis in $N$ equidistant points and the $t$ axis in $P$ equidistant points. Let $\Delta Z = \frac{Z_{\max}-b}{M}$, $\Delta S = \frac{S_{\max}-b}{N}$ and $\Delta t = \frac{T}{P}$.

Special attention needs to be paid to the boundary conditions: are they equal to those of the Asian option?
Clearly the time condition is not affected, since the size of the grid is only changed in the $(Z, S)$ dimension. Additionally the boundary condition stays the same for $S \to \infty$, since the changes in the grid are made for small $S$.
The boundary at $S = 0$ changes to a boundary condition at $S = b$:

$$\begin{aligned} V_P(Z,b,t) &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}(K - \frac{Z}{T})_+ | S_t = b, Z_t = Z\Big] \\ &= e^{-r(T-t)}(K - \frac{Z}{T})_+ \text{ with } Z > b \\ V_C(Z,b,t) &= e^{-r(T-t)}(\frac{Z}{T} - K)_+ \text{ with } Z > b \end{aligned}$$

The boundary condition at $Z = KT$ remains valid for the conditional Asian option, since only the grid boundary of $Z$ moves from 0 to $b$.

Let $S_j = b + j\Delta S$, $Z_i = b + i\Delta Z$ and $t_h = h\Delta t$. Discretizing the Black-Scholes equation (3.11) results in:

- $$\begin{aligned} rS\frac{dV}{dS} &= rS_j \frac{V^h_{i,j+1} - V^h_{i,j-1}}{2\Delta S} + O(\Delta S^2) \\ &= r(b + j\Delta S)\frac{V^h_{i,j+1} - V^h_{i,j-1}}{2\Delta S} + O(\Delta S^2) \\ &= (\frac{rb}{2\Delta S} + \frac{1}{2}rj)(V^h_{i,j+1} - V^h_{i,j-1}) + O(\Delta S^2) \end{aligned}$$

- $$\begin{aligned} \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} &= \frac{1}{2}\sigma^2 S_j^2 \frac{V^h_{i,j+1} - 2V^h_{i,j} + V^h_{i,j-1}}{\Delta S^2} + O(\Delta S^2) \\ &= \frac{1}{2}\sigma^2 (b + j\Delta S)^2 \frac{V^h_{i,j+1} - 2V^h_{i,j} + V^h_{i,j-1}}{\Delta S^2} + O(\Delta S^2) \\ &= \frac{1}{2}\sigma^2 (\frac{b^2}{\Delta S^2} + \frac{2jb}{\Delta S} + j^2)(V^h_{i,j+1} - 2V^h_{i,j} + V^h_{i,j-1}) + O(\Delta S^2) \end{aligned}$$

- $$\begin{aligned} S\frac{dV}{dZ} &= S_j \frac{V^h_{i+1,j} - V^h_{i,j}}{\Delta Z} + O(\Delta Z) \\ &= (b + j\Delta S)\frac{V^h_{i+1,j} - V^h_{i,j}}{\Delta Z} + O(\Delta Z) \\ &= (\frac{b}{\Delta Z} + \frac{j\Delta S}{\Delta Z})(V^h_{i+1,j} - V^h_{i,j}) + O(\Delta Z) \end{aligned}$$

-

$$rV = rV^h_{i,j}$$

Therefore:

$$\begin{aligned}
L^h_{i,j} &= (\frac{rb}{2\Delta S} + \frac{1}{2}rj)(V^h_{i,j+1} - V^h_{i,j-1}) + \frac{1}{2}\sigma^2(\frac{b^2}{\Delta S^2} + \frac{2jb}{\Delta S} + j^2)(V^h_{i,j+1} - 2V^h_{i,j} + V^h_{i,j-1}) \\
&+ (\frac{b}{\Delta Z} + \frac{j\Delta S}{\Delta Z})(V^h_{i+1,j} - V^h_{i,j}) - rV^h_{i,j} + O(\Delta S^2) + O(\Delta Z) \\
&= (\frac{b^2\sigma^2}{2\Delta S^2} + \frac{bj\sigma^2}{\Delta S} + \frac{1}{2}\sigma^2 j^2 + \frac{rb}{2\Delta S} + \frac{1}{2}rj)V^h_{i,j+1} \\
&+ (-r - \frac{\sigma^2 b^2}{\Delta S^2} - \frac{2\sigma^2 jb}{\Delta S} - \sigma^2 j^2 - \frac{b}{\Delta Z} - \frac{j\Delta S}{\Delta Z})V^h_{i,j} \\
&+ (\frac{b^2\sigma^2}{2\Delta S^2}\frac{\sigma^2 jb}{\Delta S} + \frac{1}{2}\sigma^2 j^2 - \frac{rb}{2\Delta S} - \frac{1}{2}rj)V^h_{i,j-1} \\
&+ (\frac{b}{\Delta Z} + \frac{j\Delta S}{\Delta Z})V^h_{i+1,j}
\end{aligned}$$

In the same way as in section 3.2.2 the Crank-Nicolson scheme can be applied, which results in:

$$V^h_{i,j} - \frac{\Delta t}{2}L^h_{i,j} = V^{h+1}_{i,j} + \frac{\Delta t}{2}L^{h+1}_{i,j}$$

### 3.2.4 Asian tail option

The valuation of the Asian tail option with finite difference method is straightforward once the discretization matrices of the European and Asian options are known. Recall that for the Asian option the "averaging feature" is only active for the last part of the lifetime of the option [5]. To value the Asian tail option, we simply iterate over time from final time $T$ to 0 and use the Asian discretization matrix for the tail part of the option and the European discretization matrix for the other part of the option. For both discretizations the regular boundary conditions apply, with the exception that the time condition for the "European part" of the option is given by:

$$V_{\text{Eu}}(S, t_{\text{tail}}) = V_{\text{Asian}}(Z, S, t_{\text{tail}})$$

for fixed $Z$ with $t_{\text{tail}}$ equal to the time where the tail of the option is activated. Here $V_{\text{Eu}}$ denotes the value of the European part of the Asian tail option and $V_{\text{Asian}}$ denotes the value of the Asian part.

A remark is that at the transition point of Asian discretization to European discretization there is a mismatch in dimension: the Asian discretization has a 3-dimensional solution whereas the European discretization has a 2-dimensional solution. This is solved by iterating over the discrete values of $Z_i$ when computing the European discretization.

### 3.2.5 Jump condition

Another way of valuing the Asian options is by using the Black-Scholes equation (3.1) for European options. This equation does not contain a variable to represent the average, but this variable is taken care of through the boundary conditions and the *jump condition*.

We have a look at the arithmetic average. Let $A_n = \frac{1}{n}\sum_{i=1}^{n} S_i$. We see that

$$A_n = A_{n-1} + \frac{S_n - A_{n-1}}{n} \tag{3.12}$$

The proof is given below:

$$\begin{aligned}
A_{n-1} + \frac{S_n - A_{n-1}}{n} &= \frac{1}{n-1}\sum_{i=1}^{n-1} S_i + \frac{1}{n}(S_n - \frac{1}{n-1}\sum_{i=1}^{n-1} S_i) \\
&= \frac{1}{n-1}\sum_{i=1}^{n-1} S_i + \frac{1}{n}S_n - \frac{1}{n}\frac{1}{n-1}\sum_{i=1}^{n-1} S_i \\
&= \frac{1}{n-1}(\sum_{i=1}^{n-1} S_i - \frac{1}{n}\sum_{i=1}^{n-1} S_i) + \frac{1}{n}S_n \\
&= \frac{1}{n-1}\sum_{i=1}^{n-1}(1 - \frac{1}{n})S_i + \frac{1}{n}S_n \\
&= \frac{n-1}{n^2-n}\sum_{i=1}^{n-1} S_i + \frac{1}{n}S_n \\
&= \frac{1}{n}\sum_{i=1}^{n-1} S_i + \frac{1}{n}S_n \\
&= A_n
\end{aligned}$$

It is clear that the recursive process described in (3.12) is discontinuous for every $t_i$, since there is a jump from $A_{i-1}$ to $A_i$. However, the realized option price $V$ needs to be continuous. If not, an arbitrage opportunity would arise, which is in contradiction with no arbitrage principle described in chapter 1. The arbitrage opportunity would arise, since people could either buy the option just before the sampling and sell right after if the jump in value is upwards and vice verse if the jump in value is downwards. Whether the jump in value will be up or down can be derived from the jump in the average, which is derived from the stock price.

Let $t_i^+$ denote the time just after $t_i$ and $t_i^-$ just before. To ensure a continuous option price at $t = t_i$, an extra condition needs to be imposed.

$$V(S_i, A_i, t_i^+) = V(S_i, A_{i-1}, t_i^-) \tag{3.13}$$

Using (3.12), (3.13) can be rewritten as:

$$V(S_i, A_{i-1} + \frac{S - A_{i-1}}{i}, t_i^+) = V(Si, A_{i-1}, t_i^-) \tag{3.14}$$

Since $A_{i-1}$ does not change between $t_{i-1}^+$ to $t_i^-$, the suffix $i-1$ can be dropped.

$$V(S_i, A + \frac{S - A}{i}, t_i^+) = V(Si, A, t_i^-) \tag{3.15}$$

This condition is called the *jump condition.*

The boundary conditions for this type of valuating the Asian option are different from those described in section 3.2.2.

Let A denote the arithmetic average of the stock price. The initial condition stays the same, the value of the Asian put ($V_P$) and Asian call option ($V_C$) is known at $t = T$:

$$V_C(S, A, T) = (A - K)_+ \tag{3.16}$$
$$V_P(S, A, T) = (K - A)_+ \tag{3.17}$$

The boundary conditions depend on the average stock price as well.
Notice that for $S = 0$, the Black-Scholes equation (3.1) changes to the equation:

$$\frac{dV}{dt} - rV = 0$$

This is an ordinary differential equation w.r.t. $t$ and can easily be solved (using (3.16) and (3.17) as initial conditions), resulting in

$$V_C(0, A, t) = e^{-r(T-t)}(A - K)_+ \tag{3.18}$$
$$V_P(0, A, t) = e^{-r(T-t)}(K - A)_+ \tag{3.19}$$

These conditions for $S = 0$ do not necessarily need to be enforced here, they follow automatically through the finite difference method.
Suppose $S \to \infty$. The value for $S \to \infty$ at $t = T$ is given by (3.16) and (3.17). For every time the average $A$ is updated, the value of the option adapts according to the jump condition [9].

# Chapter 4

# Results

Now that the Monte Carlo method and finite difference method have been defined in chapter 3, they can be used to generate results.

## 4.1 Asian option

We take a look at the Asian put option. This option has the payoff function $\Lambda(A) = (K - A)_+$, were for $A$ the arithmetic average is taken. In table 4.1 the Monte Carlo method is used to provide 95% confidence intervals for the Asian put options for various values of $r$, $\sigma$ and the starting stock price and average. In order to validate the results, the value of the Asian put options are also computed by the finite difference method (see chapter 3.2). Let $K$=4, $M = 10^4$, $\Delta t = 0.02$ (time step) and $T = 1$. Repeat the Monte Carlo method 10 times and take the average.

| | | $V(S_0 = 4, A_0 = 4, 0)$ | | $V(S_0 = 2, A_0 = 2, 0)$ | |
|---|---|---|---|---|---|
| r | $\sigma$ | Finite difference | Monte Carlo | Finite difference | Monte Carlo |
| 0.03 | 0.2 | 0.1685 | [0.1548, 0.1638] | 1.9115 | [1.9089, 1.9180] |
| | 0.3 | 0.2528 | [0.2427, 0.2560] | 1.9115 | [1.9077, 1.9215] |
| | 0.4 | 0.3526 | [0.3328, 0.3501] | 1.9122 | [1.9067, 1.9252] |
| | 0.5 | 0.4266 | [0.4226, 0.4435] | 1.9159 | [1.9088, 1.9317] |
| 0.05 | 0.2 | 0.1493 | [0.1373, 0.1459] | 1.8541 | [1.8511, 1.8602] |
| | 0.3 | 0.2318 | [0.2214, 0.2341] | 1.8541 | [1.8499, 1.8636] |
| | 0.4 | 0.3168 | [0.3099, 0.3265] | 1.8549 | [1.8490, 1.8673] |
| | 0.5 | 0.4026 | [0.3983, 0.4185] | 1.8589 | [1.8515, 1.8742] |
| 0.08 | 0.2 | 0.1208 | [0.1088, 0.1164] | 1.7704 | [1.7668, 1.7758] |
| | 0.3 | 0.2028 | [0.1926, 0.2045] | 1.7704 | [1.7657, 1.7792] |
| | 0.4 | 0.2851 | [0.2746, 0.2902] | 1.7714 | [1.7649, 1.7830] |
| | 0.5 | 0.3686 | [0.3682, 0.3875] | 1.7758 | [1.7681, 1.7904] |

Table 4.1: Values of Asian put options for various values of $r$ and $\sigma$.

Table 4.1 shows that the Monte Carlo method appears to be a decent method to compute the value of Asian options when compared to the finite difference method, since most of the confidence intervals contain the value given by the finite difference method.

## 4.2 Effect of a threshold

As mentioned in chapter 2 there are several types of Asian options available on the market. The conditional Asian option uses a threshold to determine which stock values are taken into account when computing the average. This type of option can also be evaluated using the Monte Carlo method. In this section the impact of the value of the threshold on the value of the conditional Asian put option is observed.

Let $K = 4,\ r = 0.03,\ \sigma = 0.3,\ T = 1,\ \Delta t = 0.02$ and $M = 10^4$. Let $b$ represent the value of the threshold. Notice that $b = 0$ results in a regular Asian option.
The value of the conditional Asian put option can be simulated with the Monte Carlo methods and its results are shown in table 4.2. Repeat the Monte Carlo method 10 times and take the average.

| $S_0$ | b=0 | b=1 | b=2 | b=3 | b=4 |
|---|---|---|---|---|---|
| $S_0 = 1$ | 2.8959 ± 0.0018 | 2.7864 ± 0.0012 | 1.8257 ± 0.0011 | - | - |
| $S_0 = 2$ | 1.9100 ± 0.0035 | 1.9087 ± 0.0035 | 1.6911 ± 0.0023 | 0.7496 ± 0.0020 | 0 ± 0 |
| $S_0 = 3$ | 0.9417 ± 0.0048 | 0.9417 ± 0.0048 | 0.9167 ± 0.0045 | 0.6190 ± 0.0028 | 0 ± 0 |
| $S_0 = 4$ | 0.2441 ± 0.0034 | 0.2441 ± 0.0034 | 0.2414 ± 0.0033 | 0.1696 ± 0.0022 | 0 ± 0 |
| $S_0 = 5$ | 0.0282 ± 0.0011 | 0.0282 ± 0.0011 | 0.0280 ± 0.0011 | 0.0145 ± 0.0006 | 0 ± 0 |
| $S_0 = 6$ | 0.0015 ± 0.0002 | 0.0015 ± 0.0002 | 0.0015 ± 0.0002 | 0.0004 ± 0.0000 | 0 ± 0 |

Table 4.2: Value conditional Asian put for various values of $b$ and $S_0$

Table 4.2 shows an estimate of the value of the option and the standard error[1], to provide an indication of the accuracy. A number of interesting features of the conditional Asian put can be observed.

First, notice that there is no value of the option for the starting value $S_0 = 1$ for $b = 3$ and $b = 4$. This is plausible when the payoff of the option is a bit closer inspected at $t = T$. If the price path starts at $S_0 = 1$, it is unlikely that the price will ever rise above $S = 3$ or above $S = 4$. Since the value of the put option at $t = T$ is equal to $(K - A)_+$ and the average $A$ depends on the values of $S > b$, it makes sense that the average cannot be computed if there are no values of $S > b$.

Furthermore, table 4.2 shows that for a high value of $b$ (e.g. $b = 4$), the value of the option equals 0. To understand this value, the payoff of the conditional Asian put needs to be observed:

$$V(S,T) = \Big(K - \frac{\sum_{i=1}^{n} S_i I_{\{S_i > b\}}}{\sum_{i=1}^{n} I_{\{S_i > b\}}}\Big)_+$$

In this situation $K = 4$. This means that the value of the conditional Asian put will definitely be equal to zero if $A \geq 4$. For $b = 4$, the average is computed over values of $S$ which are all larger than 4. This will therefore certainly result in the option having a value of zero.
More generally, one can state that:

$$V(S,T) = 0 \iff b \geq K \tag{4.1}$$

[1]Standard error is another name for standard deviation and this term is used here since in this case the value is an indication of the error of the estimate.

Additionally the observation can be made that the option value decreases as the starting value $S_0$ increases. This can easily be explained by looking at the payoff at final time $T$: $V(S,T) = (K-A)_+$. For the given interest and volatility ($r = 0.03$ and $\sigma = 0.3$ respectively) the stock price in the price path will not differ substantially, so if the starting price increases, the average is likely to increase as well, resulting in a decrease in payoff at final time $T$.

Moreover, the option value does not seem to differ much comparing larger values of $S$ for different values of the threshold. The reason for this is that the higher starting value of the price of the stock is, the smaller the chance that its price will drop below the threshold, which means that the average for those option (and thereby the value of the option) is likely to stay the same.

## 4.3 Number of updates of the average

In section 4.2 the role of the threshold value was looked into, where the average was taken over every value of $S$ in the price path, i.e. the average was "updated" at every time step in the price path. However, this is not necessarily always the case. The average can also be taken over every second time step, or even every third.
Let $t = ki\Delta t$ with $k \in \{1, ..., \frac{T}{\Delta t}\}$ fixed and $i \in \{1, ..., \frac{T}{k\Delta t}\}$. Here $\frac{T}{k\Delta t}$ must be a round number and is allowed to be rounded off.

Let $K = 4,\ r = 0.03,\ \sigma = 0.3,\ M = 10^4,\ T = 1$ and $\Delta t = 0.02$. This results in $\frac{T}{\Delta t} = 50$. To make the situation more straightforward, we take for the barrier $b = 0$, resulting in a regular Asian option. Repeat the Monte Carlo method 10 times and take the average.

| k | $V(S_0 = 4, 0)$ | $V(S_0 = 2, 0)$ |
|---|---|---|
| 1 | 0.2441 ± 0.0034 | 1.9100 ± 0.0035 |
| 2 | 0.2407 ± 0.0033 | 1.9106 ± 0.0035 |
| 4 | 0.2383 ± 0.0033 | 1.9106 ± 0.0034 |
| 8 | 0.2339 ± 0.0032 | 1.9105 ± 0.0034 |
| 16 | 0.2269 ± 0.0031 | 1.9105 ± 0.0033 |
| 32 | 0.1784 ± 0.0024 | 1.9203 ± 0.0026 |

Table 4.3: Value Asian put option for various number of samples

Table 4.3 shows the option value where the amount of times that the average is "updated" is divided by two in the next row. A few details attract attention.

The standard error seems to decline slightly as the number of times the average is "updated" decreases. The standard error measures the amount of dispersion of the set of option values. This is based on the underlying distribution of the samples and is therefore expected to be constant. The reason for the slight change in standard error is the fact that for larger number of updates, the error in determining the standard error is smaller. The standard error is therefore "more accurate" for larger number of updates.

An important detail to consider when implementing the Matlab code is to keep the seed the same. In order to generate the option values for different amounts of sampling, the program needs to run several times. Matlab has a pseudo-random generator to produce samples from the standard normal distribution, which will produce different samples every run unless specified otherwise. In order to accurately compare the different amount of samples, they need to be based on the same asset path, i.e. the program needs to use the same samples every run.

## 4.4 Strike price

Another element that most certainly influences the value of the option is the strike price. Let $r = 0.03,\ \sigma = 0.3,\ b = 3,\ T = 1,\ \Delta t = 0.02$ and $M = 10^4$. Repeat the Monte Carlo algorithm 10 times and take the average.

| K | $V(S_0 = 2, 0)$ | $V(S_0 = 4, 0)$ |
|---|---|---|
| 1 | $0 \pm 0$ | $0 \pm 0$ |
| 2 | $0 \pm 0$ | $0 \pm 0$ |
| 3 | $0 \pm 0$ | $0 \pm 0$ |
| 4 | $0.7496 \pm 0.0020$ | $0.1696 \pm 0.0022$ |
| 5 | $1.7176 \pm 0.0021$ | $0.8811 \pm 0.0051$ |
| 6 | $2.6880 \pm 0.0021$ | $1.8088 \pm 0.0060$ |
| 7 | $3.6584 \pm 0.0021$ | $2.7746 \pm 0.0061$ |
| 8 | $4.6289 \pm 0.0021$ | $3.7446 \pm 0.0062$ |
| 9 | $5.5993 \pm 0.0021$ | $4.7150 \pm 0.0062$ |
| 10 | $6.5698 \pm 0.0021$ | $5.6855 \pm 0.0062$ |

Table 4.4: Value conditional Asian put option for various strike values

Table 4.4 presents the option value of the conditional Asian put option for several values of the strike price. A few elements stand out.

First, notice that for both $S_0 = 2$ and $S_0 = 4$ the option value of the conditional Asian put equals zero for any strike price below 4. This can be explained by having a look at the threshold value. A threshold value of $b = 3$ has been used, which means that only values of $S \geq 3$ will be used to compute the average, automatically resulting in $A \geq 3$. Since the payoff at final time $T$ is of the form $V(S, A, T) = (K - A)_+$, this will result in $V(S, A, T) = 0$ for $K \leq 3$. More generally:

$$K \leq b \Longrightarrow V(S, A, T) = 0 \tag{4.2}$$

On the other hand $V(S, A, T) = 0 \Longrightarrow K \leq b$ is not necessarily true. $V(S, A, T) = 0$ can have many reasons, for example the case where $S = 0$.

Table 4.4 also reveals that the option value of the conditional Asian put increases as the strike price increases. The reason for this is that an increase in the strike price does not affect the average, which will therefore result in a higher value of $K - A$.

## 4.5 Conditional Asian option

As mentioned in section 3.2.3 it is difficult to find the finite difference discretization to approximate the value of the conditional Asian option. However, we were able to find the discretization of the value of the conditional Asian option where the sum of the stock price was divided by the maturity time. Let us call this type of option the conditional Asian 1 option.

Let $K = 4$, $\sigma = 0.3$, $r = 0.03$, $T = 1$, $\Delta t = 0.02$, $M = 10^4$. Repeat the Monte Carlo method 10 times and take the average. Table 4.5 shows the comparison between the conditional Asian option (computed by the Monte Carlo method) and the value of the conditional Asian 1 option (computed through the finite difference method) with a threshold of $b = 3$.

| $S$ | Conditional Asian option | Conditional Asian 1 option |
|---|---|---|
| $S = 1$ | - | 1.6390 |
| $S = 2$ | $0.7496 \pm 0.0020$ | 1.0014 |
| $S = 3$ | $0.6190 \pm 0.0028$ | 0.6199 |
| $S = 4$ | $0.1696 \pm 0.0022$ | 0.3870 |
| $S = 5$ | $0.0145 \pm 0.0006$ | 0.2430 |

Table 4.5: Value conditional Asian tail put option for various starting values $S_0$

Table 4.5 shows that the value of the value of the conditional Asian option 1 is larger than the value of the regular conditional Asian option. The reason for this is that the arithmetic average of the conditional Asian 1 option is smaller, since the sum of the stock prices is divided by the total number of time steps instead of the number of time steps where the stock price is larger than the threshold.

## 4.6 Asian tail option

As mentioned in chapter 2 there exists a type of Asian option where the averaging feature is only active for a part of the lifetime of the option. For Asian tail options, this part is the final part of the option.
Let $K = 4$, $r = 0.03$, $\sigma = 0.3$, $b = 0$ (regular Asian put option), $T = 1$, $\Delta t = 0.02$, $N_t = 50$ (number of time steps) and $M = 10^4$. Repeat the Monte Carlo algorithm 10 times and take the average.

| Tail | $V(S_0 = 2, 0)$ | $V(S_0 = 4, 0)$ |
|---|---|---|
| [1,50] | $1.9107 \pm 0.0035$ | $0.2456 \pm 0.0034$ |
| [10,50] | $1.9039 \pm 0.0041$ | $0.2853 \pm 0.0039$ |
| [20,50] | $1.9043 \pm 0.0046$ | $0.3191 \pm 0.0044$ |
| [30,50] | $1.8948 \pm 0.0051$ | $0.3577 \pm 0.0048$ |
| [40,50] | $1.8900 \pm 0.0056$ | $0.3820 \pm 0.0051$ |

Table 4.6: Value conditional Asian tail put option for various starting values $S_0$

Table 4.6 shows the value of the Asian tail put option for several lengths of the tail (measured in number of time steps). Notice that implementing a tail of [1,50] results in a regular Asian put option.

It is noteworthy that the standard error increases as the length of the tail decreases. This can be explained by looking at the average. If the tail of the Asian option is shorter, this means that fewer samples of the stock price were used to compute the average, creating a more volatile average. This in turn creates a more volatile value of the Asian tail option.

## 4.7 Geometric average

In all paragraphs above, the Asian option was based on the arithmetic average $A_n = \frac{1}{n}\sum_{i=1}^{n} S_i$. It is known that for this type of average the Asian option does not have an exact solution.
However, there exist a type of average where the Asian option does have an exact solution: *geometric average*. The discretely sampled geometric average is given by: [15]

$$A_n = \Big(\prod_{i=1}^{n} S_i\Big)^{\frac{1}{n}}$$

There is an interesting relation between Asian options with discrete arithmetic average and discrete geometric average.

**Proposition 2.** [17]
Let $V_P^a, V_C^a$ denote the Asian put and call option with arithmetic average and let $V_P^g, V_C^g$ denote the Asian put and call option with geometric average. The following inequalities hold:

$$V_P^a(S_0,t) \leq V_P^g(S_0,t) \tag{4.3}$$
$$V_C^g(S_0,t) \leq V_C^a(S_0,t) \tag{4.4}$$

*Proof.* The difference in option value is due to the difference in value of the types of averages. It can be proven that the arithmetic average taken over positive real numbers is always larger or equal to the geometric average taken over those same numbers:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \prod_{i=1}^{n} x_i \;\forall n \in \mathbb{N},\, \forall x \in \mathbb{R}^+$$

Since the value of the put and call at time $t$ is equal to $V_P(S_0,t) = e^{-r(T-t)}(K-A)_+$ and $V_C(S_0,t) = e^{-r(T-t)}(A-K)_+$ respectively, the inequalities (4.3) and (4.4) follow. □

The exact value of the Asian call and put option with geometric average can be written as: [11] [17]

$$\begin{aligned} V_P^g(S_0,0) &= e^{-rT}\Big(KN(-\hat{d}_2) - S_0 e^{\hat{u}T} N(-\hat{d}_1)\Big) \\ V_C^g(S_0,0) &= e^{-rT}\Big(S_0 e^{\hat{u}T} N(\hat{d}_1) - KN(\hat{d}_2)\Big) \end{aligned}$$

where $N$ is the $N(0,1)$ distribution function and

$$\begin{aligned} \hat{\mu} &= \frac{1}{2}\hat{\sigma}^2 + (r - \frac{1}{2}\sigma^2)\frac{n+1}{2n} \\ \hat{\sigma}^2 &= \sigma^2 \frac{(n+1)(2n+1)}{6n^2} \\ \hat{d}_1 &= \frac{\log(S_0/K) + (\hat{\mu} + \frac{1}{2}\hat{\sigma}^2)T}{\hat{\sigma}\sqrt{T}} \\ \hat{d}_2 &= \hat{d}_1 - \hat{\sigma}\sqrt{T} \end{aligned}$$

Let $K = 4$, $\sigma = 0.3$, $r = 0.03$, $T = 1$, $\Delta t = 0.02$ and $M = 10^4$. Table 4.7 shows the value of the Asian put and call options with arithmetic average (computed with the Monte Carlo method) and geometric average (computed with both the Monte Carlo method and exact).

| $S$ | $V_P^a(S,0)$ (MC) | $V_P^g(S,0)$ (MC) | $V_P^g(S,0)$ | $V_C^a(S,0)$ (MC) | $V_C^g(S,0)$ (MC) | $V_C^g(S,0)$ |
|---|---|---|---|---|---|---|
| $S=1$ | 2.8982 ± 0.0018 | 2.9057 ± 0.0017 | 2.9037 | 0 ± 0 | 0 ± 0 | -1e-16 |
| $S=2$ | 1.9146 ± 0.0035 | 1.9296 ± 0.0035 | 1.9257 | 0 ± 0 | 0 ± 0 | 4e-06 |
| $S=3$ | 0.9481 ± 0.0049 | 0.9670 ± 0.0049 | 0.9610 | 0.0171 ± 9e-04 | 0.0135 ± 8e-04 | 0.0134 |
| $S=4$ | 0.2494 ± 0.0034 | 0.2622 ± 0.0035 | 0.2559 | 0.3020 ± 0.0048 | 0.2849 ± 0.0045 | 0.2863 |
| $S=5$ | 0.0290 ± 0.0011 | 0.0341 ± 0.0013 | 0.0329 | 1.0652 ± 0.0084 | 1.0329 ± 0.0081 | 1.0413 |
| $S=6$ | 0.0017 ± 2e-04 | 0.0026 ± 3e-04 | 0.0025 | 2.0215 ± 0.0105 | 1.9775 ± 0.0103 | 1.9890 |

Table 4.7: Value Asian call and put option with geometric and arithmetic average

The results of table 4.7 agree with proposition 2.

# Chapter 5

# Conclusion

In this thesis, we introduced the problem of option valuation and its assumptions. We described European options and several types of Asian options and their use in the financial market.

In chapter 2 the European option and several types of Asian options were introduced. The asset price model described the Geometric Brownian motion for the stock price and the Black-Scholes model used the Black-Scholes partial differential equation to describe the movement of the value of the option. The Black-Scholes PDE differs per type of option, its application to the European and Asian were discussed, just like their boundary conditions. Furthermore, the Feynman-Kac formula indicated that the solution of the Black-Scholes PDE is equal to the expected value of the payoff under the risk neutral measure.

The next chapter described the numerical methods used to implement the described models. The Monte Carlo method uses the asset price model to simulate many asset price paths and computes the option value at $t = 0$ by taking the expectation of the discounted option values at $t = T$.
The finite difference method can be applied to the Black-Scholes PDE by approximating the derivatives in the equation. The discretization of the European, Asian and Asian tail option were described and it was noted that the application to the conditional Asian option was not straightforward. A sidestep was made describing how the Asian option could be valuated by using the Black-Scholes PDE for European options and introducing a jump condition.

The results of the application of both numerical methods to option valuation were given in chapter 4. In this chapter the Asian option value of the Monte Carlo method and the finite difference method were compared and the effect of a threshold was discussed. Additionally, the effect of the number of samples on the average and the value of the strike price were considered, after which the value of the Asian tail option was approximated and the difference in value by using geometric average and arithmetic average was discussed.

# Chapter 6

# Discussion and further research

Numerous elements in this report are open to discussion. For instance, one might question whether the models give the correct option value, given the many assumptions they have to adhere to (see section 1.2). The fourth assumption says that the bid-ask spread equals zero, this is not necessarily the case in the financial market. The exchange listed American and European options work with supply and demand and it might happen that a stock trader asks a price that is too high for other stock traders to want to buy it. This may cause friction when trying to hedge the risk. However, the Asian options are not listed on the exchange, they are so called *over-the-counter options* (OTC). This means that there is no bid-ask spread, making the fourth assumption quite applicable in this situation.

Additionally, the assumption is made that the interest rate is equal for borrowing and lending money and equal over time. Naturally this is not the case in the real world. The interest rate is used as a constant parameter in both the asset price model and the Black-Scholes model, so removing this assumption could result in the need for a different model. Another SDE would be needed which includes the description of the movement of the interest rate.

In the chapter "Numerical methods" it was already noted that the Monte Carlo method is easier to adapt to exotic options than the finite difference method applied to the Black-Scholes method. Since additionally the computing time of the Monte Carlo method is considerably smaller than that of the finite difference method, one could argue that the Monte Carlo method is preferable when computing these types of Asian options, or exotic options in general. As mentioned in section 3.1.1 there might arise issues with the size of the variance. This problem could be resolved by using antithetic variates or control variates.

More research could be conducted on the finite difference method of the conditional Asian option, since the discretization proposed in this thesis is the discretization of a slightly adapted version of the conditional Asian option.

# Appendix A

# Discretisation matrix Asian option

The discretisation matrices for the finite difference method for Asian options are given below.

A=

$$\begin{bmatrix} (\frac{\Delta t}{2}(-r-\sigma^2 1^2-\frac{1\Delta S}{\Delta Z})+1 & \frac{\Delta t}{2}(\frac{1}{2}\sigma^2 2^2+\frac{1}{2}2r) & 0 & \dots & \dots & 0 \\ \frac{\Delta t}{2}(\frac{1}{2}\sigma^2 1^2-\frac{1}{2}r) & \frac{\Delta t}{2}(-r-\sigma^2 2^2-\frac{2\Delta S}{\Delta Z})+1 & \frac{\Delta t}{2}(\frac{1}{2}\sigma^2 3^2+\frac{1}{2}3r) & \dots & \vdots & \vdots \\ 0 & \frac{\Delta t}{2}(\frac{1}{2}\sigma^2 2^2-\frac{1}{2}2r) & \frac{\Delta t}{2}(-r-\sigma^2 3^2-\frac{3\Delta S}{\Delta Z})+1 & \dots & \vdots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots & \vdots \\ \vdots & \dots & \dots & \dots & \frac{\Delta}{2}(-r-\sigma^2(N-1)^2-\frac{(N-1)\Delta S}{\Delta Z} & \frac{\Delta t}{2}(\frac{1}{2}\sigma^2(N-1)^2-\frac{1}{2}r(N-1) \end{bmatrix}$$

B=

$$\begin{bmatrix} (-\frac{\Delta t}{2}(-r-\sigma^2 1^2-\frac{1\Delta S}{\Delta Z})+1 & \frac{-\Delta t}{2}(\frac{1}{2}\sigma^2 2^2+\frac{1}{2}2r) & 0 & \dots & \dots & 0 \\ \frac{-\Delta t}{2}(\frac{1}{2}\sigma^2 1^2-\frac{1}{2}r) & \frac{-\Delta t}{2}(-r-\sigma^2 2^2-\frac{2\Delta S}{\Delta Z})+1 & \frac{-\Delta t}{2}(\frac{1}{2}\sigma^2 3^2+\frac{1}{2}3r) & \dots & \vdots & \vdots \\ 0 & \frac{-\Delta t}{2}(\frac{1}{2}\sigma^2 2^2-\frac{1}{2}2r) & \frac{-\Delta t}{2}(-r-\sigma^2 3^2-\frac{3\Delta S}{\Delta Z})+1 & \dots & \vdots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots & \vdots \\ \vdots & \dots & \dots & \dots & \frac{-\Delta}{2}(-r-\sigma^2(N-1)^2-\frac{(N-1)\Delta S}{\Delta Z} & \frac{-\Delta t}{2}(\frac{1}{2}\sigma^2(N-1)^2-\frac{1}{2}r(N-1) \end{bmatrix}$$

# Bibliography


[1] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 1973.

[2] P.P. Boyle. Options: A monte carlo approach. *Journal of Financial Economics*, 4(3), 1977. URL https://EconPapers.repec.org/RePEc:eee:jfinec:v:4:y:1977:i:3:p:323-338.

[3] J. Chen. Asian option, 2018. URL https://www.investopedia.com/terms/a/asianoption.asp. Accessed: 17-09-2018.

[4] J. Chen. Risk neutral probability measure, 2018. URL https://www.investopedia.com/terms/r/risk-neutral-measures.asp. Accessed: 10-12-2018.

[5] J. Chen. Asian tail, 2018. URL https://www.investopedia.com/terms/a/asiantail.asp. Accessed: 02-11-2018.

[6] P. Del Moral. *Feynman-Kac Formulae*. Springer New York, 2004. ISBN 978-1-4684-9393-1. URL https://doi.org/10.1007/978-1-4684-9393-$1_2$.

[7] W. Falloon and D. Turner. *Managing Energy Price Risks Financial Engineering*. Risk books, 1999. ISBN 978-1-899-33254-0.

[8] R. Feng and H. Volkmer. Conditional asian options. 18, March 2015.

[9] K. R. Forsyth, P. A.and Vetzal and R. Zvan. Convergence of numerical methods for valuing path-dependent options using interpolation. *Review of Derivatives Research*, 5, Oct 2002. doi: 10.1023/A:1020823700228. URL https://doi.org/10.1023/A:1020823700228.

[10] J.M. Hammersley and D.C. Handscomb. *Monte Carlo Methods*. Methuen's monographs on applied probability and statistics. Methuen, 1964. ISBN 9780416523409. URL https://books.google.nl/books?id=Kk4OAAAAQAAJ.

[11] D.J. Higham. *An introduction to Financial Option Valuation*. Cambridge University Press, 2004. ISBN 978-0-521-54757-4.

[12] A. G. Z. Kemna and T. Vorst. A pricing method for options based on average asset values. *Journal of Banking Finance*, 1990.

[13] A. Klenke. *Probability Theory: A Comprehensive Course*. Universitext. Springer London, 2013. ISBN 9781447153603.

[14] S.E. Shreve. *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model: Binomial Asset Pricing Model*. Springer-Verlag, 2003. ISBN 9780387401003.

[15] P. Wilmott, S. Howison, and J. Dewynne. *The mathematics of Financial derivatives: a student introduction*. Cambridge University Press, 1996. ISBN 978-0-521-49699-3.

[16] L.T Yueng. Crank-nicolson scheme for asian option. Master's thesis, University Tunku Abdul Rahman, 8 2012.

[17] H. Zhang. Pricing asian options using monte carlo methods. Master's thesis, Uppsala University, 9 2007.

[18] R. Zvan, P. A. Forsyth, and K. Vetzal. Robust numerical methods for pde models of asian options. *Journal of Computational Finance*, 1:39–78, 1998.