“Eigenvalue analysis of the Timoshenko Beam theory with a damped boundary condition”

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Contents

1 Introduction

In this report several theories of beam equations will be treated. The aim for this report is to get a better understanding of the beam equations and its applications. The goal is to solve a non-tensioned beam with one damped boundary and one simply supported. This will be done with the help of the Timoshenko beam theory.

Before this goal is reached, first the Euler-Bernoulli and Rayleigh theories will be treatend. This is as an introduction to the Timoshenko beam theory, since this theory is an extention of the previous two.

The idea behind this report came from the Erasmus bridge in Holland, Rotterdam. Under some weather conditions, rain fall and a heavy wind, the stay cables of this bridge began to resonate. Which could have collapsed the bridge. Luckily this was prevented by adding damping on the end of these stay cables.

The question is how much damping does one stay cable need in order to be stable under those conditions. Since a stay cable is under tension, it can be assumed that this behaves like a beam under tension. For simplicity, in this report the beam is un-tensioned.

First three beam theories (Euler-Bernoulli, Rayleigh and Timoshenko) will be explained. Then the damped boundary conditions will be introduced and an attempt will be made in solving the un-tensioned Timoshenko beam.

2 Notation

When introducing the beam theories several constants/functions will be used. Here is a summary of these

$w(x,t)$	$-$	transverse displacement of the beam compared to the centerline.
u, v, and w	$-$	represent the components of displacement parallel to x, y, and z, directions respectively.
$f(x,t)$	$-$	external force.
$M(x,t)$	$-$	bending moment.
$F(x,t)$	$-$	shear force.
V	$[m^3]-$	volume of the beam
A	$[m^2]-$	cross sectional area $\mathrm{A} = \mathrm{y} \cdot \mathrm{z}$.
ρ	$[N/m^3]-$	density of the beam.
δ	$-$	variation inintegrating, used in Hamilton's Principle.
G	$[Pa]-$	shear modulus
k	$-$	shear correction factor
E	$[Pa]-$	Young's modulus
I	$[m^4]-$	moment of inertia with respect to the $\mathrm{y}-\mathrm{axis}$.
g	$[m/s^2]-$	gravitational acceleration
l	$[m]$	length of the beam
π	$-$	strain energy
ε_{ij}	$-$	strain component
σ	$-$	stress component

3 Equations of motion

The equation of motion of a vibrating beam can be derived by using the dynamic equilibrium approach, variational method, or integral equation formulation [1]. In the following section the variational method will be used to derive the Euler-Bernoulli equation.

3.1 Euler-Bernoulli beam theory

This theory is the most basic theory for beams. To derive the equation of motion for a beam that is *slender*, a small piece of the beam will be analysed.

The rotation of cross sections of the beam is neglected compared to the translation. In addition, the angular distortion due to shear is considered negligible compared to the bending deformation.

The transverse displacement of the centerline of the beam is given by w, the displacement components of any points in the cross section, when plane sections remain plane and normal to the centerline, are given by

$$u = -z\frac{\partial w(x,t)}{\partial x} \quad v = 0 \quad w = w(x,t). \tag{3.1}$$

This can be seen from the following figure:

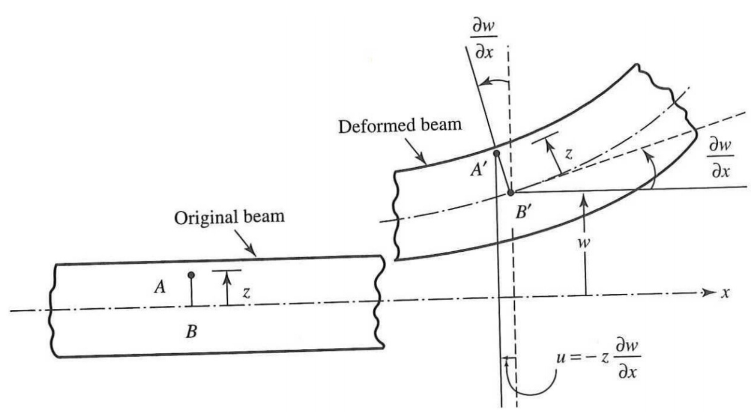

Figure 1: Deformed beam

Where it is assumed that the displacements are small, such that $\tan\alpha \approx \alpha$.

The component of strain and stress corresponding to this displacement field are given by:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \quad \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0, \\ \sigma_{xx} &= -Ez\frac{\partial^2 w}{\partial x^2} \quad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0. \end{aligned} \tag{3.2}$$

The strain energy of the system can be expressed as

$$\pi = \frac{1}{2}\iiint\limits_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx})dV$$

$$= \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx. \tag{3.3}$$

Here I denotes the area moment of inertia of the cross section of the beam about the y axis, which stands orthogonal on the x and z axis in Figure 1

$$I = I_y = \iint\limits_A z^2 dA. \tag{3.4}$$

More information about the strain energy can be found in the Appendix.

The kinetic energy T of the beam is given by

$$T = \frac{1}{2}\int_0^l \iint\limits_A \rho\left(\frac{\partial w}{\partial t}\right)^2 dAdx = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial w}{\partial t}\right)^2 dx. \tag{3.5}$$

The work done by the transverse load $f(x,t)$ is given by

$$W = \int_0^l f(x,t)w(x,t)dx. \tag{3.6}$$

Thus, Hamilton's principle states that:

$$\delta\int_{t_1}^{t_2}(\pi - T - W)dt = 0, \tag{3.7}$$

where δ is the variation between two moments of time t_1 and t_2. Detailed information about the Hamilton Principle can be found in the Appendix.

Making use of the defined variables, this is rewritten to

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{2} \int_0^l EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx - \frac{1}{2} \int_0^l \rho A \left(\frac{\partial w}{\partial t} \right)^2 dx - \int_0^l f(x,t) w(x,t) dx \right\} dt = 0. \tag{3.8}$$

Thus the generalized Hamilton's principle gives[1]

$$\delta \int_{t_1}^{t_2} (\pi - T - W) dt = \int_{t_1}^{t_2} \left\{ EI \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \bigg|_0^l - \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) \delta w \bigg|_0^l \right\} dt +$$
$$\left\{ \int_0^l \left[\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + \rho A \frac{\partial^2 w}{\partial t^2} - f \right] \delta w dx \right\} dt. \tag{3.9}$$

From this the transverse vibration beam equation can be obtained, together with the boundary conditions

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t), \tag{3.10}$$

$$EI \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \bigg|_0^l = 0, \tag{3.11}$$

$$-\frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) \delta w \bigg|_0^l = 0. \tag{3.12}$$

The first equation describes the motion of a *slender* beam. Equation (3.12),(3.11) give the boundary conditions.

Equation (3.11) gives us the options, at $x = l$:

1. $\frac{\partial w}{\partial x}$ = constant, in that way, the variation on w, $\delta \left(\frac{\partial w}{\partial x} \right)$, equals zero, or
2. $EI \frac{\partial^2 w}{\partial x^2} = 0$.

and equation (3.12) gives the following at $x = l$:

1. $w =$ constant , or
2. $-\frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right)$.

Any combination between these two sets of boundary conditions will result in a solvable problem at $x = l$.

The same applies at $x = 0$. Since the problem is a fourth order problem, there are also four boundary conditions needed. So there are two boundary conditions needed at $x = l$ and two at $x = 0$.

[1]derivation can be found in the appendix.

3.2 Rayleigh beam theory

The second theory that we will consider is Rayleigh's theory. In this theory the inertia due to the axial displacement of the beam is included. This effect is called *rotary inertia*.
The reason is that since the cross section remains plane during motion, the axial motion of points located in any cross section undergoes rotary motion about the y axis. Using $u = -z\left(\frac{\partial w}{\partial x}\right)$, see (3.1) from the Euler-Bernoulli derivation, the axial velocity is given by:

$$\frac{\partial u}{\partial t} = -z\frac{\partial^2 w}{\partial t \partial x}, \tag{3.13}$$

and hence the kinetic energy associated with the axial motion is given by:

$$T_a = \frac{1}{2}\int_0^l \iint_A \rho\left(\frac{\partial u}{\partial t}\right)^2 dA dx = \frac{1}{2}\int_0^l \left(\iint_A z^2 dA\right)\rho\left(\frac{\partial^2 w}{\partial t\partial x}\right) dx =$$

$$= \frac{1}{2}\int_0^l \rho I\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 dx. \tag{3.14}$$

The term associated with T_a in Hamilton's principle can be evaluated as

$$I_a = \delta\int_{t_1}^{t_2} T_a dt = \delta\int_{t_1}^{t_2}\frac{1}{2}\int_0^l \rho I\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 dx dt =$$

$$= \int_{t_1}^{t_2}\int_0^l \rho I\frac{\partial^2 w}{\partial t\partial x}\delta\left(\frac{\partial^2 w}{\partial t\partial x}\right) dx dt. \tag{3.15}$$

Using integration by parts with respect to time, (3.15) gives

$$I_a = -\int_{t_1}^{t_2}\int_0^l \rho I\frac{\partial^3 w}{\partial t^2\partial x}\delta\left(\frac{\partial w}{\partial x}\right) dx dt. \tag{3.16}$$

Using integration by parts with respect to x of (3.16) yields

$$I_a = \int_{t_1}^{t_2}\left[-\rho I\frac{\partial^3 w}{\partial t^2\partial x}\delta w\Big|_0^l + \int_0^l \frac{\partial}{\partial x}\left(\rho I\frac{\partial^3 w}{\partial t^2\partial x}\right)\delta w dx\right] dt. \tag{3.17}$$

Because the theory of Rayleigh is an extention of the Euler-Bernoulli theory, the energy term I_a can be used in the derivation of the Euler-Bernoulli theorem.

To get the equation of motion of a beam with rotary inertia, e.g. Rayleigh's theory, add $-I_a$ to the equation of (3.9). Which results in

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(\rho I\frac{\partial^3 w}{\partial t^2 \partial x}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = f(x,t), \tag{3.18}$$

$$EI\,\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^l - \left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) - \rho I\,\frac{\partial^3 w}{\partial t^2 \partial x}\right]\delta w\Bigg|_0^l = 0. \tag{3.19}$$

For a uniform beam, the equation of motion and the boundary conditions can be expressed as

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial t^2 \partial^2 x} = f(x,t), \tag{3.20}$$

$$EI\,\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^l = 0, \tag{3.21}$$

$$\left(EI\frac{\partial^3 w}{\partial x^3} - \rho I\,\frac{\partial^3 w}{\partial t^2 \partial x}\right)\delta w\Bigg|_0^l = 0. \tag{3.22}$$

The boundary conditions are dealt with in the same way as with the Euler-Bernoulli beam theory. Further, remark that when the rotary inertia is neglected the original Euler-Bernoulli theorem remains.

3.3 Timoshenko beam theory

The effect of shear deformation, in addition to the effect of rotary inertia, is considered in this theory. To include the effect of shear deformation, first consider a beam undergoing only shear deformation as indicated in Figure 2:

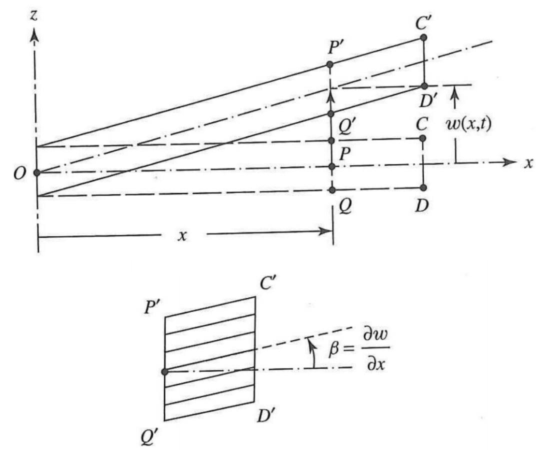

Figure 2: Shear deformation

Here a vertical section, such as PQ, before deformation remains vertical (P'Q') after deformation but moves by a distance w in the z direction. Thus, the components of displacement of a point in the beam are given by:

$$u = 0 \quad v = 0 \quad w = w(x,t). \tag{3.23}$$

The components of strain can be found as:

$$\begin{aligned} \varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \quad \varepsilon_{yy} &= \frac{\partial v}{\partial y} = 0 \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0 \\ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \varepsilon_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \\ \varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial x} \end{aligned} \tag{3.24}$$

The shear strain ε_{zx} is the same as the rotation $\beta(x,t) = \frac{\partial w}{\partial x}$ experienced by any fiber located parallel to the centerline of the beam, as shown in figure 2. The components of stress corresponding to the strains indicated by (3.24) are given by:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0, \quad \sigma_{zx} = \sigma_{zx} = G\frac{\partial w}{\partial x}. \tag{3.25}$$

Equation (3.25) states that the shear stress σ_{zx} is the same (uniform) at every point in the cross section of the beam. Since this is not true in reality, Timoshenko used a constant k, known as the *shear correction factor*, in the expression for σ_{zx} as:

$$\sigma_{zx} = kG\frac{\partial w}{\partial x}.$$

The total transverse displacement of the centerline of the beam is given by (see figure 4):

$$w = w_s + w_b, \tag{3.26}$$

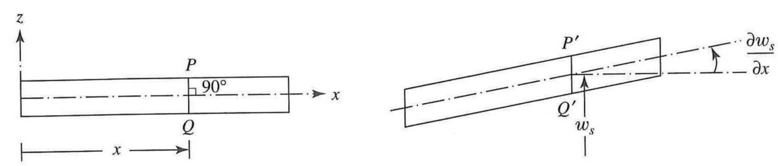

Figure 3: Shear deformation

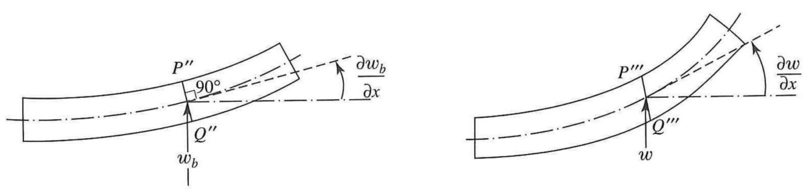

Figure 4: Rotary deformation and total deformation

and hence the total slope of the deflected centerline of the beam is approximated by:

$$\frac{\partial w}{\partial x} = \frac{\partial w_s}{\partial x} + \frac{\partial w_b}{\partial x}. \tag{3.27}$$

Since the cross section of the beam undergoes rotation due only to bending, the rotation of the cross section can be expressed as:

$$\phi = \frac{\partial w_b}{\partial x} = \frac{\partial w}{\partial x} - \frac{\partial w_s}{\partial x} = \frac{\partial w}{\partial x} - \beta, \tag{3.28}$$

where $\beta = \dfrac{\partial w_s}{\partial x}$ is the shear deformation or shear angle. An element of fiber located at a distance z from the centerline undergoes axial displacement due only to the rotation of the cross section (shear deformation does not cause any axial displacement), and hence the components of displacement can be expressed as:

$$u = -z\left(\frac{\partial w}{\partial x} - \beta\right) = -z\phi(x,t), \quad v = 0, \quad w = w(x,t). \tag{3.29}$$

Thus now we have added the equations for the motion of a particle under shear and bending deformation. Because of that, the stress- and strain components will be different. These will change in the following way:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = -z\frac{\partial \phi}{\partial x}, \\ \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \quad \varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0, \\ \varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \\ \varepsilon_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \\ \varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= -\phi + \frac{\partial w}{\partial x}. \end{aligned} \tag{3.30}$$

The components of stress corresponding to the strains of (3.30) are given by

$$\begin{aligned} \sigma_{xx} &= -Ez\frac{\partial \phi}{\partial x}, \\ \sigma_{zx} &= kG\left(\frac{\partial w}{\partial x} - \phi\right), \\ \sigma_{yy} = \sigma_{zz} &= \sigma_{xy} = \sigma_{yz} = 0. \end{aligned} \tag{3.31}$$

Thus the strain energy of the beam can be determined as

$$\begin{aligned} \pi = \frac{1}{2}\iiint\limits_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx})dV = \\ = \frac{1}{2}\int_0^l \iint\limits_A \left[Ez^2(\frac{\partial \phi}{\partial x})^2 + kG(\frac{\partial \phi}{\partial x} - \phi)^2\right] dAdx = \\ = \frac{1}{2}\int_0^l \left[EI\left(\frac{\partial \phi}{\partial x}\right)^2 + kAG\left(\frac{\partial \phi}{\partial x} - \phi\right)^2\right] dx \end{aligned} \tag{3.32}$$

The kinetic energy of the beam, including rotary inertia, is given by

$$T = \frac{1}{2}\int_0^l \left[\rho A\left(\frac{\partial w}{\partial t}\right)^2 + \rho I\left(\frac{\partial \phi}{\partial t}\right)^2\right] dx. \tag{3.33}$$

The work done by the external distributed load $f(x,t)$ is given by

$$W = \int_0^l f(x,t)w(x,t)dx. \tag{3.34}$$

Application of the extended Hamilton's principle gives

$$\delta \int_{t_1}^{t_2} (\pi - T - W)dt = 0.$$

when substituting the strain- and kinetic energy and the work done, will result in

$$\int_{t_1}^{t_2} \left\{ \int_0^l \left[EI\frac{\partial \phi}{\partial x}\delta\left(\frac{\partial \phi}{\partial x}\right) + kAG\left(\frac{\partial \phi}{\partial x} - \phi\right)\delta\left(\frac{\partial w}{\partial x}\right) - kAG\left(\frac{\partial w}{\partial x} - \phi\right)\delta\phi\right] dx \right.$$
$$\left. - \int_0^l \left[\rho A\frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right) + \rho I\frac{\partial \phi}{\partial t}\delta\left(\frac{\partial \phi}{\partial t}\right)\right] dx - \int_0^l f\delta w dx \right\} dt = 0 \tag{3.35}$$

The integrals in (3.35) can be evaluated using integration by parts (with respect to x or t). This work is quite cumbersome and will be left for the Appendix. In the end the following differential equations of motions are derived for w and ϕ:

$$\begin{aligned} -\frac{\partial}{\partial x}\left[kAG\left(\frac{\partial w}{\partial x} - \phi\right)\right] + \rho A\frac{\partial^2 w}{\partial t^2} &= f(x,t) \\ -\frac{\partial}{\partial x}\left(EI\frac{\partial \phi}{\partial x}\right) - kAG\left(\frac{\partial w}{\partial x} - \phi\right) + \rho I\frac{\partial^2 \phi}{\partial t^2} &= 0 \end{aligned} \tag{3.36}$$

With these boundary conditions

$$kAG\left(\frac{\partial w}{\partial x} - \phi\right)\delta w\Big|_0^l = 0, \tag{3.37}$$

$$EI\ \frac{\partial \phi}{\partial x}\delta\phi\Big|_0^l = 0. \tag{3.38}$$

When assuming that the beam is uniform, then the set of formulas (3.36) become:

$$\frac{\partial \phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{\rho}{kG}\frac{\partial^2 w}{\partial t^2} + \frac{f}{kAG}, \tag{3.39}$$

$$-EI\frac{\partial^2 \phi}{\partial x^2} - kAG\frac{\partial w}{\partial x} + kAG\phi + \rho I\frac{\partial^2 \phi}{\partial t^2} = 0. \tag{3.40}$$

Modifying these equations results in

$$\rho A\frac{\partial^2 w}{\partial t^2} = kAG(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x}) + f \tag{3.41}$$

$$\rho I\frac{\partial^2 \phi}{\partial t^2} = EI\frac{\partial^2 \phi}{\partial x^2} + kAG(\frac{\partial w}{\partial x} - \phi) \tag{3.42}$$

From this form it is easy to see what these equations mean. The first one describes the forces that act on the beam, the second one makes clear what affects the angle of the beam.

Although this form is quite convenient, it is still a coupled system which could be simplified. This can be done by differentiating (3.40) with respect to x. Then a term $\left(\frac{\partial \phi}{\partial x}\right)$ appears and equation (3.39) can be used.

$$-EI\frac{\partial^2}{\partial x^2}\left(\frac{\partial \phi}{\partial x}\right) - kAG\frac{\partial w}{\partial x} + kAG\phi + \rho I\frac{\partial^2}{\partial t^2}\left(\frac{\partial \phi}{\partial x}\right) = 0. \tag{3.43}$$

Which leads to

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{kG}\right)\frac{\partial^4 w}{\partial t^2 \partial x^2}+\frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4}+\frac{EI}{kAG}\frac{\partial^2 f}{\partial x^2}-\frac{\rho I}{kAG}\frac{\partial^2 f}{\partial t^2}-f=0 \tag{3.44}$$

When analysis free vibrations then $f(x,t) = 0$, thus (3.44) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1+\frac{E}{kG}\right)\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{3.45}$$

The terms in this formula are not that trivial at first sight. Therefore all terms will be discussed:

- $EI\frac{\partial^4 w}{\partial x^4} + \rho A w_{tt}$,
  - this term is also present in the Euler-Bernoulli theory;
- $-\rho I\frac{\partial^2 w}{\partial t^2}$,
  - this term denotes the effect of rotary inertia, this comes from the Rayleigh theory;
- $-\frac{E}{kG}\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4}$
  - The last two terms with the factor involving kG in the denominator, represent the influence of shear deformation.
  - The last term involves the fourth order derivative to the time. Rotary inertia is eliminated by setting terms containing ρI equal to zero (but not $EI\rho$). Shear flexibility is eliminated by letting $G \to \infty$. Thus, this last term is a coupling term which exists only if *both* effects are present.[5, p. 154]

4 Separation of variables

4.1 Timoshenko beam theory

In this section the general solution of the Timoshenko beam theory will be derived:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} = 0. \tag{4.1}$$

The method of separation will be used by setting $w(x,t) = X(x)T(t)$.

The method of separation can also be used with the equations (3.39) and (3.40), where the external force $f \equiv 0$.
By defining $w(x,t) = X(x)T(t)$ and $\phi(x,t) = Y(x)T(t)$ the following equations are obtained

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}(x) + a_1 X(x) - \frac{\mathrm{d}Y}{\mathrm{d}x}(x) = 0, \tag{4.2}$$

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2}(x) + a_2 Y(x) + a_3\frac{\mathrm{d}X}{\mathrm{d}x}(x) = 0, \tag{4.3}$$

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}(t) + \lambda T(t) = 0, \tag{4.4}$$

$$a_1 = \frac{\lambda\rho}{kG}, \quad a_2 = \frac{\rho\lambda}{E} - a_3, \quad a_3 = \frac{kAG}{EI}.$$

Here the external force $f(x,t)$, that is present in (3.39) and (3.40), is already set to zero to simplify the equations. The factor λ is the separation factor, which will be larger than zero. This is done so that the time factor $T(t)$ in the solution $w(x,t)$ will be an oscilliating function.
By displaying the Timoshenko theory in this way, the spatial factor $Y(x)$ of the bending function $\phi(x,t)$ can be written as a function of $X(x)$

$$Y(x) = -\frac{1}{a_3}\left[\frac{\mathrm{d}^3 X}{\mathrm{d}x^3}(x) + (a_1 + a_3)\frac{\mathrm{d}X}{\mathrm{d}x}(x)\right]. \tag{4.5}$$

This is very convenient when dealing with a damped boundary condition later.

For now equation (4.1) will be used. Substituting $w(x,t) = X(x)T(t)$ in this equation gives

$$EI\frac{\mathrm{d}^4 X}{\mathrm{d}x^4}T + \rho A\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}X - \rho I\left(1 + \frac{E}{kG}\right)\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{\rho^2 I}{kG}\frac{\mathrm{d}^4 T}{\mathrm{d}t^4}X = 0. \tag{4.6}$$

Dividing by $X(x)T(t)$ will simplify the formula, resulting in

$$\frac{EI\frac{\mathrm{d}^4 X}{\mathrm{d}x^4}}{X} + \frac{\rho A\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}}{T} - \frac{\rho I\left(1 + \frac{E}{kG}\right)\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}}{TX} + \frac{\frac{\rho^2 I}{kG}\frac{\mathrm{d}^4 T}{\mathrm{d}t^4}}{T} = 0. \tag{4.7}$$

The mixed derivative term is causing trouble. By differentiating the equation with respect to x, the terms depended on time can be separated.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{EI\frac{\mathrm{d}^4X}{\mathrm{d}x^4}}{X}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\rho A\frac{\mathrm{d}^2T}{\mathrm{d}t^2}}{T}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\rho I\left(1+\frac{E}{kG}\right)\frac{\mathrm{d}^2T}{\mathrm{d}t^2}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}}{TX}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\frac{\rho^2I}{kG}\frac{\mathrm{d}^4T}{\mathrm{d}t^4}}{T}\right)=0, \tag{4.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{EI\frac{\mathrm{d}^4X}{\mathrm{d}x^4}}{X}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\rho I\left(1+\frac{E}{kG}\right)\frac{\mathrm{d}^2T}{\mathrm{d}t^2}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}}{TX}\right)=0, \tag{4.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{EI\frac{\mathrm{d}^4X}{\mathrm{d}x^4}}{X}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\rho I\left(1+\frac{E}{kG}\right)\frac{\mathrm{d}^2X}{\mathrm{d}x^2}}{X}\right)\frac{\frac{\mathrm{d}^2T}{\mathrm{d}t^2}}{T}=0, \tag{4.10}$$

$$\frac{\frac{\mathrm{d}^2T}{\mathrm{d}t^2}}{T}=\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{EI\frac{\mathrm{d}^4X}{\mathrm{d}x^4}}{X}\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\rho I\left(1+\frac{E}{kG}\right)\frac{\mathrm{d}^2X}{\mathrm{d}x^2}}{X}\right)}=-\lambda, \tag{4.11}$$

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2}=-\lambda T(t). \tag{4.12}$$

This result can be used to remove the mixed partial derivative term. Note that $\frac{\mathrm{d}^4T}{\mathrm{d}t^4}=\lambda^2T(t)$, which simplifies the separated equation to

$$\frac{EI\frac{\mathrm{d}^4X}{\mathrm{d}x^4}}{X}-\lambda\rho A+\lambda\frac{\rho I\left(1+\frac{E}{kG}\right)\frac{\mathrm{d}^2X}{\mathrm{d}x^2}}{X}+\lambda^2\frac{\rho^2I}{kG}=0. \tag{4.13}$$

Multiply by $X(x)$, divide by EI, rearranging terms will result in

$$\frac{\mathrm{d}^4X}{\mathrm{d}x^4}+\frac{\lambda\rho I}{EI}\left(1+\frac{E}{kG}\right)\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+\frac{\lambda}{EI}\left(\lambda\frac{\rho^2I}{kG}-\rho A\right)X=0. \tag{4.14}$$

4.2 Eigenvalue analysis

Assuming that $X(x) = \sum_{i=1}^{4} c_i e^{r_i x}$, this equation will transform to a fourth order polynomal which we want to solve for r. Since all the powers of r are even, the substitution $r^2 = s$ is made. That way the quadratic formula can be applied to

$$s^2 + bs + c = 0 \Rightarrow s_1 = \frac{-b+\sqrt{\Delta}}{2}, \; s_2 = \frac{-b-\sqrt{\Delta}}{2}. \tag{4.15}$$

Where

$$b = \frac{\lambda \rho I \left(1 + \frac{E}{kG}\right)}{EI},$$

$$c = \frac{\lambda}{EI}\left(\lambda \frac{\rho^2 I}{kG} - \rho A\right).$$

and $\Delta = b^2 - 4c$.

Before proceeding further, remark that the term Δ can be simplified.

$$\Delta = b^2 - 4c = \frac{\lambda^2 \rho^2 I^2 \left(1 + \frac{E}{kG}\right)^2}{(EI)^2} - 4\frac{\lambda}{EI}\left(\lambda \frac{\rho^2 I}{kG} - \rho A\right) =$$

$$= \lambda^2 \rho I \left(1 - \frac{E}{kG}\right)^2 + 4\lambda EA. \tag{4.16}$$

Define $r_1 = \sqrt{s_1}, r_2 = -\sqrt{s_1}, r_3 = \sqrt{s_2}$ and $r_4 = -\sqrt{s_2}$.

4.2.1 Real-valued eigenvalues

It is usefull to know how the function $X(x)$ is affected by the value of λ. First, assume that $\lambda \in \mathbb{R}$ and is larger than zero.

It is clear that $\Delta > 0$ and $b > 0$ for all values of λ. The root s_1 is positive if

$$\sqrt{\Delta} > b \Rightarrow b^2 - 4c > b^2 \Rightarrow -4c > 0.$$

In order to have $0 > c$, λ must satisfy $\frac{kGA}{\rho I} > \lambda$. It is obvious that the root s_1 will be negative if $\lambda > \frac{kGA}{\rho I}$.

By definition of s_2, this root will always be negative if $\lambda > 0$ and realvalued. Therefore, the roots r_3 and r_4 will be complex-valued which implies that these can be written as a combination of cos and sin.

Thus when $\frac{kGA}{\rho I} > \lambda$, then r_1, r_2 are positive, the function $X(x)$ is

$$X(x) = C_1 e^{r_1 x} + C_2 e^{-r_1 x} + C_3 \cos(r_3 x) + C_4 \sin(r_3 x). \tag{4.17}$$

When $\lambda > \frac{kGA}{\rho I}$ then all the roots r_i are complex, thus

$$X(x) = C_1 \cos(r_1 x) + C_2 \sin(r_1 x) + C_3 \cos(r_3 x) + C_4 \sin(r_3 x). \tag{4.18}$$

When $\lambda = 0$ then the differential equation $\frac{\mathrm{d}^3 X}{\mathrm{d}x^3} = 0$ has to be solved. The solution for this is

$$X(x) = \frac{1}{6} c_1 x^3 + \frac{1}{2} c_2 x^2 + c_3 x + c_4. \tag{4.19}$$

4.2.2 Complex-valued eigenvalues

When the eigenvalues are complex, then the square root over an complex number has to be calculated. This is defined as

$$\sqrt{i} = \frac{\sqrt{2}}{2}(1 + i).$$

Thus the result of the square root of a complex number, is still a complex number.
In the previous situation, all roots r_i were also complex-valued which gives us, in most cases, the same version of $X(x)$. Since it could happen that one complex value of λ cancels out all the complex part.

First assume that this situation is possible. Defining $\lambda = \alpha + \beta i$ and setting $\alpha = -\dfrac{2EA}{\rho I \left(1 - \frac{E}{kG}\right)^2}$, then the square root $\sqrt{\Delta}$ will have no complex part.

Since Δ has no complex part left, it has to be negative, such that it becomes pure imaginairy. This is because the other term in the root s_i, b, will always have a complex part which needs to be cancelled out by the complex part of $\sqrt{\Delta}$.

With the assumption on α and for β the assumption $\dfrac{-4E^2 A^2 k^4 G^4}{(\rho^2 I^2 \beta^2 (-kG + E)^4)} > \beta$, the term $\sqrt{\Delta}$ is now pure imaginairy and has to cancel out with the term $\mathfrak{Im}(b) = \dfrac{\beta \rho I (1 + \frac{E}{kG})}{EI}$.

But remark that the assumption on β makes it less than zero. Which is not desired, since $\lambda > 0$. Thus there is no complex value of λ such that s_1 or s_2 becomes positive real-valued.

Thus if λ is complex then the general solution will always be of the form

$$X(x) = D_1 \cos(r_1 x) + D_2 \sin(r_1 x) + D_3 \cos(r_3 x) + D_4 \sin(r_3 x). \tag{4.20}$$

5 Damped boundary condition

The type of boundary affects the shape of the solution. Here are some boundary conditions that can occur

- clamped/fixed end(s)
- pinned/hinged end(s)
- free end(s)
- damped end(s)

under all condition different mathematical restrictions are applied. In this section the restrictions for a damped cantileverd beam will be considered. For completeness there is a mass attached to both ends [7]. The damped boundary is located at $x = l$, the boundary conditions for that point are given by

$$m\left(\frac{\partial^2 w}{\partial t^2}(l,t)\right)^2 = -F(l,t) - \alpha_1 \frac{\partial w}{\partial t}(l,t), \tag{5.1}$$

$$I_m\left(\frac{\partial^2 \phi}{\partial t^2}(l,t)\right)^2 = -M(l,t) - \beta_1 \frac{\partial \phi}{\partial t}(l,t). \tag{5.2}$$

Here m and I_m represent the mass and area moment of inertia of m respectively. The factors α_1 and β_1 respresent the damping constants, which are nonnegative.

The shear force is defined as $F(x,t) = kAG(\frac{\partial w}{\partial x}(x,t) - \phi(x,t))$, the moment is $M(x,t) = EI\frac{\partial \phi}{\partial x}$, where k is the shear coefficient.

The boundary at the point $x = 0$, where the beam is clamped, is given by

$$w(x,t) = 0, \tag{5.3}$$

$$\frac{\partial w}{\partial x}(x,t) = 0. \tag{5.4}$$

In order to use these boundary conditions to solve the system, separation of variables must be used on these conditions. Therefore, set $w(x,t) = X(x)T(t)$ and $\phi(x,t) = Y(x)T(t)$, then the boundary conditions result in

$$m\left(X(l)\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}(t)\right)^2 = -kAG\left(\frac{\mathrm{d}X}{\mathrm{d}x}(l)T(t) - Y(l)T(t)\right) - \alpha_1 X(l)\frac{\mathrm{d}T}{\mathrm{d}t}(t), \tag{5.5}$$

$$I_m\left(Y(l)\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}\right)^2 = -EI\frac{\mathrm{d}Y}{\mathrm{d}x}(l)T(t) - \beta_1 Y(l)\frac{\mathrm{d}T}{\mathrm{d}t}(t), \tag{5.6}$$

$$X(0) = 0, \tag{5.7}$$

$$\frac{\mathrm{d}X}{\mathrm{d}x}(0) = 0. \tag{5.8}$$

As defined earlier, $Y(x) = -\frac{1}{a_2}\left[\frac{\mathrm{d}^3X}{\mathrm{d}x^3} + (a_1 + a_3)\frac{\mathrm{d}X}{\mathrm{d}x}\right]$. Using this in the boundary conditions will make sure that everything is in terms of $X(x)$.

Letting the mass m tend to zero will simplify the boundary conditions, but the difficulty lies in the fact that there is a dependency on $T(t)$. When the terms are reorganized and a new separation constant C is introduced. Eventually, the following boundary conditions apply

$$\frac{kAG}{\alpha_1}\left(\frac{\mathrm{d}X}{\mathrm{d}x}(l) - Y(l)\right) + CX(l) = 0, \tag{5.9}$$

$$EI\frac{\mathrm{d}Y}{\mathrm{d}x}(l) + C\beta_1 Y(l) = 0, \tag{5.10}$$

$$\frac{\mathrm{d}X}{\mathrm{d}x}(0) = 0, \tag{5.11}$$

$$X(0) = 0. \tag{5.12}$$

where $T(t)$ has to satisfy

$$\frac{\mathrm{d}T}{\mathrm{d}t}(t) = CT(t).$$

Remark that $\frac{\mathrm{d}T}{\mathrm{d}t} = CT \Rightarrow \frac{\mathrm{d}^2T}{\mathrm{d}t^2} = C\frac{\mathrm{d}T}{\mathrm{d}t} = C^2T$ and that $\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = -\lambda T$. Thus $C^2 = -\lambda$. Which results in $C \in \mathbb{C}$

The way in which the boundary conditions will be implemented are as followed

$$kAG\left(\frac{1}{a_2}\frac{\mathrm{d}^3X}{\mathrm{d}x^3}(l) + (1 + \frac{a_1 + a_3}{a_2})\frac{\mathrm{d}X}{\mathrm{d}x}(l)\right) + \alpha_1\sqrt{-\lambda}X(l) = 0, \tag{5.13}$$

$$-\sqrt{-\lambda}\frac{\beta_1}{a_2}\frac{\mathrm{d}^3X}{\mathrm{d}x^3}(l) + EI\frac{\mathrm{d}^2X}{\mathrm{d}x^2}(l) - \sqrt{-\lambda}\beta_1\frac{a_1 + a_3}{a_2}\frac{\mathrm{d}X}{\mathrm{d}x}(l) + a_1EIX(l) = 0, \tag{5.14}$$

$$\frac{\mathrm{d}X}{\mathrm{d}x}(0) = 0, \tag{5.15}$$

$$X(0) = 0. \tag{5.16}$$

6 Specific beam

In this section the unknown material constants will be given the following value

1. $E = 207 \cdot 10^9 \ Pa$
2. $I = \frac{1}{12}(0,05) \cdot (0,15)^3 = 14,063 \cdot 10^{-6} \ m^4$
3. $G = 79,3 \cdot 10^9 \ Pa$
4. $\rho = 76,5 \cdot 10^3 \ N/m^3$
5. $k = \frac{5}{6}$
6. $A = 0,05 \cdot 0,15 = 0,0075 \ m^2$
7. $L = 1 \ m$

These values were found in [1, p. 379]. With these values the Timoshenko beam theory can be explicitly analysed, especially the eigenvalues.
The Timoshenko beam will be analysed in two different situations. One with a clamped and a free end and one with, in addition, a damped boundary. With the given values, the Timoshenko beam equation becomes

$$2,911041 \cdot 10^6 \frac{\partial^4 w}{\partial x^4} + 573,75 \frac{\partial^2 w}{\partial t^2} - 4,445725727 \frac{\partial^4 w}{\partial^2 x \partial^2 t} + 1,245400127 \cdot 10^{-6} \frac{\partial^4 w}{\partial t^4} = 0 \tag{6.1}$$

After seperation of variables, substituting $X(x) = e^{rx}$ and $r^2 = s$ into those equations will result in

$$s^2 + 1,527194473 \cdot 10^{-6} \lambda s + 4,278195076 \cdot 10^{-13} \lambda^2 - 1,970944414 \cdot 10^{-4} \lambda = 0. \tag{6.2}$$

In which s can be solved and thus the general solution of $X(x)$ can be determined.

$$X(x) = e^{r_1 x} + e^{r_2 x} + e^{r_3 x} + e^{r_4 x} \tag{6.3}$$

where

$$r_1(\lambda) = 1 \cdot 10^{-13} \sqrt{-7.635972 \cdot 10^{19} \lambda + 10 \cdot \sqrt{1.552612 \cdot 10^{37} \lambda^2 + 1.970944 \cdot 10^{46} \lambda}},$$

$$r_2(\lambda) = -r_1(\lambda),$$

$$r_3(\lambda) = 1 \cdot 10^{-13} \sqrt{-7.635972 \cdot 10^{19} \lambda - 10 \cdot \sqrt{1.552612 \cdot 10^{37} \lambda^2 + 1.970944 \cdot 10^{46} \lambda}},$$

$$r_4(\lambda) = -r_3(\lambda).$$

The relation between r_1, r_2 and r_3, r_4 is quite obvious. But could there also be a relation between r_1 and r_3? The reason to investigate this is to simplify the matrix that is obtained after introducing the boundary conditions.

Since the eigenvalues are assumed to be positive, the following graphs shows the result of r_1 and r_3 from $\lambda \in [1; 5 \cdot 10^8]$

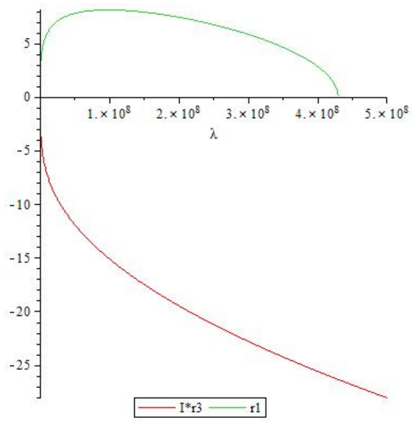

Figure 5: values of r_1, r_3 with positive real lambda's

Remark here that r_3 is fully imaginairy.

The following situation that can occur is that λ is fully imaginairy. In that case the next figure shows the relation between r_1 and r_3

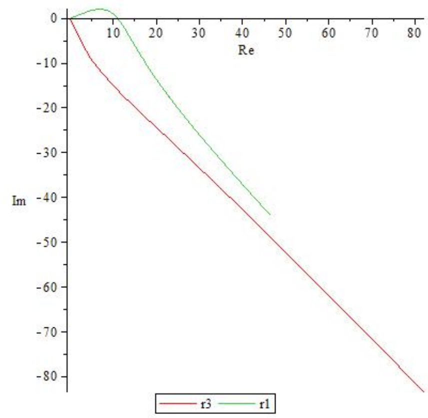

Figure 6: values of r_1, r_3 with pure positive complex lambda's

Notice how r_1 and r_3 converge to eachother. But this does not mean that there is a value λ_0 such that $r_1(\lambda_0) = r_3(\lambda_0)$. They do obtain the same value at some point, but not on the same time.

In conclussion, for real/complex positive eigenvalues there is no connection between the two roots r_1 and r_3.

6.1 Boundary - Cantilevered

Before the damping is attached to the beam, the beam will have one end free and one clamped. Investigating this situation will be usefull when the damped beam is analysed. If the damping coefficient is neglected with the damped beam, then the solution must be exactly the same as with the clamped/free-end.

The boundary conditions for the clamped/free-end beam are given by [9]

$$\begin{array}{ll} w(0,t)=0, & \frac{\partial \phi}{\partial x}(l,t)=0, \\ \phi(0,t)=0, & \frac{1}{L}\frac{\partial w}{\partial x}(l,t)-\phi(l,t)=0. \end{array}$$

Set $w(x,t)=X(x)T(t), \phi(x,t)=Y(x)T(t)$ and substituting the function (4.5), $X(x)$ must satisfy

$$\begin{array}{ll} X(0)=0, & \frac{\mathrm{d}^2X}{\mathrm{d}x^2}(l)+a_1X(l)=0, \\ \frac{\mathrm{d}^3X}{\mathrm{d}x^3}(0)+(a_1+a_3)\frac{\mathrm{d}X}{\mathrm{d}x}(0)=0, & \frac{\mathrm{d}^3X}{\mathrm{d}x^3}(l)+(a_1+a_2+a_3)\frac{\mathrm{d}X}{\mathrm{d}x}(l)=0. \end{array}$$

Substituting the function $X(x)$ in the boundary conditions will result in a system with unknowns c_1, c_2, c_3, c_4 and λ. Since this is a linear problem, this will be written in matrix form

$$\begin{bmatrix} 1 & 1 & 1 & m_1 \\ r_1^3+r_1a_1+r_1a_3 & -r_1a_1-r_1a_3-r_1^3 & r_3^3+r_3a_1+r_3a_3 & m_2 \\ r_1e^{r_1}(a_1+a_2+a_3+r_1^2) & -r_1e^{-r_1}(a_1+a_2+a_3+r1^2) & r_3e^{r_3}(r_3^2+a_1+a_2+a_3) & m_3 \\ e^{r_1}(r_1^2+a_1) & e^{-r_1}(r_1^2+a_1) & e^{r_3}(r_3^2+a_1) & m_4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.4}$$

$$m_1=1 \quad m_2=-r_3a_1-r_3a_3-r_3^3$$

$$m_3=-r_3e^{-r_3}(r_3^2+a_1+a_2+a_3) \quad m_4=e^{-r_3}(r_3^2+a_1)$$

Where r_1 and r_3 are the roots of equation (6.2). The constants a_1, a_2 and a_3 are determined by the properties of the material, the values of these are in this case given by

$$a_1=1.164444\cdot 10^{-7}\lambda \quad a_2=3.742857\cdot 10^{-8}\lambda-602.678571$$

$$a_3=602.678571 \quad a_1+a_2+a_3=1.538730\cdot 10^{-7}\lambda$$

In order to obtain a system that is solvable, the determinant of this matrix must be zero. In order to do this, there are several options

1. Construct this matrix in Maple, then calculate the whole (symbolic) determinant,
2. Construct this matrix in Matlab, then try different values of λ to calculate the determinant,
3. Choose a value for λ such that one whole row, or column, will be zero,
4. Simplify the determinant by organising all terms and factors and try to find easier solutions,

In all cases, values for λ will be positive and real-valued. This is due the boundary conditions, since there is no damping.
The first option gives the following result

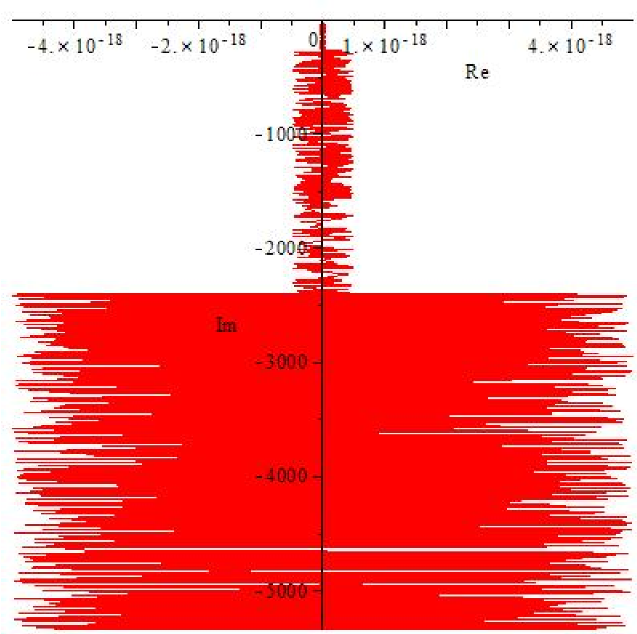

Figure 7: Complex plot of the determinant of the matrix

this is obtained by letting Maple do all the work. The figure shows that there is only one solution for the determinant, that is $\lambda = 0$.
The values for lambda range from 1 to 10^4 in this case. Notice, however, that the difference between the real and complex values are very large.
To check wether Maple was working properly, the program Matlab was also used

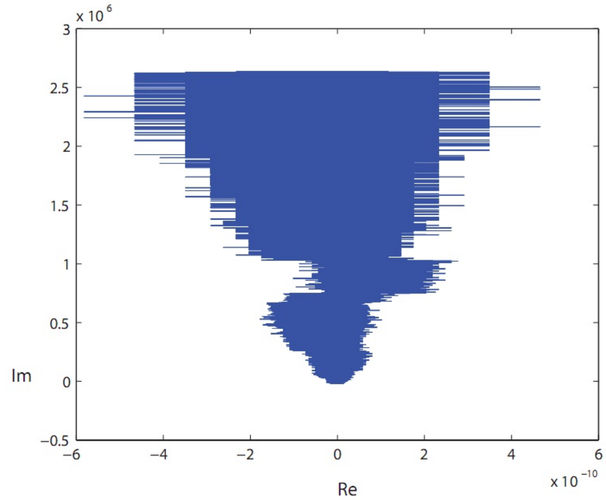

Figure 8: Complex plot of the determinant of the matrix

Again, the difference between the real and complex values are almost the same as with the program Maple.
But still there is no non-trivial root found with these methods.

The next method is to find a way to obtain all zero entries in one matrix row or column. By inspection it is obvious that it is not possible to obtain a zero column. Since there is always a 1 on the first row.

But could it be possible to obtain one row with zeros? One of the following system of equations needs to be solved

$$r_1 = 0 \quad \text{and} \quad r_3 = 0 \tag{6.5}$$

$$r_1^2 + a_1 = 0 \quad \text{and} \quad r_3^2 + a_1 = 0, \text{or} \tag{6.6}$$

$$a_1 + a_2 + a_3 + r_1^2 = 0 \quad \text{and} \quad a_1 + a_2 + a_3 + r_3^2 = 0, \text{or} \tag{6.7}$$

$$r_1^2 + a_1 + a_3 = 0 \quad \text{and} \quad r_3^2 + a_1 + a_3 = 0. \tag{6.8}$$

The first option is possible, but then a very trivial solution has been obtained. The other three options do not have a solution either. Because $r_1 = h_1 + \sqrt{h_2}$ and $r_3 = h_1 - \sqrt{h_2}$. When analysing the second option, this means that the following equation has to be satisfied

$$2h_1 + 2a_1 + \sqrt{h_2} - \sqrt{h_2} = 0 \Rightarrow h_1 + a_1 = 0.$$

Because both h_1 and a_1 only have one factor λ, the only solution that is availible is $\lambda = 0$. This situation also applies to the last two options.

The last method to try and solve this determinant is to organise the whole determinant in a simpler form. This has been tried, but failed aswell. Eventually there was a simpler form for the determinant, but this did not help to find a solution for lambda.

6.1.1 Boundary - Cantileverd and Damped

In this section the damping is attached to the free end of the boundary. Thus the following system needs to be solved

$$X(x) = D_1 \cos(r_1 x) + D_2 \sin(r_1 x) + D_3 \cos(r_3 x) + D_4 \sin(r_3 x). \tag{6.9}$$

Subject to

$$kAG\left(\frac{1}{a_2}\frac{\mathrm{d}^3 X}{\mathrm{d}x^3}(l) + (1 + \frac{a_1 + a_3}{a_2})\frac{\mathrm{d}X}{\mathrm{d}x}(l)\right) + \alpha_1\sqrt{-\lambda}X(l) = 0 \tag{6.10}$$

$$-\sqrt{-\lambda}\frac{\beta_1}{a_2}\frac{\mathrm{d}^3 X}{\mathrm{d}x^3}(l) + EI\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}(l) - \sqrt{-\lambda}\beta_1\frac{a_1 + a_3}{a_2}\frac{\mathrm{d}X}{\mathrm{d}x}(l) + a_1 EIX(l) = 0 \tag{6.11}$$

$$\frac{\mathrm{d}X}{\mathrm{d}x}(0) = 0 \tag{6.12}$$

$$X(0) = 0 \tag{6.13}$$

The mathematical details of the damped boundary is already treated in a previous section. Writing the boundary conditions in matrix form, a matrix dependend on λ is obtained. Just like before, the determinant of this system needs to be zero.

Again, the same problem arrises... Which totally blocked the way this report was going..

7 Conclussion

After several months of working on this report, the obtained result is not what I expected. I can come up with much reasons why I could not obtain the result I wanted. But in the end I think I am just not good enough to reach my goals.

What I wanted is to learn how the beam theories were made up. Followed by choosing the right boundary conditions. The next stop was to solve this determinant, such that the frequencies can be determined and the PDE can be solved.

The next step for me was to apply the damped boundary condition and after that an external force. With this system I wanted to have some function for the external force that should represent the weather conditions rain and wind.

With that I could variate the damping in such a way that the beam would not begin to resonate. The whole time the beam would be un-tensioned. In the end the term for a tensioned beam would be added.

But this whole idea stopped when I was not able to solve the determinant. Until today I still have no idea how to fix this, while others seem to have figured it out, I just could not find out how.

I really am ashamed that I do not have the results I wanted. But apparently research does not always the way you want.
One last thing is that I have truely learned much during this project, but I still need to learn a lot more.

8 Appendix

8.1 Real valued constants

To confirm that these values are realistic in the section with the special beam. Looking around on the internet the following values of some materials were found

	Density(kg/m^3)	Shear modulus(Pa)	Elasticity modulus(Pa)
aluminium	2700	$25,5$	69
copper	8940	$44,7$	$103-124$
steel	$7750-8050$	$79,3$	210

8.2 Shear stress correction factor k

In the derivation of Timoshenko's theorem there has been assumed that the shear stress σ_{zx} is not the same at every point in the cross section of the beam. For this a constant k is used, such that: $\sigma_{zx} = kG\frac{\partial w}{\partial x}$.

This constant k is called the shear stress correction factor. In recent decades, considerable study of this coefficient has been made. A summary of some of the various methods developed for selecting k may be found in the article by Cowper [6]. He also developed a procedure for determining values of k for low frequency vibrations which are quite consistent with the static, 3-dimensional theory of elasticity. Some of his results are summarized in the following table

	shape	k
Rectangle	plaatje	$\frac{10(1+\nu)}{12+11\nu}$
Circle	plaatje	$\frac{6(1+\nu)}{7+6\nu}$
Hollow circle	plaatje	$\frac{6(1+\nu)(1+m^2)^2}{(7+6\nu)(1+m^2)^2+(20+12\nu)m^2}$
Ellipse	plaatje	$\frac{12(1+\nu)n^2(3n^2+1)^2}{(40+37\nu)n^4+(16+10\nu)n^2+\nu}$
Semicircle	plaatje	$\frac{(1+\nu)}{1.305+1.273\nu}$
Thin-walled circular tube	plaatje	$\frac{2(1+\nu)}{4+3\nu}$
Thin-walled square tube	plaatje	$\frac{20(1+\nu)}{48+39\nu}$

Table 1: Shear stress correction factors k according to Cowper[6]

where $m = b\backslash a$, $n = a/b$ and ν is Poisson's ratio.

8.3 Potential Energy

The potential energy of an elastic body (U) is defined as

$$U = \pi - W_P, \tag{8.1}$$

where π is the strain energy and W_P is the work done on the body by the external forces ($-W_P$ is also called the potential energy of the applied loads). If the potential energy is expressed in terms of the displacement components u, v, and w, the principle of minimum potential energy gives, at the equilibrium state,

$$\delta U(u, v, w) = \delta\pi(u, v, w) - \delta W_P(u, v, w) = 0. \tag{8.2}$$

Where the variation is to be taken with respect to the displacement in (8.2), while the forces and stresses are assumed constant. The *strain* energy of a linear elastic body is given by:

$$\pi = \frac{1}{2} \iiint\limits_V \vec{\varepsilon}^T \vec{\sigma} dV \tag{8.3}$$

where $\vec{\varepsilon}^T$ is the transposed strain vector, $\vec{\sigma}$ the stress vector and V is the volume of the body. By using the stress-strain relations

$$\vec{\sigma} = [D]\vec{\varepsilon} \tag{8.4}$$

where $[D]$ is the elasticity matrix, thus the above equation 8.3 can be expressed as

$$\pi = \frac{1}{2} \iiint\limits_V \vec{\varepsilon}^T [D]\vec{\varepsilon} dV \tag{8.5}$$

If there were some initial strains in the problem, equation 8.5 is substracted with the term $\frac{1}{2}\int_V \iint \vec{\varepsilon}^T [D]\vec{\varepsilon}_0 dV$.

The work done by the external forces can be expressed as

$$W_P = \iiint\limits_V \vec{\bar{\phi}}^T \vec{u} dV + \iiint\limits_{S_2} \vec{\bar{\Phi}}^T \vec{u} dS_2 \tag{8.6}$$

where the vectors are defined as followed:

$$\vec{\bar{\phi}} = \begin{Bmatrix} \bar{\phi}_x \\ \bar{\phi}_y \\ \bar{\phi}_z \end{Bmatrix} \quad \vec{\bar{\Phi}} = \begin{Bmatrix} \bar{\Phi}_x \\ \bar{\Phi}_y \\ \bar{\Phi}_z \end{Bmatrix} \quad \vec{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

Using 8.5 and 8.6 the potential energy can be expressed as

$$U(u, v, w) = \frac{1}{2} \iiint\limits_V \vec{\varepsilon}^T [D](\vec{\varepsilon} - 2\vec{\varepsilon}_0) dV - \iiint\limits_V \vec{\bar{\phi}}^T \vec{u} dV - \iiint\limits_{S_2} \vec{\bar{\Phi}}^T \vec{u} dS_2 \tag{8.7}$$

As you can see, the initial strain $\vec{\epsilon_0}$ is implemented. Thus, according to the principle of minimum potential energy, the displacement field $\vec{u}(x, y, z)$ that minimizes U and satisfies all the boundary conditions is the one that satisfies the equilibrium equations. In the principle of minimum potential energy, we minimize the functional U, and the resulting equations denote the equilibrium equations while compatibility conditions are satisfied identically.

8.4 Hamilton's Principle

For an elastic body *in motion*, the equation of dynamic equilibrium for an element of the body can be written, using Cartesian tensor notation, as:

$$\sigma_{ij,j} + \phi_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad i = 1, 2, 3 \tag{8.8}$$

where ρ is the density of the material, ϕ_i is the body force per unit volume acting along the x_i direction, u_i is the component of displacement along the x_i direction, the $\sigma_{ij,j}$ denotes the stress tensor:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \tag{8.9}$$

and

$$\sigma_{ij,j} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} \tag{8.10}$$

with $x_1 = x, x_2 = y, x_3 = z$ and $u_1 = u, u_2 = v, u_3 = w$.
The solid body is assumed to have a volume V with a bounding surface S. This surface is assumed to be composed of two parts S_1, S_2. Where the displacements u_i are prescribed on S_1 and surface forces (tractions) are prescribed on S_2.

Now consider a set of virtual displacements δu_i of the vibrating body which vanishes over the boundary surface S_1, where values of displacements are prescribed, but are arbitrary over the rest of the boundary surface S_2, where surface tractions are prescribed. The virtual work done by the body and surface forces is given by

$$\iiint_V \phi_i \delta u_i dV + \iint_S \Phi_i \delta u_i dS \tag{8.11}$$

where Φ_i indicates the prescribed surface force along the direction u_i. Although the surface integral is expressed over S in equation (8.11), it needs to be integrated only over S_2, since δu_i vanishes over the surfce S_1, where the boundary displacements are prescribed. The surface forces Φ_i can be represented as

$$\Phi_i = \sigma_{ij}\nu_j \equiv \sum_{j=1}^{3} \sigma_{ij}\nu_j \quad i = 1, 2, 3 \tag{8.12}$$

where $\vec{\nu} = [\nu_1 \nu_2 \nu_3]^T$ is the unit vector along the outward normal of the surface S with ν_1, ν_2 and ν_3 as its components along the $x, y,$ and z directions, respectively.

By substituting (8.12), the second term of (8.11) can be written as

$$\iint_S \sigma_{ij} \delta u_i \nu_j dS \tag{8.13}$$

Using Gauss's theorem, expression (8.13) can be rewritten in terms of the volume integral as

$$\iint\limits_S \sigma_{ij}\delta u_i \nu_j dS = \iiint\limits_V (\sigma_{ij}\delta u_i)_{,j} dV = \iiint\limits_V \sigma_{ij,j}\delta u_i dV + \iiint\limits_V \sigma_{ij}\delta u_{i,j} dV \tag{8.14}$$

Because of the symmetry of the stress tensor, the last term in equation (8.14) can be written as

$$\iiint\limits_V \sigma_{ij}\delta u_i, jdV = \iiint\limits_V \sigma_{ij}[\frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})]dV = \iiint\limits_V \sigma_{ij}\delta\varepsilon_{ij} dV \tag{8.15}$$

where ε_{ij} denotes the strain tensor:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{8.16}$$

In view of the equations of dynamic equilibrium, (8.9), the first integral on the right hand side of (8.14), can be expressed as

$$\iiint\limits_V \sigma_{ij,j}\delta u_i dV = \iiint\limits_V \left(\rho\frac{\partial^2 u_i}{\partial t^2} - \phi_i\right)\delta u_i dV \tag{8.17}$$

Thus, the second term of expression (8.11) can be written as

$$\iint\limits_S \Phi_i\delta u_i dS = \iiint\limits_V \sigma_{ij}\delta\varepsilon_{ij} dV + \iiint\limits_V \left(\rho\frac{\partial^2 u_i}{\partial t^2} - \phi_i\right)\delta u_i dV \tag{8.18}$$

This gives the variational equation of motion

$$\iiint\limits_V \sigma_{ij}\delta\varepsilon_{ij} dV = \iiint\limits_V \left(\phi_i - \rho\frac{\partial^2 u_i}{\partial t^2}\right)\delta u_i dV + \iint\limits_S \Phi_i\delta u_i dS \tag{8.19}$$

This equation can be stated more concisely by introducing different levels of restrictions. If the body is perfectly elastic, (8.19) can be stated in terms of the strain energy density π_0 as

$$\delta\iiint\limits_V \pi_0 dV = \iiint\limits_V \left(\phi_i - \rho\frac{\partial^2 u_i}{\partial t^2}\right)\delta u_i dV + \iint\limits_S \Phi_i\delta u_i dS \tag{8.20}$$

or

$$\delta\iiint\limits_V \left(\pi_0 + \rho\frac{\partial^2 u_i}{\partial t^2}\delta u_i\right) dV = \iiint\limits_V \phi_i\delta u_i dV + \iint\limits_S \Phi_i\delta u_i dS \tag{8.21}$$

If the variations δu_i are identified with the actual displacements $\dfrac{\partial u_i}{\partial t} dt$ during a small time interval dt, equation (8.21) states that in an arbitrary time interval, the sum of the energy of deformation and the kinetic energy increases by an amount that is equal to the work done by the external forces during the same time interval.

Treating the virtual displacements δu_i as functions of time and space not identified with the actual displacements, the variational equation of motion, equation (8.20), can be integrated between two arbitrary instants of time t_1 and t_2 and we obtain:

$$\int_{t_1}^{t_2} \iiint_V \delta\pi_0 dV dt = \int_{t_1}^{t_2} dt \iiint_V \phi_i \delta u_i dV + \int_{t_1}^{t_2} dt \iint_S \Phi_i \delta u_i dS - \int_{t_1}^{t_2} dt \iiint_V \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i dV \tag{8.22}$$

Denoting the last term in equation (8.22) as A, inverting the order of integration, and integrating by parts leads to

$$A = \iiint_V = \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i dV \Big|_{t_1}^{t_2} - \iint_V dV \int_{t_1}^{t_2} \frac{\partial u_i}{\partial t} \left(\rho \frac{\partial \delta u_i}{\partial t} + \frac{\partial \rho}{\partial t} \delta u_i \right) dt \tag{8.23}$$

In most problems, the time rate of change of density of the material, $\frac{\partial \rho}{\partial t}$, can be neglected. Also, we consider δu_i to be zero at all points of the body at initial and final time t_1 and t_2, so that $\delta u_i(t_1) = \delta u_i(t_2) = 0$.

With this information, equation (8.23) can be rewritten as

$$A = -\int_{t_1}^{t_2} \iint_V \rho \frac{\partial u_i}{\partial t} \frac{\partial \delta u_i}{\partial t} dV dt = -\int_{t_1}^{t_2} \iint_V \rho \frac{\partial u_i}{\partial t} \delta \frac{\partial u_i}{\partial t} dV dt \tag{8.24}$$

$$= -\int_{t_1}^{t_2} \delta \iint_V \frac{1}{2} \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} dV dt = -\int_{t_1}^{t_2} \delta T dt \tag{8.25}$$

where

$$T = \frac{1}{2} \iint_V \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} dV \tag{8.26}$$

is the kinetic energy of the vibrating body. Thus equation (8.22) can be expressed as

$$\int_{t_1}^{t_2} \delta(\pi - T) dt = \int_{t_1}^{t_2} \iiint_V \phi_i \delta u_i dV dt + \int_{t_1}^{t_2} \iint_S \Phi_i \delta u_i dS dt \tag{8.27}$$

where π denotes the total strain energy of the solid body

$$\pi = \iiint\limits_V \pi_0 dV \tag{8.28}$$

If the external forces acting on the body are such that the sum of the integrals on the right-hand side of equation (8.27) denotes the variation of a single function W (known as the potential energy of loading), we have

$$\iiint\limits_V \phi_i \delta u_i dV + \iint\limits_S \Phi_i \delta u_i dS = -\delta W \tag{8.29}$$

Then equation (8.27) can be expressed as

$$\delta \int\limits_{t_1}^{t_2} L dt = \int\limits_{t_1}^{t_2} (\pi - T + W) dt = 0 \tag{8.30}$$

where

$$L = \pi - T + W \tag{8.31}$$

is called the Lagrangian function and equation (8.30) is known as Hamilton's principle. Note that a negative sign is included, as indicated in equation (8.29), for the potential energy of loading (W). *Hamilton's principle* can be stated in words as follows:

The time integral of the Lagrangian function between the initial time t_1 and the final time t_2 is an extremum for the actual displacements (motion) with respect to all admissible virtual displacements that vanish throughout the entire time interval: first, at all points of the body at the instants t_1 and t_2 , and second, over the surface S_1, where the displacements are prescribed.

Hamilton's principle can be interpreted in another way by considering the displacements $u_i(x_1, x_2, x_3, t), i = 1, 2, 3,$ to constitute a dynamic path in space. Then *Hamilton's principle* states: Among all admissible dynamic paths that satisfy the prescribed geometric boundary conditions on S_1 at all times and the prescribed conditions at two arbitrary instants of time t_1 and t_2 at every point of the body, the actual dynamic path (solution) makes the Lagrangian function an extremum.

8.5 Derivation Timoshenko Integral calculations

8.5.1 1

The following was already defined

$$T = \frac{1}{2}\int_0^l \left[\rho A\left(\frac{\partial w}{\partial t}\right)^2 + \rho I\left(\frac{\partial \phi}{\partial t}\right)^2\right] dx, \tag{8.32}$$

$$\pi = \frac{1}{2}\int_0^l \left[EI\left(\frac{\partial \phi}{\partial x}\right)^2 + kAG\left(\frac{\partial \phi}{\partial x} - \phi\right)^2\right] dx, \tag{8.33}$$

$$W = \int_0^l f(x,t)w(x,t)dx. \tag{8.34}$$

All that was left was to calculate the following integral

$$\delta\int_{t_1}^{t_2}(\pi - T - W)dt = 0. \tag{8.35}$$

or the integral

$$\int_{t_1}^{t_2}\left\{\int_0^l \left[EI\frac{\partial \phi}{\partial x}\delta\left(\frac{\partial \phi}{\partial x}\right) + kAG\left(\frac{\partial \phi}{\partial x} - \phi\right)\delta\left(\frac{\partial w}{\partial x}\right) - kAG\left(\frac{\partial w}{\partial x} - \phi\right)\delta\phi\right] dx \right.$$
$$\left. - \int_0^l \left[\rho A\frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right) + \rho I\frac{\partial \phi}{\partial t}\delta\left(\frac{\partial \phi}{\partial t}\right)\right] dx - \int_0^l f\delta w dx\right\} dt = 0 \tag{8.36}$$

This integral will be evaluated part by part with the help of partial integration with respect to t and x as stated before

$$\int_{t_1}^{t_2}\int_0^l EI\frac{\partial \phi}{\partial x}\delta\left(\frac{\partial \phi}{\partial x}\right) dxdt = \int_{t_1}^{t_2}\left[EI\ \frac{\partial \phi}{\partial x}\delta\phi\Big|_0^l - \int_0^l \frac{\partial}{\partial x}\left(EI\frac{\partial \phi}{\partial x}\right)\delta\phi dx\right] dt, \tag{8.37}$$

$$\int_{t_1}^{t_2}\int_0^l kAG\left(\frac{\partial w}{\partial x} - \phi\right)\delta\left(\frac{\partial w}{\partial x}\right) dxdt = \int_{t_1}^{t_2}\left[kAG\left(\frac{\partial w}{\partial x} - \phi\right)\delta w\Big|_0^l - \int_0^l kAG\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} - \phi\right)\delta w dx\right] dt, \tag{8.38}$$

$$-\int_{t_1}^{t_2}\int_0^l \rho A\frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right) dxdt = -\int_{t_1}^{t_2}\int_0^l \rho A\frac{\partial^2 w}{\partial t^2}\delta w dxdt, \tag{8.39}$$

$$-\int_{t_1}^{t_2}\int_0^l \rho I \frac{\partial \phi}{\partial t}\delta\left(\frac{\partial \phi}{\partial t}\right) dxdt = -\int_{t_1}^{t_2}\int_0^l \rho A \frac{\partial^2 \phi}{\partial t^2}\delta\phi dxdt. \tag{8.40}$$

Substitution of the equations (8.37)-(8.40) into equation (8.36) will result in the following

$$\int_{t_1}^{t_2}\left\{ kAG\left(\frac{\partial w}{\partial x}-\phi\right)\delta w\Big|_0^l + EI\,\frac{\partial \phi}{\partial x}\delta\phi\Big|_0^l + \right.$$

$$+\int_0^l \left[\frac{\partial}{\partial x}\left\langle kAG\left(\frac{\partial w}{\partial x}-\phi\right)\right\rangle + \rho A\frac{\partial^2 w}{\partial t^2} - f\right]\delta w dx +$$

$$\left. +\int_0^l \left[-\frac{\partial}{\partial x}\left(EI\frac{\partial \phi}{\partial x}\right) - kAG\left(\frac{\partial w}{\partial x}-\phi\right) + \rho I\frac{\partial^2 \phi}{\partial t^2}\right]\delta\phi dx \right\} dt = 0 \tag{8.41}$$

From this the boundary conditions and differential equations for w and ϕ can be determined.

$$\begin{aligned} -\frac{\partial}{\partial x}\left[kAG\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}-\phi\right)\right] + \rho A\frac{\partial^2 w}{\partial t^2} &= f(x,t) \\ -\frac{\partial}{\partial x}\left(EI\frac{\partial \phi}{\partial t}\right) - kAG\left(\frac{\partial w}{\partial x}-\phi\right) + \rho I\frac{\partial^2 \phi}{\partial t^2} &= 0 \end{aligned} \tag{8.42}$$

Which is the desired result.

8.5.2 2

The variations in (3.8) can be evaluated using integration by parts. This is done for each integral with respect to x separately, to obtain for the first integral:

$$\int_{t_1}^{t_2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx =$$

$$\int_{t_1}^{t_2}\left[EI\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^l - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w\Big|_0^l + \int_0^l \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w dx\right] dt. \tag{8.43}$$

Here integration by parts is used twice, such that the factor $\dfrac{\partial^2 \delta w}{\partial x^2}$ disappears and turns into δw, which is desired since we want the variation in w.

The second integral

$$\delta \int_{t_1}^{t_2} \left[\int_0^l \rho A \left(\frac{\partial w}{\partial t} \right)^2 dx \right] dt,$$

becomes

$$= \int_0^l \left(\rho A \frac{\partial w}{\partial t} \delta w \Big|_0^l \right) dx - \int_0^l \left(\int_{t_1}^{t_2} \rho A \frac{\partial^2 w}{\partial t^2} \delta w dt \right) dx$$

$$= - \int_{t_1}^{t_2} \left(\int_0^l \rho A \frac{\partial^2 w}{\partial t^2} \delta w dx \right) dt. \tag{8.44}$$

Note that here the integration by parts is done with respect to time, along with the fact that $\delta w = 0$ at $t = t_1$ and $t = t_2$ to obtain the result of (8.44).

The last integral on which the variations works gives

$$\delta \int_{t_1}^{t_2} \left(\int_0^l f w dx \right) dt = \int_{t_1}^{t_2} \int_0^l f \delta w \; dx dt. \tag{8.45}$$

Here the factor δw is already present so there is no need of doing any kind of integration.

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