ACKNOWLEDGMENT

After the submission of this correspondence, we were made aware (by L. L. Scharf) of the thesis [2]. Similar conclusions to ours were independently drawn in [2, ch. 10], namely, that the rank reduced estimator is inadmissible, whereas a suitably modified James–Stein estimator (a "James–Stein rank reduced estimator" in [2]) dominates the traditional maximum likelihood estimator. Our correspondence differs in that we assume C in (1) to be known, whereas [2, ch. 10] assumes (1) holds only approximately with C unknown. The latter assumption is relevant when $\boldsymbol{x}$ in (1) models a nonstationary time series.

REFERENCES

[1] M. E. Bock, "Minimax estimators of the mean of a multivariate normal distribution," *Ann. Stat.*, vol. 3, no. 1, pp. 209–218, 1975.
[2] K. A. Duell, "Statistical techniques for processing nonstationary signals: Wavelets and decision theory," Ph.D. dissertation, Univ. Colorado, Boulder, 1994.
[3] B. Efron, *Introduction to James and Stein (1961) Estimation with Quadratic Loss*. New York: Springer-Verlag, 1991, vol. 1, pp. 437–442.
[4] B. Efron and C. Morris, "Stein's estimation rule and its competitors—An empirical Bayes approach, *J. Amer. Stat. Assoc.*, vol. 68, pp. 117–130, Mar. 1973.
[5] E. Greenberg and C. E. Webster, *Advanced Econometrics: A Bridge to the Literature*. New York: Wiley, 1983.
[6] Y. I. Y. Guo and N. Pal, "A sequence of improvements over the James-Stein estimator," *J. Multivariate Anal.*, vol. 42, pp. 302–317, 1992.
[7] Hoerl and Kennard, "Ridge regression: Biased estimation for nonorthogonal problems," *Technometr.*, vol. 12, pp. 55–77, 1970.
[8] W. James and C. M. Stein, "Estimation with quadratic loss," in *Proc. 4th Berkeley Symp. Math. Statist. Prob.*, 1961, vol. 1, pp. 311–319.
[9] S. Kotz and N. L. Johnson, Eds., *Breakthroughs in Statistics*. New York: Springer-Verlag, vol. 1, 1991.
[10] T. Kubokawa, "An approach to improving the James-Stein estimator," *J. Multivariate Anal.*, vol. 36, pp. 121–126, 1991.
[11] E. L. Lehmann, *Theory of Point Estimation*. New York: Wiley, 1983.
[12] J. H. Manton, "Optimal estimation and identification of linear systems—Stochastic and algebraic approaches," Ph.D. dissertation, Univ. Melbourne, Parkville, Australia, Feb. 1998.
[13] J. H. Manton, V. Krishnamurthy, and H. V. Poor, "James–Stein state space filter," in *Proc. 36th IEEE Conf. Decision Contr.*, Dec. 1997, pp. 3454–3459.
[14] ——, "James–Stein state filtering algorithms," *IEEE Trans. Signal Processing*, vol. 46, pp. 2431–2447, Sept. 1998.
[15] L. L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time Series Analysis*. Reading, MA: Addison-Wesley, 1991.
[16] L. L. Scharf and D. W. Tufts, "Rank reduction for modeling stationary signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 350–355, Mar. 1987.
[17] W. E. Strawderman, "Proper Bayes minimax estimators of the multivariate normal mean," *Annu. Math. Stat.*, vol. 42, pp. 385–388, 1971.

Direction-of-Arrival Estimation for Constant Modulus Signals

Amir Leshem and Alle-Jan van der Veen

***Abstract*— In many cases where direction finding is of interest, the signals impinging on an antenna array are known to be phase modulated and, hence, to have a constant modulus (CM). This is a strong property; by itself, it is already sufficient for source separation and can be used to construct improved direction finding algorithms. We first derive the relevant Cramér–Rao bounds (CRB's) for arbitrary array configurations and specialize to uniform linear arrays. We then propose a simple suboptimal direction estimation algorithm in which the signals are separated using the CM property followed by direction finding on the decoupled signals. Compared with the ESPRIT algorithm and the CRB for arbitrary signals, the algorithm shows good results.**

***Index Terms*— Constant modulus, Cramér–Rao bound, DOA estimation.**

I. INTRODUCTION

Direction-of-arrival (DOA) estimation of multiple signals impinging on an antenna array is a well-studied problem in signal processing. "Traditional" methods exploit knowledge of the array manifold or its structure without using information on the signals. Example algorithms are MUSIC [1], ESPRIT [2], MLE [3], WSF [4], and MODE [5]. For signals with known waveforms, an algorithm is derived in [6]. Other methods exploit properties of the signals such as nonGaussianity [7] or cyclostationarity [8]. These methods are more robust to array manifold errors due to the extra information they use. Although phase-modulated signals are ubiquitous in the communication field, no detailed study of the exploitation of the constant modulus property for multiple-source DOA estimation has been done thus far. As we show here, a large improvement can be achieved by exploiting this information.

Since the pioneering work of Treichler and Agee [9], it is known that the constant modulus (CM) property is a strong property that, by itself, is already sufficient for source separation. After separation of the signals, the DOA estimation problem is decoupled and can be done for each source individually. Such a scheme is proposed in [10], where the CM signals are sequentially separated using the so-called CM array. Weak points of this and related iterative CM algorithms are their initialization, the problematic recovery of all signals, and their unpredictable convergence, which may require several hundred samples per signal. To counter these problems, Mathur *et al.* [11] propose to initialize each stage of the algorithm by a weight vector found by the MUSIC algorithm. However, it is well known that sequential DOA estimation yields poor performance for the weak sources when the stronger sources are not completely removed.

Recently, Van der Veen and Paulraj [12] have found an analytic solution to the CM source separation problem in which all weight vectors are found simultaneously and reliably from a small number of samples and without initialization problems. Thus, this algorithm is

Manuscript received June 30, 1998; revised April 26, 1999. The work of A. Leshem was supported by the NOEMI project of the STW under Contract DEL77-476. The associate editor coordinating the review of this paper and approving it for publication was Dr. Jean Jacques Fuchs.

The authors are with Department of Electrical Engineering, Delft University of Technology, Delft, The Netherlands.

Publisher Item Identifier S 1053-587X(99)08314-2.

very attractive for use as a first step in the DOA estimation problem. Moreover, it is applicable to any array geometry.

Knowing that there is a good algorithm, it becomes interesting to study the performance bounds for DOA estimation of CM signals. Here, our aim is to derive such bounds. We also give explicit bounds for the signal phase estimates. We demonstrate by simulations that the proposed algorithm almost achieves the CRB for constant modulus signals, which is below the bound for arbitrary signals. Hence, the algorithm outperforms any algorithm that does not use the CM property. We also demonstrate the robustness of the algorithm to various model errors.

II. Data Model

Consider an array with p sensors receiving q narrowband constant modulus signals. Under standard assumptions for the array manifold, we can describe the received signal as an instantaneous linear combination of the source signals, i.e.,

$$\boldsymbol{x}(t) = \boldsymbol{ABs}(t) + \boldsymbol{n}(t) \tag{1}$$

where

- $\boldsymbol{x}(t) = [x_1(t), \cdots, x_p(t)]^T$ is a $p \times 1$ vector of received signals at time t;
- $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \cdots, \boldsymbol{a}(\theta_q)]$, where $\boldsymbol{a}(\theta)$ is the array response vector for a signal from direction θ, and $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_q]$ is the DOA vector of the sources;
- $\boldsymbol{B} = \mathrm{diag}(\boldsymbol{\beta})$ is the channel gain matrix with parameters $\boldsymbol{\beta} = [\beta_1, \cdots, \beta_q]^T$, where $\beta_i \in \mathbb{R}^+$ is the amplitude of the ith signal as received by the array;
- $\boldsymbol{s}(t) = [s_1(t), \cdots, s_q(t)]^T$ is a $q \times 1$ vector of source signals at time t;
- $\boldsymbol{n}(t)$ is the $p \times 1$ additive noise vector, which is assumed spatially and temporally white Gaussian distributed with covariance matrix $\nu \boldsymbol{I}$, where $\nu = \sigma^2$ is the noise variance.

In our problem, the array is assumed to be calibrated so that the array response vector $\boldsymbol{a}(\theta)$ is a known function. As usual, we require that the array manifold satisfies the uniqueness condition, i.e., every collection of p vectors on the manifold are linearly independent [13].

We further assume that all sources have constant modulus. This is represented by the assumption that for all t, $|s_i(t)| = 1$ $(i = 1, \cdots, q)$. Unequal source powers are absorbed in the gain matrix $\boldsymbol{B}$. Phase offsets of the sources after demodulation are part of the s_i. Thus, we can write $s_i(t) = e^{j\phi_i(t)}$, where $\phi_i(t)$ is the unknown phase modulation for source i, and we define $\boldsymbol{\phi}(t) = [\phi_1(t), \cdots, \phi_q(t)]^T$ as the phase vector for all sources at time t.

Finally, we assume that N samples $[\boldsymbol{x}(1), \cdots, \boldsymbol{x}(N)]$ are available.

III. Cramér–Rao Bounds

The Cramér–Rao bound (CRB) provides a lower bound on parameter estimation variance for any unbiased estimator. We present CRB's for DOA and signal phase estimation of multiple CM signals, postponing the derivations to the Appendix.

The likelihood function is given by

$$L(\boldsymbol{x}|\boldsymbol{s}, \boldsymbol{\theta}, \boldsymbol{\beta}, \nu) = \frac{1}{(2\pi)^N \left(\frac{\nu}{2}\right)^{pN}} \cdot \exp\left\{ -\frac{1}{\nu} \sum_{k=1}^{N} (\boldsymbol{x}(k) - \boldsymbol{ABs}(k))^* (\boldsymbol{x}(k) - \boldsymbol{ABs}(k)) \right\}.$$

Let $\mathcal{L}(\boldsymbol{x}|\boldsymbol{s}, \boldsymbol{\theta}, \nu) = \log L(\boldsymbol{x}|\boldsymbol{s}, \boldsymbol{\theta}, \boldsymbol{\beta}, \nu)$. After omitting constants, we obtain

$$\mathcal{L}(\boldsymbol{x}|\boldsymbol{s}, \boldsymbol{\theta}, \boldsymbol{\beta}, \nu) = -pN \log \nu - \frac{1}{\nu} \sum_{k=1}^{N} (\boldsymbol{x}(k) - \boldsymbol{ABs}(k))^* (\boldsymbol{x}(k) - \boldsymbol{ABs}(k)).$$

Following [14], the estimation of the noise variance is decoupled from all other parameters, and its bound can be computed separately as $\mathrm{CRB}_N(\nu) = (\nu^2/pN)$. The remaining parameters are collected in the vector $[\boldsymbol{\phi}(1)^T, \cdots, \boldsymbol{\phi}(N)^T, \boldsymbol{\theta}^T, \boldsymbol{\beta}^T]^T$. Define

$$\boldsymbol{S}_k = \mathrm{diag}\ (\boldsymbol{s}(k)) \qquad \text{and} \qquad \boldsymbol{D} = \left[\frac{d\boldsymbol{a}}{d\theta}(\theta_1), \cdots, \frac{d\boldsymbol{a}}{d\theta}(\theta_q) \right].$$

The Fisher information matrix associated with the estimation of the parameter vector can be derived as (see the Appendix)

$$\mathbf{FIM}_N = \left[\begin{array}{ccc|cc} \boldsymbol{H}_1 & & 0 & \boldsymbol{\Delta}_1^T & \boldsymbol{E}_1^T \\ & \ddots & & \vdots & \vdots \\ 0 & & \boldsymbol{H}_N & \boldsymbol{\Delta}_N^T & \boldsymbol{E}_N^T \\ \hline \boldsymbol{\Delta}_1 & \cdots & \boldsymbol{\Delta}_N & \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^T \\ \boldsymbol{E}_1 & \cdots & \boldsymbol{E}_N & \boldsymbol{\Lambda} & \boldsymbol{\Upsilon} \end{array} \right] \tag{2}$$

where

$$\begin{aligned}
\boldsymbol{H}_k &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \right)^T = \frac{2}{\nu}\ \mathrm{Re}(\boldsymbol{S}_k^* \boldsymbol{B}^* \boldsymbol{A}^* \boldsymbol{ABS}_k) \\
\boldsymbol{\Delta}_k &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \right)^T = -\frac{2}{\nu}\ \mathrm{Im}\ (\boldsymbol{S}_k^* \boldsymbol{B}^* \boldsymbol{D}^* \boldsymbol{ABS}_k) \\
\boldsymbol{E}_k &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} \right)^T = -\frac{2}{\nu}\ \mathrm{Im}(\boldsymbol{S}_k^* \boldsymbol{A}^* \boldsymbol{ABS}_k) \\
\boldsymbol{\Gamma} &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \right)^T = \frac{2}{\nu} \sum_{k=1}^{N} \mathrm{Re}\ (\boldsymbol{S}_k^* \boldsymbol{B}^* \boldsymbol{D}^* \boldsymbol{DBS}_k) \\
\boldsymbol{\Lambda} &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \right)^T = \frac{2}{\nu} \sum_{k=1}^{N} \mathrm{Re}\ (\boldsymbol{S}_k^* \boldsymbol{A}^* \boldsymbol{DBS}_k) \\
\boldsymbol{\Upsilon} &:= E \frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} \right)^T = \frac{2}{\nu} \sum_{k=1}^{N} \mathrm{Re}\ (\boldsymbol{S}_k^* \boldsymbol{A}^* \boldsymbol{AS}_k).
\end{aligned} \tag{3}$$

$\boldsymbol{H}_k^{-1}$ would be the CRB on the estimation of the unknown source phases at time k in the case where the DOA's and amplitudes are known. Similarly, $\boldsymbol{\Gamma}^{-1}$ and $\boldsymbol{\Upsilon}^{-1}$ provide bounds on the estimation of the DOA's and amplitudes, respectively, when other parameters are known. The matrices $\boldsymbol{\Delta}_k$, $\boldsymbol{E}_k$, and $\boldsymbol{\Lambda}$ represent the couplings between the parameters.

The bounds on the individual parameters are obtained after inversion of the Fisher information matrix. This can be carried out in block-partitioned form (using Schur complement formulas and the Woodbury identity), which leads to more explicit expressions. Thus, assuming that the $\boldsymbol{H}_k$ are invertible (an assumption that follows from the independence condition on the array manifold and the independence of the sources), let

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} \boldsymbol{\Delta}_k \boldsymbol{H}_k^{-1} \boldsymbol{\Delta}_k^T & \sum_{k=1}^{N} \boldsymbol{\Delta}_k \boldsymbol{H}_k^{-1} \boldsymbol{E}_k^T \\ \sum_{k=1}^{N} \boldsymbol{E}_k \boldsymbol{H}_k^{-1} \boldsymbol{\Delta}_k^T & \sum_{k=1}^{N} \boldsymbol{E}_k \boldsymbol{H}_k^{-1} \boldsymbol{E}_k^T \end{bmatrix}$$

and define the $q \times q$ matrix

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^T \\ \boldsymbol{\Lambda} & \boldsymbol{\Upsilon} \end{bmatrix} - \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix}.$$

Using the Schur complement formula twice, the CRB for DOA's and amplitudes can be written more explicitly as

$$\begin{aligned}\mathrm{CRB}_N(\boldsymbol{\theta}) &= (\boldsymbol{\Psi}^{-1})_{11} = \mathrm{diag}\,[(\boldsymbol{\Gamma} - \boldsymbol{\Xi}_{11}) \\ &\quad - (\boldsymbol{\Lambda}^T - \boldsymbol{\Xi}_{12})(\boldsymbol{\Upsilon} - \boldsymbol{\Xi}_{22})^{-1}(\boldsymbol{\Lambda} - \boldsymbol{\Xi}_{21})]^{-1} \\ \mathrm{CRB}_N(\boldsymbol{\beta}) &= (\boldsymbol{\Psi}^{-1})_{22} = \mathrm{diag}\,[(\boldsymbol{\Upsilon} - \boldsymbol{\Xi}_{22}) \\ &\quad - (\boldsymbol{\Lambda} - \boldsymbol{\Xi}_{21})(\boldsymbol{\Gamma} - \boldsymbol{\Xi}_{11})^{-1}(\boldsymbol{\Lambda}^T - \boldsymbol{\Xi}_{12})]^{-1}. \end{aligned} \tag{4}$$

Similarly, using the Woodbury identity, the bound on the estimation variance of the signal phases follows as

$$\begin{aligned}&\mathrm{CRB}_N(\boldsymbol{\phi}(k)) \\ &\quad = \mathrm{diag}\left\{\boldsymbol{H}_k^{-1}\left[\boldsymbol{I} + [\boldsymbol{\Delta}_k^T \quad \boldsymbol{E}_k^T]\boldsymbol{\Psi}^{-1}\begin{bmatrix}\boldsymbol{\Delta}_k \\ \boldsymbol{E}_k\end{bmatrix}\boldsymbol{H}_k^{-1}\right]\right\}. \end{aligned} \tag{5}$$

Note that the number of samples and the quality of DOA estimation affects the bound only through the matrix $\boldsymbol{\Psi}^{-1}$.

Single CM Source

To obtain more insight into the CRB's, we consider the case of DOA estimation of a single CM source. We omit all derivation due to space limitations. The CRB on the DOA is, in this case, given by

$$\mathrm{CRB}_N(\theta) = \frac{1}{2N\ \mathrm{SNR}\ \|\boldsymbol{P}_{\boldsymbol{a}(\theta)}^{\perp}\boldsymbol{d}(\theta)\|^2}$$

where $\boldsymbol{d}(\theta) = (d\boldsymbol{a}(\theta)/d\theta)$, $\boldsymbol{P}_{\boldsymbol{a}}^{\perp} = I - \boldsymbol{a}(\boldsymbol{a}^*\boldsymbol{a})^{-1}\boldsymbol{a}^*$, and $\mathrm{SNR} = \beta^2/\nu$. This conforms with the results of [6], which obtain an identical asymptotic expression for the case of a known signal with unknown amplitude and initial phase. We can also obtain that the phase estimation variance is given by

$$\mathrm{CRB}_N(\phi(k)) = \frac{1}{2\ \mathrm{SNR}\ \|\boldsymbol{a}(\theta)\|^2}\left(1 + \frac{1}{N}c(\theta)\right)$$

where

$$c(\theta) = \frac{(\mathrm{Im}\ (\boldsymbol{d}(\theta)^*\boldsymbol{a}(\theta)))^2}{\|\boldsymbol{a}(\theta)\|^2\|\boldsymbol{P}_{\boldsymbol{a}(\theta)}^{\perp}\boldsymbol{d}(\theta)\|^2}.$$

$c(\theta)$ represents the effect of channel estimation error on the signal phase estimation.

Further simplification of the above bounds is possible if we assume that the antenna array is a uniform linear array (ULA) with antennas spaced by d wavelengths. In this case

$$\begin{aligned}\mathrm{CRB}_N(\theta) &= \frac{6}{p(p^2-1)N\ \mathrm{SNR}\ (2\pi d)^2\cos^2(\theta)} \\ \mathrm{CRB}_N(\phi(k)) &= \frac{1}{2p\ \mathrm{SNR}}\left[1 + \frac{3}{N}\ \frac{p-1}{p+1}\right]. \end{aligned} \tag{6}$$

Note that the estimation quality of the signal phases is independent of the antenna spacing and the DOA and quickly becomes independent of the number of samples N.

IV. CM-DOA Estimation Algorithm

A suboptimal but simple algorithm to estimate the DOA's using the CM property is to do the following.

1) Blindly estimate a matrix $\hat{\boldsymbol{A}} = [\boldsymbol{a}_1, \cdots, \hat{\boldsymbol{a}}_q]$ using the CM assumption.
2) For each column $\hat{\boldsymbol{a}}_i$ of $\hat{\boldsymbol{A}}$, estimate the direction $\hat{\theta}_i$ that fits best.

A closed-form solution for the first step is provided by the ACMA algorithm. It is described in detail in [12] and will not be discussed here. The second step is known to be a one-dimensional (1-D) projection of each $\hat{\boldsymbol{a}}_i$ onto the array manifold given by

$$\hat{\theta}_i = \arg\max_{\theta}\ \frac{|\hat{\boldsymbol{a}}_i^*\boldsymbol{a}(\theta)|}{\|\boldsymbol{a}(\theta)\|}. \tag{7}$$

Finally, when the ESPRIT method is applicable, a similar trick can be used to reduce the computational complexity by eliminating the 1-D searches. Let $\hat{\boldsymbol{a}}_{1,i}$ and $\hat{\boldsymbol{a}}_{2,i}$ be the two parts of the estimate of the array manifold of the ith source, which are phase shifted. We can write

$$\hat{\boldsymbol{a}}_{1,i} = e^{j(2\pi d/\lambda)\sin\theta_i}\hat{\boldsymbol{a}}_{2,i} \tag{8}$$

where d is the distance between the two parts of the array. Hence, by performing LS fitting using the estimated array manifold, we obtain

$$\hat{\theta}_i = \sin^{-1}\left(\frac{\lambda}{2\pi d}\tan^{-1}\left(\frac{\mathrm{Im}\ (\hat{\boldsymbol{a}}_{1,i}^H\hat{\boldsymbol{a}}_{2,i})}{\mathrm{Re}\ (\hat{\boldsymbol{a}}_{1,i}^H\hat{\boldsymbol{a}}_{2,i})}\right)\right). \tag{9}$$

The advantage of this CM-DOA algorithm is that it is applicable to arbitrary array configurations, unlike other fast methods such as ESPRIT, which exploits a specific array structure and breaks down with multipath propagation. Although suboptimal, its estimates are usually quite close to the CRB. In the next section, we also demonstrate the robustness of the algorithm to model errors.

V. Simulation Results

It is interesting to compare the DOA CRB's for CM signals versus the usual case of arbitrary signals [14] and versus the case of known signals with unknown amplitudes (including initial phases) [6]. Because of the complex nature of the expressions, this is practical only graphically for specific examples. We also compare the CM-DOA algorithm to ESPRIT by means of simulations. We have used the method based on (7), which is more robust than the ESPRIT-type estimation.

We have used a $p = 8$ element ULA with spacing $d = \frac{1}{2}$ wavelength and $q = 2$ equipowered random phase CM signals. If not specified otherwise, we took

- $N = 50$ samples;
- SNR = 20 dB;
- first source located at 0° (boresight);
- second at 5°.

The results have been averaged over 400 Monte Carlo runs.

We have carried out four simulation cases:

1) *Varying source separation, from* 2° *to* 20°: As seen in Fig. 1, the CM-DOA algorithm is very close to its CRB and, hence, outperforms any DOA estimation that does not use the CM information.
2) *Varying SNR*: See Fig. 2. We see that the CM-DOA estimator almost achieves the CRB.
3) *Varying array model mismatch*: We have corrupted the entries of the array response vectors by white gaussian noise with variance −50 dB to 0 dB relative to the array manifold. The DOA estimation variance for the first source is presented in Fig. 3. The CM-DOA methods give uniformly better performance relative to ESPRIT. The CRB's are not tight anymore because they do not take model error into account, but it is interesting to see that up to −20 dB model error, the CM-DOA algorithm performs better than the bound for arbitrary signals.
4) *Varying CM signal model mismatch*: We have added a white Gaussian noise signal to each of the CM signals. As seen in Fig. 4, at up to 15 dB perturbation, the CM-DOA algorithm

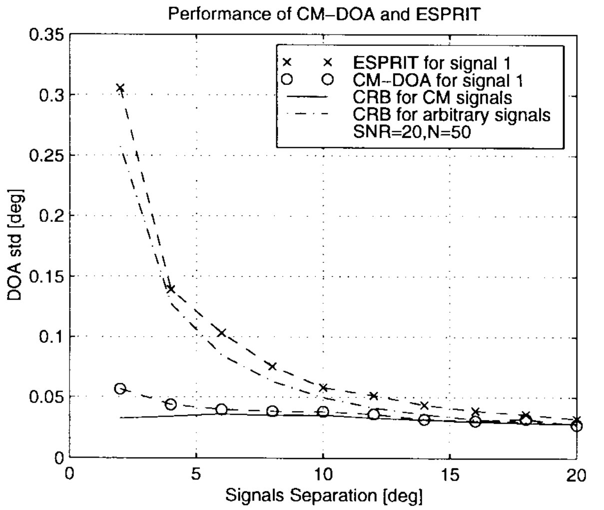

Fig. 1. DOA estimation accuracy for CM-DOA and ESPRIT. Varying source separation.

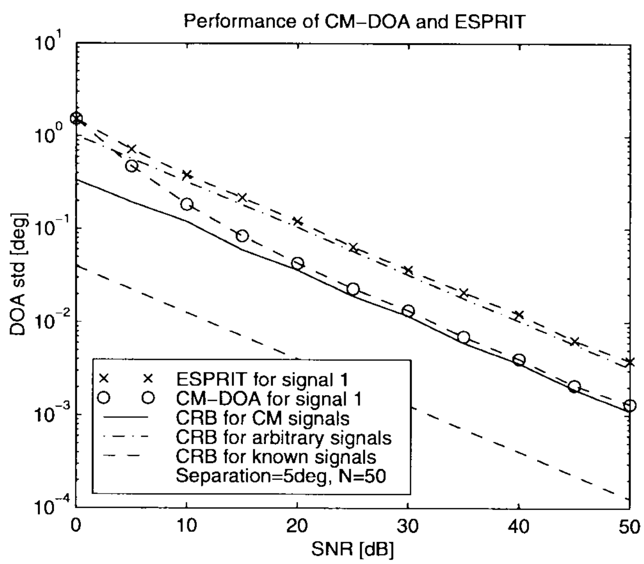

Fig. 2. DOA estimation accuracy for CM-DOA and ESPRIT. Varying SNR.

still estimates the DOA better than ESPRIT. This shows that a limited robustness to the CM assumption exists.

Finally, we have tested the performance for correlated signals in the extreme case of a small array and an angular separation of 2°. The array was a four-element ULA with half-wavelength spacings. The correlation coefficient between the signals was varied from 0.05–1. As seen in Fig. 5, the CM-DOA algorithm does not achieve the CRB in this case, but we still improve on the CRB for arbitrary signals up to correlations of 0.5.

VI. Conclusions

We have computed the Cramér–Rao bound for direction finding of constant modulus signals. The comparison to the bound for arbitrary signals shows the importance of using the constant modulus information whenever it is available. We then devised a simple two-step algorithm for the DOA estimation of CM signals using the ACMA algorithm. The algorithm is shown to outperform any algorithm that does not use signal structure over a wide range of parameters.

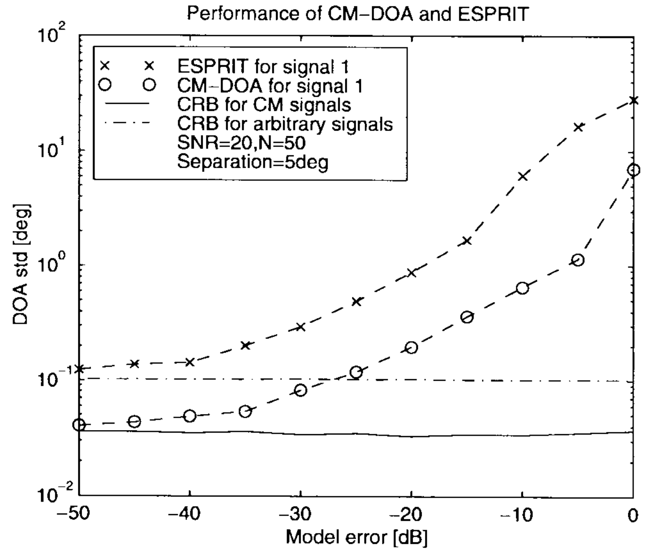

Fig. 3. DOA estimation accuracy for CM-DOA and ESPRIT. Varying array model mismatch.

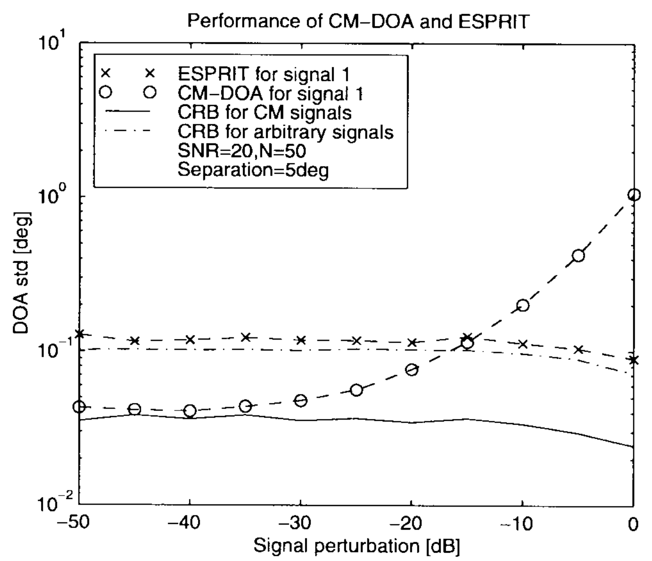

Fig. 4. DOA estimation accuracy for CM-DOA and ESPRIT. Varying signal model mismatch.

Appendix
Derivation of the Information Matrix

In this Appendix, we derive the Fisher information matrix for the DOA estimation of multiple constant modulus signals. The derivation is along the lines of [14].

Define

$$\boldsymbol{e}(k) = \boldsymbol{x}(k) - \boldsymbol{A}\boldsymbol{B}\boldsymbol{s}(k).$$

The partial derivative of $\mathcal{L}$ to ν is

$$\frac{\partial \mathcal{L}}{\partial \nu} = -\frac{pN}{\nu} + \frac{1}{\nu^2} \sum_{k=1}^{N} \boldsymbol{e}^*(k)\boldsymbol{e}(k).$$

To compute the partial derivative to $\boldsymbol{\phi}(k)$, we use

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} = \frac{\partial \overline{\boldsymbol{s}}(k)}{\partial \boldsymbol{\phi}(k)} \frac{\partial \mathcal{L}}{\partial \overline{\boldsymbol{s}}(k)} + \frac{\partial \tilde{\boldsymbol{s}}(k)}{\partial \boldsymbol{\phi}(k)} \frac{\partial \mathcal{L}}{\partial \tilde{\boldsymbol{s}}(k)}$$

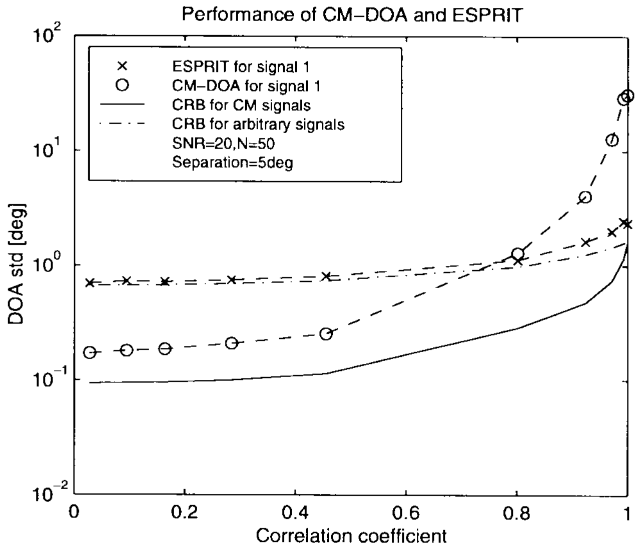

Fig. 5. DOA estimation accuracy for CM-DOA and ESPRIT for varying signal correlation.

where $\overline{\boldsymbol{s}}(k) = \text{Re } \boldsymbol{s}(k) = \cos(\boldsymbol{\phi}(k))$, and $\tilde{\boldsymbol{s}}(k) = \text{Im } \boldsymbol{s}(k) = \sin(\phi(k))$. Following [14], we obtain

$$\frac{\partial \mathcal{L}}{\partial \overline{\boldsymbol{s}}(k)} = \frac{2}{\nu} \text{ Re } (\boldsymbol{B}^* \boldsymbol{A}^* \boldsymbol{e}(k))$$
$$\frac{\partial \mathcal{L}}{\partial \tilde{\boldsymbol{s}}(k)} = \frac{2}{\nu} \text{ Im } (\boldsymbol{B}^* \boldsymbol{A}^* \boldsymbol{e}(k)).$$

In addition

$$\frac{\partial \overline{\boldsymbol{s}}(k)}{\partial \boldsymbol{\phi}(k)} = \text{diag } \{-\sin(\boldsymbol{\phi}(k))\} =: -\text{Im } (\boldsymbol{S}_k)$$
$$\frac{\partial \tilde{\boldsymbol{s}}(k)}{\partial \boldsymbol{\phi}(k)} = \text{diag } \{\cos(\boldsymbol{\phi}(k))\} =: \text{Re } (\boldsymbol{S}_k)$$

where $\boldsymbol{S}_k = \text{diag } (\boldsymbol{s}(k))$. Hence, we obtain after some easy manipulations

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}(k)} = \frac{2}{\nu} \text{ Im } (\boldsymbol{S}_k^* \boldsymbol{B}^* \boldsymbol{A}^* \boldsymbol{e}(k)).$$

In addition, from [14], we obtain

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = \frac{2}{\nu} \sum_{k=1}^{N} \text{ Re } (\boldsymbol{S}_k^* \boldsymbol{B}^* \boldsymbol{D}^* \boldsymbol{e}(k))$$

and similarly

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} = \frac{2}{\nu} \sum_{k=1}^{N} \text{ Re } (\boldsymbol{S}_k^* \boldsymbol{A}^* \boldsymbol{e}(k)).$$

Now, we are in position to compute the entries of the Fisher information matrix. Straightforward computation yields (3). Note that the variance is decoupled from all other variables so that its bound can be computed separately. The FIM of the remaining parameters then follows as (2).

REFERENCES

[1] R. Schmidt, *A Signal Subspace Approach to Multiple Emitter Location and Spectral Estimation*, Ph.D. dissertation, Stanford Univ., Stanford, CA, 1981.

[2] R. Roy, A. Paulraj, and T. Kailath, "ESPRIT—A subspace rotation approach to estimation of parameters of cisoids in noise," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 1340–1342, Oct. 1986.

[3] I. Ziskind and M. Wax, "Maximum likelihood localization of multiple sources by alternating projections," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 1553–1560, Oct. 1988.

[4] M. Viberg and B. Ottersten, "Sensor array processing based on subspace fitting," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 39, pp. 1110–1121, May 1991.

[5] P. Stoica and K. Sharman, "Maximum likelihood methods for direction estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1132–1143, Feb. 1990.

[6] J. Li, B. Halder, P. Stoica, and M. Viberg, "Computationally efficient angle estimation for signals with known waveforms," *IEEE Trans. Signal Processing*, vol. 43, pp. 2154–2163, Sept. 1995.

[7] B. Porat and B. Friedlander, "Direction finding algorithms based on higher order statistics," *IEEE Trans. Signal Processing*, vol. 37, pp. 2016–2024, Sept. 1991.

[8] G. Xu and T. Kailath, "DOA estimation via exploitation of cyclostationarity—A combination of spatial and temporal processing," *IEEE Trans. Signal Processing*, vol. 40, pp. 1775–1785, July 1992.

[9] J. Treichler and B. Agee, "A new approach to multipath correction of constant modulus signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 459–471, Apr. 1983.

[10] J. Shynk and R. Gooch, "The constant modulus array for cochannel signal copy and direction finding," *IEEE Trans. Signal Processing*, vol. 44, pp. 652–660, Mar. 1996.

[11] A. Mathur, A. Keerthi, and J. Shynk, "Cochannel signal recovery using MUSIC algorithm and the constant modulus array," *IEEE Signal Processing Lett.*, vol. 2, pp. 191–194, Oct. 1995.

[12] A. van der Veen and A. Paulraj, "An analytical constant modulus algorithm," *IEEE Trans. Signal Processing*, vol. 44, pp. 1136–1155, May 1996.

[13] M. Wax and I. Ziskind, "On unique localization of multiple sources by passive sensor arrays," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 996–1000, July 1989.

[14] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 720–743, May 1989.