Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

A study of the 1984 report *An automatic proof procedure for several geometries* by Th. Bruyn and H.L. Claasen

A thesis submitted to the
Delft Institute of Applied Mathematics
in partial fulfilment of the requirements

for the degree

BACHELOR OF SCIENCE
in
APPLIED MATHEMATICS

by

Tim Bruyn

Delft, The Netherlands
August 2017

BSc THESIS APPLIED MATHEMATICS

A study of the 1984 report *An automatic proof procedure for several geometries* by Th. Bruyn and H.L. Claasen

TIM BRUYN

Delft University of Technology

Daily supervisor and responsible professor

Prof. Dr. J.M.A.M. van Neerven

Other thesis committee members

Dr. B. van den Dries

Dr. K.P. Hart

Dr. Ir. J.F.M. Tonino

August, 2017 Delft

Preface

It has probably not escaped notice that my last name is the same as that of the author of the report studied in this thesis. This is, of course, no coincidence. My full name is Tim Theo Bruyn, named after my grandfather Theo Bruyn (1923 - 1998), whose 'hobby' of mathematics resulted after more than twenty years in the report that is the studied in this thesis.

In this preface I want to give some insight in the relationship between me and my grandfather and into the journey that culminated in this thesis.

I was only five when my grandfather passed away. Naturally, I do not have much memory of him except some vague images. Nonetheless I have always felt a strong connection to him. Being named after him, I have always been curious and the ghost of him has influenced my life in many ways. I have often been told that I resemble him, from my handwriting to my love of reading. This turned into somewhat of a self-fulfilling prophecy, as the more people would tell me I resemble him, the more I would identify myself with him. The more I interested myself in the things that he was interested in, the more my father and grandmother would give me books that my grandfather had read and in turn the more my interest was sparked. Even my choice of studying mathematics in Delft was influenced by the memory of my grandfather, as he had also studied here.

One of the few photos of me and my grandfather together, taken in 1993

When I started mathematics, my father gave me the original copy of an article that my grandfather had written in 1965, *An automatic proof procedure for straight line projective geometry in the plane* [1]. It was only five pages long and filled with the abstract mathematics that seems a kind of magic to a layman. It had been written while my grandfather, a pilot by profession, had been a part-time member of a mathematical research group lead by professor E.W. Beth (1908 - 1964). My father assured me on multiple occasions that it was indeed pure magic to him and the rest of the family, which fuelled my dedication to be the one to understand it even more. During my first year I tried studying it, but after a few months I had to give up. At that time I thought it was my lack of mathematical knowledge and decided to shelf the project.

When it was time to write my thesis, I immediately thought back to the article. With a near-complete bachelor in mathematics, my mathematical knowledge should be sufficient to finally piece his work together. I approached professor J. van Neerven whom I knew from the course History and Philosophy of Mathematics, as I thought the historical nature of the article might interest him. He agreed and for the second time during my study I faced the challenge of understanding the article.

However, again I ran into difficulty. The article was too concise, no references to literature of any kind were given and most of the definitions were postulated without any introduction or context. The writing style seemed to lack any consideration for a possible reader. I can only guess to the reason. As I later learned, the article was written two months before the closing down of the project, shortly after Beth passed away. It can be speculated that the article was rushed as members of the team were pushed to publish something before the project terminated. Whatever the reason, it turned out that even with the help of my supervisor the article was incomprehensible. After three weeks of trying various methods of visualization and trying to build understanding of the context, I had to admit that the article simply wasn't extensive enough.

Parallel to my study of the article I was looking for additional sources. This too turned out to be much harder than I had hoped. A search in the digital archives of the American Mathematical Society (MathSciNet) revealed that while the article was listed, it had zero citations from references and zero citations from reviews. It was known in the family that my grandfather had worked with Beth on a project for Euratom, the European Atomic Society. This lead to the first reference to the work of my grandfather, a single paragraph in the dissertation of Dr. P. van Ulsen on the life and work of Beth (*E.W. Beth als Logicus*, 2001 [11]). It mentioned that my grandfather had based his work on the theory of M. Carton [4].

I contacted Van Ulsen, who brought me into contact with Dr. H. Visser, a retired professor of mathematics that had studied with Beth. Although he had never met my grandfather, he had heard of him and mentioned him in one of his short articles [12]. Also, in a strange twist of fate, he happened to have played in a string quartet with my grandmother. A meeting was arranged. However, on the evening before the meeting, Visser kindly asked me to bring the reports. It was the first time I had even heard of any documented work other than the article (so needless to say, I did not have any of these reports). It was decided to postpone the meeting until I had acquired a copy these reports, which were in the possession of the Beth-archives. My grandfathers contribution turned out to be limited to one report, written in 1961 [2]. However, it turned out to be mainly a critique on the work of Carton. It described the procedure of Carton and provided insight in his methods, but for the most part it argued that much of Cartons work was not useful for the construction of an actual proof procedure. While it was very exciting to finally have another work by the hands of my grandfather, it did not actually help me to understand the article.

My search for sources got a new boost from an entirely different direction. My father had told me that years after the Euratom project my grandfather was approached by a former colleague, Dr. C. van Westrhenen, to continue his work. Van Westrhenen was thought to work at Wageningen University at that time. However, I had not managed to find any reference to Van Westrhenen and considered it a dead-end. This changed when I interviewed my uncle for creating a short biography of my grandfathers career (see appendix A) and to find additional information that could lead to new sources. My uncle recalled that Van Westrhenen was professor at Delft, rather than Wageningen. It also turned out that I had been spelling the name wrong and that it was S.C. van Westrhenen. With this new information (and some other clues) I managed to find a digital copy of the parting lecture of S.C. Van Westrhenen, written in 1993. In this document I found a list of names of people that he had worked with and decided to see if any of them still worked at the TU Delft, in the hope that they might know something of the project. I remember finding a mention of one of them on the website of the TU Delft, with a room number right above the place I was working at that moment. Excited, I rushed up to find that it was a meeting room, named after him. Eventually I got into contact with Dr. Ir. H. Tonino, who sadly informed me that Van Westrhenen had passed away in 2000.

By that time it was already June, two weeks before my final presentation and the due-date of my thesis. I had resigned myself to the idea that the conclusion of my thesis would be that there were simply not enough sources to comprehend the work of my grandfather. My supervisor and I were discussing other subjects that I could include in my thesis to at least get a passing grade. However, by the very fact that this report exists it is clear that another source was found.

By this time my family was aware that I was desperate for more sources. In something that felt nothing short of a miracle, an old box with remnants of my grandfathers work surfaced. My parents had gotten it years ago from my aunt and it was standing in a forgotten corner of the attic. The box was filled with a folder with old letters as thick as my thumb, several prints of output of the program and two copies of the report that became the subject of this thesis. My grandfather had written it in 1984, 19 years after the original article, together with an academic assistant of Van Westrhenen, H.L. Claasen. The 52-page long report provided a detailed description of the proof procedure that the 5-page article hinted at. It was this report that allowed me to continue the study of my grandfathers work and allowed me to finish my thesis in a satisfying way.

I want to thank all those I mentioned previously for helping me. I want to thank Dr. J. Spandaw, someone I have not mentioned yet, but who was perhaps the most beneficial in understanding my grandfathers work. I am grateful for his help with understanding projective geometry and embedding my grandfathers work in a larger context. Finally, I want to thank my supervisor, professor J. van Neerven. He never hesitated about supervising an unorthodox project and his enthusiasm was stimulating throughout the project, even when it seemed I would not be able to finish the project.

It has special meaning to me that the thesis for the completion of my bachelor degree is also the completion of a personal project I have had ever since I started studying mathematics. I dare say that it was a unique project and it is intensely satisfying that I could bring it to this conclusion. I hope that the reader will get as much satisfaction from reading this thesis as I got from writing it.

Contents

1 Introduction

This report is a discussion of the 1984 report *An automatic proof procedure for several geometries* by Th. Bruyn and H.L. Claasen (published at the Technical University of Delft) [3]. It was inspired by a personal desire to understand the work of Th. Bruyn (1923 - 1998), as he was the grandfather of the author of this report.

As the title suggests, the report of Bruyn and Claasen describes a method of proving geometrical theorems. This report aims to give a comprehensive explanation of the method described by Bruyn and Claasen, as well as a demonstration of their method.

Bruyn and Claasen prove that certain true propositions of the theory of intersections within the two-dimensional projective geometry over the real numbers can be formulated by use of figures. It is proven that figures obtained by manipulating these figures will also correspond to propositions. The method to do so proves that the obtained propositions are a direct consequence of the original propositions and are therefore proven to be true. One of their main results is to use the theorem of Pappus to generate the theorem of Desargues, thereby proving that Desargues follows from Pappus (something that is well known in projective geometry).

The reader is assumed to be familiar with projective geometry, as for example described by Heyting [8]. In particular; the axioms, the two-dimensional projective geometry over the real numbers (denoted in this report as $PG(2, \mathbb{R})$), the theorem of Pappus and the theorem of Desargues.

1.1 Methodology

This report studies the 1984 report *An automatic proof procedure for several geometries* by Bruyn and Claasen. This is is done by quoting sections of the report of Bruyn and Claasen and providing commentary, examples, illustrations and additional explanations. As it is a summary, not all sections of Bruyn and Claasen are discussed in this report and sections are not always discussed in the order of appearance in Bruyn and Claasen. The quoted sections of Bruyn and Claasen will mainly be definitions and theorems (and their proof).

The full report name and authors will not be mentioned for each citation. The report has a clear section numbering, so each citation will only state the section of the report it refers to. Many of the citations contain definitions or theorems and to represent that, these are written and numbered in the same way a definition or theorem would be. To distinguish between referenced definitions (or theorems) and newly introduced definitions, the referenced definitions are labelled 'Reference' with the section of the original report between brackets. If in the original report a section is labelled (e.g. definition or theorem), then the reference between brackets will reflect this and not state 'Section $i.j.k$' but, for example, 'Definition $i.j.k$'. However, it will become apparent that many sections in the report that are not specifically labelled as definition will still introduce new definitions. As such, a later reference in this report to such a section of the original will still be 'definition $i.j.k$' as it is thought to increase readability in the rest of this report. Such a reference will be the same as to a regularly introduced theorem or definition. I.e. they will not be referenced as 'the definition in section $i.j.k$', but simply as 'definition $i.j.k$'.

All referenced definitions and theorems as mentioned above are literal quotation of the original definitions. The only alterations have been corrections of spelling mistakes and typing errors. If the correction involves a typing error that could in any way be interpreted as altering the meaning of the quote, the correction will be explicitly mentioned.

1.2 A summary of each section of this report

Section 2: The theory of an automatic proof procedure for several geometries

The first two subsections of section 2 of this report will give a general introduction of notions and methods used further on. These subsections correspond directly with the first two sections of Bruyn and Claasen.

After that, the choice has been made to no longer have a single subsection in this report for each section of the report of Bruyn and Claasen. Instead, relevant parts of Bruyn and Claasen have been regrouped with the aim of forming a more coherent build-up to the eventual proof procedure.

Section 2.1 follows section 2 of Bruyn and Claasen. The notions point, line, incidence and figure will be formally introduced. A notion of 'node' is introduced as a name for both points and lines and a classification for the nodes of a figure will be given based on incidences.

Section 2.2 follows section 3 of Bruyn and Claasen. It is a more in depth study of figures. Several methods of representing figures will be introduced; the incidence matrix of a figure, the Levi graph and a graph derived from the Levi graph by splitting specific vertices of the Levi graph. These will be used to define properties of figures, most notably the definition of cells within a figure. A figure with special properties that is the main point of study of Bruyn and Claasen is introduced, which is called a proposition figure.

Section 2.3 combines parts of section 4 and 5 of Bruyn and Claasen. It is described how figures can be constructed, either by use of construction steps based on the axioms of projective geometry or by generating figures out of other figures. Based on the type of construction, a classification of figures is given; a figure is called trivial if its existence is a direct consequence of the axioms of projective geometry and non-trivial if it contains properties beyond those specified by the axioms. A proposition figure will prove to be a special case of the latter type and an in depth study of the construction of proposition figures will follow. It will conclude with an abstract method to generate new (proposition) figures based on given proposition figures.

Section 2.4 is based on sections 6 and 9 of Bruyn and Claasen. This will be the first section to specifically discuss projective geometry. It will specify which plane and theory of projective geometry that will be considered in the eventual proof procedure. The relationship between geometrical theorems and figures is studied. It is proven that a theorem of the theory of intersections in $PG(2, \mathbb{R})$ can be interpreted as a proposition figure (hence the name) and that vice versa a proposition figure can be interpreted as a theorem. The validity of such a figure-generated theorem will be discussed. Finally, it will be proven that under certain conditions the method to generate proposition figures as described in section 2.3 will yield proposition figures corresponding to true theorems.

Section 2.5 will return to section 5 of Bruyn and Claasen. The practical implications and drawbacks of the abstract method proposed in section 2.5 will be discussed. A clear procedure is formalized for generating a new figure out of a proposition figure corresponding to a true theorem of $PG(2, \mathbb{R})$. It will be shown that if the new figure is a proposition figure, it will also correspond to (and therefore prove) another true theorem of $PG(2, \mathbb{R})$.

Section 2.6 is a summary of section 2 and will explicitly list the procedure for the automatic proof procedure developed by Bruyn and Claasen as an algorithm. It does not directly correspond to any section of Bruyn and Claasen.

The astute reader will notice that several sections of Bruyn and Claasen are not mentioned. The choice has been made to focus on the parts that directly contribute to the automatic proof procedure. Section 1 is the introduction. Section 7 is very short and expands on ways to use a ruler in $PG(2, \mathbb{R})$ and is not thought to be of essence for the proof procedure. Section 8 is a discussion on the graphs that can be used to classify figures and is also not incorporated. Parts of sections 10 and 11 are incorporated in the discussion.

Section 3: A demonstration: Proving Desargues out of Pappus

Section 3 is a demonstration of the proof procedure as developed by Bruyn and Claasen, based on the example in section 9 (p.36 - p.45) of their report. It proves the theorem of Desargues out of the theorem of Pappus. It is the only demonstration Bruyn and Claasen give and while it clarifies much of the (often abstract) descriptions mentioned in the theory, it is very concise and many of the steps taken and choices made are not explicitly stated. Fortunately it takes two iterations[1] of the algorithm to prove the theorem, providing some more insight in the procedure.

The first iteration of the algorithm will follow the example with additional descriptions of the steps taken. The choices made by Bruyn and Claasen will be analysed and probable motivations will be considered.

For the second iteration of the algorithm the choice has been made to show it in a different fashion than was done by Bruyn and Claasen. While Bruyn and Claasen state that the aim of the procedure is to be able to use computers, they rely solely on (visual representations of) graphs in their example. They do provide the incidence matrices of the figures used, but do not show how the matrices can be used in their procedure. The second part of section 3 will not use the figures as given by Bruyn and Claasen, but will solely use matrices instead.

Section 4: Discussion

The discussion will include points of discussion of the report as stated by Bruyn and Claasen as well as discussions on this report. The effectiveness of the algorithm will be considered and the points of discussion that are stated by Bruyn and Claasen themselves will be analysed. A study is made of the references by Bruyn and Claasen to future reports, in an effort to determine the aspirations and ideas for development Bruyn and Claasen had at that time. This includes a discussion on results that Bruyn and Claasen claim have already been achieved, but that are not in the report. This will lead to conclusions on possible ways of reconstructing (and perhaps developing) the work of Bruyn and Claasen.

Appendix

Two appendices are included. The first appendix is a reconstruction of the academic life of Th. Bruyn, based on separate interviews with his three children. It mainly discusses the relationship between Bruyn, his co-workers and the academic world.

The second is an analysis of the proposition of Desargues. The proposition is introduced with an intuitive reasoning, after which a formal proof will be given in a Desarguesian plane and a counterexample in a non-Desarguesian plane.

Both appendices are written by the author of this report.

1.3 Conclusion

Bruyn and Claasen formulate an automatic procedure to prove theorems of the theory of intersection within the two-dimensional projective geometry over the real numbers. In the report studied, Bruyn and Claasen focus mainly on the theoretical background to prove that the proofs generated by the procedure are indeed valid. This report is a summary of their work, with added explanations and examples. The eventual procedure is condensed into a seven step algorithm for finding a new true theorem that is a direct consequent of a given true theorem. The demonstration of proving that the theorem of Desargues is a consequent of the theorem of Pappus given by Bruyn and Claasen is expanded with added figures, a detailed explanation of how the process works and an analysis of the choices made by Bruyn and Claasen.

[1] A third iteration is given after proving Desargues out of Pappus (p.45 - p.48). However, it is not stated what the purpose of this iteration is, or what proposition it corresponds to. Therefore it was not deemed to be of any help and has not been included in this report.

2 The theory of an automatic proof procedure

In this section the theory of Bruyn and Claasen will be explained and discussed. This section has roughly the same structure as the report of Bruyn and Claasen. For more on the structure and methodology, see subsection 1.1 and 1.2 of the introduction.

2.1 Starting definitions

As this report is about geometry, many of the notions used will have intuitive meaning. While this has the benefit of facilitating understanding, it also induces the risk of misinterpretation. Therefore this first section will demarcate the exact definitions of many of the notions used further on.

Reference 2.1.1 (Definition 2.1)**.** *Let P and L be two disjoint sets and let $P \times L$ be the Cartesian product of P and L. Let $I \subset P \times L$. The triple (P, L, I) is called an incidence structure. If both P and L are empty, then the incidence structure is called empty. If $n \in P$ and $m \in L$ and $(n, m) \in I$ then n and m are called incident.*
An incidence structure $B = (P^*, L^*, I^*)$ is called a substructure of A if $P^* \subset P$, $L^* \subset L$ and $I^* \subset (P^* \times L^*) \cap I$.

A is not introduced but from context it is taken that $A = (P, L, I)$ is meant.

Reference 2.1.2 (Definition 2.2)**.** *Let $A = (P, L, I)$ be an incidence structure. It is said that A is a geometrical incidence structure (abbreviated to GIS) if*

1. *two different elements of P are incident with at most one element of L;*
2. *two different elements of L are incident with at most one element of P.*

Is A empty then, by definition, A is a GIS.

The reader familiar with projective geometry will recognize the axioms of projective geometry in definition 2.1.2 (two lines are incident with one point and two points are incident with one line). Note that the reverse of definition 2.1.2 is not true: one element of P can be incident with more than two elements of L and vice versa.

Reference 2.1.3 (Section 2.4)**.** *In this report we shall only consider geometrical incidence structures. Let $A = (P, L, I)$ be a GIS. The elements of P are called points, the elements of L are called lines and the elements of I are called incidences.*

Because of the point-line duality in projective geometry, in most statements points and lines can be interchanged. Rather than defining every such statement separately for points and lines, Bruyn and Claasen have chosen to introduce a common notion for both: nodes.

Reference 2.1.4 (Section 2.5)**.** *If $A = (P, L, I)$ then $K = P \cup L$ and the elements of K are called nodes.*

If $n \in P$ and $m \in L$ then the fact that n and m are incident is, by definition, denoted by $(n, m) \in I$. Here (n, m) is an ordered pair. Often, however, we shall denote the same incidence by the unordered pair $[n, m] = [m, n]$. We still say that $[n, m] \in I$.

In this report only finite substructures of a GIS are of interest. Hence the following definition:

Reference 2.1.5 (Section 2.7)**.** *Let $F = (P_0, L_0, I_0)$ be a GIS. It is said that F is a figure if P_0 and L_0 are finite sets.*

A subfigure of a GIS $A = (P, L, I)$ is a figure which is a substructure of A.

A subfigure of a GIS A is sometimes called a figure in A

Cautionary note: In definition 2.1.5 a figure is only defined as a triplet of finite sets, with the only restrictions on these sets being the ones as stated in definition 2.1.2. It might be tempting to apply the intuitive meaning of a figure to definition 2.1.5. This is not entirely misplaced, as it is true that an intuitive figure consists of finite numbers of points, lines and incidences. As this report focusses on $PG(2, \mathbb{R})$, most figures in this report will correspond to the intuitive definition of a figure. However, the reader is advised to keep in mind that definition 2.1.5 and the intuitive definition of a figure are not necessarily the same. For example, an intuitive figure might contain parallel lines, which are not defined in projective geometry. Therefore, an intuitive figure might not be a figure by definition 2.1.5. Vice versa, a projective figure might be over a field that has no two or three dimensional representation and therefore does not correspond to the intuitive definition of a figure.

An important property of a node n in a figure F is the number of other nodes that n is incident with, called the valency of n. Bruyn and Claasen define valency as the following:

Reference 2.1.6 (Section 2.8)**.** *Let F be a figure and let n be a node of F.*
The number

$$v(F, n) = |\{m \in K : [n, m] \in I\}|$$

is called the valency of n (in F).
A figure F is said to be closed if every node of F has a valency in F of at least 3.

Note that the number three in the definition of a closed figure is not arbitrarily chosen here. The fundamental axioms of projective geometry only make statements about a node being incident with two other nodes (a point incident with two lines and a line incident with two points). The axioms allow for the construction of a new node n incident with two old ones l and m, which potentially increases the valency of l and m above two. However, the new node n would only have a valency of two. The existence of a figure where all nodes have a valency of three is therefore far from trivial. This is a first indication of how the valency of the nodes define special properties of a figure. For further study of this, a classification of nodes and incidences based on valency is introduced:

Reference 2.1.7 (Definition 2.9)**.** *Let F be a figure, let n be a node of F and let $[n, m]$ be an incidence of F.*

It is said that

i) n *is **singular (in F)** if* $v(F, n) > 3$;

ii) n *is **normal (in F)** if* $v(F, n) = 3$;

iii) n *is **abnormal (in F)** if* $v(F, n) < 3$;

iv) $[n, m]$ *is **singular (in F)** if both nodes are singular in F;*

v) $[n, m]$ *is **mixed (in F)** if one of the nodes is singular and the other one is normal in F;*

vi) $[n, m]$ *is **normal (in F)** if both nodes are normal in F;*

vii) $[n, m]$ *is **abnormal (in F)** if at least one of the nodes is abnormal in F;*

As further on multiple figures will be studied and compared, it is useful to say something about the similarities and differences of two figures. The logical way of doing this is by defining an isomorphism between two figures, which Bruyn and Claasen define as might be expected:

Reference 2.1.8 (Definition 2.11)**.** *Let $A = (P, L, I)$ and $B = (P_0, L_0, I_0)$ be two incidence structures. A and B are said to be isomorphic ($A \cong B$) if there are bijective mappings of P on P_0 and of L on L_0 such that if $n \in P$ is mapped on $n_0 \in P_0$ and $m \in L$ is mapped on $m_0 \in L_0$ then $(n, m) \in I$ if and only if $(n_0, m_0) \in I_0$.*

2.2 Some aspects of figures and operations to define them

This section corresponds to section 3 of the original report. It defines operations on a figure which can aid in a more in-depth study. This leads to new ways of analysing and classifying figures.

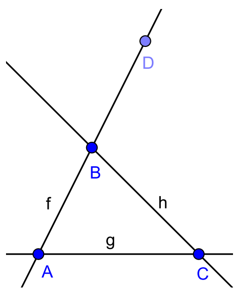
Bruyn and Claasen define a figure F in the beginning of section 3 and use the same figure throughout this section, so section 2.2 of this report will do the same. The figure F is defined as $F = (P, L, I)$, the number of points in P is denoted as $|P| = p$ and the number of lines in L is $|L| = q$. Further it is stated that $pq \neq 0$. To facilitate the reader, an example has been added. To prevent confusion between F and the figure in this specific example, the example figure is denoted by $H = (P_H, L_H, I_H)$ with the set of points $P_H = \{A, B, C, D\}$, the set of lines $L_H = \{f, g, h\}$ and the set of incidence $I_H = \{(A, f), (B, f), (D, f), (A, g), (C, g), (B, h), (C, h)\}$.

2.2.1 The incidence matrix of a figure

To begin with, it is shown how a figure F can be represented by an incidence matrix. It is defined as the following:

Reference 2.2.1 (Definition 3.2)**.** *A $(q \times p)$ matrix M is called an <u>incidence matrix of F</u> if the columns of M are labelled by the points of F and the rows of M are labelled by the lines of F, such that the entry (m, n) of M is 1 if $(n, m) \in I$ and otherwise 0.*

To demonstrate, a representation[2] of H and the corresponding incidence matrix are shown in figure 1.



I_H	A	B	C	D
f	1	1		1
g	1		1	
h		1	1	

Figure 1: The figure H and corresponding incidence matrix.

Note that the zeroes are left out, as is practice in Bruyn and Claasen[3]. The use of an incidence matrix has some great advantages over the set notation of I_H, which might feel cumbersome in comparison to the matrix. For one, Bruyn and Claasen note that the valency of a node is now much more easily determined, as it is equal to the sum of the corresponding row/column.

The triangle in figure 1 is shown in the matrix 1 as the 'ring' of six incidences on the left. A ring like this represents the smallest circuit possible in the matrix. As Bruyn and Claasen note, a rectangular pattern is not possible because of the conditions of a GIS (see definition 2.1.2).

While the nodes are arranged in alphabetical order, this is by no means necessary. It is easily seen that swapping rows of columns does not affect the incidences the matrix represents.

[2]No dimensions or coordinates have been specified in the definition of H and therefore it is impossible to draw figure H. However, as the figure 1 is isomorphic with any possible figure H, it is therefore considered a good representation of H.

[3]Bruyn and Claasen do not actually show an incidence matrix until section 9. However, when they do they leave out all zeroes. This greatly increases oversight for bigger incidence matrices.

2.2.2 Representing a figure as a graph: Levi graphs

Incidence matrices are common when considering finite geometries, but they are also often used in graph theory. Therefore it is not a far stretch to interpret the incidence matrix of F as that of a graph. This would allow the use of graph theory for further investigation of the relation between incidences and nodes. However, it is found that directly using the incidence matrix is not very helpful. A related graph called a Levi graph has proven to be more useful.

For the following definition, the reader is reminded that in definition 2.1.4 the collections of all nodes is denoted by $K = P \cup L$. Furthermore, the classification of nodes and incidences in definition 2.1.7 was taken from section 2.9 of the original report.

Reference 2.2.2 (Definition 3.6)**.** *The Levi graph $\Lambda(F)$ is the graph (K, I)*

That is: the node set of $\Lambda(F)$ are the nodes of F and the edge set of $\Lambda(F)$ are the incidences of F: two nodes of $\Lambda(F)$ are connected if and only if the two nodes in F are incident.

The edges of $\Lambda(F)$ are called incidences and we use the same characterisations for the nodes and the incidences of $\Lambda(F)$ as we did for the nodes and the incidences of F in (2.9).

The Levi graph of $\Lambda(F)$ is obtained by turning the incidence matrix of F into an adjacency matrix in the following way:

Reference 2.2.3 (Section 3.9)**.** *Let M be an incidence matrix of F and $\Lambda(F)$ the Levi graph of F. Let X be the following block matrix:*

$$X = \left(\begin{array}{c|c} 0_{q,q} & M \\ \hline M^T & 0_{p,p} \end{array} \right)$$

Then, of course, the $(p+q) \times (p+q)$ matrix X is an adjacency matrix of $\Lambda(F)$.

The Levi graph immediately shows the choice of word in definition 2.1.4: a node in a figure corresponds with a node of the Levi graph.

Bruyn and Claasen reference Levi [9] and Coxeter [5] for more information on Levi graphs.

In figure 2 this is shown for the example figure H. In the upper row of figure 2, the adjacency matrix obtained from the incidence matrix and the corresponding Levi graph is shown. In the lower row of figure 2 the rows and columns of the adjacency matrix have been rearranged. It is easily verified that no incidences have been changed in rearranging the matrix. While it is not necessary to rearrange the matrix, it makes it easier to see components and circuits in the graph. For example, the circuit corresponding to the initial triangle is clearly visible in figure 2d. While Bruyn and Claasen never explicitly mention rearranging the matrix, the example of section 9 clearly shows they do[4] and therefore the choice has been made to do the same here. Bruyn and Claasen also do not give a method to rearrange the nodes, but the best visual representation requires as many neighbouring nodes as possible to be connected. Therefore, if a Hamilton cycle exists, this is the preferred ordering. If not, a Hamilton path (as is the case with H) is the preferred ordering and otherwise simply the largest path possible.

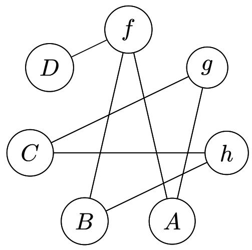
2.2.3 Derived graphs

As mentioned before, in the rearranged Levi graph in figure 2d the circuit of $\Lambda(F)$ represents the triangle in figure 1 and node D the single point not in the circuit. Purely based on this observation, it might be very informally said that D is 'different' from the other points. The eventual proof procedure relies on finding such (as of yet unspecified) 'differences' in figures and therefore a formal classification will be given. To do so, the graph will be 'split' in two components, one containing the circuit and one containing node D.

[4] It is easily seen that when the method described in section 2.2.3 is applied to the incidence matrix in section 9.15 of the original report the Levi graph obtained is different from the one shown in section 9.16 of the original report.

$\Lambda(H)$	f	g	h	A	B	C	D
f				1	1		1
g				1		1	
h					1	1	
A	1	1					
B	1		1				
C		1	1				
D	1						

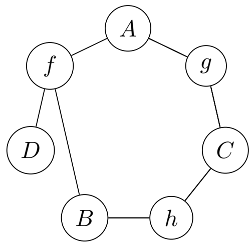
(a) The adjacency matrix of $\Lambda(H)$.



(b) The Levi graph $\Lambda(H)$.

$\Lambda(H)$	A	g	C	h	B	f	D
A		1				1	
g	1		1				
C		1		1			
h			1		1		
B				1		1	
f	1				1		1
D						1	

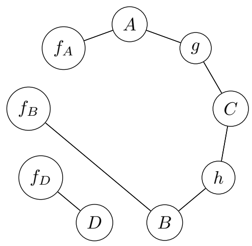
(c) The rearranged adjacency matrix of $\Lambda(H)$.



(d) The rearranged Levi graph $\Lambda(H)$.

$\Gamma^{(f)}(H)$	f_A	A	g	C	h	B	f_B	f_D	D
f_A		1							
A	1		1						
g		1		1					
C			1		1				
h				1		1			
B					1		1		
f_B						1			
f_D									1
D								1	

(e) The adjacency matrix of $\Gamma^{(f)}(H)$.



(f) The derived graph $\Gamma^{(f)}(H)$.

Figure 2: The adjacency matrix and Levi graph in two different arrangements, and the split graph (with node f split).

An intuitive way of interpreting the splitting of a node n, is to imagine that n is split such that each node m that n is incident with 'gets a part' of n. Each m is now connected to its own part of n, but the separate parts of n are not connected. This is formalized in the following way:

Reference 2.2.4 (Section 3.11)**.** *Let $\Gamma = (V, E)$ be a simple graph. Let n be a node of Γ of valency v. Let $[n, m_i]$ $(i = 1, 2, ..., v)$ be the edges to which n belongs.*

Now replace in V the node n by v new nodes $n_1, n_2, ..., n_v$ and replace in E the incidences $[n, m_i]$ by $[n_i, m_i]$ $(i = 1, 2, ..., v)$.

Then it is said that n is <u>split</u>. The graph

$$\Gamma^{(n)} = \Big(V \backslash \{n\} \cup \{n_1, n_2, ..., n_v\}, \Big(E \backslash \bigcup_{i=1}^{v} [n, m_i]\Big) \cup \bigcup_{i=1}^{v} [n_i, m_i]\Big)$$

is called the <u>graph derived from Γ by splitting n</u>.
n is said to be the <u>stem of the (split) nodes</u> $n_1, ..., n_v$ of $\Gamma^{(n)}$. $n_1, n_2, ..., n_v$ are called the <u>representatives of (the stem) n</u>.

This method applied to the example H, resulting in $\Gamma^{(f)}(H)$, is shown in figure 2f. The only node of $\Lambda(H)$ with a valency of three, f, has been split in nodes f_A, f_B, f_D according to its incidences. It is clear that $\Gamma^{(f)}(H)$ has two components. This can be seen in the adjacency matrix in figure 2e as well. Each incidence of the split node f has been placed in a row/column of its own, creating two additional rows and columns in the matrix. The incidences (D, f_D) have become the only one in their row/column.

All the nodes of the circuit are still connected, only the 'different' node D has been separated from the rest. The undefined 'difference' mentioned before can now be described by saying that in the derived graph $\Gamma^{(f)}(H)$ the node D belongs to a different component.

For the eventual proof procedure, not the nodes with a valency of three will be split, but the nodes with a valency of four or higher (the nodes classified as singular in definition 2.1.7). Hence the following definition:

Reference 2.2.5 (Definition 3.12). *Let F be a closed figure and $\Lambda(F)$ its Levi graph.*

By $\Gamma(F)$ we denote the graph which is derived from $\Lambda(F)$ by splitting the singular nodes of $\Lambda(F)$.

$\Gamma(F)$ is called the derived graph of F.

Bruyn and Claasen state that in general the derived graph $\Gamma(F)$ will not be connected. They note that for example for a singular incidence $[n, m]$ the graph $\Gamma(F)$ will have a component containing only a single incidence $[n_1, m_1]$[5]. This explains why it has been chosen to name these nodes singular.

2.2.4 Proposition figures

If the derived graph $\Gamma(F)$ is connected it has special properties. These figures will be the main point of focus of this report and the eventual proof procedure. In his 1965 article [1], Bruyn calls these figures super-cells. In the report that is currently being discussed, Bruyn and Claasen have a more specific (but not less meaningful) name for these kinds of figures:

Reference 2.2.6 (Definition 3.14). *Let F be a closed figure. F is said to be a proposition figure if $\Gamma(F)$ is connected.*

The word *proposition* has a far reaching mathematical implication and it is certainly not lightly chosen here. As the name implies, it is at the heart of the proof generating procedure that Bruyn and Claasen have developed. Its many interesting properties will be studied in the rest of this report.

2.2.5 Cells

The original figure F can be reconstructed from $\Gamma(F)$. This allows for the construction of the derived graph $\Gamma(F)$ of a figure F, the study of its components and then the use of the components to clearly define properties of F. More precisely, as each incidence of F belongs to one and only one component of $\Gamma(F)$, this will be used to classify subsets of F. Definition 2.2.7 will address the process of constructing F out of $\Gamma(F)$, definition 2.2.8 will classify subsets of I according to the components of $\Gamma(F)$ and definition 2.2.9 will use this to classify subsets of F.

For the next definitions, note that section 3.12 of the original report refers to the derived graph described in definition 2.2.5 and section 3.11 refers to splitting a graph as described in definition 2.2.4. Definition 3.14 refers to the definition of a proposition figure above.

Reference 2.2.7 (Section 3.16). *Given a closed figure F the transition of F to $\Gamma(F)$ is defined in (3.12) and (3.11).*

As in 3.14 we shall use the graph $\Gamma(F)$ to define several properties of F itself. So we sometimes have to translate properties of $\Gamma(f)$ into properties of F. This, however, comes naturally if one adds a node n' of $\Gamma(F)$ to the node n of F if n' is a representative of the (singular) node n and one adds n to itself if n is normal.

[5] This can be easily seen. If node $[n, m]$ is singular that means that both n and m are singular. Therefore they will both be split. The incidence $[n, m]$ will become a connection between two split nodes and therefore the component will contain only a single incidence.

Reference 2.2.8 (Section 3.18)**.** *Let F be a closed figure and $\Gamma(F)$ its derived graph.*

The components of $\Gamma(F)$ define an equivalence relation on the incidences of $\Gamma(F)$; that is, on the incidences of I using the correspondence described in (3.16).

A non-empty class of this equivalence relation is said to be a cell of F (so a cell is a subset of I).

We say that a node n of F belongs to a cell C of F if n (or one of its representatives) in $\Gamma(F)$ belongs to the component of $\Gamma(F)$ corresponding to C.

It can be pointed out that the components of $\Gamma(F)$ not only define an equivalence relation on the incidences of $\Gamma(F)$, but are actually the same thing. This makes definition 2.2.8 seem superfluous. It might be a mistake on behalf of Bruyn and Claasen, where they actually meant that the components of $\Gamma(F)$ define an equivalence relation on the incidences of the original figure F, instead of on the incidences of $\Gamma(F)$. This is supported by the reference to section (3.16) (definition 2.2.7), which is about how figure F can be reconstructed from $\Gamma(F)$ to define properties of figure F by use of $\Gamma(F)$, in this case to group the incidences of F into cells. However, there is not enough evidence in the report to be certain that this is what Bruyn and Claasen meant and the distinction is not very relevant in the use of cells further on in the report.

Reference 2.2.9 (Definition 3.20)**.** *Let C be a cell of a closed figure $F = (P, L, I)$. Let P^C be the set of points belonging to C and L^C the set of lines of L belonging to C then $F^C = (P^C, L^C, I^C)$. F^C is said to be a component of F.*

A component F^C of F is said to be singular if $|C| = 1$ and normal if $|C| \geq 3$.

Note that a cell C is per definition a subset of I. Therefore the use of I^C in definition 2.2.9 seems superfluous. It is assumed that Bruyn and Claasen do this for unity of notation, but that $I^C = C$ (however, Bruyn and Claasen are not entirely consistent with this, see next definition).

If a component is defined, it seems logical to also define a complement.

Reference 2.2.10 (Definition 3.22)**.** *Let C be a cell of the closed figure $F = (P, L, I)$, let P_C be the set of lines belonging to $I \backslash C$ and let L_C be the set of lines belonging to $I \backslash C$ then F_C is the figure $F_C = (P_C, L_C, I \backslash C)$.*

F_C is said to be the complement of F^C (with respect to F).

If the figure F contains k cells ($k \geq 1$) then Bruyn and Claasen denote the different cells as $C(i)$ ($i \in 0, .., k-1$), rather than C_i. As it is more common to use subscript notation when defining subsets, this might lead to some confusion. However, as Bruyn and Claasen stick to this notation, so will this report. A possible explanation for the choice of notation is that the notations of component and complement involve superscript and subscript and therefore the use of these in the notation of cells might get confusing.

These definitions can be easily applied to the example H (except that with H the nodes of valency three were split instead of with valency four or higher). The cells $C(1)$ and $C(2)$ of H contain the incidences of the different components of $\Gamma^{(f)}(H)$. The corresponding components $H^{C(1)}$ and $H^{C(2)}$ of H contain the points and lines belonging to the components. The cells and components are listed below. Note that while the cells are disjunct, the components are not. The split node f belongs to both components.

$$C(1) = \{(D, f)\}$$
$$C(2) = \{(A, f), (B, f), (A, g), (C, g), (B, h), (C, h)\}$$

$$H^{C(1)} = \Big(\{D\}, \{f\}, \{[D, f]\}\Big)$$
$$H^{C(2)} = \Big(\big\{A, B, C\big\}, \big\{f, g, h\big\}, \big\{(A, f), (B, f), (A, g), (C, g), (B, h), (C, h)\big\}\Big)$$

$$H_{C(1)} = H^{C(2)}$$
$$H_{C(2)} = H^{C(1)}$$

2.2.6 A specialization of the theorem of Desargues: an example of a derived graph

For a better understanding of the relation between proposition figures, derived graphs and cells, Bruyn and Claasen describe a specialization of the theorem of Desargues in section 3.15 of the report. This section will follow the idea of section 3.15, but with figures added and a notation deemed more consistent with the notation used further on in the report (in particular figure 9.32 of Bruyn and Claasen and figure 5b in this report).

The theorem of Desargues states the following:

Theorem 2.2.11 (Desargues)**.** *Let O, A_1, A_2 and A_3 be different points with each two point connected by a line. Let there be a point B_i ($i \in \{1,2,3\}$) on each line OA_i ($i \in \{1,2,3\}$). Then the points $C_1 = A_2A_3 \cap B_2B_3$, $C_2 = A_1A_3 \cap B_1B_3$ and $C_3 = A_1A_2 \cap B_1B_2$ are collinear.*

A representation D of the theorem of Desargues and the corresponding Levi graph $\Lambda(D)$ are shown in figure 3. It is easily verified that all nodes have a valency of 3. As there are no singular nodes, the derived graph $\Gamma(D)$ is the same as $\Lambda(D)$. It is clear that $\Gamma(D)$ is connected and this proves that D is a proposition figure.

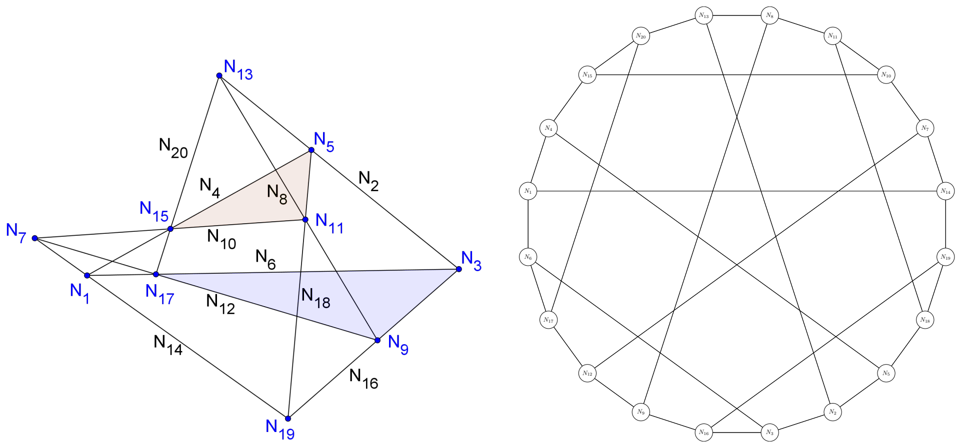
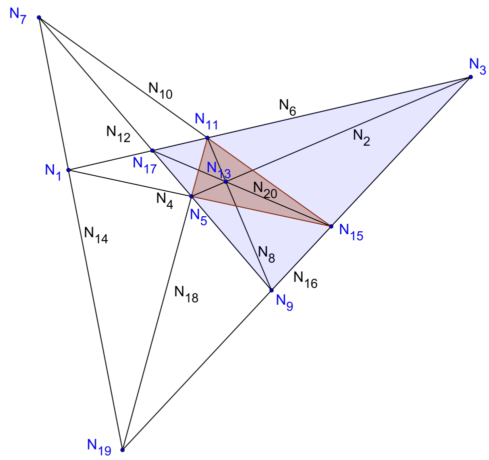


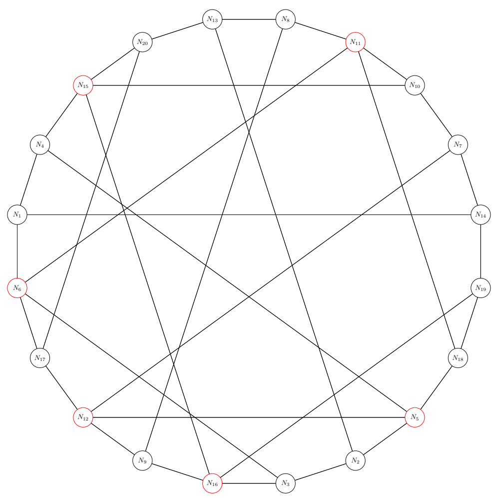
Figure 3: The configuration of the theorem of Desargues and the corresponding Levi graph.

The scenario where the points of triangle A lie on the sides of triangle B is considered a specialization of the theorem of Desargues. A representation, denoted S, can be seen in figure 4a. The red triangle A now lies inside the blue triangle B with the points of A on the sides of B. Because of this, six nodes now have a valency of 4: $\{N_5, N_6, N_{11}, N_{12}, N_{15}, N_{16}\}$, the points of triangle A and the sides of triangle B.

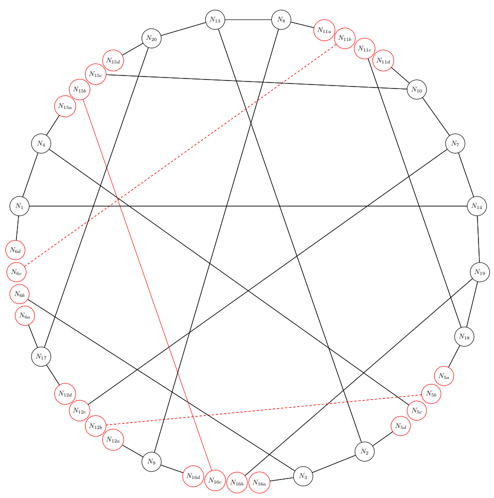
In the Levi graph $\Lambda(S)$ depicted in figure 4b these singular nodes have been coloured red. In figure 4c these nodes are split. The incidences between two split nodes are coloured red. Each incidence between



(a) The specialization of Desargues S.



(b) The Levi graph $\Lambda(S)$.



(c) The derived graph $\Gamma(S)$.

Figure 4: A specialization S of the theorem of Desargues with the Levi graph and derived graph.

two split nodes is per definition a singular cell of S. It is seen that S has three singular cells and one cell of which all the nodes (except for the split ones) have a valency of three. It is clear that S is not a proposition figure. However, when comparing the derived graph of figure 4b and the Levi graph of figure 3, it is seen that the non-singular cell of S is isomorphic to the one cell of D. S might not be a proposition figure, but one of it's cells is.

This example shows how derived graphs can be used to define properties of figures and how it can be used to distinguish a specialization of a theorem from a general theorem. It also gives an inclination of the relation between proposition figures and geometrical propositions/theorems.

2.3 Construction of figures

In section 2.2 the properties of a figure have been studied. This section will introduce a way to construct a figure F out of another figure $F(0)$ (or, if $F(0)$ is empty, introduce figure F). In the introduction Bruyn and Claasen describe section 2, 3 and 4 as a 'torrent of definitions' and nowhere that is more apt than in section 4 of Bruyn and Claasen. Definition 4.3 alone introduces no less than seven new notions. Definition 4.3 is the heart of section 4, the rest of the section mostly consists of remarks on definition 4.3. Therefore section 2.3 will give a detailed explanation of each section of the definition. First the entire definition 4.3 will be given in reference 2.3.1.

2.3.1 Definition of a construction

Reference 2.3.1 (Definition 4.3)**.** *Let $F(0) = (P_0, L_0, I_0)$ and $F = (P, L, I)$ be figures then it is said that*

a) F can be <u>elementary constructed</u> from $F(0)$ if either
$F = F(0)$ or

  a1) there is a node n such that

$$(P \cup L)\backslash(P_0 \cup L_0) = \{n\}$$

  and

  a2) there are at most two nodes m_1, m_2 in $P_0 \cup L_0$ such that $[n, m_t] \in I$ $(t = 1, 2)$;

b) F can be <u>trivially constructed from $F(0)$</u> if there is a finite sequence of figures $F(0), F(1), ..., F(t)$ such that

  b1) $F(t) = F$ and

  b2) if $t > 0$, $F(s + 1)$ can elementary be constructed from $F(s)$ $(S = 0, 1, ..., t - 1)$.

c) F is a <u>trivial figure</u> if F is trivially constructed either from the empty figure or from a trivial figure;

d) the sequence $F(0), F(1), ..., F(t)$ in b) is a <u>construction sequence between $F(0)$ and F</u>;

e) a sequence of figures $F(0), F(1), ..., F(t)$ is a <u>construction sequence (for $F(t)$)</u> if it is a construction sequence between the empty figure $F(0)$ and the (trivial) figure $F(t)$;

f) F can be <u>non-trivially constructed from $F(0)$</u> if there is an incidence i in I such that $(P, L, I\backslash\{i\})$ can be trivially constructed from $F(0)$ (F is called a <u>non-trivial figure</u>);

g) if in b) or f) $F(0)$ is the empty figure we say that F <u>can be (non-)trivially constructed</u>.

Two more definitions are useful to facilitate discussion on definition 2.3.1: the degree of freedom of a figure and the construction step.

Reference 2.3.2 (Definition 4.2)**.** *The number $\Phi(F) = 2(|P| + |L|) - |I|$ is said to be <u>the degree of freedom of F</u>.*

Note that the total number of nodes $|K|$ is defined as $|K| = |P| + |L|$. Therefore the degree of freedom $\Phi(F)$ can also be written as $\Phi(F) = 2|K| - |I|$.

Reference 2.3.3 (Definition 4.5)**.** *The composition of a figure F from a figure $F(0)$ which can be (non-)trivially constructed from $F(0)$ is sometimes called a <u>((non-)trivial) construction (of F from $F(0)$; of F (if $F(0)$ is empty))</u>.*

If F is elementary constructed from $F(0)$ and $F \neq F(0)$ then the construction of F from $F(0)$ is sometimes called a <u>construction step</u>.

2.3.2 Elementary construction steps

To begin to understand the why of the definition of an elementary construction, consider a construction step (the case where $F = F(0)$ is not of interest). A construction step involves the addition of a single node n to $F(0)$ that only uses the fundamental axioms of projective geometry.

The condition of 2.3.1-a1) that $(P \cup L)\backslash(P_0 \cup L_0) = \{n\}$ means a construction step from $F(0)$ to F can only introduce one single node n to $P_0 \cup L_0$. The reason for condition 2.3.1-a2) is that the fundamental axioms of projective geometry are statements about only two nodes (two lines have only one point of intersection and two points have only one connecting line). It is trivial that a node n can be introduced that is not incident with any node of $P_0 \cup L_0$. The node could be placed anywhere in the projective field, as can be demonstrated by the fact that n has two degrees of freedom[6]. It is also obvious that a node n can be introduced that is incident with one other node. This places a slightly bigger restriction, but n still has a single degree of freedom. The fundamental axioms state that it is possible to introduce a node n incident with two other nodes. The position of n is uniquely determined and the degree of freedom of n is zero.

Another way to explicitly define the different possible construction steps is stated in theorem 2.3.4.

Theorem 2.3.4. *A construction step to construct $F = (P, L, I)$ out of $F(0) = (P_0, L_0, I_0)$ can only be one of the following six:*

I) A line l is introduced to L_0 such that:

- *i) l is not incident with any point in P_0.*
- *ii) l is incident with one point in P_0.*
- *iii) l is incident with two points in P_0.*

II) A point p is introduced to P_0 such that:

- *i) p is not incident with any line in L_0.*
- *ii) p is incident with one line in L_0.*
- *iii) p is incident with two lines in L_0.*

Proof. By definition of a construction step, $F(i+1) \neq F(i)$. Therefore there is a node n such that $(P \cup L)\backslash(P_0 \cup L_0) = \{n\}$. Expanding on that, it is found that:

$$\begin{array}{lrl} & (P \cup L)\backslash(P_0 \cup L_0) & = \{n\} \\ \Leftrightarrow & P \cup L & = \{n\} \cup (P_0 \cup L_0) \\ \Leftrightarrow & P \cup L & = \Big(\{n\} \cup P_0\Big) \cup \Big(\{n\} \cup L_0\Big) \end{array}$$

As P and L are disjunct, this means that $P = \{n\} \cup P_0$, in which case n is renamed p, or $L = \{n\} \cup L_0$, in which case n is renamed l. There are at most two nodes m_1, m_2 in $P_0 \cup L_0$ such that $[n, m_t] \in I$ $(t = 1, 2)$. Combined with the previous step, this means that if $n = l$, there are either no points in P_0 incident with l, there is one point $p_1 \in P_0$ that is incident with l or there are two points $p_1, p_2 \in P_0$ that are incident with l. The same reasoning can be applied when $n = p$. □

2.3.3 Trivial and non-trivial figures

As these construction steps follow directly from the fundamental axioms of projective geometry, it explains why in 2.3.1-c) a figure constructed in such a manner is called a trivial figure. After all, if the existence of each construction step is trivial, then the existence of a figure constructed in such a manner is as well. Note that this does not mean that the shape and/or size of the figure are trivial. Each construction

[6]The degree of freedom of a node n can be determined by the difference between the degree of freedom of F and the degree of freedom of $F(0)$ and $\Phi(F) = 2|K| - |I| = 2(|K_0| + 1) - (|I_0| + 0) = 2|K_0| - |I_0| + 2 = \Phi\big(F(0)\big) + 2$.

sequence produces a specific figure (up to isomorphisms). It only means that the existence of such a figure is trivial.

This immediately points towards the why of the definition of a non-trivial figure in 2.3.1-f). Let $F = (P, L, I)$ be a figure and let the incidence $i \in I$ be chosen such that $F(t) = (P, L, I \backslash \{i\})$ is a trivial figure. A construction step always involves the addition of a node, while the construction of F out of $F(t)$ only involves adding an incidence. In other words, the construction step from $F(t-1)$ to $F(t)$ caused another incidence than the two[7] that it was defined by. The occurrence of such an additional incidence is far from trivial, hence the name of such a figure.

2.3.4 Reduction of a figure

If a figure F can be constructed from $F(0)$ it might be expected that this process can also be reversed and that figure F could be reduced to figure $F(0)$. This is indeed the case and will prove useful on occasion. Definition 2.3.5 formalizes this. Note that so far $F(0)$ has denoted the starting figure and F the constructed figure. As the process is reversed in definition 2.3.5, $F(0)$ will denote the constructed figure and F will denote the figure that it is reduced to.

Reference 2.3.5 (Definition 4.4)**.** *Let $F(0)$ and F be figures then it is said that*

a) <u>*$F(0)$ can be elementary reduced to F*</u> *if $F(0)$ can be elementary constructed from F;*

b) <u>*$F(0)$ can be (non-)trivially reduced to F*</u> *if $F(0)$ can be (non-)trivially constructed from F;*

c) the sequence of figures $F(0), F(1), ..., F(t) = F$ is <u>*a reduction sequence between $F(0)$ and F*</u> *if $F(t), F(t-1), ..., F(0)$ is a construction sequence between F and $F(0)$.*

d) the sequence of figures $F(0), F(1), ..., F(t)$ is a <u>*reduction sequence (for $F(0)$)*</u> *if $F(t), F(t-1), ..., F(0)$ is a construction sequence (and therefore $F(t)$ is the empty figure).*

2.3.5 Constructing a proposition figure

When the notion of a closed figure was introduced in definition 2.1.6, it was remarked that the existence of a figure where all nodes have valency of three is far from trivial. That was the first hint that every closed figure is a non-trivial figure. Section 2.2 ended with the introduction of a special closed figure, the proposition figure. The special properties of a proposition figure F are also found in the way it is constructed. Not only is there a specific incidence $i \in I$ such that $F = (P, L, I \backslash \{i\})$ is a trivial figure, it holds true for every incidence in I. This is one of the defining aspects of a proposition figure as this property will later on be used to link a proposition figure to a 'real' proposition in $PG(2, \mathbb{R}$, something that is essential in the formulation of the proof procedure.

The theorem and the proof as stated by Bruyn and Claasen is found below:

Reference 2.3.6 (Theorem 4.8)**.** *Let $F = (P, L, I)$ be a proposition figure and i an arbitrarily chosen incidence of F. Let $F_i = (P, L, I \backslash \{i\})$ then F_i is a trivial figure. F is a non-trivial figure.*

Proof. The whole proof can perhaps best be followed in $\Gamma(F)$.

Let $i = [n, n_0]$. Without loss of generality we can suppose that n is normal[8].

We compose a reduction sequence for F_i. Once this is done we can consider the "inverse" of this sequence as a construction sequence for F_i which constructs F_i (upto isomorphisms, of course). The addition of i to $I \backslash \{i\}$ completes then the non-trivial construction of F; so F is non-trivial.

The reduction of F_i consists of reduction steps of the following form:

[7] A construction step that adds a node n with a valency of one will never have this property, as the incidence i increases the valency of n by only one. This would give node n a valency of two and therefore F would still be a trivial figure.

[8] As $\Gamma(F)$ is connected, there can be no singular incidences in I. This means that every incidence has at least one normal node.

a) leave out an abnormal node p;

b) leave out the incidences (if there are any) to which p belongs.

We keep this process going on as long as there are still abnormal nodes.

Because F_i is no longer closed there is at least one abnormal node (e.g. n). So the process can start.

Every reduction step "creates" at least one new abnormal node, because every incidence is either normal or mixed.

Let m be any normal node of F. Let $[n, n_1], [n_1, n_2], ..., [n_s, n_{s+1}], ..., [n_t, m]$ be a path in $\Gamma(F)$ between n and m (such a path exists because $\Gamma(F)$ is connected). The step which leaves out n (which is certainly abnormal) makes n_1 abnormal and therefore n_1 is left out during the reduction process. The same applies to every node n_s of this path. hence m becomes eventually abnormal itself and will be left out before the reduction process stops.

If m is a singular node of F with valency v then there are v paths in $\Gamma(F)$ between n and the v representatives of m. Every representative of m is incident with a normal node which becomes abnormal during the reduction process, hence during that process the valency of m decreases from v to at most 2, but then m itself has become abnormal and will also be left out, before the reduction process stops.

So clearly the above mentioned reduction process only stops when no nodes are left. This implies the theorem. □

2.3.6 Transformation figures

This once again shows that a proposition figure has special, useful properties. As has been noted before, they are the main point of interest of the report of Bruyn and Claasen. It would be useful to have a way to construct proposition figures. A construction sequence is not of use here, as it can only be used to construct trivial figures. Therefore, Bruyn and Claasen describe a different way of constructing figures, which they call 'generating' figures. The idea is to combine two proposition figures into a new figure, called a transformation figure, containing multiple cells. These cells can then be used to find a new figure in the transformation figure.

Reference 2.3.7 (Definition 5.3)**.** *Let A be a closed figure with at least three cells. Let there be two cells $C(1)$ and $C(2)$ and two proposition figures $F(1)$ and $F(2)$ such that $A_{C(t)} \cong F(t)$ $(t = 1, 2)$.*

Let $C(0) (\neq C(1), C(2))$ be a cell of A then $A_{C(0)}$ is said to be <u>elementary generated by $F(1)$ and $F(2)$</u>.

If $F(1) \cong F(2)$ then $A_{C(0)}$ is said to be <u>elementary generated by $F(1)$</u>.

A itself is said to be <u>a transformation figure</u>.

The choice of word in 'elementary generated' seems to imply a similarity with 'elementary constructed' in definition 2.3.1-a). Just as an elementary construction is one step in a construction sequence, an elementary generated figure can be part of a sequence of figures.

Reference 2.3.8 (Definition 5.6)**.** *Let $t \in \mathbb{N} \backslash \{0, 1, 2\}$.*

Let $\big(F(1), F(2), ..., F(t)\big)$ be an ordered t-tuple of proposition figures (some of which may be isomorphic) with the following properties:

1) $F(3)$ is elementary generated by $F(1)$ and $F(2)$;

2) if $t > 3$, for $4 \leq s \leq t$ there are an s_1 and s_2, $1 \leq s_1 \leq s_2 \leq t-1$, such that $F(s)$ is elementary generated by $F(s_1)$ and $F(s_2)$;

Then $F(t)$ is said to be <u>generated by $F(1)$ and $F(2)$</u> (or <u>by $F(1)$</u> if $F(1) \cong F(2)$).

Bruyn and Claasen dedicate the next section to studying some aspects of transformation figures. The first two of the three statements in reference 2.3.9 can be regarded as corollaries of definition 2.3.7

and are almost trivial. The third observation, however, is far from trivial and shows one of the most important aspects of a transformation figure: A can be non-trivially constructed for *any* cell C of which the complement A_C is a proposition figure. This property will prove essential in later sections.

Note that section 4.8 refers to theorem 2.3.6.

Reference 2.3.9 (Section 5.5)**.** *We need a closer look into the structure of a transformation figure A.*

First, for any incidence i of A we have either $i \in A^C$ or $i \in A_C$ for any cell C of A (by definition of course).

Second, the normal and the split nodes of $A^{C(2)}$ (in the notation of (5.3)) all are nodes of $A_{C(1)}$ (and also the nodes of $A^{C(1)}$ are nodes of $A_{C(2)}$).

Third, A can be non-trivially constructed from A_C for any cell C such that A_C is a proposition figure. this can be seen as follows.

Consider $\Gamma(A)$, let $C(s)$ $(s = 0, 1, ..., t)$ be the cells of A and let $\Gamma(s)$ be the component of A corresponding to $C(s)$ $(s = 0, 1, ..., t)$. Let $C(0) = C$.

If we leave out $\Gamma(0)$ then the rest of $\Gamma(A)$ becomes the connected graph $\Gamma(A_{C(0)})$. There is at least one (singular) node n belonging to $\Gamma(0)$ as well as to $\bigcup_{s=1}^{t} \Gamma(s)$, for A has at least 3 cells and $\Gamma(A_{C(0)})$ is a connected graph.

Because $\Gamma(0)$ is connected there is a trivial construction sequence Σ, starting with n, which constructs $A^{C(0)}$ (apart from perhaps one incidence, if $A^{C(0)}$ itself is closed).

Now let us begin with $A_{C(0)}$ and let us augment $A_{C(0)}$ using the construction sequence Σ. Because A itself is closed but $\Gamma(0)$ is connected we still need exactly one non-trivial construction step to complete A. This can be shown by a reasoning which is similar to the one we used in the proof of (4.8).

We see therefore that indeed A can be non-trivially constructed from $A_{C(0)}$.

Definition 2.3.7 and corollary 2.3.9 both show the importance of cells whose complements in A are proposition figures. Therefore if figure $A_{C(0)}$ generated by proposition figures $F(1)$ and $F(2)$ is also a proposition figure, it is doubly special: not only can A be non-trivially constructed from $A_{C(0)}$, it also means A can be used to generate new proposition figures. Therefore Bruyn and Claasen have a special name for a transformation figure that generates a proposition figure:

Reference 2.3.10 (Definition 5.4)**.** *In the notation of (5.3). The transformation figure A is said to be a deduction figure if there is a cell $C \neq C(1), C(2)$ such that A_C is a proposition figure.*

As with a proposition figure, the name 'deduction figure' seems to imply a meaning beyond a geometrical figure. Deduction figures are connected to geometrical proofs in the same way that proposition figures are connected to geometrical propositions. The next section will expand on these connections.

2.4 The relationship between theorems and figures

The ultimate purpose of defining a deduction figure is to use it to 'deduce' a geometrical theorem $T(3)$ from two other geometrical theorems $T(1)$ and $T(2)$, thereby proving that $T(3)$ follows from $T(1)$ and $T(2)$. However, a theorem is obviously not the same as a figure. Therefore in this section the relation between theorems and figures is investigated. This section is based on parts of section 6 and section 8 of Bruyn and Claasen.

2.4.1 A demarcation of the projective plane used in Bruyn and Claasen

As this section will be about geometrical proofs in projective geometry, it is important to first demarcate the projective plane that is being talked about and the theories that will be considered. Bruyn and Claasen state the following:

Reference 2.4.1 (Section 6.6)**.** *In this report we not only restrict ourselves to $PG(2,\mathbb{R})$, but moreover to the so-called* <u>*theory of intersections in $PG(2,\mathbb{R})$*</u>*.*

While the reader is assumed to be familiar with projective geometry, it is never the less useful to specify what exactly is meant by $PG(2,\mathbb{R})$ and the theory of intersections.

For the definition of $PG(2,\mathbb{R})$ below, Bruyn and Claasen refer to the book Theory of Groups by M. Hall jr. [6].

Reference 2.4.2 (Section 6.4)**.** *As is known (see Hall's book) every projective plane can be coordinized by a so-called ternary ring. This ternary ring uniquely determines the plane.*

As one adds properties to the ternary ring the projective plane gets a richer structure.

In this report we consider the projective plane the ternary ring of which is the real field $\mathbb{R}$. In other words we consider the ordinary 2-dimensional projective geometry over the reals. We denote this projective geometry by $PG(2,\mathbb{R})$.

Reference 2.4.3 (Section 6.5)**.** *Of course, this synthetic definition of $PG(2,\mathbb{R})$ implies the following analytic description of $PG(2,\mathbb{R}) = (P, L, I)$*

A point (of P) is a class op proportional (3×1)-matrices over $\mathbb{R}$ $X = (x_1, c_2, x_3)^T \neq (0,0,0)^T$ and a line (of L) is a class of proportional (1×3)-matrices over $\mathbb{R}$ $U = (u_1, u_2, u_3) \neq (0,0,0)$. $(X, U) \in I$ if and only if $UX = u_1x_1 + u_2x_2 + u_3x_3 = 0$.

In $PG(2,\mathbb{R})$ the theorem of Pappus and the principle of duality hold.

The coordinate system specified in definition 2.4.3 is not mentioned again by Bruyn and Claasen, and it will not be mentioned again in this report. However, it was deemed useful to mention to provide a clear perspective on the geometry used. Furthermore, the opening statement that $PG(2,\mathbb{R}) = (P, L, I)$ and the closing statement that in $PG(2,\mathbb{R})$ the theorem of Pappus hold will both prove useful.

Bruyn and Claasen also specify what is meant with the theory of intersections. A general definition of the theory of intersections is given, as well as what this means in regard to $PG(2,\mathbb{R})$. Bruyn and Claasen do not give a specific reference as to the source of these statements.

Note that section 3 of Bruyn and Claasen was described in section 2.2 of this report. Section 7 of Bruyn and Claasen is about how a ruler and compass can be used to construct figures. Its relevance to the bigger storyline was deemed minimal and it is therefore not described in this report. Reference 2.4.5 was thought to be understandable without knowledge of section 7.3 of Bruyn and Claasen.

Reference 2.4.4 (Section 6.7)**.** <u>*The theory of intersections*</u> *in a projective plane PG bears upon axioms, definitions and theorems which are assertions on substructures of PG and upon proofs and constructions which are (non-)trivial constructions (in the sense of section 3).*

Reference 2.4.5 (Section 6.9)**.** *Related to $PG(2,\mathbb{R})$ one can say (in analytic terms) that in the theory of intersections one considers sets of linear equations and their solutions.*

Also, as remarked in (7.3), in the theory of intersections of $PG(2,\mathbb{R})$ every construction can be performed by the use of a ruler alone.

Of course, Bruyn and Claasen do not use a ruler for constructions, but construction steps as defined in definition 2.3.1. However, as an elementary construction step as defined by Bruyn and Claasen only uses the axioms of projective geometry and as in $PG(2,\mathbb{R})$ the lines that the axioms refer to are defined to be straight, a construction by use of the axioms is comparable to a construction by use of a ruler. A non-trivial construction step does not follow from the axioms and therefore has to follow from any additional axioms (or a theorem proven from these additional axioms).

In section 9 the link is made between constructions within the theory of intersections to constructions as defined Bruyn and Claasen.

Reference 2.4.6 (Definition 9.2)**.** *It is said that a proof of a theorem of $PG(2,\mathbb{R})$ belongs to the theory of intersections of $PG(2,\mathbb{R})$ based on a set Σ of theorems and axioms (of $PG(2,\mathbb{R})$) if every step of the proof can be formulated either as an elementary construction step or as a non-trivial construction, which is allowed on the basis of the theorems and axioms of Σ.*

2.4.2 The connection between figures and propositions, axioms and theorems

Now that it is specified what theory will be considered in this report and therefore the eventual proof procedure, the relationship between theorems and figures can be studied. The reader is reminded that section 2 of Bruyn and Claasen corresponds to section 2.1 of this report and in this context refers to definition 2.1.5.

Reference 2.4.7 (Section 6.11)**.** *Let us consider a theorem (or axiom) T of the theory of intersections and suppose that T is an assertion on a finite set of points P_0, a finite set of lines L_0 and a finite set of incidences $I_0 \subset P_0 \times L_0$.*

Then $F(0) = (P_0, L_0, I_0)$ is a figure in the sense of section 2.

It is said that T is a finite theorem (axiom).

$F(0)$ is said to be the figure of T.

If $F(0)$ is a closed substructure then it is said that T is a closed theorem (axiom).

The Levi graph $\Lambda(F_0)$ is said to be the Levi graph of T and the derived graph $\Gamma(F(0))$ is said to be the derived graph of T.

Bruyn and Claasen note that if a theorem T is a finite closed theorem or axiom, then the simplest case which can occur is that the figure of T is a proposition figure[9].

As each finite theorem or axiom has a figure, theorems and axioms of the theory of intersections can be classified the following:

Reference 2.4.8 (Definition 6.14)**.** *A theorem of axiom in the theory of intersections is said to be a proposition if the figure of the theorem or axiom is a proposition figure.*

A theorem or axiom is said to be a specialization of a proposition P if the figure of the theorem or axiom consists of a proposition figure of P and one or more figures the cells of which are singular.

In the rest of the report, Bruyn and Claasen consistently denote a proposition with T rather than P. This is most likely done to avoid confusion with the set of points of a figure $F = (P, L, I)$.

A proposition is normally described in words. However, this definition is not very suited for creating a proof procedure. Therefore Bruyn and Claasen have found another way of formulating a proposition.

Note that all theorems and axioms have two parts:

[9]This can be seen by considering that a figure with nodes of only valency three is per definition a proposition figure.

Antecedent:	a statement on certain conditions.
Consequent:	a statement that is true if the conditions of the antecedent are fulfilled.

As remarked in definition 2.4.7, a theorem or axiom of the theory of intersections is an assertion on a set of points, lines and their incidences.

Reference 2.4.9 (Section 6.18)**.** *Let T be a true proposition in $PG(2, \mathbb{R})$. By virtue of the properties of a proposition this proposition T consists of two parts.*

1) *the antecedent: a statement on certain conditions related to points, lines and their incidences;*

2) *the consequent: a statement that if the conditions of the antecedent are fulfilled then a certain line and a certain point are incident.*

Bruyn and Claasen use this and theorem 2.3.6 (section 4.8 in Bruyn and Claasen) to find the following formulation for a proposition:

Reference 2.4.10 (Section 6.19)**.** *Using a (proposition) figure of T we can derive from (4.8) that the antecedent of T can be formulated as a trivial construction sequence and that the consequent of T can be formulated as a non-trivial construction.*

Interpreting a proposition in the way of definition 2.4.10 is one of the key aspects that allow a proof procedure to be created. By describing a proposition using only a construction sequence and an incidence, the use of words is no longer required for stating a proposition. This opens the door to introduce the notion of 'proposition' to a computer and vice versa, to interpret output of a computer as a proposition.

2.4.3 A comparison of two definitions of a proposition

Before going into some of the consequences of formulating propositions in the manner described in definition 2.4.10, a note on the definition of a proposition as given in definition 2.4.8.

Bruyn and Claasen define a theorem or axiom as a proposition based on the properties of its figure. However, in mathematical theory there is already a clear distinction between proposition, axiom and theorem, as is for example described by Heyting (1963, p.4) [8]:

> A proposition in an axiomatic theory $\mathfrak{S}$ is a sentence in which no other notions occur than the fundamental notions of $\mathfrak{S}$ and logical notions [...] It is essential to remark that a proposition in $\mathfrak{S}$ need not be valid in $\mathfrak{S}$. A proposition is *valid* in $\mathfrak{S}$ if it is an axiom or if it can be deduced from the axioms. A valid proposition is also called a *theorem* of $\mathfrak{S}$. Every theorem of $\mathfrak{S}$ is true for every model of $\mathfrak{S}$, but a proposition in $\mathfrak{S}$ can be true for a certain model of $\mathfrak{S}$, without being a theorem of $\mathfrak{S}$.

With definition 2.4.8 Bruyn and Claasen seem to imply that these definitions correspond[10], but the relation between these two definitions is not mentioned in the report. Therefore it would be interesting to compare these definitions.

Let $F = (P, L, I)$ be the figure of a proposition T as defined by Bruyn and Claasen. It consists of points, lines and incidences, which are the fundamental notions of projective geometry. As can be seen in definition 2.4.9, T is a (logical) assertion on F. Therefore T is a sentence in which no other notions occur than the fundamental notions of projective geometry and logical notions and it can be concluded that T is indeed a proposition as defined by Heyting. Furthermore, F is a non-trivial figure (of which, per definition, the existence is not guaranteed by the axioms) so T is not valid for every model of projective geometry. Therefore at first glance a proposition as defined by Bruyn and Claasen seems to match the definition of a proposition given by Heyting.

[10]It may be expected that Bruyn and Claasen had some familiarity with this definition, particularly because Bruyn worked with Heyting close to the time when the book was published (Bruyn worked with Heyting from at least 1964 to 1969).

However, the definition of Bruyn and Claasen implies that there are no other propositions in the theory of intersections than the ones whose figure is a proposition figure. Further study would be required to prove whether or not this is the case.

2.4.4 Formulating a proposition based on its figure

Theorem 2.4.10 shows how a true theorem T with proposition figure F can be described in terms of the construction sequence of $F_i = (P, L, I\backslash\{i\})$ and the incidence i. However, it is possible to start with figure F and formulate a corresponding proposition.

Reference 2.4.11 (Section 6.23)**.** *Let j be any incidence of $F = (P, L, I)$ and $F_j = (P, L, I\backslash\{j\})$. Let Σ be a trivial construction sequence constructing F_j. Then Σ can be considered as the antecedent of a proposition T^* and the non-trivial construction which adds j to F_j as the consequent of T^*.*

However, Bruyn and Claasen note that this does raise the question whether or not the newly formulated proposition T^* is true or not. As T is a true proposition, it might be expected that T^* is as well. Theorem 2.4.12 and its proof will show that this is indeed the case.

Theorem 2.4.12 refers to the notation of (6.18) through (6.23). As not all sections have been described and the numbering in this report is different, a summary of the notations used is given in table 1.

Section	Notation	Refers to
(6.18)	T	a true proposition
	F	the figure of T
(6.20)	$F_i = (P, L, I\backslash\{i\})$	the figure created by the antecedent of T
	i	the consequent of T
	j	any incidence of F
(6.23)	Σ	the construction sequence of $F_j = (P, L, I\backslash\{j\})$
	T^*	the proposition with the antecedent Σ and the consequent j.

Table 1: The notation used in section (6.18) through (6.23) of Bruyn and Claasen.

Reference 2.4.12 (Theorem 6.24)**.** *In the notation of (6.18) through (6.23). T and T^* are equivalent propositions of $PG(2, \mathbb{R})$.*

Proof. Let T be a true proposition. $i = [n_0, n_1]$ and let $j = [n_{t-1}, n_t]$ $(t \in \mathbb{N}\backslash\{0\})$.

Suppose n_1 and n_{t-1} are normal (this can be assumed without loss of generality).

There is a path Π in $\Gamma(F)$ between n_0 and n_t of the following form:

$$[n_0, n_1], [n_1, n_2], ..., [n_{t-1}, n_t].$$

n_1 is incident with n_0, n_2 and another node m, say.

Let $F^{(1)} = (P, L, I\backslash\{[n_1, n_2]\})$ and let $\Sigma^{(1)}$ be a trivial construction sequence constructing $F^{(1)}$.

Then $\Sigma^{(1)}$ can be considered as the antecedent of a proposition $T^{(1)}$ which has to have as a consequent the conclusion that n_1 and n_2 are incident.

However, the (possibly new) node m_1, incident with n_2 and m (an elementary construction step) has to be incident with n_0, according to T (see also (6.20)). But then $[m_1, n_0] \in I$ and $[m_1, m] \in I$, so $m_1 = n_1$ which yields $[n_1, n_2] \in I$ and $T^{(1)}$ is true.

Remark that in this part of the proof we need not demand that either n_0 or n_2 is normal.

Inductively we can proceed this way along the path Π which in the end leads to the conclusion that T^* is true.

Hence T implies T^*. Now reverse this process and we can conclude that T^* also implies T which proves the theorem. □

In what could be described as a corollary, Bruyn and Claasen state the following about the number of propositions corresponding to a proposition figure and the ways to characterize them. The latter leads to the definition of a class of propositions.

Reference 2.4.13 (Section 6.25)**.** *We now turn the matter around. Let $F = (P, L, I)$ be a proposition figure of a true proposition T of $PG(2, \mathbb{R})$.*

Then, according to (6.24) with every incidence j of F there corresponds a true proposition T^* in the sense of (6.20).

Indeed T^* is equivalent to T, but its wording could be quite different. So, in principle, we have $|I|$ propositions connected with one proposition figure F and we see that we can characterize the class of these equivalent theorems by one figure F and also by the graphs $\Lambda(F)$ and $\Gamma(F)$.

Reference 2.4.14 (Definition 6.26)**.** *In the notation of (6.25). The class of equivalent propositions characterized by the same graph $\Gamma(F)$ is said to be a graph-class (of propositions) (connected with $\Gamma(F)$).*

It is said that two propositions of the same graph-class are graph-equivalent.

2.4.5 The use of true propositions to generate new true propositions

With the definition of a graph-class, all pieces are in place to get to the heart of the proof generating procedure. Theorem 2.4.16 puts these pieces together by stating that if the proposition figures used in a transformation figure (definition 2.3.7, section 5.3 of Bruyn and Claasen) correspond to true propositions, then the generated figure will also correspond to a true proposition (if, of course, it is also a proposition figure, which would make A a deduction figure).

Theorem 2.4.16 refers to many other sections of Bruyn and Claasen. These references have been listed in table 2, with the reference in this article and a short explanation of what is being referred to. Not all sections referenced to are also included in this report. section 6.2 of Bruyn and Claasen lists the axioms of projective geometry and section 6.22 is a counterexample to demonstrate that not all proposition figures correspond to true theorems.

Section	**Reference**	**Meaning**
(4.8)	Theorem 2.3.6	Any incidence of a proposition figure can be removed to make it a trivial figure
(5.3)	Definition 2.3.7	Transformation figure
(5.4)	Definition 2.3.10	Deduction figure
(5.5)	Corollary 2.3.9	A can be non-trivially constructed from A_C for any cell C such that A_C is a proposition figure
(5.6)	Definition 2.3.8	(Non-elementary) generation of figures
(6.2)		The axioms of projective geometry
(6.22)		Not every proposition figure corresponds with a true proposition

Table 2: The references used in theorem 2.4.16.

Reference 2.4.15 (Section 9.7)**.** *We start with two propositions $T(1)$ and $T(2)$ with figures $F(1)$ and $F(2)$, respectively. Then we compose a figure G which, in the sense of (5.6), is generated by $F(1)$ and $F(2)$.*

Reference 2.4.16 (Theorem 9.8)**.** *In the notation of (9.7).*

If G is a proposition figure and if $T(1)$ and $T(2)$ are true propositions of $PG(2, \mathbb{R})$ then G represents a graph-class of propositions which are true in $PG(2, \mathbb{R})$ and the proof of every proposition of that class belongs to the theory of intersections of $PG(2, \mathbb{R})$ based on $T(1)$ and $T(2)$ and the axioms implied by (6.2).

Proof. We use the notations of (5.3), (5.4) and (5.6). A is a deduction figure, taking $A_{C(0)} \cong G$.

The proposition figure G can be non-trivially constructed according to theorem (4.8). let $i = [n_i, m_i]$ be the incidence which is added by the (last) non-trivial construction step.

Suppose i belongs to $C(1)$.
(this assumption is allowed because according to (4.8) we can add any incidence of G in the non-trivial construction step and because $C(1) \subset I \backslash C(0)$, every incidence of $C(1)$ is an incidence of G, G being a proposition figure. I is the incidence set of A).

According to (5.5) A can be non-trivially constructed from G (whether or not G corresponds to a true proposition; compare with (6.22)).

Let $j = [n_2, m_2]$ be the incidence of the non-trivial construction step in the completion of A. Then $j \in C(0)$.

Now consider in A the figure $A_{C(1)} \cong F(1)$.

This represents the class of the true proposition $T(1)$. Therefore n_2 and m_2 (nodes of j) are incident indeed.

As the last step of the proof of (9.8) consider $A_{C(2)} \cong F(2)$. This figure represents the class of $T(2)$. Because $T(2)$ is also true n_1 and m_2 are incident.

Hence the class of $T(0)$, corresponding to G, has been proved and the proof belongs to the theory of intersections of $PG(2, \mathbb{R})$ based on $T(1)$ and $T(2)$. $\square$

It might seem that with theorem 2.4.16 the proof procedure is complete. Indeed, theorem 2.4.16 finishes the theoretical framework. However, to translate this theoretical framework into actual proofs some additional theory will be required. The next section will deal with this and formulate the complete proof procedure.

2.5 The procedure for proofs based on a given true proposition

To this point, the theoretical framework has been studied. Section 2.2 has provided tools for analysing figures, section 2.3 has discussed how figures are constructed and how to classify figures based on that, and section 2.4 has studied how figures correspond to theorems. This section will no longer focus on general properties of figures, but go into detail about how to construct the figures that will be used for the proof procedure.

2.5.1 The limitations

Theorem 2.4.16 seems to imply a clear proof procedure:

1. Choose your starting propositions $T(1)$ and $T(2)$.
2. Construct the corresponding proposition figures $F(1)$ and $F(2)$.
3. Construct the deduction figure D.
4. Find the cell C in D of which complement D_C is a proposition figure ($D_C \ncong F(1), F(2)$).
5. Figure D_C corresponds to a new true proposition.

Unfortunately, a closer look reveals that it is not as simple as that. There is as of yet no procedure given to construct a transformation figure A for two given proposition figures $F(1)$ and $F(2)$. Even if a transformation figure A is found, there is no guarantee that it will be a deduction figure.

Bruyn and Claasen do not answer these questions in full. Only a method is given for generating a transformation figure for a single given proposition figure and the second matter is mostly postponed to a later report.

Reference 2.5.1 (Section 5.8)**.** *We shall discuss a method to compose at least a transformation figure starting from a given proposition figure F. This method will give (a) figure(s) elementary generated by F. The second question then is whether those generated figures are proposition figures or not. In general this question has to be dealt with by the computer (see section 10 and a later report).*

However, section 10 (titled: The use of the computer) has mostly the appearance of a discussions section. It consists of a single page and mainly poses questions and intentions for future reports. The question posed in reference 2.5.1 is not answered.

Unfortunately, no reports other than this one are known. This does pose a problem for both issues. However, as stated in reference 2.5.1, Bruyn and Claasen do provide a method for constructing transformation figures for a given proposition figure.

2.5.2 The procedure

In essence, the method copies a part of the figure F (not a component, a proposition figure does not have components) and adds it to itself. With the addition of the copied part F will no longer be a proposition figure and contain multiple cells, one of which will be the copied part. Obviously, removing the copied part will make F a proposition figure once again, as will removing the original of the copied part. Thereby, the figure F with the added copy will be a transformation figure.

This is formalized by introducing a disconnecting set of incidences Ω that will disconnect the copied nodes, the starter Δ that is the 'part' to be copied and the copy Δ^*. For more on disconnecting sets, Bruyn and Claasen refer to Wilson [13]. The reference to Wilson is reference 7 in the report of Bruyn and Claasen.

Reference 2.5.2 (Section 5.10)**.** *Let $F = (P, L, I)$ be a proposition figure and let $\Gamma(F)$ be its derived graph.*

Let $V = V_1 \cup V_2$ be a partition of the node set V of $\Gamma(F)$ into non-empty disjoint subsets. If Ω is a non-empty subset of incidences, which have one node in V_1 and one node in V_2, then we say Ω is a <u>disconnecting set in $\Gamma(F)$</u> (see Wilson [7]).

Reference 2.5.3 (Definition 5.11). *Let Ω be a disconnecting set in $\Gamma(F)$ containing only normal incidences with the following properties:*

1) after removal of Ω the resulting graph has two components;

2) the node set of at least one of the components contains only nodes which are normal in $\Gamma(F)$.

Let $\Gamma(1)$ be such a component and let $\Omega = \{[n_s, m_s],\ s = 1, 2, ..., t\}$, the nodes n_s belonging to $\Gamma(1)$.

Let Δ be the graph the node set of which is the union of $\{m_1, m_2, ..., m_t\}$ and the node set of $\Gamma(1)$ and the incidence set of which is the union of Ω and the incidence set of $\Gamma(1)$.

It is said that the graph Δ is a <u>starter (for $\Gamma(F)$)</u>, that the nodes m_s, ($s = 1, 2, ..., t$) are the <u>boundary nodes of Δ</u> and that Ω is the <u>disconnecting set for Δ</u>.

Bruyn and Claasen note that a starter Δ is determined by the disconnecting set Ω and the boundary nodes. However, in the demonstration of this method in section 3 it will prove useful to have a name for the nodes of Δ that are not boundary nodes. Therefore in this report the notion of interior nodes is introduced in addition to the boundary nodes and the disconnecting set Ω.

Definition 2.5.4. *In the notation of definition 2.5.3.*

Let d be a node of Δ that is not a boundary node. d is called an <u>interior node (of Δ)</u>. The set D of all interior nodes of Δ is called the <u>interior (of Δ)</u>.

Bruyn and Claasen also note that each node of Δ has a valency of either 3 (the interior nodes, belonging to the normal component $\Gamma(1)$) or 1 (the boundary nodes).

Next, the copy of the starter Δ is defined. First, however, a new notation is introduced.

Reference 2.5.5 (Section 5.14). *Let Δ be a starter for $\Gamma(F)$. Let $\Gamma_\Delta(F)$ be the graph derived from $\Gamma(F)$ by splitting the boundary nodes of Δ.*

Note that in definition 2.5.3, the boundary nodes were denoted by m_s ($s = 1, 2, ..., t$).

Reference 2.5.6 (Section 5.16). *Let Δ^* be a graph isomorphic to Δ with nodes p_s and q_u which do not belong to P. The nodes p_s correspond 1-1 to the normal nodes of Δ and the nodes q_u correspond 1-1 to the nodes of valency 1 in Δ. For $u = 1, 2, ..., t$ the node q_u is considered as a representative of the stem m_u. The graph Δ^* is called a <u>copy of Δ</u>.*

In reference 2.5.7 Bruyn and Claasen join the copy to the original by use of a derived graph and in reference 2.5.8 Bruyn and Claasen verify that the figure obtained by adding the copy is indeed a transformation figure.

Reference 2.5.7 (Section 5.18). *Now consider the figure A with the derived graph $\Gamma_\Delta(F) \cup \Delta^*$. The figure A is well-defined, as one easily checks.*

Reference 2.5.8 (Section 5.20). *$\Gamma(A)$ and therefore A has at least three components.*

Let $\{C(0), C(1), C(2), ...\}$ be the cell-partition of A.

Suppose $A^{C(1)}$ corresponds to Δ and $A^{C(2)}$ to Δ^*. Clearly $A_{C(2)}$ (the complement of $A^{C(2)}$ with respect to A; see(3.22)) is F, a proposition figure, and $A_{C(1)}$ is a figure $F(1)$ isomorphic to F and therefore also a proposition figure.

Therefore A is a transformation figure.

Now there is another cell, $C(0)$ say. In the formulation of (5.3) and (5.6) the figure $A_{C(0)}$ is (elementary) generated by F.

It still remains to be determined whether or not $A_{C(0)}$ is a proposition figure.

With the guarantee that A is indeed a transformation figure, the full proof procedure can be formulated.

2.6 Conclusion: The procedure in seven steps

While Bruyn and Claasen formulate all the different theorems used for the proof procedure and give an example to illustrate how it works, they never explicitly list all the steps taken. In this section a seven step algorithm is given that clearly states all the individual steps.

Let $T(0)$ be a true proposition in $PG(2, \mathbb{R})$ and $F(0) = (P_0, L_0, I_0)$ the corresponding proposition figure. Let the sequence $\{T_0, T_1, ...\}$ be the sequence of true propositions following from $T(1)$.

Iterate over $i \in \mathbb{N}$:

1. Construct the derived graph $\Gamma(i)$ of $F(i)$.
2. Choose a disconnecting set Ω and a set of boundary nodes to generate a starter $\Delta(i)$
3. Create $\Gamma_{\Delta(i)}(F)$ by splitting the boundary nodes of $\Delta(i)$.
4. Construct the copy $\Delta^*(1)$ by introducing new nodes for the nodes of $\Delta(i)$ according to definition 2.5.6.
5. Construct the transformation figure $A(i)$ from its derived graph $\Gamma(A(i)) = \Gamma_{\Delta(i)} \cup \Delta^*(i)$.
6. Analyse the cells $C(j)$ $(j = 1, 2, 3, ...)$ of $A(i)$. If no cell has the property that $A_{C(j)}$ $(A_{C(j)} \ncong F(i))$ is a proposition figure, return to step 2.
7. Let $C(j)$ be the cell such that $A_{C(j)}$ is a proposition figure. Define $F(i + 1) = A_C$. According to theorem 2.4.16, $F(i + 1)$ corresponds to a true proposition $T(i + 1)$ in $PG(2, \mathbb{R})$.

3 A demonstration: Proving the theorem of Desargues out of the theorem of Pappus

This demonstration will follow the example of section 9.11 through 9.35 of Bruyn and Claasen, with the addition of the steps described in the algorithm of section 2.6 clearly labelled, added explanations and added figures. The figures that are also displayed in Bruyn and Claasen will have the section numbers in the caption. All colouring is added by the author to facilitate ease of understanding, but is not necessary for understanding of this section.

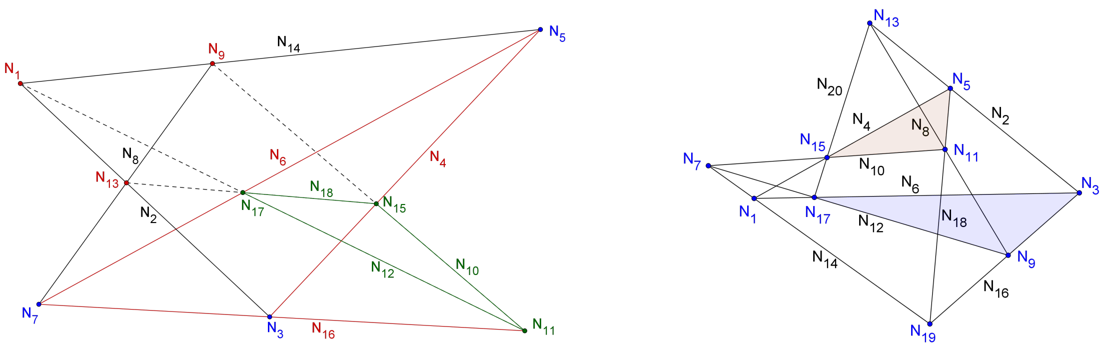
Also the choice has been made to alter the notation of points and lines in the figures. Bruyn and Claasen label all points with odd natural numbers and all lines with even natural numbers. It was thought possibly confusing to denote nodes with only a number and therefore in the figures in this report all points are denoted by N_i with i an odd natural number and all lines are denoted by N_j with j an even natural number.

3.1 The theorems of Pappus and Desargues

An important result of Bruyn and Claasen was proving the theorem of Desargues out of the theorem of Pappus. Bruyn and Claasen deemed it noteworthy enough to state this in the introduction (p.1) and to use it for the only example given in the report.

Below, the definitions of the theorem of Pappus is stated. The theorem of Desargues is described in theorem 2.2.11. The line-point diagrams are shown[11] in figure 5.

Theorem 3.1.1 (Pappus)**.** *Let l and m be two different lines with three different points A_1, A_2 and A_3 on l and three different points B_1, B_2 and B_3 on m (but not on l). Then the points $C_1 = A_2B_3 \cap A_3B_2$, $C_2 = A_1B_3 \cap A_3B_1$ and $C_3 = A_1B_2 \cap A_2B_3$ are collinear.*



(a) (Section 9.14 of Bruyn and Claasen) *(b) (Section 9.32 of Bruyn and Claasen)*

Figure 5: Representations of the theorem of Pappus and the theorem of Desargues.

The theorems of Pappus and Desargues are two of the most fundamental theorems in projective geometry. So much so that planes in projective geometry classified as (non-)Pappian and (non-)Desarguesian planes. In a Pappian plane, the underlying coordinate system is commutative and in a Desarguesian plane the underlying coordinate system is associative. It was first proven that the theorem of Desargues is a consequence of the theorem of Pappus in 1905 by Hessenberg [7]. This means that all Pappian planes are also Desarguesian planes. However, the reverse is not true. The quaternionic projective plane, for example, is Desarguesian but not Pappian, as the quaternions are associative but non-commutative. Appendix B provides a more in depth study of the theorem of Desargues, as well as an example of a Moulton plane [10], which is non-Desarguesian.

[11]Note that actually the lines of the diagrams in figure 5 should extend beyond the points, but for this example they are not used beyond the segments between the points and therefore left out. Furthermore, some nodes are differently coloured. The reason for that will become apparent in section 3.2.2.

By proving that Desargues is a consequence of Pappus with a proof procedure of their own, Bruyn and Claasen give a demonstration of the applicability of their procedure and an idea of the possibilities that it brings.

3.2 Iteration 1: A demonstration using graphs

Sections 9.11 and 9.12 of Bruyn and Claasen describe how the different propositions and figures will be labelled.

Reference 3.2.1 (Section 9.11)**.** *To demonstrate the generating method of section 5 we generate the graph-class of propositions $S(2)$, $S(3)$, and $S(4)$ based upon the proposition of Pappus, which we label $S(1)$.*

We shall show that $S(3)$ is in fact the theorem of Desargues (see 9.35).

Reference 3.2.2 (Section 9.12)**.** *Let $t \in \{1, 2, 3, 4\}$. We shall label the figure of $S(t)$ by $H(t)$ and the transformation figure constructed from $H(t)$ by $A(t)$. The starter in $H(t)$ shall be labelled by $\Delta(t)$ and its copy by $\Delta^*(t)$.*

Note that the diagram in figure 5a is a representation of figure $H(1)$. The incidence matrix of $H(1)$ is shown in table 3 (section 9.15).

	N_1	N_3	N_5	N_7	N_9	N_{11}	N_{13}	N_{15}	N_{17}
N_2	1	1					1		
N_4		1	1					1	
N_6			1	1					1
N_8				1	1		1		
N_{10}					1	1		1	
N_{12}	1					1			1
N_{14}	1		1		1				
N_{16}		1		1		1			
N_{18}							1	1	1

Table 3: (section 9.15) The incidence matrix of $H(1)$.

3.2.1 Step 1: Construct the derived graph

First the Levi graph Λ is created by using the method described in theorem 2.2.3. The adjacency matrix is shown in table 4. The rows and columns have already been rearranged so that the nodes are in order of appearance in a Hamilton cycle.

Note that the sum of the elements of a row (or column) is equal to the valency of the corresponding node. It is easily seen that this is 3 for all points and lines. Therefore each node and each incidence of $H(1)$ is normal. Therefore, the derived graph is the same as the Levi graph. This shows that by definition 2.2.6 $H(1)$ is indeed a proposition figure.

The graph is shown in figure 6a. Again, the colouring will be explained in step 2.

3.2.2 Step 2: Construct a starter

In this step a suitable starter $\Delta(1)$ has to be chosen such that the transformation figure $A(1)$ will be a deduction figure, i.e. it will contain a cell C such that $A_C(1)$ is a proposition figure. In section 9.17 Bruyn and Claasen state the chosen disconnecting set and the set of boundary nodes. No motivation for

	N_1	N_2	N_{13}	N_{18}	N_{15}	N_{10}	N_{11}	N_{12}	N_{17}	N_6	N_5	N_4	N_3	N_{16}	N_7	N_8	N_9	N_{14}
N_1		1						1										1
N_2	1		1										1					
N_{13}		1		1												1		
N_{18}			1		1				1									
N_{15}				1		1						1						
N_{10}					1		1										1	
N_{11}						1		1						1				
N_{12}	1						1		1									
N_{17}				1				1		1								
N_6									1		1				1			
N_5										1		1						1
N_4					1						1		1					
N_3		1										1		1				
N_{16}							1						1		1			
N_7										1				1		1		
N_8			1												1		1	
N_9						1										1		1
N_{14}	1										1						1	

Table 4: Adjacency matrix of $\Gamma(H(1))$, which is the same as $\Lambda(H(1))$.

the choice of this starter is given, but some observations can be made in an attempt to understand the reasoning.

Reference 3.2.3 (Section 9.17)**.** *Now consider for $\Gamma(H(1))$ the starter $\Delta(1)$ (see 5.11) determined by the disconnecting set $\Omega = \{[N_{13}, N_{18}], [N_4, N_{15}], [N_9, N_{10}], [N_{16}, N_{11}], [N_1, N_{12}], [N_6, N_{17}]\}$ and the set of boundary nodes $\{N_{13}, N_4, N_9, N_{16}, N_1, N_6\}$.*

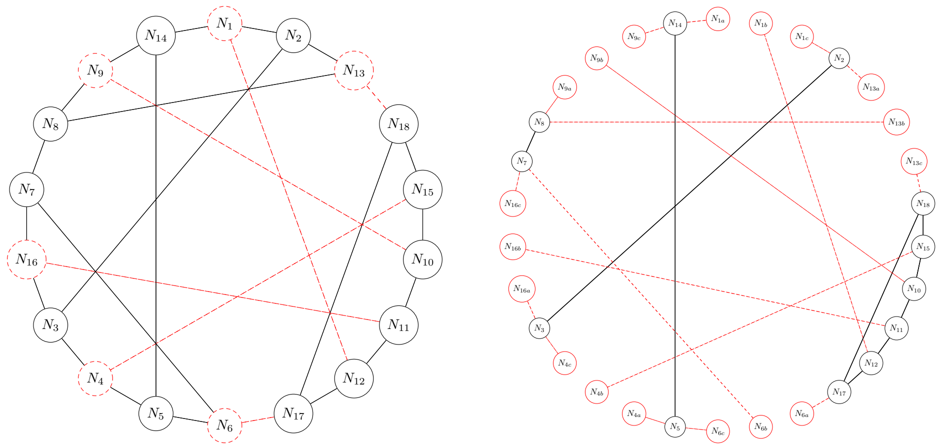
Correction of Bruyn and Claasen
The astute reader might note that in section 9.17 the disconnecting set is denoted with Λ instead of Ω. This is thought to be a typo, as disconnecting sets are always denoted with Ω and Λ is used for Levi graphs. Furthermore, the disconnecting set Ω proposed in section 9.17 contains the incidence $[N_1, N_2]$ instead of $[N_1, N_{12}]$. This is also thought to be a typo. A disconnecting set containing $[N_1, N_2]$ is in contradiction with definition 2.5.2, section 5.13 referring to section 9.17 states that the incidence $[N_1, N_{12}]$ is used in the disconnecting set for $\Delta(1)$ and the starter $\Delta(1)$ depicted in the figure of section 9.19 also clearly uses the incidence $[N_1, N_{12}]$.

Note that in the definition of a starter (definition 2.5.3) it is stated that the disconnecting set Ω needs to split $\Gamma(H(1))$ in two components and that the component used for the starter needs to have only normal incidences in $\Gamma(H(1))$. The easiest way to fulfil that condition is to pick a triangle in the configuration in figure 5a as interior nodes, such as the nodes $D = \{N_{18}, N_{15}, N_{10}, N_{11}, N_{12}, N_{17}\}$. In the figure these nodes are depicted in green. As the name implies, the disconnecting set Ω needs to disconnect these nodes from the rest of the figure. It can be easily seen that the boundary set contains the points that are incident with a side of the chosen triangle and the lines that are incident with a vertex of the triangle. The boundary nodes are depicted in red. The disconnecting set Ω simply contains the incidences between the boundary nodes and the interior nodes.

In addition to the boundary nodes, in figure 6a also the disconnecting incidences are depicted in red (as well as dashed). It is clearly visible that removing the incidences of Ω yields two components, of which one contains only normal incidences in $\Gamma(H(1))$ (aside from disconnecting incidences). Therefore the chosen starter $\Delta(1)$ fulfils the conditions of definition 2.5.3.

3.2.3 Step 3: Split the boundary nodes

The next step is to split the boundary nodes of $\Delta(1)$. This leads to the derived graph $\Gamma_{\Delta(1)}(H(1))$ depicted in figure 6b. In this figure all split nodes and incidences with split nodes are red. Only non-split



(a) (section 9.16) $\Gamma\big(H(1)\big)$. *(b)* $\Gamma_{\Delta(1)}(H(1))$.

Figure 6: The derived graph $\Gamma(H(1))$ *and the derived graph* $\Gamma_{\Delta(1)}(H(1))$*. All boundary nodes, disconnecting incidences and incidences with split nodes are red.*

nodes and incidences between two non-split nodes are black. It follows directly from definition 2.5.3 that a boundary node can never be incident with another boundary node and therefore a split node can never be incident with another split node. This means that each component has to contain at least one incidence between two non-split nodes. It is easy to see that $\Gamma_{\Delta(1)}(H(1))$ has four components. One circuit corresponding to $\Delta(1)$ and three components containing a single non-split incidence.

3.2.4 Step 4: Construct a copy

The fourth step is to construct a copy $\Delta^*(1)$ of $\Delta(1)$ according to definition 2.5.6. The new nodes N_i ($i = 19, ..., 24$) are introduced. The boundary nodes of Δ^* are the same as for the starter $\Delta(1)$. The disconnecting set for $\Delta^*(1)$, denoted by $\Omega^*(1)$, is

$$\Omega^*(1) = \{[N_{13}, N_{24}], [N_4, N_{21}], [N_9, N_{20}], [N_{16}, N_{19}], [N_1, N_{22}], [N_6, N_{23}]\}$$

3.2.5 Step 5: Introduce the transformation figure

Construct the derived graph $\Gamma(A(1)) = \Gamma_{\Delta(1)} \cup \Delta^*(1)$ to define the transformation figure $A(1)$. The incidence matrix of $A(1)$ is shown in table 5. The incidence matrix of $H(1)$ is seen in the top left corner inside the box. The red (dashed) circled nodes are the boundary nodes of $\Delta(1)$ and the black circled nodes the interior nodes of $\Delta(1)$. Correspondingly, the red circled incidences are the incidences of Ω and the black circled incidences the normal incidences of $\Delta^*(1)$.

Note that after adding the copy, the boundary nodes have a valency of four and have therefore become singular. This is a direct consequence of the definition of the copy and exactly the reason why $\Gamma_{\Delta(1)} \cup \Delta^*(1)$ can be seen as a derived graph (note that in the table the boundary nodes aren't split, as to not overcrowd the matrix).

3.2.6 Step 6: Analyse the cells of the transformation figure

As the only reason to construct A is to find a cell whose complement A_C is a proposition figure, there is no need to actually construct the transformation figure other than $\Gamma(A(1))$. $A(1)$ is shown in figure 7. The incidences of the disconnecting set Ω are marked with a red dashed line. A has five cells, the top four cells corresponding to the components of $\Gamma_{\Delta(1)}(H(1))$ of the original figure $H(1)$ and the bottom component corresponding with the copy $\Delta^*(1)$.

	N_1	N_3	N_5	N_7	N_9	N_{11}	N_{13}	N_{15}	N_{17}	N_{19}	N_{21}	N_{23}
N_2	1	1					1					
N_4		1	1					1			1	
N_6			1	1					1			1
N_8				1	1		1					
N_{10}					1	1		1				
N_{12}	1					1			1			
N_{14}	1		1		1							
N_{16}		1		1		1				1		
N_{18}							1	1	1			
N_{20}					1					1	1	
N_{22}	1									1		1
N_{24}							1				1	1

Table 5: (section 9.20) The incidence matrix of transformation figure $A(1)$. The disconnecting set and the boundary nodes are in red and dashed.

Now the task at hand is to analyse the cells $\{C(1), C(2), C(3), C(4), C(5)\}$ of $A(1)$. $C(4)$ and $C(5)$ correspond with $\Delta(1)$ and $\Delta^*(1)$ respectively and are therefore out of consideration. For each of the remaining cells it needs to be checked whether $A(1)_{C(i)}$ $(i = 1, 2, 3)$ is a proposition figure. The reader is reminded that by definition 2.4.8 this means that $A(1)_{C(i)}$ needs to be closed (each node needs to have a valency of at least three) and $A(1)_{C(i)}$ needs to be connected. To facilitate this process, the boundary nodes are each assigned their own colour.

It is easily verified that each of the $A(1)^{C(i)}$ have all their boundary nodes in common (share a colour) with $C(4)$ and $C(5)$ and are therefore connected with $A(1)^{C(4)}$ and $A(1)^{C(5)}$. This means removing any $A(1)^{C(i)}$ will still leave a connected figure. It is also easily verified that each internal node (black node) of $A(1)$ is connected to three other nodes and that all the boundary nodes have a valency of four (there are four nodes of each color). As the components are only connected in the boundary nodes this means that if any $A(1)^{C(i)}$ is removed no internal nodes of $A(1)_{C(i)}$ will go down in valency. The boundary nodes connected to $A(1)^{C(i)}$ will go down in valency by one, which means they still have valency of three. Therefore, $A(1)_{C(i)}$ is a closed figure whose derived graph is connected. By definition 2.2.6 this means that $A(1)_{C(i)}$ is a proposition figure and therefore it is concluded that $A(1)$ is a deduction figure.

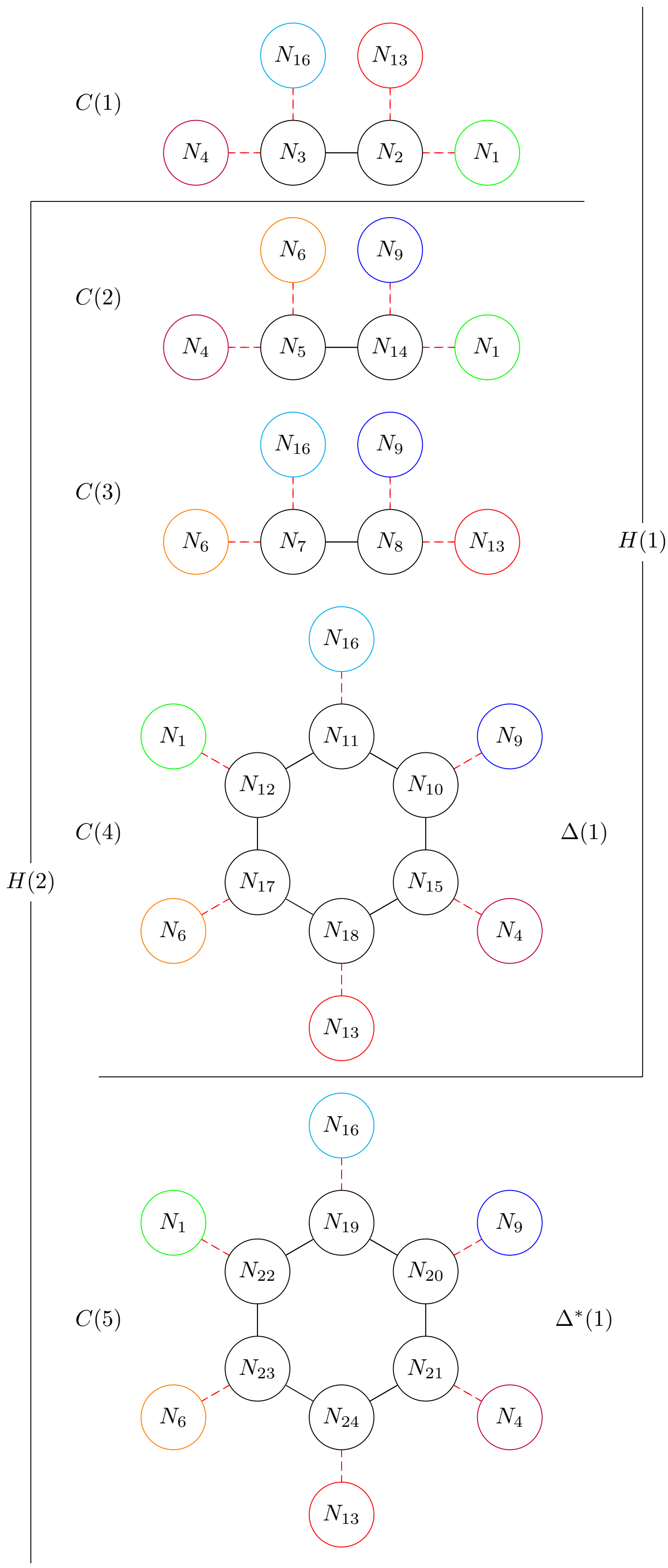


Figure 7: (section 9.19) $\Gamma(A(1))$*, with the different cells of transformation figure* $A(1)$ *shown.*

3.2.7 Step 7: Define the figure

For each of the cells in $\{C(1), C(2), C(3)\}$ the complement is a proposition figure and could therefore be chosen for the figure $H(2)$. Bruyn and Claasen have chosen to define $H(2) = A(1)_{C(1)}$ (section 9.22). $H(2)$ is a proposition figure and corresponds to a proposition $S(2)$ which, according to theorem 2.4.16, is true. This incidence matrix is the same as in table 5, except with row N_2 and column N_3 removed. The next step would be to analyse $H(2)$ and work out what properties proposition $S(2)$ has. However, the goal was not to generate an arbitrary true proposition. The purpose of this section was to demonstrate that this method could prove Desargues out of Pappus. Therefore another iteration of the procedure is required.

3.3 Iteration 2: A demonstration using matrices

The previous section relied heavily on (visual representations of) graphs. However, part of the motivation for Bruyn and Claasen to developing this proof procedure in the first place was to generate the proofs with a computer. Therefore the procedure should be able to work purely by using matrices. To demonstrate that this can be done, this iteration will only use matrices. However, a hybrid is chosen as visual aides will be used to facilitate understanding of the matrices.

The first step is to take the incidence matrix of $H(2)$ and prepare it for the coming process. With the removal of nodes N_2 and N_3, the node numbering of $H(2)$ has become inconsistent. In the incidence matrix shown in section 9.23 Bruyn and Claasen rearrange and renumber the nodes, as is shown in table 6. The outer numbering is the same numbering as used previous. The inner numbering is the new numbering used henceforth. This means that node N_9 becomes node N_1, node N_5 becomes node $N3$ and so forth. The idea behind this is unclear, but it is pointed out that by doing so a matrix is obtained that is diagonally symmetrical. Note that while for an adjacency matrix that is trivial, it is far from trivial for an incidence matrix.

Note that nodes N_1 and N_2 both have a valency of 4 and are therefore singular. This means that in $\Gamma(H(2))$ these nodes will become split. However, for the moment the construction of $\Gamma(H(2))$ will be postponed.

Old		N_9	N_5	N_7	N_{17}	N_{23}	N_1	N_{19}	N_{11}	N_{13}	N_{21}	N_{15}
	New	N_1	N_3	N_5	N_7	N_9	N_{11}	N_{13}	N_{15}	N_{17}	N_{19}	N_{21}
N_6	N_2		1	1	1	1						
N_8	N_4	1		1						1		
N_{14}	N_6	1	1				1					
N_{20}	N_8	1						1			1	
N_{10}	N_{10}	1							1			1
N_{16}	N_{12}			1				1	1			
N_{12}	N_{14}				1		1		1			
N_{22}	N_{16}					1	1	1				
N_4	N_{18}		1								1	1
N_{18}	N_{20}				1					1		1
N_{24}	N_{22}					1				1	1	

Table 6: (section 9.23) The incidence matrix of proposition figure $H(2)$. To start the process of finding figure $H(3)$, the nodes are rearranged and renumbered.

The next step is to choose a starter $\Delta(2)$. Bruyn and Claasen choose the following:

Reference 3.3.1 (Section 9.25). *In the following step we consider in $\Gamma(H(2))$ the starter $\Delta(2)$ determined by the disconnecting set $\Omega = \{[N_9, N_{22}], [N_4, N_{17}], [N_7, N_{20}], [N_{10}, N_{21}], [N_3, N_{18}], [N_8, N_{19}]\}$ and the set of boundary nodes $\{N_9, N_4, N_7, N_{10}, N_3, N_8\}$.*

Again, Bruyn and Claasen do not explain their choice. However, in the same way that in the previous iteration a possible explanation was found based on the figure, a possible explanation will be found based on the matrix.

Note that the interior set chosen by Bruyn and Claasen, $\{N_{17}, N_{18}, N_{19}, N_{20}, N_{21}, N_{22}\}$, are all the nodes at the far ends. When looking at table 6, the incidences between the interior nodes are seen in the cluster of six incidences at the bottom right. To facilitate the reader, these incidences have been circled in black. Clusters of similar shape have been seen in previous tables (most notably in section 2.2) and such a cluster always forms a circuit corresponding to a triangle. This is a first indication of why Bruyn and Claasen might have chosen this. The other incidences of the interior nodes are the incidences of Ω, (circled in red and dashed) and the boundary nodes are the nodes connected to those incidences. It can be seen that Ω is a disconnecting set by considering the incidence matrix as a block matrix with on the diagonal the incidences belonging two components and on the sides the incidences of Ω. The partition between the blocks is shown by a dashed line. It is clear that the removal of Ω would leave two components.

	N_1	N_3	N_5	N_7	N_9	N_{11}	N_{13}	N_{15}	N_{17}	N_{19}	N_{21}
N_2		1	1	1	1						
N_4	1		1						1		
N_6	1	1				1					
N_8	1						1			1	
N_{10}	1							1			1
N_{12}			1				1	1			
N_{14}				1		1		1			
N_{16}					1	1	1				
N_{18}		1								1	1
N_{20}				1					1		1
N_{22}					1				1	1	

Table 7: The incidence matrix of proposition figure $H(2)$. The boundary nodes and disconnecting set Ω of $\Delta(2)$ are circled red (dashed), the internal points and normal incidences circled black.

To create the copy $\Delta^*(2)$, the nodes N_i ($i = 23, ..., 28$) are introduced with incidences as shown in table 8. The incidence matrix is still a block matrix. The diagonal blocks are only connected to each other by the boundary nodes. This means that each block will form at least one component in $A(2)$.

	N_1	N_3	N_5	N_7	N_9	N_{11}	N_{13}	N_{15}	N_{17}	N_{19}	N_{21}	N_{23}	N_{25}	N_{27}
N_2		1	1	1	1									
N_4	1		1						1			1		
N_6	1	1				1								
N_8	1						1			1			1	
N_{10}	1							1			1			1
N_{12}			1				1	1						
N_{14}				1		1		1						
N_{16}					1	1	1							
N_{18}		1								1	1			
N_{20}				1					1		1			
N_{22}					1				1	1				
N_{24}		1											1	1
N_{26}				1								1		1
N_{28}					1							1	1	

Table 8: The incidence matrix of $A(2)$, obtained by adding the copy $\Delta^*(2)$.

Per definition, with the addition of $\Delta^*(2)$ the boundary nodes have become singular. The reader is reminded that nodes N_1 and N_2 are also singular. Therefore, when constructing the derived graph $\Gamma(A(2))$ of A, the boundary points, N_1 and N_2 must be treated the same. As they are similar, it is convenient to reorder the rows and columns of the incidence matrix so all singular nodes are clustered. By doing so, table 9 is obtained. Furthermore, lines have been drawn through the singular nodes (red, dotted lines for boundary nodes and blue, dashed lines for the original singular nodes).

	N_1	N_3	N_7	N_9	N_5	N_{11}	N_{13}	N_{15}	N_{17}	N_{19}	N_{21}	N_{23}	N_{25}	N_{27}
N_2		1	1	1	1									
N_4	1				1				1			1		
N_8	1						1			1			1	
N_{10}	1							1			1			1
N_6	1	1				1								
N_{12}					1		1	1						
N_{14}			1			1		1						
N_{16}				1		1	1							
N_{18}		1								1	1			
N_{20}			1						1		1			
N_{22}				1					1	1				
N_{24}		1											1	1
N_{26}			1									1		1
N_{28}				1								1	1	

Table 9: The cells of $A(2)$ visualized in the matrix: By moving rows and columns, all singular nodes have been placed at the top left corner. The three non-singular cells of $A(2)$ are in the blocks on the diagonal and the six singular cells are the incidences between singular nodes and boundary nodes.

By splitting the singular nodes, it means that all the incidences that lie on that line are no longer connected. The lines through the incidences of the boundary nodes have become in effect the separators of the cells. It is clear that $A(2)$ has three non-singular cells, corresponding with the blocks on the diagonal.

While in the role of cell separator the original singular incidences and the boundary nodes must be treated the same, there is an important distinction: Boundary nodes cannot be incident with each other, but they can be incident with original singular cells. Therefore, $A(2)$ also has six singular cells corresponding with the intersections of red and blue lines.

This means that in total $A(2)$ has nine cells. Three singular cells corresponding to the blocks on the diagonal and six singular cells corresponding with the intersections of singular nodes and boundary nodes.

It now needs to be decided whether $A(2)$ is a deduction figure. Fortunately this time, there is only one cell to consider, as there is no use removing cells corresponding with the starter $\Delta(2)$ and the copy $\Delta^*(2)$ and the boundary nodes cannot be removed either. This leaves the cell in the middle as the only possible option. Let $C(1)$ denote that particular cell. $A_{C(1)}$ contains only the starter, the copy and the singular cells. This means each cell is connected to a boundary node and therefore $A_{C(1)}$ is connected. It is concluded that A is a deduction figure.

Old		N_{17}	N_{19}	N_{21}	N_{23}	N_{25}	N_{27}	N_3	N_7	N_9	N_1
	New	N_1	N_3	N_5	N_7	N_9	N_{11}	N_{13}	N_{15}	N_{17}	N_{19}
N_{18}	N_2		1	1				1			
N_{20}	N_4	1		1					1		
N_{22}	N_6	1	1							1	
N_{24}	N_8					1	1	1			
N_{26}	N_{10}				1		1		1		
N_{28}	N_{12}				1	1				1	
N_4	N_{14}	1			1						1
N_8	N_{16}		1			1					1
N_{10}	N_{18}			1			1				1
N_2	N_{20}							1	1	1	

Table 10: (section 2.29) The incidence matrix of $H(3)$ corresponding with the theorem of Desargues (see figure 5b).

Let $H(3)$ denote the proposition figure $A_{C(1)}$ corresponding with the true theorem $S(3)$. The incidence matrix of $H(3)$ is shown in table 10. As before, the rows and columns are reordered and renumbered to prevent a gap in numbering. Table 10 is the same as the figure in section 2.29 of Bruyn and Claasen. This means the construction of the derived graph of $A(2)$ as shown in figure 9.27 of Bruyn and Claasen was not essential to the procedure; the same has been achieved with only the matrices. Not only that, but only the incidence matrices. While the idea behind the steps taken was based on the Levi graph and the derived graph, no construction of those matrices was necessary to obtain the matrix of 10.

As desired, the incidence matrix in table 10 corresponds to the theorem of Desargues. The nodes are numbered the same as in the line point diagram in figure 5b at the beginning of this section. It is concluded that the theorem of Desargues is a consequence of the theorem of Pappus.

This demonstrates that the proof procedure of Bruyn and Claasen works. Moreover, it can be done without the drawing of graphs, purely by manipulating the matrix. The effectiveness of this procedure will be discussed in the discussion.

4 Discussion

The effectiveness of the procedure used by Bruyn and Claasen is limited by the fact that it is only able to find new theorems, but not able to prove a given theorem. Bruyn and Claasen formulate this in the following way in the introduction:

> Our proof procedure is in essence a theorem generating one. To put it in the negative: for the moment we are not able to put to a test a given (unproved) hypothesis but are "only" able to "generate" new theorems from given theorems. "Nice" theorems only come by chance. That chance, however, can be steered by the brute force of the computer and intelligently devised algorithms.

This severely reduces the applicability of the method, as most of the times the question of whether a given theorem is true is of greater interest. However, the existence of the proof procedure found is remarkable in its own right. The proof that certain theorems can be formulated as a construction sequence and an incidence is an interesting perspective, as is the use of figures to construct new theorems.

The word 'automatic' in the title of the report hinted at the use of the computer for the proof procedure. While the report made mention of the use of computers, no reference was made as to how this could actually be programmed. It is hinted (but never explicitly stated) that a combination human intuition (for example in the choice of the starter) and computer efficiency (for example in finding the results of such a starter) might prove the best approach. This was seen in the demonstration, as it relied on human intuition for interpreting the graphs and choosing the appropriate starters and cells. This is one of the other great drawbacks of the procedure, as no method was given to make such a choice (nor any explanations for the choices made in the demonstration). Bruyn and Claasen address these issues as follows:

Reference 4.0.1 (Section 10.5)**.** *Analysing the methods described in this report one could come up with the following questions.*

Given the incidence matrix of a figure A.

1) *how does one find the component partition of A and as a sub-problem of this question how does one establish whether or not A is a proposition figure (remark that this also includes the problem to determine whether or not a given transformation figure is a deduction figure)?*
2) *how does one find appropriate starters in A for the generating method described in chapter 5 and which of [the] newly found components of the constructed transformation figure must be left out to come to useful (in whatever sense) propositions?*

Ans also, given the incidence matrices of two figures $F(1)$ and $F(2)$. How does one establish whether or not $F(1) \cong F(2)$?

To answer this, Bruyn and Claasen refer to (a) future report(s). Future reports are mentioned repeatedly and all such references have been listed below.

Introduction ... we hope to consider (in later reports) affine geometries and even parts of the Euclidean geometry.

Introduction Because our methods are not completely discussed in this report it seemed better to refrain from giving computer programs here.

Section 5.2 ... we shall only discuss here the method to generate figures which are the consequent of only one figure and of that method, we shall only give some special cases. In a subsequent report we shall discuss the method in full.

Section 5.8 The second question then is whether those generated figures are proposition figures or not. In general this question has to be dealt with by the computer (see section 10 and a later report).

Section 5.9 In this report we shall only deal with one special case: every node in the component which is to be copied has to belong to a circuit [...] In a later report we shall deal with the general case.

Section 6.28 On the other hand the above discussed theory [section 2.4.12] is not limited to $PG(2,\mathbb{R})$, but it seems that for any projective plane a similar theory can be developed. We intend to return to these problems in a later report.

Section 8.3 Therefore we set out in subsequent reports to describe propositions of $PG(2,\mathbb{R})$ by the properties of the corresponding derived graphs.

Section 9.36 In a later report we shall, after extending the theory of this report, show the equivalence of the theorem of Pappus, proposition $S(2)$ and the second quadrangle proposition and also the equivalence of the theorem of Desargues, the first quadrangle proposition and a proposition $S(4)$ which we shall presently discuss.

Section 10.8 It seemed to us, for systematic reasons mostly, that we did better to postpone the answering of the questions put forward in (10.5) to a later report. We want to describe our method to more extent first before we discuss the computer programs which underlie our whole deduction method.

Some of these statements hint at things that have already been accomplished, but have simply been left out. Based on this, it might be speculated that all of the following has partly or completely been accomplished by Bruyn and Claasen, but has not been included in the report:

1. Computer programs have been (partially) developed (to be of aid) for
   (a) generating a Levi graph and a derived graph based on a given incidence matrix;
   (b) determining whether a given figure is a proposition figure;
   (c) finding an appropriate starter in a proposition figure;
   (d) deciding what components to leave out of a constructed transformation figure;
   (e) executing the proof procedure.
2. Theory has been developed (to some extend) for
   (a) generating figures out of multiple proposition figures;
   (b) using graph theory to classify propositions based on the corresponding derived graphs;
   (c) extending this procedure to projective planes similar to $PG(2,\mathbb{R})$.
3. The proof procedure has been used to prove the equivalence of Pappus, proposition $S(2)$ and the second quadrangle proposition, and the equivalence of Desargues, the first quadrangle proposition and proposition $S(4)$[12].

However, after an extensive search to the best efforts of the author of this report, no other reports were found. This, combined with the fact that in separate interviews each of the children of Bruyn stated that he quit his work prematurely, makes it seem a safe assumption that no other reports were published. Therefore the existence of these results is, and will probably remain, unconfirmed.

This does not mean that this speculation is pointless. If one were to try to further develop the work of Bruyn and Claasen, it would seem a good starting point to first attempt to replicate the results Bruyn and Claasen already hint at. For example, it seems well possible to develop the programs that are listed.

As program 1a and 1b only involve the manipulation of graphs, it is likely that programs to do these or similar things already exist. Once these are written, program 1e could be written based on the algorithm described in section 2.6. If necessary, 1b and 1b could rely on brute force, although insight gained from developing 2b in combination with the use of graph theory might produce ways to devise more intelligent algorithms for 1b and 1b.

For further developments of the theory, the path is less clear. It would take more knowledge of projective geometry to make any feasible claims as to how that could be done.

[12] Proposition $S(4)$ has not been shown in this report.

Bruyn and Claasen also state some of their hopes for future developments. Apart from the hope of extending their method to affine geometries and parts of the Euclidean geometry mentioned in the introduction, the hope to incorporate circles in their method is mentioned. In section 7, Bruyn and Claasen note that once the entity 'circle' and its 'centre' have been introduced, not only constructions with ruler, but also those with compass fall within the boundaries of the theory of intersection. This would open up a whole new class of theorems that could be proven using this method. It is an interesting notion and would be worthy of further study.

References

[1] Bruyn, Th. (1965). An automatic proof procedure for straight line projective geometry in the plane. Nieuw archief voor de wiskunde XIII, 35-39.

[2] Bruyn, Th. (1961). Observations Concerning Computation, Deduction and Heuristics. Serie: Compte rendu des travaux effectues par l'Université d'Amsterdam dans le cadre du contrat Euratom, rapport No.9, Euratom contract 010-60-12, Amsterdam.

[3] Bruyn, Th. and Claasen, H.L. (1984). An automatic proof procedure for several geometries. Reports of the Department of Mathematics and Informatics no. 84-42, Delft University of Technology, Delft.

[4] Carton, M. (1960). Un procédé de mécanisation de la géométrie. Cybernetica 3, 83-116, 301-311.

[5] Coxeter, H.S.M. (1950). Self-dual configurations and regular graphs, Bull. Amer. Math. Soc. 56, 413-455.

[6] Hall, M jr.(1959). Theory of groups, The Macmillan Company, New York.

[7] Hessenberg, G. (1905). Beweis des Desarguesschen Satzen aus dem Pascalschen, Mathematische Annalen, Berlin.

[8] Heyting, A. (1963), Axiomatic projective geometry, Bibliotheca Mathematica, A series of monographs and applied mathematics Vol. 5, Noordhoff, Groningen-Amsterdam.

[9] Levi, F.W. (1942). Finite geometrical systems, University of Calcutta, Calcutta.

[10] Moulton, F.R. (1902) A Simple Non Desarguesian Plane Geometry, Transactions of the American Mathematical Society, Volume 3 no. 2, 192-195

[11] Ulsen, P. van (2001). E.W. Beth als logicus. University of Amsterdam, Amsterdam.

[12] Visser, H. (2010). Menselijke en kunstmatige intelligentie, http://henkxvisser.nl/Miscellaneous.html.

[13] Wilson, R.J. (1972). Introduction to graph theory, Longman, London.

Appendix A: Th. Bruyn, the person and his academic career

Little is known about the academic career of Th. Bruyn. There are very few sources available. Most of the information in this section was obtained from interviewing his children, who were still quite young at the time. His children have been interviewed separately and the stories cross-referenced in an effort to get an accurate image of events. They remembered very little about his actual work, but did remember fragments about the people involved. Because of this, this section will mostly be about the relationships of Bruyn to his co-workers and the academic world, based on the interviews.

Bruyns early years

Theo Bruyn was born in 1923. Due to heavy dyslexia, he could not come along at primary school and for a while it was even believed that he was cognitively disabled. This lasted until the seventh grade, when the principle of his school started teaching the children a little mathematics, more out of hobby than as a part of normal curriculum. It was discovered that he was extremely good at it and the story goes that he remarked 'Finally a language I can understand!' His love and fascination with mathematics never abated.

However, when he was ready to choose a study, he chose aviation engineering in Delft instead of mathematics. He came to the conclusion early on in his study that he made the wrong choice, but by then the second world war had started. During the war he received training as a fighter pilot and when the war ended, he decided to become a full time civil pilot. As the KLM was looking for pilots with knowledge of aeroplane mechanics, he also continued studying aviation engineering, which he finished in 1953.

It isn't clear when exactly he decided to start pursuing mathematics again, but around 1960 he was playing with the thought of studying mathematics. For reasons unknown, he came into contact with professor E.W. Beth (1908 - 1964), working at the University of Amsterdam. The story goes that Beth gave Bruyn one of the books he had written to read. When later Beth asked Bruyn what he thought of it, Bruyn responded that he had enjoyed studying the book. However, there was one part of the book he didn't understand. To the surprise of Beth, it turned out there had been a publishing error that had previously gone unnoticed. From that moment on, Beth and Bruyn developed an excellent working relationship. Beth convinced Bruyn to skip the studying and move straight to research.

Beth introduced Bruyn to a project they were trying to start with Euratom, the European Atomic Society. The anecdote goes that Beth told Bruyn that the team was still short an engineer. Bruyn explained to Beth that although he had originally studied engineering, he was by no means qualified to work in that field, but Beth told him that for outward appearances, it was more important to have someone with an engineering degree on the payroll, than to actually have an engineer on the team.

The Euratom project

This section is based on the section *Euratom-project* of the dissertation of Dr. P. van Ulsen, on the life and work of professor Beth (*E.W. Beth als logicus*, 2001, p.201-p.213) [11]. The work of Van Ulsen was written in Dutch, all translations have been done by the author.

The Euratom project was part of a greater project headed from Ispra in Italy. Beth's previous research had connections to other groups working within Euratom that were studying the relationship between natural languages and mathematics. These projects aimed to develop translating machines, which was deemed very useful for an international cooperating research society and for the growing European government in Brussels. Furthermore, there was an interest in automating the storage and processing of documents. In 1960 Beth was approached to do research on decision procedures.

The most important goal as stated in his first research proposal (Ulsen, p.203) was: 'The theoretical basis for the construction of a 'reasoning machine', including a study of the special conditions for an electronic computer that is capable of executing the necessary operations in an efficient manner'[13]. After deliberation with Euratom, three projects were started[14]:

1. Think machine
2. Modal logic, intuitionism and model theory.
3. Linguistic research.

The project 'Think machine' had two subtopics of study:

1. Geometry (the automation of proofs, pattern recognition and finding heuristics).
2. Reduction of proof and proof length.

Beth was convinced that based on the elementary geometry developed by Tarski, it should be possible to create a geometrical 'reasoning machine' (Ulsen p.207) that would be able to answer any appropriate geometrical question.

Bruyn was appointed for this research. He based his work on Carton (1960) [4]. Carton formulated a projective geometry with point, line and intersection as fundamental notions. Carton thought it possible to prove the theorem of Desargues from his axioms, which included the theorem of Pappos. There was also a technical aspect to the work of Bruyn, the interest in a perceptron (a parallel computer that was able to read several inputs at once). In 1961 Bruyn got in contact with the Technical University of Eindhoven for the reading and processing of geometrical constellations. In Eindhoven, as well as Paris, a study of perceptrons was in progress (Ulsen p.208).

For the other aspect of 'Think machine', the reduction of proof and proof length, S.C. van Westrhenen was appointed. Van Westrhenen used numerical analysis on the length of proof procedures within decidable classes of elementary logic formulas. Van Westrhenen would later be the one to commission the report of Bruyn and Claasen that is studied in this thesis. It is most likely that it was during this period that Bruyn and Van Westrhenen met.

In 1961 Bruyn was co-author of a report [2] where he analysed the possibility of using the axiomatic system developed by Carton for proving geometrical propositions. Apart from that, little is known about this period.

The team made several trips to Ispra, to attend conferences, meet with others and also because in Ispra there was an IBM 7090. The IBM 7090 was the first commercial transistor computer. It was crucial to much scientific development. For example, it was the first computer to calculate more than hundred thousand decimals of π (see H. Visser [12], 2010, p.7). Access to this computer was an important asset to the team.

In April 1964 Beth passed away. This was a huge blow to the project. He was succeeded by professor A. Heyting, who wrote in 1964 that Beth was the only one who saw the bigger picture of the project. He thought it possible that there were some explanations on the project that only Beth would have been able to give (Ulsen p.206). Bruyn, who had great admiration for Beth, was shattered and his productivity dropped.

In January of 1965, just before the Euratom project was shut down on March 23rd of 1965, he published a five page article in Nieuw Archief voor de Wiskunde titled *An automatic proof procedure for straight line projective geometry in the plane* [1]. In hindsight[15], much of the theory discussed in this thesis can be found in the article (it was the original inspiration for this thesis). The shutting down of the project was partly due to the passing away of Beth, but mostly because of budgetary reasons and the decision of Euratom to narrow its field of research.

[13] The original quote: 'La base théorique de la construction d'une 'machine à raisonner', y compris l'étude des conditions spéciales à un computeur électronique capable de réaliser les opérations nécessaires d'une manière efficiente'

[14] This list is a summary of the headers of p.206-p.211 of Ulsen, translated to English.

[15] The article was written down in such a concise way that without knowing what to look for, the article is near incomprehensible. The article is only five pages, no references to literature are provided, definitions are not introduced and no context was provided.

Figure 1: IBM conference in Ispra at 4, 5 and 6 october 1961. From right to left, S.C. van Westrhenen (furthest right), Th. Bruyn (next to Van Westrhenen), E.W. Beth (fourth from the right at the front) and K. de Bouvère (the second person standing above the woman with the blonde hair next to Beth).

Bruyn after Euratom

This section is again based on separate interviews with the children of Bruyn.

Before his death, Beth had asked Heyting to take personal supervision of Bruyns work. After the Euratom project finished, Bruyn continued to work with Heyting until 1969.

However, Heyting and Bruyn had very conflicting personalities and their relationship was a struggle from the beginning. This clash was emphasized by the contrast between Heyting and Beth, who had become something of a mentor for Bruyn. In Beth he felt a kindred spirit in logic, philosophy and general interest for the world. Beth was supportive of Bruyn and patient with his lack of formal mathematical training. Bruyn, dyslexic as he was, had little interest in properly writing down his ideas. The few things he did write down were typed out by his wife. As a result of that, many of his ideas weren't properly defined and/or proven. Heyting, on the other hand, placed a lot of importance in proper notation and proof. This caused a lot of tension between the two, as Heyting was of the opinion that Bruyn's ideas were interesting but vague, they were not well defined and that Bruyn's methods were unorthodox. In turn, Bruyn didn't feel Heyting appreciated the scope of his work.

Beth, who was known to have his issues with the academic climate at the University of Amsterdam, might have thought it refreshing to work with an outsider. It is known that Beth stimulated Bruyn to go straight to doing research, rather than become a student again. By the stories Bruyn's children tell, it seems like Beth encouraged Bruyn to work in his own style. By skipping the study and working in a sheltered environment, Bruyn was never forced to adapt to the academic climate at the University of Amsterdam. This didn't only show in the way he wrote down his work, but also in his general approach to his work and his approach to his co-workers and supervisors.

A story that illustrated this was when Bruyn had constructed some kind of visual aid out of one of his children's toy boxes (figure 2). Proud of his work he showed Heyting, to which Heyting remarked that he was not a child and this was not kindergarten, this was mathematics.

Figure 2: January 1964: Bruyn is creating diagrams using the toybox of one of his children. Note that the diagram in red resembles the incidence matrix in table 10.

Little is known of his work between 1965 and 1969. It is known that in 1967 he and K. de Bouvère travelled to North America to meet Alfred Tarski and Richard Montague. However, the content of those meetings is also unknown. In 1969 he got divorced and in this turbulent period, he quit the project.

In the early 1980's his former colleague S.C. van Westrhenen, then professor at the Technical University of Delft, asked him to pick up his work and was willing to locate resources to it. In 1984 he published a report titled *An automatic proof procedure for several geometries (report 1)* [3] in cooperation with H.L. Claasen, an academic assistant of S.C. van Westrhenen. It is this report that this study is based on. The report mentions ambitions for other reports, but there are as of yet none found. All that is known is that his work at the Technical University of Delft was short lived and the project ended soon after. He never picked up his work again. Theo Bruyn passed away in 1998.

Appendix B: The proposition of Desargues

In this section, all geometry follows the three fundamental axioms of projective geometry:

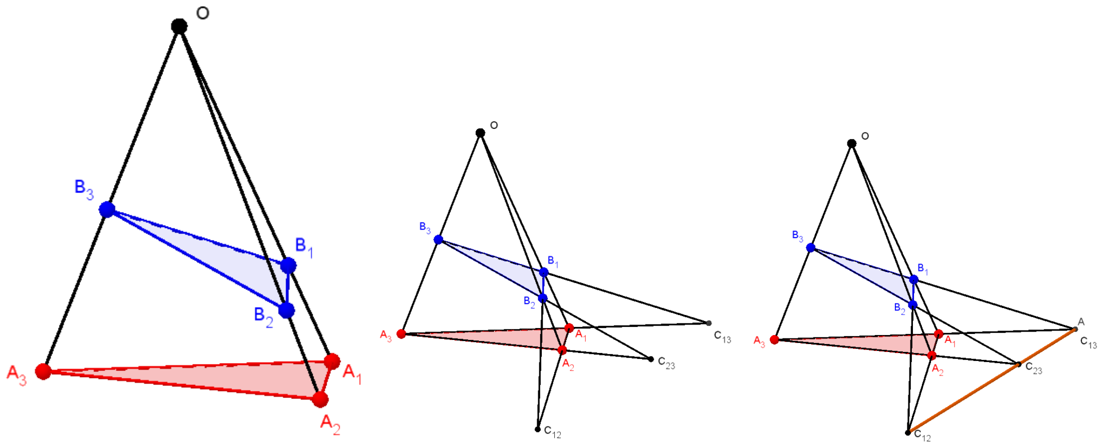
1. Given two different points, there is one and only one line with which both are incident.
2. Given two different lines, there is one and only one point with which both are incident.
3. There are at least four points such that no three of them are incident with one and the same line.

In this section, the proposition of Desargues will be examined. First, a graphical example will be given to build intuition. Then a formal (algebraic) proof will be given, which holds for most geometrical planes. After that, an example will be given where the proposition does not hold. Note that it is referred to as a proposition as opposed to a theorem. as in axiomatic geometry a theorem is a statement which can be directly deduced from the fundamental axioms of geometry (and is therefore inherently true). A proposition is a statement in which fundamental notions and logic notions occur. A proposition is valid if it is an axiom or if it can be deduced from the axioms. In other words, a proposition is valid if it is either an axiom or a theorem.

Proposition 4.0.2 (Desargues)**.** *Let A and B denote triangles, where vertices A_i and B_i ($i = 1, 2, 3$) are called corresponding vertices and sides A_iA_j and B_iB_j ($i, j = 1, 2, 3$, $i \neq j$) are called corresponding sides. If the lines connecting corresponding vertices pass through a point O, then the points of intersection of corresponding sides are collinear.*

Building intuition

Before starting on the proof, an intuition for this theorem will be built by providing a graphical example. In figure 1a two triangles are depicted, satisfying the condition that the lines connecting corresponding vertices pass through a point O. In geometry, this is referred to as triangles A and B being *perspective* from point O.



(a) (b) (c)

Figure 1: The theorem of Desargues: If the lines through corresponding vertices of the triangles are concurrent, the intersections of corresponding sides will be collinear.

The theorem continues by constructing the points of intersections C_{ij} of corresponding sides A_iA_j and B_iB_j by extending the sides of the triangle as shown in figure 1b. As we are only interested in the intersection only the segment from triangle side to intersection is shown, while in truth these extensions are true lines in the sense that they're of infinite length.

The essence of Desargues' theorem is the statement that the three intersection points C_{12}, C_{23} and C_{13} are collinear, that is to say, the points lie on the same line (as shown in figure 1c).

The way the diagrams in figure 1a, 1b and 1c are constructed may make it seem that these figures are three dimensional. Indeed, Desargues' theorem holds in $\mathbb{R}^3$ and in fact it is even so strong in $\mathbb{R}^3$ that the proof is trivial: When properly seen in 3D as depicted in figure 2, all points of intersection of corresponding sides naturally lie on the intersection of the two planes on which the triangles lie, which evidently is a straight line.

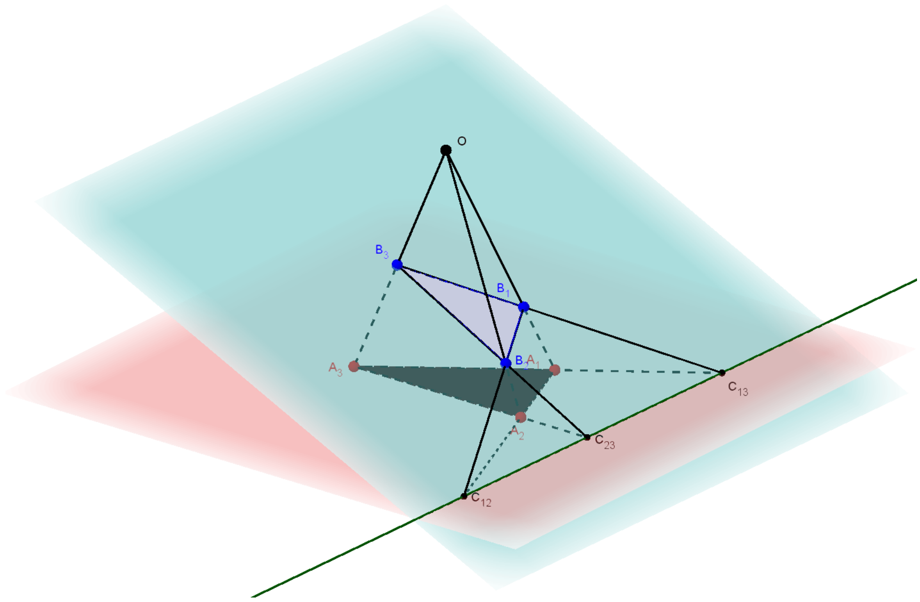


Figure 2: Desargues in 3D. The proof is trivial, as the intersection of two planes in $\mathbb{R}^3$ will always be a straigt line.

However, it is far from trivial in other fields. In the two dimensional Euclidean field, the proof is more complex and there are even some fields where Desargues' theorem does not hold. The next section will explore Desargues' theorem in other fields, starting with a purely algebraic proof.

Algebraic proof

This proof is based on the proof given in the book Axiomatic Projective Geometry (1965, p.17) by A. Heyting [8]. As the aim of this proof is to be as generic as possible, an explicit definition of points and lines will not be given. This proof relies only on the assumption that a line can be defined as a linear combination of two points.

Let A and B denote triangles, where vertices A_i and B_i ($i = 1, 2, 3$) are corresponding vertices and sides A_iA_j and B_iB_j ($i, j = 1, 2, 3$, $i \neq j$) are corresponding sides, such that the lines connecting corresponding vertices meet in a point O.

As every point B_i lies on the line through A_i and O, we can define them as a linear combination of points A_i and O. This gives us the following:

$$B_1 = O + \lambda_1 A_1$$
$$B_2 = O + \lambda_2 A_2$$
$$B_3 = O + \lambda_3 A_3$$

We define the point C_{ij} as the point of intersection of lines A_iA_j and B_iB_j. C_{ij} lies on line A_iA_j, so we can write it as a linear combination of A_i and A_j. .

$$C_{ij} = \mu_i A_i + \mu_j A_j$$

At the same time, C_{ij} also lies on line B_iB_j so it can be written as a linear combination of B_i and B_j (which are in their own right defined as a combination of O and A)

$$\begin{aligned} C_{ij} &= \nu_i B_i + \nu_j B_j \\ &= \nu_i(O + \lambda_i A_i) + \nu_j(O + \lambda_j A_j) \\ &= (\nu_i + \nu_j)O + \nu_i\lambda_i A_i + \nu_j\lambda_j A_j \end{aligned}$$

Combining both gives us the equation:

$$\begin{aligned} (\nu_i + \nu_j)O + \nu_i\lambda_i A_i + \nu_j\lambda_j A_j &= \mu_i A_i + \mu_j A_j \\ (\nu_i + \nu_j)O &= \mu_i A_i - \nu_i\lambda_i A_i + \mu_j A_j - \nu_j\lambda_j A_j \\ (\nu_i + \nu_j)O &= (\mu_i - \nu_i\lambda_i)A_i + (\mu_j - \nu_j\lambda_j)A_j \end{aligned}$$

As all λ, μ and ν are real numbers, the above states that if $\nu_i + \nu_j \neq 0$, then O is a linear combination of A_i and A_j. It is not explicitly specified that O is not on a side of the triangle, but if it is it would imply that one pair of corresponding sides lie on the same line and in that case Desargues' theorem would be trivial. So we may assume that O is not in line with any side of the triangle. This can only be the case if $\nu_i + \nu_j = 0$ (so $\nu_i = -\nu_j$).

$$\begin{aligned} C_{ij} &= (\nu_i + \nu_j)O + \nu_i\lambda_i A_i + \nu_j\lambda_j A_j \\ &= \nu_i\lambda_i A_i + \nu_j\lambda_j A_j \\ &= \nu_i\lambda_i A_i - \nu_i\lambda_j A_j \\ &= \nu_i(\lambda_i A_i - \lambda_j A_j) \end{aligned}$$

In summary we find:

$$\begin{aligned} C_{12} &= \nu_1(\lambda_1 A_1 - \lambda_2 A_2) \\ C_{23} &= \nu_2(\lambda_2 A_2 - \lambda_3 A_3) \\ C_{13} &= \nu_1(\lambda_1 A_1 - \lambda_3 A_3) \end{aligned}$$

Note that $\nu_1 + \nu_2 = 0$, so $\nu_2 = -\nu_1$. We find the following:

$$\begin{aligned} C_{12} &= \nu_1(\lambda_1 A_1 - \lambda_2 A_2) \\ &= \nu_1(\lambda_1 A_1 - \lambda_3 A_3 + \lambda_3 A_3 - \lambda_2 A_2) \\ &= \nu_1(\lambda_1 A_1 - \lambda_3 A_3) + \nu_1(\lambda_3 A_3 - \lambda_2 A_2) \\ &= \nu_1(\lambda_1 A_1 - \lambda_3 A_3) - \nu_1(\lambda_2 A_2 - \lambda_3 A_3) \\ &= \nu_1(\lambda_1 A_1 - \lambda_3 A_3) + \nu_2(\lambda_2 A_2 - \lambda_3 A_3) \\ &= C_{13} + C_{23} \end{aligned}$$

We have proven that the point C_{12} is a linear combination of the points C_{13} and C_{23}, which means that they are all on the same line.

A counter example

In this section we will define a model M for the fundamental axioms of projective geometry in which Desargues' proposition does not hold. To facilitate intuitive understanding, this model can be depicted in the Cartesian plane completed by points at infinity.

We define three kinds of M-lines:

1. Lines not parallel with the y-axis. These are given by a pair of real numbers $[a, b]$, where the first number might be interpreted as the slope and the second number as the of the y-intercept.
2. Lines parallel with the y-axis. These are given by a real number $[c]$

3. A line ω at infinity.

Similarly, three kinds of points are defined:

1. Points in the plane. These are given by a pair of real numbers (x, y).
2. Points at infinity not incident with the lines parallel with the y-axis. These are given by the real number (p) and correspond with the slope
3. A point Y at infinity incident with all lines parallel with the y-axis.

M-incidence I_M is defined as:

i)	(x,y)	I_M	$[a,b]$	$\leftrightarrow$	$y = ax + b$	$a \leq 0$	
					$y = ax + b$	$a > 0$	$x \leq 0$
					$y = 2ax + b$	$a > 0$	$x > 0$
ii)	(p)	I_M	$[a,b]$	$\leftrightarrow$	$p = a$		
iii)	(x,y)	I_M	$[c]$	$\leftrightarrow$	$x = c$		
iv)	Y	I_M	$[c]$	$\leftrightarrow$	$\forall c$		
v)	(p)	I_M	ω	$\leftrightarrow$	$\forall p$		
vi)	Y	I_M	ω				

While the definition above gives a clear definition of I in mathematical fashion, it is not very intuitive. An intuitive interpretation of I is as follows:

i)	Lines are defined in the traditional sense of $y = ax + b$, except that lines with a positive slope have a bend in them at the y-axis.
ii)	Parallel lines meet in infinity in the point $[p]$
iii)	Lines that are parallel to the y-axis are defined as the x coordinate of the points that they are incident with.
iv)	All lines parallel to the y-axis meet in infinity in the point Y.
v & *vi*)	The line at infinity ω goes through all the points at infinity.

To show that Desargues' proposition is not valid in M, we consider the following triangles A and B and the point O, as shown in figure 3.

$A_1 = (0,0)$	$B_1 = (2,0)$	$O = (0)$
$A_2 = (-1,1)$	$B_2 = (1,1)$	
$A_3 = (0,2)$	$B_3 = (2,2)$	

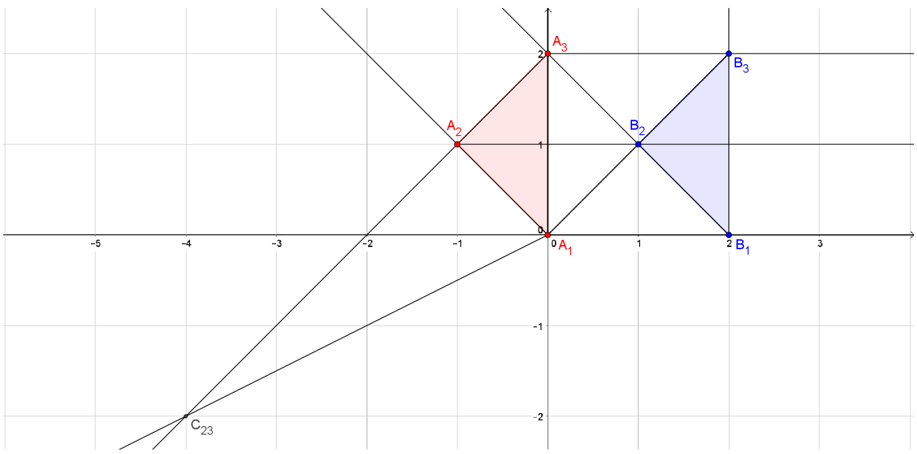


Figure 3: An example of a model where Desargues' proposition does not hold.

Note that O is defined as the point in infinity where all lines with a slope of $a = 0$ meet, in other words all lines parallel to the x-axis.

The lines A_1A_2 and B_1B_2 have the same slope of $a = -1$. This means they meet in the point (-1). The lines A_1A_3 and B_1B_3 have the same slope as they're both parallel to the y-axis (A_1A_3 *is* the y-axis) and the lines meet in point Y. The line connecting the points Y and (-1) is the line ω. However, the lines A_2A_3 and B_2B_3 meet in the point $C_{23} = (-4, -2)$, which is not incident with the line ω. Therefore, Desargues' proposition is not valid in M.

M is called a Moulton plane [10]. Because of the counterexample mentioned, Moulton planes are called non-Desarguesian planes. The existence of these planes are proof that the proposition of Desargues is not a consequence of the axioms. The 'proof' of the previous section reveals that underlying assumptions have been made about the plane and geometry used. This counterexample is a reminder that projective geometry can be very different from 'regular' geometry and that there are risks to applying spacial intuition to mathematical proofs.