A CONTRIBUTION TO THE EXTENSIONS OF ABELIAN GROUPS

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CONTENTS

KORT OVERZICHT VAN DE INHOUD

In de uitbreidingstheorie voor groepen is het probleem, om voor twee gegeven groepen K en L een expliciete methode aan te geven ter bepaling van alle mogelijke factorstelsels van L in K, nog onopgelost. Beperken we ons echter tot abelse uitbreidingen van een gegeven abelse groep K door een gegeven abelse groep L, dan is het mogelijk (zie hoofdstuk II) om een constructie aan te geven voor alle mogelijke factorstelsels in de volgende gevallen: (i) L een cyclische groep, (ii) L een quasi-cyclische groep, (iii) L de additieve groep van rationale getallen, terwijl K een willekeurige abelse groep is. Wij brengen het probleem terug tot de constructie van zekere functies van L in K, die aan zekere voorwaarden zijn gebonden.

Hoofdstuk III is gewijd aan een onderzoek van de volgende problemen:

(1) Welke zijn de torsievrije abelse groepen L, waarvoor $\mathrm{Ext}(L, T) = 0$ is voor elke abelse torsiegroep T?

(2) Als $\mathrm{Ext}(L, C(\infty)) = 0$ is, is dan L noodzakelijk vrij?

Wij leiden voor beide gevallen sommige eigenschappen af van de groepen L die aan (1) (resp. (2)) voldoen.

CHAPTER I

THE EXTENSION PROBLEM

§ 1. Introduction

The problem of group extensions consists in giving a complete survey of all groups G, such that G contains a normal subgroup A, isomorphic to a given group K with G/A isomorphic to a given group L. (Usually A is identified with K, and G is called *an extension of* K *by* L.) This problem was first proposed by O. Hölder [13][1]) and studied only for finite groups K and L.

Later the same problem was approached by O. Schreier ([18] and [19]) who considered arbitrary groups K and L. The method of approach, used by Schreier, was essentially the same as that used by Hölder.

In the general theory of Hölder and Schreier, the description of all possible extensions G (with respectively K and L as given normal subgroup and factor group) consists in finding:

(i) certain systems of automorphisms of K;

(ii) functions of $L \times L$ [2]) into K (the so-called factor sets of L into K) such that certain relations are satisfied.

The classification of the extensions of K by L is usually carried out up to equivalence: Two extensions G and $\overline{G}$ of K by L are called *equivalent* if there exists an isomorphism

$$\varphi : G \rightarrow \overline{G}$$

of G onto $\overline{G}$ leaving the elements of K and L fixed.

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[1]) Numbers in square brackets refer to the bibliography at the end of this thesis.

[2]) If A and B are sets, $A \times B$ denotes their cartesian product.

Further progress in solving the extension problem was made by R. Baer [1]. The fundamental idea used by Baer is the fact that to every extension of K by L there corresponds a well-defined homomorphism of L into the group of automorphism classes $\mathfrak{A}$(K) of K. ($\mathfrak{A}$(K) is the factor group of the group of automorphisms of K, with respect to the normal subgroup of inner automorphisms of K). He reduced the problem to the case in which K is abelian.

By making use of the fact that L can be represented as a factor group of a free group, S. Eilenberg and S. MacLane[7] reduced the study of the extensions of abelian groups to the study of homomorphisms. In the extension theory of abelian groups significant progress has been made by the use of homological methods.

The problem of giving an explicit method of construction of the possible non-equivalent extensions of a given group K by a given group L, still remains open. In chapter II we take up this problem for arbitrary abelian groups K and for L (i) a cyclic group, (ii) a quasi-cyclic group and (iii) the additive group of rational numbers. We reduce the problem to the construction of certain functions of L into K, subject to certain conditions. In chapter III we consider the following problems:

(1) Which are the torsion free abelian groups L such that Ext (L, T) $= 0$ for every torsion abelian group T? (See[2]).
(2) If Ext (L, C(∞)) $= 0$ is L necessarily free? (See[6]).

We derive some of the properties of the groups L and we also discuss problems related to (1) and (2).

§ 2. The Hölder-Schreier construction of extensions [3])

An extension G $=$ (G, $\varkappa$) of a group K by a group L is a pair consisting of a group G which contains K and a homomorphism

$$\varkappa : G \rightarrow L$$

of G onto L with kernel K. For given K and L there always exists an extension of K by L namely the direct product of K and L. Let K and L be arbitrary groups and denote their elements respectively

[3]) See [13], especially §§ 9-18, and also [18], [19] and [14].

by e, a, b, c, ... and $i, u, v, w, \ldots$ where e is the identity of K and i the identity of L. A given extension (G, χ) of K by L can be described in terms of representatives for elements of L. To each $u \in L$ we select in G a fixed representative $g(u)$ such that $g(u)\chi = u$. Since every element of G lies in some coset of K, it can be written uniquely in the form $g(u)a$ with $a \in K$. For the product of any two representatives $g(u)$ and $g(v)$ we have

$$g(u)g(v) = g(uv)f(u, v)$$

where $f(u, v) \in K$ for each pair of elements $u, v \in L$. Each representative $g(u)$ induces an automorphism φ_u in K:

$$\varphi_u : a \rightarrow a\varphi_u = g(u)^{-1}ag(u), \; a \in K.$$

Let us denote the inner automorphism of K, induced by an element $a \in K$, by $\tau(a)$. Then we have

$$(a\varphi_u)\varphi_v = (a\varphi_{uv})\tau(f(u, v)) \text{ for every } a \in K,$$

that is

$$\varphi_u\varphi_v = \varphi_{uv}\,\tau(f(u, v)) \qquad (1)$$

The associative law in the group G implies the identity

$$f(u, vw)f(v, w) = f(uv, w)(f(u, v))\,\varphi_w\,, \; u, v, w \in L \qquad (2)$$

Further, if $g(u)a$ and $g(v)b$ are two arbitrary elements of G, then

$$g(u)ag(v)b = g(uv)f(u, v)(a\varphi_v)b \qquad (3)$$

Hence from a given extension of K by L we obtain a *factor set*, i.e. a set of elements of K, $[f(u, v)]_{u, v \in L}$, and a set of automorphisms of K, $[\varphi_u]_{u \in L}$, in such a way that conditions (1) and (2) are satisfied.

Conversely, suppose that a function f of $L \times L$ into K is given and a function φ of L into the automorphism group $\mathfrak{A}(K)$ of K

$$\varphi : u \rightarrow \varphi_u, \; u \in L, \; \varphi_u \in \mathfrak{A}(K)$$

so that conditions (1) and (2) hold. Then the corresponding extension (G, χ) of K by L is the group of all pairs (u, a), $u \in L$, $a \in K$ subject to the conditions

I $(u, a) = (u', a')$ if, and only if $u = u', a = a'$;

II $(u, a)(v, b) = (uv, f(u, v)(a\varphi_v)b)$.

The element $(i, f(i, i)^{-1})$ is the identity of G and $(u^{-1}, (a\varphi_{u^{-1}})^{-1}f(u, u^{-1})^{-1}f(i, i)^{-1})$ is the inverse of (u, a).

If we map the element $a \in K$ onto the element $(i, f(i,i)^{-1}a)$ of G, we obtain an isomorphic mapping of K into G. These elements form a normal subgroup of G and every coset of K contains exactly one element of the form (u, e). Transformation of K by (u, e) induces an automorphism in K, which coincides with φ_u. The mapping

$$\chi : (u, a) \to u$$

is clearly a homomorphic mapping of G onto L with kernel K. Finally, if we choose the elements (u, e) as representatives of the left cosets of K, the factor set of this extension coincides with $[f(u, v)]_{u, v \in L}$. The extension thus constructed is equivalent to the original extension (G, χ).

Since a factor set obtained from a given extension is completely dependent on the choice of the representatives $g(u)$, we obtain: *Two extensions* (G, χ) and (G', χ') *of* K *by* L, *given by the factor sets and automorphisms* $[f(u, v)]_{u, v \in L}$, $[\varphi_u]_{u \in L}$ and $[f'(u, v)]_{u, v \in L}$, $[\varphi'_u]_{u \in L}$ *respectively, are equivalent if, and only if there exists a function* k *of* L *into* K *such that*

(a) $f'(u, v) = k(uv)^{-1} f(u, v) (k(u)) \varphi_v\, k(v)$

(b) $\varphi'_u = \varphi_u\, \tau(k(u)),\ u, v \in L.$

§ 3. Reduction to the abelian case [4])

Each extension (G, χ) of K by L determines a homomorphism of L into the group of automorphism classes $\mathfrak{A}(K)$ of K, if we map the element $u \in L$ onto the element $\varphi_u \mathcal{I}(K)$ of $\mathfrak{A}(K) = \bar{\mathfrak{A}}(K)/\mathcal{I}(K)$ where $\bar{\mathfrak{A}}(K)$ denotes the group of automorphisms of K, $\mathcal{I}(K)$ the normal subgroup of inner automorphisms of K. It is called the *homomorphism associated with the extension* (G, χ); the same homomorphism of L into $\mathfrak{A}(K)$ is associated with all the extensions of K by L, which are equivalent to (G, χ). Hence the classification of all non-equivalent extensions of K by L can be restricted to those non-equivalent extensions of K by L, which correspond to a given associated homomorphism of L into $\mathfrak{A}(K)$.

If K is non-commutative, not every homomorphism of L into

4) See [1] and [9].

$\mathfrak{A}(K)$ is associated with an extension of K by L. R. Baer constructed a counter-example. He reduced the extension problem essentially to the case in which K is abelian.

For a given K and L and a homomorphism η of L into $\mathfrak{A}(K)$, Baer constructed a group H, consisting of all those pairs $h = (u, \alpha)$ in the direct product $L \times \bar{\mathfrak{A}}(K)$ for which $\alpha \in u\eta$, $u \in L$, $\alpha \in \bar{\mathfrak{A}}(K)$.

Baer proved: *Let* (G, χ) *be an extension of K by L with the associated homomorphism* η *and let* H *be constructed as above. Then every extension* (G, χ) *of* K *by* L *associated with* η *induces a unique extension* (G, ζ) *of the centre* Z *of* K *by* H. *If two extensions* (G, χ) *and* (G', χ') *of* K *by* L *are equivalent, then the induced extensions* (G, ζ) *and* (G', ζ') *of* Z *by* H *are also equivalent.*

In the set of all non-equivalent extensions of a commutative group K by a group L with a given associated homomorphism, Baer introduced a "multiplication" and he showed that this set forms an abelian group with respect to this "multiplication".

§ 4. The method of cohomology groups [5])

If K is an (additively written) abelian group, the group of automorphism classes of K coincides with the group of automorphisms of K. Since we are considering those non-equivalent extensions of K by L (written multiplicatively) which have a given associated homomorphism, we can regard L as a *group of operators* on K. This means: K is a group with an operation $au \in K$ for $a \in K$, $u \in L$ such that

(i) $(a + b)u = au + bu$,

(ii) $a(uv) = (au)v$,

(iii) $ai = a$, $a, b \in K$, $i, u, v \in L$.

Since condition (1) of § 2 coincides with (ii) as a result of the commutativity of K, we see that an extension of an abelian group K by a group of operators L, is completely determined by a factor set i.e. a function f of $L \times L$ into K subject to condition (2), § 2. In

[5]) See [8], [9] and [16].

the set of all factor sets of the group of operators L into K, an "addition" can be introduced. If f and g are two factor sets, we define

$$(f + g)(u, v) = f(u, v) + g(u, v), \; u, v \in L.$$

With this "addition" the factor sets form an abelian group F(L, K). If k is a function of L into K, then

$$f(u, v) = -k(uv) + k(u)v + k(v), \; u, v \in L$$

is a factor set. All such factor sets form a subgroup T(L, K) of F(L, K). From the results of § 2 it follows that there is a one-to-one correspondence between the non-equivalent extensions of K by the group of operators L and the cosets of T(L, K) in F(L, K). F(L, K)/T(L, K) is called the *group of extensions* of the abelian group K by the group of operators L.

Every function f of n variables, defined on L with values in K, will be called an *n-dimensional cochain* ($n = 0, 1, 2, \ldots$) of L over K. Given two n-dimensional cochains f_1 and f_2, their sum $f_1 + f_2$, defined as

$$(f_1 + f_2)(u_1, \ldots, u_n) = f_1(u_1, \ldots, u_n) + f_2(u_1, \ldots, u_n)$$

is also a cochain. With this operation of addition the n-dimensional cochains form an abelian group denoted by $C^n(L, K)$, $n = 0, 1, 2, \ldots$ We define $C^0(L, K) = K$.

With every n-dimensional cochain f we associate an $(n + 1)$-dimensional cochain δf, the *coboundary* of f and defined as follows:

$$(\delta f)(u_1, \ldots, u_n, u_{n+1}) = f(u_2, \ldots, u_{n+1}) + \\ + \sum_{k=1}^{n} (-1)^k f(u_1, \ldots, u_{k-1}, u_k u_{k+1}, u_{k+2}, \ldots, u_{n+1}) + \\ + (-1)^{n+1} f(u_1, \ldots, u_n)u_{n+1}.$$

It follows that

$$\delta(f_1 + f_2) = \delta f_1 + \delta f_2$$

so that the mapping $f \rightarrow \delta f$ is a homomorphism of $C^n(L, K)$ into $C^{(n+1)}(L, K)$. The following important relation holds:

$$\delta(\delta f) = 0.$$

The n-dimensional cochains f with $\delta f = 0$ are called *n-dimensional cocycles.* They form a subgroup $Z^n(L, K)$ of $C^n(L, K)$. If $n > 0$, the n-dimensional cochains f such that $f = \delta f'$ for some $f' \in C^{(n-1)}(L, K)$ are coboundaries; they form a subgroup $B^n(L, K)$ of $C^n(L, K)$. For $n = 0$ we put $B^0(L, K) = 0$. Since every coboundary is a cocycle, $B^n(L, K)$ is a subgroup of $Z^n(L, K)$. The factor group

$$H^n(L, K) = Z^n(L, K)/B^n(L, K)$$

is called the *n-th cohomology group of* L *over* K.

If we consider the group of non-equivalent extensions of the abelian group K by the group of operators L, we obtain the following fundamental result [8]:

The second cohomology group $H^2(L, K)$ *coincides with the group of extensions of* K *by the group of operators* L.

Let L be represented as a factor group $L \cong F/H$ of a free group F. Transformation of H by an arbitrary element of F induces an automorphism in H. F can also be considered as a group of operators for K, by virtue of the homomorphism $F \to L$. An *operator homomorphism* of H into K is a homomorphism $\varphi : H \to K$ such that $(x^{-1}hx)\varphi = (h\varphi)x$ for all $x \in F$, $h \in H$. A *crossed homomorphism* of F into K is a one-dimensional cocycle of F in K, that is, a function ψ of F into K such that

$$(xy)\psi = (x\psi)y + y\psi \quad x, y \in F.$$

The sum of two operator homomorphisms φ_1 and φ_2 of H into K, defined by

$$(h)(\varphi_1 + \varphi_2) = h\varphi_1 + h\varphi_2, \quad h \in H$$

is an operator homomorphism of H into K. Under this addition the operator homomorphisms of H into K form an abelian group.

Now we get the following important result of Eilenberg and MacLane([8] and [16]):

The second cohomology group of L *over* K *is isomorphic to the group of operator homomorphisms of* H *into* K *modulo the subgroup of those operator homomorphisms of* H *into* K *induced by crossed homomorphisms of* F *into* K.

§ 5. Abelian extensions [6]

We assume that K and L are abelian and we shall now consider only abelian extensions of K by L. This is a special case of the previously mentioned, but since this thesis will only be concerned with the abelian extensions of abelian groups, it will be discussed in greater detail here. As we have seen above, each individual group extension can be described either by a suitable factor set or by a certain homomorphism. Let (G, η) be an (abelian) extension of K by L and choose to each $u \in L$ a representative $g(u)$ in G, $g(u)\eta = u$, and suppose, moreover, $g(0) = 0$. Then a factor set f is a function of $L \times L$ into K satisfying

(i) $f(u, 0) = f(0, v) = f(0, 0) = 0$

(ii) $f(u, v) = f(v, u)$

(iii) $f(u, v) + f(u + v, w) = f(u, v + w) + f(v, w)$, $u, v, w \in L$.

If two groups K and L are given and a factor set f of L into K, the corresponding extension (G, η) of K by L is the group of all pairs (u, a) $(u \in L,\ a \in K)$ subject to

(a) $(u, a) = (u', a')$ if, and only if $u = u'$, $a = a'$

(b) $(u, a) + (v, b) = (u + v, f(u, v) + a + b)$.

If k is a function of L into K with $k(0) = 0$, the function

$$f'(u, v) = k(u) + k(v) - k(u + v)$$

is clearly a factor set. Such a factor set is called a *transformation set*.

The direct sum of K and L is called a *splitting extension* of K by L.

The correspondence between group extensions and factors sets follows from the following theorem [7]:

Every factor set of L *in* K *corresponds to an extension of* K *by* L *which is uniquely determined up to equivalence. Conversely, every extension of* K *by* L *can be given by a factor set. Two factor sets* f_1 *and* f_2 *of* L *into* K *determine equivalent group extensions of* K *by* L *if, and only if they differ by a transformation set. The group*

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[6]) See [7] and [11].

extension determined by f is a splitting extension if, and only if f is a transformation set.

A splitting extension of K by L is therefore equivalent to the direct sum of K and L.

For fixed K and L, the sum of two factor sets f_1 and f_2 is a third factor set, defined by

$$(f_1 + f_2)(u, v) = f_1(u, v) + f_2(u, v), \; u, v \in \mathrm{L}.$$

Under this addition the factor sets form a group Fact(L, K) and the transformation sets form a subgroup Trans(L, K). From the previous theorem is follows that there is a one-to-one correspondence between the cosets of Trans(L, K) in Fact(L, K) and the classes of equivalent extensions of K by L. The group of extensions of K by L is thus defined as

$$\mathrm{Ext}(\mathrm{L}, \mathrm{K}) = \mathrm{Fact}(\mathrm{L}, \mathrm{K})/\mathrm{Trans}(\mathrm{L}, \mathrm{K}).$$

If we represent L as a factor group of a free group, the connection between homomorphisms and factor sets is given by the following theorem [7]:

If $\mathrm{L} \cong \mathrm{F}/\mathrm{H}$ *is a factor group of a free abelian group* F *while* K *is an arbitrary abelian group, then*

$$\mathrm{Ext}(\mathrm{L}, \mathrm{K}) \cong \mathrm{Hom}(\mathrm{H}, \mathrm{K})/\mathrm{Hom}(\mathrm{F} \,|\, \mathrm{H}, \mathrm{K}).$$

Hom(H, K) denotes the abelian group of all homomorphisms of H into K, the sum of two homomorphisms φ_1 and φ_2 of H into K being defined by

$$(x)(\varphi_1 + \varphi_2) = x\varphi_1 + x\varphi_2, \quad x \in \mathrm{H}$$

and Hom(F | H, K) denotes the subgroup of all homomorphisms in Hom(H, K) which can be extended to a homomorphism of F into K.

CHAPTER II

CONSTRUCTION OF EXTENSIONS

§ 1. Introduction

Throughout this and the following chapter, group will always denote an additively written abelian group [7]. In this chapter we consider the problem of constructing all factor sets of a group L into another group K. We give here a method of construction for the following cases:

(a) L a finite or infinite cyclic group
(b) L a quasi-cyclic group
(c) L the additive group of rational numbers,

K being an arbitrary group. We reduce the problem to the construction of certain functions of L into K, subject to certain conditions.

§ 2. Definitions and notation [8]

Groups in which every element has finite order, are called *torsion* groups. Those in which the elements except *0* have infinite order are called *torsion free*.

If A_λ is a set of groups, $\sum_{\lambda \in \Lambda} A_\lambda$ denotes their *direct sum* (almost all components zero) and $\sum^*_{\lambda \in \Lambda} A_\lambda$ denotes their *complete direct*

[7]) For the basic concepts concerning abelian groups we refer to [11].
[8]) See [11].

sum (where the requirement that almost all components are zero, is omitted).

R will always denote the additive group of rational numbers, $C(\infty)$ will denote the infinite cyclic group and $C(n)$ will denote a cyclic group of order n. $C(p^{\infty})$ is the quasi-cyclic group, where of course p is a prime number. We have $R/C(\infty) \cong \sum C(p^{\infty})$ where the direct sum is taken over all prime p. A group G is *divisible*, if $nG = G$ for every integer $n \neq 0$. A divisible group is isomorphic to a direct sum of groups R and groups $C(p^{\infty})$ for various primes p. A group is *reduced*, if it has no divisible subgroup $\neq 0$. Every group G has a direct decomposition $G = D + B$ where D is divisible and B is reduced.

A subgroup S of G is called a *pure subgroup* of G if $nS = S \cap nG$ for every natural integer n. If G is torsion free, S is a pure subgroup of G if, and only if G/S is torsion free. If G is torsion free, then the intersection $\bigcap_{\lambda \in \Lambda} S_\lambda$ of any set of pure subgroups S_λ of G is again pure in G.

By a *primary group* or *p-group* is meant a group in which the orders of the elements are powers of one and the same prime p. Every torsion group G can be decomposed in a unique way into the direct sum of p-groups G_p, belonging to different primes p. A subgroup B of a p-group G is called a *basic subgroup* of G, if it satisfies the following conditions:

(i) B is a direct sum of cyclic groups

(ii) B is pure in G

(iii) G/B is divisible.

Every p-group G contains a basic subgroup.

If S and T are sets, $S \times T$ will denote their cartesian product and $|S|$ will denote the power of S.

§ 3. Construction of factor sets

(a) Let L be a cyclic group, that is, either an infinite cyclic group $C(\infty) = \{\bar{u}\}$ or a finite group of prime power order, $C(p^r) = \{\bar{u}\}$. (See [11, p. 243 and p. 244.]). Suppose that f is a factor set of L into an arbitrary group K. In other words, f is a function of $L \times L$ into K such that

(i) $f(u, 0) = f(0, v) = f(0, 0) = 0, u, v \in L.$

(ii) $f(u, v) = f(v, u), u, v \in L.$

(iii) $f(u, v) + f(u + v, w) = f(u, v + w) + f(v, w), u, v, w \in L.$

Following F. Loonstra [15] we put

$$f(u, \bar{u}) = \varphi(u), u \in L$$

so that φ is a function of L into K. From (i) it follows that

$$\varphi(0) = 0 \tag{A}$$

Successive substitution in (iii) yield

$$f(u, 2\bar{u}) = \varphi(u) + \varphi(u + \bar{u}) - \varphi(\bar{u})$$

$$f(u, 3\bar{u}) = \varphi(u) + \varphi(u + \bar{u}) + \varphi(u + 2\bar{u}) - \varphi(\bar{u}) - \varphi(2\bar{u})$$

Hence by induction

$$f(u, m\bar{u}) = \sum_{k=0}^{m-1} [\varphi(u + k\bar{u}) - \varphi(k\bar{u})], (m > 0), u \in L. \tag{B}$$

Similarly, if $L = C(\infty) = \{\bar{u}\}$

$$f(u, -m\bar{u}) = \sum_{k=1}^{m} [\varphi(-k\bar{u}) - \varphi(u - k\bar{u})], (m > 0), u \in L. \tag{C}$$

Hence, to every factor set of a finite or infinite cyclic group L into K there corresponds a function φ of L into K satisfying (A).

Conversely, if φ is an arbitrary function of a cyclic group L into a group K, satisfying (A), then we construct a factor set of L into K as follows:

1) If $L = C(p^r) = \{\bar{u}\}$ we define

$$f(u, v) = f(u, m\bar{u}) = \sum_{k=0}^{m-1} [\varphi(u + k\bar{u}) - \varphi(k\bar{u})],$$

$$u, v \in C(p^r), (m > 0).$$

Clearly, f is a function of L $\times$ L into K, since

$$f(u, p^r u) = f(u, 0) = \sum_{k=0}^{p^r-1} [\varphi(u + k\bar{u}) - \varphi(k\bar{u})] = 0,$$

$$u, \in C(p^r),$$

for if k runs from 0 to $p^r—1$, then $k\bar{u}$ runs through all the elements of $C(p^r)$ and consequently $u + k\bar{u}$ will also run through all the elements of $C(p^r)$. We define $f(u, 0) = 0$ and we see that $f(0, v) = 0$, u, $v \in C(p^r)$. Hence condition (i) of a factor set is satisfied. It follows easily that $f(u, v) = f(v, u)$ and that

$$f(u, v) + f(u + v, w) = f(u, v + w) + f(v, w), \ u, v, w \in C(p^r).$$

Thus f is a factor set of $L = C(p^r)$ into K.

We still have to introduce an equivalence relation in the set of all functions of $C(p^r)$ into K subject to condition (A). Let f_1 and f_2 be two equivalent factor sets of $C(p^r)$ into K. Then there exists a function ψ of $C(p^r)$ into K, $\psi(0) = 0$, such that

$$f_1(u, v) = f_2(u, v) + \psi(u) + \psi(v) - \psi(u + v).$$

In particular

$$f_1(u, \bar{u}) = f_2(u, \bar{u}) + \psi(u) + \psi(\bar{u}) - \psi(u + \bar{u}),$$

that is

$$\varphi_1(u) = \varphi_2(u) + \psi(u) + \psi(\bar{u}) - \psi(u + \bar{u}) \qquad \text{(D)}$$

Now suppose that φ_1 and φ_2 are functions of $C(p^r)$ into K, satisfying (A) and suppose that ψ is a function of $C(p^r)$ into K, $\psi(0) = 0$, such that condition (D) holds, then the factor sets f_1 and f_2 constructed respectively from φ_1 and φ_2 are equivalent, for direct calculation shows that

$$f_1(u, v) = f_2(u, v) + \psi(u) + \psi(v) - \psi(u + v), \ u, v, w \in C(p^r).$$

2) If $L = C(\infty) = \{\bar{u}\}$ we define

$$f(u, v) = f(u, m\bar{u}) = \sum_{k=0}^{m-1} [\varphi(u + k\bar{u}) - \varphi(k\bar{u})],$$

$$m > 0. \ u, v \in C(\infty)$$

and

$$f(u, v) = f(u, -m\bar{u}) = \sum_{k=1}^{m} [\varphi(-k\bar{u}) - \varphi(u-k\bar{u})], \; m > 0,$$
$$u, v \in C(\infty).$$

Obviously, f is a function of $C(\infty) \times C(\infty)$ into K. Furthermore $f(0, v) = 0$ and we define $f(u, 0) = 0$, $u, v \in C(\infty)$ so that condition (i) of a factor set is satisfied. Straightforward calculations show that $f(u,v) = f(v, u)$ and that $f(u, v) + f(u + v, w) = f(u,v + w) + f(v, w)$, $u, v, w \in C(\infty)$ so that f is indeed a factor set.

Finally two functions φ_1 and φ_2 of $C(\infty)$ into K, satisfying (A) are equivalent if, and only if there exists a function ψ of $C(\infty)$ into K, $\psi(0) = 0$, such that

$$\varphi_1(u) = \varphi_2(u) + \psi(u) + \psi(\bar{u}) - \psi(u + \bar{u}) \qquad (D')$$

Let φ_1 and φ_2 be two arbitrary functions of $C(\infty)$ into K, satisfying (A). Then they are equivalent. The existence of a function ψ which is such, that (D′) holds, can easily be verified: We define $\psi(0) = 0$ and let $\psi(\bar{u})$ be an arbitrary element of K. Then ψ can easily be determined by means of (D′).

In particular, every function φ of $C(\infty)$ into K ($\varphi(0) = 0$) is equivalent to the function g of $C(\infty)$ into K, $g(u) = 0$ for all $u \in C(\infty)$. The factor set f' constructed from g is such, that $f'(u, v) = 0$ for all $u, v \in C(\infty)$, and the corresponding extension of K by $C(\infty)$ is the direct sum of K and $C(\infty)$. Hence we see that this is in accordance with the fact that Ext(L, K) = 0 for all groups K if L is free (see [11, p. 238]).

(b) We now proceed to the construction of all factor sets of L into K if L is a quasi-cyclic group $C(p^\infty)$ and K an arbitray group.

It is well known that the group $C(p^\infty)$ can be given by the generators

$$u_1, u_2, \ldots\ldots, u_n, \ldots\ldots,$$

and the defining relations

$$pu_1 = 0, \; pu_2 = u_1, \ldots\ldots, pu_{n+1} = u_n, \ldots\ldots$$

Let f be a factor set of $C(p^{\infty})$ into K. We put

$$f(u, u_n) = \varphi_n(u),\ u \in C(p^{\infty}).$$

Substitutions in (iii) yield

$$f(u, 2u_n) = \varphi_n(u) + \varphi_n(u + u_n) - \varphi_n(u_n),$$

$$f(u, 3u_n) =$$
$$\varphi_n(u) + \varphi_n(u + u_n) + \varphi_n(u + 2u_n) - \varphi_n(u_n) - \varphi_n(2u_n).$$

Hence by induction

$$f(u, mu_n) = \sum_{k=0}^{m-1} [\varphi_n(u + ku_n) - \varphi_n(ku_n)],\ m > 0. \qquad (E)$$

Considering the fact that $pu_1 = 0$ and $pu_{n+1} = u_n$ $(n = 1, 2, \ldots.)$ we have $f(u, u_n) = f(u, pu_{n+1}) = f(u, p^2u_{n+2}) = \ldots.. = = f(u, p^t u_{n+t}) = \ldots..$; therefore the following relations must hold:

$$\left.\begin{aligned} \varphi_n(u) &= \sum_{k=0}^{p-1} [\varphi_{n+1}(u + ku_{n+1}) - \varphi_{n+1}(ku_{n+1})] \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \sum_{k=0}^{p^t-1} [\varphi_{n+t}(u + ku_{n+t}) - \varphi_{n+t}(ku_{n+t})] \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} (n = 1, 2, \ldots) \quad (F)$$

We know, moreover, that

$$f(u, p^n u_n) = f(u, 0) = 0,$$

hence

$$\sum_{k=0}^{p^n-1} [\varphi_n(u + ku_n) - \varphi_n(ku_n)] = 0,\ (n = 1, 2, \ldots..,), \qquad (G)$$

and since $f(0, u_n) = 0$ for all u_n, we have

$$\varphi_n(0) = 0,\ (n = 1, 2, \ldots..). \qquad (H)$$

Hence, to every factor set f of $C(p^{\infty})$ into K there corresponds a sequence of functions

$$\varphi_1, \varphi_2, \ldots\ldots, \varphi_n, \ldots\ldots$$

of $C(p^{\infty})$ into K subject to the conditions (F), (G) and (H).

Conversely, suppose that $[\varphi_n]_{1 \leq n < \infty}$ is a sequence of functions of $C(p^{\infty})$ into K, satisfying (F), (G) and (H). Then we construct a factor set in the following way: We define

$$f(u, v) = f(u, mu_n) = \sum_{k=0}^{m-1} [\varphi_n(u + ku_n) - \varphi_n(ku_n)], \qquad (I)$$
$$(n = 1, 2, \ldots\ldots)$$

where $v = mu_n \in \{u_n\}$, the minimal subgroup of $C(p^{\infty})$ containing v. In order to verify that f is a factor set, we must first prove that

$$f(u, mu_n) = f(u, mp^t u_{n+t}), \ (t = 1, 2, \ldots\ldots).$$

Now

$$f(u, mp^t u_{n+t}) = \sum_{k=0}^{mp^t - 1} [\varphi_{n+t}(u + ku_{n+t}) - \varphi_{n+t}(ku_{n+t})] \qquad (1)$$

and by making use of (F) we obtain

$$f(u, mu_n) = \sum_{k=0}^{m-1} \Big\{ \sum_{l=0}^{p^t - 1} [\varphi_{n+t}(u + ku_n + lu_{n+t} - \\ - \varphi_{n+t}(ku_n + lu_{n+t})] \Big\} \qquad (2)$$

and the right hand sides of (1) and (2) are equal. We still have to verify that f is a factor set of $C(p^{\infty})$ into K.

From (I) it follows directly that $f(0, v) = 0$. Furthermore $f(u, p^n u_n) = f(u, 0) = 0$ by (G) and we define $f(u, 0) = 0$. That f also satisfies $f(u, v) = f(v, u)$ and $f(u, v) + f(u + v, w) = = f(u, v + w) + f(v, w)$, $u, v, w \in C(p^{\infty})$ follows directly from (F), (G) and (H).

Now we introduce an equivalence relation in the set of all sequences of functions of $C(p^{\infty})$ into K, subject to the conditions (F), (G) and (H). If f_1 and f_2 are two equivalent factor sets of $C(p^{\infty})$ into K, then there exists a function ψ of $C(p^{\infty})$ into K, $\psi(0) = 0$, such that

$$f_1(u, v) = f_2(u, v) + \psi(u) + \psi(v) - \psi(u+v),\ u, v \in C(p^\infty).$$

In particular

$$f_1(u, u_n) = f_2(u, u_n) + \psi(u) + \psi(u_n) - \psi(u+u_n), \quad (n = 1, 2, \ldots\ldots).$$

in other words

$$\varphi_n(u) = \varphi'_n(u) + \psi(u) + \psi(u_n) - \psi(u+u_n), \quad (n = 1, 2, \ldots\ldots). \tag{J}$$

Conversely, if we are given two sequences of functions of $C(p^\infty)$ into K, $[\varphi_n]_{1 \leqq n < \infty}$ and $[\varphi'_n]_{1 \leqq n < \infty}$ satisfying (F), (G), (H) and if there exists a function ψ of $C(p^\infty)$ into K, $\psi(0) = 0$, satisfying (J), then the factor sets f_1 and f_2 constructed respectively from the given sequences of functions, are equivalent, that is

$$f_1(u, v) = f_2(u, v) + \psi(u) + \psi(v) - \psi(u+v),\ u, v \in C(p^\infty).$$

(c) We shall now give a method of construction of all factor sets of the additive group R of rational numbers into an arbitrary group K. Our method follows the same pattern as in (b).

R has a generating system

$$u_1, u_2, \ldots\ldots, u_n, \ldots\ldots$$

and defining relations

$$2u_2 = u_1,\ 3u_3 = u_2, \ldots\ldots, (n+1)u_{n+1} = u_n, \ldots.$$

Assume that f is a factor set of R into K and put

$$f(u, u_n) = \varphi_n(u),\ u, u_n \in R,\ n = 1, 2, \ldots\ldots$$

φ_n is a function of R into K. From (i) we deduce

$$\varphi_n(0) = 0,\ n = 1, 2, \ldots\ldots \tag{M}$$

In a way similar to that used in (b) we obtain

$$f(u, mu_n) = \sum_{k=0}^{m-1} [\varphi_n(u + ku_n) - \varphi_n(ku_n)],\ m > 0, \quad n = 1, 2, \ldots\ldots \tag{N}$$

and similarly

$$f(u,-mu_n)=\sum_{k=1}^{m}\left[\varphi_n(-ku_n)-\varphi_n(u-ku_n)\right],\ m>0, \qquad \text{(O)}$$
$$n=1,2,\ldots\ldots$$

Since

$$f(u,u_n)=f(u,(n+1)u_{n+1})=f(u,(n+1)(n+2)\,u_{n+2})=\ldots.$$
$$\ldots\ldots=f(u,(n+1)(\ldots\ldots)(n+t)u_{n+t})=\ldots\ldots$$

we obtain the following relations

$$\left.\begin{aligned}\varphi_n(u)&=\sum_{k=0}^{n}\left[\varphi_{n+1}(u+ku_{n+1})-\varphi_{n+1}(ku_{n+1})\right]\\&=\sum_{k=0}^{(n+1)(n+2)-1}\left[\varphi_{n+2}(u+ku_{n+2})-\varphi_{n+2}(ku_{n+2})\right]\\&\quad\ldots\ldots\ldots\ldots\ldots\\&=\sum_{k=0}^{(n+1)(\ldots)(n+t)-1}\left[\varphi_{n+t}(u+ku_{n+t})-\varphi_{n+t}(ku_{n+t})\right]\\&\quad\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}(n=1,2,\ldots)\quad\text{(P)}$$

Hence, from a given factor set f of R into K we obtain a sequence of functions $[\varphi_n]_{1\leqq n<\infty}$ satitsfying the relations (M) and (P).

Conversely, suppose that $[\varphi_n]_{1\leqq n<\infty}$ is a sequence of functions of R into K satisfying the conditions (P) and (M). By means of these functions we shall now construct a factor set f such that $f(u,u_n)=\varphi_n(u)$, $(n=1,2,\ldots\ldots)$. To this end we define

$$f(u,v)=f(m'u_q,-mu_n)=\sum_{k=0}^{m-1}\left[\varphi_n(u+ku_n)-\varphi_n(ku_n)\right], \qquad (3)$$
$$m>0,$$

and

$$f(u,-v)=f(m'u_q,-mu_n)=\sum_{k=1}^{m}\left[\varphi_n(-ku_n)-\varphi_n(u-ku_n)\right], \qquad (4)$$
$$m>0,$$

where u_q and u_n are respectively the generators with the smallest indices q and n such that $u\in\{u_q\}$, $v\in\{u_n\}$, $u,v\in$ R.

First we show that f is a function of R $\times$ R into K, in other words we have to show that

$$f(u, v) = f(m'u_q, mu_n) = f(m'u_q, m(n+1)\ldots(n+t)u_{n+t}), \quad m > 0 \tag{5}$$

and

$$f(u, -v) = f(m'u_q, -mu_n) = $$
$$= f(m'u_q, -m(n+1)\ldots(n+t)u_{n+t}), \ m > 0. \tag{6}$$

This is indeed the case since (5) and (6) are easy consequences of the relations (M) and (P).

We show now that f is a factor set, in other words that it satisfies (i), (ii) and (iii). From our definitions it follows at once that

$$f(0, v) = 0, f(0, -v) = 0;$$

we define $f(u, 0) = 0$, $u, v \in$ R.

Direct calculations show that

$$f(u, v) = f(m'u_q, mu_n) = f(v, u);$$
$$f(u, -v) = f(m'u_q, -mu_n) = f(-v, u),$$

and that $f(u, v) + f(u+v, w) = f(u, v+w) + f(v, w)$ where $u = m'u_q$, $v = mu_n$, $w = m''u_r$, m', m, m'' integers.

On introducing an equivalence relation in the set of all sequences of functions of R into K subject to the conditions (M) and (P) we obtain the following result:

Two sequences of functions of R *into* K, $[\varphi_n]_{1 \leqq n < \infty}$ *and* $[\varphi'_n]_{1 \leqq n < \infty}$ *satisfying* (M) *and* (P) *are equivalent if, and only if there exists a function* ψ *of* R *into* K, $\psi(0) = 0$, *such that*

$$\varphi_n(u) = \varphi'_n(u) + \psi(u) + \psi(u_n) - \psi(u + u_n) \ (n = 1, 2, \ldots), \ u \in \text{R}.$$

The necessity being obvious, we note that easy calculations show that

$$f(u, v) = f'(u, v) + \psi(u) + \psi(v) - \psi(u + v), \ u, v \in \text{R}$$

where f and f' are respectively the factor sets constructed by means of $[\varphi_n]_{1 \leqq n < \infty}$ and $[\varphi'_n]_{1 \leqq n < \infty}$.

CHAPTER III

SPECIAL EXTENSION GROUPS

§ 1. Introduction

The present chapter is devoted to considering the following problems:

(1) Which are the torsion free groups L, such that Ext(L, K) = 0 for every torsion group K?

(2) If Ext(L, C(∞)) = 0, is L necessarily free?

The first problem was proposed by R. Baer [2]; he showed that a group L which satisfies the stated condition is $\aleph_1$-free. The second problem was proposed by J. H. C. Whitehead. (See [6]). A. Ehrenfeucht [6] showed that for countable L the answer is in the affirmative.

In § 3 we shall study the above mentioned groups L and we shall derive some of their properties. We shall also discuss some problems related to (1) and (2).

§ 2. Homological methods

In this section we give a brief outline of some results of homological algebra. For the proofs we refer to [4].

A sequence of groups and homomorphisms

$$L_1 \rightarrow L_2 \rightarrow \ldots\ldots \rightarrow L_n \ (n \geqq 3)$$

is called *exact* if the image of L_i in L_{i+1} under $L_i \to L_{i+1}$ coincides with the kernel of $L_{i+1} \to L_{i+2}$ ($i = 1, 2, \ldots, n-2$). Note that $L \to M$ is an isomorphism of L into M if, and only if $0 \to L \to M$ is exact, whereas $M \to N$ is a homomorphism of M onto N if, and only if $M \to N \to 0$ is exact.

If L and K are two arbitrary groups, the set of all homomorphic mappings of L into K forms, under a suitable operation, a commutative group Hom(L, K), the *homomorphism group* of L into K. If L is the direct sum of groups L_λ ($\lambda \in \Lambda$) then

$$\mathrm{Hom}\,(\sum_\lambda L_\lambda, K) \cong \sum_\lambda{}^* \mathrm{Hom}\,(L_\lambda, K)$$

and if K is the complete direct sum of the groups K_μ ($\mu \in M$) then

$$\mathrm{Hom}(L, \sum_\mu{}^* K_\mu) \cong \sum_\mu{}^* \mathrm{Hom}(L, K_\mu).$$

We have also

$$\mathrm{Ext}(\sum_\lambda L_\lambda, K) \cong \sum_\lambda{}^* \mathrm{Ext}(L_\lambda, K)$$

and

$$\mathrm{Ext}\,(L, \sum_\mu{}^* K_\mu) \cong \sum_\mu{}^* \mathrm{Ext}(L, K_\mu).$$

Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence. This sequence gives rise to the following exact sequences

$$0 \to \mathrm{Hom}(C, K) \to \mathrm{Hom}(B, K) \to \mathrm{Hom}(A, K) \to$$
$$\to \mathrm{Ext}(C, K) \to \mathrm{Ext}(B, K) \to \mathrm{Ext}(A, K) \to 0$$

and

$$0 \to \mathrm{Hom}(L, A) \to \mathrm{Hom}(L, B) \to \mathrm{Hom}(L, C) \to$$
$$\to \mathrm{Ext}(L, A) \to \mathrm{Ext}(L, B) \to \mathrm{Ext}(L, C) \to 0$$

for all groups K and L.

Let L and K be arbitrary abelian groups. Consider the free abelian group X which has as its basis the set of all ordered pairs (u, a), $u \in L$, $a \in K$. The tensor product $L \otimes K$ of L and K is defined as the factor group X/Y of X with respect to the subgroup Y generated by elements of the form

$$(u+v, a) - (u, a) - (v, a)$$

$$(u, a+b) - (u, a) - (u, b)$$

with $u, v \in L$ and $a, b \in K$. If $u \in L$, $a \in K$ we write $u \otimes a$ for the coset $(u, a) + Y$. We mention that $L \otimes K$ consists of all finite sums of the form $\sum (u_i \otimes a_i)$, $u_i \in L$, $a_i \in K$ subject to

$$(u_1 + u_2) \otimes a = u_1 \otimes a + u_2 \otimes a,$$
$$u \otimes (a_1 + a_2) = u \otimes a_1 + u \otimes a_2$$

Clearly, $L \otimes K \cong K \otimes L$.

If

$$0 \to A \to B \to C \to 0$$

is an exact sequence, then, for any group L, the sequence

$$L \otimes A \to L \otimes B \to L \otimes C \to 0 \qquad (1)$$

is also exact. If L and B are torsion free, we may add $0 \to$ to the left of (1).

§ 3. **The main theorems**

Concerning the second problem mentioned in § 1, we prove the following theorems [9]):

THEOREM 1. *If* $\mathrm{Ext}(L, C(\infty)) = 0$, *then* L *is reduced and torsion free.*

---

[9]) After I had completed the proofs of the first four theorems and Theorem 6, Professor L. Fuchs kindly pointed out that Theorem 6, Theorem 4, Theorem 2 and part of Theorem 1 had already been obtained by Rotman [17]. I am indebted to him for this reference.

Proof. Suppose the contrary. Let T denote the maximal torsion subgroup of L and D the maximal divisible subgroup of L. Then D is a direct summand of L, L = D + J where J is reduced. If D is mixed then D is a direct sum of groups R and quasi-cyclic groups,

$D = \sum R + \sum_{p_i} \sum_{\mathfrak{m}_i} C(p_i^{\infty})$. Since

$$\operatorname{Ext}(\sum_{\lambda} L_{\lambda}, K) \cong \sum_{\lambda}{}^{*} \operatorname{Ext}(L_{\lambda}, K) \tag{1}$$

we have

$$\operatorname{Ext}(L, C(\infty)) \cong \operatorname{Ext}(J, C(\infty)) + \\ + \operatorname{Ext}(\sum R, C(\infty)) + \operatorname{Ext}(\sum_{p_i} \sum_{\mathfrak{m}_i} C(p_i^{\infty}), C(\infty)).$$

We next make use of the following result [11, p. 244]: *If* B *is a torsion free and* A *a torsion group, then*

$$\operatorname{Ext}(A, B) \cong \operatorname{Hom}(A, D'/B) \tag{2}$$

where D′ *is a minimal divisible group containing* B.

However,

$$\operatorname{Ext}(\sum_{p_i} \sum_{\mathfrak{m}_i} C(p_i^{\infty}), C(\infty)) \cong \sum_{p_i}{}^{*} \sum_{\mathfrak{m}_i}{}^{*} \operatorname{Hom}(C(p_i^{\infty}), R/C(\infty)) \neq 0.$$

Hence $\operatorname{Ext}(L, C(\infty)) \neq 0$, a contradiction. Consequently the torsion subgroup T of L is reduced. This means T contains elements of order p (p prime) and of finite height and by a known theorem. [11, p. 80] T and consequently L contains a direct summand of the form $C(p^k)$ $(1 \leqq k < \infty)$. In other words $L = C(p^k) + L'$. Since there exists a non splitting extension isomorphic to $C(\infty)$ of $C(\infty)$ by $C(p^k)$, $\operatorname{Ext}(L, C(\infty)) \neq 0$ by (1), contrary to our assumption. Hence we conclude that L is torsion free.

If L is not reduced, $L = D + J = \sum R + J$ where J is torsion free and reduced. If we can show that $D = 0$ the lemma will follow. We have

$$\operatorname{Ext}(L, C(\infty)) \cong \operatorname{Ext}(J, C(\infty)) + \sum{}^{*} \operatorname{Ext}(R, C(\infty)).$$

Now $\mathrm{Ext}(R, C(\infty)) \neq 0$. To see this we represent R as a factor group of a free group, $R \cong F/H$. R is given by a set of generators $u_1, u_2, \ldots\ldots, u_n, \ldots\ldots$ and defining relations $(n+1)u_{n+1} = u_n$ $(n = 1, 2, \ldots\ldots)$. Let $F = \sum_{n=1}^{\infty} \{x_n\}$. Then H is generated by the elements $x_1 - 2x_2, x_2 - 3x_3, \ldots\ldots, x_n - (n+1)x_{n+1}, \ldots\ldots$ Evidently these elements form an independent subset in F so that $H = \sum_{n=1}^{\infty} \{x_n - (n+1)x_{n+1}\}$. Since

$$\mathrm{Ext}(R, C(\infty)) \cong \mathrm{Hom}(H, C(\infty))/\mathrm{Hom}(F|H, C(\infty))$$

let us consider the following homomorphism φ of H into $C(\infty) = \{1\}$:

$$(x_1 - 2x_2)\varphi = 1,\ (x_2 - 3x_3)\varphi = 1, \ldots\ldots,$$
$$(x_n - (n+1)x_{n+1})\varphi = 1, \ldots.$$

It follows readily that this homomorphism cannot be extended to a homomorphism of F into $C(\infty)$, whence $\mathrm{Ext}(L, C(\infty)) \neq 0$. This contradiction completes the proof of the theorem.

THEOREM 2. *If* $\mathrm{Ext}(L, C(\infty)) = 0$, *then* L *is* $\aleph_1$-*free.*

Proof. We intend to show that every countable subgroup of L is free. If N′ is any subgroup of L, the exact sequence

$$0 \to N' \to L$$

implies the exactness of

$$\mathrm{Ext}(L, C(\infty)) \to \mathrm{Ext}(N', C(\infty)) \to 0$$

and since $\mathrm{Ext}(L, C(\infty)) = 0$ it follows that $\mathrm{Ext}(N', C(\infty)) = 0$ and this holds for every subgroup N′ of L. Let M be any countable subgroup of L. By Pontrjagin's criterion [11, p. 51] M is free if, and only if any of its subgroups of finite rank is free. Let N be a subgroup of M of finite rank n, $\mathfrak{r}(N) = n$ and V_f a free subgroup of N generated by a maximal independent set of elements of N.

Then N/V_f is torsion. The exact sequence

$$0 \to V_f \overset{i}{\to} N \overset{\eta}{\to} N/V_f \to 0$$

gives rise to an exact sequence

$$0 \to \mathrm{Hom}(N/V_f, C(\infty)) \overset{\eta^*}{\to} \mathrm{Hom}(N, C(\infty)) \overset{i^*}{\to}$$

$$\overset{i^*}{\to} \mathrm{Hom}(V_f), C(\infty)) \overset{\theta}{\to} \mathrm{Ext}(N/V_f, C(\infty)) \to 0$$

that is, the sequence

$$0 \to \mathrm{Hom}(N, C(\infty)) \overset{i^*}{\to} \mathrm{Hom}(V_f, C(\infty)) \overset{\theta}{\to}$$

$$\overset{\theta}{\to} \mathrm{Hom}(N/V_f, R/C(\infty)) \to 0$$

is exact since N/V_f is torsion. Now $\mathrm{Hom}(V_f, C(\infty)) \cong V_f$ and is countable. Since θ maps $\mathrm{Hom}(V_f, C(\infty))$ onto $\mathrm{Hom}(N/V_f, R/C(\infty))$ it follows that $\mathrm{Hom}(N/V_f, R/C(\infty))$ is at most countable.

N/V_f is reduced and bounded for suppose the contrary. If D denotes the maximal divisible subgroup of N/V_f, then $N/V_f = D + J$ where J is reduced and D is a direct sum of quasi-cyclic groups. Since

$$\mathrm{Hom}\,(\sum_\lambda L_\lambda, K) \cong \sum_\lambda{}^* \mathrm{Hom}\,(L_\lambda, K)$$

we have

$$\mathrm{Hom}(N/V_f, R/C(\infty)) \cong \mathrm{Hom}(J, R/C(\infty)) +$$

$$+ \mathrm{Hom}(\sum_{p_i} \sum_{\mathfrak{m}_i} C(p_i^\infty), R/C(\infty)).$$

Now $\mathrm{Hom}(C(p^\infty), R/C(\infty))$ is nothing else but the additive group of p-adic integers [11, p. 211] and is of the power of the continuum, $\aleph$. Hence $\mathrm{Hom}(D, R/C(\infty))$ and consequently $\mathrm{Hom}(N/V_f, R/C(\infty))$ is of power $\geqq \aleph$, a contradiction. Consequently $D = 0$ and N/V_t is reduced.

If N/V_f is not bounded, then either it contains an infinity of

p-components or an unbounded p-component. In the first case each p-component contains a finite cyclic direct summand $C(p^k)$ $(1 \leqq k < \infty)$ and hence N/V_f contains an infinity of cyclic direct summands $C(p_i^{k_i})$, $(1 \leqq k_i < \infty)$, $(i \in I, |I| = \aleph_0)$,

$$N/V_f = \sum_{i \in I} C(p_i^{k_i}) + J$$

Thus

$$\mathrm{Hom}(N/V_f, R/C(\infty)) \cong \sum_{i \in I}{}^{*} \mathrm{Hom}(C(p_i^{k_i}), R/C(\infty)) + \\ + \mathrm{Hom}(J, R/C(\infty))$$

and since [11, p. 210]

$$\mathrm{Hom}(C(p_i^{k_i}), R/C(\infty)) \cong C(p_i^{k_i})$$

it follows that

$$\sum_{i \in I}{}^{*} \mathrm{Hom}(C(p_i^{k_i}), R/C(\infty))$$

is of the power of the continuum and consequently $\mathrm{Hom}(N/V_f, R/C(\infty))$ is of power $\geqq \aleph$, a contradiction.

If N/V_f contains an unbounded p-component $(N/V_f)_p$, let B denote a basic subgroup of $(N/V_f)_p$. Then $(N/V_f)_p/B$ is divisible.

The exact sequence

$$0 \to B \to (N/V_f)_p \to (N/V_f)_p/B \cong \sum C(p^\infty) \to 0$$

gives rise to the exact sequence

$$0 \to \mathrm{Hom}(\sum C(p^\infty), R/C(\infty)) \to \\ \to \mathrm{Hom}((N/V_f)_p, R/C(\infty)) \to \mathrm{Hom}(B, R/C(\infty)) \to 0$$

since $R/C(\infty)$ is divisible. $\sum^{*} \mathrm{Hom}(C(p^\infty), R/C(\infty))$ is of power $\geqq \aleph$, hence $\mathrm{Hom}((N/V_f)_p, R/C(\infty))$ and consequently $\mathrm{Hom}(N/V_f, R/C(\infty))$ is of power $\geqq \aleph$, again a contradiction.

We conclude that N/V_f is bounded. In other words, there exists a natural integer n such that $n(N/V_f) = 0$ or, equivalent to it, $nN \subseteq V_f$. Since V_f is free it follows that N is also free [11, p. 46]. This concludes the proof of the theorem.

From the above theorem follows the obvious

COROLLARY. *If* $\mathrm{Ext}(L, C(\infty)) = 0$ *and* L *is countable, then* L *is free.*

THEOREM 3. *If* $\mathrm{Ext}(L, C(\infty)) = 0$, *then for each* $0 \neq u \in L$ *there exists a homomorphism* $\varphi \in \mathrm{Hom}(L, C(\infty))$ *such that* $u\varphi \neq 0$.

Proof. Let us consider $\mathrm{Hom}(L, C(\infty))$ and $\mathrm{Hom}(\mathrm{Hom}(L, C(\infty)), C(\infty))$ and a homomorphism

$$\alpha : L \to \mathrm{Hom}(\mathrm{Hom}(L, C(\infty)), C(\infty))$$

where $(\varphi)[(u)\alpha] = (u)\varphi$, $u \in L$, $\varphi \in \mathrm{Hom}(L, C(\infty))$. The kernel of this homomorphism

$$\ker \alpha = \bigcap_{\varphi \in \mathrm{Hom}(L, C(\infty))} \ker \varphi .$$

If we can show that $\ker \alpha = 0$, the theorem will follow immediately. Now suppose that $\ker \alpha \neq 0$. Obviously, $\ker \alpha$ is a pure subgroup of L and consequently $L/\ker \alpha$ is torsion free. From the exact sequence

$$0 \to \ker \alpha \xrightarrow{i} L \xrightarrow{\eta} L/\ker \alpha \to 0$$

we derive the exact sequence

$$0 \to \mathrm{Hom}(L/\ker \alpha, C(\infty)) \xrightarrow{\eta^*} \mathrm{Hom}(L, C(\infty)) \xrightarrow{i^*}$$

$$\xrightarrow{i^*} \mathrm{Hom}(\ker \alpha, C(\infty)) \xrightarrow{\theta} \mathrm{Ext}(L/\ker \alpha, C(\infty)) \to 0$$

where the image of $\mathrm{Hom}(L, C(\infty))$ under i^* consists of all homomorphisms of $\ker \alpha$ into $C(\infty)$ which are induced by a homomorphism of L into $C(\infty)$. Clearly $[\mathrm{Hom}(L, C(\infty))]i^* = 0$ since every homomorphism of L into $C(\infty)$ maps $\ker \alpha$ into the zero of $C(\infty)$. In other words the sequences

$$0 \to \mathrm{Hom}(L/\ker \alpha, C(\infty)) \xrightarrow{\eta^*} \mathrm{Hom}(L, C(\infty)) \to 0$$

and

$$0 \to \mathrm{Hom}(\ker \alpha, C(\infty)) \xrightarrow{\theta} \mathrm{Ext}(L/\ker \alpha, C(\infty)) \to 0$$

are exact, or, equivalent to it

$$\mathrm{Hom}(L/\ker\alpha, C(\infty)) \cong \mathrm{Hom}(L, C(\infty))$$

and

$$\mathrm{Hom}(\ker\alpha, C(\infty)) \cong \mathrm{Ext}(L/\ker\alpha, C(\infty)).$$

Since $L/\ker\alpha$ is torsion free, $\mathrm{Ext}(L/\ker\alpha, C(\infty))$ is divisible, [3]. But [4, p. 116][10]).

$$\mathrm{Ext}(R, \mathrm{Hom}(\ker\alpha, C(\infty))) + \mathrm{Hom}(R, \mathrm{Ext}(\ker\alpha, C(\infty))) \cong$$
$$\cong \mathrm{Ext}\,(R \otimes \ker\alpha, C(\infty)) + \mathrm{Hom}(\mathrm{Tor}(R, \ker\alpha), C(\infty))$$

and $\mathrm{Ext}(\ker\alpha, C(\infty)) = 0$, $\mathrm{Tor}(R, \ker\alpha) = 0$ since R and $\ker\alpha$ are torsion free and $\mathrm{Ext}(R, \mathrm{Hom}(\ker\alpha, C(\infty))) = 0$ since $\mathrm{Hom}(\ker\alpha, C(\infty))$ is divisible. Consequently

$$\mathrm{Ext}(R \otimes \ker\alpha, C(\infty)) = 0.$$

Since $\ker\alpha$ is torsion free, $R \otimes \ker\alpha$ is a minimal divisible group containing $\ker\alpha$ [11, p. 256], that is $R \otimes \ker\alpha = \sum R$ whence

$$\mathrm{Ext}(R \otimes \ker\alpha, C(\infty)) \cong \sum{}^{*}\, \mathrm{Ext}(R, C(\infty)) \neq 0$$

since $\mathrm{Ext}(R, C(\infty)) \neq 0$. The arising contradiction shows that $\ker\alpha = 0$ and consequently α is an isomorphism into, that is

$$0 \to L \xrightarrow{\alpha} \mathrm{Hom}(\mathrm{Hom}(L, C(\infty)), C(\infty))$$

is exact. Q.E.D.

From the above theorem we obtain the following:

COROLLARY. *If* $\mathrm{Ext}(L, C(\infty)) = 0$ *then* L *is a subdirect sum of infinite cyclic groups.*

Proof. The mapping

$$u \to (\ldots\ldots, u + \ker\varphi_\nu, u + \ker\varphi_{\nu+1}, \ldots\ldots)$$

$(\nu \in N)$, $u \in L$, $\varphi_\nu \in \mathrm{Hom}(L, C(\infty))$ is manifestly an isomorphic mapping of L onto a subdirect sum of infinite cyclic groups. Q.E.D.

---

10) For the definition and properties of the torsion product we refer to [4].

THEOREM 4. *If* $\mathrm{Ext}(L, C(\infty)) = 0$, *then every element of* L *can be imbedded in a cyclic direct summand of* L.

Proof. Let u be an arbitrary element of L and $\{u\}$ the cyclic subgroup generated by it. Let $\{u'\}$ denote the pure cyclic subgroup of L generated by $\{u\}$. Such a subgroup certainly exists since L is $\aleph_1$-free. Consider the exact sequence

$$0 \to \{u'\} \xrightarrow{i} L \xrightarrow{\eta} L/\{u'\} \to 0$$

where $L/\{u'\}$ is torsion free since $\{u'\}$ is pure. The sequence

$$0 \to \mathrm{Hom}(L/\{u'\}, C(\infty)) \xrightarrow{\eta\theta} \mathrm{Hom}(L, C(\infty)) \xrightarrow{i\theta}$$

$$\xrightarrow{i\theta} \mathrm{Hom}(\{u'\}, C(\infty)) \xrightarrow{\theta} \mathrm{Ext}(L/\{u'\}, C(\infty)) \to 0$$

is also exact. From theorem 3 it follows that $[\mathrm{Hom}(L/\{u'\}, C(\infty))]\eta^*$ is a proper subgroup of $\mathrm{Hom}(L, C(\infty))$. Now assume that the image of $\mathrm{Hom}(L, C(\infty))$ under i^* is a proper subgroup of $\mathrm{Hom}(\{u'\}, C(\infty)) \cong C(\infty))$. Then it follows that $[\mathrm{Hom}(\{u'\}, C(\infty))]\theta$ is a finite cyclic group. But $\mathrm{Ext}(L/\{u'\}, C(\infty))$ is divisible since $L/\{u'\}$ is torsion free [3]. The desired contradiction shows that $\mathrm{Ext}(L/\{u'\}, C(\infty)) = 0$ and that the sequence

$$0 \to \mathrm{Hom}(L/\{u'\}, C(\infty)) \xrightarrow{\eta\theta} \mathrm{Hom}(L, C(\infty)) \xrightarrow{i\theta}$$

$$\xrightarrow{i\theta} \mathrm{Hom}(\{u'\}, C(\infty)) \to 0$$

is exact. $\mathrm{Hom}(\{u'\}, C(\infty)) \cong C(\infty)$ implies

$$\mathrm{Hom}(L, C(\infty)) \cong \mathrm{Hom}(L/\{u'\}, C(\infty)) + \mathrm{Hom}(\{u'\}, C(\infty)).$$

Let us consider the image of L under the mapping

$$\alpha : L \to \mathrm{Hom}(\mathrm{Hom}(L, C(\infty)), C(\infty)) \cong$$

$$\cong \mathrm{Hom}(\mathrm{Hom}(L/\{u'\}\ C(\infty)), C(\infty)) +$$

$$+ \mathrm{Hom}(\mathrm{Hom}(\{u'\}, C(\infty)), C(\infty)).$$

Under this mapping exactly the elements of $\{u'\}$ are mapped onto

$\mathrm{Hom}(\mathrm{Hom}(\{u'\}, C(\infty)), C(\infty)) \cong C(\infty)$. Hence $\{u'\}$ is a direct summand of L. This establishes the proof of the theorem.

Quite recently Chase [5] has shown that if $\mathrm{Ext}(L, C(\infty)) = 0$ then $|\mathrm{Hom}(L, C(\infty))| = 2^{|L|}$. We shall now give another simple proof of this fact for those groups L which satisfy $|L|^{\aleph_0} = |L|$. First we observe that if L is a group of cardinality $\mathfrak{m}$ and K is a group of cardinality $\mathfrak{n}$, then

$$|\mathrm{Ext}(L, K)| \leqq \mathfrak{n}^{\mathfrak{m}} \text{ and } |\mathrm{Hom}(L, K)| \leqq \mathfrak{n}^{\mathfrak{m}}. \qquad (A)$$

In the sequel we shall frequently make use of these inequalities. We shall need the following: [11])

LEMMA 5. $|\mathrm{Ext}(R, C(\infty))| = \aleph$. ($\aleph$ denotes the power of the continuum).

Proof. From our remarks above it follows that $|\mathrm{Ext}(R, C(\infty))| \leqq$ $\leqq \aleph_0^{\aleph_0} = \aleph$.

We have the exact sequence

$$0 \to C(\infty) \to R \to R/C(\infty) \to 0$$

from which we obtain the exact sequence

$$0 \to \mathrm{Hom}(C(\infty), C(\infty)) \to$$
$$\to \mathrm{Ext}(R/C(\infty), C(\infty)) \to \mathrm{Ext}(R, C(\infty)) \to 0$$

since R and $R/C(\infty)$ is divisible and $C(\infty)$ is free. From (2) it follows that

$$0 \to C(\infty) \to \mathrm{Hom}(R/C(\infty), R/C(\infty)) \to \mathrm{Ext}(R, C(\infty)) \to 0$$

is exact. Hence our assertion follows since $\mathrm{Hom}(R/C(\infty), R/C(\infty))$ is of the power $\aleph$.

THEOREM 5. *If* $\mathrm{Ext}(L, C(\infty)) = 0$ *and* $|L| = \mathfrak{m} > \aleph_0$, $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *then* $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$.

---

11) We shall assume in the sequel that the Generalized Continuum Hypothesis holds.

Proof. If L is countable, then by theorem 2 it is free and consequently $\mathrm{Hom}(L, C(\infty)) \cong L$ if L has finite rank. If L has countably infinite rank, then obviously we have $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$. Consequently we may assume that $|L| = \mathfrak{m} > \aleph_0$.

Now $|\mathrm{Hom}(L, C(\infty))| \leqq \aleph_0^{\mathfrak{m}} = 2^{\mathfrak{m}}$. Assume that $|\mathrm{Hom}(L, C(\infty))| = \mathfrak{n} < 2^{\mathfrak{m}}$. We have the following isomorphism [4, p. 116]:

$$\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty))) + \mathrm{Hom}(R, \mathrm{Ext}(L, C(\infty))) \cong$$
$$\cong \mathrm{Ext}(R \otimes L, C(\infty)) + \mathrm{Hom}(\mathrm{Tor}(R, L), C(\infty))$$

that is, we have

$$\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty))) \cong \mathrm{Ext}(R \otimes L, C(\infty))$$

since $\mathrm{Ext}(L, C(\infty)) = 0$ and $\mathrm{Tor}(R, L) = 0$, because R and L are torsion free (see [4, Chapter VII]). Let us consider $|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))|$. We have $|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))| \leqq \leqq \mathfrak{n}^{\aleph_0}$. If $\mathfrak{n} = \aleph_0$, then $|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))| \leqq \aleph$ and if $\mathfrak{n} > \aleph_0$, we have $|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))| \leqq \mathfrak{n}^{\aleph_0} < 2^{\mathfrak{m}}$. At any rate, we see that our assumption on $|\mathrm{Hom}(L, C(\infty))|$ implies that $|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))| < 2^{\mathfrak{m}}$.

However, $R \otimes L$ is a minimal divisible group containing L [11, p. 256], that is $R \otimes L \cong \sum_{\mathfrak{m}} R$ and

$$\mathrm{Ext}(R \otimes L, C(\infty)) \cong \mathrm{Ext}(\sum_{\mathfrak{m}} R, C(\infty)) \cong \sum_{\mathfrak{m}}{}^{*} \mathrm{Ext}(R, C(\infty)).$$

From lemma 5 and our assumption on $|L| = \mathfrak{m}$ we obtain $|\mathrm{Ext}(R \otimes L, C(\infty))| = 2^{\mathfrak{m}}$.

The contradiction thus obtained shows that $|\mathrm{Hom}(L, C(\infty))| = = 2^{\mathfrak{m}}$. This establishes the proof of the theorem.

From theorem 5 we obtain the following (see [5, p. 698]):

COROLLARY 5.1. *If* $\mathrm{Ext}(L, C(\infty)) = 0$, $|L| = \mathfrak{m} > \aleph_0$, $\mathfrak{m}^{\aleph_0} = = \mathfrak{m}$, *then* $\mathfrak{r}(L/pL) = \mathfrak{r}(L)$[12] *for every prime p.*

---

[12]) If G is a group, $\mathfrak{r}(G)$ denotes its rank.

Proof. Let $R^{(p)}$ denote the subgroup of R generated by $p^{-1}, p^{-2}, \ldots, p^{-n}, \ldots$. Then $R^{(p)}/C(\infty) \cong C(p^{\infty})$. From the exact sequence

$$0 \to C(\infty) \to R^{(p)} \to R^{(p)}/C(\infty) \cong C(p^{\infty}) \to 0$$

we obtain the exact sequence

$$0 \to L \otimes C(\infty) \cong L \to L \otimes R^{(p)} \to L \otimes C(p^{\infty}) \to 0.$$

Since $pR^{(p)} = R^{(p)}$, clearly $p(L \otimes R^{(p)}) = L \otimes R^{(p)}$. This exact sequence induces exactness of

$$0 \to \mathrm{Hom}(L, C(\infty)) \to \mathrm{Ext}(L \otimes C(p^{\infty}), C(\infty)) \to$$
$$\to \mathrm{Ext}(L \otimes R^{(p)}, C(\infty)) \to 0$$

that is, (by (2) and [11, p. 255])

$$0 \to \mathrm{Hom}(L, C(\infty)) \to \mathrm{Hom}(L \otimes C(p^{\infty}), R/C(\infty)) \to$$
$$\to \mathrm{Ext}(L \otimes R^{(p)}, C(\infty)) \to 0$$

is exact. But $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$ and hence it follows from the above exact sequence that $|\mathrm{Hom}(L \otimes C(p^{\infty}), R/C(\infty))| \geqq 2^{\mathfrak{m}}$. However, from the definition of the tensor product it follows that $|L \otimes C(p^{\infty})| \leqq \mathfrak{m}$ and from (A) it follows that $|\mathrm{Hom}(L \otimes C(p^{\infty}), R/C(\infty))| \leqq 2^{\mathfrak{m}}$. Consequently

$$|\mathrm{Hom}(L \otimes C(p^{\infty}), R/C(\infty))| = 2^{\mathfrak{m}}.$$

Moreover, it follows now also that $|L \otimes C(p^{\infty})| = \mathfrak{m}$.

On the other hand, it is known that $L \otimes C(p^{\infty}) \cong \sum_{\mathfrak{n}} C(p^{\infty})$ where $\mathfrak{n}$ denotes the rank of L/pL, $\mathfrak{n} = \mathfrak{r}(L/pL)$ [2, p. 255]. Thus we have

$$|L \otimes C(p^{\infty})| = |\sum_{\mathfrak{n}} C(p^{\infty})| = \mathfrak{n}\aleph_0 = \max(\mathfrak{n}, \aleph_0) = \mathfrak{n}.$$

Hence $\mathfrak{n} = \mathfrak{m}$ since obviously $\mathfrak{n} < \mathfrak{m}$ leads to a contradiction. This completes the proof since $\mathfrak{r}(L) = \mathfrak{m}$.

COROLLARY 5.2. *If* L *is torsion free,* $|L| = \mathfrak{m} > \aleph_0$, $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *and* $|\mathrm{Ext}(L, C(\infty))| < 2^{\aleph_0}$ *then* $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$ *and* $\mathfrak{r}(L/pL) = \mathfrak{r}(L)$ *for every prime p.*

Proof. It is known [5, p. 694, Theorem 4.4] that if L is torsion free and $|\mathrm{Ext}(L, C(\infty))| < 2^{\aleph_0}$, then L is $\aleph_1$-free. Hence we may assume that $|L| > \aleph_0$.

We make us of the following isomorphism [4, p. 116]:

$$\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty))) + \mathrm{Hom}(R, \mathrm{Ext}(L, C(\infty))) \cong$$
$$\cong \mathrm{Ext}(R \otimes L, C(\infty)) + \mathrm{Hom}(\mathrm{Tor}(R, L), C(\infty)).$$

Because R and L are torsion free, $\mathrm{Tor}(R, L) = 0$, hence
$$\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty))) + \mathrm{Hom}(R, \mathrm{Ext}(L, C(\infty))) \cong$$
$$\cong \mathrm{Ext}(R \otimes L, C(\infty))$$
and it is known (see the proof of theorem 5) that

$$|\mathrm{Ext}(R \otimes L, C(\infty))| = |\sum_{\mathfrak{m}}{}^{*} \mathrm{Ext}(R, C(\infty))| = 2^{\mathfrak{m}}.$$

Put $|\mathrm{Hom}(L, C(\infty))| = \mathfrak{n}$. Since $|\mathrm{Ext}(L, C(\infty))| < 2^{\aleph_0}$ it follows from (A) that $|\mathrm{Hom}(R, \mathrm{Ext}(L, C(\infty)))| \leqq \aleph_0^{\aleph_0} = 2^{\aleph_0}$. It follows also from (A) that $|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))| \leqq \mathfrak{n}^{\aleph_0}$. From the above isomorphism it follows that

$$|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))| + |\mathrm{Hom}(R, \mathrm{Ext}(L, C(\infty)))| = 2^{\mathfrak{m}}.$$

Since we have assumed $\mathfrak{m} > \aleph_0$, it is also clear that $|\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty)))| = 2^{\mathfrak{m}}$. We conclude (by (A)) that

$$|\mathrm{Hom}(L, C(\infty))| = \mathfrak{n} = 2^{\mathfrak{m}}$$

since, obviously, $\mathfrak{n} < 2^{\mathfrak{m}}$ gives rise to a contradiction.

From the exact sequence (see corollary 5.1)

$$0 \to L \to L \otimes R^{(p)} \to L \otimes C(p^{\infty}) \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}(L, C(\infty)) \to \mathrm{Ext}(L \otimes C(p^{\infty}), C(\infty)) \cong$$
$$\cong \mathrm{Hom}(L \otimes C(p^{\infty}), R/C(\infty)).$$

From this exact sequence we obtain, similarly as in corollary 5.1, that $|\mathrm{Ext}(L \otimes C(p^{\infty}), C(\infty))| = 2^{\mathfrak{m}}$. The proof that $\mathfrak{r}(L/pL) = \mathfrak{r}(L) = \mathfrak{m}$ follows now in exactly the same way as in the corresponding part of the proof of corollary 5.1. This establishes the proof of the corollary.

REMARK. If L is torsion free, $|L| > \aleph_0$, and $|\mathrm{Ext}(L, C(\infty))| <$ $< 2^{\aleph_0}$, let U_f denote a pure subgroup of L of finite rank. Then U_f is free and L/U_f is torsion free. The exact sequence

$$0 \to U_f \to L \to L/U_f \to 0$$

implies the exactness of

$$\mathrm{Hom}(U_f, C(\infty)) \to \mathrm{Ext}(L/U_f, C(\infty)) \to \mathrm{Ext}(L, C(\infty)) \to 0.$$

Since $\mathrm{Hom}(U_f, C(\infty))$ as well as $\mathrm{Ext}(L, C(\infty))$ is countable, it follows that $|\mathrm{Ext}(L/U_f, C(\infty))| < 2^{\aleph_0}$.

This remark and corollary 5.2 lead to the following plausible conjecture:

If L is torsion free and $|\mathrm{Ext}(L, C(\infty))| < 2^{\aleph_0}$, *then* $\mathrm{Ext}(L, C(\infty)) = 0$.

Suppose that every torsion free group L such that $|\mathrm{Ext}(L, C(\infty))| < 2^{\aleph_0}$, satisfies the following condition:

Every pure subgroup of L of finite rank is a direct summand of L.

Then clearly $\mathrm{Ext}(L, C(\infty)) = 0$. (See also [17, p. 251]).

COROLLARY 5.3. *If* $\mathrm{Ext}(L, C(\infty)) = 0$ *and* $|L| = \mathfrak{m} > \aleph_0$, *then* $|\mathrm{Hom}(L \otimes L, C(\infty))| = 2^{\mathfrak{m}}$ *and* $\mathfrak{r}((L \otimes L)/p(L \otimes L)) =$ $= \mathfrak{r}(L \otimes L) = \mathfrak{m}$ *for every prime p.*

Proof. That $|\mathrm{Hom}(L \otimes L, C(\infty))| = 2^{\mathfrak{m}}$ is evident, while on the other hand $\mathfrak{r}((L \otimes L)/p(L \otimes L)) = \mathfrak{r}(L \otimes L) = \mathfrak{m}$ follows from the fact that $\mathfrak{r}(L/pL) = \mathfrak{r}(L)$ for every prime p and the following theorem [12, p. 13]:

If U *and* V *are torsion free groups, then* $U \otimes V$ *is again torsion free and for any prime p we have* $\mathfrak{r}((U \otimes V)/p(U \otimes V)) =$ $= \mathfrak{r}(U/pU)\,\mathfrak{r}(V/pV)$.

REMARK. The following statements are easy consequences of theorem 5 and its corollaries:

(i) *If* L *is torsion free and* $|L| = \mathfrak{m} > \aleph_0, \mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *then either* $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$ *or* $|\mathrm{Ext}(L, C(\infty))| = 2^{\mathfrak{m}}$ *or both.*

(ii) *If* L *is any* $\aleph_1$*-free group,* $|L| = \mathfrak{m} > \aleph_0$ *and* $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$, *then* $\mathfrak{r}(L/pL) = \mathfrak{r}(L)$ *for every prime p.*

(iii) If $L = \sum_{\aleph_0}^{*} C(\infty)$ then L is $\aleph_1$-free but not free, [11, p. 168]. However, $\mathrm{Ext}(L, C(\infty)) \neq 0$. (See [17]). This follows also easily from our previous observations:

As a matter of fact, it is known that $|\mathrm{Hom}(L, C(\infty))| = \aleph_0$ [11, p. 228], and from the isomorphism

$$\mathrm{Ext}(R, \mathrm{Hom}(L, C(\infty))) + \mathrm{Hom}(R, \mathrm{Ext}(L, C(\infty))) \cong \mathrm{Ext}(R \otimes L, C(\infty))$$

and the fact that $|\mathrm{Ext}(R \otimes L, C(\infty))| = 2^{\aleph}$, we deduce $|\mathrm{Hom}(R, \mathrm{Ext}(L, C(\infty)))| = 2^{\aleph}$ whence $|\mathrm{Ext}(L, C(\infty))| = 2^{\aleph}$, indeed.

COROLLARY 5.4. *If* L *is reduced,* $|L| = \mathfrak{m} > \aleph_0$, $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *and* $\mathrm{Ext}(L, C(\infty))$ *is torsion free and if, moreover,* $\mathfrak{r}(L/pL) = \mathfrak{r}(L)$ *for at least one prime p, then* $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$. *Conversely, if* L *is reduced,* $|L| = \mathfrak{m} > \aleph_0$, *and* $\mathrm{Ext}(L, C(\infty))$ *is torsion free and if* $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$, *then* $\mathfrak{r}(L/pL) = \mathfrak{r}(L)$ *for every prime p.*

Proof. It is known [5, p. 693] that if L is reduced and $\mathrm{Ext}(L, C(\infty))$ is torsion free, then L is $\aleph_1$-free and every pure subgroup of L of finite rank is a direct summand of L. Assume that $\mathfrak{r}(L/pL) = \mathfrak{r}(L)$ for a prime *p*. Then it follows from [4, p.116] that

$$\mathrm{Ext}(C(p^{\infty}), \mathrm{Hom}(L, C(\infty))) \cong \mathrm{Ext}(C(p^{\infty}) \otimes L, C(\infty))$$

since $\mathrm{Ext}(L, C(\infty))$ is torsion free and L is torsion free. By [11, p. 255] and our assumption it follows that $C(p^{\infty}) \otimes L \cong \sum_{\mathfrak{m}} C(p^{\infty})$ and hence $|\mathrm{Ext}(\sum_{\mathfrak{m}} C(p^{\infty}), C(\infty))| = |\mathrm{Hom}(\sum_{\mathfrak{m}} C(p^{\infty}), R/C\infty)|$ $= 2^{\mathfrak{m}}$. Thus $|\mathrm{Ext}(C(p^{\infty}), \mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$, whence $|\mathrm{Hom}(L, C(\infty))| = 2^{\mathfrak{m}}$.

The converse of the corollary follows from (ii) above, completing the proof.

Now we turn our attention to the first problem mentioned in § 1. Baer proved that groups L for which $\mathrm{Ext}(L, T) = 0$ for every torsion group T, are necessarily $\aleph_1$-free, and he also raised the following question: Is $\mathrm{Ext}(L, T) = 0$ for every torsion group T, if L is the complete direct sum of a countable set of infinite cyclic groups? (See [2]).

A negative answer to this question was given by Baer [3], Erdös [10], and Sasiada [11]. We shall give here another simple proof of this fact.

THEOREM 6. *If* $L = \sum_{\aleph_0}^{*} C(\infty)$, *then there exist non-splitting extensions of torsion groups T by* L.

Proof. Let $T = \sum_{n=1}^{\infty} C(p^n)$. We shall verify that $\mathrm{Ext}(L, T) \neq 0$. First we show that $\mathrm{Ext}(R, T) \neq 0$. To this end we put $T^* = \sum_{n=1}^{\infty}{}^{*} C(p^n)$ and let T′ denote the maximal torsion subgroup of T^*. Then T is a basic subgroup of T′. Since T′/T is divisible, it follows that $T^*/T = T'/T + M/T$ where M/T is torsion free. It can easily be shown that M/T contains a non zero torsion free divisible subgroup. We conclude that M does not split since T and T^* are reduced.

Now $L = \sum_{\aleph_0}^{*} C(\infty)$ and put $N = \sum_{\aleph_0} C(\infty)$. Then L/N contains a divisible subgroup of rank $\aleph$, [11, p. 182], that is $L/N = \sum_{\aleph} R + J$. The exact sequence

$$0 \rightarrow N \rightarrow L \rightarrow L/N \rightarrow 0$$

gives rise to the exact sequence

$$\mathrm{Hom}(L, T) \rightarrow \mathrm{Hom}(N, T) \rightarrow \mathrm{Ext}(L/N, T) \rightarrow \mathrm{Ext}(L, T) \rightarrow 0$$

since N is free. But

$$\mathrm{Hom}(N, T) \cong \sum_{\aleph_0}^{*} \mathrm{Hom}(C(\infty), T) \cong \sum_{\aleph_0}^{*} T$$

and is of power $\aleph$. Furthermore

$$\mathrm{Ext}(L/N, T) \cong \sum_{\aleph}^{*} \mathrm{Ext}(R, T) + \mathrm{Ext}(J, T)$$

and is of power $\geqq 2^{\aleph}$. Since Hom(N, T) is of power $\aleph$ we conclude that $\mathrm{Ext}(L, T) \neq 0$. Q.E.D.

THEOREM 7. *If* Ext(L, T) $= 0$ *for every torsion group T,* $|L| = \mathfrak{m} > \aleph_0$, $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *then* $\mathfrak{r}(L/pL) = \mathfrak{r}(L) = \mathfrak{m}$ *for every prime p.*

For the proof we need the following:

LEMMA 7. *If* Ext(L, T) $= 0$ *for every torsion group* T *and* S *is a reduced and unbounded torsion group,* $|S| \leqq |L| = \mathfrak{m}$, $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *then* $|\mathrm{Hom}(L, S)| = 2^{\mathfrak{m}}$.

REMARK. Actually we prove much more than we need.

If S is not reduced, then it contains a direct summand $C(p^\infty)$. Let G be an arbitrary torsion free group and $|G| = \mathfrak{m} > \aleph_0$, $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$. Then $|\mathrm{Hom}(G, S)| = 2^{\mathfrak{m}}$. This follows from (A) and the following observation [4, p. 28]:

$$\mathrm{Hom}(R, \mathrm{Hom}(G, C(p^\infty))) \cong \mathrm{Hom}(R \otimes L, C(p^\infty)) \cong$$
$$\cong \sum_{\mathfrak{m}}{}^{*} \mathrm{Hom}(R, C(p^\infty))$$

and $|\sum_{\mathfrak{m}}^{*} \mathrm{Hom}(R, C(p^\infty))| = 2^{\mathfrak{m}}$.

Proof of lemma 7. First we prove that $\mathrm{Ext}(R, S) \neq 0$ [13]. To this end, let S be an arbitrary reduced and unbounded torsion group which satisfies $|S| \leqq \mathfrak{m}$. Because of the $\aleph_1$-freeness of L we have assumed $\mathfrak{m} > \aleph_0$.

Let $S = \sum_{i \in I} S_{p_i}$ be a decomposition of S into its primary components (S_{p_i} denotes the p_i-component of S). Since S is unbounded, either $|I| = \aleph_0$, or S contains an unbounded p-component S_p. In the first case each p-component S_{p_i} contains a cyclic direct summand $C(p_i^{k_i})$ and hence, if we put $S_{p_i} = C(p_i^{k_i}) + S'_{p_i}$, we have $S = \sum_{i \in I} C(p_i^{k_i}) + \sum_{i \in I} S'_{p_i}$, $|I| = \aleph_0$. Let $T_1 = \sum_{i \in I} C(p_i^{k_i})$ and write $T_1^* = \sum_{i \in I}^{*} C(p_i^{k_i})$. Then clearly T_1 is the maximal

13) See [11, p. 186].

torsion subgroup of T_1^* and T_1^*/T_1 is divisible and torsion free. T_1^*/T_1 does not split since both T_1^* and T_1 are reduced. Consequently $\mathrm{Ext}(R, S) \neq 0$. Moreover [14], $|\mathrm{Ext}(R, T_1)| = \aleph$ for the exact sequence

$$0 \to T_1 \to T_1^* \to T_1^*/T_1 \to 0$$

implies the exactness of

$$0 \to \mathrm{Hom}(R, T_1^*/T_1) \to \mathrm{Ext}(R, T_1) \to \sum_{i \in I}{}^* \mathrm{Ext}(R, C(p_i^{k_i})) = 0$$

since a bounded pure subgroup is a direct summand [11, p. 80]. $\mathrm{Hom}(R, T_1^*/T_1) \cong T_1^*/T_1$ for the exact sequence

$$0 \to C(\infty) \to R \to R/C(\infty) \to 0$$

gives rise to the exact sequence

$$0 \to \mathrm{Hom}(R, T_1^*/T_1) \to \mathrm{Hom}(C(\infty), T_1^*/T_1) \cong T_1^*/T_1 \to 0$$

since T_1^*/T_1 is torsion free and divisible. Because $|T_1^*/T_1| = \aleph$ it follows that $|\mathrm{Ext}(R, T_1)| = \aleph$ and hence $|\mathrm{Ext}(R, S)| \geqq \aleph$.

If S contains an unbounded p-component S_p, put $S = S_p + A$. Let B denote a basic group of S_p and write $B = \sum_{n=1}^{\infty} B_n$ where $B_n = \sum C(p^n)$ $(n = 1, 2, \ldots\ldots)$.

Let T′ denote the maximal torsion subgroup of $B^* = \sum_{n=1}^{\infty}{}^* B_n$. Then T'/B is divisible [11, p. 100] and thus

$$B^*/B = T'/B + M/B \qquad (M \cap T' = B, \{M, T\} = B^*)$$

where M/B is torsion free. It can easily be shown that M/B contains a non zero divisible subgroup. However, M/B does not split since both B^* and B are reduced. It can also B shown that $|\mathrm{Ext}(R, B)| \geqq \aleph_0$. But from the exact sequence [11, p. 106]

$$S_p \to B \to 0$$

we obtain the exact sequence

$$\mathrm{Ext}(R, S_p) \to \mathrm{Ext}(R, B) \to 0$$

whence it follows that $\mathrm{Ext}(R, S_p) \neq 0$ and $|\mathrm{Ext}(R, S_p)| \geqq \aleph_0$. Hence $\mathrm{Ext}(R, S) \neq 0$ and $|\mathrm{Ext}(R, S)| \geqq \aleph_0$.

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14) See [11, p. 248].

By making use of the isomorphism [4, p. 116]:

$$\mathrm{Ext}(R, \mathrm{Hom}(L, S)) + \mathrm{Hom}(R, \mathrm{Ext}(L, S)) \cong$$
$$\cong \mathrm{Ext}(R \otimes L, S) + \mathrm{Hom}(\mathrm{Tor}(R, L), S)$$

we obtain

$$\mathrm{Ext}(R, \mathrm{Hom}(L, S)) \cong \mathrm{Ext}(R \otimes L, S)$$

since $\mathrm{Ext}(L, S) = 0$ and $\mathrm{Tor}(R, L) = 0$ (because of the torsion freeness of R and L). Moreover, $|\mathrm{Ext}(R \otimes L, S)| = |\sum_{\mathfrak{m}}^{*}\mathrm{Ext}(R, S)| = 2^{\mathfrak{m}}$. From (A) it follows that $|\mathrm{Ext}(R, \mathrm{Hom}(L, S))| \leq |\mathrm{Hom}(L, S)|^{\aleph_0}$ whence $|\mathrm{Hom}(L, S)| = 2^{\mathfrak{m}}$. This completes the proof of the lemma.

Proof of theorem 7. Let $T = \sum_{n=1}^{\infty} C(p^n)$. Then it follows from the above lemma that $|\mathrm{Hom}(L, T)| = 2^{\mathfrak{m}}$. We assume of course that $|L| = \mathfrak{m} > \aleph_0$. Then from the exact sequence (see corollary 5.1)

$$0 \rightarrow C(\infty) \rightarrow R^{(p)} \rightarrow C(p^{\infty}) \rightarrow 0$$

we derive the exact sequence

$$0 \rightarrow L \otimes C(\infty) \cong L \rightarrow L \otimes R^{(p)} \rightarrow L \otimes C(p^{\infty}) \rightarrow 0$$

from which we obtain the exact sequence

$$0 \rightarrow \mathrm{Hom}(L, T) \rightarrow \mathrm{Ext}\,(L \otimes C(p^{\infty}), T) \rightarrow \mathrm{Ext}(L \otimes R^{(p)}, T) \rightarrow 0.$$

Clearly $\mathrm{Hom}(L \otimes R^{(p)}, T) = 0$ because of $p(L \otimes R^{(p)}) = L \otimes R^{(p)}$. From this exact sequence it follows that $|\mathrm{Ext}(L \otimes C(p^{\infty}), T)| = 2^{\mathfrak{m}}$. Thus $|L \otimes C(p^{\infty})| = \mathfrak{m}$. That $\mathfrak{r}(L/pL) = \mathfrak{r}(L) = \mathfrak{m}$ follows now in the same way as in the corresponding part of the proof of lemma 5.1.

COROLLARY 7.1. *If* $\mathrm{Ext}(L, T) = 0$ *for every torsion group* T, $|L| = \mathfrak{m} > \aleph_0, \mathfrak{m}^{\aleph_0} = \mathfrak{m}$, *then* $\mathfrak{r}((L \otimes L)/p(L \otimes L)) = \mathfrak{r}(L \otimes L) = \mathfrak{m}$ *for every prime* p.

Proof. Since $\mathfrak{r}(L/pL) = \mathfrak{r}(L) = \mathfrak{m}$, the corollary is a consequence of [12, p. 13, Theorem 2.3].

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BIOGRAPHY

The author of this thesis was born in 1936 in South Africa. Having completed his high school studies at Heilbron in 1953, he attended the University of the Orange Free State, Bloemfontein, where he obtained the degree M.Sc. in mathematics in 1958. In 1959 he was appointed lecturer at the University of Pretoria and in September 1960 he went to Delft, the Netherlands, as assistant to Professor Loonstra.

Future address of the author:

Dept. of Math., University of Pretoria, Pretoria.
Republic of South Africa.

STELLINGEN

I

De uitbreidingstheorie voor groepen wordt dikwijls — ten onrechte — toegeschreven aan O. Schreier.

O. Hölder, Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4, Math. Ann., vol. 43 (1893), 301-412.
O. Schreier, Über die Erweiterung von Gruppen, I, Monatsh. Math. Phys., vol. 34 (1926), 165-180.

II

Gebruik makende van de in dit proefschrift gebezigde notatie wordt het volgende uitgesproken: Indien $|\mathrm{Ext}(L, C(\infty))| < 2^{\aleph_0}$, dan is het vermoeden gerechtvaardigd, dat $\mathrm{Ext}(L, C(\infty)) = 0$ is.

Zie proefschrift blz. 34.

III

Het in het boek „Algebra" van B. L. van der Waerden gegeven bewijs van de „Satz vom primitiven Element" is aanvechtbaar.

B. L. van der Waerden, Algebra I, Fünfte Auflage, Springer-Verlag, Berlin (1960), blz. 138.

IV

De methoden die Dieudonné heeft gebruikt om aan te tonen, dat functies, die uniform kunnen worden benaderd door trapfuncties, een primitieve bezitten, kan worden vereenvoudigd, indien men zich beperkt tot continue functies en tot de benadering met behulp van polygonale functies.

J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York (1960).

V

Om de door Robbins en Monro ontwikkelde methode van stochastische approximatie practisch bruikbaar te maken, zou de methode gewijzigd moeten worden in die zin, dat van versnelde convergentie gebruik wordt gemaakt.

H. Robbins and S. Monro, A stochastic approximation method, Ann. Math. Statistics, vol. 22 (1951), 400-407.

VI

Het moet worden betreurd, dat Van der Walt in zijn proefschrift „Fixed and almost fixed points" naast toepassingen van de „fixed point theory" in de functionaal analyse niet meer practisch gerichte toepassingen heeft aangegeven.

T. van der Walt, „Fixed and almost fixed points", Dissertatie, Mathematisch Centrum, Amsterdam (1963).

VII

Op verschillende manieren kan men aantonen dat een publicatie van Komleva onjuistheden bevat.

E. A. Komleva, Over asymptotische eigenschappen van positieve sommeringsmethoden met reeksen van Fourier, Mededelingen van Mathematische Instituten (1959), blz. 89-93 (Russisch).

VIII

Het is van belang om in de methode bij diverse onderdelen in de Wiskunde meer analogie te onderkennen; als voorbeelden dienen:

(i) Het approximatief bepalen van die oplossingen van algebraïsche vergelijkingen en het oplossen van zekere congruenties.

(ii) Ontbindingsstellingen over lineaire differentie-operatoren en differentiaal-operatoren.

IX

Door verschillende auteurs wordt een functie verward met één van de functiewaarden.

H. Behnke, Vorlesung über Infinitesimalrechnung I, Aschendorffsche Verlagsbuchhandlung, Münster, Westf. (1961).

X

Het verdient aanbeveling om bij het onderwijs in de wiskunde ook de geschiedenis van de wiskunde te betrekken.