18

Nonlinear parabolic evolution equations in critical spaces

As we have seen in the preceding sections, in the context of inhomogeneous linear evolution equations, maximal regularity enables one to set up an isomorphism between the space of data (initial value and inhomogeneity) and the solution space. In the present section we will show how this idea can be used to study to non-linear evolution equations. Specifically, we consider a class of quasi-linear evolution equations of the form

$$\begin{cases} u'(t) + A(u(t))u(t) &= F(u(t)), \quad t \in (0,T), \\ u(0) &= u_0. \end{cases}$$

The setting is as follows. We are given a pair of Banach spaces (X_0, X_1) along with a continuous embedding $X_1 \hookrightarrow X_0$. The initial value u_0 is taken in a suitable interpolation space of X_0 and X_1, and for each v_0 in some neighbourhood Y of u_0 the operator $A(v_0)$ is a linear operator in X_0 with domain $\mathsf{D}(A(v_0)) = X_1$. As such, we interpret $A(v_0)$ as a bounded linear operator in $\mathscr{L}(X_1, X_0)$. The mapping F is defined on an interpolation space of X_0 and X_1, takes values in X_0, and is assumed to satisfy suitable local Lipschitz conditions; the precise assumptions will be formulated later. Our aim is to present several local well-posedness results, and to discuss a blow-up criterion which can be used to derive global well-posedness.

Before we start with this programme, we first explain the difference between semi-linear and quasi-linear evolution equations. In the *quasi-linear case*, the typical situation is that

$$A : Y \to \mathscr{L}(X_1, X_0) \text{ and } F : Z \to X_0$$

are Lipschitz continuous on bounded subsets of Y and Z, where Y and Z are (subsets of) suitable interpolation spaces between X_0 and X_1. In the *semi-linear case* one typically has that

$$A \in \mathscr{L}(X_1, X_0) \text{ and } F : Z \to X_0,$$

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_8

where A is a fixed operator in $\mathscr{L}(X_1, X_0)$ and F is a locally Lipschitz continuous mapping on bounded subsets of Z, where Z is as before. Clearly, every semi-linear equation is quasi-linear, but the converse is not true. In principle one can allow A and F to be also time-dependent, but in order to keep the presentation within reasonable bound we will not consider this additional level of generality.

When analysing evolution equations with maximal L^p-regularity methods, one usually takes Y equal to (or a subset of) the real interpolation space $(X_0, X_1)_{1-\frac{1}{p},p}$, at least in the absence of weights. The reason for taking Y of this form is that one has a continuous embedding (see Corollary L.4.6)

$$L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0) \hookrightarrow C([0,T];(X_0,X_1)_{1-\frac{1}{p},p}). \tag{18.1}$$

The space on the left-hand side is the usual space in which solutions lie when maximal L^p-regularity techniques are applicable. For the space Z one can take either take Y, or more generally $(X_0, X_1)_{\beta,1}$ with $\beta \in [1-\frac{1}{p}, 1)$, the latter requiring polynomial growth restrictions on F. In practice, we often split F into two parts $F = F_{\mathrm{Tr}} + F_{\mathrm{c}}$, where

$$F_{\mathrm{Tr}} : Y \to X_0, \quad F_{\mathrm{c}} : Z \to X_0 \tag{18.2}$$

with Y and Z as before. Here the subscript Tr stands for *trace space* and c stands for *critical*. The word *critical* is also used in the title of the chapter. In Section 18.2 we will give a definition of a criticality using only evolution equation terminology. Surprisingly, this often coincides with criticality from a PDE perspective.

The following simple example explains why the additional flexibility in choosing Z may be expected to be useful.

Example 18.0.1. On $\mathbb{R}^d$ consider the equation

$$\begin{cases} \partial_t u - a(u)\Delta u &= -u^3, \\ u(0) &= u_0, \end{cases} \tag{18.3}$$

where $a : \mathbb{R} \to [0,\infty)$ is a given locally Lipschitz continuous function. If a is non-constant, then (18.3) leads to a quasi-linear evolution equation, and if a is constant it leads to a semi-linear evolution equation. In both cases, the spaces X_0 and X_1 need to be chosen as function spaces relative to which the definitions $A(u)v := a(u)\Delta v$ and $F(u) := -u^3$ admit meaningful interpretations. A possible choice is to take

$$X_0 := L^q(\mathbb{R}^d), \quad X_1 := W^{2,q}(\mathbb{R}^d).$$

With these choices, $Y := (X_0, X_1)_{1-\frac{1}{p},p}$ equals the Besov space $B^{2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$ (see Theorem 14.4.31). If we assume that $2 - \frac{2}{p} - \frac{d}{q} > 0$, then we have the continuous embedding (see Corollary 14.4.27)

$$B^{2-\frac{2}{p}}_{q,p}(\mathbb{R}^d) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}^d).$$

The space Y then consists of bounded continuous functions, and consequently for $u \in Y$ we can interpret $a(u)$ as a bounded continuous function. The operator $A(u)$ is then well defined as an element of $\mathscr{L}(X_1, X_0)$. On the other hand, since $F(u) = -u^3$ belongs to X_0 if and only if $u \in L^{3q}(\mathbb{R}^d)$, for $u \in Y$ we can interpret $F(u)$ as an element of X_0 as soon as we have a continuous embedding

$$B^{2-\frac{2}{p}}_{q,p}(\mathbb{R}^d) \hookrightarrow L^{3q}(\mathbb{R}^d).$$

This embedding holds under the (much weaker) condition $2 - \frac{2}{p} - \frac{d}{q} > -\frac{d}{3q}$, where even equality is allowed if $p \leqslant 3q$ (see (14.22), Proposition 14.6.13, and Theorem 14.6.26). To optimally exploit this fact in situations where the more stringent condition $2 - \frac{2}{p} - \frac{d}{q} > 0$ mentioned earlier is not needed, e.g., in the semi-linear case arising when $a \equiv 1$, we may admit functions F defined on a space Z that is larger than Y. Even more flexibility is created if we take time integrability into account as by maximal L^p-regularity methods we actually only need

$$W^{1,p}(0,T;X_0) \cap L^p(0,T;X_1) \hookrightarrow L^{3p}(0,T;L^{3q}(\mathbb{R}^d))$$

in order to define $F(u)$. Conditions for this are given by Corollary L.4.7 which in the current situation $\alpha = 0$, $h = 3$ and thus $\theta = 1 - \frac{2}{3p}$ lead to the requirement $H^{2\theta,q} \hookrightarrow L^{3q}$, which holds if and only if $\frac{d}{q} + \frac{2}{p} \leqslant 3$, which is even weaker than what we saw before. We will come back to this point in Examples 18.1.3 and 18.3.1.

In Section 18.1 we start with the study of local existence and uniqueness for semi-linear equations, where the function F is defined on the trace space $Y = (X_0, X_1)_{1-\frac{1}{p}}$ with $p \in (1, \infty)$. Here we can admit initial values u_0 which belong to the space Y. We present this setting separately, as it allows us to introduce some important techniques in the simplest possible setting.

In Sections 18.2 we turn to the study of local well-posedness in the technically more demanding quasi-linear setting. At the same time, we improve on the assumptions needed to make things work: it is possible to allow exponents $p \in [1, \infty]$ and functions F of the form $F_{\mathrm{Tr}} + F_{\mathrm{c}}$ as in (18.2), with F_{Tr} defined on Y as before and F_{c} on a larger space Z. Furthermore, we work in a weighted setting in time. This has at least three important advantages:

(i) it allows initial data u_0 belonging to the space $(X_0, X_1)_{1-\alpha-\frac{1}{p}}$, where $\alpha > 0$ is a parameter associated with the weight;
(ii) global existence of solutions can be proved under milder blow-up criteria;
(iii) it allows the inclusion of the endpoint $p = \infty$ (inclusion of the endpoint $p = 1$ is possible for different reasons).

Blow-up criteria will be discussed in Section 18.2.d. After presenting an illustrating example in Section 18.3, the final Section 18.4 presents long term and even global well-posedness results for small initial data in the case $F = F_{\mathrm{c}}$ (i.e., $F_{\mathrm{Tr}} = 0$).

18.1 Semi-linear evolution equations with $F = F_{\mathrm{Tr}}$

In this section we study local well-posedness of semi-linear evolution equations of the form

$$\begin{cases} u'(t) + Au(t) &= F(u(t)), \quad t \in (0,T), \\ u(0) &= u_0, \end{cases} \tag{18.4}$$

where $T > 0$ is fixed. Parabolic partial differential equations of evolution type can often be cast into this form. Some examples will be encountered below.

Our standing assumptions are as follows. We let X_0 and X_1 be Banach spaces, with X_0 continuously embedded into X_0, we fix $p \in (1,\infty)$, and make the following assumptions:

(1) $A : X_1 \to X_0$ is a bounded linear operator;
(2) $F : (X_0, X_1)_{1-\frac{1}{p},p} \to X_0$ is a locally Lipschitz function;
(3) u_0 belongs to $(X_0, X_1)_{1-\frac{1}{p},p}$.

For the sake of brevity, in what follows we will write

$$X_{1-\frac{1}{p},p} := (X_0, X_1)_{1-\frac{1}{p},p}$$

and refer to this space as the *trace space* associated with the problem (18.4).

The following definition extends the notion of L^p-solutions to the present setting.

Definition 18.1.1. *A function $u \in L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0)$ is called an* L^p-solution *to* (18.4) *if for all $t \in [0,T]$ we have*

$$u(t) - u_0 + \int_0^t Au(s)\,\mathrm{d}s = \int_0^t F(u(s))\,\mathrm{d}s.$$

The assumptions imply that Au belongs to $L^p(0,T;X_0)$, and therefore the first integral is well defined as a Bochner integral in X_0. To prove the Bochner integrability of $F(u) : s \mapsto F(u(s))$ in X_0, we note that the assumptions and (18.1) imply that $u \in C([0,T];X_{1-\frac{1}{p},p})$. Consequently, $F(u)$ is well defined as a function in $C([0,T];X_0)$.

In order to get acquainted with the type of arguments involved, we begin by proving a preliminary local existence and uniqueness result. Later on, in Section 18.2, this result will be further improved in several ways, and continuous dependence on the initial data and conditions for global well-posedness will be discussed.

Theorem 18.1.2 (Local well-posedness for semi-linear problems). *Let $X_1 \hookrightarrow X_0$ as stated, let $p \in (1,\infty)$, and assume the conditions (1), (2), (3) to be satisfied. Suppose that*

(1) *The operator A, viewed as a linear operator in X_0 with domain $\mathsf{D}(A) = X_1$, has maximal L^p-regularity on bounded time intervals;*
(2) *There exists an non-decreasing function $\psi : (0,\infty) \to (0,\infty)$ such that for all $r > 0$ and all $x, y \in X_{1-\frac{1}{p},p}$ satisfying*

$$\|x\|_{X_{1-\frac{1}{p},p}} \leqslant r \quad \textit{and} \quad \|y\|_{X_{1-\frac{1}{p},p}} \leqslant r$$

one has

$$\|F(x) - F(y)\|_{X_0} \leqslant \psi(r)\|x - y\|_{X_{1-\frac{1}{p},p}}. \tag{18.5}$$

Then for all $R > 0$ there exists a $T > 0$ such that for all $u_0 \in X_{1-\frac{1}{p},p}$ satisfying $\|u_0\|_{X_{1-\frac{1}{p},p}} \leqslant R$ the problem (18.4) has a unique L^p-solution u. Moreover,

$$u \in L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0).$$

The bound (18.5) is a quantified local Lipschitz assumption, where "local" refers to bounded subsets of $X_{1-\frac{1}{p},p}$. We note that by (18.1) the L^p-solution u satisfies

$$u \in C([0,T];X_{1-\frac{1}{p},p}).$$

As a preparation for the proof, we first explain how the maximal L^p-regularity of A will be used to prove the theorem. By maximal L^p-regularity, for all $f \in L^p(0,T;X_0)$ the problem

$$\begin{cases} u' + Au &= f \quad \text{on } (0,T) \\ u(0) &= u_0 \end{cases}$$

admits a unique L^p-solution. Moreover, there exists a constant $C_{p,A,T}$, independent of f and u_0, such that

$$\|u\|_{L^p(0,T;X_1)\cap W^{1,p}(0,T;X_0)} \leqslant C_{p,A,T}(\|f\|_{L^p(0,T;X_0)} + \|u_0\|_{X_{1-\frac{1}{p},p}}). \tag{18.6}$$

This follows from Proposition 17.2.14 and a repetition of the argument in (17.10) and (17.11). For the optimal choice of these constants, using (17.25) one can check that $C_{p,A,T} \leqslant C_{p,A,T'}$ whenever $T \leqslant T' < \infty$.

Proof of Theorem 18.1.2. The theorem will be established by applying the Banach fixed point theorem on a suitable bounded closed subset of the *maximal regularity space*

$$\mathrm{MR}^p(0,T) := L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0).$$

The boundedness of the embedding $\mathrm{MR}^p(0,T) \hookrightarrow C([0,T];(X_0,X_1)_{1-\frac{1}{p},p})$ (see (18.1)) enables us to use the local Lipschitz assumption on F.

Let $R>0$ and fix an $u_0 \in X_{1-\frac{1}{p},p}$ satisfying $\|u_0\|_{X_{1-\frac{1}{p},p}} \leqslant R$. In order to define a suitable subset of $\mathrm{MR}^p(0,T)$ on which the fixed point argument can be performed, we introduce the *reference solution* z_{u_0} as the unique L^p-solution to

$$\begin{cases} z'(t) + Az(t) &= 0, \quad t \geqslant 0, \\ z(0) &= u_0. \end{cases}$$

Note that $z_{u_0} \in \mathrm{MR}^p(0,T)$ for every $T<\infty$. By (18.1) (and its proof) and (18.6) (with $f=0$) we have

$$\begin{aligned} \sup_{t\in[0,1]} \|z_{u_0}(t)\|_{X_{1-\frac{1}{p},p}} &\leqslant C_{p,T}\|z_{u_0}\|_{\mathrm{MR}^p(0,1)} \\ &\leqslant C_{p,T}C_{p,A,T}\|u_0\|_{X_{1-\frac{1}{p},p}} \leqslant C_{p,T}C_{p,A,T}R =: M_R. \end{aligned}$$

Fix an arbitrary $T\in(0,1]$, and consider the closed ball

$$B_1^T(u_0) := \{u \in \mathrm{MR}^p(0,T) : u(0)=u_0,\ \|u - z_{u_0}\|_{\mathrm{MR}^p(0,T)} \leqslant 1\}.$$

Let $\Phi : B_1^T(u_0) \to \mathrm{MR}^p(0,T)$ be defined by $\Phi(v) := u$, where u is the unique L^p-solution to the problem

$$\begin{cases} u' + Au &= F(v), \\ u(0) &= u_0. \end{cases}$$

This unique solution exists by the discussion preceding the proof; note that $F(v) \in C([0,T];X_0)$ by the continuity of F and (18.1).

For later purpose we observe that for all $v_1, v_2 \in B_1^T(u_0)$, Corollary L.4.6 (using $v_1(0)-v_2(0)=0$ to get T-independent constants) implies that

$$\|v_1 - v_2\|_{C([0,T];X_{1-\frac{1}{p},p})} \leqslant C_p\|v_1-v_2\|_{\mathrm{MR}^p(0,T)} \leqslant 2C_p. \tag{18.7}$$

In particular, since $T\leqslant 1$, upon taking $v_1 = v \in B_1^T(u_0)$ and $v_2 := z_{u_0} \in B_1^T(u_0)$, we find that

$$\begin{aligned} \|v\|_{C([0,T];X_{1-\frac{1}{p},p})} &\leqslant \|v - z_{u_0}\|_{C([0,T];X_{1-\frac{1}{p},p})} + \|z_{u_0}\|_{C([0,T];X_{1-\frac{1}{p},p})} \\ &\leqslant 2C_p + M_R =: N_R. \end{aligned} \tag{18.8}$$

To be able to apply the Banach fixed point theorem to Φ, we need to check that Φ maps the closed ball $B_1^T(u_0)$ into itself and is uniformly contractive on it. For both assertions it will be necessary to choose $T\in(0,1]$ small enough.

First we check that Φ maps $B_1^T(u_0)$ into itself. For all $v \in B_1^T(u_0)$ one has

$$\begin{aligned}\|F(v(t))\|_{X_0} &\leqslant \|F(v(t)) - F(z_{u_0}(t))\|_{X_0} + \|F(z_{u_0}(t)) - F(0)\|_{X_0} + \|F(0)\|_{X_0} \\ &\leqslant \psi(N_R)(\|v(t) - z_{u_0}(t)\|_{X_{1-\frac{1}{p},p}} + \|z_{u_0}(t)\|_{X_{1-\frac{1}{p},p}}) + \|F(0)\|_{X_0} \\ &\leqslant \psi(N_R)(2C_p + M_R) + \|F(0)\|_{X_0} =: C_{R,F},\end{aligned}$$

where we used (18.5) and (18.8). Thus

$$\|F(v)\|_{L^p(0,T;X_0)} \leqslant T^{1/p} C_{R,F}.$$

Therefore, letting $u = \Phi(v)$ and using the maximal L^p-regularity estimate (18.6) for the equation which $u - z_{u_0}$ satisfies, we find that

$$\|u - z_{u_0}\|_{\mathrm{MR}^p(0,T)} \leqslant C_{p,A,T}\|F(v)\|_{L^p(0,T;X_0)} \leqslant C_{p,A,1} T^{1/p} C_{R,F}.$$

Therefore, for $0 < T \leqslant (C_{p,A,1}C_{R,F})^{-p} \wedge 1$ we find that $u \in B_1^T(u_0)$.

To check that Φ is a uniform contraction, let $v_i \in B_1^T(u_0)$ for $i \in \{1, 2\}$. Using the maximal L^p-regularity estimate (18.6) for the equation which $u_1 - u_2$ satisfies, and (18.5), we find that

$$\begin{aligned}\|\Phi(v_1) - \Phi(v_2)\|_{\mathrm{MR}^p(0,T)} &\leqslant C_{p,A,T}\|F(v_1) - F(v_2)\|_{L^p(0,T;X_0)} \\ &\leqslant C_{p,A,1} T^{1/p}\psi(N_R)\|v_1 - v_2\|_{C([0,T];X_{1-\frac{1}{p},p})} \\ &\leqslant C_{p,A,1} T^{1/p}\psi(N_R) C_p\|v_1 - v_2\|_{\mathrm{MR}^p(0,T)},\end{aligned}$$

where in the last step we used (18.7). Therefore, combining both conditions on T it follows that for $T = \frac{1}{2}((C_{p,A,1}\psi(N_R)C_p)^{-p} \wedge (C_{p,A,1}C_{R,F})^{-p} \wedge 1)$ the mapping Φ is a uniform contraction on $B_1^T(u_0)$ with

$$\|\Phi(v_1) - \Phi(v_2)\|_{\mathrm{MR}^p(0,T)} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\mathrm{MR}^p(0,T)}.$$

By the Banach fixed point theorem, the restriction of Φ to $B_1^T(u_0)$ has a unique fixed point $u \in B_1^T(u_0)$. From the definition of Φ, it is immediate that u is an L^p-solution to (18.4).

It remains to prove the uniqueness. Uniqueness is clear on $B_1^T(u_0)$, but we still need to prove uniqueness in the larger set $\mathrm{MR}^p(0,T)$. Let $u_1, u_2 \in \mathrm{MR}^p(0,T)$ be L^p-solutions to (18.4). Then for every $t \in [0,T]$, by Corollary L.4.6 (with t-independent constant), (18.6), and the remarks below it, and (18.5),

$$\begin{aligned}\|u_1(t) - u_2(t)\|_{X_{1-\frac{1}{p},p}} &\leqslant C_p\|u_1 - u_2\|_{\mathrm{MR}^p(0,t)} \\ &\leqslant C_p C_{p,A,T}\|F(u_1) - F(u_2)\|_{L^p(0,t;X_0)} \\ &\leqslant C_p C_{p,A,T}\psi(N)\|u_1 - u_2\|_{L^p(0,t;X_{1-\frac{1}{p},p})},\end{aligned}$$

where N is such that $\|u_i\|_{C([0,T];X_{1-\frac{1}{p},p})} \leqslant N$ for $i \in \{1,2\}$. Therefore, applying Gronwall's inequality to $\|u_1(t) - u_2(t)\|^p_{X_{1-\frac{1}{p},p}}$, we find that $u_1 \equiv u_2$ on $[0,T]$. □

Here is a simple example to which Theorem 18.1.2 can be applied. Further examples will be given in Section 18.3.

Example 18.1.3. Let $A \in \mathscr{L}(X_1, X_0)$, where

$$X_0 = H^{s,q}(\mathbb{R}^d) \quad \text{and} \quad X_1 = H^{s+2,q}(\mathbb{R}^d)$$

with $s \in (-2,0]$ and $q \in (1,\infty)$. In the present situation, Theorem 14.4.31 shows that $X_{1-\frac{1}{p},p} = B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$. In order to have a concrete equation in mind note that one for instance could take A to be a second order differential operator such as $-\Delta$, and we could consider the PDE

$$\begin{cases} \partial_t u - \Delta u &= f(u), \\ u(0) &= u_0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given locally Lipschitz function satisfying $f(0) = 0$.

Suppose now that $s + 2 - \frac{2}{p} - \frac{d}{q} > 0$. Then, by the Sobolev embedding in Corollary 14.4.27, we have a continuous embedding

$$X_{1-\frac{1}{p},p} \hookrightarrow C_{\mathrm{b}}(\mathbb{R}^d). \tag{18.9}$$

We claim that the so-called *Nemitskii map*

$$F(u)(x) = f(u(x)), \quad x \in \mathbb{R}^d,$$

is well defined and locally Lipschitz as a mapping from $X_{1-\frac{1}{p},p}$ into X_0. To prove this, fix $N > 0$ and elements $u, v \in X_{1-\frac{1}{p},p}$ satisfying $\|u\|_{X_{1-\frac{1}{p},p}} \leqslant N$ and $\|v\|_{X_{1-\frac{1}{p},p}} \leqslant N$. Then, we obtain that for some constant L depending on f, N, and the embedding constant of (18.9),

$$\begin{aligned} \|F(u) - F(v)\|_{X_0} &\leqslant \Big(\int_{\mathbb{R}^d} |f(u(x)) - f(v(x))|^q \,\mathrm{d}x \Big)^{1/q} \\ &\leqslant L \Big(\int_{\mathbb{R}^d} |u(x) - v(x)|^q \,\mathrm{d}x \Big)^{1/q} \\ &\leqslant LC \|u - v\|_{X_{1-\frac{1}{p},p}}, \end{aligned}$$

where in the last step we used (14.22) and Proposition 14.4.18. Taking $v = 0$ and using the assumption $f(0) = 0$, one also obtains

$$\|F(u)\|_{X_0} \leqslant LC \|u\|_{X_{1-\frac{1}{p},p}}.$$

These two estimates prove the claim.

The above estimates on F are not optimal and the condition on the exponents, namely, $s+2-\frac{2}{p}-\frac{d}{q}>0$ turns out to be far from sharp. We also notice that in the example we can only treat rather smooth initial values $u_0 \in B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$ (in particular they need to be Hölder continuous). This turns out to be far from sharp. Both these sharpness issues will be addressed in the next section.

18.2 Local well-posedness for quasi-linear evolution equations

In the present section and the next, we will study local well-posedness for quasi-linear evolution equations of the form introduced at the beginning of Chapter 18,

$$\begin{cases} u'(t) + A(u(t))u(t) & = F(u(t)), \quad t \in (0,T), \\ u(0) & = u_0. \end{cases}$$

We will make several changes to the simple setting considered in Section 18.1. Besides the fact that the operator A now depends on the solution u, the changes are as follows:

- The non-linearity is of the form

$$F = F_{\mathrm{Tr}} + F_{\mathrm{c}},$$

  where F_{Tr} plays a similar role as in Section 18.1, and F_{c} is the so-called *critical part* of F. We assume that both F_{Tr} and F_{c} are defined on a suitable subset of $X_{\sigma,p}$ (see (18.10) below) with $\sigma \in [0, 1-\frac{1}{p}]$, and that F_{c} satisfies a suitable polynomial growth condition.
- Weights in time are added (see Corollaries 17.2.37 and 17.2.48). This will enable us to reduce the smoothness conditions on the initial data. At the same time, this makes it possible to formulate flexible conditions for global existence.
- The full range $p \in [1, \infty]$ will be considered.

In Example 18.3.1 we will see that the new setting takes care of the issues raised in the discussion after Example 18.1.3.

18.2.a Setting

Turning to the details, as before we make the standing assumption that we have a continuous embedding of Banach spaces

$$X_1 \hookrightarrow X_0.$$

Without loss of generality we will always assume that the constant in the embedding is $\geqslant 1$.

We further fix

$$p \in [1, \infty]$$

and

$$\alpha \in [0, \tfrac{1}{p'}) \cup \{0\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$; we take $\alpha > 0$ if $p = \infty$. The exponent α enters into the weight $w_\alpha(t) := t^\alpha$ that will be used later. In applications, the choice of α determines which initial condition u_0 can be allowed; larger values of α permit initial conditions with less smoothness. The exponent

$$\sigma := 1 - \alpha - \frac{1}{p}$$

has already been encountered in Corollary 17.2.37, and will occur frequently in what follows.

For the sake of notational brevity, we will use the conventions that

$$\begin{aligned} (X_0, X_1)_{0,r} &:= X_0 && \text{for } r \in [1, \infty], \\ X_{\theta,r} &:= \overline{X_1}^{(X_0,X_1)_{\theta,r}} && \text{for } \theta \in (0,1) \text{ and } r \in [1, \infty], \\ X_\theta &:= X_{\theta,1} && \text{for } \theta \in (0,1). \end{aligned} \tag{18.10}$$

Note that $X_{\theta,r} = (X_0, X_1)_{\theta,r}$ if $\theta \in (0,1)$ and $r \in [1, \infty)$, because in these ranges $X_0 \cap X_1 = X_1$ is dense in $(X_0, X_1)_{\theta,r}$ by Corollary C.3.15. For $\theta = 0$, $X_{\theta,r} = (X_0, X_1)_{\theta,r}$ holds for all $r \in [1, \infty]$ by definition.

Remark 18.2.1. There is some flexibility with regard to the choice of the spaces X_θ in (18.10). These spaces will appear only in the assumptions on the non-linearity F_c through (18.11) below. The only requirement needed is that $X_{\theta,1}$ continuously embeds into this space. In the above definition one could for instance take X_θ to be $X_{\theta,r}$, $[X_0, X_1]_\theta$, or $\mathsf{D}((\omega + A(u_0))^\theta)$ for $\omega \in \mathbb{R}$ large enough.

In addition to the above-stated assumptions on the spaces X_0, X_1 and the parameters p, α, σ, we make the following structural assumptions on the operator A and the non-linearity F.

Assumption 18.2.2. *We fix an open set $O_{\sigma,p} \subseteq X_{\sigma,p}$ and assume:*

(1) *the initial condition u_0 belongs to $O_{\sigma,p}$;*
(2) *there exists a constant $L \geqslant 0$ such that the mapping $A : O_{\sigma,p} \to \mathscr{L}(X_1, X_0)$ satisfies*

$$\|A(u) - A(v)\|_{\mathscr{L}(X_1,X_0)} \leqslant L\|u - v\|_{X_{\sigma,p}}, \quad u, v \in O_{\sigma,p};$$

(3) *the mapping $F : X_1 \cap O_{\sigma,p} \to X_0$ admits a decomposition $F = F_{\mathrm{Tr}} + F_c$, where*

(i) $F_{\mathrm{Tr}} : X_1 \cap O_{\sigma,p} \to X_0$ *and there exists an* $L_{\mathrm{Tr}} \geqslant 0$ *such that*

$$\|F_{\mathrm{Tr}}(u) - F_{\mathrm{Tr}}(v)\|_{X_0} \leqslant L_{\mathrm{Tr}} \|u - v\|_{X_{\sigma,p}}, \quad u, v \in X_1 \cap O_{\sigma,p};$$

(ii) $F_{\mathrm{c}} : X_1 \cap O_{\sigma,p} \to X_0$ *and there exist* $m \geqslant 1$, $\beta_j \in (\sigma, 1)$, $\rho_j > 0$ *for* $j \in \{1, \dots, m\}$, *and* $L_{\mathrm{c}} \geqslant 0$ *such that*

$$\|F_{\mathrm{c}}(u) - F_{\mathrm{c}}(v)\|_{X_0} \leqslant L_{\mathrm{c}} \sum_{j=1}^{m} (1 + \|u\|_{X_{\beta_j}}^{\rho_j} + \|v\|_{X_{\beta_j}}^{\rho_j}) \|u - v\|_{X_{\beta_j}} \tag{18.11}$$

for all $u, v \in X_1 \cap O_{\sigma,p}$, *and where*

$$\beta_j \leqslant \frac{1 + \rho_j \sigma}{1 + \rho_j}, \qquad j \in \{1, \dots, m\}. \tag{18.12}$$

Several clarifying comments are in order.

In typical applications, the set $O_{\sigma,p}$ is a bounded subset of $X_{\sigma,p}$. In situations where A, F_{Tr}, and F_{c} are defined on all of $X_{\sigma,p}$ (in case of A and F_{Tr}) and X_1 (in case of F_{c}), the constants L, L_{Tr} and L_{c} will increase with $O_{\sigma,p}$. Thus, although some of the above Lipschitz estimates are formulated as global Lipschitz conditions on $O_{\sigma,p}$, they should actually be thought of as local Lipschitz conditions on $X_{\sigma,p}$.

The quasi-linear operator A is Lipschitz on the same space as F_{Tr}. In the semi-linear case the operator A can be taken constant on $O_{\sigma,p}$.

The assumptions on the non-linearity F_{Tr} are very similar to the ones in Theorem 18.1.2 in case $\alpha = 0$ and $p \in (1, \infty)$, but for simplicity we chose to let F_{Tr} be defined on the full space $(X_0, X_1)_{1-\frac{1}{p},p}$ in that result. Taking larger values of α leads to more restrictive conditions on F_{Tr}. However, at the same time it will lead to less conditions on the initial data. The mapping F_{Tr} uniquely extends to a continuous mapping on $O_{\sigma,p}$.

A central role is played by the non-linear mapping F_{c}, where c stands for "critical". Let $\beta = \max_{j \in \{1,\dots,m\}} \beta_j$. By (18.11) and the density of X_1 in X_β we find that F_{c} uniquely extends to a locally Lipschitz function $F_{\mathrm{c}} : \overline{O_{\sigma,p} \cap X_1}^{X_\beta} \to X_0$.

The restriction (18.12) should be seen as a balance between the polynomial growth rate ρ_j of the local Lipschitz constant and the regularity exponent β_j. The larger ρ_j is, the smaller β_j needs to be. The case of equality plays a special role:

Definition 18.2.3 (Criticality). *Let Assumption 18.2.2 hold. The space* $X_{\sigma,p}$ *and the parameter* σ *are called* critical *if equality holds in* (18.12) *for some* $j \in \{1, \dots, m\}$. *In case of strict inequality in* (18.12) *for all* $j \in \{1, \dots, m\}$, *the space* $X_{\sigma,p}$ *and the parameter* σ *are called* sub-critical.

In applications to concrete non-linear PDEs, the parameters β_j and ρ_j are determined by spatial Sobolev embedding and the growth order of the polynomial non-linearity. Often, one can choose a minimal σ for which at least one of the inequalities becomes an equality. After σ has been determined, one can choose α and p such that $\sigma = 1 - \alpha - \frac{1}{p}$ holds. Here, p is usually chosen large (and thus $\alpha \in [0, 1/p') \cup \{0\}$ is close to $1-\sigma$), as this leads to the best time regularity results. Quite often, the critical space (for the initial values) $X_{\sigma,p}$ has some scaling behaviour which fits well to the scaling behaviour of solutions to the corresponding PDE. If $p = \infty$, the unweighted case $\alpha = 0$ cannot be considered due to a technical reason: in the proofs below we need $\alpha + \frac{1}{p} > 0$.

In the case $p = 1$, which is allowed in our setting, the other assumptions enforce $\alpha = 0$ and $\sigma = 0$. In particular, there is not much flexibility for the function $F_{\rm Tr}$ and it needs to be locally Lipschitz on X_0. On the other hand, (18.11) is still quite flexible: for instance if $m = 1$ one can allow $F_{\rm c} : X_{1/(1+\rho)} \to X_0$ where growth of power ρ is allowed for the Lipschitz constant.

Remark 18.2.4 (Time-dependent and inhomogeneous settings). It is possible to extend the above setting to time-dependent mappings $A : [0,T] \times O_{\sigma,p} \to \mathscr{L}(X_1, X_0)$ and $F : [0,T] \times X_1 \cap O_{\sigma,p} \to X_0$. This does not lead to any major changes as long as the mapping properties of A and F and estimates are uniform in $t \in [0,T]$ (or the constants in the estimates satisfy a suitable integrability condition). Usually, A is assumed to be continuous in time, so that maximal regularity of $A(0, u_0)$ can be used in local well-posedness results in a similar way as we did in Theorem 17.2.51. Continuity in time can be avoided by introducing a suitable notion of maximal regularity for the case of time-dependent A.

One can also allow a further inhomogeneity by allowing non-linearities of the form $F = F_{\rm Tr} + F_{\rm c} + f$, where $f : (0,T) \to X_0$ satisfies appropriate integrability assumptions.

We will now proceed to the main theorems on local well-posedness for the quasi-linear problem

$$\begin{cases} u' + A(u)u & = F(u), \quad \text{on } (0,T), \\ u(0) & = v_0, \end{cases} \tag{18.13}$$

where $v_0 \in O_{\sigma,p}$ can be taken as the given u_0 or close to u_0 in $X_{\sigma,p}$-norm. Allowing v_0 to be taken from a neighbourhood of u_0 will be important as we will also give prove continuous dependence on the initial data. Moreover, it will be used to obtain criteria for global well-posedness.

Define

$$\mathrm{MR}^p_\alpha(0,T) := \begin{cases} L^p_{w_\alpha}(0,T;X_1) \cap W^{1,p}_{w_\alpha}(0,T;X_0) & \text{if } p < \infty; \\ C_{w_\alpha,0}((0,T];X_1) \cap C^1_{w_\alpha,0}((0,T];X_0) & \text{if } p = \infty, \end{cases}$$

where we recall that the Banach space $C_{w_\alpha,0}((0,T];X_1)$ was defined before Corollary 17.2.48, and is a closed subspace of $L^\infty_{w_\alpha}(0,T;X_1)$. Similar assertions hold for its C^1-variant.

Definition 18.2.5. *Let Assumption 18.2.2 hold. A function $u \in \mathrm{MR}^p_\alpha(0,T)$ is called a $L^p_{w_\alpha}$-*solution *to (18.13) on $(0,T)$ if u takes values in $O_{\sigma,p}$, $A(u)u, F(u) \in L^1(0,T;X_0)$, and for all $t \in [0,T]$ we have*

$$u(t) - v_0 + \int_0^t A(u(s))u(s)\,\mathrm{d}s = \int_0^t F(u(s))\,\mathrm{d}s.$$

Later on in Lemma 18.2.8, we will see that the integrability assumptions on $A(u)u$ and $F(u)$ are actually redundant, and that one even has $A(u)u, F(u) \in L^p_{w_\alpha}(0,T;X_0)$.

18.2.b Main local well-posedness result

The main result of this section is the following local well-posedness for quasi-linear equations.

Theorem 18.2.6 (Local well-posedness for quasi-linear problems). *Let Assumption 18.2.2 hold. If, for some $u_0 \in O_{\sigma,p}$, the operator $A(u_0)$ has maximal L^p-regularity (maximal C-regularity if $p = \infty$) on finite time intervals, then there exist $T > 0$ and $\varepsilon > 0$ such that for all*

$$v_0 \in B_{X_{\sigma,p}}(u_0,\varepsilon) \subseteq O_{\sigma,p}$$

the problem (18.13) has a unique $L^p_{w_\alpha}$-solution $u_{v_0} \in \mathrm{MR}^p_\alpha(0,T)$. Moreover, there exists a constant $C \geqslant 0$ such that for all $v_0, v_1 \in B_{X_{\sigma,p}}(u_0,\varepsilon)$ we have

$$\|u_{v_0} - u_{v_1}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C\|v_0 - v_1\|_{X_{\sigma,p}}. \tag{18.14}$$

From Corollary L.4.6 we additionally see that

$$u_{v_0} \in C([0,T];X_{1-\alpha-\frac{1}{p},p}) \cap C((0,T];X_{1-\frac{1}{p},p}). \tag{18.15}$$

This shows that for $\alpha > 0$, the solution u instantaneously (that is, for $t \in (0,T]$) regularises from $X_{1-\alpha-\frac{1}{p},p}$ to $X_{1-\frac{1}{p},p}$. By similar arguments, an analogous continuous dependence as in (18.14) holds in $C([0,T];X_{1-\alpha-\frac{1}{p},p})$ and in the weighted space $C_{w_\alpha}((0,T];X_{1-\frac{1}{p},p})$.

The parameters T, ε, and C in Theorem 18.2.6 depend on the choice of u_0 in general. The parameters T and ε need to be small enough for the conclusions of the theorem to hold. This has several reasons. First of all, ε must be small because we need $B_{X_{\sigma,p}}(u_0,\varepsilon)$ to be contained in $O_{\sigma,p}$. More importantly, the proof uses the maximal regularity of $A(u_0)$ to obtain local well-posedness of (18.13) with initial value v_0, via a perturbation argument involving the smallness of $\|u_0 - v_0\|_{X_{\sigma,p}}$.

The time T must be small for two reasons. First of all, we need to assure that u maps $[0,T]$ to $O_{\sigma,p}$. Secondly, in the proof of the theorem we also need T to be small in order to be able to use fixed point arguments. This is hardly surprising: already in the familiar setting of ordinary differential equations, blow-up can occur in the presence of locally Lipschitz continuous non-linearities F. Theorem 18.2.15 will provide conditions under which one can extend the time interval of existence and uniqueness to the full interval $[0,\infty)$. For a special class of semi-linear equations, Theorem 18.2.17 will give large-time well-posedness for small initial data.

18.2.c Proof of the main result

The proof Theorem 18.2.6 uses a fixed point argument similar to the one of Theorem 18.1.2. However, the proof is technically more demanding due to the quasi-linear structure of the problem, the presence of the additional term $F_{\rm c}$, the use of weights, and the admission of the full range $p \in [1,\infty]$; some new ideas are needed to deal with these difficulties.

We will use the following abbreviations to keep the formulas at a reasonable length. For $k \in \{0,1\}$ and $j \in \{1,\dots,m\}$ we let, with notation introduced earlier,

$$E_k := \begin{cases} L^p_{w_\alpha}(0,T;X_k) & \text{if } p<\infty \\ C_{w_\alpha,0}((0,T];X_k) & \text{if } p=\infty \end{cases}$$

$$Y_j := \begin{cases} L^{(\rho_j+1)p}_{w_{\alpha/(\rho_j+1)}}(0,T;X_{\beta_j^*}) & \text{if } p<\infty \\ C_{w_{\alpha/(\rho_j+1)},0}((0,T];X_{\beta_j^*}) & \text{if } p=\infty, \end{cases}$$

where $\beta_j^* := \frac{1+\rho_j\sigma}{1+\rho_j}$. Assumption 18.2.2 implies that $\beta_j \leqslant \beta_j^*$.

Lemma 18.2.7. *Let Assumption 18.2.2 hold. Then for all $T>0$ we have continuous embeddings*

$$\begin{aligned} &\mathrm{MR}^p_\alpha(0,T) \hookrightarrow C([0,T];X_{\sigma,p}), \\ &\mathrm{MR}^p_\alpha(0,T) \hookrightarrow Y_j, \quad j\in\{1,\dots,m\}, \end{aligned}$$

and there exists a constant $M_{1,T} \geqslant 0$ such that for all $u \in \mathrm{MR}^p_\alpha(0,T)$ and $j \in \{1,\dots,m\}$ we have

$$\|u\|_{C([0,T];X_{\sigma,p})} + \|u\|_{Y_j} \leqslant M_{1,T}\|u\|_{\mathrm{MR}^p_\alpha(0,T)}. \tag{18.16}$$

These constants may be chosen so that $\sup_{T\geqslant 1} M_{1,T} < \infty$. For functions $u \in \mathrm{MR}^p_\alpha(0,T)$ satisfying $u(0)=0$, the constants $M_{1,T}$ can be replaced by a constant M_1 independent of $T>0$.

Proof. For $p\in[1,\infty)$, the embeddings and estimates follow from Corollaries L.4.6 and L.4.7, where for $p=1$ we additionally use Remark L.4.2 and Proposition L.4.5.

For $p=\infty$, the same can be done if Y_j is replaced by $C_{w_{\alpha/(\rho_j+1)}}((0,T];X_{\beta_j^*})$. To get the embedding into its closed subspace Y_j, recall $u\in \mathrm{MR}^\infty_\alpha(0,T) = C_{w_{\alpha,0}}((0,T];X_1)\cap C^1_{w_{\alpha,0}}((0,T];X_0)$. Then, for all $j\in\{1,\dots,m\}$, by (L.19),

$$\|u(t)\|_{X_{\beta_j^*}} \leqslant (1-\sigma)^{-1}\|u(t)\|^{\lambda}_{X_{\sigma,\infty}}\|u(t)\|^{1-\lambda}_{X_1},$$

where $\lambda = \frac{1-\beta_j^*}{\alpha} = \frac{\rho_j}{1+\rho_j}$. Hence

$$t^{\alpha/(\rho_j+1)}\|u(t)\|_{X_{\beta_j^*}} \leqslant (1-\sigma)^{-1}\|u\|^{\lambda}_{C([0,T];X_{\sigma,\infty})}(t^\alpha\|u(t)\|_{X_1})^{1-\lambda},$$

and the latter tends to zero as $t\downarrow 0$ since $u\in E_1$. This shows that $u\in Y_j$. □

Lemma 18.2.8. *Let Assumption 18.2.2 hold. Let $u,v,z\in \mathrm{MR}^p_\alpha(0,T)$ be given, and assume that u and v take values in $O_{\sigma,p}$. Then we have $A(u)z\in E_0$, $F_{\mathrm{c}}(u)\in E_0$, and $F_{\mathrm{Tr}}(u)\in C([0,T];X_0)$. Moreover,*

$$\|A(u)z - A(v)z\|_{E_0} \leqslant L\|u-v\|_{C([0,T];X_{\sigma,p})}\|z\|_{E_1}$$

and

$$\begin{aligned}\|F_{\mathrm{Tr}}(u)-F_{\mathrm{Tr}}(v)\|_{C([0,T];X_0)} &\leqslant L_{\mathrm{Tr}}\|u-v\|_{C([0,T];X_{\sigma,p})},\\ \|F_{\mathrm{c}}(u)-F_{\mathrm{c}}(v)\|_{E_0} &\leqslant \sum_{j=1}^m C^{\rho_j}_{\beta_j,X}L_{\mathrm{c}}\big[T^{\delta_j}+\|u\|^{\rho_j}_{Y_j}+\|v\|^{\rho_j}_{Y_j}\big]\|u-v\|_{Y_j},\end{aligned} \tag{18.17}$$

where $\delta_j = \frac{\alpha\rho_j}{1+\rho_j}+\frac{\rho_j}{(1+\rho_j)p}$.

This lemma asserts in particular that the integrability assumptions on $A(u)u$ and $F(u)$ in Definition 18.2.5 are automatically satisfied for functions $u\in \mathrm{MR}^p_\alpha(0,T)$.

Proof. First consider the case $p<\infty$. By Assumption 18.2.2(2),

$$\|A(u(t))z(t)-A(v(t))z(t)\|_{X_0} \leqslant L\|u(t)-v(t)\|_{X_{\sigma,p}}\|z(t)\|_{X_1}. \tag{18.18}$$

This gives the required estimate for A. Taking $v\equiv u_0\in X_{\sigma,p}$ fixed, one also sees that the function $t\mapsto A(u(t))z(t)$ belongs to E_0.

By Assumption 18.2.2(3),

$$\|F_{\mathrm{Tr}}(u(t))-F_{\mathrm{Tr}}(v(t))\|_{X_0} \leqslant L_{\mathrm{Tr}}\|u(t)-v(t)\|_{X_{\sigma,p}}.$$

This implies the estimate for F_{Tr} in (18.17); the assumptions on F_{Tr} and the continuity of $u:[0,T]\to X_{\sigma,p}$ (see Lemma 18.2.7) imply that $t\mapsto F_{\mathrm{Tr}}(u(t))$ belongs to $C([0,T];X_0)$.

Next, we have $u,v\in Y_j$ by Lemma 18.2.7. Moreover, for all $j\in\{1,\dots,m\}$,

$$
\begin{aligned}
&\Big\|(1+\|u\|_{X_{\beta_j^*}}^{\rho_j}+\|v\|_{X_{\beta_j^*}}^{\rho_j})\|u-v\|_{X_{\beta_j^*}}\Big\|_{L^p_{w_\alpha}(0,T)}\\
&\overset{\text{(i)}}{\leqslant} \Big\|(1+\|u\|_{X_{\beta_j^*}}^{\rho_j}+\|v\|_{X_{\beta_j^*}}^{\rho_j})\|_{L^{(\rho_j+1)p/\rho_j}_{w_{\alpha\rho_j/(\rho_j+1)}}(0,T)}\|u-v\|_{L^{(\rho_j+1)p}_{w_{\alpha/(\rho_j+1)}}(0,T;X_{\beta_j^*})}\\
&\overset{\text{(ii)}}{\leqslant} \big[T^{\delta_j}+\|u\|_{Y_j}^{\rho_j}+\|v\|_{Y_j}^{\rho_j}\big]\|u-v\|_{Y_j}
\end{aligned}
$$

where in (i) we applied Hölder's inequality with $\frac{1}{(1+\rho_j)}+\frac{\rho_j}{(1+\rho_j)}=1$ and in (ii) the definition of Y_j and the triangle inequality. The estimate for $F_{\rm c}$ in (18.17) now follows from Assumption 18.2.2 and the inequality $\|x\|_{X_{\beta_j}}\leqslant C_{\beta_j,X}\|x\|_{X_{\beta_j^*}}$ (see Proposition L.1.1(2)), where we used that the embedding constant in $X_1\hookrightarrow X_0$ is $\geqslant 1$, and thus $C_{\beta_j,X}\geqslant 1$.

The estimate for $F_{\rm Tr}$ immediately extends to $p=\infty$. The estimates for A and $F_{\rm c}$ also extend to $p=\infty$ if we replace $E_0=C_{w_\alpha,0}((0,T];X_0)$ by $L^\infty_{w_\alpha}(0,T;X_0)$. In order to obtain the estimates in the E_0-norm, it remains to prove that $t\mapsto A(u(t))z(t)$ and $t\mapsto F_{\rm c}(u(t))$ are continuous on $(0,T]$ and $t^\alpha\|A(u(t))z(t)\|_{X_0}$ and $t^\alpha\|F_{\rm c}(u(t))\|_{X_0}$ are bounded and tend to zero as $t\downarrow 0$.

To prove continuity for A, we observe that for $s,t\in(0,T]$

$$
\begin{aligned}
&\|A(u(t))z(t)-A(u(s))z(s)\|_{X_0}\\
&\leqslant \|(A(u(t))-A(u(s))z(t)\|_{X_0}+\|A(u(s))(z(t)-z(s))\|_{X_0}\\
&\leqslant L\|u(t)-u(s)\|_{X_{\sigma,\infty}}\|z(t)\|_{X_1}+\|A(u(s))\|_{\mathscr{L}(X_1,X_0)}\|(z(t)-z(s))\|_{X_1}\\
&\leqslant \frac{L}{t^\alpha}\|u(t)-u(s)\|_{X_{\sigma,\infty}}\|z\|_{E_1}+\|A(u(s))\|_{\mathscr{L}(X_1,X_0)}\|(z(t)-z(s))\|_{X_1}.
\end{aligned}
$$

The latter tends to zero if $t\to s$, and the desired continuity follows. To prove the bound and convergence of $t^\alpha\|A(u(t))z(t)\|_{X_0}$, we observe that by (18.18), applied with $v\equiv x\in X_1\cap O_{\sigma,p}$,

$$
\begin{aligned}
\|A(u(t))z(t)\|_{X_0}&\leqslant \|A(x)z(t)\|_{X_0}+\|A(u(t))z(t)-A(x)z(t)\|_{X_0}\\
&\leqslant \|A(x)\|_{\mathscr{L}(X_1,X_0)}\|z(t)\|_{X_1}+L\|u-x\|_{C([0,T];X_{\sigma,p})}\|z(t)\|_{X_1}.
\end{aligned}
$$

Since $z\in C_{w_\alpha,0}((0,T];X_1)$, this implies the desired boundedness and convergence.

To prove continuity for $F_{\rm c}$, note that by Assumption 18.2.2, for $s,t\in(0,T]$ we have

$$
\|F_{\rm c}(u(t))-F_{\rm c}(u(s))\|_{X_0}\leqslant L_{\rm c}\sum_{j=1}^m(1+\|u(t)\|_{X_{\beta_j}}^{\rho_j}+\|u(s)\|_{X_{\beta_j}}^{\rho_j})\|u(t)-u(s)\|_{X_{\beta_j}}.
$$

The latter tends to zero as $t\to s$. Indeed, since $u\in Y_j$, $u:(0,T]\to X_{\beta_j^*}\hookrightarrow X_{\beta_j}$ is continuous for each $j\in\{1,\dots,m\}$. To prove the bound and the convergence for $F_{\rm c}$, note that as already mentioned, we have

$$
\|F_{\rm c}(u)-F_{\rm c}(v)\|_{C_{w_\alpha}((0,T];X_0)}\leqslant L_{\rm c}\sum_{j=1}^m C_{\beta_j,X}^{\rho_j}(T^{\delta_j}+\|u\|_{Y_j}^{\rho_j}+\|v\|_{Y_j}^{\rho_j})\|u-v\|_{Y_j}.
$$

From the definitions of δ_j and Y_j it follows that $T^{\delta_j}, \|u\|_{Y_j}, \|v\|_{Y_j} \to 0$ as $T \downarrow 0$. In particular, the estimate implies that $t^\alpha \|F_c(u(t)) - F_c(v(t))\|_{X_0} \to 0$ as $t \downarrow 0$. For $v \equiv x$ with $x \in X_1 \cap O_{\sigma,p}$ it is clear that $t^\alpha \|F_c(v)\|_{X_0} \to 0$ as $t \downarrow 0$. Therefore, $t^\alpha \|F(u(t))\|_{X_0} \to 0$ as $t \downarrow 0$, and hence $F(u) \in E_0$. □

For each $v_0 \in X_{\sigma,p}$ and $T > 0$, we define the *reference solution* $z_{v_0} \in \mathrm{MR}^p_\alpha(0,T)$ as the $L^p_{w_\alpha}$-solution to the following linear problem (see Corollaries 17.2.37 and 17.2.48):

$$\begin{cases} u' + A(u_0)u &= 0, \quad \text{on } \mathbb{R}_+, \\ u(0) &= v_0. \end{cases}$$

Clearly, the mapping $v_0 \mapsto z_{v_0}$ is linear.

Let $\varepsilon, r, T > 0$ be fixed for the moment; these parameters will be chosen small enough shortly. For $v_0 \in B_{X_{\sigma,p}}(u_0, \varepsilon) \subseteq O_{\sigma,p}$ consider the following subset of $\mathrm{MR}^p_\alpha(0,T)$:

$$B^T_r(v_0) = \{v \in \mathrm{MR}^p_\alpha(0,T) : \, v(0) = v_0, \, \|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant r\}. \tag{18.19}$$

Note that $B^T_r(v_0)$ is a closed subset of $\mathrm{MR}^p_\alpha(0,T)$ by the continuity of the trace at zero (see Lemma 18.2.7).

To prove local well-posedness for (18.13), we will apply the Banach fixed point theorem to the mapping $\varPhi_{v_0} : B^T_r(v_0) \to B^T_r(v_0)$ defined by $\varPhi_{v_0}(v) = u$, where u is the $L^p_{w_\alpha}$-solution to

$$\begin{cases} u' + A(u_0)u &= (A(u_0) - A(v))v + F(v), \quad \text{on } (0,T), \\ u(0) &= v_0. \end{cases} \tag{18.20}$$

Below we will first ensure that $B^T_r(v_0) \subseteq O_{\sigma,p}$ for $\varepsilon, r > 0$ small enough, so that $A(v)$ and $F(v)$ are well-defined. Then from its definition, it is clear that $\varPhi_{v_0}$ maps $B^T_r(v_0)$ to $\mathrm{MR}^p_\alpha(0,T)$. Below we will check that for $\varepsilon, r, T > 0$ small enough, $\varPhi_{v_0}$ is well defined as a mapping from $B^T_r(v_0)$ to itself by using the maximal regularity assumption on $A(u_0)$ and the mapping properties of A and F. Note that a function u is an $L^p_{w_\alpha}$-solution to (18.13) if and only if u is an $L^p_{w_\alpha}$-solution (C_{w_α}-solution if $p = \infty$) to (18.20) with $u = v$. Before we turn to the fixed point argument we need several preparatory lemmas.

Choose $\varepsilon_0 > 0$ such that $\overline{B_{X_{\sigma,p}}(u_0, \varepsilon_0)} \subseteq O_{\sigma,p}$. Fix $T_1 > 0$ such that

$$\|z_{u_0} - u_0\|_{C([0,T_1];X_{\sigma,p})} < \varepsilon_0/3. \tag{18.21}$$

By Corollaries 17.2.37 and 17.2.48, there is a constant C_{T_1} such that for every $v_0 \in X_{\sigma,p}$ we have

$$\|z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)} \leqslant C_{T_1} \|v_0\|_{X_{\sigma,p}}. \tag{18.22}$$

The constant C_{T_1} will depend on T_1 in general, but this will not create problems since T_1 is fixed.

In order to show that $A(v)$ and $F(v)$ in (18.20) are well defined, we need to check that $v(t) \in O_{\sigma,p}$ for all $t \in (0,T)$ when $\varepsilon \in (0,\varepsilon_0)$, $r \in (0,1]$, and $T \in (0,T_1]$ are small enough. This is taken care of in the next lemma.

Lemma 18.2.9. *Let Assumption 18.2.2 hold, and let $\varepsilon_0 > 0$ be chosen as before (18.21). For small enough $r \in (0,1]$ and $\varepsilon \in (0,\varepsilon_0)$ the following holds: For all $v_0 \in B_{X_{\sigma,p}}(u_0,\varepsilon)$, all $T \in (0,T_1]$, and all $v \in B_r^T(v_0)$, one has $\|v - u_0\|_{C([0,T];X_{\sigma,p})} < \varepsilon_0$, and thus $v(t) \in O_{\sigma,p}$ for all $t \in [0,T]$.*

Proof. For notational convenience we write $\|\cdot\|_{\infty,T} = \|\cdot\|_{C([0,T];X_{\sigma,p})}$. For all $v \in B_r^T(v_0)$,

$$\begin{aligned}\|v - z_{v_0}\|_{\infty,T} &\leqslant M_1\|v - z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T)} && \text{(by (18.16))}\\ &\leqslant M_1\|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} + M_1\|z_{u_0} - z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}\\ &\leqslant M_1 r + M_1 C_{T_1}\|u_0 - v_0\|_{X_{\sigma,p}} && \text{(by (18.22))}.\end{aligned}$$

Therefore, by (18.16), (18.21), and (18.22)

$$\begin{aligned}\|v - u_0\|_{\infty,T} &\leqslant \|v - z_{v_0}\|_{\infty,T} + \|z_{v_0} - z_{u_0}\|_{\infty,T_1} + \|z_{u_0} - u_0\|_{\infty,T_1}\\ &\leqslant M_1 r + C_{T_1}(M_1 + M_{1,T_1})\|u_0 - v_0\|_{X_{\sigma,p}} + \|z_{u_0} - u_0\|_{\infty,T_1}\\ &\leqslant M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \varepsilon_0/3.\end{aligned} \tag{18.23}$$

This implies the required result for all $r, \varepsilon > 0$ small enough. □

In the next lemma we collect some estimates for A, F_{Tr}, and F_{c}, which will be used to ensure that Φ_{v_0} maps $B_r^T(v_0)$ to itself.

Lemma 18.2.10 (Smallness). *Let Assumption 18.2.2 hold. Fix $T \in (0,T_1]$ and let $\varepsilon \in (0,\varepsilon_0)$ and $r \in (0,1]$ be as in Lemma 18.2.9. Then for all $v_0 \in B_{X_{\sigma,p}}(u_0,\varepsilon)$ and $v \in B_r^T(v_0)$ we have*

$$\begin{aligned}&\|(A(v) - A(u_0))v\|_{E_0}\\ &\qquad\leqslant \big(M_1 r + M_{2,T_1}\varepsilon + \|z_{u_0} - u_0\|_{C([0,T];X_{\sigma,p})}\big)(r + \|z_{u_0}\|_{E_1}),\\ &\|F_{\mathrm{Tr}}(v)\|_{E_0} \leqslant T^{\alpha+\frac{1}{p}}\big(L_{\mathrm{Tr}}\varepsilon_0 + \|F_{\mathrm{Tr}}(u_0)\|_{X_0}\big),\\ &\|F_{\mathrm{c}}(v)\|_{E_0} \leqslant C_{\varepsilon,r,T}(u_0) r + C_{\varepsilon,T}(u_0),\end{aligned}$$

where $C_{\varepsilon,r,T}(u_0)$ and $C_{\varepsilon,T}(u_0)$ are independent of v_0 and v, $C_{\varepsilon,r,T}(u_0)$ and $C_{\varepsilon,T}(u_0)$ are non-decreasing in each of the variables ε, r, and T, and satisfy $C_{\varepsilon,r,T}(u_0), C_{\varepsilon,T}(u_0) \to 0$ as $\varepsilon, r, T \downarrow 0$.

Proof. We use the short-hand notation $\|\cdot\|_{\infty,T} := \|\cdot\|_{C([0,T];X_{\sigma,p})}$.

As in (18.23), one sees that

$$\|v - u_0\|_{\infty,T} \leqslant M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \|z_{u_0} - u_0\|_{\infty,T}.$$

Therefore, by Lemma 18.2.8,

$$\begin{aligned}
&\|(A(v) - A(u_0))v\|_{E_0} \\
&\quad \leqslant L\|v - u_0\|_{\infty,T}\|v\|_{E_1} \\
&\quad \leqslant \big(M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \|z_{u_0} - u_0\|_{\infty,T}\big)\|v\|_{E_1}. \\
&\quad \leqslant \big(M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \|z_{u_0} - u_0\|_{\infty,T}\big)(r + \|z_{u_0}\|_{E_1}).
\end{aligned}$$

For $F_{\rm Tr}$ we have pointwise estimate

$$\begin{aligned}
\|F_{\rm Tr}(v)\|_{X_0} &\leqslant \|F_{\rm Tr}(v) - F_{\rm Tr}(u_0)\|_{X_0} + \|F_{\rm Tr}(u_0)\|_{X_0} \\
&\leqslant L_{\rm Tr}\|v - u_0\|_{X_{\sigma,p}} + \|F_{\rm Tr}(u_0)\|_{X_0} \\
&\leqslant L_{\rm Tr}\varepsilon_0 + \|F_{\rm Tr}(u_0)\|_{X_0},
\end{aligned}$$

where in the last step we used Lemma 18.2.9. Taking $L^p_{w_\alpha}$-norms, we obtain

$$\|F_{\rm Tr}(v)\|_{E_0} \leqslant T^{\alpha+\frac{1}{p}}(L_{\rm Tr}\varepsilon_0 + \|F_{\rm Tr}(u_0)\|_{X_0}).$$

The estimate for $F_{\rm c}$ is more difficult to obtain. By the second estimate in (18.17),

$$\|F_{\rm c}(v) - F_{\rm c}(z_{u_0})\|_{E_0} \leqslant \sum_{j=1}^m C^{\rho_j}_{\beta_j,X} L_{\rm c}\big(T^{\delta_j} + \|v\|^{\rho_j}_{Y_j} + \|z_{u_0}\|^{\rho_j}_{Y_j}\big)\|v - z_{u_0}\|_{Y_j}.$$

It remains to estimate $\|v\|_{Y_j}$ and $\|v - z_{u_0}\|_{Y_j}$. By (18.16),

$$\begin{aligned}
\|v - z_{u_0}\|_{Y_j} &\leqslant \|v - z_{v_0}\|_{Y_j} + \|z_{v_0} - z_{u_0}\|_{Y_j} \\
&\leqslant M_1\|v - z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T)} + M_{1,T_1}\|z_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)} \\
&\leqslant M_1\|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} + (M_1 + M_{1,T_1})\|z_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)} \\
&\leqslant M_1 r + (M_1 + M_{1,T_1})C_{T_1}\|v_0 - u_0\|_{X_{\sigma,p}} \\
&\leqslant M_1 r + (M_1 + M_{1,T_1})C_{T_1}\varepsilon,
\end{aligned}$$

applying (18.22) in the penultimate estimate. Similarly,

$$\|v\|_{Y_j} \leqslant \|v - z_{u_0}\|_{Y_j} + \|z_{u_0}\|_{Y_j} \leqslant M_1 r + (M_1 + M_{1,T_1})C_{T_1}\varepsilon + \|z_{u_0}\|_{Y_j}.$$

Combining things, we obtain the estimate

$$\begin{aligned}
&\|F_{\rm c}(v)\|_{E_0} \\
&\leqslant \|F_{\rm c}(v) - F_{\rm c}(z_{u_0})\|_{E_0} + \|F_{\rm c}(z_{u_0})\|_{E_0} \\
&\leqslant \sum_{j=1}^m C^{\rho_j}_{\beta_j,X} L_{\rm c}\big(T^{\delta_j} + 2(M_1 r + \widetilde{C}_{T_1}\varepsilon + k_{j,T}(u_0))^{\rho_j}\big)(M_1 r + \widetilde{C}_{T_1}\varepsilon) + k_{c,T}(u_0),
\end{aligned}$$

where have set $\widetilde{C}_{T_1} = (M_1 + M_{1,T_1})C_{T_1}$, $k_{j,T}(u_0) = \|z_{u_0}\|_{Y_j}$, and $k_{c,T}(u_0) = \|F_{\rm c}(z_{u_0})\|_{E_0}$. Note that $k_{j,T}(u_0) \to 0$ and $k_{c,T}(u_0) \to 0$ as $T \downarrow 0$ since $z_{u_0} \in \mathrm{MR}^p_\alpha(0,T) \subseteq Y_j$ and since $F_{\rm c}(z_{u_0}) \in E_0$ by Lemma 18.2.8.

The estimate $\|F_c(v)\|_{E_0} \leqslant C_{\varepsilon,r,T}(u_0)r + C_{\varepsilon,T}(u_0)$ in the statement of the lemma now follows, with constants

$$C_{\varepsilon,r,T}(u_0) = \sum_{j=1}^{m} C^{\rho_j}_{\beta_j,X} L_c\big(T^{\delta_j} + 2(M_1 r + \widetilde{C}_{T_1}\varepsilon + k_{j,T}(u_0))^{\rho_j}\big)M_1$$
$$C_{\varepsilon,T}(u_0) = \widetilde{C}_{T_1}\varepsilon M_1^{-1} C_{\varepsilon,1,T}(u_0) + k_{c,T}(u_0),$$

where we used that $r \in (0,1]$. □

Remark 18.2.11. In the last part of the proof one does not have $k_{j,T}(u_0) \to 0$ and $k_{c,T}(u_0) \to 0$ as $T \downarrow 0$ if one were to use maximal $L^\infty_{w_\alpha}$-regularity or data u_0 in $(X_0, X_1)_{\sigma,\infty}$ rather than in the closed subspace $X_{\sigma,\infty}$. This is one of the reasons for working with maximal C_{w_α}-regularity and data in $X_{\sigma,\infty}$. It is also clear from the above proof that $\alpha = 0$ leads to difficulties if $p = \infty$. For example, the estimate for $F_{\mathrm{Tr}}(v)$ in Lemma 18.2.10 contains a factor $T^{\alpha+\frac{1}{p}}$ which does not vanish in the limit $T \downarrow 0$ if $\alpha = 0$ and $p = \infty$.

The final lemma contains Lipschitz variations of the above estimates, which will be used to show that Φ_{v_0} is a uniform contraction.

Lemma 18.2.12 (Lipschitz estimates). *Let Assumption 18.2.2 hold. Fix $T \in (0, T_1]$ and let $\varepsilon \in (0, \varepsilon_0)$ and $r \in (0, 1]$ be as in Lemma 18.2.9. Then for all $v_{1,0}, v_{2,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$, and all $v_1 \in B^T_r(v_{1,0})$ and $v_2 \in B^T_r(v_{2,0})$,*

$$\|(A(v_1) - A(v_2))v_1\|_{E_0} + \|(A(u_0) - A(v_2))(v_1 - v_2)\|_{E_0} + \|F_{\mathrm{Tr}}(v_1) - F_{\mathrm{Tr}}(v_2)\|_{E_0} + \|F_c(v_1) - F_c(v_2)\|_{E_0}$$

can be estimated from above by

$$L_{\varepsilon,r,T}(u_0)(\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} + \|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}),$$

where $L_{\varepsilon,r,T}(u_0)$ is a constant independent of $v_{1,0}, v_{2,0}, v_1, v_2$, non-decreasing in each of the variables ε, r, T, and satisfying $L_{\varepsilon,r,T}(u_0) \to 0$ as $\varepsilon, r, T \downarrow 0$.

Proof. We use the short-hand notation $\|\cdot\|_{\infty,T} := \|\cdot\|_{C([0,T];X_{\sigma,p})}$.

First we provide an estimate for $\|v\|_{\infty,T}$ and $\|v - u_0\|_{\infty,T}$ for $v \in B^T_r(v_0)$ and $v_0 \in B_{X_{\sigma,p}}(u_0, \varepsilon)$. By (18.22) and (18.16),

$$\begin{aligned} \|v\|_{\infty,T} &\leqslant \|z_{v_0}\|_{\infty,T} + \|v - z_{v_0}\|_{\infty,T} \\ &\leqslant M_{1,T_1} C_{T_1}\|v_0\|_{X_{\sigma,p}} + M_1\|v - z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T)} \\ &\leqslant (M_{1,T_1} + M_1)C_{T_1}\|v_0\|_{X_{\sigma,p}} + M_1\|v\|_{\mathrm{MR}^p_\alpha(0,T)}. \end{aligned} \tag{18.24}$$

Similarly, setting $k_T(u_0) := \|z_{u_0} - u_0\|_{\infty,T}$,

$$
\begin{aligned}
&\|v-u_0\|_{\infty,T}\\
&\quad\leqslant \|v-z_{v_0}\|_{\infty,T}+\|z_{v_0}-z_{u_0}\|_{\infty,T}+k_T(u_0)\\
&\quad\leqslant M_1\|v-z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T)}+M_{1,T_1}\|z_{v_0}-z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}+k_T(u_0)\\
&\quad\leqslant M_1\|v-z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)}+(M_1+M_{1,T_1})\|z_{v_0}-z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}+k_T(u_0)\\
&\leqslant M_1 r+(M_1+M_{1,T_1})C_{T_1}\varepsilon+k_T(u_0).
\end{aligned}
\tag{18.25}
$$

To estimate the first A-term, by Lemma 18.2.8 we obtain

$$
\begin{aligned}
\|(A(v_1)-A(v_2))v_1\|_{E_0}&\leqslant L\|v_1-v_2\|_{\infty,T}\|v_1\|_{E_1}\\
&\leqslant L\|v_1-v_2\|_{\infty,T}(r+\|z_{u_0}\|_{E_1}).
\end{aligned}
$$

Therefore, the required estimate follows from (18.24) with $v_0=0$.

For the second A-term, we again use Lemma 18.2.8 and obtain

$$
\|(A(u_0)-A(v_2))(v_1-v_2)\|_{E_0}\leqslant L\|u_0-v_2\|_{\infty,T}\|v_1-v_2\|_{E_1}.
$$

Therefore, the required estimate follows from (18.25).

For the F_{Tr}-term, we use Lemma 18.2.8 to obtain

$$
\|F_{\mathrm{Tr}}(v_1)-F_{\mathrm{Tr}}(v_2)\|_{E_0}\leqslant T^{\alpha+\frac1p}L_{\mathrm{Tr}}\|v_1-v_2\|_{\infty,T}.
$$

Therefore, the estimate follows from (18.24) again.

The F_{c}-term is more difficult to estimate. In the same way as in (18.24) and (18.25) one shows that

$$
\|v\|_{Y_j}\leqslant (M_{1,T_1}+M_1)C_{T_1}\|v_0\|_{X_{\sigma,p}}+M_1\|v\|_{\mathrm{MR}^p_\alpha(0,T)}
\tag{18.26}
$$

and

$$
\|v\|_{Y_j}\leqslant M_1 r+(M_1+M_{1,T_1})C_{T_1}\varepsilon+\|z_{u_0}\|_{Y_j}.
\tag{18.27}
$$

By the second estimate in (18.17),

$$
\|F_{\mathrm{c}}(v_1)-F_{\mathrm{c}}(v_2)\|_{E_0}\leqslant\sum_{j=1}^m C^{\rho_j}_{\beta_j,X}L_{\mathrm{c}}\big(T^{\delta_j}+\|v_1\|^{\rho_j}_{Y_j}+\|v_2\|^{\rho_j}_{Y_j}\big)\|v_1-v_2\|_{Y_j}.
$$

Using (18.26), we find

$$
\|v_1-v_2\|_{Y_j}\leqslant (M_{1,T_1}+M_1)C_{T_1}\|v_{1,0}-v_{2,0}\|_{X_{\sigma,p}}+M_1\|v_1-v_2\|_{\mathrm{MR}^p_\alpha(0,T)}.
$$

The required estimate for F_{c} now follows by applying (18.27) to estimate $\|v_1\|^{\rho_j}_{Y_j}$ and $\|v_2\|^{\rho_j}_{Y_j}$. □

After these preparations we are ready to turn to the proof of Theorem 18.2.6. It will be useful to recall the maximal regularity estimate which follows from Corollaries 17.2.37 and 17.2.48: for all $f \in E_0$ and $v_0 \in X_{\sigma,p}$ there exists a unique $L^p_{w_\alpha}$-solution (C_{w_α}-solution if $p = \infty$) to the problem

$$\begin{cases} u' + A(u_0)u &= f \quad \text{on } (0,T), \\ u(0) &= v_0, \end{cases}$$

and there exists a constant $C_T \geqslant 0$, independent of f and v_0, such that

$$\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C_T \|f\|_{E_0} + C_T \|v_0\|_{X_{\sigma,p}}. \tag{18.28}$$

This constant C_T also depends on $A(u_0)$ and p, but we can choose it in such a way that $C_T \leqslant C_{T_1}$ whenever $T < T_1$; this follows from a weighted version of (17.25).

Proof of Theorem 18.2.6. Fix $\varepsilon \in (0, \varepsilon_0)$ and $r \in (0,1]$ be as in Lemma 18.2.9, and let $T \in (0, T_1]$. Let $B^T_r(v_0)$ be as in (18.19). Let $\varPhi_{v_0} : B^T_r(v_0) \to \mathrm{MR}^p_\alpha(0,T)$ be defined by $\varPhi_{v_0}(v) := u$, where u is the $L^p_{w_\alpha}$-solution (C_{w_α}-solution if $p = \infty$) to the problem

$$\begin{cases} u' + A(u_0)u &= (A(u_0) - A(v))v + F(v), \\ u(0) &= v_0. \end{cases} \tag{18.29}$$

Then v takes values in $O_{\sigma,p}$ by Lemma 18.2.9, and we have $(A(v) - A(u_0))v \in E_0$ and $F(v) \in E_0$ by Lemma 18.2.10. Below Theorem 18.2.6 we have already observed that local existence and uniqueness follow if we can show that $\varPhi_{v_0}$ has a unique fixed point.

Since $u - z_{u_0}$ satisfies (18.29) with v_0 replaced by $v_0 - u_0$, by the maximal regularity estimate (18.28) applied on $(0, T_1)$ (see (18.21) for the definition of T_1) we have

$$\begin{aligned} \|u - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} &\leqslant C_{A,T_1}\big(\|u_0 - v_0\|_{X_{\sigma,p}} + \|(A(u_0) - A(v))v + F(v)\|_{E_0}\big) \\ &\leqslant C_{A,T_1}\varepsilon + \widetilde{C}_{\varepsilon,r,T}\, r + \widetilde{C}_{\varepsilon,T}, \end{aligned}$$

applying Lemma 18.2.10 in the last step, and where $\widetilde{C}_{\varepsilon,r,T}$ and $\widetilde{C}_{\varepsilon,T}$ are constants such that $\widetilde{C}_{\varepsilon,r,T} \to 0$ as $\varepsilon, r, T \downarrow 0$. Therefore, for $r, \varepsilon, T > 0$ small enough we obtain $\|u - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant r$, and thus $u \in B^T_r(v_0)$.

Next, fix $v_{j,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$ and $v_j \in B^T_r(v_{j,0})$ for $j \in \{1,2\}$. Then $u = \varPhi_{v_{1,0}}(v_1) - \varPhi_{v_{2,0}}(v_2)$ solves the problem

$$\begin{cases} u' + A(u_0)u &= (A(u_0) - A(v_1))v_1 - (A(u_0) - A(v_2))v_2 + F(v_1) - F(v_2), \\ u(0) &= v_{1,0} - v_{2,0}. \end{cases}$$

Therefore, by the maximal regularity estimate (18.28),

$$\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C_{A,T_1}(R_A + R_F) + C_{A,T_1}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}},$$

where

$$\begin{aligned} R_A &:= \|(A(u_0) - A(v_1))v_1 - (A(u_0) - A(v_2))v_2\|_{E_0} \\ &\leqslant \|(A(v_1) - A(v_2))v_1\|_{E_0} + \|(A(u_0) - A(v_2))(v_1 - v_2)\|_{E_0} \end{aligned}$$

and

$$\begin{aligned} R_F &:= \|F(v_1) - F(v_2)\|_{E_0} \\ &\leqslant \|F_{\mathrm{Tr}}(v_1) - F_{\mathrm{Tr}}(v_2)\|_{E_0} + \|F_{\mathrm{c}}(v_1) - F_{\mathrm{c}}(v_2)\|_{E_0}. \end{aligned}$$

From Lemma 18.2.12 we deduce that

$$\begin{aligned} \|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant\ & C_{A,T_1} L_{\varepsilon,r,T}(u_0)\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ &+ C_{A,T_1}(L_{\varepsilon,r,T}(u_0) + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}. \end{aligned}$$

Choosing $\varepsilon > 0$, $r > 0$, and $T > 0$ so small that $C_{A,T_1} L_{\varepsilon,r,T}(u_0) \leqslant 1/2$, we obtain

$$\begin{aligned} \|\varPhi_{v_{1,0}}(v_1) - \varPhi_{v_{2,0}}(v_2)\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant\ & \frac{1}{2}\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ &+ (C_{A,T_1} + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}. \end{aligned} \tag{18.30}$$

The estimate (18.30) allows us to finish the proof of local well-posedness. By (18.30), $\varPhi_{v_0} : B^T_r(v_0) \to B^T_r(v_0)$ is a uniform contraction, and thus it has a unique fixed point $u_{v_0} \in B^T_r(v_0)$. This is the required solution to (18.13). Moreover, (18.30) implies that for all $v_{1,0}, v_{2,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$,

$$\begin{aligned} \|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant\ & \frac{1}{2}\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \\ &+ (C_{A,T_1} + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}} \end{aligned}$$

which implies

$$\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant 2(C_{A,T_1} + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}.$$

This gives (18.14).

It remains to prove uniqueness. Uniqueness does hold if we only consider solutions in $B^T_r(v_0)$. In order to derive uniqueness for the larger set $\mathrm{MR}^p_\alpha(0,T)$, we will replace ε and T by suitable smaller values $\widetilde{\varepsilon}$ and $\widetilde{T}$. The above estimates then show that $\varPhi_{v_0} : B^{\widetilde{T}}_r(v_0) \to B^{\widetilde{T}}_r(v_0)$ and (18.30) holds with T replaced by $\widetilde{T}$.

Let $\widetilde{\varepsilon} := \min\big\{\varepsilon, \frac{r}{8(C_{A,T_1}+1)}\big\}$ and set

$$\widetilde{T} := \inf\Big\{t \in [0,T] : \|u_{u_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,t)} \geqslant \frac{r}{2}\Big\},$$

where $\inf\varnothing := T$. Then, for all $v_0 \in B_{X_{\sigma,p}}(u_0, \widetilde{\varepsilon})$,

$$\|u_{v_0} - u_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant 2(C_{A,T_1}+1)\|v_0 - u_0\|_{X_{\sigma,p}} \leqslant \frac{r}{4}.$$

In particular,

$$\|u_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} \leqslant \|u_{v_0} - u_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} + \|u_{u_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} \leqslant \frac{3r}{4}.$$

We claim that for every $v_0 \in B_{X_{\sigma,p}}(u_0, \widetilde{\varepsilon})$, the element $u_{v_0} \in \mathrm{MR}^p_\alpha(0,\widetilde{T})$ is the unique $L^p_{w_\alpha}$-solution to (18.13). To show this, we will prove the slightly stronger result (which will play a key role in the construction of the maximal solution in Section 18.2.d) that, for an $\tau > 0$, if $v \in \mathrm{MR}^p_\alpha(0,\tau)$ is an $L^p_{w_\alpha}$-solution to (18.13), then $v \equiv u_{v_0}$ on $[0, \widetilde{T}\wedge\tau]$. This will give the theorem for $\widetilde{T}$ instead of T.

Let

$$\tau_v := \inf\{t \in [0,\widetilde{T}\wedge\tau] : \|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,t)} \geqslant r\},$$

setting $\inf\varnothing := \widetilde{T}\wedge\tau$. Then $v|_{[0,\tau_v]}$ belongs to $B^{\tau_v}_r(v_0)$, and since $\tau_v \leqslant T$ it follows that $v|_{[0,\tau_v]} = u_{v_0}|_{[0,\tau_v]}$ by uniqueness of the fixed point in $B^{\tau_v}_r(v_0)$. Thus we obtain

$$\|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\tau_v)} = \|u_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\tau_v)} \leqslant \|u_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} < r,$$

and therefore $\tau_v = \widetilde{T}\wedge\tau$. This gives the claimed result. □

18.2.d Maximal solutions

Having established local well-posedness in Theorem 18.2.6, we will now extend the time interval on which the solution exists to a maximal time interval $[0, T_{\max}(v_0))$.

Definition 18.2.13. *Let Assumption 18.2.2 hold and assume that $v_0 \in O_{\sigma,p}$. A pair $(v, T_{\max}(v_0))$ is called a* maximal $L^p_{w_\alpha}$-solution to (18.13) *if $T_{\max}(v_0) \in (0,\infty]$ and $v : [0, T_{\max}(v_0)) \to X_0$ are such that*

- *for all $T \in (0, T_{\max}(v_0))$, $v|_{(0,T)}$ belongs to $\mathrm{MR}^p_\alpha(0,T)$ and is an $L^p_{w_\alpha}$-solution to (18.13) on $(0,T)$;*
- *whenever $u \in \mathrm{MR}^p_\alpha(0,T)$ is a unique $L^p_{w_\alpha}$-solution to (18.13) for some $T > 0$, one has $T \leqslant T_{\max}(v_0)$ and $u \equiv v$ on $(0,T)$.*

Note that maximal $L^p_{w_\alpha}$-solutions are unique. An even stronger uniqueness assertion will be derived in Remark 18.2.16 under further restrictions. We will now show that the solution to (18.13) provided by Theorem 18.2.6 can be extended to a maximal $L^p_{w_\alpha}$-solution.

Theorem 18.2.14 (Maximal solutions). *Let Assumption 18.2.2 hold, let $u_0 \in O_{\sigma,p}$, and suppose that $A(u_0)$ has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Let $\varepsilon > 0$ be as in Theorem 18.2.6, and let $v_0 \in O_{\sigma,p}$ be such that $\|u_0 - v_0\|_{X_{\sigma,p}} < \varepsilon$. Then there exists a maximal $L^p_{w_\alpha}$-solution $(u, T_{\max}(v_0))$ to (18.13).*

Proof. Let us say that an $L^p_{w_\alpha}$-solution v to (18.13) on $(0,T)$ *has the uniqueness property* if for any $\tau > 0$ and any $L^p_{w_\alpha}$-solution u to (18.13) on $(0,\tau)$, we have $v \equiv u$ on $[0, T \wedge \tau]$. Let $T_{\max}(v_0)$ be the supremum of all $T > 0$ such that there exists an $L^p_{w_\alpha}$-solution to (18.13) on $(0,T)$ with the uniqueness property. Then $T_{\max}(v_0) > 0$ by Theorem 18.2.6. Note that the uniqueness property was established as part of the uniqueness proof. It follows that there exists a maximal $L^p_{w_\alpha}$-solution $u : [0, T_{\max}(v_0)) \to X_0$ to (18.13). □

Theorem 18.2.15 (Global well-posedness for quasi-linear equations). *Let Assumption 18.2.2 hold, and suppose that for all $u_0 \in O_{\sigma,p}$ the operator $A(u_0)$ has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Let $v_0 \in O_{\sigma,p}$ and let $v : [0, T_{\max}(v_0)) \to X_0$ be the maximal solution provided by Theorem 18.2.14. If $T_{\max}(v_0) < \infty$, then either*

- $\lim_{t\uparrow T_{\max}(v_0)} v(t)$ *does not exist in $X_{\sigma,p}$, or*
- $v_* := \lim_{t\uparrow T_{\max}(v_0)} v(t)$ *exist in $X_{\sigma,p}$, but $v_* \notin O_{\sigma,p}$.*

The final assertion in the theorem is called a *blow-up criterion*. Blow-up criteria can be used to prove global well-posedness. In typical applications, assuming $T_{\max}(v_0) < \infty$, energy estimates can be used to show that $v_* := \lim_{t\uparrow T_{\max}(v_0)} v(t)$ exists in $O_{\sigma,p}$. This contradicts Theorem 18.2.15 and thus leads to $T_{\max}(v_0) = \infty$, i.e., global existence. Further blow-up criteria are discussed in the Notes.

Proof. Assuming that $T_0 := T_{\max}(v_0) < \infty$ and that $v_* := \lim_{t\uparrow T_0} v(t)$ exists in $X_{\sigma,p}$ with $v_* \in O_{\sigma,p}$, a contradiction will be derived.

The idea is to restart the problem at time T_0 with initial value v_* and apply Theorem 18.2.6 to extend v to a larger time interval $[0, T_0 + \delta]$. However, it is not self-evident that $v \in \mathrm{MR}^p_\alpha(0, T_0 + \delta)$. This problem will be overcome by using a compactness argument.

From the continuity of v and the assumption that the limit v_* at $t = T_0$ exist, it follows that the set

$$K := \{v(t) : t \in [0, T_0)\} \cup \{v_*\}$$

is compact in $X_{\sigma,p}$. By Theorem 18.2.6, for all $x \in K$ there exists an open ball $B(x, \varepsilon_x) \subseteq O_{\sigma,p}$ such that for initial values from $B(x, \varepsilon_x)$ we can find an $L^p_{w_\alpha}$-solution in $\mathrm{MR}^p_\alpha(0, t_x)$ for some $t_x > 0$. Since K is compact, the open cover $\{B(x, \varepsilon_x) : x \in K\}$ has a finite sub-cover $\{B(x_n, \varepsilon_{x_n}) : n = 1, \dots N\}$. Let $\delta := \min_{n=1,\dots,N} t_{x_n}$. Then for all $x \in K$ there exists a unique $L^p_{w_\alpha}$-solution $u_x \in \mathrm{MR}^p_\alpha(0, \delta)$ to the problem

$$\begin{cases} u' + A(u)u &= F(u), \\ u(0) &= x. \end{cases} \tag{18.31}$$

Now we are ready to define a suitable extension of v. Let $x := v(T_0 - \frac{1}{2}\delta)$, and let $u_x \in \mathrm{MR}^p_\alpha(0,\delta)$ be as above. Then $t \mapsto v(T_0 - \frac{1}{2}\delta + t)$ belongs to $\mathrm{MR}^p_\alpha(0,\gamma)$ for all $\gamma \in (0,\delta)$ and is an $L^p_{w_\alpha}$-solution to (18.31). Therefore, uniqueness gives that $v(T_0 - \frac{1}{2}\delta + t) = u_x(t)$ for all $t \in [0,\delta/2)$. Now one can check that the function $v_{\mathrm{ext}} : [0, T_0 + \delta/2] \to X_{\sigma,p}$ defined by

$$v_{\mathrm{ext}}(t) = \begin{cases} v(t), & t \in [0,T_0); \\ u_x(t - T_0 + \frac{1}{2}\delta), & t \in [T_0 - \frac{1}{2}\delta, T_0 + \frac{1}{2}\delta]. \end{cases}$$

is well defined, belongs to $\mathrm{MR}^p_\alpha(0, T + \frac{1}{2}\delta)$, and is an $L^p_{w_\alpha}$-solution to (18.13) on $(0, T_0 + \frac{1}{2}\delta)$. This contradicts the maximality of T_0. ☐

Under the conditions of Theorem 18.2.15, one can leave out the uniqueness from the second bullet in Definition 18.2.13. This excludes the existence of an (non-unique) $L^p_{w_\alpha}$-solution $u \in \mathrm{MR}^p_\alpha(0,T)$ which extends v.

Remark 18.2.16. Let Assumption 18.2.2 hold, and suppose that for all $u_0 \in O_{\sigma,p}$ the operator $A(u_0)$ has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Let $v_0 \in O_{\sigma,p}$ and let $v : [0, T_{\max}(v_0)) \to X_0$ be the maximal solution provided by Theorem 18.2.14. Now suppose that $u \in \mathrm{MR}^p_\alpha(0,T)$ is an $L^p_{w_\alpha}$-solution to (18.13) for some $T > 0$. We claim that $T \leqslant T_{\max}(v_0)$ and $u \equiv v$ on $(0,T)$. To see this, first note that by the uniqueness property of the proof of Theorem 18.2.14 one has $u = v$ on $[0, T \wedge T_{\max}(v_0))$. Thus it remains to show $T \leqslant T_{\max}(v_0)$. Suppose that $T > T_{\max}(v_0)$. Since $u \in \mathrm{MR}^p_\alpha(0,T)$, it follows from Lemma 18.2.7 that

$$v_* := \lim_{t \uparrow T_{\max}(v_0)} v(t) = \lim_{t \uparrow T_{\max}(v_0)} u(t) = u(T_{\max}(v_0)) \ \text{ exists in } X_{\sigma,p},$$

and $v_* \in O_{\sigma,p}$. This contradicts Theorem 18.2.15 and thus the claim follows.

As a consequence of Theorem 18.2.15 we obtain the following criteria for global well-posedness for (18.13) in the semi-linear case.

Theorem 18.2.17 (Global well-posedness for semi-linear equations). *Let Assumption 18.2.2 hold for any bounded open set $O_{\sigma,p}$, and that $A \in \mathscr{L}(X_1, X_0)$ has maximal L^p-regularity (maximal C-regularity if $p = \infty$) on finite time intervals. Then for every $v_0 \in X_{\sigma,p}$ there exists a maximal $L^p_{w_\alpha}$-solution $(v, T_{\max}(v_0))$ to (18.13) with $T_{\max}(v_0) > 0$. Moreover, if either one of the following holds:*

(1) $p < \infty$, $\sup_{t \in [0, T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,p}} + \|v\|_{L^p_{w_\alpha}(0, T_{\max}(v_0); X_1)} < \infty$;

(2) $p = \infty$, $\sup_{t \in [0, T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,\infty}} + t^\alpha \|v(t)\|_{X_1} < \infty$;

(3) $\sup_{t\in[0,T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,p}} < \infty$ *and Assumption 18.2.2 holds in the sub-critical case,*

then $T_{\max}(v_0) = \infty$, *and thus the* $L^p_{w_\alpha}$*-solution* v *exists globally.*

Proof. The existence of the maximal solution has already been observed in Theorem 18.2.14.

We start with a preliminary observation. Fix $\rho > 0$ and $T \in (0,\infty)$, and set $\beta^* := 1 - (\alpha + \frac{1}{p})(1 - \frac{1}{\rho+1})$. We claim that for all $\beta \in (\sigma, \beta^*]$ and $u \in L^\infty(0,T;X_{\sigma,p}) \cap L^p_{w_\alpha}(0,T;X_1)$ we have

$$\|u\|_{L^{hp}_{w_{\alpha/h}}(0,T;(X_0,X_1)_{\beta,1})} \leqslant C_T \|u\|^{\lambda}_{L^\infty(0,T;X_{\sigma,p})} \|u\|^{1-\lambda}_{L^p_{w_\alpha}(0,T;X_1)}, \tag{18.32}$$

where $h = \rho + 1$, $\lambda \in (0,1)$ is given by $\lambda = \frac{1-\beta}{\alpha+\frac{1}{p}}$, and where C_T also depends on α, h, p and is non-decreasing in T. From the assumption on β it follows that $\lambda \in [1 - \frac{1}{1+\rho}, 1)$. Moreover, if $\beta < \beta^*$, one even has $\lambda > 1 - \frac{1}{\rho+1}$. To prove (18.32), note that by (C.6), Theorem L.3.1, and (L.2),

$$\begin{aligned}\|u(t)\|_{\beta,1} &\leqslant C\|u(t)\|_{((X_0,X_1)_{\sigma,p},X_1)_{1-\lambda,1}} \\ &\leqslant C\|u(t)\|^{\lambda}_{\sigma,p}\|u(t)\|^{1-\lambda}_{X_1},\end{aligned}$$

with the understanding that $\|u(t)\|_{\sigma,p}$ needs to be replaced by $\|u(t)\|_{X_0}$ in the case $p = 1$. Taking $L^{hp}_{w_{\alpha/h}}(0,T)$-norms on both sides gives

$$\begin{aligned}\|u\|_{L^{hp}_{w_{\alpha/h}}(0,T;(X_0,X_1)_{\beta,1})} &\leqslant C\|u\|^{\lambda}_{L^\infty(0,T;X_{\sigma,p})}\|u\|^{1-\lambda}_{L^{hp(1-\lambda)}_{w_{\alpha/(hp(1-\lambda))}}(0,T;X_1)} \\ &\leqslant C_T\|u\|^{\lambda}_{L^\infty(0,T;X_{\sigma,p})}\|u\|^{1-\lambda}_{L^p_{w_\alpha}(0,T;X_1)},\end{aligned}$$

where we used $h(1-\lambda) = (1+\rho)(1-\lambda) \leqslant 1$.

(1): Suppose, for a contradiction, that $T_{\max}(v_0) < \infty$. Let $O_{\sigma,p} \subseteq X_{\sigma,p}$ be a bounded open set such that $\overline{v([0,T_{\max}(v_0)))} \subseteq O_{\sigma,p}$. Taking $\beta = \beta^*_j$ and $h = \rho_j + 1$ in (18.32), we obtain $u \in Y_j$ for every j, and thus $F_{\rm c}(v) \in L^p_{w_\alpha}(0,T_{\max}(v_0);X_0)$ by Lemma 18.2.8. Since $F_{\rm Tr} : O_{\sigma,p} \to X_0$ has linear growth, it is straightforward to check that

$$F_{\rm Tr}(v) \in L^\infty(0,T_{\max}(v_0);X_0) \subseteq L^p_{w_\alpha}(0,T_{\max}(v_0);X_0).$$

Therefore, maximal L^p-regularity of A implies that $v \in \mathrm{MR}^p_\alpha(0,T_{\max}(v_0))$. In particular, $\lim_{t\uparrow T_{\max}(v_0)} v(t)$ exists in $X_{\sigma,p}$ (see Lemma 18.2.7). This contradicts Theorem 18.2.14. It follows that $T_{\max}(v_0) = \infty$.

(2): This can be proved similarly, this time using maximal C-regularity.

(3): Suppose, for a contradiction, that $T_{\max}(v_0) < \infty$. Let $O_{\sigma,p}$ be as in the proof of (1), and let $T \in (0,T_{\max}(v_0))$. As before, it suffices to prove $v \in \mathrm{MR}^p(0,T_{\max}(v_0))$.

By maximal regularity (see Corollaries 17.2.37 and 17.2.48) we can estimate

$$\|v\|_{\mathrm{MR}^p(0,T)} \leqslant C(\|v_0\|_{X_{\sigma,p}} + \|F_{\mathrm{c}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)} + \|F_{\mathrm{Tr}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)}), \tag{18.33}$$

where the constant C depends on $T_{\max}(v_0)$, but not on T. As before, $\|F_{\mathrm{Tr}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)}$ can be estimated above by $K(1+\|v\|_{L^\infty(0,T_{\max}(v_0);X_{\sigma,p})})$. The F_{c}-term is more complicated to handle; this is where the subcriticality enters. Set

$$\overline{Y}_j := \begin{cases} L^{(\rho_j+1)p}_{w_{\alpha/(\rho_j+1)}}(0,T;X_{\beta_j}) & \text{if } p<\infty; \\ C_{w_{\alpha/(\rho_j+1)},0}((0,T];X_{\beta_j}) & \text{if } p=\infty. \end{cases}$$

Fix $x \in O_{\sigma,p} \cap X_1$. Repeating the proof of the second estimate in (18.17) with β_j^* replaced by β_j, we obtain

$$\begin{aligned}
\|F_{\mathrm{c}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)} &\leqslant \|F_{\mathrm{c}}(v)-F_{\mathrm{c}}(x)\|_{L^p_{w_\alpha}(0,T;X_0)} + \|F_{\mathrm{c}}(x)\|_{L^p_{w_\alpha}(0,T;X_0)} \\
&\leqslant L_{\mathrm{c}} \sum_{j=1}^m \big(T^{\delta_j} + \|v\|^{\rho_j}_{\widetilde{Y}_j} + \|x\|^{\rho_j}_{\widetilde{Y}_j}\big)\|v-x\|^{\rho_j}_{\widetilde{Y}_j} \\
&\leqslant L_{\mathrm{c}} \sum_{j=1}^m C_{j,x} + \|v\|^{\rho_j+1}_{\widetilde{Y}_j},
\end{aligned}$$

where in the last step we used Young's inequality in the form $a^\rho b \leqslant a^{\rho+1} + b^{\rho+1}$, and the constant $C_{j,x}$ depends on $T_{\max}(v_0)$ but not on T. Let

$$M := \sup_{t\in[0,T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,p}}.$$

By (18.32) with $h=\rho_j+1$, $\beta=\beta_j$, and $\lambda_j = \frac{1-\beta_j}{\alpha+\frac{1}{p}}$, we find that

$$\begin{aligned}
\|v\|^{\rho_j+1}_{\widetilde{Y}_j} &\leqslant C_T^{\rho_j+1} M^{\lambda_j(\rho_j+1)} \|v\|^{(1-\lambda_j)(\rho_j+1)}_{L^p_{w_\alpha}(0,T;X_1)} \\
&\leqslant C_T^{\rho_j+1} C_{j,\varepsilon} M^{(\lambda_j(\rho_j+1))/(1-\beta_j)} + \varepsilon\|v\|_{L^p_{w_\alpha}(0,T;X_1)},
\end{aligned}$$

where we used $\beta_j = (1-\lambda_j)(\rho_j+1) \in (0,1)$ by subcriticality, and we used Young's inequality in the form $ab^{\beta_j} \leqslant \varepsilon^{-\beta_j/(1-\beta_j)} a^{1/(1-\beta_j)} + \varepsilon b$ for arbitrary $\varepsilon>0$. Taking $\sum_{j=1}^m$ this results in the estimate

$$\|F_{\mathrm{c}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)} \leqslant C_{M,\varepsilon} + L_{\mathrm{c}} m\varepsilon \|v\|_{L^p_{w_\alpha}(0,T;X_1)}.$$

Combining this estimate with (18.33), we obtain

$$(1-C\varepsilon L_{\mathrm{c}} m)\|v\|_{\mathrm{MR}^p(0,T)} \leqslant C(\|v_0\|_{X_{\sigma,p}} + \|F_{\mathrm{Tr}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)}).$$

Setting $\varepsilon = (2CL_{\mathrm{c}}m)^{-1}$ and letting T tend to $T_{\max}(v_0)$, it follows that $v \in \mathrm{MR}^p(0,T_{\max}(v_0))$. □

18.3 Examples and comparison

In order to understand the assumptions on the non-linearity F_c in Assumption 18.2.2, we will now discuss in detail a standard situation, and make a comparison with Example 18.1.3 which involved only the non-linearity F_{Tr}.

Example 18.3.1 (Critical spaces and non-linearities). Let

$$X_0 := H^{s,q}(\mathbb{R}^d), \quad X_1 := H^{s+2,q}(\mathbb{R}^d)$$

with $s \in (-2, 0]$ and $q \in (1, \infty)$. Since $s + 2 > 0$, powers of functions in X_1 are well defined. Notice that X_1 features two more derivatives than X_0; this is the typical situation encountered in applications to PDEs with a leading term of second order. Note that (see Theorem 5.6.9)

$$[X_0, X_1]_\beta = H^{s+2\beta,q}(\mathbb{R}^d)$$

and

$$X_{\sigma,p} = B^{s+2\sigma}_{q,p}(\mathbb{R}^d),$$

where $p \in (1, \infty)$ (extensions to the end-points are possible, but not considered here for simplicity) and $\sigma \in (0, 1/p']$ are arbitrary but fixed for the moment.

Suppose now that $f \in C^1(\mathbb{R})$ satisfies

$$f(0) = 0 \quad \text{and} \quad |f'(t)| \leqslant \ell |t|^\rho, \quad t \in \mathbb{R}, \tag{18.34}$$

for a suitable exponent $\rho > 0$ and constant $\ell \geqslant 0$. Let $F_c : X_1 \to X_0$ be given by

$$(F_c(u))(x) := f(u(x)), \quad x \in \mathbb{R}^d.$$

Then F_c is well-defined and Lipschitz on bounded subsets of X_β under suitable conditions. Indeed, for all $u, v \in X_1$,

$$\begin{aligned}
\|F_c(u)-F_c(v)\|_{X_0} & \\
&= \|f(u) - f(v)\|_{H^{s,q}} \\
&\leqslant C\|f(u) - f(v)\|_r && \text{(Sobolev embedding)} \\
&\leqslant C\ell\|(|u|^\rho + |v|^\rho)(u - v)\|_r && \text{(mean value theorem)} \\
&\leqslant C\ell(\|u\|^\rho_{(\rho+1)r} + \|v\|^\rho_{(\rho+1)r})\|u - v\|_{(\rho+1)r} && \text{(Hölder inequality)} \\
&\leqslant C\ell(\|u\|^\rho_{X_\beta} + \|v\|^\rho_{X_\beta})\|u - v\|_{X_\beta} && \text{(Sobolev embedding),}
\end{aligned}$$

provided we impose some further restrictions in order to justify the application of the Sobolev embeddings. Specifically, the first Sobolev embedding can be applied if $-\frac{d}{r} = s - \frac{d}{q}$ and $1 < r \leqslant q$, which leads to the condition

$$s > -\frac{d}{q'}. \tag{18.35}$$

The second Sobolev embedding can be applied if

$$s + 2\beta - \frac{d}{q} = -\frac{d}{(\rho+1)r}, \quad \text{and} \quad q \leqslant (\rho+1)r,$$

which after substitution of the identity $-\frac{d}{r} = s - \frac{d}{q}$ leads to the condition

$$s + 2\beta - \frac{d}{q} = \frac{1}{\rho+1}\Big(s - \frac{d}{q}\Big) \quad \text{and} \quad q \leqslant \frac{dq(\rho+1)}{d-qs}.$$

Thus we arrive at the conditions

$$\beta = \frac{\rho}{2(\rho+1)}\Big(\frac{d}{q} - s\Big) \quad \text{and} \quad s \geqslant -\frac{d\rho}{q}. \tag{18.36}$$

Sobolev embeddings can also be applied in sub-optimal cases, but here we wish to demonstrate certain optimality and scaling behaviour which is present only if all Sobolev embeddings are sharp.

Combining (18.36) with the (sub)criticality condition (18.12), we obtain

$$\rho\Big(\frac{d}{q} - s\Big) \leqslant 2 + 2\rho\sigma,$$

and criticality holds if

$$\sigma = \frac{1}{2}\Big(\frac{d}{q} - s\Big) - \frac{1}{\rho}.$$

Since $\sigma \in (0, 1/p']$ we arrive the following condition on (q, s) to obtain a critical setting:

$$0 < \frac{1}{2}\Big(\frac{d}{q} - s\Big) - \frac{1}{\rho} \leqslant \frac{1}{p'}. \tag{18.37}$$

If (18.37) holds for some p, then it also holds for all larger values of p, and one can take the limit $p \to \infty$. Thus (18.35), (18.36), (18.37), and the assumption $s \in (-2, 0]$ imply

$$\max\Big\{-2 + \frac{d}{q} - \frac{2}{\rho}, -2, -\frac{d}{q'}\Big\} < s < \frac{d}{q} - \frac{2}{\rho}, \text{ and } -\frac{d\rho}{q} \leqslant s \leqslant 0. \tag{18.38}$$

In the converse direction, if (18.38) holds, then (18.37) holds for large enough p, so the existence of a triple (p, q, s) satisfying the aforementioned conditions is equivalent to (18.38).

Elementary computations show that we can find pairs (s, q) satisfying these conditions holds if and only if

$$\rho > \frac{2}{d} \quad \text{and} \quad \frac{2}{\rho(\rho+1)} < \frac{d}{q}. \tag{18.39}$$

In this case, the corresponding critical space for the initial data is given by

$$X_{\sigma,p} = B^{s+2\sigma}_{q,p}(\mathbb{R}^d) = B^{\frac{d}{q}-\frac{2}{\rho}}_{q,p}(\mathbb{R}^d). \tag{18.40}$$

An interesting feature of (18.40) is that the parameter s does not appear in the critical space $X_{\sigma,p}$ and the smoothness parameter is independent of p.

Remark 18.3.2. The homogeneous variant of $B^{\frac{d}{q}-\frac{2}{\rho}}_{q,p}(\mathbb{R}^d)$ scales as $\|u(\lambda\cdot)\| \eqsim \lambda^{\frac{2}{\rho}}\|u\|$. It follows from this that if u is a solution to a PDE with leading second order differential operator in the space variables, with non-linearity $f(u) = k|u|^{\rho+1}$ (or similar scaling behaviour), and with initial data u_0, then $(t,x) \mapsto \lambda^{\frac{2}{\rho}}u(\lambda^2 t, \lambda x)$ is a solution to the same equation with initial data $\lambda^{\frac{2}{\rho}}u_0(\lambda\cdot)$. This shows that the scaling of the space we encountered in (18.40) is the correct one (up to being a inhomogeneous Besov space).

Specialising to the case $\frac{d}{q} - \frac{2}{\rho} = 0$ and taking p large enough, we also see that one can consider initial data from $L^q(\mathbb{R}^d)$, as this spaces embeds into $B^0_{q,p}(\mathbb{R}^d)$. This space has the same scaling behaviour as just discussed.

In (18.40) the limiting case where $q = \frac{1}{2}d\rho(\rho+1)$ shows that we can 'almost' treat initial data from the space $B^{-2/(\rho+1)}_{q,p}(\mathbb{R}^d)$. The less important so-called microscopical tuning parameter p in (18.40) needs to be so large that (18.37) holds.

Unlike in Example 18.1.3, it now becomes possible to take the special structure of f into account. The space of initial data which we could consider in the example was $B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$ with $s \in (-2, 0]$ and $s+2-\frac{2}{p}-\frac{d}{q} > 0$. Under these restrictions, the smoothness parameter satisfies $s+2-\frac{2}{p} > \frac{d}{q}$, which leads to a much smaller class of initial data than considered in (18.40). Introducing weights in the set-up of Example 18.1.3, does not change anything.

Remark 18.3.3. When $\mathbb{R}^d$ is replaced by a bounded domain, the condition (18.34) on f in Example 18.3.1 can be weakened to

$$|f'(t)| \leqslant \ell(1 + |t|^\rho), \quad t \in \mathbb{R}.$$

Indeed, the step where Hölder's inequality is used can then be replaced by

$$\|(1 + |u|^\rho + |v|^\rho)(u - v)\|_r \leqslant C\ell(1 + \|u\|^\rho_{(\rho+1)r} + \|v\|^\rho_{(\rho+1)r})\|u - v\|_{(\rho+1)r}.$$

Similarly, one can check that $f(0)$ is allowed to be non-zero.

We finish this section with an example illustrating how Theorems 18.2.6 and 18.2.15 can be applied to obtain local and global well-posedness for certain concrete PDEs.

Example 18.3.4 (Local well-posedness for the Allen-Cahn equation). On $\mathbb{R}^d$ with $d \geqslant 2$ (the case $d = 1$ can be included by making subcritical choices) we consider the so-called *Allen-Cahn equation*

$$\begin{cases} \partial_t u - \Delta u & = -u^3 + u, \\ u(0) & = u_0. \end{cases} \tag{18.41}$$

This equation fits into the setting discussed in Example 18.3.1 with $X_0 = H^{s,q}(\mathbb{R}^d)$ and $X_1 = H^{s+2,q}(\mathbb{R}^d)$ for suitable (q,s). Indeed, taking $\rho = 2$, one checks that (18.39) holds if $1 < q < 3d$. Let $s \in (-2,0]$ be such that (18.38) holds with $\rho = 2$, and set $\sigma := \frac{1}{2}(\frac{d}{q} - s) - \frac{1}{2}$. Choose $p \in (1,\infty)$ so large that (18.37) holds. Then, by Example 18.3.1, $F(u) = -u^3$ satisfies the Assumption 18.2.2. We choose to include the linear part of $-u^3 + u$ into the operator A. Another possibility would be to put it into F as well, and consider $\rho_1 = 2$ and $\rho_2 > 0$ arbitrary small.

From Example G.5.6 it follows that for $s = 0$ the operator $Au = -\Delta u - u$ on X_0, with domain X_1, is sectorial of angle zero. Moreover, by Theorems 17.4.1 and 17.2.26, A has maximal L^p-regularity on finite time intervals for all $p \in (1,\infty)$. Since the Bessel potentials $(1-\Delta)^{t/2}$ commute with Δ, the maximal L^p-regularity extends to the full range $s \in \mathbb{R}$.

From now on we view (18.41) as an abstract problem of the form (18.13). In particular, we say that (18.41) admits a *(maximal)* (p,q,s,σ)*-solution* if (18.13) has a (maximal) $L^p_{w_\alpha}$-solution. Applying Theorems 18.2.6 and 18.2.14, it follows that for every $u_0 \in O_{\sigma,p} = X_{\sigma,p} = B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)$ (see (18.40)), the problem (18.41) admits a maximal (p,q,s,σ)-solution $(u, T_{\max}(u_0))$. Moreover,

$$\begin{aligned} u \in W^{1,p}_{w_\alpha}(0,T;H^{s,q}(\mathbb{R}^d)) \cap L^p_{w_\alpha}(0,T;H^{s+2,q}(\mathbb{R}^d)) \\ \cap C([0,T];B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)) \cap C([\tau,T];B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)) \end{aligned} \tag{18.42}$$

for all $0 < \tau < T < T_{\max}(u_0)$, where we used the instantaneous regularisation stated in (18.15).

Global well-posedness can often be obtained via Theorem 18.2.17, but to apply it to the rough initial data considered in the above example requires first performing a (weighted) bootstrap argument to obtain enough regularity in space and time. After that, suitable energy estimate can be applied. Bootstrapping regularity will not be discussed here (a concise discussion of this technique is included in the Notes). Instead, we will only prove global well-posedness for sufficiently smooth initial data. This is done in the next example. In particular, all initial data $u_0 \in L^q(\mathbb{R}^d)$ for $q \in (d,2d)$ are covered if $d \in \{2,3,4,5,6\}$.

Example 18.3.5 (Global well-posedness for the Allen-Cahn equation). Consider again the problem (18.41) in dimension $d \geqslant 2$. In order to obtain that u takes values in $H^{1,q}(\mathbb{R}^d)$, the smallest value of s which we can allow (without bootstrapping) is $s = -1$. Let $q \in (\frac{d}{2}, 2d)$ and $p \in (2,\infty)$ are such that $\frac{d}{q} + \frac{2}{p} \leqslant 2$ (see (18.37)), and set $\rho := 2$, $\sigma := \frac{d}{2q}$, and $\alpha := 1 - \frac{1}{p} - \sigma$. These choices form a special case of Example 18.3.4, and in particular they lead to a critical setting.

Let $u_0 \in B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)$; note that this space contains $L^q(\mathbb{R}^d)$ if $q > d$. By the result of Example 18.3.4, the problem (18.41) admits a (unique) maximal (p, q, s, σ)-solution, and for all $0 < \tau < T < T_{\max}(u_0)$ we have

$$u \in L^p(\tau, T; H^{1,q}(\mathbb{R}^d)) \cap C([\tau, T]; B^{1-\frac{2}{p}}_{q,p}(\mathbb{R}^d)).$$

We will show global existence, i.e., $T_{\max}(u_0) = \infty$, under the more restrictive conditions

$$\max\{d, 2d-6\} < q < 2d \quad \text{and} \quad 2 < p \leqslant \frac{2q}{2d-q}. \tag{18.43}$$

For $d = 2$ we can take $q \in (2, 4)$ and $p \in (2, 2q/(4-q)]$. For $d = 3$ we can take $q \in (3, 6)$ and $p \in (2, 2q/(6-q)]$. We do not claim this is optimal, and we expect that by further bootstrapping some of these conditions can be omitted.

Step 1 – Assuming that $T_{\max}(u_0) < \infty$, we will derive a contradiction with Theorem 18.2.17(1). For the latter it suffices to use Step 2 below. However, we prefer to show the techniques to check Theorem 18.2.17(1) since this can be useful for other situations. This boils down to showing that $u \in L^p_{w_\alpha}(0, T; H^{1,q}(\mathbb{R}^d))$ and

$$\sup_{t\in[0,T_{\max}(v_0))} \|u(t)\|_{B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)} < \infty.$$

By (18.42), both assertions are clear on $[0, \tau]$ for any $\tau < T_{\max}(v_0)$. Thus it suffices to show that, for some $\tau > 0$,

$$u \in L^p(\tau, T_{\max}(u_0); H^{1,q}(\mathbb{R}^d)) \quad \text{and} \quad \sup_{t\in[\tau,T_{\max}(u_0))} \|u(t)\|_{B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)} < \infty. \tag{18.44}$$

Step 2 – We show the second part of (18.44). Since $\frac{d}{q} - 1 < 0$, by the easy embeddings of (14.23) and Proposition 14.4.18, it is enough to show that

$$\sup_{t\in[\tau,T_{\max}(u_0))} \|u(t)\|_{L^q(\mathbb{R}^d)} < \infty.$$

The idea will be to apply the chain rule of Lemma 18.3.6 below. For this we need that $u^3 \in L^1(\tau, T; L^q)$ for $0 < \tau < T < T_{\max}(u_0)$. To see this, note that by Sobolev embedding with $\theta - \frac{d}{q} = -\frac{d}{3q}$ and interpolation,

$$\|u^3\|_{L^q} = \|u\|^3_{L^{3q}} \leqslant C\|u\|^3_{H^{\theta,q}} \leqslant C'\|u\|^{3(1-\theta)}_{L^q}\|u\|^{3\theta}_{H^{1,q}}.$$

As observed before, the L^q-norm of u is uniformly bounded on $[\tau, T]$. Thus for the integrability of $\|u^3\|_{L^q}$ in time it remains to note that $u \in L^p(\tau, T; H^{1,q}) \hookrightarrow L^{3\theta}(\tau, T; H^{1,q})$ since $p > 2 > \frac{2d}{q} = 3\theta$.

Applying the chain rule to the identity

$$u(t) - u(\tau) = \int_\tau^t \Delta u(r) dr + \int_\tau^t -u^3(r) + u(r)\,\mathrm{d}r, \quad t \in [\tau, T],$$

we see that

$$\begin{aligned} \|u(t)\|^q_{L^q(\mathbb{R}^d)} &= \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} - q(q-1)\int_\tau^t \int_{\mathbb{R}^d} |u|^{q-2}|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}r \\ &\qquad + q\int_\tau^t \int_{\mathbb{R}^d} |u|^{q-2}(-u^4 + u^2)\,\mathrm{d}x\,\mathrm{d}r \\ &\leqslant \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} + q\int_\tau^t \|u(r)\|^q_{L^q(\mathbb{R}^d)}\,\mathrm{d}r. \end{aligned} \tag{18.45}$$

Therefore, by Gronwall's lemma applied to $t \mapsto \|u(t)\|^q_{L^q(\mathbb{R}^d)}$,

$$\|u(t)\|^q_{L^q(\mathbb{R}^d)} \leqslant \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} e^{q(t-\tau)}.$$

Since we assumed $T_{\max}(u_0) < \infty$, this implies the desired bound

$$N := \sup_{t\in[\tau, T_{\max}(u_0))} \|u(t)\|^q_{L^q(\mathbb{R}^d)} \leqslant \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} e^{qT_{\max}(u_0)} < \infty. \tag{18.46}$$

As a consequence of (18.45), we also find that

$$\int_\tau^{T_{\max}(u_0)} \int_{\mathbb{R}^d} |u|^{q-2}|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}r \leqslant C_{q,T_{\max}(u_0)} \|u(\tau)\|^q_{L^q(\mathbb{R}^d)}, \tag{18.47}$$

where $C_{q,T_{\max}(u_0)} = \frac{(1+qT_{\max}(u_0))}{q(q+1)} e^{q(T_{\max}(u_0))}$.

Step 3 – By (18.42) we have $u(\tau) \in B^{1-\frac{2}{p}}_{q,p} = (X_0, X_1)_{1-\frac{1}{p},p}$. Therefore, if we can show that $-u^3 + u$ belongs to $L^p(\tau, T_{\max}(u_0); H^{-1,q}(\mathbb{R}^d))$, the first part of (18.44) follows from maximal L^p-regularity applied on the interval $(\tau, T_{\max}(u_0))$ with inhomogeneity $u - u^3$.

It is clear from Step 2 that u has the required regularity, so it remains to consider the term u^3. By Sobolev embedding,

$$\begin{aligned} \|u^3\|_{L^p(\tau,T_{\max}(u_0);H^{-1,q}(\mathbb{R}^d))} &\leqslant C\|u^3\|_{L^p(\tau,T_{\max}(u_0);L^{\frac{qd}{q+d}}(\mathbb{R}^d))} \\ &= C\|u\|^3_{L^{3p}(\tau,T_{\max}(u_0);L^{q_0}(\mathbb{R}^d))}, \end{aligned}$$

where $q_0 = \frac{3qd}{q+d}$. To prove that the latter is finite, note that by Sobolev embedding with $\theta - \frac{d}{2} = -\frac{dq}{2q_0}$ (then $\theta \in (0,1]$ by (18.43) and $2q_0/q > 2$ since $q < 2d$),

$$\begin{aligned} \|u\|^{q/2}_{L^{q_0}(\mathbb{R}^d)} &= \||u|^{q/2}\|_{L^{2q_0/q}} \\ &\leqslant C_0\||u|^{q/2}\|_{H^{\theta,2}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant C_1 \| |u|^{q/2} \|_{L^2(\mathbb{R}^d)}^{1-\theta} \| |u|^{q/2} \|_{W^{1,2}(\mathbb{R}^d)}^{\theta} \\
&\leqslant C_2 \big[\| |u|^{q/2} \|_{L^2(\mathbb{R}^d)} + \| |u|^{q/2} \|_{L^2(\mathbb{R}^d)}^{1-\theta} \| \nabla |u|^{q/2} \|_{L^2(\mathbb{R}^d)}^{\theta} \big] \\
&= C_2 \Big[\|u\|_{L^q(\mathbb{R}^d)}^{q/2} + \|u\|_{L^q(\mathbb{R}^d)}^{q(1-\theta)/2} \frac{q^\theta}{2^\theta} \Big(\int_{\mathbb{R}^d} |u|^{q-2} |\nabla u|^2 \, \mathrm{d}x \Big)^{\theta/2} \Big]. \\
&\leqslant C_2 \Big[N^{1/2} + N^{(1-\theta)/2} \frac{q^\theta}{2^\theta} \Big(\int_{\mathbb{R}^d} |u|^{q-2} |\nabla u|^2 \, \mathrm{d}x \Big)^{\theta/2} \Big],
\end{aligned}
$$

where we used (18.46). Therefore, $u \in L^{3p}(\tau, T_{\max}(u_0); L^{q_0}(\mathbb{R}^d))$ follows if we can check that

$$\int_\tau^{T_{\max}(u_0)} \Big(\int_{\mathbb{R}^d} |u|^{q-2} |\nabla u|^2 \, \mathrm{d}x \Big)^{3p\theta/q} \mathrm{d}t < \infty.$$

The latter follows from (18.47) since our choice of θ satisfies $\theta \leqslant \frac{q}{3p}$, which follows from (18.43).

The following chain rule was used in Example 18.3.5.

Lemma 18.3.6 (Chain rule in the weak setting). *Let $q \in [2, \infty)$ and $p \in (1, \infty)$. Suppose that $u \in C([\tau, T]; L^q(\mathbb{R}^d)) \cap L^p(\tau, T; H^{1,q}(\mathbb{R}^d))$, $G \in L^{p'}(\tau, T; L^q(\mathbb{R}^d; \mathbb{R}^d))$, and $g \in L^1(\tau, T; L^q(\mathbb{R}^d))$ are such that for all $t \in [\tau, T]$*

$$u(t) = u(\tau) + \int_\tau^t \nabla \cdot G(s) \, \mathrm{d}s + \int_\tau^t g(s) \, \mathrm{d}s, \tag{18.48}$$

where the equality is meant in the space $H^{-1,q}(\mathbb{R}^d)$. Then

$$
\begin{aligned}
\|u(t)\|_{L^q(\mathbb{R}^d)}^q = \|u(\tau)\|_{L^q(\mathbb{R}^d)}^q - q(q-1) \int_\tau^t \langle G(s), |u(s)|^{q-2} \nabla u(s) \rangle \, \mathrm{d}s \\
+ q \int_\tau^t \langle g(s), |u(s)|^{q-2} u(s) \rangle \, \mathrm{d}s,
\end{aligned}
\tag{18.49}
$$

where the duality pairing is in $(L^q, L^{q'})$ in both cases.

In view of the Mihlin multiplier theorem (see Theorem 5.5.10),

$$\|\nabla \cdot G\|_{H^{-1,q}(\mathbb{R}^d)} = \Big\| \mathscr{F}^{-1} [\xi \mapsto \frac{2\pi i \xi}{(1+|\xi|^2)^{1/2}} \cdot \widehat{G}(\xi)] \Big\|_{L^q(\mathbb{R}^d)} \leqslant C_{p,d} \|G\|_{L^q(\mathbb{R}^d; \mathbb{R}^d)},$$

and therefore the integral of $\nabla \cdot G$ exists as a Bochner integral in $H^{-1,q}(\mathbb{R}^d)$.

Proof. Without loss of generality we may assume that $\tau = 0$. First we establish some boundedness properties which also show the well-definedness of the integrals appearing in (18.49). For all $v \in L^\infty(0, T; L^q(\mathbb{R}^d))$ and $w \in L^p(0, T; L^q(\mathbb{R}^d))$, by Hölder's inequality in the space variables with $\frac{1}{q} + \frac{q-2}{q} + \frac{1}{q} = 1$, and subsequently in the time variable with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |G|\,|v|^{q-2}|w|\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \leqslant \int_0^T \|G\|_{L^q(\mathbb{R}^d;\mathbb{R}^d)}\|v\|_{L^q(\mathbb{R}^d)}^{q-2}\|w\|_{L^q(\mathbb{R}^d)}\,\mathrm{d}s \\
&\qquad \leqslant \|v\|_{L^\infty(0,T;L^q(\mathbb{R}^d))}^{q-2}\|G\|_{L^{p'}(0,T;L^q(\mathbb{R}^d;\mathbb{R}^d))}\|w\|_{L^p(0,T;L^q(\mathbb{R}^d))}.
\end{aligned} \tag{18.50}$$

In a similar way one proves that

$$\int_0^T \int_{\mathbb{R}^d} |g|\,|v|^{q-1}\,\mathrm{d}x\,\mathrm{d}s \leqslant \|g\|_{L^1(0,T;L^q(\mathbb{R}^d))}\|v\|_{L^\infty(0,T;L^q(\mathbb{R}^d))}^{q-1}.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d}\varphi\,\mathrm{d}x = 1$, and let $\varphi_n := n^d\varphi(n\cdot)$. By Theorem 2.3.8, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have

$$\varphi_n * f \to f \text{ and } |\varphi_n * f| \leqslant Mf \ \text{ almost everywhere,} \tag{18.51}$$

where M denotes the Hardy–Littlewood maximal operator.

Taking convolutions in (18.48), we obtain

$$u_n(t) - u_n(0) = \int_0^t \nabla\cdot G_n(s)\,\mathrm{d}s + \int_0^t g_n(s)\,\mathrm{d}s,$$

where $u_n = \varphi_n * u$, $\nabla G_n = \nabla\cdot(\varphi_n * G) = \varphi_n * (\nabla\cdot G)$, and $g_n = \varphi_n * g$. Fix $x \in \mathbb{R}^d$ and let $R > 0$ be so large that $|u(s,x)| \leqslant R$ for all $s \in [0,T]$. Let $\zeta \in C_c^2(\mathbb{R})$ be such that $\zeta(y) = |y|^q$ for $|y| \leqslant R$. Note that $\zeta'(y) = |y|^{q-2}y$ and $\zeta''(y) = |y|^{q-2}$ for $|y| \leqslant R$. Applying the chain rule for weak derivatives in time to the function $t \mapsto \zeta(u(t,x))$, we obtain

$$\begin{aligned}
|u_n(t,x)|^q = |u_n(0,x)|^q &+ q\int_0^t |u_n(s,x)|^{q-2}u_n(s,x)\nabla\cdot G_n(s,x)\,\mathrm{d}s \\
&+ q\int_0^t q|u_n(s,x)|^{q-2}u_n(s,x)g_n(s,x)\,\mathrm{d}s.
\end{aligned}$$

Integrating over $\mathbb{R}^d$ and using Fubini's theorem and integrating by parts, we obtain

$$\begin{aligned}
\|u_n(t)\|_{L^q(\mathbb{R}^d)}^q = \|u_n(0)\|_{L^q(\mathbb{R}^d)}^q &- q(q-1)\int_0^t \langle G_n(s), |u_n(s)|^{q-2}\nabla u_n(s)\rangle\,\mathrm{d}s \\
&+ \int_0^t q\langle g_n(s), |u_n(s)|^{q-2}u_n(s)\rangle\,\mathrm{d}s.
\end{aligned}$$

From the observation (18.51) we deduce that $u_n \to u$ in $L^q(\mathbb{R}^d)$ pointwise in $[0,T]$, $u_n \to u$ in $L^p(0,T;H^{1,q}(\mathbb{R}^d))$, $G_n \to G$ in $L^{p'}(0,T;L^q(\mathbb{R}^d))$, and $g_n \to g$ in $L^1(0,T;L^q(\mathbb{R}^d))$. Thus it remains to let $n \to \infty$ in the above identity and use the boundedness/continuity properties from the beginning of the proof to obtain convergence. Indeed, after extracting almost everywhere convergent subsequences and relabelling, convergence follows by dominated convergence. For instance, for the first term,

$$\int_0^T \int_{\mathbb{R}^d} |u_n|^{q-2} G_n \cdot \nabla u_n \,\mathrm{d}x \,\mathrm{d}s \to \int_0^T \int_{\mathbb{R}^d} |u|^{q-2} G \cdot \nabla u \,\mathrm{d}x \,\mathrm{d}s$$

follows since $|u_n|^{q-2} G_n \cdot \nabla u_n$ is dominated by $|Mu|^{q-2} MG\, M|\nabla u|$, which is an integrable function by (18.50) and the boundedness of M on $L^q(\mathbb{R}^d)$. ☐

18.4 Long-time existence for small initial data and $F = F_{\mathrm{c}}$

Short-time existence and uniqueness has been proved in Theorems 18.1.2 and 18.2.6 In this section we prove that, under suitable conditions, for initial values with small norm in $X_{\sigma,p}$ one can obtain well-posedness on *arbitrary long time intervals* $[0, T]$. This result is typical for the semi-linear setting. The assumptions on F will be similar to the ones of Section 18.2. However, we will assume that $F = F_{\mathrm{c}}$, $F(0) = 0$, and replace (18.11) by the slightly more restrictive condition (18.52) below. Moreover, we assume $A \in \mathscr{L}(X_1, X_0)$, and thus we only consider the semi-linear setting.

Theorem 18.4.1 (Semi-linear equations with small initial data). *Let $p \in [1, \infty]$ and $\alpha \in [0, \frac{1}{p'}) \cup \{0\}$, where we take $\alpha > 0$ if $p = \infty$. Let $\sigma = 1 - \alpha - \frac{1}{p} \in [0, 1/p'] \cap [0, 1)$. Let X_0 and X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ with embedding constant $C_X \geqslant 1$. Let $O_{\sigma,p} \subseteq X_{\sigma,p}$ be an open set and suppose that $0 \in O_{\sigma,p}$. Let $A \in \mathscr{L}(X_1, X_0)$ and suppose that A has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Suppose that $F_{\mathrm{c}} : X_1 \cap O_{\sigma,p} \to X_0$ is such that $F_{\mathrm{c}}(0) = 0$ and*

$$\|F_{\mathrm{c}}(u) - F_{\mathrm{c}}(v)\|_{X_0} \leqslant L_{\mathrm{c}} \sum_{j=1}^{m} (\|u\|_{X_{\beta_j}}^{\rho_j} + \|v\|_{X_{\beta_j}}^{\rho_j}) \|u - v\|_{X_{\beta_j}} \tag{18.52}$$

for all $u, v \in X_1 \cap O_{\sigma,p}$, where $\beta_j \in (\sigma, 1)$, $\rho_j > 0$ are such that $\beta_j \leqslant \frac{1+\rho_j \sigma}{1+\rho_j}$ for $j \in \{1, \ldots, m\}$. Then for every $T \in (0, \infty)$ there exist $\varepsilon > 0$ such that for each $\|v_0\|_{X_{\sigma,p}} \leqslant \varepsilon$, the problem

$$\begin{cases} u' + Au &= F(u), \quad \text{on } (0, T), \\ u(0) &= v_0, \end{cases} \tag{18.53}$$

has a unique $L^p_{w_\alpha}$-solution $u_{v_0} \in \mathrm{MR}^p_\alpha(0, T)$. Moreover, there is a $C \geqslant 0$ such that for all $\|v_0\|_{X_{\sigma,p}}, \|v_1\|_{X_{\sigma,p}} \leqslant \varepsilon$,

$$\|u_{v_0} - u_{v_1}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C \|v_0 - v_1\|_{X_{\sigma,p}}. \tag{18.54}$$

If additionally, A has maximal L^p-regularity (C-regularity if $p = \infty$) on $\mathbb{R}_+$ and $0 \in \varrho(A)$, then the above holds with $(0, T)$ replaced by $\mathbb{R}_+$.

Proof. In the proof we use the notation $E_j = L^p_{w_\alpha}(0,T;X_j)$. Let $u_0 = 0$ and set $T_1 = T$. Without loss of generality we may assume $T \geqslant 1$ and $r \leqslant 1$. Let $\Phi_{v_0} : B^T_r(v_0) \to \mathrm{MR}^p_\alpha(0,T)$ be defined by $\Phi_{v_0}(v) := u$, where u is the unique $L^p_{w_\alpha}$-solution to

$$\begin{cases} u' + Au &= F(v), \\ u(0) &= v_0. \end{cases}$$

Note that for $r \in (0,1]$ and $\varepsilon > 0$ small enough, v takes values in $O_{\sigma,p}$ by Lemma 18.2.9, and by Lemma 18.2.10 we have $F(v) \in E_0$. Below Theorem 18.2.6, it has already been observed that local existence and uniqueness follow if we can show that Φ_{v_0} has a unique fixed point.

By the maximal regularity estimate (18.28) we have $u \in \mathrm{MR}^p_\alpha(0,T)$, $u(0) = v_0$, and

$$\begin{aligned} \|u\|_{\mathrm{MR}^p_\alpha(0,T)} &\leqslant C_{A,T}\|v_0\|_{X_{\sigma,p}} + C_{A,T}\|F(v)\|_{E_0} \\ &\leqslant C_{A,T}\varepsilon + C_{A,T}CL_{\mathrm{c}}\sum_{j=1}^m \|v\|^{\rho_j+1}_{\mathrm{MR}^p_\alpha(0,T)} \\ &\leqslant C_{A,T}\varepsilon + C_{A,T}CL_{\mathrm{c}}\sum_{j=1}^m r^{\rho_j+1}, \end{aligned}$$

where the estimate for $F(v)$ follows from Lemmas 18.2.7 and 18.2.8, the constant C can be taken T-independent since $T \geqslant 1$, and we used (18.19) with $u_0 = 0$ and $z_{u_0} = 0$. Note that the terms T^{δ_j} can be avoided due to the more restrictive condition (18.52). The above estimate shows that for $r, \varepsilon > 0$ small enough, $\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant r$, and thus $u \in B^T_r(v_0)$.

Next, fix $v_{j,0} \in B_{X_{\sigma,p}}(u_0,\varepsilon)$ and $v_j \in B^T_r(v_{j,0})$ for $j \in \{1,2\}$. Then $u = \Phi_{v_{1,0}}(v_1) - \Phi_{v_{2,0}}(v_2)$ solves the problem

$$\begin{cases} u' + Au &= F(v_1) - F(v_2), \\ u(0) &= v_{1,0} - v_{2,0}. \end{cases}$$

Therefore, by the maximal regularity estimate (18.28),

$$\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C_{A,T}\|F(v_1) - F(v_2)\|_{E_0} + C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}},$$

From Lemmas 18.2.7 and 18.2.8 we obtain that

$$\begin{aligned} &\|F(v_1) - F(v_2)\|_{E_0} \\ &\qquad \leqslant CL_{\mathrm{c}}\sum_{j=1}^m \big[\|v_1\|^{\rho_j}_{\mathrm{MR}^p_\alpha(0,T)} + \|v_2\|^{\rho_j}_{\mathrm{MR}^p_\alpha(0,T)}\big]\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ &\qquad \leqslant 2CL_{\mathrm{c}}\sum_{j=1}^m r^{\rho_j}\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)}. \end{aligned}$$

Therefore, by choosing $r > 0$ small enough,

$$\begin{aligned}\|\Phi_{v_{1,0}}(v_1) - \Phi_{v_{2,0}}(v_2)\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ + C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}.\end{aligned} \tag{18.55}$$

By (18.55), $\Phi_{v_0} : B^T_r(v_0) \to B^T_r(v_0)$ is a strict contraction, and thus it has a unique fixed point $u_{v_0} \in B^T_r(v_0)$. This is the required solution to (18.53). Moreover, (18.55) implies that for $v_{1,0}, v_{2,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$

$$\begin{aligned}\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant \frac{1}{2}\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \\ + C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}},\end{aligned}$$

and thus

$$\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant 2C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}.$$

which gives (18.54).

In case A has maximal regularity on $\mathbb{R}_+$ and $0 \in \varrho(A)$, then (18.28) holds with $(0,T)$ replaced by $\mathbb{R}_+$. Moreover, one can check that Lemma 18.2.9 holds with $(0,T_1)$ replaced by $\mathbb{R}_+$. Therefore, one can repeat the above argument on the half line. □

18.5 Notes

The theory of abstract non-linear parabolic evolution equations has a long history going back to the work of the Japanese school in the 1960s, with contributions of Fujita, Kato, Tanabe, and others. Excellent monographs on the subject are available, including Amann [1995], Friedman [1969], Henry [1981], Lunardi [1995], Lions [1969], Pazy [1983], Prüss and Simonett [2016], Tanabe [1979], Yagi [2010]. For the purpose of this chapter we chose to limit ourselves to the maximal L^p-regularity approach to quasi-linear evolution equations, mostly focussing on local well-posedness. Other approaches, including maximal Hölder regularity, the so-called Kato approach, and the theory monotone operators, are treated in some of the references just mentioned. Maximal regularity techniques have important applications to a number of topics not covered in this volume, such as linearised stability, semi-flows, higher order regularity, sharp conditions for global well-posedness, numerical analysis, and applications to concrete PDEs. Maximal L^p-regularity for stochastic evolution equations will be covered in Volume IV.

The maximal L^p-regularity approach to quasi-linear evolution equations was initiated by the influential paper Clément and Li [1993/94], and further investigated and extended in Prüss [2002] and Amann [2005]. The semi-linear setting of Theorem 18.1.2 is a special case of the results in these works, and is presented here as a warm-up to the later results.

A local well-posedness result under the assumption of maximal C-regularity was obtained by Clément and Simonett [2001]. The use of weights in time seems essential in the latter (see Remark 18.2.11). Based on the weighted maximal L^p-regularity result of Prüss and Simonett [2004], the L^p-setting was extended to a weighted setting in time by Köhne, Prüss, and Wilke [2010]. The use of weights has several advantages:

- well-posedness in case of rough initial data
- instantaneous regularisation
- compactness properties of orbits

and has become standard in the theory of evolution equations. All of the above works have found applications to concrete quasi- and semi-linear PDEs, many of which are collected and mentioned in the influential monograph Prüss and Simonett [2016]. Since the number of applications is too large to discuss here, we will mainly focus on applications of the theory of critical spaces in these notes.

For parabolic equations it is often possible to bootstrap regularity in time and space. Sometimes one can even derive real analyticity via use the so-called parameter trick of Angenent [1990b,a]; see also Prüss and Simonett [2016, Section 5.2] for a presentation in the setting of abstract quasi-linear evolution equations. Applications of maximal L^p-regularity techniques to the study of linearised stability for non-linear parabolic evolution equation can be found in Lunardi [1995], Prüss [2002], Prüss, Simonett, and Zacher [2009], Matioc and Walker [2020], and references therein.

Critical spaces

In the present abstract evolution equations framework, the splitting $F = F_{\mathrm{Tr}} + F_{\mathrm{c}}$ was first introduced in LeCrone et al. [2014]. In this paper, local well-posedness in the subcritical case was proved using maximal L^p-regularity for $1 < p < \infty$. Shortly afterwards, it was realised in Prüss and Wilke [2017] that under additional conditions on A and (X_0, X_1), local well-posedness can even be obtained in the critical case. Consequences for the Navier–Stokes equations were discussed in Prüss and Wilke [2018]. Further results and applications to concrete and abstract problems were given in Prüss, Simonett, and Wilke [2018]. In particular, this paper discusses the relationship between scaling invariance and criticality for several concrete PDEs. It is remarkable that an abstract definition for criticality can be given which leads to new insights for many concrete PDEs. In the same paper, by way of an example it is shown that the sub-criticality condition (18.12) cannot be improved. The L^p-framework was extended to maximal C-regularity in LeCrone and Simonett [2020].

Theorem 18.2.6 unifies and extends several of the results mentioned in the preceding discussion. For simplicity, here we only considered the case where A and F are time-independent, but this restriction can be avoided easily (see Remark 18.2.4). The unification lies in the fact that one proof is presented which works for all $p \in [1, \infty]$ and all admissible weights, with $p = \infty$ corresponding to maximal C-regularity. Moreover, we do not need geometric conditions on X_0 such as the UMD property, or further conditions on $A(u_0)$ besides maximal L^p- or C-regularity. In part of the existing literature, the spaces X_{β_j} appearing in (18.11) are taken as the complex interpolation spaces $[X_0, X_1]_{\beta_j}$. Taking the real interpolation spaces $(X_0, X_1)_{\beta_j,1}$ leads to a less restrictive condition on F_c and is easier to work with in the proofs.

The case $p = 1$ of Theorem 18.2.6 seems to be new. It is important to observe that for $p = 1$ one is forced to take $\sigma = \alpha = 0$, which in turn forces the X_0-valued trace part $F_{\rm Tr}$ to be defined on an open subset $O_{\sigma,p}$ of the same space X_0. For non-linearities of the form $F = F_{\rm Tr}$, this requirement rules out many interesting examples of non-linearities. However, by allowing non-linearities with a critical part, i.e., non-linearities of the form $F = F_{\rm Tr} + F_c$, many interesting examples can be covered even when $p = 1$, the point being that it suffices to have F_c locally Lipschitz with respect to the norms of the smaller spaces $X_{1/(1+\rho_j)}$ (with the ρ_j's as in Assumption 18.2.2). On the other hand, according to Theorem 17.4.5, operators with maximal L^1-regularity are rare. An exception is the case where X_0 itself is a real interpolation space in which case the Da Prato–Grisvard theorem applies (see Corollary 17.3.20).

It should be observed that a more flexible condition on F_c could be used in (18.11), namely

$$\|F_c(u) - F_c(v)\|_{X_0} \leqslant L_c \sum_{j=1}^{m} (1 + \|u\|_{X_{\varphi_j}}^{\rho_j} + \|v\|_{X_{\varphi_j}}^{\rho_j}) \|u - v\|_{X_{\beta_j}}, \qquad (18.56)$$

with $\varphi_j \in (\sigma, 1)$, $\beta_j \in (\sigma, \varphi_j]$, along with the subcriticality condition

$$\rho_j(\varphi_j - \sigma) + \beta_j \leqslant 1, \qquad j \in \{1, \ldots, m\}. \qquad (18.57)$$

The formulation (18.56) allows for different space regularity for u, v, and $u-v$ on the right-hand side (see Agresti and Veraar [2022a] and Prüss, Simonett, and Wilke [2018]). However, in all known examples, it suffices to take $\varphi_j = \beta_j$ (as we do in the main text) in order to obtain the sharpest results. Note that by taking $\varphi_j = \beta_j$, (18.57) reduces to the sub-criticality condition (18.12).

Global well-posedness and blow-up criteria

The existence of a maximal time interval in Theorem 18.2.14 is a standard result. Often it is only stated and proved under the more restrictive assumption that $A(v_0)$ have maximal L^p- or C-regularity for all $v_0 \in O_{\sigma,p}$. The present formulation only uses maximal regularity of $A(u_0)$. In a slightly different set-up it appears in Agresti and Veraar [2022a].

The global well-posedness result of Theorem 18.2.15 is also standard. The statement and proof closely follow Prüss and Simonett [2016, Corollary 5.1.2]. The weight t^α can be helpful in proving global well-posedness, as estimates in the space $X_{\sigma,p}$ are easier to obtain for smaller values of σ (i.e., for higher values of α). In the semi-linear case, the blow-up criteria can be further weakened as was done in Theorem 18.2.17. In case of semi-linear functions F of quadratic type, blow-up criteria appear in Prüss, Simonett, and Wilke [2018, Section 2.1]. Some of these were extended, for a more general class of semi-linearities F, in to a stochastic setting in Agresti and Veraar [2022b, Theorem 4.11]. Simplifying this to the deterministic setting, one arrives at the following result:

Theorem 18.5.1 (Serrin-type blow-up criteria). *Let $p \in (1, \infty)$, suppose that Assumption 18.2.2 holds, and let $A \in \mathscr{L}(X_1, X_0)$ have maximal L^p-regularity on finite time intervals. Let $(u, T_{\max}(u_0))$ denote the maximal $L^p_{w_\alpha}$-solution to*

$$\begin{cases} u' + Au &= F(u), \quad \text{on } (0,T), \\ u(0) &= u_0. \end{cases}$$

Suppose that for each $j \in \{1, \dots, m\}$ we have

$$\rho_j < 1 + \alpha p \quad \text{or} \quad (\alpha = 0 \text{ and } \rho_j \leqslant 1).$$

Then the following assertions hold:

- *for all $T < T_{\max}(u_0)$ one has $\|u\|_{L^p(0,T;X_{1-\alpha})} < \infty$;*
- *if $T_{\max}(u_0) < \infty$, then $\|u\|_{L^p(0,T_{\max}(u_0);X_{1-\alpha})} = \infty$.*

In the case of (sub-)quadratic semi-linearity F, one has $\rho_j \leqslant 1$ and the above condition always holds.

Applications

The theory of quasi-linear evolution equations in critical spaces as presented in this chapter has been applied to models in several scientific areas which include fluid dynamics, chemistry, neuroscience, free boundary problems, and differential geometry. For details we refer to the founding papers and books LeCrone, Prüss, and Wilke [2014], Prüss and Simonett [2016], Prüss and Wilke [2018], Prüss, Simonett, and Wilke [2018], LeCrone and Simonett [2020], and for further applications to the more recent papers Hieber and Prüss [2018], Mazzone, Prüss, and Simonett [2019a,b], Simonett and Prüss [2019], Binz, Hieber, Hussein, and Saal [2020], Giga, Gries, Hieber, Hussein, and Kashiwabara [2020], Hieber, Hussein, and Saal [2023], Hieber, Kress, and Stinner [2021], Mazzone [2021], Prüss, Simonett, and Wilke [2021], Court and Kunisch [2022], Simonett and Wilke [2022b]. This list is likely to expand in the near future, as the splitting $F = F_{\mathrm{Tr}} + F_{\mathrm{c}}$ of Theorem 18.2.6 has proved to be very powerful in applications to concrete non-linear parabolic equations of semi- and quasi-linear type. It leads to new insights for many PDEs to which the original framework of Clément and Li [1993/94] was applicable. Moreover, some of the new blow-up criteria can make it possible to obtain global well-posedness results.

The examples considered in Section 18.3 are very basic, and local/global well-posedness is well known for a broad class of initial values. The examples are merely chosen to demonstrate the abstract theorems of Section 18.2 in a simple setting. The method to check the blow criteria in Example 18.3.5 is taken from Agresti and Veraar [2023a], where these techniques are used in several examples.

An extension of the results of Section 18.2 to stochastic quasi-linear evolution equations in critical spaces was recently obtained in Agresti and Veraar [2022a,b], where completely new proofs where required. Applications to stochastic PDE can be found in these works, as well as in Agresti and Veraar [2021, 2022c, 2023b,a], Agresti [2022], Agresti, Hieber, Hussein, and Saal [2022a,b].