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THE ANALYSIS OF 3-PHASE SQUIRREL-CAGE INDUCTION MOTORS INCLUDING SPACE HARMONICS AND MUTUAL SLOTTING IN TRANSIENT AND STEADY STATE

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ABSTRACT - From general equations which describe the transient electromechanical behavior of the asynchronous squirrel-cage motor, and which include the influence of space harmonics and mutual slotting, simplified models are derived and compared. The models derived are demonstrated in examples where special attention is paid to the influence of the place of the harmonics in the mutual inductance matrix and the influence of mutual slotting. Further, the steady-state equations are derived and the back-transformation for the stator and rotor currents is given. One example is compared with the result of measurements.

Keywords - asynchronous machines, space harmonics, mutual slotting, steady state and transients.

1. INTRODUCTION

In [1] general equations which include the influence of space harmonics and mutual slotting have been derived for the 3-phase squirrel-cage induction motor. These equations which are the result of many transformations and which include all space harmonics are respectively:

$$\frac{d}{dt}(\psi_{s0}) = u_{s0} - R_s i_{s0}$$
$$\frac{d}{dt}(\psi_{sf}) = u_{sf} - R_s i_{sf} - j\frac{d\gamma}{dt}\psi_{sf} \tag{1}$$
$$\frac{d}{dt}(\psi''_{rm}) = -\tilde{R}_{rm} i_{rm} \quad , \quad m = 0 \text{ or } K$$
$$\frac{d}{dt}(\psi''_{rk}) = -\tilde{R}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) - e_k\psi''_{rk} - f_k\psi''^{*}_{rk} \quad , \quad k=1,2,\ldots,K-1$$

where the first two equations describe the zero-sequence and the forward-sequence stator component respectively. γ is an angle which can be chosen freely. The third equation describes the zero-sequence component of the rotor (m=0) or the rotor circuit K. The equation m=K only exists if the mutual inductance matrix L'_{sr} after the group and symmetrical-component transformations contains a "middle" column filled with harmonics. K = N/2z, where N is the number of rotor slots and z is the greatest common divisor of N and the pole pair p. The fourth equation describes the voltage equations for the remaining rotor circuits. The factors e_k and f_k are the result of the time-dependent rotor transformation and they cause rotational EMFs. The maximum number of rotor-voltage equations is K, while in the L'_{sr}- matrix whether m=0 or m=K exists.

The flux-current relations are:

$$\psi_{s0} = \tilde{L}_{s0} i_{s0} + M_{1m} i_{rm} + \sum_{i=1}^{K-1}[M_{0i} i''_{ri} + M^{*}_{0i} i''^{*}_{ri}]$$

$$\psi_{sf} = \tilde{L}_s\, i_{sf} + M_{2m} i_{rm} + \sum_{i=1}^{K-1} M_i\, i''_{ri} \tag{2}$$

$$\psi''_{rm} = M_{1m} i_{s0} + M^{*}_{2m} i_{sf} + M_{2m} i^{*}_{sf} + (\tilde{L}_{rm} + \Delta_{mm}) i_{rm} + \sum_{i=1}^{K-1}[\Delta_{mi} i''_{ri} + \Delta_{m-i} i''^{*}_{ri}]$$

$$\psi''_{rk} = M^{*}_{0k} i_{s0} + M^{*}_{k} i_{sf} + \tilde{L}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + \Delta_{km} i_{rm} + \sum_{i=1}^{K-1}[\Delta_{ki} i''_{ri} + \Delta_{k-i} i''^{*}_{ri}]$$

where the symbols $\tilde{L}$ denote the self-inductances of the transformed stator and rotor circuits, M the resultant mutual inductances and Δ the additional self- and mutual rotor inductances due to stator slotting.

The electromagnetic torque equation is:

$$T_e(t) = \mathrm{Re}\Bigg[\Bigg[2.\sum_{i=1}^{K-1}[(\frac{\partial M^{*}_{0i}}{\partial\theta} + g_i M^{*}_{0i} + h_i M_{0i}) i''^{*}_{ri}] + i_{rm}\frac{\partial M_{1m}}{\partial\theta}\Bigg] i_{s0} +$$
$$+2\Bigg[\sum_{i=1}^{K-1}([\frac{\partial M^{*}_{i}}{\partial\theta} + (g_i - j\frac{\partial\gamma}{\partial\theta})M^{*}_{i}] i''^{*}_{ri} + h^{*}_{i} M^{*}_{i} i''_{ri}) + i_{rm}(\frac{\partial M^{*}_{2m}}{\partial\theta} - j\frac{\partial\gamma}{\partial\theta}M^{*}_{2m})\Bigg] i_{sf}$$
$$+\sum_{k=1}^{K-1}\Bigg[(\frac{\partial\Delta_{mk}}{\partial\theta} i''_{rk} + \frac{\partial\Delta_{m-k}}{\partial\theta} i''^{*}_{rk}) i^{*}_{rm} + \sum_{i=1}^{K-1}(\frac{\partial\Delta_{ik}}{\partial\theta} i''_{rk} + \frac{\partial\Delta_{i-k}}{\partial\theta} i''^{*}_{rk}) i''^{*}_{ri}\Bigg]$$
$$+2\sum_{i=1}^{K-1}\Bigg[(g_i\Delta_{im} + h_i\Delta^{*}_{m-i}) i_{rm} + \sum_{k=1}^{K-1}[(g_i\Delta_{ik} + h_i\Delta^{*}_{k-i}) i''_{rk} + (g_i\Delta_{i-k} + h_i\Delta^{*}_{ik}) i''^{*}_{rk}]\Bigg] i''^{*}_{ri}\Bigg] \tag{3}$$

The factors g_i and h_i are the result of the time-dependent rotor transformation. θ displays the rotor-position angle. The mechanical equations of motion are:

$$\frac{ds}{dt} = \frac{1}{J}[-\frac{p}{\omega}(T_e - T_{me}) + d_{me}(1-s)] \quad , \quad \frac{d(p\theta)}{dt} = \omega(1-s) \tag{4}$$

where s is the slip of the rotor, T_{me} the load torque, J the inertia of the complete rotor mass and d_{me} the coefficient of mechanical damping.

From (1)-(3) general equations can be derived for the two types of armature reaction [1].

2. FIRST TYPE OF ARMATURE REACTION

For the first type of armature reaction it holds that the zero-sequence stator component has no influence on the electromechanical behavior and can be removed. The elements Δ_{i-k} in $\Delta L''_{rr}$ are not relevant for the equations of the first type and can be removed. The factors $k_k = f_k = h_i = 0$ and $M_{2K} = M_{20} = 0$.

When, further, the stator winding is connected to a three-phase symmetrical voltage source:

$$u_{si} = U\sqrt{2}\cos(\omega t + \phi - 2\pi i/3) \quad , \quad i = a,b,c \; (=0,1,2) \tag{5}$$

the transformed voltages for a star-connected stator winding are:

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$$u_{sf} = \frac{U}{\sqrt{2}} e^{j(\omega t + \phi' - \gamma)} \text{ and } u_{s0} = 0 \quad \text{where } \phi' = \phi - \pi/6. \tag{6}$$

Upon assuming now $\gamma = \omega t + \phi'$ yields:

$$u_{sf} = U/\sqrt{2}, \quad \frac{d\gamma}{dt} = \omega \text{ and } \frac{\partial \gamma}{\partial \theta} = 0.$$

Insertion of these properties into the general equations (1)- (3) yields:

$$\frac{d}{dt}\psi_{sf} = u_{sf} - R_s i_{sf} - j\omega\psi_{sf} \; ; \; \psi_{sf} = \tilde{L}_s i_{sf} + \sum_{i=1}^{K-1} \hat{M}_i i''_{ri} \tag{7}$$

$$\frac{d}{d\tau}\psi''_{rk} = -\tilde{R}_{rk} i''_{rk} - e_k \psi''_{rk} \; ; \; \psi''_{rk} = \hat{M}_k i_{sf} + \tilde{L}_{rk} i''_{rk} + \sum_{i=1}^{K-1} \Delta_{ki} i''_{ri}$$

$k=1,2,..,K-1$

and the electromagnetic torque equation:

$$T_e = \text{Real}\left[\sum_{i=1}^{K-1} \left[2\left(\frac{\partial \hat{M}_i}{\partial \theta} + g_i \hat{M}_i\right) i_{sf} + \sum_{k=1}^{K-1} \left(\frac{\partial \Delta_{ik}}{\partial \theta} + 2 g_i \Delta_{ik}\right) i''^{*}_{rk}\right] i''^{*}_{ri}\right] \tag{8}$$

where

$$g_i = -j\frac{\partial \eta_i}{\partial \theta} \text{ and } e_i = j\left[\omega - \frac{d\eta_i}{dt}\right] \; ; \; \eta_i = \alpha_i \text{ or } \beta_i$$

The angle η_i as well as the mutual inductances $\hat{M}_i$, which are real and equal to the resultant inductances $\hat{M}_{2i}$ or $\hat{M}_{3i}$ in the L'_{sr}-matrix, are normally composed of more inductances (see fig 1). The derivatives and the derivative $\partial\Delta_{ik}/\partial\theta$ are given in appendix I.

column number

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
m_{15} m_{15}^*			m_3			m_9^*			m_9			m_3^*		
	m_1 m_{31} m_{-29}			m_{19} m_{-11}			m_7 m_{-23}			m_{25} m_{-5}			m_{13} m_{-17}	
		m_{-17}^* m_{13}^*			m_{-5}^* m_{25}^*			m_{-23}^* m_7^*			m_{-11}^* m_{19}^*			m_{-29}^* m_1^* m_{31}^*

a) First type: N = 30, z = 2, N/z = 15

Fig. 1. Distribution of harmonics in L'_{sr}-matrix

Simplified models

If no stator slotting is taken into account ($\Delta_{ik}=0$) the currents can be solved from the flux-current equations analytically:

$$i''_{ri} = \frac{\psi''_{ri} - \hat{M}_i i_{sf}}{\tilde{L}_{ri}} \; ; \; i_{sf} = \frac{\psi_{sf} - \Sigma \hat{M}_i \psi''_{ri}/\tilde{L}_{ri}}{\tilde{L}_s - \Sigma \hat{M}_i^2/\tilde{L}_{ri}}$$

If no armature reaction is taken into account, this means only one single harmonic inductance in one element of the L'_{sr}-matrix, the equations become:

$$\frac{d}{dt}\psi_{sf} = u_{sf} - R_s i_{sf} - j\omega\psi_{sf}; \quad \psi_{sf} = \tilde{L}_s i_{sf} + \sum_{i=1}^{K-1} \hat{M}_i i''_{ri} \tag{9}$$

$$\frac{d}{dt}\psi''_{ri} = -\tilde{R}_{ri} i''_{ri} - j\omega s_{\nu_i} \psi''_{ri}; \quad \psi''_{ri} = \hat{M}_i i_{sf} + \tilde{L}_{ri} i''_{ri}$$

$i=1,2,...,K-1$

$$T_e(t) = 2p.\text{Real}\left[\sum_{i=1}^{K-1} -j\nu_i \hat{M}_i i''^{*}_{ri} i_{sf}\right] \tag{10}$$

$\hat{M}_i$ = constant, $g_i = -jp\nu_i$, $e_i = j\omega s_{\nu_i}$, $s_{\nu_i} = 1 - \nu_i(1-s)$

3. SECOND TYPE OF ARMATURE REACTION

For the second type, the equations in their most general form (1)-(3) have to be used when all harmonics have been taken into account. However, for normally constructed machines, usually a characteristic set of harmonics can be selected which represents the most significant influence of the space harmonics on the electromechanical behavior of the machine.

Simplified models

First-order approximation for the star-connected stator with ungrounded neutral.

For a first-order approximation a set of harmonics has to be selected that fulfills the constraint:

$$\gamma = (\alpha_i + \beta_i)/2 \; (= \alpha_K \text{ or } \alpha_0) \tag{11}$$

where α_i, β_i, α_K or α_0 denotes the angle in the exponential function in the resultant mutual inductance in one element of the L'_{sr}-matrix after symmetrical component transformation.

0	1	2	3	4	5	6	7	8	9	10	11	12	13
	m_{15} m_{27}^*		m_3 m_{39}^*		m_{33} m_9^*		m_{21} m_{21}^*		m_9 m_{33}^*		m_{39} m_3^*		m_{27} m_{15}^*
	m_1		m_{31} m_{-11}		m_{19} m_{-23}		m_7 m_{-35}		m_{37} m_{-5}		m_{25} m_{-17}		m_{13} m_{-29}
	m_{-29}^* m_{13}^*		m_{-17}^* m_{25}^*		m_{-5}^* m_{37}^*		m_{-35}^* m_7^*		m_{-23}^* m_{19}^*		m_{-11}^* m_{31}^*		m_1^*

b) Second type: N = 28, z = 2, N/z = 14

0	1	2	3	4	5	6	7	8	9	10
m_{33} m_{33}^*	m_{45} m_{21}^*	m_9^*	m_3	m_{15} m_{51}^*	m_{27} m_{39}^*	m_{39} m_{27}^*	m_{51} m_{15}^*	m_3^*	m_9	m_{21} m_{45}^*
m_{-11} m_{55}	m_1	m_{13}	m_{25} m_{-41}	m_{37} m_{-29}	m_{49} m_{-17}	m_{-5}	m_7	m_{19} m_{-47}	m_{31} m_{-35}	m_{43} m_{-23}
m_{-11}^* m_{55}^*	m_{-23}^* m_{43}^*	m_{31}^* m_{-35}^*	m_{19}^* m_{-47}^*	m_7^*	m_{-5}^*	m_{49}^* m_{-17}^*	m_{37}^* m_{-29}^*	m_{25}^* m_{-41}^*	m_{13}^*	m_1^*

c) Second type: N = 22, z = 2, N/z = 11

Fig. 2 Distribution of harmonics in L'_{sr}-matrix

To fulfill (11) at the most one harmonic inductance is allowed in one element. For example, in fig.2, where the mutual inductance matrix for N=28, z=2 is given, the following sets can be depicted:

$$\begin{vmatrix} 1 & -11 & 19 & 7 \\ 13 & 25 & -5 & 7 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 31 & 19 & 7 \\ 13 & -17 & -5 & 7 \end{vmatrix}$$

$\gamma = \alpha_K = 7p\theta$ and therefore it holds for the other columns: α_i(second row) + β_i(third row) = $2\gamma = 14p\theta$.

With the selected set the inductances M_i are constant and referring to the star connection with ungrounded neutral the zero-sequence component and the related θ-dependent mutual inductances can be removed from the equations. Most elements in the $\Delta L''_{rr}$-matrix are constant and the few remaining θ-dependent elements will be ignored as they are of minor importance. This results in the following system of equations with constant coefficients.

$$\frac{d}{dt}(\Psi_{sf}) = u_{sf} - R_s i_{sf} - j\frac{d\gamma}{dt}\Psi_{sf} \tag{12}$$

$$\frac{d}{dt}\Psi''_{rk} = -\tilde{R}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) - e_k \Psi''_{rk} - f_k \Psi''^{*}_{rk} \; , \; k=1,2,..,K-1$$

$$\frac{d}{dt}(\Psi''_{rm}) = -R_{rm} i_{rm}\ , \quad m=K \text{ or } 0$$

$$\Psi_{sf} = \tilde{L}_s i_{sf} + \hat{M}_{2m} i_{rm} + \sum_{i=1}^{K-1} \hat{M}_i i''_{ri} \tag{13}$$

$$\Psi''_{rm} = \hat{M}_{2m}(i_{sf}+i^*_{sf}) + (\tilde{L}_{rm} + \Delta_{mm}) i_{rm} + \sum_{i=1}^{K-1}[\Delta_{mi} i''_{ri} + \Delta_{m-i} i''^*_{ri}]$$

$$\Psi''_{rk} = \hat{M}_k i_{sf} + \tilde{L}_{rk}(i''_{rk} + k_k i''^*_{rk}) + \Delta_{km} i_{rm} + \sum_{i=1}^{K-1}[\Delta_{ki} i''_{ri} + \Delta_{k-i} i''^*_{ri}]$$

The complete L-matrix defined by the relation Ψ=LI is a real and constant matrix which has to be inverted only once. The torque equation becomes:

$$T_e = 2.\mathrm{Re}\Bigg[\sum_{i=1}^{K-1}[g_i i''^*_{ri} + h_i^* i''_{ri}]\hat{M}_i i_{sf} - j\frac{\partial\gamma}{\partial\theta}[\hat{M}_{2m} i_{rm} + \sum_{i=1}^{K-1}\hat{M}_i i''^*_{ri}] i_{sf} + \sum_{i=1}^{K-1}[(g_i\Delta_{im} + h_i\Delta_{m-i}) i_{rm} + \sum_{k=1}^{K-1}((g_i\Delta_{ik} + h_i\Delta_{k-i}) i''_{rk} + (g_i\Delta_{i-k} + h_i\Delta_{ki}) i''^*_{rk})\ ]\ i''_{ri}\Bigg] \tag{14}$$

The factors e_i, g_i, f_i and h_i are:

$$e_i = g_i \frac{d\theta}{dt} \quad ; \quad g_i = j\ \left[\frac{\partial\gamma_i}{\partial\theta} - \frac{\frac{\partial\alpha_i}{\partial\theta}\hat{M}^2_{2i} - \frac{\partial\beta_i}{\partial\theta}\hat{M}^2_{3i}}{E_i^2}\right]$$

$$f_i = h_i \frac{d\theta}{dt} \quad ; \quad h_i = -j\left(\frac{\partial\alpha_i}{\partial\theta} - \frac{\partial\beta_i}{\partial\theta}\right)\frac{\hat{M}_{2i}\hat{M}_{3i}}{E_i^2}$$

The symmetrical voltage source (6) can be written as:

$$u_{sf} = \frac{1}{\sqrt{2}}\ \underline{U}_{sf} e^{j(s_\nu \omega t + \phi')} \tag{15}$$

where $s_\nu = 1-\gamma'(1-s)$ and $\gamma = \gamma' p\theta = \gamma'(1-s)\omega t$

When no stator slotting is taken into account $\Delta_{ki} = 0$ and the inversion of the complete L-matrix can be done analytically [2] . When the multiple armature reaction is ignored the same equations as derived for the first type (9) and (10) can be used.

<u>Second-order approximation for the star-connected stator with ungrounded neutral.</u>

In column 1 of the L'_{sr}-matrix more inductances may be necessary to yield a satisfactory result, while this column always contains the fundamental. In the next model two inductances are taken into account in column 1, row 3. For the corresponding terms in the set of the foregoing equations (k=1) the general expressions have to be used now. The influence of this extra inductance, which introduces an θ-dependence in the resultant inductance of column 1, is <u>not</u> considered in the $\Delta L''_{rr}$-matrix.

For k=1 the equations (12) and (13) become:

$$\frac{d}{dt}\Psi''_{r1} = -\tilde{R}_{r1}(i''_{r1} + k_1 i''^*_{r1}) - e_1\Psi''_{r1} - f_1\Psi''^*_{r1} \tag{16}$$

$$\Psi''_{r1} = M^*_1 i_{sf} + \tilde{L}_{r1}(i''_{r1} + k_1 i''^*_{r1}) + \Delta_{1n} i_{rn} + \sum_{i=1}^{K-1}[\Delta_{ki} i''_{ri} + \Delta_{k-i} i''^*_{ri}]$$

For i=1 the terms in the torque equation (14) which contain i_{sf} have to be replaced by:

$$\left[\frac{\partial M^*_1}{\partial\theta}\ i''^*_{r1} + (g_1 i''^*_{r1} + h^*_1 i''_{r1}) M^*_1 - j\ \frac{\partial\gamma}{\partial\theta}\ M^*_1 i''^*_{r1}\right]\ i_{sf} \tag{17}$$

where

$M_1 = \hat{M}_1 e^{j(\gamma-\gamma_1)}$ and e_1, f_1, h_1, k_1, g_1 conform the general expressions as presented in [1]

4. <u>THE STEADY-STATE EQUATIONS</u>

From the general equations for the second type of armature reaction with the first order approximation (12) and (13) the particular solution (for s=constant) can be determined, while the rotor position angle is not present in the parameters but only in the source voltage. When the stator winding is connected to a symmetrical voltage source with the transformed voltage (15) the formal solution of the other variables can be written in the following form:

$$i_{sf} = I_{sf} e^{j(s_\nu\omega t + \phi')} + I^*_{sb} e^{-j(s_\nu\omega t + \phi')}$$

$$i''_{ri} = I^+_{ri} e^{j(s_\nu\omega t + \phi')} + I^{-*}_{ri} e^{-j(s_\nu\omega t + \phi')} \tag{18}$$

$$i_{rm} = I_{rm} e^{j(s_\nu\omega t + \phi')} + I^*_{rm} e^{-j(s_\nu\omega t + \phi')}$$

Substitution of these quantities into (12) and (13) gives the steady-state voltage equations for the second type of armature reaction:

$$\begin{aligned}
U_{sf} &= R_s I_{sf} + j\omega[\ \gamma'(1-s) + s_\nu]\ \Psi_{sf} \\
0 &= R_s I_{sb} - j\omega[\ \gamma'(1-s) - s_\nu]\ \Psi_{sb} \\
0 &= \tilde{R}_{rm} I_{rm} + j\omega\ s_\nu\ \Psi_{rm} \\
0 &= \tilde{R}_{ri}(I^+_{ri} + k_i I^-_{ri}) + (e_i + j\omega s_\nu)\ \Psi^+_{ri} + f_i\ \Psi^-_{ri} \\
0 &= \tilde{R}_{ri}(I^-_{ri} + k_i I^+_{ri}) + (e^*_i + j\omega s_\nu)\ \Psi^-_{ri} + f^*_i\ \Psi^+_{ri}
\end{aligned} \tag{19}$$

where

$$\Psi_{sf} = [\tilde{L}_s I_{sf} + \hat{M}_{2m} I_{rm} + \sum_{i=1}^{K-1}\hat{M}_i\ I^+_{ri}]$$

$$\Psi_{sb} = [\tilde{L}_s I_{sb} + \hat{M}_{2m} I_{rm} + \sum_{i=1}^{K-1}\hat{M}_i\ I^-_{ri}]$$

$$\Psi_{rm} = [\tilde{L}_{rm} I_{rm} + \hat{M}_{2m}(I_{sf} + I_{sb}) + \sum_{i=1}^{K-1}(\Delta_{mi} I^+_{ri} + \Delta_{m-i} I^-_{ri})]$$

$$\Psi^+_{ri} = [\tilde{L}_{ri}(I^+_{ri} + k_i I^-_{ri}) + \hat{M}_i I_{sf} + \Delta_{im} I_{rm} + \sum_{j=1}^{K-1}(\Delta_{ij} I^+_{rj} + \Delta_{i-j} I^-_{rj})]$$

$$\Psi^-_{ri} = [\tilde{L}_{ri}(I^-_{ri} + k_i I^+_{ri}) + \hat{M}_i I_{sb} + \Delta_{im} I_{rm} + \sum_{j=1}^{K-1}(\Delta_{ij} I^-_{rj} + \Delta_{i-j} I^+_{rj})]$$

$$e_i = g_i \frac{\omega}{p}(1-s)\ ;\ f_i = h_i \frac{\omega}{p}(1-s)$$

The electromagnetic torque equation which follows from (14) after substitution of (18) can be split up into a constant and a pulsating part is given in appendix I.

5. <u>BACK TRANSFORMATION OF CURRENTS</u>

To find the currents in the a,b,c-domain the inverse transformations have to be executed.

<u>Back transformation of stator currents</u>

The stator currents were transformed to symmetrical components and by the matrix C [1] respectively using the relations:

$I_s = S_s\ I'_s$ and $I'_s = C\ I''_s$ or contracted: $I_s = S_s\ C\ I''_s$

where

$$S_s\ C = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & e^{j\gamma} & e^{-j\gamma} \\ 1 & a^{-1}e^{j\gamma} & a^{-2}e^{-j\gamma} \\ 1 & a^{-2}e^{j\gamma} & a^{-1}e^{-j\gamma}\end{bmatrix} \quad ; \quad a = e^{j2\pi/3}$$

After these back transformation the original currents for the 3-phase winding are:

$$i_{sn} = \frac{1}{\sqrt{3}}\ [\ i_{s0} + 2\ \mathrm{Re}(\ a^{-(n-1)} i_{sf} e^{j\gamma})]\quad ;\ n = \{^{1,2,3}_{a,b,c}$$

Back transformation of rotor currents

The rotor currents were transformed by the group transformation, to symmetrical components and finally by the time-dependent transformation matrix B [1, 2, 3] respectively which gives the following relations:

$$I_r = (G\, S_r)\, B\, I''_r \text{ or } I_r = G\, I^{\#}_r\,,\; I^{\#}_r = S_r\, I'_r \text{ and } I'_r = B\, I''_r$$

Upon considering the last transformation the relations between the '- and "-quantities are:

$$i'_{ri} = c_i i''_{ri} + d_i i''^{*}_{ri} \quad \text{for } i=1,2,..,K-1$$

For i=0 and i=K the currents and related quantities were not transformed.

The factors c_i and d_i are given in [1]. For the second type of armature reaction and first-order approximation the amplitudes of c_i and d_i are constant. The exponential functions in c_i and d_i introduce the influence of the rotor position angle and the connected frequencies. The next step is the back transformation with help of matrix S_r which gives the following relation:

$$i^{\#}_{ri} = \frac{\sqrt{z}}{\sqrt{N}}\left(i_{r0} + 2.\sum_{k=1}^{K} \mathrm{Re}[\, b^{-k(i-1)} i'_{ri}\,] + b^{-K(i-1)} i_{rK}\right)$$

for i=1,2,..,N/z ; K=N/(2z) and $b = e^{j2\pi p/N}$.

The last back transformation is nothing more than multiplying the currents with a factor $1/\sqrt{z}$, so

$$i_{ri} = 1/\sqrt{z}\; i^{\#}_{ri} \quad \text{where } i=1,2,\ldots\ldots,N/z$$

This procedure gives only the N/z currents which were related to the first group. The other (z-1)N/z currents which are related to those groups that gave, after group transformation, only the trivial solution can be found by the following relation:

$$i_{r(i+k(N/z))} = i_{ri} \quad \text{where } k=1,2,\ldots.(z-1)$$

Upon combining the last two back transformations we find:

$$i_{ri} = \frac{1}{\sqrt{N}}\left(i_{r0} + 2.\sum_{k=1}^{K-1} \{\mathrm{Re}[\, b^{-k(i-1)} i'_{ri}]\} + b^{-K(i-1)} i_{rK}\right) \quad (20)$$

for i=1,2,..,N/z

6. NUMERICAL CALCULATIONS

A general computer program, written in FORTRAN 77, has been developed for use on an IBM personal computer. This program is divided into four sections: PARAM, STAAM, DYNAM and PLOT.

1) PARAM determines the parameters for the reduced set of equations. The machine data concerning the dimensions of the machine, winding layout, slots, air gap, etc. are stored in data files. In the program the values of slot openings, air gap, skew factor, winding pitch chord, saturation factor and terminal voltage can be changed. The type of armature reaction is determined by the program as well as the place of the different harmonics in the L'_{sr}-matrix and displayed on the terminal. From here a choice can be made as to which harmonics are to be considered in the calculation and whether to take the influence of stator slotting into account. For the latter there are three possibilities:
* $\Delta L_{rr} = 0$ (for test) * ΔL_{rr} = constant * $\Delta L_{rr} = f(\theta)$

In the determination of the L''_{sr}- and the ΔL_{rr}-matrices the rotor slotting is always taken into account.

2) STAAM calculates the steady-state characteristics for given s_0 and s_{end}, where s_0 is the start value of the slip and s_{end} the end value in the case of:
* second and first type without armature reaction.
* second type of armature reaction with a first-order approximation with or without the influence of stator slotting (ΔL_{rr} should be constant).

In this program the influence of skin effects in the rotor bars can be taken into account by changing the bar resistance and leakage inductance as a function of the slip.

3) DYNAM calculates the transient state for all options. The rotor position angle θ, the phase angle ϕ and the amplitude of the terminal voltage, and, for instance, the load torque T_{me} and the mechanical damping d_{me} can be set. Finally, it is also possible to calculate the quasi-steady state by setting ds/dt = constant = a very small value. The used integration routines are the fourth-order Runga Kutta method or Hamming's modified predictor-corrector method (also fourth order).

4) PLOT is a plot program which facilitates an analysis of the calculated results. All quantities can be plotted as function of the slip S or as function of the time T (reduced time $T = \omega\, t$). Some options are:
* The electromagnetic torque TE (for the steady state also the pulsating torque TEW). For the torques TE and TEW an analysis can be made of the influence of the different currents.
* The stator currents (ISA, ISB, ISC, as well as the transformed currents ISF and ISB)
* The rotor current i_{ri} as mesh current or bar current or as contribution of a separate transformed rotor circuit to the current in mesh 1.

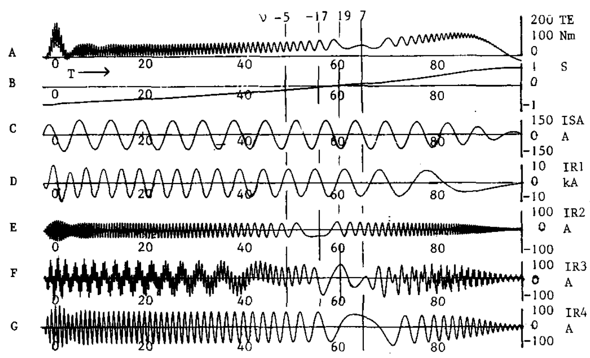


Fig. 3: The reverse rotation test for the 36/28 motor as a function of the time T for ν=1,-5,7,13,-17,19,31. IR1 is mesh current, IR2 .. IR4 are the components in IR1.

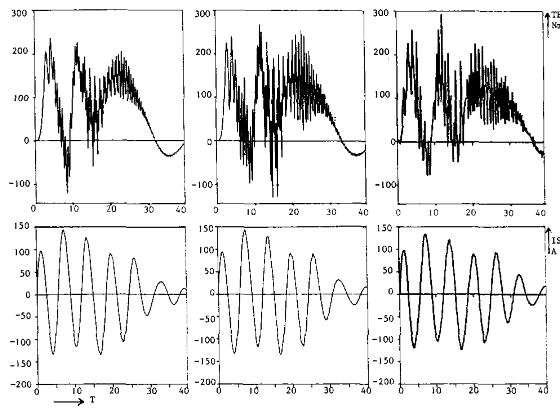
In fig. 3 the characteristics for the reverse rotation test are given for a 7.5 kW squirrel-cage motor. The stator has a chorded 3-phase winding (s/τ_p = 7/9) with 36 slots in the stator and 28 in the rotor. This motor displays the second type of armature reaction and with the following set of selected harmonics:

$$\begin{vmatrix} 1 & 31 & 19 & 7 \\ 13 & -17 & -5 & 7 \end{vmatrix} \quad \text{(See also fig. 2.)}$$

the model with a first-order approximation, (12)-(14), is used. Figs. 3a and b give the torque TE and the number of revolutions 1-S, fig. 3c the stator-phase current ISA and fig. 3d the rotor current in mesh 1 IR1 as time functions in reduced time T.

As IR1, by back transformation, is composed of the addition of the transformed currents, see (20)for i=1, the influence of these separate currents on IR1 can be drawn. In figs. 3e,f,g the currents IR2,IR3 and IR4 represent the contributions of the rotor circuits 2,3,4 to IR1. The values of the slip where the frequency of the separate harmonics becomes equal to zero are s =

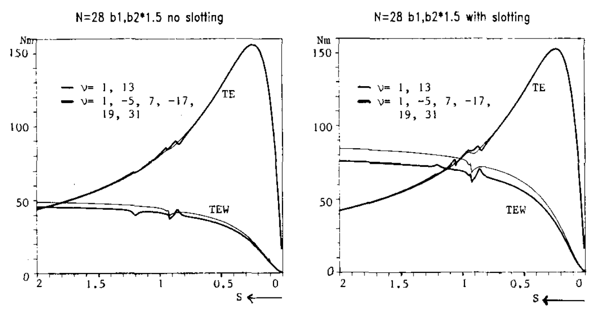
$(\nu-1)/\nu$. The synchronous torque appears when $s_\gamma = s_7 = 0$. To obtain realistic results the influence of $\nu=-29$ has to be investigated, which leads us to a second-order approximation. The result is given in fig. 4a, where the influence of the saturation of the leakage flux due to the currents which appear during the running up will be approximated by an increase in the slot openings of stator and rotor of 1.5 times the geometric one. It can be mentioned that insofar as no stator slotting is considered, the contribution of the column containing the fundamental is sufficient for a good representation of the dynamic curves, and especially the pulsating torques.



a) b) c)

Fig. 4 Starting curves for the 36/28 motor as a function of the time T and enlarged slot openings (1.5 times). a) No stator slotting, $\nu=1,13,-29$. b) stator slotting, $\nu=1,-5,7,13,-17,19,-29,31$. c) measured curves.

The model (16)-(17), fig. 4b, which takes the stator slotting and as many harmonics as possible for this type into account indeed yields the best result when compared with the measurements in fig.4c, but its time consumption on a personal computer is 2.3 hours. However, for this motor the results as given in fig.4a are satisfactory enough, taking into account a calculation time of 11 minutes.

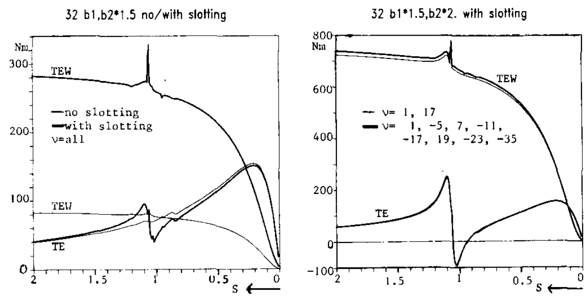


a) b)

Fig. 5 Steady-state curves of the electromagnetic torque for the 36/28 motor (constant part TE, pulsating part TEW). a) No stator slotting. b) with stator slotting.

Upon investigating the presence of asynchronous torques the steady-state characteristics were calculated and are presented in fig. 5. For this machine the parasitic asynchronous torques are of minor importance, even when the stator slotting is taken into account. This contrasts with machines where in the L'_{sr}-matrix the fundamental and one of the stator-slot harmonics belonging to the first pair: $\nu = \pm N_s/p + 1$, appear in one column. This appears in machines where:

$$N = \frac{z}{p}\,\frac{N_s \pm (i2p)}{k} \quad \text{where } i=0,1 \quad \text{and} \quad k=1,2,3,\ldots\ldots$$



a) b)

Fig. 6 Steady-state torque curves for the 36/32 motor as a function of the slip s. a) with/without stator slotting. b) with stator slotting, enlarged rotor slot openings.

These kinds of machines can not normally be used without skewing the rotor bars over one stator-slot pitch.

In fig. 6a the torque-slip curves are given for a rotor with N=32 with effective slot openings of 150% and without and with stator slotting respectively. The harmonics considered in the L'_{sr}-matrix are:

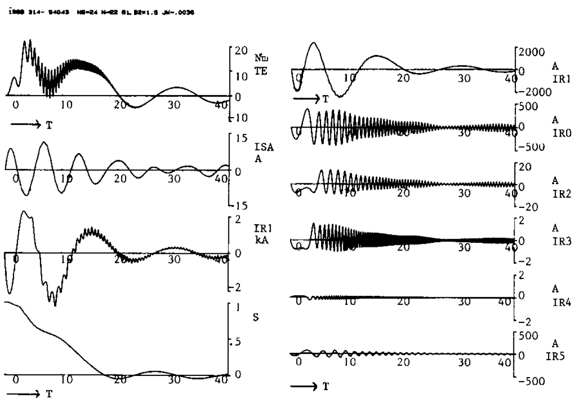
$$\begin{vmatrix} 1 & 19 & -11 & 7 \\ -17 & -35 & -5 & -23 \end{vmatrix}$$

A further increase of the rotor slot opening is demonstrated in fig.6b, as well as the influence of the harmonics. Obviously only one rotor equation is sufficient. The final example for the second type of armature reaction treats the combination $N_s=24$, N=22 for a 0.55kW squirrel-cage motor. This example has been chosen while here a large number of stator-slot harmonics are represented in the L'_{sr}-matrix and even one of the first pair appears in the zero column, see fig. 2, where the stator-slot harmonics are $\nu=-11,13,-23,25$,etc.

In fig. 7 the no-load starting curves are given when no stator slotting is taken into account and the L'_{sr}-matrix contains the following harmonics:

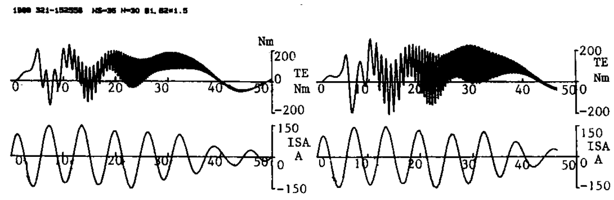
$$\begin{vmatrix} -11 & 1 & 13 & 25 & 37 & -17 \\ -11 & -23 & -35 & & & -5 \end{vmatrix}$$

Upon comparing the rotor current in mesh 1 (IR1 in fig. 7a) with the contributing currents due to circuit 1 ($\nu=1$ and $\nu=-23$) and due to circuit 0 ($\nu=-11$), see fig. 7b, it is clear that $\nu=-11$ has a significant influence on the performance of the motor. The influence of the other rotor circuits (IR2 .. IR5) is of minor importance.The influence of stator slotting is given in fig. 8a. In this case the motor will run with a slip of s= 1.09 because of the big parasitic asynchronous torque due to $\nu=-11$. When, however, the rotor bars are skewed over one rotor-tooth pitch, the influence of $\nu=-11$ is diminished to a small value.



a) b)

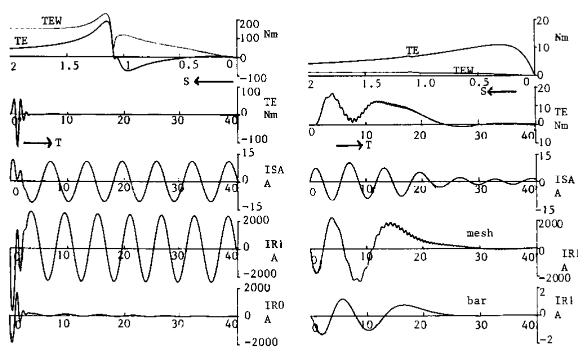
Fig. 7 No-load starting curves of the 24/22 motor as a function of the time T with enlarged slot openings and ν=1,-5,-11,13,-17,-23,25,-35,37.
a) IR1 is the current in mesh 1. b) IR0 -- IR5 are the components of the mesh current IR1.



a) b)

Fig. 9: No-load starting curves for the 36/30 motor. ν=1,-5,7,....31. a) without stator slotting b) stator slotting included.

See fig. 8b where, besides the mesh current IR1, also the bar current IR1 is given, which confirms that ν=-11 gives rise to currents only in the ring of the cage.



a) b)

Fig. 8 24/22 motor. Influence of stator slotting. Harmonics taken into account ν=1,-11,-23. Steady-state and starting curves. a) for enlarged stator- and rotor-slot openings. b) same as a) however skewed rotor bars.

For three-phase induction motors with the first type of armature reaction only one example is given as these motors show great similarity in their performance. This type of armature reaction, which also holds for the slip-ring motor, appears when N/z is a multiple of 3.

In fig. 9a, the no-load starting curves are given for a 7.5 kW. motor with N_s=36 and N=30 and with enlarged slot openings up to 150%. The harmonics which are taken into account in the L'_{sr}-matrix (see fig. 1a) are:

1	-11	7
-29	19	23
31		
13	-5	
-17	25	

Also for this type the result will be the same when only ν=1,-29,31 are taken into account.

When the stator slotting is taken into account the difference between the model including all harmonics and only ν=1,-29,31 is also very small, which confirms that one rotor voltage equation only gives quite good results.

For all motors with the first type of armature reaction it holds that they have only one synchronous torque which appears at s=1,[1].

7. CONCLUSIONS

From the viewpoint of multiple armature reaction, all symmetrical induction machines can be divided into two types. The first type of armature reaction, for which N/z is a multiple of 3, contains neither mutual connections between symmetrical components of stator currents nor connections between the positive and negative sequence rotor currents.

For the second type of armature reaction, when N/z is not divisible by 3 in integer numbers, there are mutual connections between symmetrical components of stator and rotor currents.

Mathematical models of machines with the first type of armature reaction, reviewed after the time-dependent transformation, are much simpler in practical use, in spite of their coefficients being θ-dependent functions, while the set of equations has a very simple flux-current relation and is easy to prepare for computers. In principle there is no restriction in respect to the number of harmonics which can be taken into account.

The mathematical models of machines with the second type of armature reaction are more restricted in their usage because the forward and backward component of the stator currents, as well as the positive and negative sequence components of the rotor currents, are coupled in the equations. However, the proper choice of space harmonics to be taken into account could minimize the costs of computations.

In both cases of armature reaction the introduced mathematical model of induction machines gives a much better insight into the influence of higher space harmonics on dynamics and steady state, while it provides an effective method of computing and analyzing this influence in conjunction with any desirable number of harmonics to be taken into account.

When starting with the transient behavior, it holds that only the transformed rotor circuit which contains the fundamental has to be taken into account for most cases.

For both types it holds that the influence of the stator slotting has always to be taken into account when stator-slot harmonics are present in column 1 of the L'_{sr}-matrix. For the second type it may be necessary to introduce more rotor circuits into the model to include a correct description of the internal rotor couplings.

For the steady-state characteristics (asynchronous torques) the influence of the other rotor circuits has to be examined. These characteristics which can be obtained in a relatively short calculation time give good information about the influence of space harmonics on the performance of the machine with respect to the parasitic asynchronous torques and the pulsating torques. With these results it can be decided whether taking more rotor circuits into account could improve the shape of the starting characteristics.

The model has been prepared to be implemented on an IBM personal computers with plotting facilities. The computation time differs from machine to machine and from model to model. Here a global indication of the computation times is given, related to the number of rotor circuits taken into account, the order of approximation and the stator slotting. The time concerning starting characteristics is normalized on 1000 calculation steps (included screen output).

First type of armature reaction:
no stator slotting: 2 - 5 min (1 - 5 rotor circuits)
stator slotting: 3 - 20 min (1*3 harm. ; 4*2 harm.)
Second type of armature reaction:
1st order,no stator slotting:1 - 3 min (1-6 rotor circ)
2nd order,no stator slotting:2 - 6 min (1-6 rotor circ)
1st order, stator slotting:2 - 8 min (1-6 rotor circ)
2nd order, stator slotting:4 -30 min (1-4 rotor circ)
Steady-state characteristics: approximately 3 minutes for 300 steps and 6 rotor circuits.

With respect to the geometric data of the machine the following remarks can be made:
The influence of the slot openings on the pulsating torques and on the parasitic asynchronous torques due to stator slot harmonics is strongly determined by the Carter factors which appear in the slot-factors of stator and rotor. Whether this influences the parasitic torques significantly depends on the places of the relevant harmonics in the L'_{sr}-matrix. If they are in the same column as the fundamental, or present in the 0-column, their influence has to be considered as significant. These situations should be avoided, but if for other reasons such a situation is chosen, an adequate tool to minimize their effects is the skewing of the rotor bars over one stator-tooth pitch. Other parasitic phenomena, which are beyond the scope of this paper, such as cross-currents in the rotor iron could spoil the results accomplished by this measure.

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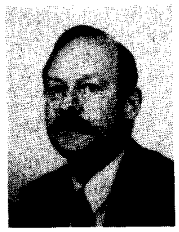
Gerardus Chr. Paap was born in Rotterdam, the Netherlands on February 2, 1946. He received his M.SC. from Delft University of Technology in 1972 and his Ph.D. from the Technical University of Lodz in 1988. From 1973 he has been with the Faculty of Electrical Engineering of the Delft University of Technology.



First, from 1973 to 1985, with the Electrical Machines and Drives Division where he lectured on the fundamentals of electrical machines and from 1985 with the Power Systems Laboratory as Assistant Professor. His research interests include power systems transients and the dynamics of electrical machines.

APPENDIX I

Derivatives in equation (8) are:

$$\frac{\partial\eta}{\partial\theta} = \frac{p}{\hat{M}}\left[\sum_{k=1}^{n}\nu_k\hat{m}_k^2 + \sum_{k=1}^{n}\sum_{i=1\neq k}^{n}\hat{m}_i\hat{m}_k\cos((\nu_k-\nu_i)p\theta)\right]$$

$$\frac{\partial\hat{M}}{\partial\theta} = -\frac{p}{\hat{M}}\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\left[(\nu_i-\nu_k)\hat{m}_i\hat{m}_k\sin((\nu_i-\nu_k)p\theta)\right]$$

$$\frac{\partial\Delta_{ik}}{\partial\theta} = j\,p\,\Delta\tilde{L}_{rh}\,\cdot$$

$$\Big[\,[(N'_s+\alpha'_i-\alpha'_k)e^{j(\eta+\alpha_i-\alpha_k)}-(N'_s+\beta'_i-\beta'_k)e^{-j(\eta+\beta_i-\beta_k)}]\,q_{ik}+$$
$$+[-(N'_s-\alpha'_i+\alpha'_k)e^{-j(\eta-\alpha_i+\alpha_k)}+(N'_s-\beta'_i+\beta'_k)e^{j(\eta-\beta_i+\beta_k)}]\,q_{ki}+$$
$$+[-(N'_s-\alpha'_k+\beta'_i)e^{j(\eta-\alpha_k+\beta_i)}+(N'_s-\alpha'_i+\beta'_k)e^{-j(\eta-\alpha_i+\beta_k)}]\,q_{N/z-k,i}$$
$$+[(N'_s+\alpha'_k-\beta'_i)e^{-j(\eta+\alpha_k-\beta_i)}-(N'_s+\alpha'_i-\beta'_k)e^{j(\eta+\alpha_i-\beta_k)}]\,q_{i,N/z-k}\Big]$$

where $\eta = N_s\theta$, $N'_s = N_s/p$, $\alpha'_i = \partial\alpha_i/\partial p\theta$, $\beta'_i = \partial\beta_i/\partial p\theta$

The steady-state torque equation: a) constant part

$$T_{ec} = 2.\mathrm{Re}\Big[\sum_{i=1}^{K-1}[\hat{M}_i(g_i-jp\gamma')(I^{+*}_{ri}I_{sf}+I^{-}_{ri}I^{*}_{sb}) + h^*_i\hat{M}_i(I^{+}_{ri}I^{*}_{sb}+$$
$$+I^{-*}_{ri}I_{sf})] - jp\gamma'\hat{M}_{2m}(I^{*}_{rm}I_{sf}+I_{rm}I^{*}_{sb}) + \sum_{i=1}^{K-1}[(g_i\Delta_{im}+h_i\Delta_{m-i})$$
$$(I_{rm}I^{+*}_{ri}+I^{*}_{rm}I^{-}_{ri})] + \sum_{i=1}^{K-1}\sum_{j=1}^{K-1}[(g_i\Delta_{ij}+h_i\Delta_{j-i})(I^{+}_{rj}I^{+*}_{ri}+I^{-}_{rj}I^{-*}_{ri})$$
$$+(g_i\Delta_{j-i}+h_i\Delta_{ji})(I^{+*}_{rj}I^{-}_{ri}+I^{-}_{rj}I^{+*}_{ri})]\Big]$$

b) pulsating part:

$$T_e = 2.\mathrm{Real}\Big[\Big[\sum_{i=1}^{K-1}\hat{M}_i[(g_i-jp\gamma')I^{+*}_{ri}I^{*}_{sb}+h^*_iI^{-*}_{ri}I^{*}_{sb}] -$$
$$-jp\gamma'\hat{M}_{2m}I^{*}_{rm}I^{*}_{sb}+\sum_{i=1}^{K-1}((g_i\Delta_{im}+h_i\Delta_{m-i})I^{*}_{rm}I^{+*}_{ri}+\sum_{j=1}^{K-1}[(g_i\Delta_{ij}+$$
$$+h_i\Delta_{j-i})I^{-*}_{rj}I^{+*}_{ri}+(g_i\Delta_{j-i}+h_i\Delta_{ji})I^{+*}_{rj}I^{+*}_{ri}])\Big]e^{-2j\alpha}+$$
$$+\Big[\sum_{i=1}^{K-1}\hat{M}_i[(g_i-jp\gamma')I^{-}_{ri}I_{sf}+h^*_iI^{+}_{ri}I_{sf}] - jp\gamma'\hat{M}_{2m}I_{rm}I_{sf}+$$
$$+\sum_{i=1}^{K-1}((g_i\Delta_{im}+h_i\Delta_{m-i})\underline{I}_{rm}\underline{I}^{-}_{ri}+\sum_{j=1}^{K-1}[(g_i\Delta_{ij}+h_i\Delta_{j-i})\underline{I}^{+}_{rj}\underline{I}^{-}_{ri}+$$
$$+(g_i\Delta_{j-i}+h_i\Delta_{ji})\underline{I}^{-}_{rj}\underline{I}^{-}_{ri}])\Big]e^{2j\alpha}\Big]$$

where $\alpha = s_\nu\omega t + \phi'$

ON THE THEORY OF 3-PHASE SQUIRREL-CAGE INDUCTION MOTORS INCLUDING SPACE HARMONICS AND MUTUAL SLOTTING

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ABSTRACT - In this paper general equations for the asynchronous squirrel-cage motor which contain the influence of space harmonics and the mutual slotting are derived by using among others the power-invariant symmetrical component transformation and a time-dependent transformation with which, under certain circumstances, the rotor-position angle can be removed from the coefficient matrix. The developed models implemented in a machine-independent computer program form powerful tools, with which the influence of space harmonics in relation to the geometric data of specific motors can be analyzed for steady-state and transient performances. Simulations and measurements are presented in a companion paper.

Keywords - asynchronous machines, general theory, space harmonics, mutual slotting, transients.

1. INTRODUCTION

In general the most difficult problems in the theory and modeling of induction motors are saturation of the magnetic circuit and all "parasitic effects" caused by higher harmonics in the magnetic field in the air gap. Both phenomena become even more complicated if the influence of the slotting of stator and rotor surfaces are considered.

Gradually, mathematical models of induction machines have been worked out which take more details of the real geometrical construction into account and enable increasingly closer insight into their influence. At first models were developed which included space harmonics but which ignored those combinations of harmonics which could cause the multiple armature reaction [1,2,3]. These models include only the asynchronous torques and do not include the synchronous and pulsating torques. In [5] the general equations for the squirrel-cage induction motor are derived by means of harmonic analysis. Although these equations describe the dynamic behavior, for numerical calculations, a relatively large computation time is required because of the problem of the rotor position angle. In [6] this model is extended by the addition of the influence of stator slotting. In [7] and [8] dynamic induction motor models have been developed where the squirrel cage and the 3-phase stator winding are represented by equivalent polyphase windings. In the model presented in [7] it is possible to simplify the set of equations when no more than two harmonics per phase group are taken into account, but it is not general for all squirrel-cage motors. In [9] a transformation has been developed which simplifies the general set of equations and under certain circumstances transforms the set of equations in such a way that the ever-present influence of the rotor position angle, when considering the multiple armature reaction, can be removed from the parameters and only appears in the source voltage. This enables a particular solution of the differential equations. In [9], however, a smooth air gap which could have a significant influence on the results was assumed. Further, the zero-sequence component was not included in the equations.

In this paper the model presented in [9] is extended by the addition of the mutual slotting and the zero-sequence component. The influence of saturation due to the main field can, as a linear magnetic circuit is presupposed, be taken into account by making an increase in the air gap. The influence of the saturation due to the slot-leakage fluxes can be dealt with by additional widening of the slot openings.

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2. MATHEMATICAL MODEL

After applying the group transformation to the rotor equations and the symmetrical component transformation to both the stator and the rotor equations, the following set of matrix equations arises [6,10]:

$$U'_s = R'_s\, I'_s + d/dt(\, L'_{ss}\, I'_s + L'_{sr}\, I'_r\,)$$
$$0 = R'_r\, I'_r + d/dt(\, L'_{rs}\, I'_s + L'_{rr}\, I'_r\,) \qquad (1)$$

$$T_e = \frac{1}{2}(\, I'^{*T}_r \frac{\partial L'_{rs}}{\partial\theta} I'_s + I'^{*T}_s \frac{\partial L'_{sr}}{\partial\theta} I'_r + I'^{*T}_r \frac{\partial \Delta L'_{rr}}{\partial\theta} I'_r) \qquad (2)$$

where θ is the rotor-position angle. A list of symbols is provided in section 6.

The elements of the voltage and current matrices are:

$$U'_s = [u_s^0, u_s^+, u_s^-]^T \;;\quad I'_s = [i_s^0, i_s^+, i_s^-]^T$$
$$I'_r = [i_{r0}, i'_{r1}, \ldots, i_{rK}, \ldots, i'^{\,*}_{r1}]^T \text{ ,subscript } K=N/2z$$

where N is the number of rotor slots and z the highest common factor of N and the number of pole pairs p.

The influence of the rotor slotting on the self-inductance of the stator is only taken into account by the Carter factor [10] and therefore no ΔL_{ss}-matrix appears in expression (2). In the rotor self-inductance and the mutual inductance the slotting is taken into account completely.

The transformed stator parameters are:

$$R'_s = R_s = \text{diag}[R_s, R_s, R_s] \;;\; L'_{ss} = \text{diag}[\tilde{L}_{s0}, \tilde{L}_s, \tilde{L}_s]$$

where the zero-sequence inductance $\tilde{L}_{s0}$ contains the harmonics ν=3,9,15,.... and the positive sequence inductance $\tilde{L}_s$ contains the harmonics ν=1,-5,7,-11,...

For the rotor the transformed parameters are:

$$R'_r = \text{diag}[\tilde{R}_{r0}, \tilde{R}_{r1}, \ldots, \tilde{R}_{rK}, \ldots \tilde{R}_{r1}],$$
$$L'_{rr} = \text{diag}[\tilde{L}_{r0}, \tilde{L}_{r1}, \ldots, \tilde{L}_{rK}, \ldots, \tilde{L}_{r1}] + \Delta L'_{rr}$$

where $\Delta L'_{rr} = \sum_n \Delta\tilde{L}_{rn}(e^{jnN\theta} Q + e^{-jnN\theta} Q^{*T})$; n=1,2,3,...

After these transformations the complex mutual inductance matrix becomes as given in equation (3) where g=0,±1,±2,.. ν = 1,-5,7,-11,13,.. μ'=±μ , μ=3,9,15,... In each element of (3) only those harmonics appear that fulfill all the constraints that are given in this matrix. In [6,10] a comprehensive determination of the

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separate parameters is provided. Appendix I gives a short overview of the resultant parameters.

$$L'_{sr}=\begin{bmatrix} \sum\limits_{\mu'=gN/z} m_{\mu'} & \sum\limits_{\mu'=gN/z+1} m_{\mu'} & \cdots & \sum\limits_{\mu'=gN/z+(N/z-1)} m_{\mu'} \\ \sum\limits_{\nu=gN/z} m_{\nu} & \sum\limits_{\nu=gN/z+1} m_{\nu} & \cdots & \sum\limits_{\nu=[gN/z+(N/z-1)]} m_{\nu} \\ \sum\limits_{\nu=-gN/z} m^{*}_{\nu} & \sum\limits_{\nu=-(gN/z+1)} m^{*}_{\nu} & \cdots & \sum\limits_{\nu=-[gN/z+(N/z-1)]} m^{*}_{\nu} \end{bmatrix} \quad (3)$$

<u>The structure of the L'_{sr}-matrix</u>

The behavior of the squirrel-cage induction motor is strongly determined by the magnetic coupling between stator and rotor [9]. The presence of a harmonic in an element of the mutual inductance matrix is directly connected to the number of rotor bars N and by z. On the grounds of the present symmetry it is possible to write the L'_{sr}-matrix as follows:

$$L'_{sr}=\begin{bmatrix} M_{10} & M_{11}\cdots M_{1(K-1)} & M_{1K} & M^{*}_{1(K-1)}\cdots & M^{*}_{11} \\ M_{20} & M_{21}\cdots M_{2(K-1)} & M_{2K} & M_{3(K-1)}\cdots & M_{31} \\ M^{*}_{20} & M^{*}_{31}\cdots M^{*}_{3(K-1)} & M^{*}_{2K} & M^{*}_{2(K-1)}\cdots & M^{*}_{21} \end{bmatrix} \quad (4)$$

where

$M_{1i}=2\hat{M}_{1i}\cos\epsilon_i$ (i=0 or K) ; $M_{1i}=\hat{M}_{1i}e^{j\epsilon_i}$ (i=1,....K-1)

$M_{2i}=\hat{M}_{2i}e^{j\alpha_i}$ (i=0,...K) ; $M_{3i}=\hat{M}_{3i}e^{j\beta_i}$ (i=1,....K-1)

and in general $\hat{M}_{ki}e^{j\eta}=\sum\limits_{\nu} m_{\nu}=\sum\limits_{\nu}^{n}\hat{m}_{\nu}e^{j\nu p\theta}$,k=1,2,3 (5)

<u>Time-dependent transformations</u>

The rotor and stator variables are transformed with complex, power-invariant, time-dependent transformations, as presented in [9]. These transformations change the L'_{sr}-matrix in such a way that the mutual connections between the positive and negative sequence components of the stator currents disappear. The transformation matrices for the rotor voltage and current are:

$$A=\begin{bmatrix} 1 & . & . & . & . & 0 \\ . & a_1 & . & . & . & b_1 \\ . & . & a_2 & . & b_2 & . \\ . & . & . & 1 & . & . \\ . & . & b_2^{*} & . & a_2^{*} & . \\ 0 & b_1^{*} & . & . & . & a_1^{*} \end{bmatrix};\quad B=\begin{bmatrix} 1 & . & . & . & . & 0 \\ . & c_1 & . & . & . & d_1 \\ . & . & c_2 & . & d_2 & . \\ . & . & . & 1 & . & . \\ . & . & d_2^{*} & . & c_2^{*} & . \\ 0 & d_1^{*} & . & . & . & c_1^{*} \end{bmatrix}$$

which satisfy the condition: $A^{T}.B^{*}=U$ or $A^{-1}=B^{*T}$,where U is the unity matrix, and where

$a_i=D_i\hat{M}_{2i}/E_i^2\ e^{j(\gamma_i-\alpha_i)}$ $\quad b_i=D_i\hat{M}_{3i}/E_i^2\ e^{-j(\gamma_i-\beta_i)}$

$c_i=\hat{M}_{2i}/D_i\ e^{j(\gamma_i-\alpha_i)}$ $\quad d_i=-\hat{M}_{3i}/D_i\ e^{-j(\gamma_i-\beta_i)}$

$D_i=\sqrt{(\hat{M}_{2i}^2+\hat{M}_{3i}^2)}$ $\quad E_i=\sqrt{(\hat{M}_{2i}^2-\hat{M}_{3i}^2)}$ if $(\hat{M}_{2i}\neq\hat{M}_{3i})$

for i = 1,2,..,K-1.

If $\hat{M}_{2i}=\hat{M}_{3i}=0$ then $a_i=c_i=1,\ b_i=d_i=0$.

The relations between the old and the new variables are given by

$$U'_r=A.U''_r\ ;\ I'_r=B.I''_r \quad (6)$$

Because of the structure of the transformation matrix B the transformed currents can be written as follows:

$$I''_r=[\ i_{r0},i''_{r1},i''_{r2},\ldots,i_{rK},\ldots,i''^{*}_{r2},i''^{*}_{r1}]$$

The equations are further developed by transforming the stator variables, using the following relations between the old and new quantities:

$$U'_s=C\ U''_s \text{ and } I'_s=C\ I''_s \text{ where } C=\text{diag}[\ 1,\ e^{j\gamma},\ e^{-j\gamma}] \quad (7)$$

The new stator variables are defined as:

$$U''_s=[\ u_{s0},u_{sf},u_{sb}]^{T} \text{ and } I''_s=[\ i_{s0},i_{sf},i_{sb}]^{T}$$

where $u_{sb}=u^{*}_{sf}$ and $i_{sb}=i^{*}_{sf}$.

The introduced angles γ, and $\gamma_1,\ldots\ldots,\gamma_{K-1}$ will later be determined in relation to specific motor data.

The transformation of equations (1) and (2) using relations (6) and (7) yields:

$$U''_s=R''_sI''_s+C^{-1}\frac{dC}{dt}(L''_{ss}I''_s+L''_{sr}I''_r)+\frac{d}{dt}(\ L''_{ss}I''_s+L''_{sr}I''_r) \quad (8)$$

$$0=R''_rI''_r+A^{-1}\frac{dA}{dt}(L''_{rs}I''_s+L''_{rr}I''_r)+\frac{d}{dt}(\ L''_{rs}I''_s+L''_{rr}I''_r\ )$$

$$T_e=\text{Re}[I''^{*T}_r\frac{\partial L''_{rs}}{\partial\theta}I''_s+I''^{*T}_rA^{-1}\frac{\partial A}{\partial\theta}L''_{rs}I''_s+I''^{*T}_rL''_{rs}\frac{\partial C^{-1}}{\partial\theta}CI''_s$$
$$+I''^{*T}_rA^{-1}\frac{\partial A}{\partial\theta}\Delta L''_{rr}I''_r+\frac{1}{2}I''^{*T}_r\frac{\partial\Delta L''_{rr}}{\partial\theta}I''_r] \quad (9)$$

where

$L''_{ss}=C^{-1}L'_{ss}C=L'_{ss}$, $R''_s=C^{-1}R'_sC=R'_s$, $L''_{sr}=C^{-1}L'_{sr}B=L''^{*T}_{rs}$

and $L''_{rr}=A\ L'_{rr}B$ and $R''_{rr}=A\ R'_{rr}B$.

In the derivation of the electromagnetic torque equation the fact that $\Delta L''_{rr}=\Delta L''^{*T}_{rr}$ was used.

The mutual inductance matrix becomes:

$$L''_{sr}=\begin{bmatrix} M_{10} & M_{01}\cdots M_{0(K-1)} & M_{1K} & M^{*}_{0(K-1)}\cdots & M^{*}_{01} \\ M_{20} & M_{1}\ \cdots M_{(K-1)} & M_{2K} & 0\ \cdots & 0 \\ M^{*}_{20} & 0\ \cdots 0 & M^{*}_{2K} & M^{*}_{(K-1)}\cdots & M^{*}_{1} \end{bmatrix} \quad (10)$$

where $M_{1i}=2\hat{M}_{1i}\cos\epsilon_i$ and $M_{2i}=\hat{M}_{2i}e^{j(\alpha_i-\gamma)}$ for i=0 or K

$M_i=\hat{M}_ie^{j(\gamma_i-\gamma)}$ for i=1,....K-1

$\hat{M}_i=E^2/D_i$ (if $\hat{M}_{2i}=\hat{M}_{3i}=0$ then $\hat{M}_i=0$).

$M_{0i}=\frac{1}{D_i}[M_{1i}\hat{M}_{2i}e^{j(\gamma_i-\alpha_i+\epsilon_i)}-\hat{M}_{1i}\hat{M}_{3i}e^{j(\gamma_i-\beta_i-\epsilon_i)}]$

for i=1,....K-1

if $\hat{M}_{2i}=\hat{M}_{3i}=0$, $M_{0i}=\hat{M}_{1i}e^{j\epsilon_i}$

In the second and third row, the mutual inductance matrix now contains inductances which consist of a real part $\hat{M}$ and a complex exponential function. $\hat{M}$ depends, in general, on the rotor position angle θ . The angles γ_i and γ in the complex functions are still to be determined. The free angles γ_i also appear in the zero sequence components in the first row.

The new rotor matrices become:

$$L''_{rr} = \begin{bmatrix} \tilde{L}_{r0} & \cdot & & \cdot & 0 \\ \cdot & \tilde{L}_{r1} & & \cdot & k_i\tilde{L}_{r1} \\ & \cdot & \cdot & \cdot & \\ & & \ddots & & \\ \cdot & \cdots & \tilde{L}_{rK} & \cdots & \cdot \\ & & & \ddots & \\ 0 & k_i^*\tilde{L}_{r1} & & & \tilde{L}_{r1} \end{bmatrix} + \Delta L''_{rr} ; \quad R''_r = \begin{bmatrix} \tilde{R}_{r0} & \cdot & & \cdot & 0 \\ \cdot & \tilde{R}_{r1} & & \cdot & k_i\tilde{R}_{r1} \\ & \cdot & \cdot & \cdot & \\ & & \ddots & & \\ \cdot & \cdots & \tilde{R}_{rK} & \cdots & \cdot \\ & & & \ddots & \\ 0 & k_i^*\tilde{R}_{r1} & & & \tilde{R}_{r1} \end{bmatrix} \quad (11)$$

where

$$k_i = \frac{-2\hat{M}_{2i}\hat{M}_{3i}}{D_i^2} e^{j(\alpha_i+\beta_i-2\gamma_i)}, \text{ if } \hat{M}_{2i} = \hat{M}_{3i} = 0\text{: } k_i = 0 \quad (12)$$

Matrix $\Delta L''_{rr}$ can be written as: $\Delta L''_{rr}(i,k) = \sum_n \Delta\tilde{L}_{rn}\Delta_n(i,k)$

where

$$\Delta_n(i,k) = (c_k c_i^* e^{j\kappa} + d_k^* d_i e^{-j\kappa})q_{ik} + (c_k c_i^* e^{-j\kappa} + d_k^* d_i e^{j\kappa})q_{ki} + (c_k d_i e^{j\kappa} + d_k^* c_i^* e^{-j\kappa})q_{N/z-k,i} + (c_k d_i e^{-j\kappa} + d_k^* c_i^* e^{j\kappa})q_{i,N/z-k}$$

$\kappa = nN_s\theta$.

For $\Delta L''_{rr}$ it holds that:

$$\Delta L''_{rr}(i,k) = \Delta L''^{*}_{rr}(k,i) \text{ and } \Delta L''_{rr}(i,k) = \Delta L''_{rr}(N/z-k, N/z-i)$$

The factors $A^{-1}\frac{dA}{dt}$ and $A^{-1}\frac{\partial A}{\partial\theta}$ in equations (8) and (9) can be written as:

$$A^{-1}\frac{dA}{dt} = \begin{bmatrix} 0 & \cdot & \cdot & & \cdot & \cdot & 0 \\ \cdot & e_1 & \cdot & & \cdot & \cdot & f_1 \\ \cdot & \cdot & e_2 & & \cdot & f_2 & \cdot \\ & & & \ddots & & & \\ \cdot & \cdot & \cdot & 0 & \cdots & & \cdot \\ & & & & \ddots & & \\ \cdot & \cdot & f_2^* & & \cdot & e_2^* & \cdot \\ 0 & f_1^* & \cdot & & \cdot & \cdot & e_1^* \end{bmatrix} ; \quad A^{-1}\frac{\partial A}{\partial\theta} = \begin{bmatrix} 0 & \cdot & \cdot & & \cdot & \cdot & 0 \\ \cdot & g_1 & \cdot & & \cdot & \cdot & h_1 \\ \cdot & \cdot & g_2 & & \cdot & h_2 & \cdot \\ & & & \ddots & & & \\ \cdot & \cdot & \cdot & 0 & \cdots & & \cdot \\ & & & & \ddots & & \\ \cdot & \cdot & h_2^* & & \cdot & g_2^* & \cdot \\ 0 & h_1^* & \cdot & & \cdot & \cdot & g_1^* \end{bmatrix}$$

where

$$g_i = -j\left[\frac{\partial\gamma_i}{\partial\theta} - \frac{\frac{\partial\alpha_i}{\partial\theta}\hat{M}_{2i}^2 - \frac{\partial\beta_i}{\partial\theta}\hat{M}_{3i}^2}{E_i^2}\right] + \left[\frac{1}{D_i}\frac{\partial D_i}{\partial\theta} - \frac{1}{E_i}\frac{\partial E_i}{\partial\theta}\right],$$

$$e_i = -j\left[\frac{d\gamma_i}{dt} - \frac{\frac{d\alpha_i}{dt}\hat{M}_{2i}^2 - \frac{d\beta_i}{dt}\hat{M}_{3i}^2}{E_i^2}\right] + \left[\frac{1}{D_i}\frac{dD_i}{dt} - \frac{1}{E_i}\frac{dE_i}{dt}\right],$$

$$(13)$$

$$h_i = \left[-j\left(\frac{\partial\alpha_i}{\partial\theta} - \frac{\partial\beta_i}{\partial\theta}\right)\frac{\hat{M}_{2i}\hat{M}_{3i}}{E_i^2} + \frac{1}{E_i^2}\left(\hat{M}_{2i}\frac{\partial\hat{M}_{3i}}{\partial\theta} - \hat{M}_{3i}\frac{\partial\hat{M}_{2i}}{\partial\theta}\right)\right]e^{j\theta_i}$$

$$f_i = \left[-j\left(\frac{d\alpha_i}{dt} - \frac{d\beta_i}{dt}\right)\frac{\hat{M}_{2i}\hat{M}_{3i}}{E_i^2} + \frac{1}{E_i^2}\left(\hat{M}_{2i}\frac{d\hat{M}_{3i}}{dt} - \hat{M}_{3i}\frac{d\hat{M}_{2i}}{dt}\right)\right]e^{j\theta_i}$$

$\theta_i = \alpha_i + \beta_i - 2\gamma_i$.

If $\hat{M}_{2i} = \hat{M}_{3i} = 0$ then $e_i = f_i = g_i = h_i = 0$.

The matrices $C^{-1}\frac{dC}{dt}$ and $\frac{\partial C}{\partial\theta}^{-1} C$ are:

$$C^{-1}\frac{dC}{dt} = -j\,\frac{d\gamma}{dt}\text{diag}[\,0,1,-1] \text{ and } \frac{\partial C}{\partial\theta}^{-1} C = -j\,\frac{\partial\gamma}{\partial\theta}\,\text{diag}[\,0,1,-1]$$

As follows from equations (10) and (11), mutual connections between positive and negative sequence stator currents are replaced by additional mutual connections between positive and negative sequence rotor currents.

From the general equations in matrix form it appears that in general:

L''_{sr}, L''_{rr}, $\Delta L''_{rr}$ and R''_{rr} depend on θ.

L''_{rr} and R''_{rr} are functions of $\hat{M}_{2i}.\hat{M}_{3i}.e^{j(\alpha_i+\beta_i-2\gamma_i)}$.

$A^{-1}dA/dt$ depends on $\hat{M}_{2i}.\hat{M}_{3i}.e^{j(\alpha_i+\beta_i-2\gamma_i)}$, $d\gamma_i/dt$, $d\alpha_i/dt$ and $d\beta_i/dt$.

$A^{-1}\partial A/\partial\theta$ depends on $\hat{M}_{2i}.\hat{M}_{3i}.e^{j(\alpha_i+\beta_i-2\gamma_i)}$, $\partial\gamma_i/\partial\theta$, $\partial\alpha_i/\partial r$ and $\partial\beta_i/\partial\theta$.

$\Delta L''_{rr}$ contains exponential functions with $N_s\theta$, α_i, α_k, β_i, β_k, γ_i and γ_k.

$C^{-1}dC/dt$ and $C^{-1}\partial C/\partial\theta$ depend on $d\gamma/dt$ and $\partial\gamma/\partial\theta$ respectively.

In order to see the advantages of these transformations the structures of the mutual inductance matrix and the rotor self-inductance matrix have to be examined in order to develop simpler equations. In [9] it is shown that 3-phase induction motors can be divided into two types, each with a specific kind of multiple armature reaction. Both types can be recognized by the structure of their mutual-inductance matrix. This also holds when the mutual slotting is taken into account. In Table 1. representatives of these two types are given.

3. THE FIRST TYPE OF ARMATURE REACTION

The first type of armature reaction occurs in 3-phase induction motors where $N/z = k.3$ and this can be seen in the structure of the L'_{sr}-matrix.

An example of this type is given in Table 1.a from which the following properties can be derived:

either $M_{2i}=0$ or $M_{3i}=0$ or $M_{2i}=M_{3i}=0$ and $M_{20}=M_{2K}=0$.

From this it follows that $A = B$ and that A itself becomes unitary. The transformation is then only an angle transformation and does not influence the amplitudes. The elements in the matrices containing $M_{2i}M_{3i}$ disappear. Further, it is easy to see in equations (12) and (13) that for this type $k_i = f_i = h_i = 0$. This means that all exponential terms in the general equations (8) and (9) disappear, except in the mutual inductances M_i in expression (10). The exponential functions in the second and third row of the mutual inductance matrix can now be removed by introducing the following constraint:

$$\gamma_i = \gamma.$$

column number

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
m_{15} m^*_{15}			m_3			m^*_9			m_9			m^*_3		
	m_1 m_{31} m_{-29}			m_{19} m_{-11}			m_7 m_{-23}			m_{25} m_{-5}			m_{13} m_{-17}	
		m^*_{-17} m^*_{13}			m^*_{-5} m^*_{25}			m^*_{-23} m^*_7			m^*_{-11} m^*_{19}			m^*_{-29} m^*_1 m^*_{31}

a) First type: $N = 30$, $z = 2$, $N/z = 15$

0	1	2	3	4	5	6	7	8	9	10	11	12	13
	m_{15} m^*_{27}		m_3 m^*_{39}		m_{33} m^*_9		m_{21} m^*_{21}		m_9 m^*_{33}		m_{39} m^*_3		m_{27} m^*_{15}
	m_1		m_{31} m_{-11}		m_{19} m_{-23}		m_7 m_{-35}		m_{37} m_{-5}		m_{25} m_{-17}		m_{13} m_{-29}
	m^*_{-29} m^*_{13}		m^*_{-17} m^*_{25}		m^*_{-5} m^*_{37}		m^*_{-35} m^*_7		m^*_{-23} m^*_{19}		m^*_{-11} m^*_{31}		m^*_1

b) Second type: $N = 28$, $z = 2$, $N/z = 14$

0	1	2	3	4	5	6	7	8	9	10
m_{33} m^*_{33}	m_{45} m^*_{21}	m^*_9	m_3	m_{15} m^*_{51}	m_{27} m^*_{39}	m_{39} m^*_{27}	m_{51} m^*_{15}	m^*_3	m_9	m_{21} m^*_{45}
m_{-11} m_{55}	m_1	m_{13}	m_{25} m_{-41}	m_{37} m_{-29}	m_{49} m_{-17}	m_{-5}	m_7	m_{19} m_{-47}	m_{31} m_{-35}	m_{43} m_{-23}
m^*_{-11} m^*_{55}	m^*_{-23} m^*_{43}	m^*_{31} m^*_{-35}	m^*_{19} m^*_{-47}	m^*_7	m^*_{-5}	m^*_{49} m^*_{-17}	m^*_{37} m^*_{-29}	m^*_{25} m^*_{-41}	m^*_{13}	m^*_1

c) Second type: $N = 22$, $z = 2$, $N/z = 11$

Table 1. Distribution of harmonics in the L'_{sr}-matrix.

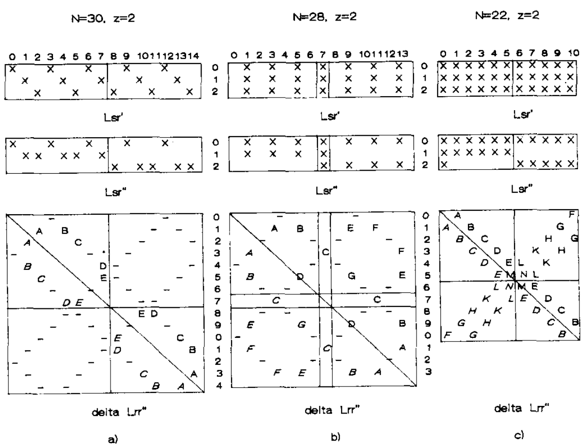

Table 2. The structure of L'_{sr}, L''_{sr} and $\Delta L''_{rr}$ for both types of armature reaction. The italic symbols mean complex conjugate values.

From Table 2.a, which gives a general impression of the structure of the L'_{sr}-, L''_{sr}- and $\Delta L''_{rr}$-matrices for the first type of armature reaction, it can be concluded that for this type the sequence components of the stator are linked neither by the L''_{sr}- nor by the $\Delta L''_{rr}$-matrix. This means that the zero-sequence component does not have any influence on the electro-mechanical behavior of the motor and therefore yields the conclusion:

> Motors with the first type of armature reaction do not display any difference in electro-mechanical behavior for a star-connected or delta-connected stator winding.

As the positive-sequence and negative-sequence rotor currents are not linked, the final equations only contain the forward-sequence component of the stator variables and the positive-sequence components of the rotor variables. This leads to the following voltage equations:

$$u_{sf} = R_s i_{sf} + j\frac{d\gamma}{dt}\Psi_{sf} + \frac{d}{dt}\Psi_{sf}$$

$$0 = \tilde R_{rk} i''_{rk} + e_k \Psi''_{rk} + \frac{\tilde d}{dt}\Psi''_{rk} \qquad ,k = 1,2,\dots K-1$$

the flux-current relations:

$$\Psi_{sf} = \tilde L_s i_{sf} + \sum_{i=1}^{K-1} \hat M_i i''_{ri}$$

$$\Psi''_{rk} = \hat M_k i_{sf} + \tilde L_{rk} i''_{rk} + \sum_{i=1}^{K-1} \Delta L''_{rr}(k,i) i''_{ri}$$

and the electromagnetic torque equation:

$$T_e = -2\,\mathrm{Re}\left[\sum_{i=1}^{K-1}\left[i''^*_{ri}\left(\frac{\partial \hat M_i}{\partial\theta} + g_i \hat M_i\right) i_{sf}\right] + \sum_{i=1}^{K-1}\sum_{k=1}^{K-1}\left[\frac{1}{2}\frac{\partial \Delta L''_{rr}(i,k)}{\partial\theta} + g_i \Delta L''_{rr}(i,k)\right] i''_{rk} i''^*_{ri}\right]$$

where

$\hat M_i = \hat M_{2i}$, $e_i = j(\omega - d\alpha_i/dt)$, $g_i = -j\,\partial\alpha_i/\partial\theta$ when $\hat M_{3i} = 0$

$\hat M_i = -\hat M_{3i}$, $e_i = j(\omega - d\beta_i/dt)$, $g_i = -j\,\partial\beta_i/\partial\theta$ when $\hat M_{2i} = 0$

For this type it holds that no equation for a 0-column or a middle-column K, as far as it exists, can be present. As the maximum number of voltage equations can be counted:
For the stator: 1
For the rotor : K-1, where $K = N/2z$ when N/z is even
$K = N/2z + 1/2$ when N/z is odd.

A further reduction in the number of equations is achieved when the mutual inductance matrix contains columns which are not filled with inductances. These potential equations yield trivial solutions and can therefore be omitted. If the motor is connected to a symmetrical power source then the transformed voltage becomes

for the star connection: $u_{sf} = \frac{1}{\sqrt{2}} U_N e^{j(\omega t + \phi' - \gamma)}$

and

for the delta connection: $u_{sf} = \frac{1}{2}\sqrt{6}\, U_N e^{j(\omega t + \phi - \gamma)}$

where $\phi' = \phi - \pi/6$ and U_N is the terminal voltage (rms).
Upon taking the remaining free angle

$$\gamma = \omega t + \phi' \qquad (\text{or } \omega t + \phi) ,$$

$\frac{d\gamma}{dt} = \omega$ and u_{sf} becomes a DC voltage.

This possibility to transform u_{sf} into a DC voltage here enables the formulation of the conclusions [9]:

* The starting characteristics of this type of motor do not depend on the moment of switching, but only on the rotor position angle at s=1 as $\hat{M}_i$ and $\Delta L''_{rr}(i,k)$ still remain functions of θ.

* This type has only one synchronous torque which appears at s=1. When the rotor moves, only pulsating torques exist.

4. THE SECOND TYPE OF ARMATURE REACTION

Three-phase squirrel-cage motors for which N/z is not a multiple of three display the multiple-armature reaction of the second type. This type possesses the following characteristic geometrical properties (see figures 1.b and c, 2.b and c):
In the mutual inductance matrix both M_{2i} and M_{3i} can have values not equal to zero. Further, M_{20} and M_{2K} never occur together. From Table 2.b and c, which gives a general impression of the structures of the L'_{sr}-, L''_{sr}- and $\Delta L''_{rr}$-matrices for the second type of armature reaction, it can be concluded that for this type the sequence components of the stator and the rotor are linked by the L''_{sr}- and by the $\Delta L''_{rr}$-matrix respectively.
Therefore, for the second type, the equations in their most general form have to be used when all harmonics have to be taken into account. These equations (8) and (9) can be written as:

$$u_{s0} = R_s i_{s0} + \frac{d}{dt}(\psi_{s0})$$

$$u_{sf} = R_s i_{sf} + \frac{d}{dt}(\psi_{sf}) + j\frac{d\gamma}{dt}\psi_{sf}$$

$$0 = R_{rm} i_{rm} + \frac{d}{dt}(\psi''_{rm}) \quad , \quad m = 0 \text{ or } K$$

$$0 = \tilde{R}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + e_k \psi''_{rk} + f_k \psi''^{*}_{rk} + \frac{d}{dt}(\psi''_{rk}), \quad k=1,2,\ldots,K-1$$

with the flux-current relations:

$$\psi_{s0} = \tilde{L}_{s0} i_{s0} + M_{1m} i_{rm} + \sum_{i=1}^{K-1}[M_{0i} i''_{ri} + M^{*}_{0i} i''^{*}_{ri}]$$

$$\psi_{sf} = \tilde{L}_s i_{sf} + M_{2m} i_{rm} + \sum_{i=1}^{K-1} M_i i''_{ri}$$

$$\psi''_{rm} = M_{1m} i_{s0} + M^{*}_{2m} i_{sf} + M_{2m} i^{*}_{sf} + (\tilde{L}_{rm} + \Delta_{mm}) i_{rm} + \sum_{i=1}^{K-1}[\Delta_{mi} i''_{ri} + \Delta_{m-i} i''^{*}_{ri}]$$

$$\psi''_{rk} = M^{*}_{0k} i_{s0} + M^{*}_{k} i_{sf} + \tilde{L}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + \Delta_{km} i_{rm} + \sum_{i=1}^{K-1}[\Delta_{ki} i''_{ri} + \Delta_{k-i} i''^{*}_{ri}]$$

and the electromagnetic torque equation:

$$T_e(t) = \mathrm{Re}\Bigg[\Big[2.\sum_{i=1}^{K-1}[(\frac{\partial M^{*}_{0i}}{\partial\theta} + g_i M^{*}_{0i} + h_i M_{0i}) i''^{*}_{ri}] + i_{rm}\frac{\partial M_{1m}}{\partial\theta}\Big] i_{s0} +$$
$$+2\Big[\sum_{i=1}^{K-1}([\frac{\partial M^{*}_{i}}{\partial\theta} + (g_i - j\frac{\partial\gamma}{\partial\theta}) M^{*}_{i}] i''^{*}_{ri} + h^{*}_{i} M^{*}_{i} i''_{ri}) + i_{rm}(\frac{\partial M^{*}_{2m}}{\partial\theta} - j\frac{\partial\gamma}{\partial\theta} M^{*}_{2m})\Big] i_{sf}$$
$$+\sum_{k=1}^{K-1}\Big[(\frac{\partial\Delta_{mk}}{\partial\theta} i''_{rk} + \frac{\partial\Delta_{m-k}}{\partial\theta} i''^{*}_{rk}) i^{*}_{rm} + \sum_{i=1}^{K-1}(\frac{\partial\Delta_{ik}}{\partial\theta} i''_{rk} + \frac{\partial\Delta_{i-k}}{\partial\theta} i''^{*}_{rk}) i''^{*}_{ri}\Big]$$
$$+2\sum_{i=1}^{K-1}\Big[(g_i\Delta_{im} + h_i\Delta^{*}_{m-i}) i_{rm} + \sum_{k=1}^{K-1}[(g_i\Delta_{ik} + h_i\Delta^{*}_{k-i}) i''_{rk} +$$
$$+ (g_i\Delta_{i-k} + h_i\Delta^{*}_{ik}) i''^{*}_{rk}]\Big] i''^{*}_{ri}\Bigg]$$

In these equations the following short notations are used:

$$\Delta_{ki} = \Delta L''_{rr}(k,i) \quad ; \quad \Delta_{k-i} = \Delta L''_{rr}(k, N/z-i)$$

However for normally constructed machines, usually a characteristic set of harmonics can be selected which represents the most significant influence on the behavior of the machine.
When in every column in the mutual inductance matrix this set of harmonics fulfills the following constraint:

$$\gamma = \gamma_i = [\alpha_i + \beta_i]/2 \quad (= \alpha_K \text{ or } \alpha_0) \qquad (14)$$

the exponential functions in the second and third row of the L'_{sr}-matrix, as well as in the factors k_i, h_i and f_i disappear. Constraint (14) can only be fulfilled if every element of the L'_{sr}-matrix contains at the most only one inductance.
The inductances in the first row remain functions of θ, however, in consequence of the constraint (14) the exponential functions in M_{0i} in expression (10) become each other's complex conjugate:

$$M_{0i} = \frac{1}{D_i}\hat{M}_{1i}(\hat{M}_{2i} e^{j\epsilon} - \hat{M}_{3i} e^{-j\epsilon}) \text{ where } \epsilon = \gamma_i - \alpha_i + \epsilon_i = -\gamma_i + \beta_i + \epsilon_i$$

Normally this results in one single angle for all columns ($\pm 3\gamma$ for combinations of the lower zero-sequence harmonics and $\pm g.3\gamma$ for the higher, where g=0,1,2,...).
After having analyzed the behavior of the squirrel-cage motor with the second type of armature reaction, from the general model some simplified models can be derived. In this paper only the equations for a first-order approximation for the star connection with ungrounded neutral will be given which provides a system of differential equations with constant coefficients. The θ-dependent elements in the first row of the L'_{sr}-matrix disappear from the equations. ($i_{s0}=0$). The selected set of harmonics has to fulfill constraint (14). The $\Delta L''_{rr}$-matrix contains, in principle, inductances which depend on θ, however, in this case, most elements become constant and, in analogy with the elements in the first row of the L'_{sr}-matrix, the few remaining θ-dependent elements in the $\Delta L''_{rr}$-matrix acquire the same angle dependence: $\pm g.6\gamma$.
The contribution of these θ-dependent terms is most often a second-order effect and will be ignored in this model. This results in the following system of differential equations with constant coefficients.

$$u_{sf} = R_s i_{sf} + \frac{d}{dt}(\Psi_{sf}) + j\frac{d\gamma}{dt}\Psi_{sf}$$

$$0 = R_{rm} i_{rm} + \frac{d}{dt}(\Psi''_{rm}) \quad , \quad m = 0 \text{ or } K$$

$$0 = \tilde{R}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + e_k \Psi''_{rk} + f_k \Psi''^{*}_{rk} + \frac{d}{dt}\Psi''_{rk}, \quad k=1,2,\ldots,K-1$$

and the flux-current relations:

$$\Psi_{sf} = \tilde{L}_s i_{sf} + \hat{M}_{2m} i_{rm} + \sum_{i=1}^{K-1}\hat{M}_i i''_{ri}$$

$$\Psi''_{rm} = \hat{M}_{2m}(i_{sf} + i^{*}_{sf}) + (\tilde{L}_{rm} + \Delta_{mm}) i_{rm} + \sum_{i=1}^{K-1}[\Delta_{mi} i''_{ri} + \Delta_{m-i} i''^{*}_{ri}]$$

$$\Psi''_{rk} = \hat{M}_k i_{sf} + \tilde{L}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + \Delta_{km} i_{rm} + \sum_{i=1}^{K-1}[\Delta_{ki} i''_{ri} + \Delta_{k-i} i''^{*}_{ri}]$$

The complete *L*-matrix is real and constant. In calculations it has to be inverted only once, which saves much computer application time.

The torque equation becomes:

$$T_e=2.\mathrm{Re}\Bigg[\sum_{i=1}^{K-1}[g_i i''^{*}_{ri}+h^{*}_i i''_{ri}]\hat{M}_i i_{sf}-j\frac{\partial\gamma}{\partial\theta}[\hat{M}_{2m}i_{rm}+\sum_{i=1}^{K-1}\hat{M}_i i''^{*}_{ri}]i_{sf}+$$

$$+\sum_{i=1}^{K-1}[(g_i\Delta_{im}+h_i\Delta_{m-i})i_{rm}+\sum_{k=1}^{K-1}\{(g_i\Delta_{ik}+h_i\Delta_{k-i})i''_{rk}+$$

$$+(g_i\Delta_{i-k}+h_i\Delta_{ki})i''^{*}_{rk}\}]i''_{ri}\Bigg]$$

where the factors e_i, g_i, f_i and h_i become:

$$e_i=g_i\frac{d\theta}{dt}\quad ; \quad g_i=j\left[\frac{\partial\gamma_i}{\partial\theta}-\frac{\frac{\partial\alpha_i}{\partial\theta}\hat{M}^2_{2i}-\frac{\partial\beta_i}{\partial\theta}\hat{M}^2_{3i}}{E_i^2}\right]$$

$$f_i=h_i\frac{d\theta}{dt}\quad ; \quad h_i=-j\left(\frac{\partial\alpha_i}{\partial\theta}-\frac{\partial\beta_i}{\partial\theta}\right)\frac{\hat{M}_{2i}\hat{M}_{3i}}{E_i^2}$$

Note that when not more then one harmonic inductance is present in every element of the mutual inductance matrix L'_{sr}, see expression (4), the factors g_i and h_i are constant too.

As this system has constant coefficients a particular solution is possible when the slip s is constant [10]. Upon assuming a symmetric voltage system between the terminals of the star-connected stator winding the transformed stator voltage can be written as:

$$u_{sf}=\frac{1}{\sqrt{2}}U_N\, e^{j(\omega t+\phi'-\gamma)}$$

For a constant value of the slip s, γ can be written as: $\gamma=\gamma' p\theta=\gamma'(1-s)\omega t$ and after introducing $s_\nu=1-\gamma'(1-s)$, the transformed voltage becomes:

$$u_{sf}=\underline{U}_{sf}\, e^{j(s_\nu\omega t+\phi')}\ ,\ \text{where } \underline{U}_{sf}=\frac{1}{\sqrt{2}}U_N \ \text{ and}$$

U_N is the rms-value of the terminal voltage.

From here it is clear that, because of the presence of groups of harmonics that result in different γ's, the synchronous torques which change into pulsating torques with a frequency of $2s_\nu\omega t$ outside their synchronous speeds always cause synchronous torques to appear simultaneously with pulsating torques. This contrasts with the first type where only one synchronous torque exists at stand-still.

For the delta connection similar models can be derived. In these models the L''_{sr}- matrix remains θ dependent because of the elements in row 1.

One difficulty in this development of a system with a maximum of constant elements is constituted by the practical fact that for column 1 it may be necessary to consider more inductances in element 3 in the interests of obtaining more accurate results [9].

The complete set of equations, which describes the transient state includes the mechanical equation of motion:

$$T_e-T_{me}=J\frac{d^2\theta}{dt^2}+d_{me}\frac{d\theta}{dt}$$

where T_{me} is the load torque, J the inertia of the complete rotor mass and d_{me} the damping coefficient.

5. CONCLUSIONS

In this paper general equations for squirrel-cage induction motors have been derived based on the real geometry of the motor. The squirrel cage has been described by its meshes; no equivalent windings have been used. By means of complex time-dependent transformations free angles are introduced which are very helpfully simplifying the set of equations when the specific geometrical properties of the both types, in which the asynchronous machines fundamentally can be divided, are taken into account. The equations derived are general in the sense that all space harmonics are taken into account, due to the MMF as well as to the double slotting. This provides a better calculation of the synchronous, pulsating and asynchronous torques. The final equations enable the formulation of some specific properties of both types in connection to their electromechanical behavior. Further they are valid for star and delta connection and for any arbitrary source voltage.

6. ABBREVIATIONS AND SYMBOLS

i(t), u(t)	instantaneous values (complex or real)
$i^*(t)$, $u^*(t)$	conjugate complex values
I,*U*,*L*	matrices
I^T,U^T,etc	transposed matrices
I,U	rms values (complex or real)
$\hat{L}$, $\hat{M}$, etc	magnitudes (real)
$\tilde{L}$, $\tilde{M}$, etc	inductances after group- and symmetrical-component transformation.
i',*I*',L'_{sr},etc	quantities after group- and symmetrical-component transformation.
i",*I*",L''_{sr},etc	quantities after the time-dependent transformation.

Symbols (in SI-units)

d_{me}	coefficient of mechanical damping
i,I,*I*	current
j	complex number j=(0,1)
J	inertia of rotor mass
K	integer number K=N/2z or N/2z+1/2
L,*L*	inductance
m_k	mutual inductance
M	mutual inductance
N	number of rotor slots
N_s	number of stator slots
p	number of pole pairs
R,*R*	resistance
s	slip
T_e	electromagnetic torque
T_{me}	load torque
u,U,*U*	voltage
U	unit matrix
z	greatest common divisor of N and p
α, β, ϵ	angle in composed inductance of one element in the mutual inductance matrix.
γ	angle introduced by transformation
θ	rotor position angle
ν,μ	integer numbers
ϕ	phase angle of stator phase voltage
ϕ'	phase angle of stator terminal voltage
Ψ,Ψ,ψ	flux linkage
ω	angular frequency of the source voltage
Δ_{ki}	short notation for $\Delta L_{rr}(k,i)$
ΔL_{rr}	rotor inductance matrix due to stator slotting

Subscripts

i_{sf}	forward component of stator current
i_{sb}	backward component of stator current
i_r	rotor mesh current

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BIOGRAPHY

Gerardus Chr. Paap was born in Rotterdam, the Netherlands on February 2, 1946. He received his M.SC. from Delft University of Technology in 1972 and his Ph.D. from the Technical University of Lodz in 1988. From 1973 he has been with the Faculty of Electrical Engineering of the Delft University of Technology. First, from 1973 to 1985, with the Electrical Machines and Drives Division where he lectured on the fundamentals of electrical machines and from 1985 with the Power Systems Laboratory as Assistant Professor. His research interests include power systems transients and the dynamics of electrical machines.

APPENDIX I

The resultant parameters which appear in the equations after group and symmetrical component transformation are:

the zero sequence inductance:

$$\tilde{L}_{s0} = \mu_0 \frac{6\, w_s^2\, D\, l}{\pi\, p^2\, \delta} \sum_{\nu} \frac{\xi_{s\nu}^2}{\nu^2} \zeta_{s\nu} + \tilde{L}_{s0\sigma} \quad \text{with } \nu=3,9,15,\ldots.$$

the positive sequence inductance :

$$\tilde{L}_{s} = \mu_0 \frac{3\, w_s^2\, D\, l}{\pi\, p^2\, \delta} \sum_{\nu} \frac{\xi_{s\nu}^2}{\nu^2} \zeta_{s\nu} + \tilde{L}_{s\sigma} \quad \text{with } \nu=1,-5,7,-11,13,.$$

where δ the effective air gap, w_s the number of turns per stator phase in series, D diameter of stator bore and l the effective iron length. $\tilde{L}_{s0\sigma}$ and $\tilde{L}_{s\sigma}$ are the zero-sequence and positive-sequence leakage inductances.

For the rotor the transformed parameters are:

$$\tilde{R}_{rk} = 2R_R + R_{rb}\left(2.\sin\frac{k\pi p}{N}\right)^2, \quad k=0,1,\ldots,(N/z-1)$$

$$\tilde{L}_{r0} = \frac{\beta}{N+\beta}\tilde{L}_N + \tilde{L}_{r0\sigma};\ \tilde{L}_{rk} = \tilde{L}_N + \tilde{L}_{rk\sigma},\quad k=1,2,\ldots,(N/z-1)$$

$$\tilde{L}_{rk\sigma} = 2L_{R\sigma} + L_{rb\sigma}\left(2.\sin\frac{k\pi p}{N}\right)^2 \quad k=0,1,\ldots,(N/z-1)$$

where R_R the resistance of a rotor-ring segment and R_{rb} the resistance of a bar. $L_{R\sigma}$ and $L_{rb\sigma}$ are the leakage inductances of a ring segment and a bar respectively.
β is the rate between the conductance of the unipolar flux path with cross-section A_0 and the effective air gap Δ and one rotor tooth with cross-section A_t.

$$\Delta\tilde{L}_{rn} = \frac{1}{2}\frac{\hat{\Lambda}_{sn}}{\Lambda_{s0}} \zeta_{srn}\, \xi_{\Delta skn}\, \tilde{L}_N, \quad \text{where } \tilde{L}_N = \mu_0 \frac{\pi D l}{N\delta}$$

For the n-th harmonic of the stator-conductance wave it holds for the elements in the matrix $Q[q(i,k)]$: $n.N_s/N \neq 1,2,3,\ldots$.

$q(i,k) = 0$	if $i = k$
$q(i,k) \neq 0$	if $n.N_s + (i-k)p = gN$; $g=0,1,2,..$
$q(i,k) = \beta/(N+\beta)$	if $i = 1$ or $k = 1$
$q(i,k) = 1$	if $i \neq k$ and $k \neq 1$

The elements in the resultant mutual inductance in expression (5) are:

$$\hat{M}_{ij} = \sqrt{\left[\sum_{k=1}^{n} \hat{m}_k^2 + 2\sum_{l=1}^{n-1}\sum_{k=l+1}^{n} \hat{m}_l\, \hat{m}_k \cos((\nu_l-\nu_k)p\theta)\right]}$$

$$\eta = \arctan\left[\sum_{k=1}^{n} m_k \sin(\nu_k p\theta) / \sum_{k=1}^{n} \hat{m}_k \cos(\nu_k p\theta)\right]$$

and the original inductances in expression (3):

$$m_\nu = \hat{m}_\nu\, e^{j\nu p\theta}\ ;\ \hat{m}_\nu = \sqrt{(3N)}.\tilde{L}_N \frac{w_s}{\pi p}\frac{1}{\nu}\, \xi_{s\nu}\xi_{r\nu}\xi_{sk\nu}\zeta_{s\nu}$$

where $\nu = \nu,\mu$

$\xi_{s\nu}$	is the winding factor of the stator.
$\zeta_{s\nu}$	is the stator slot factor.
$\xi_{sk\nu}$	is the skew factor
$\xi_{r\nu}$	is the rotor slot factor
ζ_{srn}	is the factor of mutual slotting
$\xi_{\Delta skn}$	is the skew factor in ΔL_{rr}