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ON THE BUCKLING OF IMPERFECT ANISOTROPIC SHELLS WITH ELASTIC EDGE SUPPORTS UNDER COMBINED LOADING

Part I: Theory and Numerical Analysis

J. Arbocz, J. de Vries and J.M.A.M. Hol

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Delft University of Technology

Faculty of Aerospace Engineering

Delft, The Netherlands

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List of Symbols

A_{ij}	extensional stiffness matrix (see Eq. (A.7))
A_{ij}^*	semi-inverted extensional stiffness matrix (see Eq. (A.9))
$\bar{A}_{ij}^*$	nondimensional A_{ij}^* ($\bar{A}_{ij}^* = EtA_{ij}^*$)
B_{ij}	bending-stretching coupling matrix (see Eq. (A.7))
B_{ij}^*	semi-inverted bending-stretching coupling matrix (see Eq. (A.9))
$\bar{B}_{ij}^*$	nondimensional B_{ij}^* ($\bar{B}_{ij}^* = (2c/t) B_{ij}^*$)
c	$= \sqrt{3(1-\nu^2)}$
D_{ij}	flexural stiffness matrix (see Eq. (A.7))
D_{ij}^*	semi-inverted flexural stiffness matrix (see Eq. (A.11))
$\bar{D}_{ij}^*$	nondimensional D_{ij}^* ($\bar{D}_{ij}^* = (4c^2/Et^3) D_{ij}^*$)
E	arbitrarily chosen reference Young's modulus
f_o	axisymmetric Airy stress function (see Eq. (10))
f_1, f_2, f_3, f_4	asymmetric Airy stress functions (see Eq. (10))
F	Airy stress function
h_k	thickness of the k^{th} layer
$L_{A^*}()$	linear operator defined by Eq. (4)
$L_{B^*}()$	linear operator defined by Eq. (5)
$L_{D^*}()$	linear operator defined by Eq. (6)
$L_{NL}()$	nonlinear operator defined by Eq. (7)
M_x, M_y, M_{xy}	moment resultants
n	number of full waves in the circumferential direction (see Eqs. (8)-(10))
N_x, N_y, N_{xy}	stress resultants
p	external pressure
$\bar{p}$	nondimensional external pressure $\bar{p} = (cR^2/Et^2)p$
p_c	buckling load of the "perfect" structure
q	axial load eccentricity measured from the midsurface of the shell wall - positive inward
$\bar{q}$	nondimensional load eccentricity ($\bar{q} = 4cRq/t^2$)
Q_{ij}	specially orthotropic laminar stiffness matrix (see Eq. (A.5))
$\bar{Q}_{ij}$	generally orthotropic laminar stiffness matrix (see Eq. (A.2))
R	shell radius
$\hat{R}$	nondimensional constant ($= \lambda/\bar{p}_e$)
	for hydrostatic pressure $\hat{R} = 0.5$, for external lateral pressure $\hat{R} = 0$
t	shell wall-thickness
u, v	displacement components in the x and y directions, respectively
w_o	axisymmetric radial displacement function (see Eq. (9))

w_1, w_2	asymmetric radial displacement functions (see Eq. (9))
W	radial displacement (positive inward)
$\bar{W}$	initial radial imperfection $(\bar{W}=\xi\hat{W})$ - positive inward
$\hat{W}$	shape of the initial radial imperfection
W_ν	axial Poisson's effect $(W_\nu=\bar{A}^*_{12}\lambda/c)$
$\hat{W}_\nu$	$=\bar{A}^*_{12}/c$
W_p	radial Poisson's effect $(W_p=\bar{A}^*_{22}\bar{p}/c)$
$\hat{W}_p$	$=\bar{A}^*_{22}/c$
W_t	circumferential Poisson's effect $(W_t=-\bar{A}^*_{26}\bar{\tau}/c)$
$\hat{W}_t$	$=-\bar{A}^*_{26}/c$
x, y	axial and circumferential coordinates on the middle surface of the shell, respectively
$\bar{x}, \bar{y}$	nondimensional coordinates $(\bar{x}=x/R,\ \bar{y}=y/R)$
$\underset{\sim}{Y}$	unified vector variable (see Eq. (145))
z	coordinate normal to the middle surface of the shell (positive inward)
$\bar{Z}$	modified Batdorf parameter $(\bar{Z}=L^2/Rt)$
γ_{xy}	shearing strain
Δ	generalized displacement
ε_M	membrane strain which corresponds to the applied variable load
$\varepsilon_x, \varepsilon_y$	normal strains
θ	circumferential coordinate $(\theta=y/R)$
$\tilde{\theta}_c$	slope of the fundamental path
$\tilde{\theta}^*_c$	initial slope of the postbuckling path
$\kappa_x, \kappa_y, \kappa_{xy}$	curvature components
λ	nondimensional axial load parameter $\lambda=(cR/Et^2)N_o)$
Λ	nondimensional variable load factor
Λ_c	nondimensional variable load factor evaluated at the bifurcation point
Λ_s	nondimensional variable load factor evaluated at the limit point
ν	arbitrarily chosen reference Poisson's ratio
ρ_s	normalized variable load factor $(\rho_s=\Lambda_s/\Lambda_c)$
σ_x, σ_y	normal stresses
$\bar{\tau}$	nondimensional torque parameter $(\bar{\tau}=(cR/Et^2)N_{xy})$ - positive counter-clockwise
τ_{xy}	shearing stress

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Part I: Theory and Numerical Analysis

by

J. Arbocz[1], J. de Vries[2] and J.M.A.M. Hol[2]
Faculty of Aerospace Engineering
Delft University of Technology
Delft, The Netherlands

ABSTRACT

A rigorous solution is presented for the case of stiffened anisotropic cylindrical shells with general imperfections under combined loading, where the edge supports are provided by symmetrical or unsymmetrical elastic rings. The circumferential dependence is eliminated by a truncated Fourier series. The resulting nonlinear 2-point boundary value problem is solved numerically via the "Parallel Shooting Method". The changing deformation patterns resulting from the different degrees of interaction between the given initial imperfections and the specified end rings are displayed. Recommendations are made as to the minimum ring stiffnesses required for optimal load carrying configurations.

1. Professor of Aircraft Structures
2. Assistant Professor

1. INTRODUCTION

All modern aerospace structures are subject to weight restrictions and must satisfy severe reliability criteria. Thus the aerospace community is always looking for ways to produce lighter and more reliable structures. Because thin walled stiffened and unstiffened shells exhibit very favorable strength over weight ratios they are often used in aerospace applications. Initially they were made mostly of light-weight metal alloys but recently bonded sandwich constructions with honeycomb core and advanced composite designs have also been proposed. Unfortunately, thin walled shells are prone to buckling instabilities.

Numerous investigations in the past have shown that the load carrying capacity of orthotropic and of anisotropic shells can be significantly affected by a variety of factors such as the presence of initial geometric imperfections[1,2] and of different in-plane and out-of-plane boundary conditions[3,4], by the nonlinear prebuckling deformations caused by the edge constraints[5,6], by eccentricities of the load application point[7,8] and by inelastic effects[9].

Measuring and recording initial geometric imperfections has become a standard experimental procedure both for laboratory scale[10,11] and full scale structures[12]. Recently Singer and his coworkers[13,14] have developed an experimental technique which makes it possible to estimate the degree of elastic support present in a particular test set up.

When investigating the effect of initial geometric imperfections, in the beginning it has been a standard practice to neglect the effects caused by the edge restraint and the different boundary conditions, in order to simplify the mathematical problem that had to be solved. Later Arbocz and Sechler[15,16] and Simitses and his coworkers[17,18] have presented solutions for isotropic, orthotropic and anisotropic shells with both axisymmetric and asymmetric imperfections under different external loads, where a rigorous satisfaction of the specified boundary conditions was included.

In this paper a rigorous solution is presented for the case of stiffened layered composite shells with general axisymmetric and asymmetric imperfections under combined axial compression, internal or external pressure and torsion, where the edge supports are provided by symmetrical or unsymmetrical elastic rings. The analysis is based on a combination of the nonlinear Donnell type anisotropic imperfect shell equations[19,20] with the ring equations of Cohen[21].

2. THEORETICAL ANALYSIS

In an effort to gain insight into the possible nonlinear interaction between elastic boundary conditions and the initial imperfections the following analytical investigation is carried out, whereby the elastic boundary conditions are modeled by attaching rings of general cross-sectional shape eccentrically at the shell edges. The sign convention used for shell and ring analysis is shown in Fig. 1. Whenever necessary the corresponding variables will be distinguished by superscripts $(\ )^s$ for shell variables and by superscripts $(\ )^r$ for rings variables.

2.1 Anisotropic Shell Equations

Using the sign convention defined in Fig. 1 and introducing an Airy stress function F such that

$$N_x = F_{,yy} \quad ; \quad N_y = F_{,xx} \quad ; \quad N_{xy} = -F_{,xy} \tag{1}$$

then the Donnell type nonlinear governing equations for imperfect, anisotropic shells[19,20] can be written as

$$L_{A^*}(F) - L_{B^*}(W) = -\frac{1}{R} W_{,xx} - \frac{1}{2} L_{NL}(W, W + 2\bar{W}) \tag{2}$$

$$L_{B^*}(F) + L_{D^*}(W) = \frac{1}{R} F_{,xx} + L_{NL}(F, W + \bar{W}) + p_e \tag{3}$$

The linear operators are

$$L_{A^*}(\;) = A^*_{22}(\;)_{,xxxx} - 2A^*_{26}(\;)_{,xxxy} + (2A^*_{12} + A^*_{66})(\;)_{,xxyy} - 2A^*_{16}(\;)_{,xyyy} + A^*_{11}(\;)_{,yyyy} \tag{4}$$

$$L_{B^*}(\;) = B^*_{21}(\;)_{,xxxx} + (2B^*_{26} - B^*_{61})(\;)_{,xxxy} + (B^*_{11} + B^*_{22} - 2B^*_{66})(\;)_{,xxyy} + (2B^*_{16} - B^*_{62})(\;)_{,xyyy} + B^*_{12}(\;)_{,yyyy} \tag{5}$$

$$L_{D^*}(\;) = D^*_{11}(\;)_{,xxxx} + 4D^*_{16}(\;)_{,xxxy} + 2(D^*_{12} + 2D^*_{66})(\;)_{,xxyy} + 4D^*_{26}(\;)_{,xyyy} + D^*_{22}(\;)_{,yyyy} \tag{6}$$

and the nonlinear operator is

$$L_{NL}(S,T) = S_{,xx}T_{,yy} - 2S_{,xy}T_{,xy} + S_{,yy}T_{,xx} \tag{7}$$

Commas in the subscripts denote repeated partial differentiation with respect to the independent variables following the comma. The stiffness parameters A^*_{11}, B^*_{11}, D^*_{11}, A^*_{12}, ... etc. including smeared stiffeners are defined in Appendix A. $\bar{W}$ is the component of the initial stress free imperfection and W is the component of the displacement normal to the shell midsurface, here both positive inward.

These equations, together with the appropriate boundary conditions provided by the elastic end-rings, govern the behavior of circular cylindrical shells

- in the prebuckling stress- and deformation state;
- at the limit or bifurcation point (if there is one);
- in the postbuckling stress and deformation state.

2.2 Reduction to an Equivalent Set of Ordinary Differential Equations

If the initial imperfections are represented by

$$\bar{W}(\bar{x},\theta) = tA_o(\bar{x}) + tA_1(\bar{x}) \cos n\theta + tA_2(\bar{x}) \sin n\theta \tag{8}$$

where $A_o(\bar{x})$, $A_1(\bar{x})$ and $A_2(\bar{x})$ are known functions of $\bar{x} = x/R$, then Eqs. (2) and (3) admit separable solutions of the following form

$$W(\bar{x},\theta) = t(W_v + W_{p_e} + W_t) + tw_o(\bar{x}) + tw_1(\bar{x}) \cos n\theta + tw_2(\bar{x}) \sin n\theta \tag{9}$$

$$F(\bar{x},\theta) = \frac{ERt^2}{c} \{ -\frac{1}{2}\lambda\theta^2 - \frac{1}{2}\bar{p}_e\bar{x}^2 - \bar{\tau}\,\bar{x}\theta + f_o(\bar{x}) + f_1(\bar{x}) \cos n\theta + f_2(\bar{x}) \cos 2n\theta + f_3(\bar{x}) \sin n\theta + f_4(\bar{x}) \sin 2n\theta \} \tag{10}$$

where $\bar{x} = x/R$ and $\theta = y/R$.

Assuming the axial dependence of the response to be an unknown function of $\bar{x}$ will reduce the stability problem to the solution of a set nonlinear ordinary differential equations. This will allow the rigorous enforcing of the boundary conditions. The values of the Poisson's expansions

$$W_v = \frac{\bar{A}^*_{12}}{c}\lambda\,; \quad W_{p_e} = \frac{\bar{A}^*_{22}}{c}\bar{p}_e\,; \quad W_t = -\frac{\bar{A}^*_{26}}{c}\bar{\tau} \tag{11}$$

are obtained by enforcing the circumferential periodicity condition (see Appendix B).

Substituting the expressions for $\bar{W}$, W and F into the compatibility Eq. (2), using some trigonometric identities, and finally equating coefficients of like terms, results in the following system of five nonlinear ordinary differential equations

$$\bar{A}^*_{22}f_o^{iv} - \frac{1}{2}\frac{t}{R}\bar{B}^*_{21}w_o^{iv} + cw_o'' - \frac{c}{4}\frac{t}{R}n^2\{ w_1''(w_1 + 2A_1) + 2w_1'(w_1' + 2A_1') + w_1(w_1'' + 2A_1'') + w_2''(w_2 + 2A_2) + 2w_2'(w_2' + 2A_2') + w_2(w_2'' + 2A_2'') \} = 0 \tag{12}$$

$$\bar{A}^*_{22}f_1^{iv} - n^2(2\bar{A}^*_{12} + \bar{A}^*_{66})f_1'' + n^4\bar{A}^*_{11}f_1 - 2n\bar{A}^*_{26}f_3''' + 2n^3\bar{A}^*_{16}f_3' - \frac{1}{2}\frac{t}{R}\{ \bar{B}^*_{21}w_1^{iv} - n^2(\bar{B}^*_{11} + \bar{B}^*_{22} - 2\bar{B}^*_{66})w_1'' + n^4\bar{B}^*_{12}w_1 + n(2\bar{B}^*_{26} - \bar{B}^*_{61})w_2''' - n^3(2\bar{B}^*_{16} - \bar{B}^*_{62})w_2' \} + cw_1'' - \frac{c}{2}\frac{t}{R}n^2\{ w_o''(w_1 + 2A_1) + w_1(w_o'' + 2A_o'') \} = 0 \tag{13}$$

$$\bar{A}^*_{22}f_2^{iv} - 4n^2(2\bar{A}^*_{12} + \bar{A}^*_{66})f_2'' + 16n^4\bar{A}^*_{11}f_2 - 4n\bar{A}^*_{26}f_4''' + 16n^3\bar{A}^*_{16}f_4' - \frac{c}{4}\frac{t}{R}n^2\{ w_1''(w_1 + 2A_1) - 2w_1'(w_1' + 2A_1') + w_1(w_1'' + 2A_1'') - [w_2''(w_2 + 2A_2) - 2w_2'(w_2' + 2A_2') + w_2(w_2'' + 2A_2'')] \} = 0 \tag{14}$$

$$\bar{A}^*_{22}f_3^{iv} - n^2(2\bar{A}^*_{12} + \bar{A}^*_{66})f_3'' + n^4\bar{A}^*_{11}f_3 + 2n\bar{A}^*_{26}f_1''' - 2n^3\bar{A}^*_{16}f_1' - \frac{1}{2}\frac{t}{R}\{\bar{B}^*_{21}w_2^{iv} - n^2(\bar{B}^*_{11} + \bar{B}^*_{22} - 2\bar{B}^*_{66})w_2'' + n^4\bar{B}^*_{12}w_2 - n(2\bar{B}^*_{26} - \bar{B}^*_{61})w_1''' + n^3(2\bar{B}^*_{16} - \bar{B}^*_{62})w_1'\} + cw_2'' - \frac{c}{2}\frac{t}{R}n^2\{w_o''(w_2 + 2A_2) + w_2(w_o'' + 2A_o'')\} = 0 \tag{15}$$

$$\bar{A}^*_{22}f_4^{iv} - 4n^2(2\bar{A}^*_{12} + \bar{A}^*_{66})f_4'' + 16n^4\bar{A}^*_{11}f_4 + 4n\bar{A}^*_{26}f_2''' - 16n^3\bar{A}^*_{16}f_2' - \frac{c}{4}\frac{t}{R}n^2\{w_2''(w_1 + 2A_1) - 2w_2'(w_1' + 2A_1') + w_2(w_1'' + 2A_1'') + w_1''(w_2 + 2A_2) - 2w_1'(w_2' + 2A_2') + w_1(w_2'' + 2A_2'')\} = 0 \tag{16}$$

Substituting in turn the expressions assumed for $\bar{W}$, W and F into the equilibrium Eq. (3) and applying Galerkin's procedure yields the following system of three nonlinear ordinary differential equations

$$\bar{B}^*_{21}f_o^{iv} + \frac{1}{2}\frac{t}{R}\bar{D}^*_{11}w_o^{iv} - 2c\frac{R}{t}f_o'' + 2c\lambda(w_o'' + A_o'') + cn^2\{f_1''(w_1 + A_1) + 2f_1'(w_1' + A_1') + f_1(w_1'' + A_1'') + f_3''(w_2 + A_2) + 2f_3'(w_2' + A_2') + f_3(w_2'' + A_2'')\} = 0 \tag{17}$$

$$\bar{B}^*_{21}f_1^{iv} - n^2(\bar{B}^*_{11} + \bar{B}^*_{22} - 2\bar{B}^*_{66})f_1'' + n^4\bar{B}^*_{12}f_1 + n(2\bar{B}^*_{26} - \bar{B}^*_{61})f_3''' - n^3(2\bar{B}^*_{16} - \bar{B}^*_{62})f_3' + \frac{1}{2}\frac{t}{R}\{\bar{D}^*_{11}w_1^{iv} - 2n^2(\bar{D}^*_{12} + 2\bar{D}^*_{66})w_1'' + n^4\bar{D}^*_{22}w_1 + 4n\bar{D}^*_{16}w_2''' - 4n^3\bar{D}^*_{26}w_2'\} - 2c\frac{R}{t}f_1'' + 2c\lambda(w_1'' + A_1'') - 2cn^2\bar{p}_e(w_1 + A_1) - 4cn\bar{\tau}(w_2' + A_2') + 2cn^2\{f_o''(w_1 + A_1) + f_1(w_o'' + A_o'')\} + cn^2\{f_2''(w_1 + A_1) + 4f_2'(w_1' + A_1') + 4f_2(w_1'' + A_1'') + f_4''(w_2 + A_2) + 4f_4'(w_2' + A_2') + 4f_4(w_2'' + A_2'')\} = 0 \tag{18}$$

$$\bar{B}^*_{21}f_3^{iv} - n^2(\bar{B}^*_{11} + \bar{B}^*_{22} - 2\bar{B}^*_{66})f_3'' + n^4\bar{B}^*_{12}f_3 - n(2\bar{B}^*_{26} - \bar{B}^*_{61})f_1''' + n^3(2\bar{B}^*_{16} - \bar{B}^*_{62})f_1' + \frac{1}{2}\frac{t}{R}\{\bar{D}^*_{11}w_2^{iv} - 2n^2(\bar{D}^*_{12} + 2\bar{D}^*_{66})w_2'' + n^4\bar{D}^*_{22}w_2 - 4n\bar{D}^*_{16}w_1''' + 4n^3\bar{D}^*_{26}w_1'\} - 2c\frac{R}{t}f_3'' + 2c\lambda(w_2'' + A_2'') - 2cn^2\bar{p}_e(w_2 + A_2) + 4cn\bar{\tau}(w_1' + A_1') + 2cn^2\{f_o''(w_2 + A_2) + f_3(w_o'' + A_o'')\} + cn^2\{f_4''(w_1 + A_1) + 4f_4'(w_1' + A_1') + 4f_4(w_1'' + A_1'') - [f_2''(w_2 + A_2) + 4f_2'(w_2' + A_2') + 4f_2(w_2'' + A_2'')]\} = 0 \tag{19}$$

The nondimensional stiffness coefficients $\bar{A}^*_{ij}$, $\bar{B}^*_{ij}$ and $\bar{D}^*_{ij}$ are listed in Appendix A and $(\ )' = d/d\bar{x}$.

Notice that Eq. (12) can be integrated twice yielding

$$\bar{A}^*_{22}f_o'' - \frac{1}{2}\frac{t}{R}\bar{B}^*_{21}w_o'' + cw_o - \frac{c}{4}\frac{t}{R}n^2\left\{w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\right\} = 0 \tag{20}$$

The constants of integration are set equal to zero in order to satisfy the circumferential periodicity condition (see Appendix B). In order to be able to use the "Parallel Shooting Method" for the numerical solution of the above system of nonlinear ordinary differential equations it is necessary to reformulate Eqs. (13) - (19) such that a single 4th order derivative appears on the left-hand side. This can be accomplished as follows:

- with the help of Eq. (18) eliminate the term w_1^{iv} from Eq. (13);
- with the help of Eq. (19) eliminate the term w_2^{iv} from Eq. (15);
- with the help of Eqs. (12) and (20) eliminate the terms f_o^{iv} and f_o'' from Eq. (17);
- with the help of Eqs. (13) and (20) eliminate the terms f_1^{iv} and f_o'' from Eq. (18);
- with the help of Eqs. (15) and (20) eliminate the terms f_3^{iv} and f_o'' from Eq. (19).

Carrying out the details makes it possible to write the resulting equations as

$$
\begin{aligned}
f_1^{iv} = {} & C_1 f_1'' - C_2 f_1 - C_3 w_1'' - C_4 \lambda (w_1'' + A_1'') + C_5 w_1 + C_{201} f_3''' + C_{202} f_3' + C_{203} w_2''' + C_{204} w_2' \\
& + C_{211} \bar{p}_e (w_1 + A_1) + C_{212} \bar{\tau} (w_2' + A_2') + C_6 [\, w_o''(w_1 + 2A_1) + w_1 (w_o'' + 2A_o'') \,] - C_7 w_o''(w_1 + A_1) \\
& + C_8 w_o (w_1 + A_1) - 2C_{10} f_1 (w_o'' + A_o'') - C_9 [\, w_1 (w_1 + 2A_1) + w_2 (w_2 + 2A_2) \,] (w_1 + A_1) \\
& - C_{10} \{ f_2''(w_1 + A_1) + 4 f_2'(w_1' + A_1') + 4 f_2 (w_1'' + A_1'') + f_4''(w_2 + A_2) + 4 f_4'(w_2' + A_2') + 4 f_4 (w_2'' + A_2'' \}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
f_2^{iv} = {} & C_{11} f_2'' - C_{12} f_2 + C_{205} f_4''' + C_{206} f_4' + C_{13} \{ w_1''(w_1 + 2A_1) - 2 w_1'(w_1' + 2A_1') + w_1 (w_1'' + 2A_1'') \\
& - [\, w_2''(w_2 + 2A_2) - 2 w_2'(w_2' + 2A_2') + w_2 (w_2'' + 2A_2'') \,] \}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
f_3^{iv} = {} & C_1 f_3'' - C_2 f_3 - C_3 w_2'' - C_4 \lambda (w_2'' + A_2'') + C_5 w_2 - C_{201} f_1''' - C_{202} f_1' - C_{203} w_1''' - C_{204} w_1' \\
& + C_{211} \bar{p}_e (w_2 + A_2) - C_{212} \bar{\tau} (w_1' + A_1') + C_6 [\, w_o''(w_2 + 2A_2) + w_2 (w_o'' + 2A_o'') \,] - C_7 w_o''(w_2 + A_2) \\
& + C_8 w_o (w_2 + A_2) - 2C_{10} f_3 (w_o'' + A_o'') - C_9 [\, w_1 (w_1 + 2A_1) + w_2 (w_2 + 2A_2) \,] (w_2 + A_2) \\
& - C_{10} \{ f_4''(w_1 + A_1) + 4 f_4'(w_1' + A_1') + 4 f_4 (w_1'' + A_1'') - [\, f_2''(w_2 + A_2) + 4 f_2'(w_2' + A_2') + 4 f_2 (w_2'' + A_2'') \,] \}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
f_4^{iv} = {} & C_{11} f_4'' - C_{12} f_4 - C_{205} f_2''' - C_{206} f_2' + C_{13} \{ w_2''(w_1 + 2A_1) - 2 w_2'(w_1' + 2A_1') + w_2 (w_1'' + 2A_1'') \\
& + w_1''(w_2 + 2A_2) - 2 w_1'(w_2' + 2A_2') + w_1 (w_2'' + 2A_2'') \}
\end{aligned}
\tag{24}
$$

$$w_o^{iv} = C_{14}w_o'' - C_{15}w_o - C_{19}\lambda(w_o'' + A_o'') + C_{17}[\, w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\,] - C_{16}\{\, w_1''(w_1 + 2A_1) + 2w_1'(w_1' + 2A_1') + w_1(w_1'' + 2A_1'') + w_2''(w_2 + 2A_2) + 2w_2'(w_2' + 2A_2') + w_2(w_2'' + 2A_2'')\,\} - C_{18}\{\, f_1''(w_1 + A_1) + 2f_1'(w_1' + A_1') + f_1(w_1'' + A_1'') + f_3''(w_2 + A_2) + 2f_3'(w_2' + A_2') + f_3(w_2'' + A_2'')\,\} \quad (25)$$

$$w_1^{iv} = C_{20}w_1'' - C_{21}w_1 + C_{22}f_1'' - C_{24}f_1 - C_{23}\lambda(w_1'' + A_1'') + C_{209}w_2''' + C_{210}w_2' + C_{207}f_3''' + C_{208}f_3' + C_{213}\bar{p}_e(w_1 + A_1) + C_{214}\bar{\tau}(w_2' + A_2') - C_{25}[\, w_o''(w_1 + 2A_1) + w_1(w_o'' + 2A_o'')\,] - C_{26}w_o''(w_1 + A_1) + C_{27}w_o(w_1 + A_1) - C_{28}[\, w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\,](w_1 + A_1) - 2C_{29}f_1(w_o'' + A_o'') - C_{29}\{\, f_2''(w_1 + A_1) + 4f_2'(w_1' + A_1') + 4f_2(w_1'' + A_1'') + f_4''(w_2 + A_2) + 4f_4'(w_2' + A_2') + 4f_4(w_2'' + A_2'')\,\} \quad (26)$$

$$w_2^{iv} = C_{20}w_2'' - C_{21}w_2 + C_{22}f_3'' - C_{24}f_3 - C_{23}\lambda(w_2'' + A_2'') - C_{209}w_1''' - C_{210}w_1' - C_{207}f_1''' - C_{208}f_1' + C_{213}\bar{p}_e(w_2 + A_2) - C_{214}\bar{\tau}(w_1' + A_1') - C_{25}[\, w_o''(w_2 + 2A_2) + w_2(w_o'' + 2A_o'')\,] - C_{26}w_o''(w_2 + A_2) + C_{27}w_o(w_2 + A_2) - C_{28}[\, w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\,](w_2 + A_2) - 2C_{29}f_3(w_o'' + A_o'') - C_{29}\{\, f_4''(w_1 + A_1) + 4f_4'(w_1' + A_1') + 4f_4(w_1'' + A_1'') - [\, f_2''(w_2 + A_2) + 4f_2'(w_2' + A_2') + 4f_2(w_2'' + A_2'')\,]\,\} \quad (27)$$

The constants C_1, C_2, ... , C_{214} are listed in Appendix C. Besides the traditional simply supported and clamped boundary conditions (following Refs. [3] and [13] usually designated as SS-1 through SS-4 and C-1 through C-4, see Appendix D for details), in this paper boundary conditions will also be modeled by attaching elastic rings of general cross-sectional shape eccentrically at the shell edges.

2.3 Equilibrium Equations of the End Rings

Cohen's ring equations[21] are based on moderate rotations and are derived using the principle of virtual work (see Appendix E for details). They can be written as

$$EI_z u_{,\theta\theta\theta\theta} - GJu_{,\theta\theta} + EI_{xz}(v_{,\theta\theta\theta} + w_{,\theta\theta\theta\theta}) - a(EI_z + GJ)\beta_{y,\theta\theta} = a^3(aF_x^r) \quad (28a)$$

$$EI_{xz}u_{,\theta\theta\theta} + (EI_x + a^2EA)v_{,\theta\theta} + (EI_x w_{,\theta\theta\theta} - a^2EAw_{,\theta}) - aEI_{xz}\beta_{y,\theta} = -a^3(aF_y^r) \quad (28b)$$

$$EI_{xz}u_{,\theta\theta\theta\theta} + (EI_x v_{,\theta\theta\theta} - a^2EAv_{,\theta}) + (EI_x w_{,\theta\theta\theta\theta} + a^2EAw) - aEI_{xz}\beta_{y,\theta\theta} = a^3(aF_z^r) \quad (28c)$$

$$(EI_z + GJ)u_{,\theta\theta} + EI_{xz}(v_{,\theta} + w_{,\theta\theta}) + a(GJ\beta_{y,\theta\theta} - EI_z\beta_y) = -a^3M_t^r \quad (28d)$$

where the definition of the variables and the sign convention used is displayed in Fig. 2.

In order to be able to satisfy the displacement compatibility conditions between the end-rings and the edges of the shell, one must express the ring displacements in the same form as the one assumed for the shell displacements. Further the expansion assumed for the load terms must not only be consistent with the terms assumed for the displacements but they must also form a self-equilibrating force system. Thus the Fourier decomposition of the ring equations will be based on the following expressions

$$
\begin{aligned}
u^r &= u_o^r + u_1^r \cos n\theta + u_2^r \cos 2n\theta + u_3^r \sin n\theta + u_4^r \sin 2n\theta \\
v^r &= v_o^r + v_1^r \sin n\theta + v_2^r \sin 2n\theta + v_3^r \cos n\theta + v_4^r \cos 2n\theta \\
w^r &= w_o^r + w_1^r \cos n\theta + w_2^r \sin n\theta \\
\beta_y^r &= \beta_{y_o}^r + \beta_{y_1}^r \cos n\theta + \beta_{y_2}^r \sin n\theta
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
F_x^r &= F_{x_1}^r \cos n\theta + F_{x_2}^r \cos 2n\theta + F_{x_3}^r \sin n\theta + F_{x_4}^r \sin 2n\theta \\
F_y^r &= F_{y_1}^r \sin n\theta + F_{y_2}^r \sin 2n\theta + F_{y_3}^r \cos n\theta + F_{y_4}^r \cos 2n\theta \\
F_z^r &= F_{z_o}^r + F_{z_1}^r \cos n\theta + F_{z_2}^r \sin n\theta \\
M_t^r &= M_{t_o}^r + M_{t_1}^r \cos n\theta + M_{t_2}^r \sin n\theta
\end{aligned}
\tag{30}
$$

Substituting these expressions into Cohen's ring equations and equating coefficients of like terms results in the following

2.3.1 Separated Set of Cohen's Ring Equations:

For n = 0

$$\mathbf{S}_1^r\ \mathbf{U}_1^r = \mathbf{F}_1^r$$

$$
\frac{1}{a^3}\begin{bmatrix} a^2EA & 0 \\ 0 & a^2EI_z \end{bmatrix}\begin{Bmatrix} w_o^r \\ \beta_{y_o}^r \end{Bmatrix} = \begin{Bmatrix} aF_{z_o}^r \\ aM_{t_o}^r \end{Bmatrix}
\tag{31}
$$

For n ≥ 2

$$
\begin{bmatrix} \mathbf{S}_2^r & 0 & 0 & 0 \\ 0 & \mathbf{S}_3^r & 0 & 0 \\ 0 & 0 & \mathbf{S}_4^r & 0 \\ 0 & 0 & 0 & \mathbf{S}_5^r \end{bmatrix}\begin{Bmatrix} \mathbf{U}_2^r \\ \mathbf{U}_3^r \\ \mathbf{U}_4^r \\ \mathbf{U}_5^r \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_2^r \\ \mathbf{F}_3^r \\ \mathbf{F}_4^r \\ \mathbf{F}_5^r \end{Bmatrix}
\tag{32}
$$

where the $\mathbf{S}_i^r$ are matrices and the $\mathbf{U}_i^r$, $\mathbf{F}_i^r$ are vectors defined as follows

$$\mathbf{S}_2^r = \frac{1}{a^3}\begin{bmatrix} n^2(n^2EI_z + GJ) & -n^3EI_{xz} & n^4EI_{xz} & an^2(EI_z + GJ) \\ -n^3EI_{xz} & n^2(EI_x + a^2EA) & -n(n^2EI_x + a^2EA) & -anEI_{xz} \\ n^4EI_{xz} & -n(n^2EI_x + a^2EA) & n^4EI_x + a^2EA & an^2EI_{xz} \\ an^2(EI_z + GJ) & -anEI_{xz} & an^2EI_{xz} & a^2(EI_z + n^2GJ) \end{bmatrix} \tag{33}$$

$$\mathbf{S}_3^r = \frac{1}{a^3}\begin{bmatrix} 4n^2(4n^2EI_z + GJ) & -8n^3EI_{xz} \\ -8n^3EI_{xz} & 4n^2(EI_x + a^2EA) \end{bmatrix} \tag{34}$$

$$\mathbf{S}_4^r = \frac{1}{a^3}\begin{bmatrix} n^2(n^2EI_z + GJ) & n^3EI_{xz} & n^4EI_{xz} & an^2(EI_z + GJ) \\ n^3EI_{xz} & n^2(EI_x + a^2EA) & n(n^2EI_x + a^2EA) & anEI_{xz} \\ n^4EI_{xz} & n(n^2EI_x + a^2EA) & n^4EI_x + EA & an^2EI_{xz} \\ an^2(EI_z + GJ) & anEI_{xz} & an^2EI_{xz} & a^2(EI_z + n^2GJ) \end{bmatrix} \tag{35}$$

$$\mathbf{S}_5^r = \frac{1}{a^3}\begin{bmatrix} 4n^2(4n^2EI_z + GJ) & 8n^3EI_{xz} \\ 8n^3EI_{xz} & 4n^2(EI_x + a^2EA) \end{bmatrix} \tag{36}$$

Further

$$\begin{aligned} (\mathbf{U}_2^r)^T &= [\, u_1^r \;\; v_1^r \;\; w_1^r \;\; \beta_{y_1}^r \,] \quad ; \quad (\mathbf{U}_3^r)^T = [u_2^r \;\; v_2^r] \\ (\mathbf{U}_4^r)^T &= [\, u_3^r \;\; v_3^r \;\; w_2^r \;\; \beta_{y_2}^r \,] \quad ; \quad (\mathbf{U}_5^r)^T = [u_4^r \;\; v_4^r] \end{aligned} \tag{37}$$

finally

$$\begin{aligned} (\mathbf{F}_2^r)^T &= [aF_{x_1}^r \;\; aF_{y_1}^r \;\; aF_{z_1}^r \;\; aM_{t_1}^r] \quad ; \quad (\mathbf{F}_3^r)^T = [aF_{x_2}^r \;\; aF_{y_2}^r] \\ (\mathbf{F}_4^r)^T &= [aF_{x_3}^r \;\; aF_{y_3}^r \;\; aF_{z_2}^r \;\; aM_{t_2}^r] \quad ; \quad (\mathbf{F}_5^r)^T = [aF_{x_4}^r \;\; aF_{y_4}^r] \end{aligned} \tag{38}$$

2.3.2 Forces and Moments Acting at the Ring Centroid

Next one must express the line loads (F_x^r, F_y^r, F_z^r) and the torsional line moment (M_t^r) acting at the ring centroid in terms of the applied external loads and the stress and moment resultant of the shell wall attached to it. Considering the free-body diagrams of the shell edges from Fig. 3 one obtains the following relationships at x = 0 (at the lower edge)

$$
\begin{aligned}
aF_x^r &= (R - q)N_o + RN_x^s \\
aF_y^r &= -(R - q_t)S_o + RN_{xy}^s \\
aF_z^r &= RH^s \\
aM_t^r &= -e_z RN_x^s + (q - e_z)(R - q)N_o + e_x RH^s + RM_x^s
\end{aligned}
\tag{39}
$$

Whereas considering Fig. 4 at the upper edge, at x = L, the following expressions hold

$$
\begin{aligned}
aF_x^r &= -(R - q)N_o - RN_x^s \\
aF_y^r &= (R - q_t)S_o - RN_{xy}^s \\
aF_z^r &= -RH^s \\
aM_t^r &= e_z RN_x^s - (q - e_z)(R - q)N_o - e_x RH^s - RM_x^s
\end{aligned}
\tag{40}
$$

Notice that at the upper edge (at x = L) one obtains the same expressions as at the lower edge (at x = 0) except for the minus sign in front of the terms on the right hand side. Further N_o is the external compressive line load applied at a distance q from the shell midsurface, whereas S_o is the external torsional line load applied at a distance q_t from the shell midsurface.

2.3.3 Displacement and Rotation Compatibility Conditions

Considering now the compatibility of the ring and shell displacements and rotations, with the help of Fig. 5a one obtains the following expressions at x = 0 (at the lower edge)

$$
\begin{aligned}
u^r &= u^s - e_z w_{,x}^s \\
v^r &= \frac{a}{R} v^s - e_x u_{,y}^s - e_z w_{,y}^s \\
w^r &= w^s + e_x w_{,x}^s \\
\beta_y^r &= -w_{,x}^s
\end{aligned}
\tag{41}
$$

Notice that the displacement v^r consists of three parts, namely

- a rotation around the centerline of the shell

$$
v^r = \frac{a}{R} v^s \tag{42a}
$$

- a rotation about the x-axis

$$
v^r = -e_z w_{,y}^s \tag{42b}
$$

- a rotation about the z-axis

$$v^r = -e_x u^s_{,y} \qquad (42c)$$

Further, by looking at Fig. 5b, it can easily be seen that at the upper edge (at x = L) one obtains identical compatibility conditions.

2.4 Fourier Decomposition in the Circumferential Direction

Using the previously shown Fourier decomposition of the ring variables (Eqs. (29)-(30)) and the following Fourier decomposition of the shell variables

$$\begin{aligned}
u^s &= t(u^s_o + u^s_1 \cos n\theta + u^s_2 \cos 2n\theta + u^s_3 \sin n\theta + u^s_4 \sin 2n\theta)\\
v^s &= t(v^s_o + v^s_1 \sin n\theta + v^s_2 \sin 2n\theta + v^s_3 \cos n\theta + v^s_4 \cos 2n\theta)\\
w^s &= t(W_v + W_p + W_t + w^s_o + w^s_1 \cos n\theta + w^s_2 \sin n\theta)\\
w^s_{,x} &= t(w^s_{,o,x} + w^s_{1,x} \cos n\theta + w^s_{2,x} \sin n\theta)
\end{aligned} \qquad (43)$$

and

$$\begin{aligned}
N^s_x &= \frac{Et^2}{cR}(N^s_{x_o} + N^s_{x_1} \cos n\theta + N^s_{x_2} \cos 2n\theta + N^s_{x_3} \sin n\theta + N^s_{x_4} \sin 2n\theta)\\
N^s_{xy} &= \frac{Et^2}{cR}(N^s_{xy_o} + N^s_{xy_1} \sin n\theta + N^s_{xy_2} \sin 2n\theta + N^s_{xy_3} \cos n\theta + N^s_{xy_4} \cos 2n\theta)\\
H^s &= \frac{t}{R^3} D (H^s_o + H^s_1 \cos n\theta + H^s_2 \sin n\theta)\\
M^s_x &= \frac{t}{R^2} D (M^s_{x_o} + M^s_{x_1} \cos n\theta + M^s_{x_2} \sin n\theta)
\end{aligned} \qquad (44)$$

where

$$D = \frac{Et^3}{4c^2} \quad ; \quad c^2 = 3(1 - \nu^2) \quad \text{and} \quad \theta = \frac{y}{R} \qquad (45)$$

one obtains upon substitution into Eqs. (39) and equating coefficients of like terms, the following

2.4.1 Separated Form of the Forces and Moments Acting at the Ring Centroid at the Lower Edge (at x = 0)

For n = 0

$$N^s_{x_o} = -\frac{cR}{Et^2} N_o\left(1 - \frac{q}{R}\right) = -\lambda\left(1 - \frac{q}{R}\right) \approx -\lambda \qquad (46)$$

$$N_{xy_o}^{s} = \frac{cR}{Et^2} S_o(1 - \frac{q_t}{R}) = \bar{\tau}(1 - \frac{q_t}{R}) \approx \bar{\tau} \tag{47}$$

and

$$\mathbf{F}_1^r = \mathbf{B}_1 \mathbf{F}_1^s + \mathbf{F}_1^e \tag{48}$$

where the $\mathbf{B}_1$ matrix and the $\mathbf{F}_1^r$, $\mathbf{F}_1^s$, $\mathbf{F}_1^e$ column vectors are defined as

$$\mathbf{B}_1 = \frac{t}{R^2} D \begin{bmatrix} 1 & 0 \\ e_x & R \end{bmatrix} \tag{49}$$

$$(\mathbf{F}_1^r)^T = [aF_{z_o}^r \quad aM_{t_o}^r] \tag{50}$$

$$(\mathbf{F}_1^s)^T = [H_o^s \quad M_{x_o}^s] \tag{51}$$

$$(\mathbf{F}_1^e)^T = \frac{t}{R^2} D[0 \quad R\lambda\bar{q}(1 - \frac{q}{R})] \quad \text{where } D = \frac{Et^3}{4c^2} \; ; \; \bar{q} = 4c\frac{R}{t^2}q \tag{52}$$

$$\begin{Bmatrix} \mathbf{F}_2^r \\ \mathbf{F}_3^r \\ \mathbf{F}_4^r \\ \mathbf{F}_5^r \end{Bmatrix} = \begin{bmatrix} \mathbf{B}_2 & 0 & 0 & 0 \\ 0 & \mathbf{B}_3 & 0 & 0 \\ 0 & 0 & \mathbf{B}_4 & 0 \\ 0 & 0 & 0 & \mathbf{B}_5 \end{bmatrix} \begin{Bmatrix} \mathbf{F}_2^s \\ \mathbf{F}_3^s \\ \mathbf{F}_4^s \\ \mathbf{F}_5^s \end{Bmatrix} \tag{53}$$

where the $\mathbf{B}_i$ are matrices and $\mathbf{F}_i^r$, $\mathbf{F}_i^s$ are column vectors defined as

$$\mathbf{B}_2 = \frac{Et^2}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4c}(\frac{t}{R})^2 & 0 \\ -e_z & 0 & \frac{1}{4c}(\frac{t}{R})^2 e_x & \frac{1}{4c}(\frac{t}{R})^2 R \end{bmatrix} \tag{54}$$

$$\mathbf{B}_3 = \frac{Et^2}{c}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{55}$$

$$\mathbf{B}_4 = \frac{Et^2}{c}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4c}(\frac{t}{R})^2 & 0 \\ -e_z & 0 & \frac{1}{4c}(\frac{t}{R})^2 e_x & \frac{1}{4c}(\frac{t}{R})^2 R \end{bmatrix} \tag{56}$$

$$\mathbf{B}_5 = \frac{Et^2}{c}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{55}$$

Further

$$\begin{aligned} (\mathbf{F}_2^r)^T &= [aF_{x_1}^r \;\; aF_{y_1}^r \;\; aF_{z_1}^r \;\; aM_{t_1}^r] \quad ; \quad (\mathbf{F}_3^r)^T = [aF_{x_2}^r \;\; aF_{y_2}^r] \\ (\mathbf{F}_4^r)^T &= [aF_{x_3}^r \;\; aF_{y_3}^r \;\; aF_{z_2}^r \;\; aM_{t_2}^r] \quad ; \quad (\mathbf{F}_5^r)^T = [aF_{x_4}^r \;\; aF_{y_4}^r] \end{aligned} \tag{58}$$

and

$$\begin{aligned} (\mathbf{F}_2^s)^T &= [N_{x_1}^s \;\; N_{xy_1}^s \;\; H_1^s \;\; M_{x_1}^s] \quad ; \quad (\mathbf{F}_3^s)^T = [N_{x_2}^s \;\; N_{xy_2}^s] \\ (\mathbf{F}_4^s)^T &= [N_{x_3}^s \;\; N_{xy_3}^s \;\; H_2^s \;\; M_{x_2}^s] \quad ; \quad (\mathbf{F}_5^s)^T = [N_{x_4}^s \;\; N_{xy_4}^s] \end{aligned} \tag{59}$$

At the upper edge (at $x = L$) one obtains the same expressions except for a minus sign in front of the terms on the right hand side. Thus

For n = 0

$$\mathbf{F}_1^r = -\mathbf{B}_1\mathbf{F}_1^s - \mathbf{F}_1^e \tag{60}$$

For ≥ 0

$$\begin{aligned} \mathbf{F}_2^r &= -\mathbf{B}_2\mathbf{F}_2^s \\ \mathbf{F}_3^r &= -\mathbf{B}_3\mathbf{F}_3^s \\ \mathbf{F}_4^r &= -\mathbf{B}_4\mathbf{F}_4^s \\ \mathbf{F}_5^r &= -\mathbf{B}_5\mathbf{F}_5^s \end{aligned} \tag{61}$$

Substituting next the corresponding Fourier expansions into Eq. (41) and equating coefficients of like terms one obtains the following

2.4.2 Separated Form of the Displacement and Rotation Compatibility Conditions at the Lower Edge (at x = 0)

For n = 0

$$\mathbf{U}_1^r = \mathbf{E}_1 \mathbf{U}_1^s + \mathbf{U}_1^e \tag{62}$$

where the $\mathbf{E}_1$ matrix and the $\mathbf{U}_1^r$, $\mathbf{U}_1^s$, $\mathbf{U}_1^e$ column vectors are defined as

$$\mathbf{E}_1 = \frac{t}{R}\begin{bmatrix} R & e_x \\ 0 & -1 \end{bmatrix} \tag{63}$$

$$(\mathbf{U}_1^r)^T = [w_o^r \quad \beta_{y_o}^r] \tag{64}$$

$$(\mathbf{U}_1^s)^T = [w_o^s \quad w_{o,\bar{x}}^s] \tag{65}$$

$$(\mathbf{U}_1^e)^T = [t(W_v + W_{pe} + W_t) \quad 0] \tag{66}$$

and $(\ )_{,\bar{x}} = \frac{1}{R}(\ )_{,x}$

For n $\geq$ 2

$$\begin{Bmatrix} \mathbf{U}_2^r \\ \mathbf{U}_3^r \\ \mathbf{U}_4^r \\ \mathbf{U}_5^r \end{Bmatrix} = \begin{bmatrix} \mathbf{E}_2 & 0 & 0 & 0 \\ 0 & \mathbf{E}_3 & 0 & 0 \\ 0 & 0 & \mathbf{E}_4 & 0 \\ 0 & 0 & 0 & \mathbf{E}_5 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_2^s \\ \mathbf{U}_3^s \\ \mathbf{U}_4^s \\ \mathbf{U}_5^s \end{Bmatrix} \tag{67}$$

where the $\mathbf{E}_i$ are matrices and $\mathbf{U}_i^r$, $\mathbf{U}_i^s$ are column vectors defined as

$$\mathbf{E}_2 = \frac{t}{R}\begin{bmatrix} R & 0 & 0 & -e_z \\ ne_x & a & ne_z & 0 \\ 0 & 0 & R & e_x \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{68}$$

$$\mathbf{E}_3 = \frac{t}{R}\begin{bmatrix} R & 0 \\ 2ne_x & a \end{bmatrix} \tag{69}$$

$$\mathbf{E}_4 = \frac{t}{R}\begin{bmatrix} R & 0 & 0 & -e_z \\ -ne_x & a & -ne_z & 0 \\ 0 & 0 & R & e_x \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{70}$$

$$\mathbf{E}_5 = \frac{t}{R}\begin{bmatrix} R & 0 \\ -2ne_x & a \end{bmatrix} \tag{71}$$

Further

$$\begin{aligned} (\mathbf{U}_2^r)^T &= [u_1^r \;\; v_1^r \;\; w_1^r \;\; \beta_{y_1}^r] \quad ; \quad (\mathbf{U}_3^r)^T = [u_2^r \;\; v_2^r] \\ (\mathbf{U}_4^r)^T &= [u_3^r \;\; v_3^r \;\; w_2^r \;\; \beta_{y_2}^r] \quad ; \quad (\mathbf{U}_5^r)^T = [u_4^r \;\; v_4^r] \end{aligned} \tag{72}$$

and

$$\begin{aligned} (\mathbf{U}_2^s)^T &= [u_1^s \;\; v_1^s \;\; w_1^s \;\; w_{1,\bar{x}}^s] \quad ; \quad (\mathbf{U}_3^s)^T = [u_2^s \;\; v_2^s] \\ (\mathbf{U}_4^s)^T &= [u_3^s \;\; v_3^s \;\; w_2^s \;\; w_{2,\bar{x}}^s] \quad ; \quad (\mathbf{U}_5^r)^T = [u_4^s \;\; v_4^s] \end{aligned} \tag{73}$$

Recall that, as stated earlier (see Figures 5a and 5b), at the upper edge (at $x = L$) the compatibility conditions are identical. Thus Eqs. (62) - (73) are valid both at $x = 0$ and $x = L$.

2.5 General Elastic Boundary Conditions

Considering the separated set of Cohen's ring equations (Eqs. (31-(32)), the separated set of equations of the forces and moments acting at the ring centroid (Eqs. (46)-(61)) and the separated set of equations for the displacement and rotation compatibility conditions (Eqs. (62)-(73)), then one has in matrix notation

at the lower edge (at x = 0)

for n = 0

$$\mathbf{S}_1^r \, \mathbf{U}_1^r = \mathbf{F}_1^r \tag{31}$$

$$F_1^r = B_1 F_1^s + F_1^e \tag{48}$$

$$U_1^r = E_1 U_1^s + U_1^e \tag{62}$$

for n ≥ 2

$$S_i^r\, U_i^r = F_i^r \tag{32}$$

$$F_i^r = B_i F_i^s \qquad i = 2,3,4,5 \tag{53}$$

$$U_i^r = E_i U_i^s \tag{67}$$

Combining these equations one obtains the general elastic boundary conditions

at the lower edge (at x = 0)

for n = 0

$$S_1^r\,(E_1 U_1^s + U_1^e) = B_1 F_1^s + F_1^e \tag{74}$$

for n ≥ 2

$$S_i^r (E_i U_i^s) = B_i F_i^s \qquad i = 2,3,4,5 \tag{75}$$

at the upper edge (at x = L)

for n = 0

$$S_1^r\,(E_1 U_1^s + U_1^e) = -B_1 F_1^s - F_1^e \tag{76}$$

for n ≥ 2

$$S_i^r (E_i U_i^s) = -B_i F_i^s \qquad i = 2,3,4,5 \tag{77}$$

Multiplying Eqs. (74) and (75) out yields the general elastic boundary conditions at the lower edge, at x = 0. These can be written in two different forms, namely the

2.5.1 General Elastic Boundary Conditions Valid in the Limit as $E_r \to 0$ (free edges)

For n = 0

$$\begin{aligned} a_{11}^o(w_o^s + W_v + W_{p_e} + W_t) + a_{12}^o w_{o,\bar{x}}^s &= H_o^s \\ a_{21}^o(w_o^s + W_v + W_{p_e} + W_t) + a_{22}^o w_{o,\bar{x}}^s &= M_{x_o}^s + \lambda\bar{q} \end{aligned} \tag{78}$$

For n $\geq$ 2

$$
\begin{aligned}
a_{11}^{1}u_{1}^{s} + a_{12}^{1}v_{1}^{s} + a_{13}^{1}w_{1}^{s} + a_{14}^{1}w_{1,\bar{x}}^{s} &= N_{x_1}^{s} \\
a_{21}^{1}u_{1}^{s} + a_{22}^{1}v_{1}^{s} + a_{23}^{1}w_{1}^{s} + a_{24}^{1}w_{1,\bar{x}}^{s} &= N_{xy_1}^{s} \\
a_{31}^{1}u_{1}^{s} + a_{32}^{1}v_{1}^{s} + a_{33}^{1}w_{1}^{s} + a_{34}^{1}w_{1,\bar{x}}^{s} &= H_{1}^{s} \\
a_{41}^{1}u_{1}^{s} + a_{42}^{1}v_{1}^{s} + a_{43}^{1}w_{1}^{s} + a_{44}^{1}w_{1,\bar{x}}^{s} &= M_{x_1}^{s}
\end{aligned}
\tag{79}
$$

and

$$
\begin{aligned}
a_{11}^{2}u_{2}^{s} + a_{12}^{2}v_{2}^{s} &= N_{x_2}^{s} \\
a_{21}^{2}u_{2}^{s} + a_{22}^{2}v_{2}^{s} &= N_{xy_2}^{s}
\end{aligned}
\tag{80}
$$

and

$$
\begin{aligned}
a_{11}^{3}u_{3}^{s} + a_{12}^{3}v_{3}^{s} + a_{13}^{3}w_{2}^{s} + a_{14}^{3}w_{2,\bar{x}}^{s} &= N_{x_3}^{s} \\
a_{21}^{3}u_{3}^{s} + a_{22}^{3}v_{3}^{s} + a_{23}^{3}w_{2}^{s} + a_{24}^{3}w_{2,\bar{x}}^{s} &= N_{xy_3}^{s} \\
a_{31}^{3}u_{3}^{s} + a_{32}^{3}v_{3}^{s} + a_{33}^{3}w_{2}^{s} + a_{34}^{3}w_{2,\bar{x}}^{s} &= H_{2}^{s} \\
a_{41}^{3}u_{3}^{s} + a_{42}^{3}v_{3}^{s} + a_{43}^{3}w_{2}^{s} + a_{44}^{3}w_{2,\bar{x}}^{s} &= M_{x_2}^{s}
\end{aligned}
\tag{81}
$$

and

$$
\begin{aligned}
a_{11}^{4}u_{4}^{s} + a_{12}^{4}v_{4}^{s} &= N_{x_4}^{s} \\
a_{21}^{4}u_{4}^{s} + a_{22}^{4}v_{4}^{s} &= N_{xy_4}^{s}
\end{aligned}
\tag{82}
$$

and the

2.5.2 General Elastic Boundary Conditions Valid in the Limit as $E_r \rightarrow \infty$ (fully clamped edges)

For n = 0

$$
\begin{aligned}
b_{11}^{o}H_{o}^{s} + b_{12}^{o}(M_{x_o}^{s} + \lambda\bar{q}) &= w_{o}^{s} + W_{v} + W_{p_e} + W_{t} \\
b_{21}^{o}H_{o}^{s} + b_{22}^{o}(M_{x_o}^{s} + \lambda\bar{q}) &= w_{o,\bar{x}}^{s}
\end{aligned}
\tag{83}
$$

For n $\geq$2

$$
\begin{aligned}
b_{11}^{1}N_{x_1}^{s} + b_{12}^{1}N_{xy_1}^{s} + b_{13}^{1}H_{1}^{s} + b_{14}^{1}M_{x_1}^{s} &= u_{1}^{s} \\
b_{21}^{1}N_{x_1}^{s} + b_{22}^{1}N_{xy_1}^{s} + b_{23}^{1}H_{1}^{s} + b_{24}^{1}M_{x_1}^{s} &= v_{1}^{s} \\
b_{31}^{1}N_{x_1}^{s} + b_{32}^{1}N_{xy_1}^{s} + b_{33}^{1}H_{1}^{s} + b_{34}^{1}M_{x_1}^{s} &= w_{1}^{s} \\
b_{41}^{1}N_{x_1}^{s} + b_{42}^{1}N_{xy_1}^{s} + b_{43}^{1}H_{1}^{s} + b_{44}^{1}M_{x_1}^{s} &= w_{1,\bar{x}}^{s}
\end{aligned}
\tag{84}
$$

and

$$b_{11}^2 N_{x_2}^s + b_{12}^2 N_{xy_2}^s = u_2^s$$
$$b_{21}^2 N_{x_2}^s + b_{22}^2 N_{xy_2}^s = v_2^s \qquad (85)$$

and

$$b_{11}^3 N_{x_3}^s + b_{12}^3 N_{xy_3}^s + b_{13}^3 H_2^s + b_{14}^3 M_{x_2}^s = u_3^s$$
$$b_{21}^3 N_{x_3}^s + b_{22}^3 N_{xy_3}^s + b_{23}^3 H_2^s + b_{24}^3 M_{x_2}^s = v_3^s$$
$$b_{31}^3 N_{x_3}^s + b_{32}^3 N_{xy_3}^s + b_{33}^3 H_2^s + b_{34}^3 M_{x_2}^s = w_2^s \qquad (86)$$
$$b_{41}^3 N_{x_3}^s + b_{42}^3 N_{xy_3}^s + b_{43}^3 H_2^s + b_{44}^3 M_{x_2}^s = w_{2,\overline{x}}^s$$

and

$$b_{11}^4 N_{x_4}^s + b_{12}^4 N_{xy_4}^s = u_4^s$$
$$b_{21}^4 N_{x_4}^s + b_{22}^4 N_{xy_4}^s = v_4^s \qquad (87)$$

Notice that in turn the **general elastic boundary conditions at the upper edge, at x = L**, are obtained by multiplying Eqs. (76) - (77) out. These operations yield the same type of expressions as Eqs. (78) - (87), with the exception that all right hand terms are preceded by a negative sign. It turns out that it is convenient to incorporate these negative signs in the definition of the ring stiffness coefficients a_{ij}^k and the ring flexibility coefficients b_{ij}^k, respectively. The components of the boundary stiffness and of the boundary flexibility matrices are listed in Appendix F.

2.6 Reduced General Elastic Boundary Conditions

Next the general elastic boundary conditions must be expressed in terms of the variables used in the anisotropic shell analysis. Recalling from Eq. (44) that

$$N_x^s = \frac{Et^2}{cR} \left(N_{x_o}^s + N_{x_1}^s \cos n\theta + N_{x_2}^s \cos 2n\theta + N_{x_3}^s \sin n\theta + N_{x_4}^s \sin 2n\theta\right) = F_{,yy} \qquad (88)$$

then substituting for $F_{,yy}$ from Eq. (10) and equating coefficients of like terms one gets

$$N_{x_o}^s = -\lambda$$
$$N_{x_1}^s = -n^2 f_1 \qquad N_{x_3}^s = -n^2 f_3 \qquad (89)$$
$$N_{x_2}^s = -4n^2 f_2 \qquad N_{x_4}^s = -4n^2 f_4$$

Recalling further from Eq. (44) that

$$N^s_{xy} = \frac{Et^2}{cR}\left(N^s_{xy_o} + N^s_{xy_1}\sin n\theta + N^s_{xy_2}\sin 2n\theta + N^s_{xy_3}\cos n\theta + N^s_{xy_4}\cos 2n\theta\right) = -F_{,xy} \qquad (90)$$

then substituting for $F_{,xy}$ from Eq. (10) and equating coefficients of like terms yields

$$\begin{aligned} &N^s_{xy_o} = \bar{\tau} \\ &N^s_{xy_1} = nf_1' \qquad N^s_{xy_3} = -\,nf_3' \\ &N^s_{xy_2} = 2nf_2' \qquad N^s_{xy_4} = -\,2nf_4' \end{aligned} \qquad (91)$$

where $(\ )' = (\ )_{,\bar{x}}$ and $\bar{x} = \frac{x}{R}$.

Using the expression derived in References [6] and [20] for the transverse shear force H^S one can write

$$H^s = \frac{t}{R^3}D\left(H^s_o + H^s_1\cos n\theta + H^s_2\sin n\theta\right) = M^s_{x,x} + (M^s_{xy} + M^s_{yx})_{,y} + (W_{,x} + \bar{W}_{,x})N^s_x + (W_{,y} + \bar{W}_{,y})N^s_{xy} \qquad (92)$$

From the semi-inverted constitutive equation (see Appendix A for details)

$$\begin{aligned} M^s_{x,x} &= (C^*_{11}N^s_x + C^*_{12}N^s_y + C^*_{16}N^s_{xy} + D^*_{11}\kappa_x + D^*_{12}\kappa_y + D^*_{16}\kappa_{xy})_{,x} \\ &= C^*_{11}F_{,xyy} + C^*_{12}F_{,xxx} - C^*_{16}F_{,xxy} - D^*_{11}W_{,xxx} - D^*_{12}W_{,xyy} - 2D^*_{16}W_{,xxy} \end{aligned} \qquad (93)$$

$$\begin{aligned} (M^s_{xy} + M^s_{yx})_{,y} &= 2(C^*_{61}N^s_x + C^*_{62}N^s_y + C^*_{66}N^s_{xy} + D^*_{16}\kappa_x + D^*_{26}\kappa_y + D^*_{66}\kappa_{xy})_{,y} \\ &= 2C^*_{61}F_{,yyy} + 2C^*_{62}F_{,xxy} - 2C^*_{66}F_{,xyy} - 2D^*_{16}W_{,xxy} - 2D^*_{26}W_{,yyy} - 4D^*_{66}W_{,xyy} \end{aligned} \qquad (94)$$

Substituting for $\bar{W}$, W and F from Eqs. (8)-(10), regrouping and equating coefficients of like terms one obtains

$$\begin{aligned} H^s_o = {} & 2\,\frac{R}{t}\,\bar{C}^*_{12}f_o''' - \bar{D}^*_{11}w_o''' - 4c\,\frac{R}{t}\,\lambda(w_o' + A_o') \\ & -2c\,\frac{R}{t}\,n^2\left[\,(w_1' + A_1')f_1 + (w_2' + A_2')f_3 + (w_1 + A_1)f_1' + (w_2 + A_2)f_3'\,\right] \end{aligned} \qquad (95)$$

$$\begin{aligned} H^s_1 = {} & 2\,\frac{R}{t}\left[\,\bar{C}^*_{12}f_1''' + (2\bar{C}^*_{62} - \bar{C}^*_{16})nf_3'' - (\bar{C}^*_{11} - 2\bar{C}^*_{66})n^2f_1' - 2\bar{C}^*_{61}n^3f_3\,\right] - 4c\,\frac{R}{t}\,\lambda(w_1' + A_1') \\ & -\left[\,\bar{D}^*_{11}w_1''' + 4\bar{D}^*_{16}nw_2'' - (\bar{D}^*_{12} + 4\bar{D}^*_{66})n^2w_1' - 2\bar{D}^*_{26}n^3w_2\,\right] + 4c\,\frac{R}{t}\,n\bar{\tau}(w_2 + A_2) \\ & -4c\,\frac{R}{t}\,n^2\left[\,(w_o' + A_o')f_1 + 2(w_1' + A_1')f_2 + 2(w_2' + A_2')f_4 + (w_1 + A_1)f_2' + (w_2 + A_2)f_4'\,\right] \end{aligned} \qquad (96)$$

$$H_2^s = 2\frac{R}{t}[\bar{C}^*_{12}f_3''' - (2\bar{C}^*_{62} - \bar{C}^*_{16})nf_1'' - (\bar{C}^*_{11} - 2\bar{C}^*_{66})n^2 f_3' + 2\bar{C}^*_{61}n^3 f_1] - 4c\frac{R}{t}\lambda(w_2' + A_2')$$
$$- [\bar{D}^*_{11}w_2''' - 4\bar{D}^*_{16}nw_1'' - (\bar{D}^*_{12} + 4\bar{D}^*_{66})n^2 w_2' + 2\bar{D}^*_{26}n^3 w_1] - 4c\frac{R}{t}n\bar{\tau}(w_1 + A_1) \tag{97}$$
$$- 4c\frac{R}{t}n^2[(w_o' + A_o')f_3 + 2(w_1' + A_1')f_4 - 2(w_2' + A_2')f_2 + (w_1 + A_1)f_4' - (w_2 + A_2)f_2']$$

But now from Eq. (20)

$$f_o''' = \frac{1}{2}\frac{t}{R}\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}w_o''' - \frac{c}{\bar{A}^*_{22}}w_o' + \frac{c}{4}\frac{t}{R}\frac{n^2}{\bar{A}^*_{22}}[w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)]'$$

Thus Eq. (95) becomes upon substitution and regrouping

$$H_o^s = -(\bar{D}^*_{11} - \frac{\bar{C}^*_{12}\bar{B}^*_{21}}{\bar{A}^*_{22}})w_o''' - 2c\frac{R}{t}\frac{\bar{C}^*_{12}}{\bar{A}^*_{22}}w_o' - 4c\frac{R}{t}\lambda(w_o' + A_o')$$
$$+ cn^2\frac{\bar{C}^*_{12}}{\bar{A}^*_{22}}(w_1w_1' + A_1w_1' + w_1A_1' + w_2w_2' + A_2w_2' + w_2A_2') \tag{98}$$
$$- 2c\frac{R}{t}n^2[(w_1' + A_1')f_1 + (w_2' + A_2')f_3 + (w_1 + A_1)f_1' + (w_2 + A_2)f_3']$$

Finally, recalling that $\bar{C}^*_{ij} = -\bar{B}^{*T}_{ij}$ one can write Eq. (98) and Eqs. (96)-(97) as

$$H_o^s = -(\bar{D}^*_{11} + \frac{\bar{B}^{*2}_{21}}{\bar{A}^*_{22}})w_o''' + 2c\frac{R}{t}\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}w_o' - 4c\frac{R}{t}\lambda(w_o' + A_o')$$
$$- cn^2\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}(w_1w_1' + A_1w_1' + w_1A_1' + w_2w_2' + A_2w_2' + w_2A_2') \tag{99}$$
$$- 2c\frac{R}{t}n^2[(w_1' + A_1')f_1 + (w_2' + A_2')f_3 + (w_1 + A_1)f_1' + (w_2 + A_2)f_3']$$

$$H_1^s = -2\frac{R}{t}[\bar{B}^*_{21}f_1''' + (2\bar{B}^*_{26} - \bar{B}^*_{61})nf_3'' - (\bar{B}^*_{11} - 2\bar{B}^*_{66})n^2 f_1' - 2\bar{B}^*_{16}n^3 f_3] - 4c\frac{R}{t}\lambda(w_1' + A_1')$$
$$- [\bar{D}^*_{11}w_1''' + 4\bar{D}^*_{16}nw_2'' - (\bar{D}^*_{12} + 4\bar{D}^*_{66})n^2 w_1' - 2\bar{D}^*_{26}n^3 w_2] + 4c\frac{R}{t}n\bar{\tau}(w_2 + A_2) \tag{100}$$
$$- 4c\frac{R}{t}n^2[(w_o' + A_o')f_1 + 2(w_1' + A_1')f_2 + 2(w_2' + A_2')f_4 + (w_1 + A_1)f_2' + (w_2 + A_2)f_4']$$

$$H_2^s = -2\frac{R}{t}[\,\bar{B}_{21}^* f_3''' - (2\bar{B}_{26}^* - \bar{B}_{61}^*)nf_1'' - (\bar{B}_{11}^* - 2\bar{B}_{66}^*)n^2 f_3' + 2\bar{B}_{16}^* n^3 f_1\,] - 4c\frac{R}{t}\lambda(w_2' + A_2')$$
$$-[\,\bar{D}_{11}^* w_2''' - 4\bar{D}_{16}^* nw_1'' - (\bar{D}_{12}^* + 4\bar{D}_{66}^*)n^2 w_2' + 2\bar{D}_{26}^* n^3 w_1\,] - 4c\frac{R}{t}n\bar{\tau}(w_1 + A_1) \quad (101)$$
$$-4c\frac{R}{t}n^2[\,(w_o' + A_o')f_3 + 2(w_1' + A_1')f_4 - 2(w_2' + A_2')f_2 + (w_1 + A_1)f_4' - (w_2 + A_2)f_2'\,]$$

Next combining the last Eq. (44) with the semi-inverted constitutive equation (see Appendix A for details) one can write

$$M_x^s = \frac{t}{R^2}D(M_{x_o}^s + M_{x_1}^s \cos n\theta + M_{x_2}^s \sin n\theta)$$
$$= C_{11}^* N_x^s + C_{12}^* N_y^s + C_{16}^* N_{xy}^s + D_{11}^* \kappa_x + D_{12}^* \kappa_y + D_{16}^* \kappa_{xy} \quad (102)$$
$$= C_{11}^* F_{,yy} + C_{12}^* F_{,xx} - C_{16}^* F_{,xy} - D_{11}^* W_{,xx} - D_{12}^* W_{,yy} - 2D_{16}^* W_{,xy}$$

Substituting for W and F from Eqs. (9)-(10), regrouping and equating coefficients of like terms one gets

$$M_{x_o}^s = -2\frac{R}{t}[\,\bar{C}_{11}^*\lambda + \bar{C}_{12}^*(\bar{p}_e - f_o'') - \bar{C}_{16}^*\bar{\tau}\,] - \bar{D}_{11}^* w_o'' \quad (103)$$

$$M_{x_1}^s = 2\frac{R}{t}(\bar{C}_{12}^* f_1'' - \bar{C}_{11}^* n^2 f_1 - \bar{C}_{16}^* nf_3') - (\bar{D}_{11}^* w_1'' - \bar{D}_{12}^* n^2 w_1 + 2\bar{D}_{16}^* nw_2') \quad (104)$$

$$M_{x_2}^s = 2\frac{R}{t}(\bar{C}_{12}^* f_3'' - \bar{C}_{11}^* n^2 f_3 + \bar{C}_{16}^* nf_1') - (\bar{D}_{11}^* w_2'' - \bar{D}_{12}^* n^2 w_2 - 2\bar{D}_{16}^* nw_1') \quad (105)$$

But now from Eq. (20)

$$f_o'' = \frac{1}{2}\frac{t}{R}\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}w_o'' - \frac{c}{\bar{A}_{22}^*}w_o + \frac{c}{4}\frac{t}{R}\frac{n^2}{\bar{A}_{22}^*}[\,w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\,]$$

thus Eq. (103) becomes upon substituting and regrouping

$$M_{x_o}^s = -\left(\bar{D}_{11}^* - \frac{\bar{C}_{12}^*\bar{B}_{21}^*}{\bar{A}_{22}^*}\right)w_o'' - 2c\frac{R}{t}\frac{\bar{C}_{12}^*}{\bar{A}_{22}^*}w_o - 2\frac{R}{t}(\bar{C}_{11}^*\lambda + \bar{C}_{12}^*\bar{p}_e - \bar{C}_{16}^*\bar{\tau})$$
$$+\frac{c}{2}n^2\frac{\bar{C}_{12}^*}{\bar{A}_{22}^*}[\,w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\,] \quad (106)$$

Finally, recalling that $\bar{C}_{ij}^* = -\bar{B}_{ij}^T$ one can write Eq. (106) and Eqs. (104)-(105) as

$$M^S_{x_0} = -\left(\bar{D}^*_{11} + \frac{\bar{B}^{*2}_{21}}{\bar{A}^*_{22}}\right) w_0'' + 2c\,\frac{R}{t}\,\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}\, w_0 + 2\,\frac{R}{t}\,(\bar{B}^*_{11}\lambda + \bar{B}^*_{21}\bar{p}_e - \bar{B}^*_{61}\bar{\tau})$$
$$- \frac{c}{2}\, n^2\, \frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}\,[\, w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\,] \tag{107}$$

$$M^S_{x_1} = -2\,\frac{R}{t}\,(\bar{B}^*_{21} f_1'' - \bar{B}^*_{11} n^2 f_1 - \bar{B}^*_{61} n f_3') - (\bar{D}^*_{11} w_1'' - \bar{D}^*_{12} n^2 w_1 + 2\bar{D}^*_{16} n w_2') \tag{108}$$

$$M^S_{x_2} = -2\,\frac{R}{t}\,(\bar{B}^*_{21} f_3'' - \bar{B}^*_{11} n^2 f_3 + \bar{B}^*_{61} n f_1') - (\bar{D}^*_{11} w_2'' - \bar{D}^*_{12} n^2 w_2 - 2\bar{D}^*_{16} n w_1') \tag{109}$$

Next one must express the in-plane shell displacements u^S and v^S in terms of the shell variables $\bar{W}$, W and F. Eliminating initially v from the strain-displacement relations

$$\varepsilon_y = v_{,y} - \frac{1}{R} W + \frac{1}{2}(W_{,y} + 2\bar{W}_{,y}) W_{,y} \tag{110}$$

$$\gamma_{xy} = u_{,y} + v_{,x} + (W_{,x} + \bar{W}_{,x}) W_{,y} + W_{,x}\bar{W}_{,y} \tag{111}$$

yields

$$u_{,yy} = \gamma_{xy,y} - \varepsilon_{y,x} - \frac{1}{R} W_{,x} - (W_{,x} + \bar{W}_{,x}) W_{,yy} - W_{,x}\bar{W}_{,yy} \tag{112}$$

Introducing the semi-inverted form of the constitutive equations (see also Appendix A) one gets

$$\begin{aligned} u_{,yy} = {} & (A^*_{16} F_{,yy} + A^*_{26} F_{,xx} - A^*_{66} F_{,xy} - B^*_{61} W_{,xx} - B^*_{62} W_{,yy} - 2B^*_{66} W_{,xy})_{,y} \\ & -(A^*_{12} F_{,yy} + A^*_{22} F_{,xx} - A^*_{16} F_{,xy} - B^*_{21} W_{,xx} - B^*_{22} W_{,yy} - 2B^*_{26} W_{,xy})_{,x} \\ & -\frac{1}{R} W_{,x} - (W_{,x} + \bar{W}_{,x}) W_{,yy} - W_{,x}\bar{W}_{,yy} \end{aligned} \tag{113}$$

Substituting for $\bar{W}$, W and F from Eqs. (8)-(10) one obtains after some regrouping

$$u_{,yy} = \frac{1}{R^2} u_{,\theta\theta} = \frac{t}{R^2}\{\tilde{U}_1(\bar{x})\cos n\theta + \tilde{U}_2(\bar{x})\cos 2n\theta + \tilde{U}_3(\bar{x})\sin n\theta + \tilde{U}_4(\bar{x})\sin 2n\theta\} \tag{114}$$

where

$$\begin{aligned} \tilde{U}_1(\bar{x}) = {} & \frac{1}{c}\{-[\bar{A}^*_{22} f_1''' - (\bar{A}^*_{12} + \bar{A}^*_{66}) n^2 f_1' - 2\bar{A}^*_{26} n f_3'' + \bar{A}^*_{16} n^3 f_3] + c\frac{t}{R} n^2 [(w_0' + A_0') w_1 + w_0' A_1] \\ & + \frac{1}{2}\frac{t}{R}[\bar{B}^*_{21} w_1''' - (\bar{B}^*_{22} - 2\bar{B}^*_{66}) n^2 w_1' + (2\bar{B}^*_{26} - \bar{B}^*_{61}) n w_2'' + \bar{B}^*_{62} n^3 w_2] - c w_1'\} \end{aligned} \tag{115}$$

$$\tilde{U}_2(\bar{x}) = \frac{1}{c}\{-[\bar{A}^*_{22}f_2''' - (\bar{A}^*_{12}+\bar{A}^*_{66})4n^2 f_2' - 4\bar{A}^*_{26}nf_4'' + 8\bar{A}^*_{16}n^3 f_4] + \frac{c}{2}\frac{t}{R}n^2[(w_1'+A_1')w_1 + w_1'A_1 - (w_2'+A_2')w_2 - w_2'A_2]\} \tag{115b}$$

$$\tilde{U}_3(\bar{x}) = \frac{1}{c}\{-[\bar{A}^*_{22}f_3''' - (\bar{A}^*_{12}+\bar{A}^*_{66})n^2 f_3' + 2\bar{A}^*_{26}nf_1'' - \bar{A}^*_{16}n^3 f_1] + c\frac{t}{R}n^2[(w_o'+A_o')w_2 + w_o'A_2] + \frac{1}{2}\frac{t}{R}[\bar{B}^*_{21}w_2''' - (\bar{B}^*_{22}-2\bar{B}^*_{66})n^2 w_2' - (2\bar{B}^*_{26}-\bar{B}^*_{61})nw_1'' - \bar{B}^*_{62}n^3 w_1] - cw_2'\} \tag{115c}$$

$$\tilde{U}_4(\bar{x}) = \frac{1}{c}\{-[\bar{A}^*_{22}f_4''' - (\bar{A}^*_{12}+\bar{A}^*_{66})4n^2 f_4 + 4\bar{A}^*_{26}nf_2'' - 8\bar{A}^*_{16}n^3 f_2] + \frac{c}{2}\frac{t}{R}n^2[(w_2'+A_2')w_1 + w_2'A_1 + (w_1'+A_1')w_2 + w_1'A_2]\} \tag{115d}$$

and where Eq. (20) has been used to obtain the following expression for f_o'''

$$f_o''' = \frac{1}{2}\frac{t}{R}\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}w_o''' - \frac{c}{\bar{A}^*_{22}}w_o' + \frac{c}{2}\frac{t}{R}\frac{n^2}{\bar{A}^*_{22}}[(w_1'+A_1')w_1 + w_1'A_1 + (w_2'+A_2')w_2 + w_2'A_2] \tag{116}$$

Integrating Eq. (114) twice with respect to θ yields

$$u = t\{-\frac{1}{n^2}\tilde{U}_1(\bar{x})\cos n\theta - \frac{1}{4n^2}\tilde{U}_2\cos 2n\theta - \frac{1}{n^2}\tilde{U}_3(\bar{x})\sin n\theta - \frac{1}{4n^2}\tilde{U}_4(\bar{x})\sin 2n\theta + \tilde{U}_5(\bar{x})\cdot\theta + \tilde{U}_6(\bar{x})\} \tag{117}$$

Notice that because of the periodicity condition

$$\int_0^{2\pi} u_{,\theta}\, d\theta = 0 \qquad \rightarrow \qquad \tilde{U}_5(\bar{x}) = 0 \tag{118}$$

In order to determine the form of the remaining function $\tilde{U}_6(\bar{x})$ one must consider an alternate derivation of u.

Using the strain-displacement equation

$$\varepsilon_x = u_{,x} + \frac{1}{2}(W_{,x} + 2\bar{W}_{,x})W_{,x} \tag{119}$$

and the appropriate semi-inverted constitutive equation one gets

$$u_{,x} = \varepsilon_x - \frac{1}{2}(W_{,x} + 2\bar{W}_{,x})W_{,x} \tag{120a}$$

$$u_{,x} = A^*_{11}F_{,yy} + A^*_{12}F_{,xx} - A^*_{16}F_{,xy} - B^*_{11}W_{,xx} - B^*_{12}W_{,yy} - 2B^*_{16}W_{,xy} - \frac{1}{2}(W_{,x} + 2\bar{W}_{,x})W_{,x} \tag{120b}$$

Substituting again for $\overline{W}$, W and F from Eqs. (8) - (10) one obtains after some regrouping

$$u_{,x}=\frac{1}{R}u'=\frac{t}{R}\{\hat{U}'_o(\bar{x})+\hat{U}'_1(\bar{x})\cos n\theta+\hat{U}'_2(\bar{x})\cos 2n\theta+\hat{U}'_3(\bar{x})\sin n\theta+\hat{U}'_4(\bar{x})\sin 2n\theta\} \tag{121}$$

where

$$\begin{aligned}\hat{U}'_o(\bar{x})&=\frac{1}{c}\{-(\bar{A}^*_{11}\lambda+\bar{A}^*_{12}\bar{p}_e-\bar{A}^*_{16}\bar{\tau})+\frac{1}{2}\frac{t}{R}(\bar{B}^*_{21}\frac{\bar{A}^*_{12}}{\bar{A}^*_{22}}-\bar{B}^*_{11})w''_o-c\frac{\bar{A}^*_{12}}{\bar{A}^*_{22}}w_o\\&+\frac{c}{4}\frac{t}{R}\frac{\bar{A}^*_{12}}{\bar{A}^*_{22}}n^2[(w_1+2A_1)w_1+(w_2+2A_2)w_2]-\frac{c}{4}\frac{t}{R}[2(w'_o+2A'_o)w'_o+(w'_1+2A'_1)w'_1+(w'_2+2A'_2)w'_2]\}\end{aligned} \tag{122a}$$

$$\begin{aligned}\hat{U}'_1(\bar{x})&=\frac{1}{c}\{(\bar{A}^*_{12}f''_1-\bar{A}^*_{11}n^2f_1-\bar{A}^*_{16}nf'_3)\\&-\frac{1}{2}\frac{t}{R}(\bar{B}^*_{11}w''_1-\bar{B}^*_{12}n^2w_1+2\bar{B}^*_{16}nw'_2)-\frac{c}{t}\frac{t}{R}[(w'_1+2A'_1)w'_o+(w'_o+2A'_o)w'_1]\}\end{aligned} \tag{122b}$$

$$\hat{U}'_2(\bar{x})=\frac{1}{c}\{(\bar{A}^*_{12}f''_2-4\bar{A}^*_{11}n^2f_2-2\bar{A}^*_{16}nf'_4)-\frac{c}{4}\frac{t}{R}[(w'_1+2A'_1)w'_1-(w'_2+2A'_2)w'_2]\} \tag{122c}$$

$$\begin{aligned}\hat{U}'_3(\bar{x})&=\frac{1}{c}\{(\bar{A}^*_{12}f''_3-\bar{A}^*_{11}n^2f_3+\bar{A}^*_{16}nf'_1)\\&-\frac{1}{2}\frac{t}{R}(\bar{B}^*_{11}w''_2-\bar{B}^*_{12}n^2w_2-2\bar{B}^*_{16}nw'_1)-\frac{c}{2}\frac{t}{R}[(w'_2+2A'_2)w'_o+(w'_o+2A'_o)w'_2]\}\end{aligned} \tag{122d}$$

$$\hat{U}_4(\bar{x})=\frac{1}{c}\{(\bar{A}^*_{12}f''_4-4\bar{A}^*_{11}n^2f_4+2\bar{A}^*_{16}nf'_2)-\frac{c}{4}\frac{t}{R}[(w'_2+2A'_2)w'_1+(w'_1+2A'_1)w'_2]\} \tag{122e}$$

and where Eq. (20) has been used to eliminate f''_o.

Integrating Eq. (120) once with respect to $\bar{x}$ yields

$$u=t\{\hat{U}_o(\bar{x})+\hat{U}_1(\bar{x})\cos n\theta+\hat{U}_2(\bar{x})\cos 2n\theta+\hat{U}_3(\bar{x})\sin n\theta+\hat{U}_4(\bar{x})\sin 2n\theta+\hat{U}_7(\theta)\} \tag{123}$$

Comparing the 2 expressions obtained for u (Eqs. (117) and (123)) one sees that they will be identical if and only if,

$$\hat{U}_o(\bar{x})=\tilde{U}_6(\bar{x})\ ;\ \hat{U}_1(\bar{x})=-\frac{1}{n^2}\tilde{U}_1(\bar{x})\ ;\ \hat{U}_2(\bar{x})=-\frac{1}{4n^2}\tilde{U}_2(\bar{x})\ ;\ \hat{U}_3(\bar{x})=-\frac{1}{n^2}\tilde{U}_3(\bar{x})\ ;\ \hat{U}_4(\bar{x})=-\frac{1}{4n^2}\tilde{U}_4(\bar{x}) \tag{124a}$$

and

$$\hat{U}_7(\theta)=0 \tag{124b}$$

Returning to Eq. (117) and substituting for $u \equiv u^s$ from Eq. (43) yields

$$\begin{aligned} t(u_o^s + u_1^s \cos n\theta + u_2^s \cos 2n\theta + u_3^s \sin n\theta + u_4^s \sin 2n\theta) \\ = t\{\tilde{U}_6(\bar{x}) - \frac{1}{n^2}\tilde{U}_1(\bar{x})\cos n\theta - \frac{1}{4n^2}\tilde{U}_2(\bar{x})\cos 2n\theta - \frac{1}{n^2}\tilde{U}_3(\bar{x})\sin n\theta - \frac{1}{4n^2}\tilde{U}_4(\bar{x})\,\mathrm{sins}2n\theta\} \end{aligned} \tag{125}$$

Equating coefficients of like terms one obtains

$$u_o^s = \tilde{U}_6(\bar{x}) = \hat{U}_o(\bar{x}) \tag{126a}$$

$$\begin{aligned} u_1^s = -\frac{1}{n^2}\tilde{U}_1(\bar{x}) = \frac{1}{cn^2}\{\bar{A}_{22}^* f_1''' - (\bar{A}_{12}^* + \bar{A}_{66}^*) n^2 f_1' - 2\bar{A}_{26}^* n f_3'' + \bar{A}_{16}^* n^3 f_3 + c w_1' \\ - \frac{1}{2}\frac{t}{R}[\bar{B}_{21}^* w_1''' - (\bar{B}_{22}^* - 2\bar{B}_{66}^*) n^2 w_1' + (2\bar{B}_{26}^* - \bar{B}_{61}^*) n w_2'' + \bar{B}_{62}^* n^3 w_2] - c\frac{t}{R} n^2 [(w_o' + A_o') w_1 + w_o' A_1]\} \end{aligned} \tag{126b}$$

$$\begin{aligned} u_2^s = -\frac{1}{4n^2}\tilde{U}_2(\bar{x}) = \frac{1}{4cn^2}\{\bar{A}_{22}^* f_2''' - (\bar{A}_{12}^* + \bar{A}_{66}^*) 4n^2 f_2' \\ - 4\bar{A}_{26}^* n f_4'' + 8\bar{A}_{16}^* n^3 f_4 - \frac{c}{2}\frac{t}{R} n^2 [(w_1' + A_1') w_1 + w_1' A_1 - (w_2' + A_2') w_2 - w_2' A_2]\} \end{aligned} \tag{126c}$$

$$\begin{aligned} u_3^s = -\frac{1}{n^2}\tilde{U}_3(\bar{x}) = \frac{1}{cn^2}\{\bar{A}_{22}^* f_3''' - (\bar{A}_{12}^* + \bar{A}_{66}^*) n^2 f_3' + 2\bar{A}_{26}^* n f_1'' - \bar{A}_{16}^* n^3 f_1 + c w_2' \\ - \frac{1}{2}\frac{t}{R}[\bar{B}_{21}^* w_2''' - (\bar{B}_{22}^* - 2\bar{B}_{66}^*) n^2 w_2' - (2\bar{B}_{26}^* - \bar{B}_{61}^*) n w_1'' - \bar{B}_{62}^* n^3 w_1] - c\frac{t}{R} n^2 [(w_o' + A_o') w_2 + w_o' A_2]\} \end{aligned} \tag{126d}$$

$$\begin{aligned} u_4^s = -\frac{1}{4n^2}\tilde{U}_4(\bar{x}) = \frac{1}{4cn^2}\{\bar{A}_{22}^* f_4''' - (\bar{A}_{12}^* + \bar{A}_{66}^*) 4n^2 f_4' \\ + 4\bar{A}_{26}^* n f_2'' - 8\bar{A}_{16}^* n^3 f_2 - \frac{c}{2}\frac{t}{R} n^2 [(w_2' + A_2') w_1 + w_2' A_1 + (w_1' + A_1') w_2 + w_1' A_2]\} \end{aligned} \tag{126e}$$

Thus once the solution of W and F has converged at a specified load level, one can then evaluate the above expressions for u. The only exception is u_o^s, the evaluation of which requires the solution of the following ordinary differential equation

$$\begin{aligned} \frac{d}{dx}u_o^s = -\frac{1}{c}(\bar{A}_{11}^*\lambda + \bar{A}_{12}^*\bar{p}_e - \bar{A}_{16}^*\bar{\tau}) + \frac{1}{2c}\frac{t}{R}(\bar{B}_{21}^*\frac{\bar{A}_{12}^*}{\bar{A}_{22}^*} - \bar{B}_{11}^*) w_o'' - \frac{\bar{A}_{12}^*}{\bar{A}_{22}^*} w_o \\ + \frac{1}{4}\frac{t}{R}\frac{\bar{A}_{12}^*}{\bar{A}_{22}^*} n^2 [(w_1 + 2A_1) w_1 + (w_2 + 2A_2) w_2] - \frac{1}{4}\frac{t}{R}[2(w_o' + 2A_o') w_o' + (w_1' + 2A_1') w_1' + (w_2' + 2A_2') w_2'] \end{aligned} \tag{127a}$$

$u_o^s(\bar{x}=0) = 0$ - to suppress rigid body motion (127b)

To obtain a similar expression for v^s one begins with the following strain-displacement relation

$$\varepsilon_y = v_{,y} - \frac{1}{R}W + \frac{1}{2}(W_{,y} + 2\bar{W}_{,y})W_{,y} \tag{110}$$

and introduces the appropriate semi-inverted form of the constitutive equation to obtain

$$\begin{aligned} v_{,y} &= \varepsilon_y + \frac{1}{R}W - \frac{1}{2}(W_{,y} + 2\bar{W}_{,y})W_{,y} \\ v_{,y} &= A_{12}^* F_{,yy} + A_{22}^* F_{,xx} - A_{26}^* F_{,xy} - B_{21}^* W_{,xx} - B_{22}^* W_{,yy} - 2B_{26}^* W_{,xy} + \frac{1}{R}W - \frac{1}{2}(W_{,y} + 2\bar{W}_{,y})W_{,y} \end{aligned} \tag{128}$$

Substituting for $\bar{W}$, W and F from Eqs. (8) - (10) one obtains after some regrouping

$$v_{,y} = \frac{1}{R}v_{,\theta} = \frac{t}{R}\{\tilde{V}_1(\bar{x})\cos n\theta + \tilde{V}_2(\bar{x})\cos 2n\theta + \tilde{V}_3(\bar{x})\sin n\theta + \tilde{V}_4(\bar{x})\sin 2n\theta\} \tag{129}$$

where

$$\tilde{V}_1(\bar{x}) = \frac{1}{c}\{(\bar{A}_{22}^* f_1'' - \bar{A}_{12}^* n^2 f_1 - \bar{A}_{26}^* n f_3') - \frac{1}{2}\frac{t}{R}(\bar{B}_{21}^* w_1'' - \bar{B}_{22}^* n^2 w_1 + 2\bar{B}_{26}^* n w_2') + c w_1\} \tag{130a}$$

$$\tilde{V}_2(\bar{x}) = \frac{1}{c}\{(\bar{A}_{22}^* f_2'' - 4\bar{A}_{12}^* n^2 f_2 - 2\bar{A}_{26}^* n f_4') + \frac{c}{4}\frac{t}{R}n^2[(w_1 + 2A_1)w_1 - (w_2 + 2A_2)w_2]\} \tag{130b}$$

$$\tilde{V}_3(\bar{x}) = \frac{1}{c}\{(\bar{A}_{22}^* f_3'' - \bar{A}_{12}^* n^2 f_3 + \bar{A}_{26}^* n f_1') - \frac{1}{2}\frac{t}{R}(\bar{B}_{21}^* w_2'' - \bar{B}_{22}^* n^2 w_2 - 2\bar{B}_{26}^* n w_1') + c w_2\} \tag{130c}$$

$$\tilde{V}_4(\bar{x}) = \frac{1}{c}\{(\bar{A}_{22}^* f_4'' - 4\bar{A}_{12}^* n^2 f_4 + 2\bar{A}_{26}^* n f_2') + \frac{c}{4}\frac{t}{R}n^2[(w_2 + 2A_2)w_1 + (w_1 + 2A_1)w_2]\} \tag{130d}$$

and where Eq. (20) has been used to eliminate f_o''.

Integrating Eq. (129) once with respect to θ yields

$$v = t\{\frac{1}{n}\tilde{V}_1(\bar{x})\sin n\theta + \frac{1}{2n}\tilde{V}_2(\bar{x})\sin 2n\theta - \frac{1}{n}\tilde{V}_3(\bar{x})\cos n\theta - \frac{1}{2n}\tilde{V}_4(\bar{x})\cos 2n\theta + \tilde{V}_5(\bar{x})\} \tag{131}$$

In order to determine the form of the unknown function $\tilde{V}_5(\bar{x})$ one must consider an alternate derivation of v. Solving for $v_{,x}$ from the strain-displacement relation for γ_{xy} (see Eq. (111)) yields

$$v_{,x} = \gamma_{xy} - u_{,y} - (W_{,x} + \bar{W}_{,x})W_{,y} - W_{,x}\bar{W}_{,y} \tag{132}$$

Introducing for γ_{xy} the appropriate semi-inverted form of the constitutive equation one gets

$$v_{,x} = A_{16}^* F_{,yy} + A_{26}^* F_{,xx} - A_{66}^* F_{,xy} - B_{61}^* W_{,xx} - B_{62}^* W_{,yy} - 2B_{66}^* W_{,xy} - u_{,y} - (W_{,x} + \bar{W}_{,x})W_{,y} - W_{,x}\bar{W}_{,y} \tag{133}$$

Substituting for $\bar{W}$, W and F from Eqs. (8) - (10) and for u from Eq. (123) one obtains after some regrouping

$$v_{,x}=\frac{1}{R}v'=\frac{t}{R}\{\hat{V}'_o(\bar{x})+\hat{V}'_1(\bar{x})\sin n\theta+\hat{V}'_2\sin 2n\theta+\hat{V}'_3(\bar{x})\cos n\theta+\hat{V}'_4(\bar{x})\cos 2n\theta\} \tag{134}$$

where

$$\hat{V}'_o(\bar{x})=\frac{1}{c}\{-(\bar{A}^*_{16}\lambda+\bar{A}^*_{26}\bar{p}_e-\bar{A}^*_{66}\bar{\tau})+\frac{1}{2}\frac{t}{R}(\bar{B}^*_{21}\frac{\bar{A}^*_{26}}{\bar{A}^*_{22}}-\bar{B}^*_{61})w''_o-c\frac{\bar{A}^*_{26}}{\bar{A}^*_{22}}w_o$$
$$+\frac{c}{4}\frac{t}{R}\frac{\bar{A}^*_{26}}{\bar{A}^*_{22}}n^2[(w_1+2A_1)w_1+(w_2+2A_2)w_2]-\frac{c}{2}\frac{t}{R}n[(w'_1+A'_1)w_2+w'_1A_2-(w'_2+A'_2)w_1-w'_2A_1]\} \tag{135a}$$

$$\hat{V}'_1(\bar{x})=\frac{1}{c}\{(\bar{A}^*_{26}f''_3-\bar{A}^*_{16}n^2f_3+\bar{A}^*_{66}nf'_1)$$
$$-\frac{1}{2}\frac{t}{R}(\bar{B}^*_{61}w''_2-\bar{B}^*_{62}n^2w_2-2\bar{B}^*_{66}nw'_1)-\frac{c}{n}\tilde{U}_1+c\frac{t}{R}n[(w'_o+A'_o)w_1+w'_oA_1]\} \tag{135b}$$

$$\hat{V}'_2(\bar{x})=\frac{1}{c}\{(\bar{A}^*_{26}f''_4-4\bar{A}^*_{16}n^2f_4+2\bar{A}^*_{66}nf'_2)$$
$$-\frac{c}{2n}\tilde{U}_2+\frac{c}{2}\frac{t}{R}n[(w'_1+A'_1)w_1+w'_1A_1-(w'_2+A'_2)w_2-w'_2A_2]\} \tag{135c}$$

$$\hat{V}'_3(\bar{x})=\frac{1}{c}\{(\bar{A}^*_{26}f''_1-\bar{A}^*_{16}n^2f_1-\bar{A}^*_{66}nf'_3)$$
$$-\frac{1}{2}\frac{t}{R}(\bar{B}^*_{61}w''_1-\bar{B}^*_{62}n^2w_1+2\bar{B}^*_{66}nw'_2)+\frac{c}{n}\tilde{U}_3-c\frac{t}{R}n[(w'_o+A'_o)w_2+w'_oA_2]\} \tag{135d}$$

$$\hat{V}'_4(\bar{x})=\frac{1}{c}\{(\bar{A}^*_{26}f''_2-4\bar{A}^*_{16}n^2f_2-2\bar{A}^*_{66}nf'_4)$$
$$+\frac{c}{2n}\tilde{U}_4-\frac{c}{2}\frac{t}{R}n[(w'_2+A'_2)w_1+w'_2A_1+(w'_1+A'_1)w_2+w'_1A_2]\} \tag{135e}$$

and where Eq. (20) has been used to eliminate f''_o.

Integrating Eq. (134) once with respect to $\bar{x}$ yields

$$v=t\{\hat{V}_o(\bar{x})+\hat{V}_1(\bar{x})\sin n\theta+\hat{V}_2(\bar{x})\sin 2n\theta+\hat{V}_3(\bar{x})\cos n\theta+\hat{V}_4(\bar{x})\cos 2n\theta+\hat{V}_6(\theta)\} \tag{136}$$

Comparing the 2 expressions obtained for v (Eqs. (131) and (136)) one observes that they will be identical if and only if,

$$\hat{V}_o(\bar{x})=\tilde{V}_5(\bar{x})\ ;\ \hat{V}_1(\bar{x})=\frac{1}{n}\tilde{V}_1(\bar{x})\ ;\ \hat{V}_2(\bar{x})=\frac{1}{2n}\tilde{V}_2(\bar{x})\ ;\ \hat{V}_3(\bar{x})=-\frac{1}{n}\tilde{V}_3(\bar{x})\ ;\ \hat{V}_4(\bar{x})=-\frac{1}{2n}\tilde{V}_4(\bar{x}) \tag{137a}$$

and

$$\hat{V}_6(\theta) = 0 \tag{137b}$$

Returning to Eq. (131) and substituting for $v \equiv v^S$ from Eq. (43)

$$\begin{aligned} & t(v_o^S + v_1^S \sin n\theta + v_2^S \sin 2n\theta + v_3^S \cos n\theta + v_4^S \cos 2n\theta) \\ & \quad = t\{\tilde{V}_5(x) + \frac{1}{n}\tilde{V}_1(\bar{x}) \sin sn\theta + \frac{1}{2n}\tilde{V}_2(\bar{x}) 2n\theta - \frac{1}{n}\tilde{V}_3(\bar{x}) \cos n\theta - \frac{1}{2n}\tilde{V}_4(\bar{x}) \cos 2n\theta\} \end{aligned} \tag{138}$$

Equating coefficients of like terms one obtains

$$v_o^S = \tilde{V}_5(x) = \hat{V}_o(\bar{x}) \tag{139a}$$

$$v_1^S = \frac{1}{n}\tilde{V}_1(\bar{x}) = \frac{1}{cn}\{\bar{A}_{22}^* f_1'' - \bar{A}_{12}^* n^2 f_1 - \bar{A}_{26}^* n f_3' - \frac{1}{2}\frac{t}{R}(\bar{B}_{21}^* w_1'' - \bar{B}_{22}^* n^2 w_1 + 2\bar{B}_{26}^* n w_2') + c w_1\} \tag{139b}$$

$$v_2^S = \frac{1}{2n}\tilde{V}_2(\bar{x}) = \frac{1}{2cn}\{\bar{A}_{22}^* f_2'' - 4\bar{A}_{12}^* n^2 f_2 - 2\bar{A}_{26}^* n f_4' + \frac{c}{4}\frac{t}{R} n^2 [(w_1 + 2A_1) w_1 - (w_2 + 2A_2) w_2]\} \tag{139c}$$

$$v_3^S = -\frac{1}{n}\tilde{V}_3(\bar{x}) = -\frac{1}{cn}\{\bar{A}_{22}^* f_3'' - \bar{A}_{12}^* n^2 f_3 + \bar{A}_{26}^* n f_1' - \frac{1}{2}\frac{t}{R}(\bar{B}_{21}^* w_2'' - \bar{B}_{22}^* n^2 w_2 - 2\bar{B}_{26}^* n w_1') + c w_2\} \tag{139d}$$

$$v_4^S = -\frac{1}{2n}\tilde{V}_4(\bar{x}) = -\frac{1}{2cn}\{\bar{A}_{22}^* f_4'' - 4\bar{A}_{12}^* n^2 f_4 + 2\bar{A}_{26}^* n f_2' + \frac{c}{4}\frac{t}{R} n^2 [(w_2 + 2A_2) w_1 + (w_1 + 2A_1) w_2]\} \tag{139e}$$

Thus once the solution of W and F has converged at a specified load level, one can then evaluate the above expressions for v. The only exception is v_o^S, the evaluation of which requires the solution of the following ordinary differential equation

$$\begin{aligned} \frac{d}{d\bar{x}} v_o^S = & -\frac{1}{c}(\bar{A}_{16}^* \lambda + \bar{A}_{26}^* \bar{p}_e - \bar{A}_{66}^* \bar{\tau}) + \frac{1}{2c}\frac{t}{R}\left(\bar{B}_{21}^* \frac{\bar{A}_{26}^*}{\bar{A}_{22}^*} - \bar{B}_{61}^*\right) w_o'' - \frac{\bar{A}_{26}^*}{\bar{A}_{22}^*} w_o \\ & + \frac{1}{4}\frac{t}{R}\frac{\bar{A}_{26}^*}{\bar{A}_{22}^*} n^2 [(w_1 + 2A_1) w_1 + (w_2 + 2A_2) w_2] - \frac{1}{2}\frac{t}{R} n [(w_1' + A_1') w_2 + w_1' A_2 - (w_2' + A_2') w_1 - w_2' A_1] \end{aligned} \tag{140a}$$

$$v_o^S(\bar{x} = 0) = 0 \qquad \text{- to suppress rigid body motion} \tag{140b}$$

Turning now to the general elastic boundary conditions, in the limiting case as $E_r \to 0$ one can regroup Eqs. (78)-(82) as follows

$$
\begin{aligned}
h_1 &= a_{11}^{o}(w_o^{s} + W_v + W_{p_e} + W_t) + a_{12}^{o} w_{o,\bar{x}}^{s} - H_o^{s} = 0 \\
h_2 &= a_{21}^{o}(w_o^{s} + W_v + W_{p_e} + W_t) + a_{22}^{o} w_{o,\bar{x}}^{s} - (M_{x_o}^{s} + \lambda\bar{q}) = 0 \\
h_3 &= a_{11}^{1}u_1^{s} + a_{12}^{1}v_1^{s} + a_{13}^{1}w_1^{s} + a_{14}^{1}w_{1,\bar{x}}^{s} - N_{x_1}^{s} = 0 \\
h_4 &= a_{21}^{1}u_1^{s} + a_{22}^{1}v_1^{s} + a_{23}^{1}w_1^{s} + a_{24}^{1}w_{1,\bar{x}}^{s} - N_{xy_1}^{s} = 0 \\
h_5 &= a_{31}^{1}u_1^{s} + a_{32}^{1}v_1^{s} + a_{33}^{1}w_1^{s} + a_{34}^{1}w_{1,\bar{x}}^{s} - H_1^{s} = 0 \\
h_6 &= a_{41}^{1}u_1^{s} + a_{42}^{1}v_1^{s} + a_{43}^{1}w_1^{s} + a_{44}^{1}w_{1,\bar{x}}^{s} - M_{x_1}^{s} = 0 \\
h_7 &= a_{11}^{2}u_2^{s} + a_{12}^{2}v_2^{s} - N_{x_2}^{s} = 0 \\
h_8 &= a_{21}^{2}u_2^{s} + a_{22}^{2}v_2^{s} - N_{xy_2}^{s} = 0 \\
h_9 &= a_{11}^{3}u_3^{s} + a_{12}^{3}v_3^{s} + a_{13}^{3}w_2^{s} + a_{14}^{3}w_{2,\bar{x}}^{s} - N_{x_3}^{s} = 0 \\
h_{10} &= a_{21}^{3}u_3^{s} + a_{22}^{3}v_3^{s} + a_{23}^{3}w_2^{s} + a_{24}^{3}w_{2,\bar{x}}^{s} - N_{xy_3}^{s} = 0 \\
h_{11} &= a_{31}^{3}u_3^{s} + a_{32}^{3}v_3^{s} + a_{33}^{3}w_2^{s} + a_{34}^{3}w_{2,x}^{s} - H_2^{s} = 0 \\
h_{12} &= a_{41}^{3}u_3^{s} + a_{42}^{3}v_3^{s} + a_{43}^{3}w_2^{s} + a_{44}^{3}w_{2,\bar{x}}^{s} - M_{x_2}^{s} = 0 \\
h_{13} &= a_{11}^{4}u_4^{s} + a_{12}^{4}v_4^{s} - N_{x_4}^{s} = 0 \\
h_{14} &= a_{21}^{4}u_4^{s} + a_{22}^{4}v_4^{s} - N_{xy_4}^{s} = 0
\end{aligned}
\tag{141}
$$

Notice that upon replacing the shell variables involved u^{s}, v^{s}, w^{s}, $w_{,\bar{x}}^{s}$ and N_x^{s}, N_{xy}^{s}, H^{s}, M_x^{s} by their equivalent expressions in terms of the variables used in the anisotropic shell analysis $\bar{W}$, W and F derived earlier, one obtains a set of highly nonlinear boundary conditions. In short these equations shall be denoted as

$$\mathbf{h}(\bar{x}, \mathbf{Y}(x) \; ; \lambda, p_e, \bar{\tau}) = 0 \tag{142}$$

An alternate set of nonlinear boundary conditions can be obtained by rewriting the general elastic boundary conditions in the limiting case as $E_r \to \infty$ given by Eqs. (83)-(87) as follows

$$g_1 = b^o_{11}H^s_o + b^o_{12}(M^s_{x_o} + \lambda\bar{q}) - (w^s_o + W_v + W_{p_e} + W_t) = 0$$

$$g_2 = b^o_{21}H^s_o + b^o_{22}(M^s_{x_o} + \lambda\bar{q}) - w^s_{o,\bar{x}} = 0$$

$$g_3 = b^1_{11}N^s_{x_1} + b^1_{12}N^s_{xy_1} + b^1_{13}H^s_1 + b^1_{14}M^s_{x_1} - u^s_1 = 0$$

$$g_4 = b^1_{21}N^s_{x_1} + b^1_{22}N^s_{xy_1} + b^1_{23}H^s_1 + b^1_{24}M^s_{x_1} - v^s_1 = 0$$

$$g_5 = b^1_{31}N^s_{x_1} + b^1_{32}N^s_{xy_1} + b^1_{33}H^s_1 + b_{34}M^s_{x_1} - w^s_1 = 0$$

$$g_6 = b^1_{41}N^s_{x_1} + b^1_{42}N^s_{xy_1} + b^1_{43}H^s_1 + b^1_{44}M^s_{x_1} - w^s_{1,\bar{x}} = 0$$

$$g_7 = b^2_{11}N^s_{x_2} + b^2_{12}N^s_{xy_2} - u^s_2 = 0$$

$$g_8 = b^2_{21}N^s_{x_2} + b^2_{22}N^s_{xy_2} - v^s_2 = 0 \qquad (143)$$

$$g_9 = b^3_{11}N^s_{x_3} + b^3_{12}N^s_{xy_3} + b^3_{13}H^s_2 + b^3_{14}M^s_{x_2} - u^s_3 = 0$$

$$g_{10} = b^3_{21}N^s_{x_3} + b^3_{22}N^s_{xy_3} + b^3_{23}H^s_2 + b^3_{24}M^s_{x_2} - v^s_3 = 0$$

$$g_{11} = b^3_{31}N^s_{x_3} + b^s_{32}N^s_{xy_3} + b^3_{33}H^s_2 + b^3_{34}M^s_{x_2} - w^s_2 = 0$$

$$g_{12} = b^3_{41}N^s_{x_3} + b^3_{42}N^s_{xy_3} + b^3_{43}H^s_2 + b^3_{44}M^s_{x_2} - w^s_{2,\bar{x}} = 0$$

$$g_{13} = b^4_{11}N^s_{x_4} + b^4_{12}N^s_{xy_4} - u^s_4 = 0$$

$$g_{14} = b^4_{21}N^s_{x_4} + b^4_{22}N^s_{xy_4} - v^s_4 = 0$$

Notice that upon replacing the shell variables involved u^s, v^s, w^s, $w^s_{,\bar{x}}$ and N^s_x, N^s_{xy}, H^s, M^s_x by their equivalent expressions in terms of the variables used in the anisotropic shell analysis $\bar{W}$, W and F derived earlier, one obtains indeed an alternate set of highly nonlinear boundary conditions. In short the equations shall be denoted as

$$\mathbf{g}(\bar{x}, \mathbf{Y}(\bar{x}) \; ; \lambda, \bar{p}_e, \bar{\tau}) = 0 \qquad (144)$$

3. NUMERICAL ANALYSIS

All the known numerical techniques for the solution of nonlinear equations involve iterative improvements of initial guesses of the solution. Working with different shell configurations[15,16] it was found that it is more efficient to adjust the initial guesses of the solution at a limited number of "matching points" rather than trying to adjust the solution at all points along the interval simultaneously, as required by the standard finite difference schemes. Thus, in the following, the problem will be cast into a form that is suited for the so-called "Parallel Shooting Method"[22] that will be used for obtaining numerical solutions.

3.1 The Governing Equations in Terms of the Vector Variable $\underset{\sim}{Y}$

Introducing now the 28-dimensional vector variable $\underset{\sim}{Y}$ defined as follows

$$
\begin{array}{llll}
Y_1 = f_1 & Y_8 = f_1' & Y_{15} = f_1'' & Y_{22} = f_1''' \\
Y_2 = f_2 & Y_9 = f_2' & Y_{16} = f_2'' & Y_{23} = f_2''' \\
Y_3 = f_3 & Y_{10} = f_3' & Y_{17} = f_3'' & Y_{24} = f_3''' \\
Y_4 = f_4 & Y_{11} = f_4' & Y_{18} = f_4'' & Y_{25} = f_4''' \\
Y_5 = w_o & Y_{12} = w_o' & Y_{19} = w_o'' & Y_{26} = w_o''' \\
Y_6 = w_1 & Y_{13} = w_1' & Y_{20} = w_1'' & Y_{27} = w_1''' \\
Y_7 = w_2 & Y_{14} = w_2' & Y_{21} = w_2'' & Y_{28} = w_2'''
\end{array}
\tag{145}
$$

where $(\ )' = d/d\bar{x}\ (\ )$, then the system of governing equations (Eqs. (21)-(27)) and the general elastic boundary conditions (Eqs. (142) and (144)) can be reduced to the following nonlinear 2-point boundary value problem

$$\frac{d}{d\bar{x}} \underset{\sim}{Y} = \underset{\sim}{f}(\bar{x}, \underset{\sim}{Y} ; \lambda, \bar{p}_e, \bar{\tau}) \qquad \text{for } 0 \le \bar{x} \le \frac{L}{R} \tag{146}$$

$$\underset{\sim}{g}(\bar{x} = 0, \underset{\sim}{Y}(o) ; \lambda, \bar{p}_e, \bar{\tau}) = 0 \qquad \text{at } \bar{x} = 0 \tag{147}$$

$$\underset{\sim}{h}(\bar{x} = \frac{L}{R}, \underset{\sim}{Y}(\frac{L}{R}) ; \lambda, \bar{p}_e, \bar{\tau}) = 0 \qquad \text{at } \bar{x} = \frac{L}{R} \tag{148}$$

where the general nonlinear boundary conditions are specified by the 14-dimensional vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$. The solution of this nonlinear 2-point boundary value problem will then locate the limit point of the prebuckling state, whereby one of the 3 possible load parameters $(\lambda, \bar{p}_e, \bar{\tau})$ is chosen as the variable load Λ. The remaining two load parameters are assigned fixed values. By definition, the value of the variable load parameter Λ corresponding to the limit point will be the theoretical buckling load Λ_S.

As can be seen from Fig. 6, using load increments $\Delta\Lambda$ the solution fails to converge close to and beyond the limit point. This situation is somewhat unsatisfactory, especially since without the appropriate starting values the nonlinear iteration scheme will fail to converge also at load levels less than the theoretical buckling load. A closer look at the solution curve presented in Fig. 6 reveals, however, that one should be able to extend the response curve around the limit point by using increments of the appropriate deformation parameter instead of increments in loading.

Using the concept of generalized load-displacement relationships defined initially by Budiansky[23], if the variable load Λ is axial compression, that is $\Lambda = \lambda$, then the appropriate generalized displacement is the unit end-shortening defined as

$$\varepsilon = \frac{-1}{2\pi RL} \int_0^{2\pi R} \int_0^{L} (u_{,x} - qW_{,xx})\, dxdy \tag{149}$$

Notice that the significance of ε is, that the product $\lambda \cdot \varepsilon$ represents the decrease in potential energy of the applied load.

Substituting for $u_{,x}$ from Eq. (120b), for $W_{,xx}$ from Eq. (9) and carrying out the y-integration one obtains after some regrouping

$$\begin{aligned}\delta = \frac{\varepsilon}{\varepsilon_{c\ell}} = \bar{A}^*_{11}\lambda + \bar{A}^*_{12}\bar{p}_e - \bar{A}^*_{16}\bar{\tau} + \frac{1}{2}\frac{t}{L}(\bar{B}^*_{11} - \bar{B}^*_{21}\frac{\bar{A}^*_{12}}{\bar{A}^*_{22}} + \frac{1}{2}\frac{t}{R}\bar{q}) \int_0^{L/R} w_o'' d\bar{x} + c\frac{R}{L}\frac{\bar{A}^*_{12}}{\bar{A}^*_{22}} \int_0^{L/R} w_o d\bar{x} \\ - \frac{c}{4}\frac{t}{L}\frac{\bar{A}^*_{12}}{\bar{A}^*_{22}} n^2 \int_0^{L/R} \{ w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2) \} d\bar{x} \\ + \frac{c}{2}\frac{t}{L} \int_0^{L/R} \{ (w_o' + 2A_o')w_o' + \frac{1}{2} [(w_1' + 2A_1')w_1' + (w_2' + 2A_2')w_2'] \} d\bar{x}\end{aligned} \tag{150}$$

where

$$\varepsilon_{c\ell} = \frac{1}{c}\frac{t}{R} \tag{151}$$

Notice that the terms involving the integrals represent the nonlinear part of the end-shortening. Once the solution of the boundary value problem (146) has converged, it is advantageous to evaluate $\delta^{n\ell}$ by solving the associated initial value problems, rather than using numerical integration schemes. Here one must solve

$$\begin{aligned}\frac{d}{d\bar{x}}\delta^{n\ell} = C_{50}w_o'' + C_{51}w_o - C_{52} [w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)] \\ + C_{53} \{ w_o'(w_o' + 2A_o') + \frac{1}{2} [w_1'(w_1' + 2A_1') + w_2'(w_2' + 2A_2')] \}\end{aligned} \tag{152}$$

$$\delta^{n\ell}(o) = 0 \tag{153a}$$

$$\delta^{n\ell}\left(\frac{L}{R}\right) = \delta_o \tag{153b}$$

where δ_o is the prescribed "end-shortening".

If, however, the variable load Λ is external pressure, that is $\Lambda = \bar{p}_e$, then following Hutchinson[24] the appropriate generalized displacement is the average normal displacement defined as

$$W_{ave} = \frac{1}{2\pi RL} \int_0^{2\pi R} \int_0^{L} W \, dxdy \tag{154}$$

Substituting for W from Eq. (9) and carrying out the y-integration one obtains after some regrouping

$$\bar{W}_{ave} = \frac{W_{ave}}{W_{c\ell}} = \bar{A}^*_{12}\lambda + \bar{A}^*_{22}\bar{p}_e - \bar{A}^*_{26}\bar{\tau} + c\,\frac{R}{L}\int_0^{L/R} w_o d\bar{x} \tag{155}$$

where

$$W_{c\ell} = \frac{t}{c} \tag{156}$$

The term involving the integral represents the nonlinear part of the average normal displacement. Notice that the significance of $\bar{W}_{ave}$ is, that the product $\bar{p}_e \cdot \bar{W}_{ave}$ represents the decrease in potential energy of the applied load. When the solution of the boundary value problem (146) has converged, it is advantageous to evaluate $\bar{W}_{ave}$ by solving the associated initial value problems, rather than using numerical integration schemes. Here one must solve

$$\frac{d}{d\bar{x}}\bar{W}_{ave} = \frac{cR}{L} w_o \tag{157}$$

$$\bar{W}_{ave}(o) = 0 \tag{158a}$$

$$\bar{W}_{ave}\left(\frac{x}{L}\right) = \bar{W}^{(o)}_{ave} \tag{158b}$$

where $\bar{W}^{(o)}_{ave}$ is the prescribed "average normal displacement".

If, on the other hand, the variable load Λ is torsion, that is $\Lambda = \bar{\tau}$, then the appropriate generalized displacement is the apparent shearing strain[23] defined as

$$\gamma = \frac{1}{2\pi RL} \int_0^{2\pi R} \int_0^{L} v_{,x} \, dxdy \tag{159}$$

Substituting for $v_{,x}$ from Eq. (134) and carrying out the y-integration one obtains after some regrouping

$$\begin{aligned} \bar{\gamma} = \frac{\gamma}{\gamma_{c\ell}} = (-\bar{A}^*_{16}\lambda - \bar{A}^*_{26}\bar{p}_e + \bar{A}^*_{66}\bar{\tau}) + \frac{1}{2}\frac{t}{L}\left(\bar{B}^*_{21}\frac{\bar{A}^*_{26}}{\bar{A}^*_{22}} - \bar{B}^*_{61}\right)\int_0^{L/R} w_o'' d\bar{x} - \frac{cR}{L}\frac{\bar{A}^*_{26}}{\bar{A}^*_{22}}\int_0^{L/R} w_o d\bar{x} \\ + \frac{c}{4}\frac{t}{L}\frac{\bar{A}^*_{26}}{\bar{A}^*_{22}} n^2 \int_0^{L/R} \{ w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2) \} d\bar{x} \\ - \frac{c}{2}\frac{t}{L} n \int_0^{L/R} \{ w_1'(w_2 + A_2) + A_1' w_2 - [w_2'(w_1 + A_1) + A_2' w_1] \} d\bar{x} \end{aligned} \tag{160}$$

where

$$\gamma_{c\ell} = \frac{1}{c}\,\frac{t}{R} \tag{161}$$

The terms involving the integrals represent the nonlinear part of the apparent shearing strain. Notice that the significance of $\bar{\gamma}$ is, that the product $\bar{\tau} \cdot \bar{\gamma}$ represents the decrease in potential energy of the applied load. Once the solution of the boundary value problem (146) has converged, it is advantageous to evaluate $\bar{\gamma}$ by solving the associated initial value problems, rather then using numerical integration schemes. Here one must solve

$$\frac{d}{dx}\bar{\gamma}^{n\ell} = C_{550}w_o'' - C_{551}w_o + C_{552}\,[\,w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)\,] \\ - C_{553}\,\{\,w_1'(w_2 + A_2) + A_1'w_2 - [\,w_2'(w_1 + A_1) + A_2'w_1\,]\,\} \tag{162}$$

$$\bar{\gamma}^{n\ell}(o) = 0 \tag{163a}$$

$$\bar{\gamma}^{n\ell}\left(\frac{L}{R}\right) = \bar{\gamma}^{o} \tag{163b}$$

where $\bar{\gamma}^{o}$ is the prescribed "apparent shearing strain".

One of these initial value problems, expressed in terms of the unified vector variable $\underset{\sim}{Y}$, represents the additional equation needed when solving the 2-point boundary value problem (Eq. (146)) using increments of the appropriate deformation parameter instead of increments in applied loading[16].

3.2 Numerical Analysis using Load Increments

Due to the highly nonlinear nature of the problem anything but a numerical solution is out of question. Trying to make use of the readily available coded subroutines for solving nonlinear initial value problems lead to the decision to employ the "shooting method" to solve the nonlinear 2-point boundary value problem represented by Eq. (146). Though, due to the numerical instabilities, it may become necessary to employ "parallel shooting" over (say) 8 intervals to carry out the integration over shell lengths greater than L/R = 1.0, for the purpose of describing the method let us consider just "parallel shooting" over 4 intervals.

Let us associate the following 4 initial value problems with the above 2-point boundary value problem

$$\frac{d}{dx}\underset{\sim}{U}_s = \underset{\sim}{f}(\bar{x}, \underset{\sim}{U}_s\,;\,\lambda, \bar{p}_e, \bar{\tau}) \qquad 0 \le \bar{x} \le \bar{x}_1 \qquad \underline{\text{Forward Integration}}$$

$$\underset{\sim}{U}_s(0) = \underset{\sim}{s} \tag{164}$$

$$\frac{d}{dx}\underset{\sim}{V}_p = \underset{\sim}{f}(\bar{x}, \underset{\sim}{V}_p\,;\,\lambda, \bar{p}_e, \bar{\tau}) \qquad \bar{x}_1 \le \bar{x} \le \bar{x}_2 \qquad \underline{\text{Backward Integration}}$$

$$\underset{\sim}{V}_p(\bar{x}_2) = \underset{\sim}{p} \tag{165}$$

$$\frac{d}{d\bar{x}} \underset{\sim}{U}_p = \underset{\sim}{f}(\bar{x}, \underset{\sim}{U}_p ; \lambda, \bar{p}_e, \bar{\tau}) \qquad \bar{x}_2 \le \bar{x} \le \bar{x}_3 \qquad \underline{\text{Forward Integration}}$$

$$\underset{\sim}{U}_p(\bar{x}_2) = \underset{\sim}{p} \tag{166}$$

$$\frac{d}{d\bar{x}} \underset{\sim}{V}_t = \underset{\sim}{f}(\bar{x}, \underset{\sim}{V}_t ; \lambda, \bar{p}_e, \bar{\tau}) \qquad \bar{x}_3 \le \bar{x} \le \frac{L}{R} \qquad \underline{\text{Backward Integration}}$$

$$\underset{\sim}{V}_t(\frac{L}{R}) = \underset{\sim}{t} \tag{167}$$

Notice that the unknown boundary conditions are specified by the 28-dimensional vectors $\underset{\sim}{s}$, $\underset{\sim}{p}$ and $\underset{\sim}{t}$. In addition, at the shell edges (at $\bar{x}=0$ and $\bar{x}=L/R$) the unknown vectors $\underset{\sim}{s}$ and $\underset{\sim}{t}$ must satisfy the general elastic boundary conditions, specified by the 14-dimensional vector equations $\underset{\sim}{g}$ or $\underset{\sim}{h}$, derived earlier.
Under appropriate smoothness conditions on the nonlinear vector function $\underset{\sim}{f}(\bar{x},\underset{\sim}{Y} ; \lambda, \bar{p}_e, \bar{\tau})$ one is assured of the existence of unique solutions of these initial value problems. As can be seen from Fig. 7 these solutions must satisfy the following matching conditions

$$\begin{aligned} \underset{\sim}{U}_s(\bar{x} = \bar{x}_1, \underset{\sim}{s} ; \lambda, \bar{p}_e, \bar{\tau}) &= \underset{\sim}{V}_p(\bar{x} = \bar{x}_1, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) \\ \underset{\sim}{U}_p(\bar{x} = \bar{x}_3, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) &= \underset{\sim}{V}_t(\bar{x} = \bar{x}_3, \underset{\sim}{t} ; \lambda, \bar{p}_e, \bar{\tau}) \end{aligned} \tag{168}$$

Introducing the new 56-dimensional vector function $\underset{\sim}{\varphi}$ these conditions can be written as

$$\underset{\sim}{\varphi}(\underset{\sim}{S}) = \begin{bmatrix} \underset{\sim}{U}_s(\bar{x} = \bar{x}_1, \underset{\sim}{s} ; \lambda, \bar{p}_e, \bar{\tau}) - \underset{\sim}{V}_p(\bar{x} = \bar{x}_1, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) \\ \underset{\sim}{U}_p(\bar{x} = \bar{x}_3, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) - \underset{\sim}{V}_t(\bar{x} = \bar{x}_3, \underset{\sim}{t} ; \lambda, \bar{p}_e, \bar{\tau}) \end{bmatrix} = 0 \tag{169}$$

where

$$\underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{s} \\ \underset{\sim}{p} \\ \underset{\sim}{t} \end{bmatrix} \qquad \text{is an 84-dimensional vector} \tag{170}$$

Finally, by introducing the 84-dimensional vector function $\underset{\sim}{\phi}$ one can combine the given nonlinear boundary vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$ and the above matching conditions into a single vector equation.

$$\underset{\sim}{\phi}(\underset{\sim}{S}) = \begin{bmatrix} g(\bar{x} = 0, \underset{\sim}{Y}(0) = \underset{\sim}{s} ; \lambda, \bar{p}_e, \bar{\tau}) \\ \underset{\sim}{\varphi}(\underset{\sim}{S}) \\ h(\bar{x} = \frac{L}{R}, \underset{\sim}{Y}(\frac{L}{R}) = \underset{\sim}{t} ; \lambda, \bar{p}_e, \bar{\tau}) \end{bmatrix} = 0 \tag{171}$$

Thus the solution of the nonlinear 2-point boundary value problem (Eqs. (146)-(148)) has been transformed to the solution of the 4 associated initial value problems (Eqs. (164)-(167)) and to the finding of the roots $\underset{\sim}{S}$ of the system of simultaneous equations defined by Eq. (171).

Using Newton's method for finding the roots of $\underset{\sim}{\phi}(\underset{\sim}{S}) = 0$ one has the following iteration scheme:

$$\underset{\sim}{S}^{\nu+1} = \underset{\sim}{S}^{\nu} + \Delta\underset{\sim}{S}^{\nu} \qquad \nu = 1,2,\ldots. \tag{172}$$

where $\Delta\underset{\sim}{S}^{\nu}$ is the solution of the 84th order linear algebraic system

$$\frac{\partial\underset{\sim}{\phi}}{\partial\underset{\sim}{S}}(\underset{\sim}{S}^{\nu}) \cdot \Delta\underset{\sim}{S}^{\nu} = -\underset{\sim}{\phi}(\underset{\sim}{S}^{\nu}) \tag{173}$$

To apply Newton's method one must be able to find the Jacobian matrix $J(\underset{\sim}{S}^{\nu})$

$$J = \frac{\partial\underset{\sim}{\phi}}{\partial\underset{\sim}{S}}(\underset{\sim}{S}^{\nu}) = \left[\begin{array}{cccccccccc}
\frac{\partial g_1}{\partial s_1} & \cdots & \frac{\partial g_1}{\partial s_{28}} & & & & & & & \\
\cdots & \cdots & \cdots & & \mathbf{0} & & & & \mathbf{0} & \\
\frac{\partial g_{14}}{\partial s_1} & \cdots & \frac{\partial g_{14}}{\partial s_{28}} & & & & & & & \\
\frac{\partial \varphi_1}{\partial s_1} & \cdots & \frac{\partial \varphi_1}{\partial s_{28}} & \frac{\partial \varphi_1}{\partial p_1} & \cdots & \frac{\partial \varphi_1}{\partial p_{28}} & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
\cdots & & \cdots & \cdots & & \cdots & & & \mathbf{0} & \\
\cdots & & \cdots & \cdots & & \cdots & & & & \\
\frac{\partial \varphi_{28}}{\partial s_1} & \cdots & \frac{\partial \varphi_{28}}{\partial s_{28}} & \frac{\partial \varphi_{28}}{\partial p_1} & \cdots & \frac{\partial \varphi_{28}}{\partial p_{28}} & & & & \\
 & & & \frac{\partial \varphi_{29}}{\partial p_1} & \cdots & \frac{\partial \varphi_{29}}{\partial p_{28}} & \frac{\partial \varphi_{29}}{\partial t_1} & \cdots & \frac{\partial \varphi_{29}}{\partial t_{28}} & \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & \mathbf{0} & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & & & \cdots & \cdots & & \cdots & \\
 & & & \frac{\partial \varphi_{56}}{\partial p_1} & \cdots & \frac{\partial \varphi_{56}}{\partial p_{28}} & \frac{\partial \varphi_{56}}{\partial t_1} & \cdots & \frac{\partial \varphi_{56}}{\partial t_{28}} & \\
 & & & & & & \frac{\partial h_1}{\partial t_1} & \cdots & \frac{\partial h_1}{\partial t_{28}} & \\
 & \mathbf{0} & & & \mathbf{0} & & \cdots & \cdots & \cdots & \\
 & & & & & & \frac{\partial h_{14}}{\partial t_1} & \cdots & \frac{\partial h_{14}}{\partial t_{28}} &
\end{array}\right] \tag{174}$$

Notice that those components of this Jacobian, which involve derivatives of the components of the specified boundary vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$, can be calculated analytically. However, the components involving derivatives of the matching conditions must be obtained by solving the appropriate variational equations. In order to solve for these components let us introduce the following new 28-dimensional vectors

$$\begin{aligned}
&\underset{\sim}{W}_i = \frac{\partial}{\partial s_i}\underset{\sim}{U}_s \quad \text{for} \quad i = 1,2,\ldots,28 && ; \quad \underset{\sim}{Z}_i = \frac{\partial}{\partial p_j}\underset{\sim}{V}_p \quad \text{for} \quad i = 29,30,\ldots,56 \\
& && \qquad\qquad\qquad\qquad j = i-28 \\
&\underset{\sim}{W}_i = \frac{\partial}{\partial p_j}\underset{\sim}{U}_p \quad \text{for} \quad i = 29,30,\ldots,56 && ; \quad \underset{\sim}{Z}_i = \frac{\partial}{\partial t_j}\underset{\sim}{V}_t \quad \text{for} \quad i = 57,58,\ldots,84 \\
&\qquad\qquad\qquad\qquad j = i-28 && \qquad\qquad\qquad\qquad j = i-56
\end{aligned} \tag{175}$$

These vectors are then found as the solutions of variational equations obtained by implicit differentiation of the corresponding associated initial value problems. Thus the following variational equations must be integrated forward

$$\begin{aligned}
&\frac{d}{d\bar{x}}\underset{\sim}{W}_i = \frac{\partial}{\partial \underset{\sim}{U}_s}\underset{\sim}{f}\,(\bar{x}, \underset{\sim}{U}_s\,;\lambda, \bar{p}_e, \bar{\tau})\cdot \underset{\sim}{W}_i && 0 \le \bar{x} \le \bar{x}_1 \\
&\underset{\sim}{W}_i(0) = \underset{\sim}{I}_i && \text{for } i = 1,2,\ldots,28
\end{aligned} \tag{176}$$

$$\begin{aligned}
&\frac{d}{d\bar{x}}\underset{\sim}{W}_i = \frac{\partial}{\partial \underset{\sim}{U}_p}\underset{\sim}{f}\,(\bar{x}, \underset{\sim}{U}_p\,;\lambda, \bar{p}_e, \bar{\tau})\cdot \underset{\sim}{W}_i && \bar{x}_2 \le \bar{x} \le \bar{x}_3 \\
&\underset{\sim}{W}_i(\bar{x}_2) = \underset{\sim}{I}_j && \text{for } i = 29,30,\ldots,56 \\
& && \quad j = i-28
\end{aligned} \tag{177}$$

and the following variational equations must be integrated backward

$$\begin{aligned}
&\frac{d}{d\bar{x}}\underset{\sim}{Z}_i = \frac{\partial}{\partial \underset{\sim}{V}_p}\underset{\sim}{f}\,(\bar{x}, \underset{\sim}{V}_p\,;\lambda, \bar{p}_e, \bar{\tau})\cdot \underset{\sim}{Z}_i && \bar{x}_1 \le \bar{x} \le \bar{x}_2 \\
&\underset{\sim}{Z}_i(\bar{x}_2) = \underset{\sim}{I}_j && \text{for } i = 29,30,\ldots,56 \\
& && \quad j = i-28
\end{aligned} \tag{178}$$

$$\begin{aligned}
&\frac{d}{d\bar{x}}\underset{\sim}{Z}_i = \frac{\partial}{\partial \underset{\sim}{V}_t}\underset{\sim}{f}\,(\bar{x}, \underset{\sim}{V}_t\,;\lambda, \bar{p}_e, \bar{\tau})\cdot \underset{\sim}{Z}_i && \bar{x}_3 \le \bar{x} \le \frac{L}{R} \\
&\underset{\sim}{Z}_i\left(\frac{L}{R}\right) = \underset{\sim}{I}_j && \text{for } i = 57,58,\ldots,84 \\
& && \quad j = i-56
\end{aligned} \tag{179}$$

Here $\underset{\sim}{I}_i \equiv \{0,...,0,1,0,...0\}^T$ is the i^{th} generalized unit vector in the n-space. Notice that the components of the Jacobian matrix J'

$$J' = \frac{\partial}{\partial \underset{\sim}{U}} \underset{\sim}{f} = \frac{\partial}{\partial \underset{\sim}{V}} \underset{\sim}{f} = \begin{bmatrix} \frac{\partial f_1}{\partial U_1} & \frac{\partial f_1}{\partial U_2} & \cdots & \frac{\partial f_1}{\partial U_{28}} \\ \frac{\partial f_2}{\partial U_1} & \cdots & & \\ \cdots & \cdots & & \\ \frac{\partial f_{28}}{\partial U_1} & \cdots & & \frac{\partial f_{28}}{\partial U_{28}} \end{bmatrix} \tag{180}$$

can be calculated analytically. For details see Appendix G.

Since the Jacobian J' is a function of $\underset{\sim}{U}$ (or $\underset{\sim}{V}$), the variational equations (176)-(177) depend step-by-step on the results of the associated initial value problems (164) and (166), whereas the variational equations (178)-(179) depend step-by-step on the results of the associated initial value problems (165) and (167). Thus the variational equations depend on the initial guess $\underset{\sim}{S}^V$. Hence it is advantageous to integrate per interval the 28 variational equations simultaneously with the corresponding associated initial value problem. This results for parallel shooting over 4 intervals in four 812-dimensional, 1st order, nonlinear ordinary differential equations.

One of the greatest difficulties in "shooting" consists of obtaining a starting estimate of the initial data which is sufficiently close to the exact initial data, so that the iteration scheme used to find a solution of the nonlinear problem will converge. Fortunally, in the cases of imperfect cylindrical shells loaded by axial compression, external pressure or torsion the nonlinear solution approaches the linearized solution asymptotically as the variable load parameter $\Lambda \rightarrow 0$. Thus for sufficiently low values of the variable load parameter Λ one can use the linearized solutions as starting values for the nonlinear iteration scheme. Solutions of the linearized problem are also obtained by the "shooting method". It is well known that for the linearized 2-point boundary value problem Newton's method yields the correct initial value $\underset{\sim}{S}$ directly without the need of iterations[22]. The solution of the associated initial value problems and the variational equations is done by the library subroutine DEQ from Caltech's Willis Booth Computing Center. DEQ uses the method of Runge-Kutta-Gill to compute starting values for an Adam-Moulton corrector-predictor scheme. The program includes an option with variable interval size and uses automatic truncation error control. For proper convergence (5 digits accuracy) at low variable load levels 2 iterations are sufficient; however, at variable load levels close to the limit point 6-12 iterations may be needed to obtain the same level of convergence. Thus, upon reaching a variable load level corresponding to point A in Fig. 6, instead of further increments in the variable load $\Delta\Lambda$ increments in the nonlinear part of the appropriate deformation parameter are used to continue the integration. As can be seen from Figure 6 this switch in increments makes it possible to integrate around the limit point at B and get converged solutions on the decreasing part of the solution curve.

3.3 Numerical Analysis using Increments of Deformation

Here one must find a solution for the nonlinear 2-point boundary value problem defined by Eqs. (146)-(148) under the restriction that the solution must satisfy also a constraint condition specified by either Eq. (149) if the variable load is axial compression ($\Lambda = \lambda$), or by Eq. (154) if the variable load is external pressure ($\Lambda = \bar{p}_e$), or by Eq. (159) if the variable load is torsion ($\Lambda = \bar{\tau}$). Thus if one increases the dimension of the unified vector variable $\underset{\sim}{Y}$ by one ($\rightarrow Y_{29} = \Delta,\ f_{29} = F_\Delta$) then one must solve the following nonlinear 2-point boundary value problem

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{Y}} = \hat{\underset{\sim}{f}}\,(\bar{x}, \underset{\sim}{Y};\ \lambda, \bar{p}_e, \bar{\tau}) \qquad \text{for } 0 \le \bar{x} \le \frac{L}{R} \qquad (181)$$

$$\underset{\sim}{g}(\bar{x} = 0, \underset{\sim}{Y}(o);\ \lambda, \bar{p}_e, \bar{\tau}) = 0 \qquad \text{at } \bar{x} = 0 \qquad (182)$$

$$\underset{\sim}{h}(\bar{x} = \frac{L}{R}, \underset{\sim}{Y}(\frac{L}{R});\ \lambda, \bar{p}_e, \bar{\tau}) = 0 \qquad \text{at } \bar{x} = \frac{L}{R} \qquad (183)$$

$$\Delta = \Delta_0 \qquad \text{prescribed deformation} \qquad (184)$$

Notice that the length of the capped vectors is incremented by one with respect to their uncapped equivalents in Eqs. (146)-(148). Thus the unified variable $\hat{\underset{\sim}{Y}}$ and the nonlinear vector function $\hat{\underset{\sim}{f}}$ are 29-dimensional, whereas the general nonlinear boundary conditions are specified by the 14-dimensional vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$.

Proceeding as before let us associate the following 4 initial value problems with the above 2-point boundary value problem

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{U}}_s = \hat{\underset{\sim}{f}}(\bar{x}, \hat{\underset{\sim}{U}}_s\ ;\ \lambda, \bar{p}_e, \bar{\tau}) \qquad 0 \le \bar{x} \le \bar{x}_1 \qquad \underline{\text{Forward integration}} \qquad (185)$$

$$\hat{\underset{\sim}{U}}_s(0) = \begin{bmatrix} \underset{\sim}{s} \\ 0 \end{bmatrix}$$

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{V}}_p = \hat{\underset{\sim}{f}}(\bar{x}, \hat{\underset{\sim}{V}}_p\ ;\ \lambda, \bar{p}_e, \bar{\tau}) \qquad \bar{x}_1 \le \bar{x} \le \bar{x}_2 \qquad \underline{\text{Backward integration}} \qquad (186)$$

$$\hat{\underset{\sim}{V}}_p(\bar{x}_2) = \begin{bmatrix} \underset{\sim}{p} \\ 0 \end{bmatrix}$$

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{U}}_p = \hat{\underset{\sim}{f}}(\bar{x}, \hat{\underset{\sim}{U}}_p\ ;\ \lambda, \bar{p}_e, \bar{\tau}) \qquad \bar{x}_2 \le \bar{x} \le \bar{x}_3 \qquad \underline{\text{Forward integration}} \qquad (187)$$

$$\hat{\underset{\sim}{U}}_p(\bar{x}_2) = \begin{bmatrix} \underset{\sim}{p} \\ 0 \end{bmatrix}$$

$$\frac{d}{dx}\hat{\underset{\sim}{V}}_t = \hat{\underset{\sim}{f}}(\bar{x}, \hat{\underset{\sim}{V}}_t ; \lambda, \bar{p}_e, \bar{\tau}) \qquad \bar{x}_3 \le \bar{x} \le \frac{L}{R} \qquad \underline{\text{Backward integration}} \qquad (188)$$

$$\hat{\underset{\sim}{V}}_t(\frac{L}{R}) = \begin{bmatrix} \underset{\sim}{t} \\ 0 \end{bmatrix}$$

As mentioned earlier, under appropriate smoothness conditions on the nonlinear vector function $\hat{\underset{\sim}{f}}(\bar{x}, \hat{\underset{\sim}{Y}} ; \lambda, \bar{p}_e, \bar{\tau})$ one is assured of the existence of unique solutions of the above initial value problems, which besides the usual matching conditions

$$\begin{aligned} \underset{\sim}{U}_s(\bar{x} = \bar{x}_1, \underset{\sim}{s} ; \lambda, \bar{p}_e, \bar{\tau}) &= \underset{\sim}{V}_p(\bar{x} = \bar{x}_1, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) \\ \underset{\sim}{U}_p(\bar{x} = \bar{x}_3, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) &= \underset{\sim}{V}_t(\bar{x} = \bar{x}_3, \underset{\sim}{t} ; \lambda, \bar{p}_e, \bar{\tau}) \end{aligned} \qquad (189)$$

must also satisfy the following constraint condition

$$\Delta_s(\bar{x}_1) - \Delta_p(\bar{x}_1) + \Delta_p(\bar{x}_3) - \Delta_t(\bar{x}_3) = \Delta_o \qquad (190)$$

Introducing the new 57-dimensional vector function $\hat{\underset{\sim}{\varphi}}$ these conditions can be written as

$$\hat{\underset{\sim}{\varphi}}(\hat{\underset{\sim}{S}}) = \begin{bmatrix} \underset{\sim}{U}_s(\bar{x} = \bar{x}_1, \underset{\sim}{s} ; \lambda, \bar{p}_e, \bar{\tau}) - \underset{\sim}{V}_p(\bar{x} = \bar{x}_1, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) \\ \underset{\sim}{U}_p(\bar{x} = \bar{x}_3, \underset{\sim}{p} ; \lambda, \bar{p}_e, \bar{\tau}) - \underset{\sim}{V}_t(\bar{x} = \bar{x}_3, \underset{\sim}{t} ; \lambda, \bar{p}_e, \bar{\tau}) \\ \Delta_s(\bar{x}_1) - \Delta_p(\bar{x}_1) + \Delta_p(\bar{x}_3) - \Delta_t(\bar{x}_3) - \Delta_o \end{bmatrix} = 0 \qquad (191)$$

where

$$\hat{\underset{\sim}{S}} = \begin{bmatrix} \underset{\sim}{s} \\ \underset{\sim}{p} \\ \underset{\sim}{t} \\ \Lambda \end{bmatrix} \qquad \text{is an 85-dimensional vector} \qquad (192)$$

Finally, by introducing the 85-dimensional vector function $\hat{\underset{\sim}{\phi}}$ one can combine the given nonlinear boundary vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$ and the above matching and constraint conditions into a single equation

$$\hat{\underset{\sim}{\phi}}(\hat{\underset{\sim}{S}}) = \begin{bmatrix} \underset{\sim}{g}(\bar{x} = 0, \underset{\sim}{Y}(o) = \underset{\sim}{s} ; \lambda, \bar{p}_e, \bar{\tau}) \\ \hat{\underset{\sim}{\varphi}}(\hat{\underset{\sim}{S}}) \\ \underset{\sim}{h}(\bar{x} = \frac{L}{R}, \underset{\sim}{Y}(\frac{L}{R}) = \underset{\sim}{t} ; \lambda, \bar{p}_e, \bar{\tau}) \end{bmatrix} = 0 \qquad (193)$$

Thus the solution of the nonlinear 2-point boundary value problem (Eqs. (181)-(184)) has been transformed to the solution of the 4 associated initial value problems (Eqs. (185)-(188)) and to the finding of the roots $\hat{\underset{\sim}{S}}$ of the system of simultaneous equations defined by Eq. (193).

Using Newton's method for finding the roots of $\hat{\underset{\sim}{\phi}}(\hat{\underset{\sim}{S}}) = 0$ one has the following iteration scheme

$$\hat{\underset{\sim}{S}}^{v+1} = \hat{\underset{\sim}{S}}^{v} + \Delta\hat{\underset{\sim}{S}}^{v} \tag{194}$$

where $\Delta\hat{\underset{\sim}{S}}^{v}$ is the solution of the 85th order linear algebraic system

$$\frac{\partial}{\partial \hat{\underset{\sim}{S}}} \hat{\underset{\sim}{\phi}} (\hat{\underset{\sim}{S}}) \cdot \Delta\hat{\underset{\sim}{S}}^{v} = - \hat{\underset{\sim}{\phi}}(\hat{\underset{\sim}{S}}^{v}) \tag{195}$$

To apply Newton's method one must be able to find the Jacobian matrix $\hat{J}(\hat{\underset{\sim}{S}}^{v})$

$$\hat{J}(\hat{\underset{\sim}{S}}^{v}) = \frac{\partial}{\partial \hat{\underset{\sim}{S}}} \hat{\underset{\sim}{\phi}} (\hat{\underset{\sim}{S}}^{v}) = \begin{bmatrix} \frac{\partial\hat{\phi}_1}{\partial S_1} & \frac{\partial\hat{\phi}_1}{\partial S_2} & \cdots & \frac{\partial\hat{\phi}_1}{\partial S_{84}} & \frac{\partial\hat{\phi}_1}{\partial \Lambda} \\ \frac{\partial\hat{\phi}_2}{\partial S_1} & \cdots & & \frac{\partial\hat{\phi}_2}{\partial S_{84}} & \frac{\partial\hat{\phi}_2}{\partial \Lambda} \\ \cdots & & \cdots & \cdots & \cdots \\ \cdots & & \cdots & \cdots & \cdots \\ \frac{\partial\hat{\phi}_{84}}{\partial S_1} & \cdots & \cdots & \frac{\partial\hat{\phi}_{84}}{\partial S_{84}} & \frac{\partial\hat{\phi}_{84}}{\partial \Lambda} \\ \frac{\partial\hat{\phi}_{85}}{\partial S_1} & \cdots & \cdots & \frac{\partial\hat{\phi}_{85}}{\partial S_{84}} & \frac{\partial\hat{\phi}_{85}}{\partial \Lambda} \end{bmatrix} \tag{196}$$

Looking at Eq. (193) it is obvious that there are components of this Jacobian which involve derivatives of the specified boundary vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$, which can be calculated analytically. However, the components involving derivatives of the matching and constraint conditions must be obtained by solving the appropriate variational equations. In order to solve for these components let us introduce the following new 29-dimensional vectors

$$\begin{aligned} &\hat{\underset{\sim}{W}}_i = \frac{\partial}{\partial s_i} \hat{\underset{\sim}{U}}_s \quad \text{for} \quad i = 1,2,\ldots,28 \quad ; \quad &&\hat{\underset{\sim}{Z}}_i = \frac{\partial}{\partial p_j} \hat{\underset{\sim}{V}}_p \quad \text{for} \quad i = 29,30,\ldots,56, \; j = i-28 \\ &\hat{\underset{\sim}{W}}_i = \frac{\partial}{\partial p_j} \hat{\underset{\sim}{U}}_p \quad \text{for} \quad i = 29,30,\ldots,56, \; j = i-28 \quad ; \quad &&\hat{\underset{\sim}{Z}}_i = \frac{\partial}{\partial t_j} \hat{\underset{\sim}{V}}_t \quad \text{for} \quad i = 57,58,\ldots,84, \; j = i-56 \end{aligned} \tag{197}$$

and

$$\hat{\underset{\sim}{W}}_s = \frac{\partial}{\partial \Lambda} \hat{\underset{\sim}{U}}_s \qquad \hat{\underset{\sim}{Z}}_p = \frac{\partial}{\partial \Lambda} \hat{\underset{\sim}{V}}_p$$

$$\hat{\underset{\sim}{W}}_p = \frac{\partial}{\partial \Lambda} \hat{\underset{\sim}{U}}_p \qquad \hat{\underset{\sim}{Z}}_t = \frac{\partial}{\partial \Lambda} \hat{\underset{\sim}{V}}_t \tag{198}$$

Notice that all capped vectors have been incremented with respect to the corresponding uncapped vectors of Eq. (175) by one component, namely by $Y_{29} = \Delta$, such that

$$\hat{\underset{\sim}{W}}_i = \left[\underset{\sim}{W}_i^T \, , \frac{\partial \Delta}{\partial S_i} \right]^T \tag{199}$$

These vectors are then found as the solutions of variational equations obtained by implicit differentiation of the corresponding associated initial value problems. Thus the following variational equations must be integrated forward

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{W}}_i = \frac{\partial}{\partial \hat{\underset{\sim}{U}}_s} \hat{\underset{\sim}{f}} \, (\bar{x}, \hat{\underset{\sim}{U}}_s \, ; \lambda, \bar{p}_e, \bar{\tau}) \cdot \hat{\underset{\sim}{W}}_i \qquad 0 \le \bar{x} \le \bar{x}_1$$

$$\hat{\underset{\sim}{W}}_i(0) = \hat{\underset{\sim}{I}}_i \qquad \text{for } i = 1,2,...,28 \tag{200}$$

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{W}}_i = \frac{\partial}{\partial \hat{\underset{\sim}{U}}_p} \hat{\underset{\sim}{f}} \, (\bar{x}, \hat{\underset{\sim}{U}}_p \, ; \lambda, \bar{p}_e, \bar{\tau}) \cdot \hat{\underset{\sim}{W}}_i \qquad \bar{x}_2 \le \bar{x} \le \bar{x}_3$$

$$\hat{\underset{\sim}{W}}_i(\bar{x}_2) = \hat{\underset{\sim}{I}}_j \qquad \text{for } i = 29,30,...,56 \quad j = i-28 \tag{201}$$

and the following variational equations must be integrated backward

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{Z}}_i = \frac{\partial}{\partial \hat{\underset{\sim}{V}}_p} \hat{\underset{\sim}{f}} \, (\bar{x}, \hat{\underset{\sim}{V}}_p \, ; \lambda, \bar{p}_e, \bar{\tau}) \cdot \hat{\underset{\sim}{Z}}_i \qquad \bar{x}_1 \le \bar{x} \le \bar{x}_2$$

$$\hat{\underset{\sim}{Z}}_i(\bar{x}_2) = \hat{\underset{\sim}{I}}_j \qquad \text{for } i = 29,30,...,56 \quad j = i-28 \tag{202}$$

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{Z}}_i = \frac{\partial}{\partial \hat{\underset{\sim}{V}}_t} \hat{\underset{\sim}{f}} \, (\bar{x}, \hat{\underset{\sim}{V}}_t \, ; \lambda, \bar{p}_e, \bar{\tau}) \cdot \hat{\underset{\sim}{Z}}_i \qquad \bar{x}_3 \le \bar{x} \le \frac{L}{R}$$

$$\hat{\underset{\sim}{Z}}_i\left(\frac{L}{R}\right) = \hat{\underset{\sim}{I}}_j \qquad \text{for } i = 57,58,...,84 \quad j = i-56 \tag{203}$$

where $\underset{\sim}{\hat{I}}_i \equiv \{0, \ldots, 0, 1, 0, \ldots 0\}^T$ is the i^{th} generalized unit vector in the n-space.
In addition, the following inhomogeneous variational equations must be integrated forward

$$\frac{d}{d\bar{x}}\underset{\sim}{\hat{W}}_s = \frac{\partial}{\partial \underset{\sim}{\hat{U}}_s}\underset{\sim}{\hat{f}}\,(\bar{x}, \underset{\sim}{\hat{U}}_s\,;\lambda, \bar{p}_e, \bar{\tau})\,\underset{\sim}{\hat{W}}_s + \frac{\partial}{\partial \Lambda}\underset{\sim}{\hat{f}}$$
$$\underset{\sim}{\hat{W}}_s(0) = \underset{\sim}{\hat{I}}_\Lambda \tag{204}$$

$$\frac{d}{d\bar{x}}\underset{\sim}{\hat{W}}_p = \frac{\partial}{\partial \underset{\sim}{\hat{U}}_p}\underset{\sim}{\hat{f}}\,(\bar{x}, \underset{\sim}{\hat{U}}_p\,;\lambda, \bar{p}_e, \bar{\tau})\,\underset{\sim}{\hat{W}}_p + \frac{\partial}{\partial \Lambda}\underset{\sim}{\hat{f}}$$
$$\underset{\sim}{\hat{W}}_p(\bar{x}_2) = \underset{\sim}{0} \tag{205}$$

and the following inhomogeneous variational equations must be integrated backward

$$\frac{d}{d\bar{x}}\underset{\sim}{\hat{Z}}_p = \frac{\partial}{\partial \underset{\sim}{\hat{V}}_p}\underset{\sim}{\hat{f}}\,(\bar{x}, \underset{\sim}{\hat{V}}_p\,;\lambda, \bar{p}_e, \bar{\tau})\,\underset{\sim}{\hat{Z}}_p + \frac{\partial}{\partial \Lambda}\underset{\sim}{\hat{f}}$$
$$\underset{\sim}{\hat{Z}}_p(\bar{x}_2) = \underset{\sim}{0} \tag{206}$$

$$\frac{d}{d\bar{x}}\underset{\sim}{\hat{Z}}_t = \frac{\partial}{\partial \underset{\sim}{\hat{V}}_t}\underset{\sim}{\hat{f}}\,(\bar{x}, \underset{\sim}{\hat{V}}_t\,;\lambda, \bar{p}_e, \bar{\tau})\,\underset{\sim}{\hat{Z}}_t + \frac{\partial}{\partial \Lambda}\underset{\sim}{\hat{f}}$$
$$\underset{\sim}{\hat{Z}}_t\left(\frac{L}{R}\right) = \underset{\sim}{\hat{I}}_\Lambda \tag{207}$$

The generalized i^{th} unit vectors $\underset{\sim}{\hat{I}}_i$ in the 29-dimensional space are obtained from the corresponding ones in the 28-dimensional space by adding a zero in the 29th row. The forcing function $\partial\underset{\sim}{\hat{f}}/\partial\Lambda$ and the unit vectors $\underset{\sim}{\hat{I}}_\Lambda$ for the inhomogeneous variational equations are summarized in Appendix H. Notice that the components of the Jacobian matrix $\hat{J}'$

$$\hat{J}' = \frac{\partial}{\partial \underset{\sim}{\hat{U}}}\underset{\sim}{\hat{f}}\,(\bar{x}, \underset{\sim}{\hat{U}}\,;\lambda, \bar{p}_e, \bar{\tau}_t) = \begin{bmatrix} \frac{\partial f_1}{\partial U_1} & \frac{\partial f_1}{\partial U_2} & \cdots & \cdots & \frac{\partial f_1}{\partial U_{28}} & \frac{\partial f_1}{\partial \Delta} \\ \frac{\partial f_2}{\partial U_1} & & & & \cdots & \cdots \\ \cdots & & & & \cdots & \cdots \\ \cdots & & & & \cdots & \cdots \\ \frac{\partial f_{28}}{\partial U_1} & \cdots & & \cdots & \frac{\partial f_{28}}{\partial U_{28}} & \frac{\partial f_{28}}{\partial \Delta} \\ \frac{\partial f_\Delta}{\partial U_1} & \cdots & & \cdots & \frac{\partial f_\Delta}{\partial U_{28}} & \frac{\partial f_\Delta}{\partial \Delta} \end{bmatrix} \tag{208}$$

can be calculated analytically. For details see Appendix G.

Notice that since the Jacobian $\hat{\tilde{J}}$ is a function of $\hat{\underset{\sim}{U}}$ (or $\hat{\underset{\sim}{V}}$), therefore the variational equations depend step-by-step on the results of the associated initial value problems. Thus the variational equations depend on the initial guess $\hat{\underset{\sim}{S}}^v$. Hence the variational equations must be integrated together with the corresponding associated initial value problems. For parallel shooting over 4 intervals this actually involves the numerical integration of four 870-dimensional, 1st order, nonlinear ordinary differential equations. The final values of these integrations are then used to assemble the vector function $\hat{\underset{\sim}{\phi}}(\hat{\underset{\sim}{S}}^v)$ and the Jacobian matrix $\frac{\partial}{\partial \hat{\underset{\sim}{S}}} \hat{\underset{\sim}{\phi}} (\hat{\underset{\sim}{S}}^v)$ given by Eqs. (193) and (196), respectively. For details please consult Appendix I.

4. DISCUSSION

To test the accuracy and reliability of the collapse load calculations, initially using Khot's unsymmetrically laid up glass-epoxy shell[26] close approximations of the bifurcation buckling loads for various boundary conditions are computed. This is done by assuming a very small asymmetric imperfection

$$\bar{W} = t\bar{\xi}_2 \sin m\pi \frac{x}{L} \cos n \frac{y}{R} \qquad ; \qquad \bar{\xi}_2 = 0.1 \cdot 10^{-5} \tag{209}$$

where m.n are integers denoting the number of axial half-waves and the number of full waves in the circumferential direction, respectively. By choosing n such that it corresponds to the critical buckling mode of the perfect shell, the limit point of the prebuckling state for a vanishingly small initial asymmetric imperfection ($\bar{\xi}_2 = 0.1 \cdot 10^{-5}$) approximates closely the bifurcation point of the perfect shell calculated by ANILISA[29].

As can be seen from the results listed in Table 1 in all cases but one (the SS-2 boundary conditions) the limit loads calculated by the present program COLLAPSE are, as expected, slightly lower than the bifurcation buckling loads of the corresponding perfect shells calculated by ANILISA. The explanation for the sole exception, the case of SS-2 boundary conditions, can be found by looking at the load vs deformation plots displayed in Figs. 10-18.

It is known from Koiter's Imperfection Sensitivity Theory[1] that if the buckling load of a structure is sensitive to small asymmetric imperfections, then buckling occurs at a limit point. The existence of a limit point implies that for the corresponding perfect structure the initial slope of the postbuckling path at the bifurcation point is negative (downward). As can be seen from Fig. 11b the load vs deformation (end shortening) curve does not have a limit point. That is, after buckling the structure continues to develop considerable postbuckling strength. Thus the loss of stability of the primary path does not result in structural collapse. The same type of behavior is displayed in Fig. 11a, where the maximum value of the asymmetric mode $w_{1,max}$ is plotted as a function of the axial load parameter $\rho_S = \lambda_S/\lambda_C$. Notice that up to close to the bifurcation point the asymmetric mode w_1 has vanishingly small amplitudes (of the order of

10^{-7}). After passing the bifurcation point the asymmetric mode w_1 grows very fast, but as can be seen distinctly, with increasing axial load level. The load vs deformation path has in this case a point of inflection in place of a limit point. Collapse occurs here at a higher load level not shown in fig. 11.

To illustrate this point further the results of Koiter's Asymptotic Theory computed with ANILISA[29] are shown in Table 2. Notice that for SS-2 boundary conditions both the second postbuckling coefficient b and $\tilde{\theta}^*_c$, the initial slope of the postbuckling path are positive. This implies that the buckling load of the perfect structure is not sensitive to small asymmetric imperfections, a prediction that is fully confirmed by the present nonlinear analysis.

Further, as can be seen in Fig. 13, for the SS-4 boundary conditions the present nonlinear computaional module COLLAPSE has encountered convergence difficulties shortly after passing the limit point. To ascertain that the solution curve has not turned back on itself, the results close to the limit point have been replotted using a blown-up scale. As can be seen in Fig. 14, the limit point is very sharp but indeed one has obtained two distinct solution branches.

The fact that in Fig. 15b the postbuckling path turns back on itself can be attributed to the fact that in this case in the deep postbuckling region the response cannot be described accurately with a single asymmetric mode with n full waves in the circumferential direction only, as is done in COLLAPSE. Due to nonlinear coupling effects additional asymmetric modes with 2n circumferential full waves are triggered. The suppression of these additional response modes is the cause of the turn-back phenomena.

The response mode shapes at the limit points calculated for the different boundary conditions are displayed in Fig. 19. Recalling Eq. (9), for each boundary condition three curves are shown, namely the

- axisymmetric part $\quad W_v + w_o(x)$
- asymmetric part $\quad w_1(x)\cos n\theta + w_2(x)\sin n\theta$
- compound shape at $\theta = 0$ $\quad \dfrac{W}{t} = W_v + w_o(x) + w_1(x)\cos n\theta + w_2(x)\sin n\theta$

Comparing the response modes at the limit points calculated with COLLAPSE with the corresponding bifurcation buckling modes calculated with ANILISA[29] shown in Fig. 20, one sees an excellent agreement as far as the mode shapes go. The amplitudes differ, since the bifurcation buckling modes in Fig. 20 are normalized to unity, whereas the corresponding collapse modes in Fig. 19 are not.

Finally some remarks about the normalization and re-normalization used are necessary. All external load parameters, λ for axial load $(=(cR/Et^2)N_o)$, $\bar{p}_e$ for external lateral pressure $(=(cR^2/Et^2)p)$ and $\bar{\tau}$ for torsion $(=(cR/Et^2)N_{xy})$ are normalized according to the conventions introduced by the Harvard School under Budiansky and Hutchinson[2,23,24]. Notice that with this notation the critical buckling load of an axially compressed isotropic shell using membrane prebuckling (the so-called clasical buckling load) is $\lambda_c = 1.0$.

In order to be able to make direct comparisons with the many published results and to assess the possible imperfection sensitivity of the buckling load of the perfect shell for both, the load scale and the deformation scale a second normalization has been introduced. For a particular boundary condition the total end-shortening given by Eq. (150) is normalized by the value of the linear part of the end-shortening

$$\delta^{lim} = \bar{A}_{11}^{*}\lambda + \bar{A}_{12}^{*}\bar{p}_e - \bar{A}_{16}^{*}\bar{\tau} \tag{210}$$

evaluated at the limit point. The re-normalized axial load ρ_S is obtained by using the value of the nondimensional perfect shell buckling load λ_c^{nl} using nonlinear prebuckling and calculated by ANILISA[29] as a normalization factor.

5. CONCLUSIONS

In order to explore the capabilities of COLLAPSE, a program which can solve the nonlinear Donnell type imperfect anisotropic shell equations under combined axial load, internal or external prressure and torsion, a series of test cases are investigated. Notice that besides the standard simply supported and clamped boundary conditions listed in Tables 1 and 2, also elastically supported edge conditions represented by elastic rings of various shapes are considered. Specifically the following topics are examined:

1. Imperfection Sensitivity Predictions for various boundary conditions
   - asymptotic vs nonlinear solutions
2. Effect of elastic boundary conditions
   - symmetric cases
   - eccentric cases
3. Interaction curves for axial compression and torsion
   - modal imperfections
   - affine imperfections

The results and their interpretaion for the stringer stiffened shell AS-2[10] and for Khot's glass-epoxy ($-40^o, 40^o, 0^o$) shell[26] are presented in Part II of this report.

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APPENDIX A: The Laminate Constitutive Equations including Smeared Stiffeners

In deriving the constitutive equations for the shell wall made out of filamentary laminae, both unidirectional or woven, it is assumed that the laminae are homogeneous orthotropic layers in a plane stress state. The constitutive equations of the individual lamina in turn are used to derive the stress-strain relations for the laminate.

Normally, as can be seen from Fig. 8, the lamina principal axes (1,2) do not coincide with the reference axes of the shell wall. When this occurs, the constitutive relations for each individual lamina must be transformed to the shell wall reference axes in order to determine the shell wall constitutive relations.

Following Ashton et al[25] one obtains the stress-strain relationship for the k^{th}-lamina in terms of the midsurface strain- and curvature tensors of the shell wall as

$$[\sigma]_k = [\bar{Q}]_k\,[\varepsilon] + z[\bar{Q}]_k\,[\kappa] \qquad \text{(A.1)}$$

where

$$[\bar{Q}]_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \qquad \text{(A.2)}$$

and

$$\begin{aligned} [\sigma]_k &= [\sigma_x \quad \sigma_y \quad \tau_{xy}]_k^T \\ [\varepsilon] &= [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]^T \\ [\kappa] &= [\kappa_x \quad \kappa_y \quad \kappa_{xy}]^T \end{aligned} \qquad \text{(A.3)}$$

with

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}s^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4) \\ \bar{Q}_{22} &= Q_{11}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}c^4 \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(c^4 + s^4) \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})c^3s + (Q_{12} - Q_{22} + 2Q_{66})cs^3 \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})cs^3 + (Q_{12} - Q_{22} + 2Q_{66})c^3s \end{aligned} \qquad \text{(A.4)}$$

and

$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}} \quad ; \quad Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}} \quad ; \quad c = \cos\theta_K \quad ; \quad s = \sin\theta_K$$

$$Q_{12} = Q_{21} = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} \quad ; \quad Q_{66} = G_{12} \tag{A.5}$$

Notice that the factor (1/2) which appears in front of the shearing strain γ_{xy} in the strain tensor has been incorporated into the $\bar{Q}$ matrix.

The laminate constitutive equations are obtained by integrating the constitutive equations for the laminae, (Eq. A.1), over the thickness t of the laminate (see Fig. 8) yielding

$$[N] = [A]\,[\varepsilon] + [B]\,[\kappa]$$
$$[M] = [B]\,[\varepsilon] + [D]\,[\kappa] \tag{A.6}$$

where

$$A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k\,(h_k - h_{k-1}) \quad ; \quad \bar{A}_{ij} = \frac{1}{Et} A_{ij}$$

$$B_{ij} = \sum_{k=1}^{N} \frac{1}{2} (\bar{Q}_{ij})_k\,(h_k^2 - h_{k-1}^2) \quad ; \quad \bar{B}_{ij} = \frac{2c}{Et^2} B_{ij} \tag{A.7}$$

$$D_{ij} = \sum_{k=1}^{N} \frac{1}{3} (\bar{Q}_{ij})_k\,(h_k^3 - h_{k-1}^3) \quad ; \quad \bar{D}_{ij} = \frac{4c^2}{Et^3} D_{ij} \qquad \text{for } i,j = 1,2,6$$

and

E = reference modulus

ν = reference Poisson's ratio

$$t = \text{laminate thickness} = \sum_{k=1}^{N} (h_k - h_{k-1})$$

$$c = \sqrt{3(1-\nu^2)}$$

Inverting the first constitutive equation one gets

$$[\varepsilon] = [A^*]\,[N] + [B^*]\,[\kappa] \tag{A.8}$$

where

$$[A^*] = [A]^{-1} \quad ; \quad [B^*] = -[A]^{-1}[B] \tag{A.9}$$

Substituting the above equation into the second constitutive equation (A.6) yields

$$[M] = [C^*][N] + [D^*][\kappa] \tag{A.10}$$

where

$$[C^*] = [B][A]^{-1} = -[B^*]^T \quad ; \quad [D^*] = [D] - [B][A]^{-1}[B] \tag{A.11}$$

Finally

$$\bar{A}_{ij}^* = Et\, A_{ij}^* \quad ; \quad \bar{B}_{ij}^* = (2c/t)\, B_{ij}^* \quad ; \quad \bar{D}_{ij}^* = (4c^2/Et^3)\, D_{ij}^* \tag{A.12}$$

The layered composite cylindrical shell body can further be reinforced by using closely spaced rings placed parallel to the y-axis and closely spaced stringers placed parallel to the x-axis (see also Fig. 9). Let

A_s, A_r	= Cross-sectional area of stringer and ring, respectively
d_s, d_r	= Distance between stringers and rings, respectively
E_s, E_r	= Young's modulus of stringer and ring, respectively
e_s, e_r	= Distance between centroid of stiffener cross-section and middle surface of the shell (positive inward)
G_s, G_r	= Shear modulus of stringer and ring, respectively
I_s, I_r	= Moment of inertia of stiffener cross-section about the middle surface of the shell
I_{t_s}, I_{t_r}	= Torsion constant of stiffener cross-section
t	= Thickness of shell
ν_s, ν_r	= Poisson's ratio of stringer and ring, respectively

then one obtains the following additional "smeared" stiffener terms that must be added to the previously derived shell wall stiffness coefficients for layered composite shells

$$A_{11} = A_{11}^s + \frac{E_s A_s}{d_s}$$

$$A_{22} = A_{22}^s + \frac{E_r A_r}{d_r}$$

$$B_{11} = B_{11}^s + e_s \frac{E_s A_s}{d_s}$$

$$B_{22} = B_{22}^s + e_r \frac{E_r A_r}{d_r}$$

$$D_{11} = D_{11}^{s} + \frac{E_s I_s}{d_s}$$

$$D_{22} = D_{22}^{s} + \frac{E_r I_r}{d_r}$$

$$D_{66} = D_{66}^{s} + \frac{1}{4}\left(\frac{G_s I t_s}{d_s} + \frac{G_r I t_r}{d_r}\right)$$

where

$$G_s = \frac{E_s}{2(1+\nu_s)} \quad ; \quad G_r = \frac{E_r}{2(1+\nu_r)}$$

APPENDIX B: The Periodicity Condition

If the solution is to satisfy periodicity in the circumferential direction then the following expression must hold

$$\int_{0}^{2\pi R} v_{,y} dy = 0 \tag{B.1}$$

where from Eq. (128)

$$v_{,y} = A_{12}^{*}F_{,yy} + A_{22}^{*}F_{,xx} - A_{26}^{*}F_{,xy} - B_{21}^{*}W_{,xx} - B_{22}^{*}W_{,yy} - 2B_{26}^{*}W_{,xy} + \frac{W}{R} - \frac{1}{2}W_{,y}(W_{,y} + 2\bar{W}_{,y})$$

Substituting for $\bar{W}$, W and F from Eqs. (8)-(10) and regrouping yields

$$\begin{aligned} v_{,y} = \frac{1}{R}v_{,\theta} = \frac{t}{cR}\Big\{ & -(\bar{A}_{12}^{*}\lambda + \bar{A}_{22}^{*}\bar{p}_e - \bar{A}_{26}^{*}\bar{\tau}) + c(W_v + W_p + W_t) + \bar{A}_{22}^{*}f_o'' - \frac{1}{2}\frac{t}{R}\bar{B}_{21}^{*}w_o'' + cw_o \\ & - \frac{c}{4}\frac{t}{R}n^2[(w_1 + 2A_1)w_1 + (w_2 + 2A_2)w_2]\Big\} \\ & + \frac{t}{cR}\Big\{\tilde{V}_1(\bar{x})\cos n\theta + \tilde{V}_2(\bar{x})\cos 2n\theta + \tilde{V}_3(\bar{x})\sin n\theta + \tilde{V}_4(\bar{x})\sin 2n\theta\Big\} \end{aligned} \tag{B.2}$$

where the functions $\tilde{V}_1(\bar{x}), \ldots, \tilde{V}_4(\bar{x})$ are given by Eqs. (130a) - (130d).

Substituting this expression into Eq. (B.1) and carrying out the y-integration one obtains

$$\begin{aligned} \Big\{ & (cW_v - \bar{A}_{12}^{*}\lambda) + (cW_p - \bar{A}_{22}^{*}\bar{p}_e) + (cW_t + \bar{A}_{26}^{*}\bar{\tau}) + \bar{A}_{22}^{*}f_o'' - \frac{1}{2}\frac{t}{R}\bar{B}_{21}^{*}w_o'' + cw_o \\ & - \frac{c}{4}\frac{t}{R}n^2[(w_1 + 2A_1)w_1 + (w_2 + 2A_2)w_2]\Big\}(2\pi R) = 0 \end{aligned} \tag{B.3}$$

Recalling that from Eq. (20)

$$\bar{A}_{22}^{*}f_o'' - \frac{1}{2}\frac{t}{R}\bar{B}_{21}^{*}w_o'' + cw_o - \frac{c}{4}\frac{t}{R}n^2[(w_1 + 2A_1)w_1 + (w_2 + 2A_2)w_2] = \tilde{C}_1\bar{x} + \tilde{C}_2 \tag{20}$$

Substituting this expression into Eq. (B.3) yields after some simplification

$$\{(cW_v - \bar{A}_{12}^{*}\lambda) + (cW_p - \bar{A}_{22}^{*}\bar{p}_e) + (cW_t + \bar{A}_{26}^{*}\bar{\tau}) + \tilde{C}_1\bar{x} + \tilde{C}_2\} = 0 \tag{B.4}$$

Notice that if one lets

$$W_v = \frac{1}{c}\bar{A}_{12}^{*}\lambda \tag{B.5a}$$

$$W_p = \frac{1}{c} \bar{A}_{22}^{*} \bar{p}_e \tag{B.5b}$$

$$W_t = -\frac{1}{c} \bar{A}_{26}^{*} \bar{\tau} \tag{B.5c}$$

and

$$\tilde{C}_1 = \tilde{C}_2 = 0 \tag{B.6}$$

then the periodicity condition specified by Eq. (B.1) is identically satisfied.

APPENDIX C: The Coefficients Used in the Governing Equations

$$C_1 = \frac{n^2}{\Delta_1}\left(\bar{D}^*_{11}\bar{A}^*_{xxyy} + \bar{B}^*_{21}\bar{B}^*_{xxyy}\right) + 2c\,\frac{R}{t}\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$C_2 = \frac{n^4}{\Delta_1}\left(\bar{D}^*_{11}\bar{A}^*_{11} + \bar{B}^*_{21}\bar{B}^*_{12}\right)$$

$$C_3 = \frac{1}{2}\frac{t}{R}\frac{n^2}{\Delta_1}\left(\bar{D}^*_{11}\bar{B}^*_{xxyy} - \bar{B}^*_{21}\bar{D}^*_{xxyy}\right) + \frac{c}{\Delta_1}\bar{D}^*_{11}$$

$$C_4 = 2c\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$C_5 = \frac{1}{2}\frac{t}{R}\frac{n^4}{\Delta_1}\left(\bar{D}^*_{11}\bar{B}^*_{12} - \bar{B}^*_{21}\bar{D}^*_{22}\right)$$

$$C_6 = \frac{c}{2}\frac{t}{R}n^2\frac{\bar{D}^*_{11}}{\Delta_1}$$

$$C_7 = c\frac{t}{R}n^2\frac{\bar{B}^*_{21}\bar{B}^*_{21}}{\bar{A}^*_{22}\Delta_1}$$

$$C_8 = 2c^2n^2\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}\Delta_1}$$

$$C_9 = \frac{c^2}{2}\frac{t}{R}n^4\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}\Delta_1}$$

$$C_{10} = cn^2\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$C_{11} = 4n^2\frac{\bar{A}^*_{xxyy}}{\bar{A}^*_{22}}$$

$$C_{12} = 16n^4\frac{\bar{A}^*_{11}}{\bar{A}^*_{22}}$$

$$C_{13} = \frac{c}{4}\frac{t}{R}\frac{n^2}{\bar{A}^*_{22}}$$

$$C_{17} = c^2\,\frac{R}{t}\frac{n^2}{\Delta_1}$$

$$C_{18} = 2c\frac{R}{t}n^2\frac{\bar{A}^*_{22}}{\Delta_1}$$

$$C_{19} = 4c\frac{R}{t}\frac{\bar{A}^*_{22}}{\Delta_1}$$

$$C_{20} = \frac{n^2}{\Delta_1}\left(\bar{A}^*_{22}\bar{D}^*_{xxyy} + \bar{B}^*_{21}\bar{B}^*_{xxyy}\right) + 2c\,\frac{R}{t}\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$C_{21} = \frac{n^4}{\Delta_1}\left(\bar{A}^*_{22}\bar{D}^*_{22} + \bar{B}^*_{21}\bar{B}^*_{12}\right)$$

$$C_{22} = 2\frac{R}{t}\frac{n^2}{\Delta_1}\left(\bar{A}^*_{22}\bar{B}^*_{xxyy} - \bar{B}^*_{21}\bar{A}^*_{xxyy}\right) + 4c\,\frac{R^2}{t^2}\frac{\bar{A}^*_{22}}{\Delta_1}$$

$$C_{23} = 4c\frac{R}{t}\frac{\bar{A}^*_{22}}{\Delta_1}$$

$$C_{24} = 2\frac{R}{t}\frac{n^4}{\Delta_1}\left(\bar{A}^*_{22}\bar{B}^*_{12} - \bar{B}^*_{21}\bar{A}^*_{11}\right)$$

$$C_{25} = cn^2\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$C_{26} = 2cn^2\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$C_{27} = 4c^2\frac{R}{t}\frac{n^2}{\Delta_1}$$

$$C_{28} = c^2\frac{n^4}{\Delta_1}$$

$$C_{29} = 2c\frac{R}{t}n^2\frac{\bar{A}^*_{22}}{\Delta_1}$$

$$C_{14}= 4c\frac{R}{t}\frac{\bar{B}_{21}^{*}}{\Delta_1}$$

$$C_{15}= 4c^2\frac{R^2}{t^2}\frac{1}{\Delta_1}$$

$$C_{16}= \frac{c}{2}n^2\frac{\bar{B}_{21}^{*}}{\Delta_1}$$

$$C_{50}= \frac{1}{2}\frac{t}{L}(\bar{B}_{11}^{*}-\bar{B}_{21}^{*}\frac{\bar{A}_{12}^{*}}{\bar{A}_{22}^{*}}+\frac{1}{2}\;\frac{t}{R}\bar{q})$$

$$C_{51}= c\frac{R}{L}\frac{\bar{A}_{12}^{*}}{\bar{A}_{22}^{*}}$$

$$C_{201}= \frac{n}{\Delta_1}(2\bar{D}_{11}^{*}\bar{A}_{26}^{*}-\bar{B}_{21}^{*}\bar{B}_{xxxy}^{*})$$

$$C_{202}= -\frac{n^3}{\Delta_1}(2\bar{D}_{11}^{*}\bar{A}_{16}^{*}-\bar{B}_{21}^{*}\bar{B}_{xyyy}^{*})$$

$$C_{203}= \frac{1}{2}\frac{t}{R}\frac{n}{\Delta_1}(\bar{D}_{11}^{*}\bar{B}_{xxxy}^{*}-B_{21}^{*}\bar{D}_{xxxy}^{*})$$

$$C_{204}= -\frac{1}{2}\frac{t}{R}\frac{n^3}{\Delta_1}(\bar{D}_{11}^{*}\bar{B}_{xyyy}^{*}-\bar{B}_{21}^{*}\bar{D}_{xyyy}^{*})$$

$$C_{205}= 4n\frac{\bar{A}_{26}^{*}}{\bar{A}_{22}^{*}}$$

$$C_{206}= -16n^3\frac{\bar{A}_{16}^{*}}{\bar{A}_{22}^{*}}$$

$$C_{207}= -2\frac{R}{t}\frac{n}{\Delta_1}(\bar{A}_{22}^{*}\bar{B}_{xxxy}^{*}+2\bar{B}_{21}^{*}\bar{A}_{26}^{*})$$

$$C_{550}= \frac{1}{2}\frac{t}{L}(\bar{B}_{21}^{*}\;\frac{\bar{A}_{26}^{*}}{\bar{A}_{22}^{*}}-\bar{B}_{61}^{*})$$

$$C_{551}= c\frac{R}{L}\frac{\bar{A}_{26}^{*}}{\bar{A}_{22}^{*}}$$

$$C_{52}= \frac{c}{4}\frac{t}{L}\frac{\bar{A}_{12}^{*}}{\bar{A}_{22}^{*}}n^2$$

$$C_{53}= \frac{c}{2}\frac{t}{L}$$

$$C_{208}= 2\frac{R}{t}\frac{n^3}{\Delta_1}(\bar{A}_{22}^{*}\bar{B}_{xyyy}^{*}+2\bar{B}_{21}^{*}\bar{A}_{16}^{*})$$

$$C_{209}= -\frac{n}{\Delta_1}(\bar{A}_{22}^{*}\bar{D}_{xxxy}^{*}+\bar{B}_{21}^{*}\bar{B}_{xxxy}^{*})$$

$$C_{210}= \frac{n^3}{\Delta_1}(\bar{A}_{22}^{*}\bar{D}_{xyyy}^{*}+\bar{B}_{21}^{*}\bar{B}_{xyyy}^{*})$$

$$C_{211}= 2cn^2\frac{\bar{B}_{21}^{*}}{\Delta_1}$$

$$C_{212}= 4cn\frac{\bar{B}_{21}^{*}}{\Delta_1}$$

$$C_{213}= 4c\frac{R}{t}n^2\frac{\bar{A}_{22}^{*}}{\Delta_1}$$

$$C_{214}= 8c\frac{R}{t}n\frac{\bar{A}_{22}^{*}}{\Delta_1}$$

$$C_{552}= \frac{c}{4}\frac{t}{L}\frac{\bar{A}_{26}^{*}}{\bar{A}_{22}^{*}}n^2$$

$$C_{553}= \frac{c}{2}\frac{t}{L}n$$

where

$$\Delta_1 = \bar{D}^*_{11}\bar{A}^*_{22}+\bar{B}^*_{21}\bar{B}^*_{21}$$

$$\bar{A}^*_{xxyy} = 2\bar{A}^*_{12}+\bar{A}^*_{66}$$

$$\bar{B}^*_{xxxy} = 2\bar{B}^*_{26}-\bar{B}^*_{61}$$

$$\bar{D}^*_{xxxy} = 2(\bar{D}^*_{16}+\bar{D}^*_{61}) = 4\bar{D}^*_{16}$$

$$\bar{B}^*_{xxyy} = \bar{B}^*_{11}+\bar{B}^*_{22}-2\bar{B}^*_{66}$$

$$\bar{D}^*_{xxyy} = \bar{D}^*_{12}+\bar{D}^*_{21}+4\bar{D}^*_{66} = 2(\bar{D}^*_{12}+2\bar{D}^*_{66})$$

$$\bar{B}^*_{xyyy} = 2\bar{B}^*_{16}-\bar{B}^*_{62}$$

$$\bar{D}^*_{xyyy} = 2(\bar{D}^*_{26}+\bar{D}^*_{62}) = 4\bar{D}^*_{26}$$

and

$$\bar{q} = 4c\frac{R}{t^2}q$$

APPENDIX D: Derivation of the Reduced Boundary Conditions

It is necessary to express the different boundary conditions in terms of the shell variables $\bar{W}$, W and F. Considering successively

1. Boundary condition: $u = u_o$

From the strain-displacement relations

$$u_{,y} = \gamma_{xy} - v_{,x} - W_{,x}W_{,y} - \bar{W}_{,x}W_{,y} - W_{,x}\bar{W}_{,y} \tag{D.1}$$

$$v_{,y} = \varepsilon_y + \frac{W}{R} - \frac{1}{2}W_{,y}^2 - \bar{W}_{,y}W_{,y} \tag{D.2}$$

Differentiating the first equation with respect to y and the second with respect to x and then eliminating the term $v_{,xy}$ between the resulting expressions yields

$$u_{,yy} = \gamma_{xy,y} - \varepsilon_{y,x} - \frac{1}{R}W_{,x} - W_{,x}W_{,yy} - \bar{W}_{,x}W_{,yy} - W_{,x}\bar{W}_{,yy} \tag{D.3}$$

Using the semi-inverted constitutive equations

$$\gamma_{xy} = A_{16}^*F_{,yy} + A_{26}^*F_{,xx} - A_{66}^*F_{,xy} - B_{61}^*W_{,xx} - B_{62}^*W_{,yy} - 2B_{66}^*W_{,xy} \tag{D.4}$$

$$\varepsilon_y = A_{12}^*F_{,yy} + A_{22}^*F_{,xx} - A_{26}^*F_{,xy} - B_{21}^*W_{,xx} - B_{22}^*W_{,yy} - 2B_{66}^*W_{,xy} \tag{D.5}$$

Recalling further the fact that if a function $\phi(x,y)$ in an orthogonal reference frame satisfies the condition

$$\phi(x,y) = C \quad \text{at} \quad x = x_o \tag{D.6}$$

where both C and x_o are constants then

$$\frac{\partial^r}{\partial y^r}\phi(x,y) = 0 \quad \text{at} \quad x = x_o \quad \text{for } r = 1,2,\ldots \tag{D.7}$$

hence since $u = u_o$ at $x = 0,L$, therefore $u_{,yy} = 0$ at the shell edges. Thus specializing Eq. (D.3) to the shell edges, one obtains upon substitution for γ_{xy} and ε_y and some regrouping

$$\begin{aligned} &\frac{1}{Et}\{ -\bar{A}_{22}^*F_{,xxx} + 2\bar{A}_{26}^*F_{,xxy} - (\bar{A}_{12}^* + \bar{A}_{66}^*)F_{,xyy} + \bar{A}_{16}^*F_{,yyy} \} \\ &+ \frac{t}{2c}\{ \bar{B}_{21}^*W_{,xxx} + (2\bar{B}_{26}^* - \bar{B}_{61}^*)W_{,xxy} + (\bar{B}_{22}^* - 2\bar{B}_{66}^*)W_{,xyy} - \bar{B}_{62}^*W_{,yyy} \} \\ &- \frac{1}{R}W_{,x} - (W_{,x} + \bar{W}_{,x})W_{,yy} - W_{,x}\bar{W}_{,yy} = 0 \end{aligned} \tag{D.8}$$

Substituting now for $\bar{W}$, W and F from Eqs. (8)-(10) and regrouping yields

$$
\begin{aligned}
&\{\, \bar{A}^*_{22} f_0''' - \frac{1}{2}\frac{t}{R}\bar{B}^*_{21} w_0''' + c w_0' - \frac{c}{2}\frac{t}{R} n^2 [w_1'(w_1 + A_1) + w_1 A_1' + w_2'(w_2 + A_2) + w_2 A_2'] \,\} \\
&+ \{\, \bar{A}^*_{22} f_1''' - (\bar{A}^*_{12} + \bar{A}^*_{66}) n^2 f_1' - 2\bar{A}^*_{26} n^3 f_3'' + \bar{A}^*_{16} n^3 f_3 + c w_1' - c\,\frac{t}{R} n^2 [w_0'(w_1 + A_1) + w_1 A_0'] \\
&\quad - \frac{1}{2}\frac{t}{R} [\bar{B}^*_{21} w_1''' - (\bar{B}^*_{22} - 2\bar{B}^*_{66}) n^2 w_1' + (2\bar{B}^*_{26} - \bar{B}^*_{61}) n w_2'' + \bar{B}^*_{62} n^3 w_2] \,\} \cos n\theta \\
&+ \{\, \bar{A}^*_{22} f_2''' - (\bar{A}^*_{12} + \bar{A}^*_{66}) 4n^2 f_2' - 4\bar{A}^*_{26} n f_4'' + 8\bar{A}^*_{16} n^3 f_4 \\
&\quad - \frac{c}{2}\frac{t}{R} n^2 [w_1'(w_1 + A_1) + w_1 A_1' - w_2'(w_2 + A_2) - w_2 A_2'] \,\} \cos 2n\theta \\
&+ \{\, \bar{A}^*_{22} f_3''' - (\bar{A}^*_{12} + \bar{A}^*_{66}) n^2 f_3' + 2\bar{A}^*_{26} n f_1'' - \bar{A}^*_{16} n^3 f_1 + c w_2' - c\,\frac{t}{R} n^2 [w_0'(w_2 + A_2) + w_2 A_0'] \\
&\quad - \frac{1}{2}\frac{t}{R} [\bar{B}^*_{21} w_2''' - (\bar{B}^*_{22} - 2\bar{B}^*_{66}) n^2 w_2' - (2\bar{B}^*_{26} - \bar{B}^*_{61}) n w_1'' - \bar{B}^*_{62} n^3 w_1] \,\} \sin n\theta \\
&+ \{\, \bar{A}^*_{22} f_4''' - (\bar{A}^*_{12} + \bar{A}^*_{66}) 4n^2 f_4' + 4\bar{A}^*_{26} n f_2'' - 8\bar{A}^*_{16} n^3 f_2 \\
&\quad - \frac{c}{2}\frac{t}{R} n^2 [w_1'(w_2 + A_2) + w_1 A_2' + w_2'(w_1 + A_1) + w_2 A_1'] \,\} \sin 2n\theta = 0
\end{aligned}
\tag{D.9}
$$

This expression must hold for all values of $\theta = y/R$, which implies that the expressions in the curly brackets must be equal to zero. This yields then 5 conditions that $u = u_o$ at $x = 0, L/R$ implies. Notice that the first of these equations does actually not represent a new (boundary) condition, since it can be obtained from Eq. (20) by a single differentiation with respect to $\bar{x} = x/R$.

2. Boundary condition: $v = v_o$

From the strain-displacement relations and using the semi-inverted constitutive equations one gets

$$
\begin{aligned}
v_{,y} &= \varepsilon_y + \frac{W}{R} - \frac{1}{2} W_{,y}(W_{,y} + 2\bar{W}_{,y}) \\
&= A^*_{12} F_{,yy} + A^*_{22} F_{,xx} - A^*_{26} F_{,xy} - B^*_{21} W_{,xx} - B^*_{22} W_{,yy} - 2B^*_{26} W_{,xy} + \frac{W}{R} - \frac{1}{2} W_{,y}(W_{,y} + 2\bar{W}_{,y})
\end{aligned}
\tag{D.10}
$$

Specializing to the shell edges at $x = 0, L/R$ implies $v_{,y} = 0$.

Hence upon substituting for $\bar{W}$, W and F from Eqs. (8)-(10) one obtains after some regrouping

$$
\begin{aligned}
&\{ (cW_v - \bar{A}^*_{12}\lambda) + (cW_{p_e} - \bar{A}^*_{22}\bar{p}_e) + (cW_t + \bar{A}^*_{26}\bar{\tau}) \\
&\underline{\quad + \bar{A}^*_{22}f_o'' - \frac{1}{2}\frac{t}{R}\bar{B}^*_{21}w_o'' + cw_o - \frac{c}{4}\frac{t}{R}n^2[w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)] \}} \\
&\quad + \{ \bar{A}^*_{22}f_1'' - \bar{A}^*_{12}n^2f_1 - \bar{A}^*_{26}nf_3' + cw_1 - \frac{1}{2}\frac{t}{R}(\bar{B}^*_{21}w_1'' - \bar{B}^*_{22}n^2w_1 + 2\bar{B}^*_{26}nw_2') \} \cos n\theta \\
&\quad + \{ \bar{A}^*_{22}f_2'' - 4\bar{A}^*_{12}n^2f_2 - 2\bar{A}^*_{26}nf_4' + \frac{c}{4}\frac{t}{R}n^2[w_1(w_1 + 2A_1) - w_2(w_2 + 2A_2)] \} \cos 2n\theta \\
&\quad + \{ \bar{A}^*_{22}f_3'' - \bar{A}^*_{12}n^2f_3 + \bar{A}_{26}nf_1' + cw_2 - \frac{1}{2}\frac{t}{R}(\bar{B}^*_{21}w_2'' - \bar{B}^*_{22}n^2w_2 - 2\bar{B}^*_{26}nw_1') \} \sin n\theta \\
&\quad + \{ \bar{A}^*_{22}f_4'' - 4\bar{A}^*_{12}n^2f_4 + 2\bar{A}^*_{26}nf_2' + \frac{c}{4}\frac{t}{R}n^2[w_2(w_1 + 2A_1) + w_1(w_2 + 2A_2)] \} \sin 2n\theta = 0
\end{aligned}
\tag{D.11}
$$

This expression must hold for all values of $\theta = y/R$, which implies that the expressions in the curly brackets must be equal to zero. This yields then 5 conditions that $v = v_o$ at $x = 0, L/R$ implies. Notice that the first of these equations does actually not represent a new (boundary) condition, since the underscored part is just Eq. (20). Thus it vanishes identically. The remaining terms are also identically equal to zero if one introduces for W_v, W_{p_e}, and W_t their values given by Eqs. (B.5).

3. Boundary condition: W = 0

Substituting for W from Eq. (9) yields

$$W_v + W_{p_e} + W_t + w_o + w_1 \cos n\theta + w_2 \sin n\theta = 0 \tag{D.12}$$

For this expression to be true for all values of $\theta = y/R$, the following boundary conditions must hold at $\bar{x} = 0, L/R$:

$$w_o = -(W_v + W_{p_e} + W_t) \tag{D.13a}$$

$$w_1 = w_2 = 0 \tag{D.13b}$$

4. Boundary condition: $W_{,x} = 0$

Substituting for $W_{,x}$ from Eq. (9) yields

$$w_o' + w_1' \cos n\theta + w_2' \sin n\theta = 0 \tag{D.14}$$

For this expression to be true for all values of $\theta = y/R$, the following conditions must hold at $\bar{x} = 0, L/R$

$$w_o' = w_1' = w_2' = 0 \tag{D.15}$$

5. Boundary condition $N_x = -N_o$

Substituting for F from Eq. (10) one obtains

$$N_x = F_{,yy} = \frac{Et^2}{cR}\left(-\lambda - n^2 f_1 \cos n\theta - 4n^2 f_2 \cos 2n\theta - n^2 f_3 \sin n\theta - 4n^2 f_4 \sin 2n\theta\right) = -N_o \tag{D.16}$$

For this expression to be true for all values of $\theta = y/R$, the following boundary conditions must hold at $\bar{x} = 0, L/R$

$$\lambda = -N_o \frac{cR}{Et^2} \tag{D.17}$$

$$f_1 = f_2 = f_3 = f_4 = 0 \tag{D.18}$$

Notice that there are only 4 new boundary conditions, since the condition given by Eq. (D.17) is just the definition of the nondimensional axial load parameter λ.

6. Boundary condition: $N_{xy} = S_o$

Substituting for F from Eq. (10) one obtains

$$N_{xy} = -F_{,xy} = -\frac{Et^2}{cR}\left(-\bar{\tau} - nf_1' \sin n\theta - 2nf_2' \sin 2n\theta + nf_3' \cos n\theta + 2nf_4' \cos 2n\theta\right) = S_o \tag{D.19}$$

For this expression to be true for all values of $\theta = y/R$, the following boundary conditions must hold at $\bar{x} = 0, L/R$

$$\bar{\tau} = S_o \frac{cR}{Et^2} \tag{D.20}$$

$$f_1' = f_2' = f_3' = f_4' = 0 \tag{D.21}$$

Notice that there are, once again, only 4 new boundary conditions, since the condition given by Eq. (D.20) is just the definition of the nondimensional torque parameter $\bar{\tau}$.

7. Boundary condition: H = 0

From the derivation of the nonlinear equilibrium equations using the stationary potential energy criterion[30], by definition

$$H = M_{x,x} + (M_{xy} + M_{yx})_{,y} + N_x(W_{,x} + \bar{W}_{,x}) + N_{xy}(W_{,y} + \bar{W}_{,y}) = 0 \tag{D.22}$$

Using the appropriate semi-inverted constitutive equations one obtains

$$H = - B^*_{11}F_{,xyy} - B^*_{21}F_{,xxx} + B^*_{61}F_{,xxy} - D^*_{11}W_{,xxx} - D^*_{12}W_{,xyy} - 2D^*_{16}W_{,xxy}$$

$$+ 2(- B^*_{16}F_{,yyy} - B^*_{26}F_{,xxy} + B^*_{66}F_{,xyy} - D^*_{16}W_{,xxy} - D^*_{26}W_{,yyy} - 2D^*_{66}W_{,xyy}) \qquad \text{(D.23)}$$

$$+ F_{,yy}(W_{,x} + \bar{W}_{,x}) - F_{,xy}(W_{,y} + \bar{W}_{,y}) = 0$$

Substituting for $\bar{W}$, W and F from Eqs. (8)-(10) and regrouping one obtains the following residual

$$\begin{aligned}
\varepsilon_H = \{ & - \bar{B}^*_{21}f_o''' - \frac{1}{2}\frac{t}{R}\bar{D}^*_{11}w_o''' - 2c\lambda(w_o' + A_o') - cn^2[f_1(w_1' + A_1') + f_3(w_2' + A_2') + f_1'(w_1 + A_1) + f_3'(w_2 + A_2)] \} \\
+ \{ & \frac{1}{2}\frac{t}{R}[- \bar{D}^*_{11}w_1''' + (\bar{D}^*_{12} + 4\bar{D}^*_{66})n^2w_1' - 4\bar{D}^*_{16}nw_2'' + 2\bar{D}^*_{26}n^3w_2] - 2c\lambda(w_1' + A_1') + 2cn\bar{\tau}(w_2 + A_2) \\
& - \bar{B}^*_{21}f_1''' + (\bar{B}^*_{11} - 2\bar{B}^*_{66})n^2f_1' + (\bar{B}^*_{61} - 2\bar{B}^*_{26})nf_3'' + 2\bar{B}^*_{16}n^3f_3 \\
& - 2cn^2[f_1(w_o' + A_o') + 2f_2(w_1' + A_1') + 2f_4(w_2' + A_2') + f_2'(w_1 + A_1) + f_4'(w_2 + A_2)] \} \cos n\theta \\
+ \{ & - \bar{B}^*_{21}f_2''' + 4(\bar{B}^*_{11} - 2\bar{B}^*_{66})n^2f_2' + 2(\bar{B}^*_{61} - 2\bar{B}^*_{26})nf_4'' + 16\bar{B}^*_{16}n^3f_4 \\
& - cn^2[8f_2(w_o' + A_o') + f_1(w_1' + A_1') - f_3(w_2' + A_2') - f_1'(w_1 + A_1) + f_3'(w_2 + A_2)] \} \cos 2n\theta \\
+ \{ & - cn^2[4f_2(w_1' + A_1') - 4f_4(w_2' + A_2') - 2f_2'(w_1 + A_1) + 2f_4'(w_2 + A_2)] \} \cos 3n\theta \\
+ \{ & \frac{1}{2}\frac{t}{R}[- \bar{D}^*_{11}w_2''' + (\bar{D}^*_{12} + 4\bar{D}^*_{66})n^2w_2' + 4\bar{D}^*_{16}nw_1'' - 2\bar{D}^*_{26}n^3w_1] - 2c\lambda(w_2' + A_2') - 2cn\bar{\tau}(w_1 + A_1) \\
& - \bar{B}^*_{21}f_3''' + (\bar{B}^*_{11} - 2\bar{B}^*_{66})n^2f_3' - (\bar{B}^*_{61} - 2\bar{B}^*_{26})nf_1'' - 2\bar{B}^*_{16}n^3f_1 \\
& - 2cn^2[f_3(w_o' + A_o') + 2f_4(w_1' + A_1') - 2f_2(w_2' + A_2') + f_4'(w_1 + A_1) - f_2'(w_2 + A_2)] \} \sin n\theta \\
+ \{ & - \bar{B}^*_{21}f_4''' + 4(\bar{B}^*_{11} - 2\bar{B}^*_{66})n^2f_4' - 2(\bar{B}^*_{61} - 2\bar{B}^*_{26})nf_2'' - 16\bar{B}^*_{16}n^3f_2 \\
& - cn^2[8f_4(w_o' + A_o') + f_3(w_1' + A_1') + f_1(w_2' + A_2') - f_3'(w_1 + A_1) - f_1'(w_2 + A_2)] \} \sin 2n\theta \\
& + \{-cn^2[4f_2(w_1' + A_1') - 4f_4(w_2' + A_2') - 2f_2'(w_1 + A_1) + 2f_4'(w_2 + A_2)]\}\cos 3n\theta \\
+ \{ & - cn^2[4f_4(w_1' + A_1') + 4f_2(w_2' + A_2') - 2f_4'(w_1 + A_1) - 2f_2'(w_2 + A_2)] \} \sin 3n\theta = 0
\end{aligned} \qquad \text{(D.24)}$$

Here the θ = y/R dependence is eliminated by using the following 3 Galerkin integrals

$$\int_0^{2\pi} \varepsilon_H d\theta = 0 \qquad \text{(D.25)}$$

$$\int_0^{2\pi} \varepsilon_H \cos n\theta \, d\theta = 0 \tag{D.26}$$

$$\int_0^{2\pi} \varepsilon_H \sin n\theta \, d\theta = 0 \tag{D.27}$$

Notice that these are the same Galerkin integrals that have been used earlier to eliminate the y-dependence from the out-of-plane equilibrium equation (3). Substituting for ε_H and carrying out the integrals involved one obtains the following boundary conditions at $\bar{x} = 0, L/R$

$$\begin{aligned} &\bar{B}^*_{21} f_o''' + \frac{1}{2}\frac{t}{R}\bar{D}^*_{11} w_o''' + 2c\lambda(w_o' + A_o') \\ &\qquad + cn^2[f_1(w_1' + A_1') + f_3((w_2' + A_2') + f_1'(w_1 + A_1) + f_3'(w_2 + A_2)] = 0 \end{aligned} \tag{D.28}$$

$$\begin{aligned} &\bar{B}^*_{21} f_1''' - (\bar{B}^*_{11} - 2\bar{B}^*_{66}) n^2 f_1' - (\bar{B}^*_{61} - 2\bar{B}^*_{26}) n f_3'' - 2\bar{B}^*_{16} n^3 f_3 + 2c\lambda(w_1' + A_1') - 2cn\bar{\tau}(w_2 + A_2) \\ &\qquad + \frac{1}{2}\frac{t}{R}[\bar{D}^*_{11} w_1''' - (\bar{D}^*_{12} + 4\bar{D}^*_{66}) n^2 w_1' + 4\bar{D}^*_{16} n w_2'' - 2\bar{D}^*_{26} n^3 w_2] \\ &\qquad + 2cn^2[f_1(w_o' + A_o') + 2f_2(w_1' + A_1') + 2f_4(w_2' + A_2') + f_2'(w_1 + A_1) + f_4'(w_2 + A_2)] = 0 \end{aligned} \tag{D.29}$$

$$\begin{aligned} &\bar{B}^*_{21} f_3''' - (\bar{B}^*_{11} - 2\bar{B}^*_{66}) n^2 f_3' + (\bar{B}^*_{61} - 2\bar{B}^*_{26}) n f_1'' + 2\bar{B}^*_{16} n^3 f_1 + 2c\lambda(w_2' + A_2') + 2cn\bar{\tau}(w_1 + A_1) \\ &\qquad + \frac{1}{2}\frac{t}{R}[\bar{D}^*_{11} w_2''' - (\bar{D}^*_{12} + 4\bar{D}^*_{66}) n^2 w_2' - 4\bar{D}^*_{16} n w_1'' + 2\bar{D}^*_{26} n^3 w_1] \\ &\qquad + 2cn^2[f_3(w_o' + A_o') + 2f_4(w_1' + A_1') - 2f_2(w_2' + A_2') + f_4'(w_1 + A_1) - f_2'(w_2 + A_2)] = 0 \end{aligned} \tag{D.30}$$

But now, differentiating Eq. (20) once with respect to $\bar{x} = x/R$ one obtains

$$f_o''' = \frac{1}{2}\frac{t}{R}\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}} w_o''' - \frac{c}{\bar{A}^*_{22}} w_o' + \frac{c}{2}\frac{t}{R}\frac{n^2}{\bar{A}^*_{22}}[w_1'(w_1 + A_1) + w_1 A_1' + w_2'(w_2 + A_2) + w_2 A_2'] \tag{D.31}$$

Thus upon substituting and regrouping the first condition, Eq. (D.28), becomes

$$\begin{aligned} &\frac{1}{2}\frac{t}{R}\left(\bar{D}^*_{11} + \frac{\bar{B}^*_{21}\bar{B}^*_{21}}{\bar{A}^*_{22}}\right) w_o''' - c\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}} w_o' + 2c\lambda(w_o' + A_o') \\ &\qquad + \frac{c}{2}\frac{t}{R}\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}} n^2[w_1'(w_1 + A_1) + w_1 A_1' + w_2'(w_2 + A_2) + w_2 A_2'] \\ &\qquad + cn^2[f_1(w_1' + A_1') + f_3(w_2' + A_2') + f_1'(w_1 + A_1) + f_3'(w_2 + A_2)] = 0 \end{aligned} \tag{D.32}$$

8. Boundary condition: $M_x = -N_o q$

Using the appropriate semi-inverted constitutive equation one gets

$$M_x = -B^*_{11}F_{,yy} - B^*_{21}F_{,xx} + B^*_{61}F_{,xy} - D^*_{11}W_{,xx} - D^*_{12}W_{,yy} - 2D^*_{16}W_{,xy} = -N_o q \qquad (D.33)$$

Substituting for W and F from Eqs. (9)-(10) and regrouping one obtains the following residual

$$\begin{aligned}
\varepsilon_M = \{ & -\bar{B}^*_{11}\lambda - \bar{B}^*_{21}\bar{p}_e + \bar{B}^*_{61}\bar{\tau} + \bar{B}^*_{21}f''_o + \frac{1}{2}\frac{t}{R}\bar{D}^*_{11}w''_o - \frac{1}{2}\frac{t}{R}\lambda\bar{q} \} \\
& + \{ \bar{B}^*_{21}f''_1 - \bar{B}^*_{11}n^2f_1 - \bar{B}^*_{61}nf'_3 + \frac{1}{2}\frac{t}{R}(\bar{D}^*_{11}w''_1 - \bar{D}^*_{12}n^2w_1 + 2\bar{D}^*_{16}nw'_2) \} \cos n\theta \\
& + \{ \bar{B}^*_{21}f''_2 - 4\bar{B}^*_{11}n^2f_2 - 2\bar{B}^*_{61}nf'_4 \} \cos 2n\theta \\
& + \{ \bar{B}^*_{21}f''_3 - \bar{B}^*_{11}n^2f_3 + \bar{B}^*_{61}nf'_1 + \frac{1}{2}\frac{t}{R}(\bar{D}^*_{11}w''_2 - \bar{D}^*_{12}n^2w_2 - 2\bar{D}^*_{16}nw'_1) \} \sin n\theta \\
& + \{ \bar{B}^*_{21}f''_4 - 4\bar{B}^*_{11}n^2f_4 + 2\bar{B}^*_{61}nf'_2 \} \sin 2n\theta = 0
\end{aligned} \qquad (D.34)$$

where $\bar{q} = 4c(R/t^2)q$ and q is the load eccentricity.

Here the $\theta = y/R$ dependence is eliminated by using the previously defined (see Eqs. (D.25)-(D.27)) 3 Galerkin integrals. Substituting for ε_M and carrying out the integrals involved yields the following boundary conditions at $\bar{x} = 0, L/R$

$$\bar{B}^*_{21}f''_o + \frac{1}{2}\frac{t}{R}\bar{D}^*_{11}w''_o - (\bar{B}^*_{11} + \frac{1}{2}\frac{t}{R}\bar{q})\lambda - \bar{B}^*_{21}\bar{p}_e + \bar{B}^*_{61}\bar{\tau} = 0 \qquad (D.35)$$

$$\bar{B}^*_{21}f''_1 - \bar{B}^*_{11}n^2f_1 - \bar{B}^*_{61}nf'_3 + \frac{1}{2}\frac{t}{R}(\bar{D}^*_{11}w''_1 - \bar{D}^*_{12}n^2w_1 + 2\bar{D}^*_{16}nw'_2) = 0 \qquad (D.36)$$

$$\bar{B}^*_{21}f''_3 - \bar{B}^*_{11}n^2f_3 + \bar{B}^*_{61}nf'_1 + \frac{1}{2}\frac{t}{R}(\bar{D}^*_{11}w''_2 - \bar{D}^*_{12}n^2w_2 - 2\bar{D}^*_{16}nw'_1) = 0 \qquad (D.37)$$

Substituting for f''_o from Eq. (20) and regrouping the first condition, Eq. (D.35), becomes

$$\begin{aligned}
& \frac{1}{2}\frac{t}{R}(\bar{D}^*_{11} + \frac{\bar{B}^*_{21}\bar{B}^*_{21}}{\bar{A}^*_{22}})w''_o - c\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}w_o - (\bar{B}^*_{11} + \frac{1}{2}\frac{t}{R}\bar{q})\lambda - \bar{B}^*_{21}\bar{p}_e + \bar{B}^*_{61}\bar{\tau} \\
& \qquad + \frac{c}{4}\frac{t}{R}\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}n^2[w_1(w_1 + 2A_1) + w_2(w_2 + 2A_2)] = 0
\end{aligned} \qquad (D.38)$$

The 8 different edge conditions discussed previously can be combined to yield besides the 4 different simply supported and the 4 different clamped boundary conditions, the free edge conditions and, if the initial imperfections are chosen accordingly the symmetry conditions at $\bar{x} = \frac{1}{2}\frac{L}{R}$.

SS-1 Boundary Condition

$N_x = -N_o \quad \rightarrow \quad f_1 = f_2 = f_3 = f_4 = 0$ (D.39)

$N_{xy} = S_o \quad \rightarrow \quad f_1' = f_2' = f_3' = f_4' = 0$ (D.40)

$W = 0 \quad \rightarrow \quad w_o = -(W_v + W_{p_e} + W_t)$ (D.41a)

$w_1 = w_2 = 0$ (D.41b)

$M_x = -N_o q \quad \rightarrow \quad w_o'' = C_{41}\lambda + C_{220}\bar{\tau}$ (D.42a)

$w_1'' = -B_1 f_1'' - B_2 w_2'$ (D.42b)

$w_2'' = -B_1 f_3'' + B_2 w_1'$ (D.42c)

where

$$B_1 = \frac{2R}{t}\frac{\bar{B}^*_{12}}{\bar{D}^*_{11}} \quad ; \quad C_{41} = \frac{1}{\Delta_1}\left\{\frac{2R}{t}(\bar{A}^*_{22}\bar{B}^*_{11} - \bar{A}^*_{12}\bar{B}^*_{21}) + \bar{A}^*_{22}\bar{q}\right\}$$

(D.43)

$$B_2 = \frac{2\bar{D}^*_{16}}{\bar{D}^*_{11}}n \quad ; \quad C_{220} = \frac{1}{\Delta_1}\frac{2R}{t}(\bar{A}^*_{26}\bar{B}^*_{21} - \bar{A}^*_{22}\bar{B}^*_{61}) \quad \text{and} \quad \Delta_1 = \bar{A}^*_{22}\bar{D}^*_{11} + \bar{B}^*_{21}\bar{B}^*_{21}$$

SS-2 Boundary Condition

$u = u_o \quad \rightarrow \quad f_1''' = B_5 f_3'' - B_6 f_3 - B_7 w_1' + B_8 w_1''' + 2B_9 A_1 w_o'$ (D.44a)

$f_2''' = 2B_{10} f_4'' - 8B_{11} f_4 + B_9(A_1 w_1' - A_2 w_2')$ (D.44b)

$f_3''' = -B_5 f_1'' + B_6 f_1 - B_7 w_2' + B_8 w_2''' + 2B_9 A_2 w_o'$ (D.44c)

$f_4''' = -2B_{10} f_2'' + 8B_{11} f_2 + B_9(A_2 w_1' + A_1 w_2')$ (D.44d)

$$N_{xy} = S_o \quad \rightarrow \quad f_1' = f_2' = f_3' = f_4' = 0 \qquad (D.45)$$

$$W = 0 \quad \rightarrow \quad w_o = -(W_v + W_{p_e} + W_t) \qquad (D.46a)$$

$$w_1 = w_2 = 0 \qquad (D.46b)$$

$$M_x = -N_o q \quad \rightarrow \quad w_o'' = C_{41}\lambda + C_{220}\bar{\tau} \qquad (D.47a)$$

$$w_1'' = -B_1 f_1'' + B_3 f_1 - B_4 w_2' \qquad (D.47b)$$

$$w_2'' = -B_1 f_3'' + B_3 f_3 + B_4 w_1' \qquad (D.47c)$$

where

$$B_3 = \frac{2R}{t} \frac{\bar{B}^*_{11}}{\bar{D}^*_{11}} n^2 \qquad B_8 = \frac{1}{2} \frac{t}{R} \frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}$$

$$B_4 = \frac{2\bar{D}^*_{16}}{\bar{D}^*_{11}} n = B_2 \qquad B_9 = \frac{c}{2} \frac{t}{R} \frac{n^2}{\bar{A}^*_{22}}$$

$$B_5 = \frac{n}{\bar{A}^*_{22}} [\, 2\bar{A}^*_{26} - \frac{\bar{B}^*_{21}}{\bar{D}^*_{11}} (2\bar{B}^*_{26} - \bar{B}^*_{61}) \,] \qquad B_{10} = 2n \frac{\bar{A}^*_{26}}{\bar{A}^*_{22}} \qquad (D.48)$$

$$B_6 = \frac{n^3}{\bar{A}^*_{22}} [\, \bar{A}^*_{16} - \frac{\bar{B}^*_{11}}{\bar{D}^*_{11}} (2\bar{B}^*_{26} - \bar{B}^*_{61}) \,] \qquad B_{11} = n^3 \frac{\bar{A}^*_{16}}{\bar{A}^*_{22}}$$

$$B_7 = \frac{1}{\bar{A}^*_{22}} \{ c + \frac{1}{2} \frac{t}{R} n^2 [\, \bar{B}^*_{22} - 2\bar{B}^*_{26} - \frac{2\bar{D}^*_{16}}{\bar{D}^*_{11}} (2\bar{B}^*_{26} - \bar{B}^*_{61}) \,] \} \qquad B_{12} = n^2 \frac{\bar{D}^*_{12}}{\bar{D}^*_{11}}$$

SS-3 Boundary Condition

$$N_x = -N_o \quad \rightarrow \quad f_1 = f_2 = f_3 = f_4 = 0 \qquad (D.49)$$

$$v = 0 \quad \rightarrow \quad f_1'' = B_{15} f_3' + B_{16} w_2' \qquad (D.50a)$$

$$f_2'' = 2B_{17} f_4' \qquad (D.50b)$$

$$f_3'' = -B_{15}f_1' - B_{16}w_1' \quad \text{(D.50c)}$$

$$f_4'' = -2B_{17}f_2' \quad \text{(D.50d)}$$

$W = 0 \quad \rightarrow$

$$w_o = -(W_v + W_{p_e} + W_t) \quad \text{(D.51a)}$$

$$w_1 = w_2 = 0 \quad \text{(D.51b)}$$

$M_x = -N_o q \quad \rightarrow$

$$w_o'' = C_{41}\lambda + C_{220}\bar{\tau} \quad \text{(D.52a)}$$

$$w_1'' = B_{18}f_3' - B_{19}w_2' \quad \text{(D.52b)}$$

$$w_2'' = -B_{18}f_1' + B_{19}w_1' \quad \text{(D.52c)}$$

where

$$B_{15} = \frac{n}{\Delta_1}(\bar{A}^*_{26}\bar{D}^*_{11} + \bar{B}^*_{21}\bar{B}^*_{61}) \qquad B_{18} = 2\,\frac{R}{t}\,\frac{n}{\Delta_1}(\bar{A}^*_{22}\bar{B}^*_{61} - \bar{A}^*_{26}\bar{B}^*_{21})$$

$$B_{16} = \frac{t}{R}\,\frac{n}{\Delta_1}(\bar{B}^*_{26}\bar{D}^*_{11} - \bar{B}^*_{21}\bar{D}^*_{16}) \qquad B_{19} = 2\,\frac{n}{\Delta_1}(\bar{A}^*_{22}\bar{D}^*_{16} + \bar{B}^*_{21}\bar{B}^*_{26}) \quad \text{(D-53)}$$

$$B_{17} = n\,\frac{\bar{A}^*_{26}}{\bar{A}^*_{22}} \qquad \Delta_1 = \bar{A}^*_{22}\bar{D}^*_{11} + \bar{B}^*_{21}\bar{B}^*_{21}$$

SS-4 Boundary Condition

$u = u_o \quad \rightarrow$

$$f_1''' = B_{28}f_1' - B_{29}f_3 + B_8 w_1''' + B_{30}w_1' + 2B_9A_1w_o' \quad \text{(D.54a)}$$

$$f_2''' = 4B_{31}f_2' - 8B_{32}f_4 + B_9A_1w_1' - B_9A_2w_2' \quad \text{(D.54b)}$$

$$f_3''' = B_{28}f_3' + B_{29}f_1 + B_8w_2''' + B_{30}w_2' + 2B_9A_2w_o' \quad \text{(D.54c)}$$

$$f_4''' = 4B_{31}f_4' + 8B_{32}f_2 + B_9A_2w_1' + B_9A_1w_2' \quad \text{(D.54d)}$$

$v = v_o \quad \rightarrow$

$$f_1'' = B_{20}f_1 + B_{15}f_3' + B_{16}w_2' \quad \text{(D.55a)}$$

$$f_2'' = 4B_{24}f_2 + 2B_{17}f_4' \quad \text{(D.55b)}$$

$$f_3'' = B_{20} f_3 - B_{15} f_1' - B_{16} w_1' \quad \text{(D.55c)}$$

$$f_4'' = 4B_{24} f_4 - 2B_{17} f_2' \quad \text{(D.55d)}$$

$$W = 0 \quad \rightarrow \quad w_o = -(W_v + W_{p_e} + W_t) \quad \text{(D.56a)}$$

$$w_1 = w_2 = 0 \quad \text{(D.56b)}$$

$$M_x = -N_o q \quad \rightarrow \quad w_o'' = C_{41}\lambda + C_{220}\bar{\tau} \quad \text{(D.57a)}$$

$$w_1'' = B_{25} f_1 + B_{18} f_3' - B_{19} w_2' \quad \text{(D.57b)}$$

$$w_2'' = B_{25} f_3 - B_{18} f_1' + B_{19} w_1' \quad \text{(D.57c)}$$

where

$$B_{20} = \frac{n^2}{\Delta_1} (\bar{A}_{12}^* \bar{D}_{11}^* + \bar{B}_{11}^* \bar{B}_{21}^*)$$

$$B_{23} = 4c \frac{R}{t} \frac{n}{\Delta_1} \bar{A}_{22}^*$$

$$B_{24} = n^2 \frac{\bar{A}_{12}^*}{\bar{A}_{22}^*}$$

$$B_{25} = 2 \frac{R}{t} \frac{n^2}{\Delta_1} (\bar{A}_{22}^* \bar{B}_{11}^* - \bar{A}_{12}^* \bar{B}_{21}^*)$$

$$B_{28} = \frac{n^2}{\bar{A}_{22}^*} \{ \bar{A}_{12}^* + \bar{A}_{66}^* - \frac{2\bar{A}_{26}^*}{\Delta_1} (\bar{A}_{26}^* \bar{D}_{11}^* + \bar{B}_{21}^* \bar{B}_{61}^*) + \frac{(2\bar{B}_{26}^* - \bar{B}_{61}^*)}{\Delta_1} (\bar{A}_{26}^* \bar{B}_{21}^* - \bar{A}_{22}^* \bar{B}_{61}^*) \}$$

$$B_{29} = \frac{n^3}{\bar{A}_{22}^*} \{ \bar{A}_{16}^* - \frac{2\bar{A}_{26}^*}{\Delta_1} (\bar{A}_{12}^* \bar{D}_{11}^* + \bar{B}_{11}^* \bar{B}_{21}^*) - \frac{(2\bar{B}_{26}^* - \bar{B}_{61}^*)}{\Delta_1} (\bar{A}_{22}^* \bar{B}_{11}^* - \bar{A}_{12}^* \bar{B}_{21}^*) \} \quad \text{(D.58)}$$

$$B_{30} = \frac{1}{\bar{A}_{22}^*} \{ \frac{t}{R} n^2 [-\frac{1}{2} (\bar{B}_{22}^* - 2\bar{B}_{66}^*) + \frac{2\bar{A}_{26}^*}{\Delta_1} (\bar{B}_{21}^* \bar{D}_{16}^* - \bar{B}_{26}^* \bar{D}_{11}^*) + \frac{(2\bar{B}_{26}^* - \bar{B}_{61}^*)}{\Delta_1} (\bar{A}_{22}^* \bar{D}_{16}^* + \bar{B}_{21}^* \bar{B}_{26}^*)] - c \}$$

$$B_{31}=\frac{n^2}{\bar{A}^*_{22}}\left(\bar{A}^*_{12}+\bar{A}^*_{66}-\frac{2\bar{A}^*_{26}\bar{A}^*_{26}}{\bar{A}^*_{22}}\right).$$

$$B_{32}=\frac{n^3}{\bar{A}^*_{22}}\left(\bar{A}^*_{16}-\frac{2\bar{A}^*_{12}\bar{A}^*_{26}}{\bar{A}^*_{22}}\right)\qquad \Delta_1=\bar{A}^*_{22}\bar{D}^*_{11}+\bar{B}^*_{21}\bar{B}^*_{21}$$

C-1 Boundary Condition

$N_x = -N_o \rightarrow f_1=f_2=f_3=f_4=0$ (D.59)

$N_{xy}=S_o \rightarrow f_1'=f_2'=f_3'=f_4'=0$ (D.60)

$W=0 \rightarrow w_o=-(W_v+W_{p_o}+W_t)$ (D.61a)

$w_1=w_2=0$ (D.61b)

$W_{,x}=0 \rightarrow w_o'=w_1'=w_2'=0$ (D.62)

C-2 Boundary Condition

$u=u_o \rightarrow f_1'''=B_{10}f_3''-B_{11}f_3+B_8w_1'''+B_{13}w_2''$ (D.63a)

$f_2'''=2B_{10}f_4''-8B_{11}f_4$ (D.63b)

$f_3'''=-B_{10}f_1''+B_{11}f_1+B_8w_2'''-B_{13}w_1''$ (D.63c)

$f_4'''=-2B_{10}f_2''+8B_{11}f_2$ (D.63d)

$N_{xy}=S_o \rightarrow f_1'=f_2'=f_3'=f_4'=0$ (D.64)

$W=0 \rightarrow w_o=-(W_v+W_{p_e}+W_t)$ (D.65a)

$w_1=w_2=0$ (D.65b)

$W_{,x} = 0 \rightarrow w_o' = w_1' = w_2' = 0$ (D.66)

C-3 Boundary Condition

$N_x = -N_o \rightarrow f_1 = f_2 = f_3 = f_4 = 0$ (D.67)

$v = v_o \rightarrow f_1'' = B_{17} f_3' + B_8 w_1''$ (D.68a)

$f_2'' = 2B_{17} f_4'$ (D.68b)

$f_3'' = -B_{17} f_1' + B_8 w_2''$ (D.68c)

$f_4'' = -2B_{17} f_2'$ (D.68d)

$W = 0 \rightarrow w_o = -(W_v + W_{p_e} + W_t)$ (D.69a)

$w_1 = w_2 = 0$ (D.69b)

$W_{,x} = 0 \rightarrow w_o' = w_1' = w_2' = 0$ (D.70)

C-4 Boundary Condition

$u = u_o \rightarrow f_1''' = B_{31} f_1' - B_{32} f_3 + B_8 w_1''' + B_{14} w_2''$ (D.71a)

$f_2''' = 4B_{31} f_2' - 8B_{32} f_4$ (D.71b)

$f_3''' = B_{31} f_3' + B_{32} f_1 + B_8 w_2''' - B_{14} w_1''$ (D.71c)

$f_4''' = 4B_{31} f_4' + 8B_{32} f_2$ (D.71d)

$v = v_o \rightarrow f_1'' = B_{24} f_1 + B_{17} f_3' + B_8 w_1''$ (D.72a)

$f_2'' = 4B_{24} f_2 + 2B_{17} f_4'$ (D.72b)

$f_3'' = B_{24} f_3 - B_{17} f_1' + B_8 w_2''$ (D.72c)

$$f_4'' = 4B_{24}f_4 - 2B_{17}f_2' \tag{D.72d}$$

$W = 0 \quad \rightarrow$

$$w_o = -(W_v + W_{p_e} + W_t) \tag{D.73a}$$

$$w_1 = w_2 = 0 \tag{D.73b}$$

$W_{,x} = 0 \quad \rightarrow$

$$w_o' = w_1' = w_2' = 0 \tag{D.74}$$

Symmetry Condition at $\bar{x} = \frac{1}{2}\frac{L}{R}$

$u = 0 \quad \rightarrow$

$$f_1''' = B_{33}f_3'' + B_{34}f_3 + B_{35}w_2'' + (B_{36} + B_{37}\bar{\tau})w_2 + B_{37}A_2\bar{\tau} \tag{D.75a}$$

$$f_2''' = 2B_{10}f_4'' - 8B_{11}f_4 \tag{D.75b}$$

$$f_3''' = -B_{33}f_1'' - B_{34}f_1 - B_{35}w_1'' - (B_{36} + B_{37}\bar{\tau})w_1 - B_{37}A_1\bar{\tau} \tag{D.75c}$$

$$f_4''' = -2B_{10}f_2'' + 8B_{11}f_2 \tag{D.75d}$$

$N_{xy} = 0 \quad \rightarrow$

$$f_1' = f_2' = f_3' = f_4' = 0 \tag{D.76}$$

$H = 0 \quad \rightarrow$

$$w_o''' = 0 \tag{D.77a}$$

$$w_1''' = -B_{38}f_3'' + B_{39}f_3 - B_{40}w_2'' + (B_{41} + B_{23}\bar{\tau})w_2 + B_{23}A_2\bar{\tau} \tag{D.77b}$$

$$w_2''' = B_{38}f_1'' - B_{39}f_1 + B_{40}w_1'' - (B_{41} + B_{23}\bar{\tau})w_1 - B_{23}A_1\bar{\tau} \tag{D.77c}$$

$W_{,x} = 0 \quad \rightarrow$

$$w_o' = w_1' = w_2' = 0 \tag{D.78}$$

where

$$B_{33} = \frac{n}{\Delta_1}\left[\, 2\bar{A}_{26}^*\bar{D}_{11}^* - \bar{B}_{21}^*(2\bar{B}_{26}^* - \bar{B}_{61}^*)\,\right] \qquad \Delta_1 = \bar{A}_{22}^*\bar{D}_{11}^* + \bar{B}_{21}^*\bar{B}_{21}^*$$

$$B_{34} = \frac{n^3}{\Delta_1}(2\bar{B}_{21}^*\bar{B}_{16}^* - \bar{A}_{16}^*\bar{D}_{11}^*)$$

$$B_{35} = \frac{1}{2}\frac{t}{R}\frac{n}{\Delta_1}\left[(2\bar{B}^*_{26} - \bar{B}^*_{61})\bar{D}^*_{11} - 4\bar{B}^*_{21}\bar{D}^*_{16} \right]$$

$$B_{36} = \frac{1}{2}\frac{t}{R}\frac{n^3}{\Delta_1}(\bar{B}^*_{62}\bar{D}^*_{11} + 2\bar{B}^*_{21}\bar{D}^*_{26}) \qquad B_{37} = 2cn\,\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$B_{38} = 2\,\frac{R}{t}\frac{n}{\bar{D}^*_{11}}\left\{ 2\bar{B}^*_{26} - \bar{B}^*_{61} + \frac{\bar{B}^*_{21}}{\Delta_1}\left[2\bar{A}^*_{26}\bar{D}^*_{11} - \bar{B}^*_{21}(2\bar{B}^*_{26} - \bar{B}^*_{61}) \right] \right\}$$

(D.79)

$$B_{39} = 2\,\frac{R}{t}\frac{n^3}{\bar{D}^*_{11}}\left[2\bar{B}^*_{16} - \frac{\bar{B}^*_{21}}{\Delta_1}(2\bar{B}^*_{12}\bar{B}^*_{16} - \bar{A}^*_{16}\bar{D}^*_{11}) \right]$$

$$B_{40} = \frac{n}{\bar{D}^*_{11}}\left\{ 4\bar{D}^*_{16} + \frac{\bar{B}^*_{21}}{\Delta_1}\left[(2\bar{B}^*_{26} - \bar{B}^*_{61})\bar{D}^*_{11} - 4\bar{B}^*_{21}\bar{D}^*_{16} \right] \right\}$$

$$B_{41} = \frac{n^3}{\bar{D}^*_{11}}\left[2\bar{D}^*_{26} - \frac{\bar{B}^*_{21}}{\Delta_1}(\bar{B}^*_{26}\bar{D}^*_{11} + 2\bar{B}^*_{21}\bar{D}^*_{26}) \right]$$

Free Edge Condition

$$N_x = -N_o \quad \rightarrow \quad f_1 = f_2 = f_3 = f_4 = 0 \qquad \text{(D.80)}$$

$$N_{xy} = S_o \quad \rightarrow \quad f_1' = f_2' = f_3' = f_4' = 0 \qquad \text{(D.81)}$$

$$H = 0 \rightarrow w_o''' = (B_{42} - B_{43}\lambda)w_o' - B_{43}A_o'\lambda - B_{44}\left[w_1'(w_1 + A_1) + w_1A_1' + w_2'(w_2 + A_2) + w_2A_2' \right] \qquad \text{(D.82a)}$$

$$w_1''' = -B_1f_1''' + B_{45}f_3'' + (B_{46} - B_{47}\lambda)w_1' - B_{47}A_1'\lambda + (B_{48} + B_{49}\bar{\tau})w_2 + B_{49}A_2\bar{\tau} \qquad \text{(D.82b)}$$

$$w_2''' = -B_1f_3''' - B_{45}f_1'' + (B_{46} - B_{47}\lambda)w_2' - B_{47}A_2'\lambda - (B_{48} + B_{49}\bar{\tau})w_1 - B_{49}A_1\bar{\tau} \qquad \text{(D.82c)}$$

$M_x = -N_oq \rightarrow$

$$w_o'' = B_{42}w_o + B_{50}\lambda + B_{51}\bar{p}_e - B_{52}\bar{\tau} - B_{53}\left[w_1(w_1 + 2A_1)\ w_2(w_2 + 2A_2) \right] \qquad \text{(D.83a)}$$

$$w_1'' = B_{12}w_1 - B_2w_2' - B_1f_1'' \qquad \text{(D.83b)}$$

$$w_{26}'' = B_{12}w_2 + B_2w_1' - B_1f_3'' \qquad \text{(D.83c)}$$

where

$$B_{42} = 2c\,\frac{R}{t}\,\frac{\bar{B}^*_{21}}{\Delta_1} \qquad B_{49} = 4c\,\frac{R}{t}\,\frac{n}{\bar{D}^*_{11}}$$

$$B_{43} = 4c\,\frac{R}{t}\,\frac{1}{\Delta_1} \qquad B_{50} = 2\,\frac{R}{t}\,\frac{\bar{A}^*_{22}}{\Delta_1}\left(\bar{B}^*_{11} + \frac{1}{2}\,\frac{t}{R}\,\bar{q}\right)$$

$$B_{44} = cn^2\,\frac{\bar{B}^*_{21}}{\Delta_1} \qquad B_{51} = 2\,\frac{R}{t}\,\frac{\bar{A}^*_{22}}{\Delta_1}\,\bar{B}^*_{21} \tag{D.84}$$

$$B_{45} = 2\,\frac{R}{t}\,\frac{n}{\bar{D}^*_{11}}\left(\bar{B}^*_{61} - 2\bar{B}^*_{26} + \frac{4\bar{B}^*_{21}\bar{D}^*_{16}}{\bar{D}^*_{11}}\right) \qquad B_{52} = 2\,\frac{R}{t}\,\frac{\bar{A}^*_{22}}{\Delta_1}\,\bar{B}^*_{61}$$

$$B_{46} = \frac{n^2}{\bar{D}^*_{11}}\left(\bar{D}^*_{12} + 4\bar{D}^*_{66} - \frac{8\bar{D}^*_{16}\bar{D}^*_{16}}{\bar{D}^*_{11}}\right) \qquad B_{53} = \frac{c}{2}\,n^2\,\frac{\bar{B}^*_{21}}{\Delta_1}$$

$$B_{47} = 4c\,\frac{R}{t}\,\frac{1}{\bar{D}^*_{11}}$$

$$B_{48} = \frac{2n^3}{\bar{D}^*_{11}}\left(\bar{D}^*_{26} - \frac{2\bar{D}^*_{12}\bar{D}^*_{16}}{\bar{D}^*_{11}}\right)$$

Symmetry or Anti-symmetry condition at $\bar{x} = \frac{1}{2}\frac{L}{R}$

It has been shown in Ref. [26] that if a layered composite shell is layed-up in such a fashion that in the stiffness matrix the coupling terms between the bending and shear strain of the middle surface are not zero, then the buckling and the postbuckling modes are skewed with respect to the shell axis. To represent this skewed pattern it is necessary to include in the circumferential Fourier representation both $\cos n\theta$ and $\sin n\theta$ terms (see also Eqs. (8) - (10)). It has been shown by Booton [27] that if the axial dependence of the $\cos n\theta$ terms is symmetric with respect to the mid-section (at x = L/2) of the shell, then the $\sin n\theta$ terms are anti-symmetric with respect to the same location or vice-versa. This implies that, if one uses the appropriate initial imperfection shapes (say, A_0 and A_1 are symmetric and A_2 is anti-symmetric with respect to x = L/2) then one can work also for the general collapse analysis with half of the shell length only by enforcing appropriate symmetry and/or anti-symmetry conditions at x = L/2 (or $\bar{x} = (1/2)L/R$).

In order to derive these conditions one must recall that symmetry at x = L/2 implies that

$u = N_{xy} = H = w_{,x} = 0$

whereas anti-symmetry at x = L/2 requires that

$N_x = -N_o,\ v = W = 0,\ M_x = -N_o q$

at the shell mid-section.

Thus if A_o and A_1 are symmetric at x = L/2 then, considering Eqs. (8) - (10) and Eqs. (34) - (44), one obtains

$u_1 = u_2 = 0$

$N_{xy_1} = N_{xy_2} = 0$

$H_1 = H_2 = 0$

$w_o' = w_1' = 0$

whereas the fact that A_2 is anti-symmetric at x = L/2 implies

$N_{x_3} = N_{x_4} = 0$

$v_3 = v_4 = 0$

$w_2 = 0$

$M_{x_2} = 0$

Next one must express these conditions in terms of the shell variables $\bar{W}$, W and F.

First boundary condition

$$u_1 = 0 \quad \rightarrow \quad \bar{A}^*_{22} f_1''' - (\bar{A}^*_{12} + \bar{A}^*_{66}) n^2 f_1' - 2\bar{A}^*_{26} n f_3'' + \bar{A}^*_{16} n^3 f_3 + c w_1' \qquad \text{(D.85a)}$$

$$- \frac{1}{2}\frac{t}{R} [\, \bar{B}^*_{21} w_1''' - (\bar{B}^*_{22} - 2\bar{B}^*_{66}) n^2 w_1' + (2\bar{B}^*_{26} - \bar{B}^*_{61}) n w_2'' + \bar{B}_{62} n^3 w_2 \,]$$

$$- c \frac{t}{R} n^2 [\, w_o'(w_1 + A_1) + w_1 A_o' \,] = 0$$

$$u_2 = 0 \quad \rightarrow \quad \bar{A}^*_{22} f_2''' - (\bar{A}^*_{12} + \bar{A}^*_{66}) 4n^2 f_2' - 4\bar{A}^*_{26} n f_4'' + 8\bar{A}^*_{16} n^3 f_4 \tag{D.85b}$$

$$- \frac{c}{t}\frac{t}{R} n^2 \left[w_1'(w_1 + A_1) + w_1 A_1' - w_2'(w_2 + A_2) - w_2 A_2' \right] = 0$$

$$N_{x_3} = 0 \quad \rightarrow \quad f_3 = 0 \tag{D.85c}$$

$$N_{x_4} = 0 \quad \rightarrow \quad f_4 = 0 \tag{D.85d}$$

Second boundary condition

$$N_{xy_1} = 0 \quad \rightarrow \quad f_1' = 0 \tag{D.86a}$$

$$N_{xy_2} = 0 \quad \rightarrow \quad f_2' = 0 \tag{D.86b}$$

$$v_3 = 0 \quad \rightarrow \quad \bar{A}^*_{22} f_3'' - \bar{A}^*_{12} n^2 f_3 + \bar{A}^*_{26} n f_1' + c w_2 - \frac{1}{2}\frac{t}{R} (\bar{B}^*_{21} w_2'' - \bar{B}^*_{22} n^2 w_2 - 2\bar{B}^*_{26} n w_1') = 0 \tag{D.87a}$$

$$v_4 = 0 \quad \rightarrow \quad \bar{A}^*_{22} f_4'' - 4\bar{A}^*_{12} n^2 f_4 + 2\bar{A}^*_{26} n f_2' + \frac{c}{4}\frac{t}{R} n^2 \left[w_2(w_1 + 2A_1) + w_1(w_2 + 2A_2) \right] = 0 \tag{D.87b}$$

Third boundary condition

$$H_1 = 0 \rightarrow \frac{1}{2}\frac{t}{R} (\bar{A}^*_{22}\bar{D}^*_{11} + \bar{B}^*_{21}\bar{B}^*_{21}) w_o''' - c\bar{B}^*_{21} w_o' + 2\bar{C}\bar{A}^*_{22}\lambda(w_o' + A_o') \tag{D.88a}$$

$$+ \frac{c}{2}\frac{t}{R} \bar{B}^*_{21} n^2 [w_1'(w_1 + A_1) + w_1 A_1' + w_2'(w_2 + A_2) + w_2 A_2']$$

$$+ c\bar{A}^*_{22} n^2 \left[f_1(w_1' + A_1') + f_3(w_2' + a_2') + f_1'(w_1 + A_1) + f_3'(w_2 + A_2) \right] = 0$$

$$H_2 = 0 \rightarrow \bar{B}^*_{21} f_1''' - (\bar{B}^*_{11} - 2\bar{B}^*_{66}) n^2 f_1 - (\bar{B}^*_{61} - 2\bar{B}^*_{26}) n f_3'' - 2\bar{B}^*_{16} n^3 f_3 + 2c\lambda(w_1' + A_1') - 2cn\bar{\tau}(w_2 + A_2)$$

$$+ \frac{1}{2}\frac{t}{R} [\bar{D}^*_{11} w_1''' - (\bar{D}^*_{12} + 4\bar{D}^*_{66}) n^2 w_1' + 4\bar{D}^*_{16} n w_2'' - 2\bar{D}^*_{26} n^3 w_2] \tag{D.88b}$$

$$+ 2cn^2 [f_1(w_o' + A_o') + 2f_2(w_1' + A_1') + 2f_4(w_2' + A_2') + f_2'(w_1 + A_1) + f_4'(w_2 + A_2)] = 0$$

$$W_2 = 0 \tag{D.88c}$$

Fourth boundary condition

$$w_o' = 0 \tag{D.89a}$$

$$w_1' = 0 \tag{D.89b}$$

$$M_{x_2} = 0 \quad \rightarrow \quad \bar{B}^*_{21} f_2'' - \bar{B}^*_{11} n^2 f_3 + \bar{B}^*_{61} n f_1' + \frac{1}{2}\frac{t}{R}\left(\bar{D}^*_{11} w_2'' - \bar{D}^*_{12} n^2 w_2 - 2\bar{D}^*_{16} n w_1'\right) = 0 \tag{D.89c}$$

Recalling further that A_o, A_1 symmetric implies $A_o' = A_1' = 0$ at x = L/2, whereas A_2 antisymmetric implies $A_2 = 0$ at x = L/2.

$$\bar{A}^*_{22} f_1''' - 2\bar{A}^*_{26} n f_3'' - \frac{1}{2}\frac{t}{R}\left[\bar{B}^*_{21} w_1''' + (2\bar{B}^*_{26} - \bar{B}^*_{61}) n w_2''\right] = 0 \tag{D.85a}$$

$$\bar{A}^*_{22} f_2''' - 4\bar{A}^*_{26} n f_4'' = 0 \tag{D.85b}$$

$$\bar{A}^*_{22} f_3'' - \frac{1}{2}\frac{t}{R}\bar{B}^*_{21} w_2'' = 0 \tag{D.87a}$$

$$\bar{A}^*_{22} f_4'' = 0 \tag{D.87b}$$

$$\frac{1}{2}\frac{t}{R}\frac{1}{\bar{A}^*_{22}}\left(\bar{A}^*_{22}\bar{D}^*_{11} + \bar{B}^*_{21}\bar{B}^*_{21}\right) w_o''' = 0 \tag{D.88a}$$

$$\bar{B}^*_{21} f_1''' - (\bar{B}^*_{61} - 2\bar{B}^*_{26}) n f_3'' + \frac{1}{2}\frac{t}{R}\left(\bar{D}^*_{11} w_1''' + 4\bar{D}^*_{16} n w_2''\right) = 0 \tag{D.88b}$$

$$\bar{B}^*_{21} f_3'' + \frac{1}{2}\frac{t}{R}\bar{D}^*_{11} w_2'' = 0 \tag{D.89b}$$

From Eqs. (D.87b), (D.88a) and (D.85b) it follows that $f_4'' = w_o''' = f_2''' = 0$.
Considering Eqs. (D.87a) and (D.89b) it can be seen by inspection that the solution is $w_2'' = f_3'' = 0$.
Taking all these results into consideration the remaining 2 equations become

$$\bar{A}^*_{22} f_1''' - \frac{1}{2}\frac{t}{R}\bar{B}^*_{21} w_1''' = 0 \tag{D.85a}$$

$$\bar{B}^*_{21} f_1''' + \frac{1}{2}\frac{t}{R}\bar{D}^*_{11} w_1''' = 0 \tag{D.88b}$$

Once again, by inspection, their solution is $f_1''' = w_1''' = 0$.

Summarizing, if A_o, A_1 are symmetric and A_2 is anti-symmetric with respect to x = L/2, then the appropriate symmetry and/or anti-symmetry conditions at $\bar{x} = (1/2)L/R$ are

$$f_3 = f_4 = w_2 = f_1' = f_2' = w_o' = w_1' = f_3'' = f_4'' = w_2'' = f_1''' = f_2''' = w_o''' = w_1''' = 0 \tag{D.90}$$

APPENDIX E Derivation of Cohen's Ring Equations[21]

When ring stiffeners are attached to the shell edges, boundary conditions for the shell equations must be generated which represent the ring behavior. Ideally the ring reactions enter the shell at a single meridional station, the ring boundary. A set of suitable ring equations based on moderate rotations are derived in this appendix by using the stationary potential energy criterion. These equations are based on the following assumptions

- all geometrical and mechanical properties are axisymmetric;
- the ring material is homogeneous and isotropic;
- the effects of non-uniform warping of ring sections, transverse shear strains and shear center eccentricity related to the section centroid are neglected.

The origin of the ring cross-sectional xyz-axis is assumed to be at the centroid of the ring cross-section. The sign convention used is indicated in Fig. 2.

For a one-dimensional theory of rings the centroidal hoop strain ε_θ is the only extensional strain of consequence. The strain-displacement relations have been derived in Ref. [21] as

$$\varepsilon_\theta = \frac{1}{a}(v' - w) + \frac{1}{2}(\beta_x^2 + \beta_z^2) \qquad \beta_x = \frac{1}{a}(v + w') \;;\; \beta_z = -\frac{u'}{a}$$

$$\kappa_x = \frac{1}{a}\beta_x' = \frac{1}{a^2}(v' + w'') \tag{E.1}$$

$$\kappa_z = \frac{1}{a}\beta_z' + \frac{1}{a}\beta_y = \frac{1}{a}\left(-\frac{u''}{a} + \beta_y\right)$$

and the twist per unit length of the curved bar is given by

$$\varphi = \frac{1}{a}\beta_y' + \frac{1}{a^2}u' \tag{E.2}$$

where $(\ )' = \frac{d}{d\theta}(\ )$.

If one includes the initial stress-free imperfection $\hat{w}(\theta)$ of the ring then ε_θ and β_x must be modified as follows

$$\varepsilon_\theta = \frac{1}{a}(v' + w) + \frac{1}{2}(\hat{\beta}_x^2 + \beta_z^2) - \frac{1}{2}\left(\frac{\hat{w}'}{a}\right)^2$$

$$\hat{\beta}_x = \beta_x + \frac{1}{a}\hat{w}' = \frac{1}{a}(v + w' + \hat{w}') \tag{E.3}$$

Neglecting the effect of nonuniform torsion for a homogeneous, one dimensional ring theory the constitutive equations are

$$
\begin{aligned}
N_\theta &= EA\varepsilon_\theta \\
M_x &= EI_x\kappa_x - EI_{xz}\kappa_z \\
M_z &= -EI_{xz}\kappa_x + EI_z\kappa_z \\
M_t &= GJ\varphi
\end{aligned}
\tag{E.4}
$$

By definition the potential energy of the ring is

$$\Pi^r = U^r - W^r \tag{E.5}$$

where

$$U^r = \frac{1}{2}\int_0^{2\pi} (N_\theta\varepsilon_\theta + M_x\kappa_x + M_z\kappa_z + M_t\varphi)a\,d\theta \tag{E.6a}$$

$$W^r = \int_0^{2\pi} (\bar{N}_x u + \bar{S}v + \bar{P}w + \bar{M}_x\beta_x + \bar{M}_y\beta_y + \bar{M}_z\beta_z)a\,d\theta \tag{E.6b}$$

and

$\bar{N}_x$, $\bar{S}$, $\bar{P}$, $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ are the external line loads applied to the ring. See Fig. 2 for the notation and sign convention used.

The variational statement of equilibrium for a homogeneous ring under the specified external line loads is

$$\delta\Pi^r = \delta\int_0^{2\pi} F(u, u', u'', v, v', w, w', w'', \beta_y, \beta_y')a\,d\theta = 0 \tag{E.7}$$

where

$$
\begin{aligned}
F = \frac{1}{2}&(EA\varepsilon_\theta^2 + EI_x\kappa_x^2 + EI_z\kappa_z^2 - 2EI_{xz}\kappa_x\kappa_z + GJ\varphi^2) \\
&- (\bar{N}_x u + \bar{S}v + \bar{P}w + \bar{M}_x\beta_x + \bar{M}_y\beta_y + \bar{M}_z\beta_z)
\end{aligned}
\tag{E.8}
$$

thus in this case the equilibrium condition

$$\delta\Pi^r = \frac{\partial\Pi^r}{\partial u}\delta u + \frac{\partial\Pi^r}{\partial v}\delta v + \frac{\partial\Pi^r}{\partial w}\delta w + \frac{\partial\Pi^r}{\partial\beta_y}\delta\beta_y = 0 \tag{E.9}$$

implies the 4 conditions

$$\frac{\partial \Pi^r}{\partial u} = \frac{\partial \Pi^r}{\partial v} = \frac{\partial \Pi^r}{\partial w} = \frac{\partial \Pi^r}{\partial \beta_y} = 0 \tag{E.10}$$

Carrying out the indicated variations, integrating by parts whenever necessary and using the fact that the variations δu, δv, δw and $\delta\beta_y$ must be periodic one obtains the expressions

$$\begin{aligned}
\frac{\partial \Pi^r}{\partial u} &= \int_0^{2\pi} \left\{ \frac{\partial F}{\partial u} - \frac{d}{d\theta}\frac{\partial F}{\partial u'} + \frac{d^2}{d\theta^2}\frac{\partial F}{\partial u''} \right\} \delta u \, a d\theta = 0 \\
\frac{\partial \Pi^2}{\partial v} &= \int_0^{2\pi} \left\{ \frac{\partial F}{\partial v} - \frac{d}{d\theta}\frac{\partial F}{\partial v'} \right\} \delta v \, a d\theta = 0 \\
\frac{\partial \Pi^r}{\partial w} &= \int_0^{2\pi} \left\{ \frac{\partial F}{\partial w} - \frac{d}{d\theta}\frac{\partial F}{\partial w'} + \frac{d^2}{d\theta^2}\frac{\partial F}{\partial w''} \right\} \delta w \, a d\theta = 0 \\
\frac{\partial \Pi^r}{\partial \beta_y} &= \int_0^{2\pi} \left\{ \frac{\partial F}{\partial \beta_y} - \frac{d}{d\theta}\frac{\partial F}{\partial \beta_y'} \right\} \delta\beta_y \, a d\theta = 0
\end{aligned} \tag{E.11}$$

For this to be true for arbitrary nonzero variations δu, δv, δw, and $\delta\beta_y$ one must satisfy the following Euler equations

$$\begin{aligned}
&\frac{\partial F}{\partial u} - \frac{d}{d\theta}\frac{\partial F}{\partial u'} + \frac{d^2}{d\theta^2}\frac{\partial F}{\partial u''} = 0 \\
&\frac{\partial F}{\partial v} - \frac{d}{d\theta}\frac{\partial F}{\partial v'} = 0 \\
&\frac{\partial F}{\partial w} - \frac{d}{d\theta}\frac{\partial F}{\partial w'} + \frac{d^2}{d\theta^2}\frac{\partial F}{\partial w''} = 0 \\
&\frac{\partial F}{\partial \beta_y} - \frac{d}{d\theta}\frac{\partial F}{\partial \beta_y'} = 0
\end{aligned} \tag{E.12}$$

Carrying out the indicated differentiations, substituting and regrouping yields the following ring equations

$$EI_z u^{iv} - GJu'' + EI_{xz}v''' + EI_{xz}w^{iv} - a(EI_z + GJ)\beta_y'' = a^4\bar{N}_x + a^3(M_z^*)' \tag{E.13a}$$

$$EI_{xz}u''' + (EI_x + a^2EA)v'' + EI_x w''' - a^2EAw' - aEI_{xz}\beta_y' = -a^4S^* - a^3M_x^* \tag{E.13b}$$

$$EI_{xz}u^{iv} + EI_x v''' - a^2EAv' + EI_x w^{iv} + a^2EAw - aEI_{xz}\beta_y'' = a^4P^* - a^3(M_x^*)' \tag{E.13c}$$

$$(EI_z + GJ)u'' + EI_{xz}v' + EI_{xz}w'' + aGJ\beta_y'' - aEI_z\beta_y = -a^3\bar{M}_y \qquad \text{(E.13d)}$$

Notice that in these equations the nonlinear terms have been considered as effective additional loads and moments applied to be linearized ring equations. These effective additional loads and moments are

$$\begin{aligned} P^* &= \bar{P} + \frac{EA}{a}\left[\frac{1}{2}(\hat{\beta}_x^2 + \beta_z^2) - \frac{1}{2}\left(\frac{\hat{w}'}{a}\right)^2\right] \\ S^* &= \bar{S} + \frac{EA}{a}\left[\frac{1}{2}(\hat{\beta}_x^2 + \beta_z^2) - \frac{1}{2}\left(\frac{\hat{w}'}{a}\right)^2\right]' \\ M_x^* &= \bar{M}_x - N_\theta\hat{\beta}_x \\ M_z^* &= \bar{M}_z - N_\theta\beta_z \end{aligned} \qquad \text{(E.14)}$$

APPENDIX F: The Boundary Stiffness and Boundary Flexibility Matrices

Substituting in the separated form of Cohen's ring equations, Eqs. (28a)-(28d), for the ring displacements and the ring forces their equivalents in terms of the shell variables from Eq. (41) and Eqs. (39) or (40), respectively one obtains the following equations at the lower edge (at x = 0)

$$
\begin{aligned}
&EI_z(u^s - e_z w^s_{,x})_{,\theta\theta\theta\theta} - GJ(u^s - e_z w^s_{,x})_{,\theta\theta} \\
&\quad + EI_{xz}\{(\tfrac{a}{R} v^s - e_x u^s_{,y} - e_z w^s_{,y})_{,\theta\theta\theta} + (w^s + e_x w^s_{,x})_{,\theta\theta\theta\theta}\} \\
&\qquad - a(EI_z + GJ)(- w^s_{,x})_{,\theta\theta} = a^3\{ (R - q) N_o + RN^s_x\}
\end{aligned}
\tag{F.1a}
$$

$$
\begin{aligned}
&EI_{xz}(u^s - e_z w^s_{,x})_{,\theta\theta\theta} + (EI_x + a^2EA)(\tfrac{a}{R} v^s - e_x u^s_{,y} - e_z w^s_{,y})_{,\theta\theta} \\
&\quad + EI_x(w^s + e_x w^s_{,x})_{,\theta\theta\theta} - a^2EA(w^s + e_x w^s_{,x})_{,\theta} \\
&\qquad - aEI_{xz}(- w^s_{,x})_{,\theta} = - a^3\{ - (R - q_t)S_o + RN^s_{xy}\}
\end{aligned}
\tag{F.1b}
$$

$$
\begin{aligned}
&EI_{xz}(u^s - e_z w^s_{,x})_{,\theta\theta\theta\theta} + EI_x(\tfrac{a}{R} v^s - e_x u^s_{,y} - e_z w^s_{,y})_{,\theta\theta\theta} \\
&\quad - a^2EA(\tfrac{a}{R} v^s - e_x u^s_{,y} - e_z w^s_{,y})_{,\theta} + EI_x(w^s + e_x w^s_{,x})_{,\theta\theta\theta\theta} \\
&\qquad + a^2EA(w^s + e_x w^s_{,x}) - aEI_{xz}(- w^s_{,x})_{,\theta\theta} = a^3\{RH^s\}
\end{aligned}
\tag{F1.c}
$$

$$
\begin{aligned}
&(EI_z + GJ)(u^s - e_z w^s_{,x})_{,\theta\theta} + EI_{xz}(\tfrac{a}{R} v^s - e_x u^s_{,y} - e_z w^s_{,y})_{,\theta} \\
&\quad + EI_{xz}(w^s + e_x w^s_{,x})_{,\theta\theta} + aGJ(- w^s_{,x})_{,\theta\theta} - aEI_z(- w^s_{,x}) = \\
&\qquad -a^2\{- e_z RN^s_x + (q - e_z)(R - q)N_o + e_x RH^s + RM^s_x\}
\end{aligned}
\tag{F.1d}
$$

Substituting for the shell variables their Fourier representations from Eqs. (43) and (44), respectively, yields after some regrouping the following expressions

$$
\begin{aligned}
&(a^1_{11}u_1 + a^1_{12}v_1 + a^1_{13}w_1 + a^1_{14}w_{1,\bar{x}} - N_{x_1})\cos n\theta + (a^2_{11}u_2 + a^2_{12}v_2 - N_{x_2})\cos 2n\theta \\
&\quad + (a^3_{11}u_3 + a^3_{12}v_3 + a^3_{13}w_2 + a^3_{14}w_{2,\bar{x}} - N_{x_3})\sin n\theta + (a^4_{11}u_4 + a^4_{12}v_4 - N_{x_4})\sin 2n\theta = 0
\end{aligned}
\tag{F.2a}
$$

$$
\begin{aligned}
&(a^1_{21}u_1 + a^1_{22}v_1 + a^1_{23}w_1 + a^1_{24}w_{1,\bar{x}} - N_{xy_1})\sin n\theta + (a^2_{21}u_2 + a^2_{22}v_2 - N_{xy_2})\sin 2n\theta \\
&\quad + (a^3_{21}u_3 + a^3_{22}v_3 + a^3_{23}w_2 + a^3_{24}w_{2,\bar{x}} - N_{xy_3})\cos n\theta + (a^4_{21}u_4 + a^4_{22}v_4 - N_{xy_4})\cos 2n\theta = 0
\end{aligned}
\tag{F.2b}
$$

$$\varepsilon_H = \{a^o_{11}(W_v + W_{p_e} + W_t + w_o) + a^o_{12} w_{o,\bar{x}} - H_o\}$$
$$+ (a^1_{31}u_1 + a^1_{32}v_1 + a^1_{33}w_1 + a^1_{34}w_{1,\bar{x}} - H_1)\cos n\theta + (\ldots)\cos 2n\theta + (\ldots)\cos 3n\theta \quad \text{(F.2c)}$$
$$+ (a^3_{31}u_3 + a^3_{32}v_3 + a^3_{33}w_2 + a^3_{34}w_{2,\bar{x}} - H_2)\sin n\theta + (\ldots)\sin 2n\theta + (\ldots)\cos 3n\theta$$

$$\varepsilon_M = \{ a^o_{21}(W_v + W_{p_e} + W_t + w_o) + a^o_{22} w_{o,\bar{x}} - [M_{x_o} + \lambda\bar{q}(1 - \frac{q}{R})] \}$$
$$- (a^1_{41}u_1 + a^1_{42}v_1 + a^1_{43}w_1 + a^1_{44}w_{1,\bar{x}} - M_{x_1})\cos n\theta + (\ldots)\cos 2n\theta \quad \text{(F.2d)}$$
$$- (a^3_{41}u_3 + a^3_{42}v_3 + a^3_{43}w_2 + a^3_{44}w_{2,\bar{x}} - M_{x_2})\sin n\theta + (\ldots)\sin 2n\theta$$

where

$$a^o_{11} = (\frac{R}{a})^2 \bar{D}_{11} \qquad a^o_{21} = -(\frac{R}{a})^2 \bar{D}_{12}$$
$$a^o_{12} = (\frac{R}{a})^2 \bar{D}_{12} \qquad a^o_{22} = -[\bar{D}_{22} + (\frac{R}{a})^2 \frac{e_x}{R} \bar{D}_{12}] \quad \text{(F.3)}$$

$$a^1_{11} = \frac{1}{4c}(\frac{t}{a})^2 n^2 \bar{D}_{33} \qquad a^1_{13} = \frac{1}{4c}(\frac{t}{a})^2 n^4 \bar{D}_{35}$$
$$a^1_{12} = -\frac{1}{4c}(\frac{t}{a})^2 n^3 \frac{a}{R} \bar{D}_{34} \qquad a^1_{14} = \frac{1}{4c}(\frac{t}{a})^2 n^2 \bar{D}_{36} \quad \text{(F.4)}$$

$$a^2_{11} = \frac{1}{c}(\frac{t}{a})^2 n^2 \bar{D}_{77} \qquad a^2_{12} = -\frac{2}{c}(\frac{t}{a})^2 n^3 \frac{a}{R} \bar{D}_{78} \quad \text{(F.5)}$$

$$a^3_{11} = \frac{1}{4c}(\frac{t}{a})^2 n^2 \bar{D}_{33} \qquad a^3_{13} = \frac{1}{4c}(\frac{t}{a})^2 n^4 \bar{D}_{35}$$
$$a^3_{12} = \frac{1}{4c}(\frac{t}{a})^2 n^3 \frac{a}{R} \bar{D}_{34} \qquad a^3_{14} = \frac{1}{4c}(\frac{t}{a})^2 n^2 \bar{D}_{36} \quad \text{(F.6)}$$

$$a^4_{11} = \frac{1}{c}(\frac{t}{a})^2 n^2 \bar{D}_{77} \qquad a^4_{12} = \frac{2}{c}(\frac{t}{a})^2 n^3 \frac{a}{R} \bar{D}_{78} \quad \text{(F.7)}$$

$$a^1_{21} = \frac{1}{4c}(\frac{t}{a})^2 n^3 \bar{D}_{43} \qquad a^1_{23} = -\frac{1}{4c}(\frac{t}{a})^2 n \bar{D}_{45}$$
$$a^1_{22} = \frac{1}{4c}(\frac{t}{a})^2 n^2 \frac{a}{R} \bar{D}_{44} \qquad a^1_{24} = \frac{1}{4c}(\frac{t}{a})^2 n \bar{D}_{46} \quad \text{(F.8)}$$

$$a_{21}^{2}= \frac{2}{c}\left(\frac{t}{a}\right)^{2} n^{3}\,\bar{D}_{87} \qquad a_{22}^{2}= \frac{1}{c}\left(\frac{t}{a}\right)^{2} n^{2}\,\frac{a}{R}\,\bar{D}_{88} \tag{F.9}$$

$$\begin{aligned} a_{21}^{3} &= -\frac{1}{4c}\left(\frac{t}{a}\right)^{2} n^{3}\,\bar{D}_{43} & a_{23}^{3} &= \frac{1}{4c}\left(\frac{t}{a}\right)^{2} n\,\bar{D}_{45} \\ a_{22}^{3} &= \frac{1}{4c}\left(\frac{t}{a}\right)^{2} n^{2}\,\frac{a}{R}\,\bar{D}_{44} & a_{24}^{3} &= -\frac{1}{4c}\left(\frac{t}{a}\right)^{2} n\,\bar{D}_{46} \end{aligned} \tag{F.10}$$

$$a_{21}^{4}= -\frac{2}{c}\left(\frac{t}{a}\right)^{2} n^{3}\,\bar{D}_{87} \qquad a_{22}^{4}= \frac{1}{c}\left(\frac{t}{a}\right)^{2} n^{2}\,\frac{a}{R}\,\bar{D}_{88} \tag{F.11}$$

$$\begin{aligned} a_{31}^{1} &= n^{2}\left(\frac{R}{a}\right)^{2}\bar{D}_{53} & a_{33}^{1} &= \left(\frac{R}{a}\right)^{2}\bar{D}_{55} \\ a_{32}^{1} &= -n\left(\frac{R}{a}\right)\bar{D}_{54} & a_{34}^{1} &= \left(\frac{R}{a}\right)^{2}\bar{D}_{56} \end{aligned} \tag{F.12}$$

$$\begin{aligned} a_{31}^{3} &= n^{2}\left(\frac{R}{a}\right)^{2}\bar{D}_{53} & a_{33}^{3} &= \left(\frac{R}{a}\right)^{2}\bar{D}_{55} \\ a_{32}^{3} &= n\left(\frac{R}{a}\right)\bar{D}_{54} & a_{34}^{3} &= \left(\frac{R}{a}\right)^{2}\bar{D}_{56} \end{aligned} \tag{F.13}$$

$$\begin{aligned} a_{41}^{1} &= n^{2}\,\frac{R}{a}\left(\bar{D}_{63}+\frac{e_z}{a}\bar{D}_{33}-\frac{e_x}{a}\bar{D}_{53}\right) & a_{43}^{1} &= \frac{R}{a}\left(n^{2}\bar{D}_{65}+n^{4}\,\frac{e_z}{a}\bar{D}_{35}-\frac{e_x}{a}\bar{D}_{55}\right) \\ a_{42}^{1} &= n\,\frac{R}{a}\left(-\frac{a}{R}\bar{D}_{64}-n^{2}\,\frac{e_z}{R}\bar{D}_{34}+\frac{e_x}{R}\bar{D}_{54}\right) & a_{44}^{1} &= \frac{R}{a}\left(\bar{D}_{66}+n^{2}\,\frac{e_z}{a}\bar{D}_{36}-\frac{e_x}{a}\bar{D}_{56}\right) \end{aligned} \tag{F.14}$$

$$\begin{aligned} a_{41}^{3} &= n^{2}\,\frac{R}{a}\left(\bar{D}_{63}+\frac{e_z}{a}\bar{D}_{33}-\frac{e_x}{a}\bar{D}_{53}\right) & a_{43}^{3} &= \frac{R}{a}\left(n^{2}\bar{D}_{65}+n^{4}\,\frac{e_z}{a}\bar{D}_{35}-\frac{e_x}{a}\bar{D}_{55}\right) \\ a_{42}^{3} &= n\,\frac{R}{a}\left(\frac{a}{R}\bar{D}_{64}+n^{2}\,\frac{e_z}{R}\bar{D}_{34}-\frac{e_x}{R}\bar{D}_{54}\right) & a_{44}^{3} &= \frac{R}{a}\left(\bar{D}_{66}+n^{2}\,\frac{e_z}{a}\bar{D}_{36}-\frac{e_x}{a}\bar{D}_{56}\right) \end{aligned} \tag{F.15}$$

and the stiffness coefficients $\bar{D}_{ij}$ are defined as

$$\bar{D}_{11}= \frac{a^{2}EA}{aD} \qquad \text{where } D = \frac{Et^{3}}{4c^{2}}\ ;\ c^{2}= 3(1-\nu^{2})$$

$$\bar{D}_{12}= \frac{e_x}{R}\left(\frac{a^{2}EA}{aD}\right)$$

$$\bar{D}_{22} = \frac{EI_z}{aD}$$

$$\bar{D}_{33} = n^2 \left[\frac{EI_z}{aD} - \frac{e_x}{R}\left(\frac{EI_{xz}}{aD}\right)\right] + \frac{GJ}{aD}$$

$$\bar{D}_{34} = \frac{EI_{xz}}{aD}$$

$$\bar{D}_{35} = \left(1 - \frac{e_z}{R}\right)\left(\frac{EI_{xz}}{aD}\right)$$

$$\bar{D}_{36} = n^2 \frac{e_x}{R}\left(\frac{EI_{xz}}{aD}\right) - \left(\frac{a}{R} + n^2 \frac{e_z}{R}\right)\left(\frac{EI_z}{aD}\right) - \left(\frac{a}{R} + \frac{e_z}{R}\right)\left(\frac{GJ}{aD}\right)$$

$$\bar{D}_{43} = \frac{e_x}{R}\left(\frac{EI_x}{aD} + \frac{a^2EA}{aD}\right) - \left(\frac{EI_{xz}}{aD}\right)$$

$$\bar{D}_{44} = \frac{EI_x}{aD} + \frac{a^2EA}{aD}$$

$$\bar{D}_{45} = n^2\left(1 - \frac{e_z}{R}\right)\left(\frac{EI_x}{aD}\right) + \left(1 - n^2 \frac{e_z}{R}\right)\left(\frac{a^2EA}{aD}\right)$$

$$\bar{D}_{46} = \left(\frac{a}{R} + n^2 \frac{e_z}{R}\right)\left(\frac{EI_{xz}}{aD}\right) - \frac{e_x}{R}\left[n^2\left(\frac{EI_x}{aD}\right) + \frac{a^2EA}{aD}\right]$$

$$\bar{D}_{53} = n^2\left[\frac{EI_{xz}}{aD} - \frac{e_x}{R}\left(\frac{EI_x}{aD}\right)\right] - \frac{e_x}{R}\left(\frac{a^2EA}{aD}\right) \qquad \text{(F.16)}$$

$$\bar{D}_{54} = n^2\left(\frac{EI_x}{aD}\right) + \frac{a^2EA}{aD}$$

$$\bar{D}_{55} = n^4\left(1 - \frac{e_z}{R}\right)\left(\frac{EI_x}{aD}\right) + \left(1 - n^2 \frac{e_z}{R}\right)\left(\frac{a^2EA}{aD}\right)$$

$$\bar{D}_{56} = \frac{e_x}{R}\left[n^4\left(\frac{EI_x}{aD}\right) + \frac{a^2EA}{aD}\right] - n^2\left(\frac{a}{R} + n^2 \frac{e_z}{R}\right)\left(\frac{EI_{xz}}{aD}\right)$$

$$\bar{D}_{63} = \frac{EI_z}{aD} + \frac{GJ}{aD} - \frac{e_x}{R}\left(\frac{EI_{xz}}{aD}\right)$$

$$\bar{D}_{64} = \frac{EI_{xz}}{aD}$$

$$\bar{D}_{65} = \left(1 - \frac{e_z}{R}\right)\left(\frac{EI_{xz}}{aD}\right)$$

$$\bar{D}_{66} = n^2 \frac{e_x}{R}\left(\frac{EI_{xz}}{aD}\right) - \left(\frac{a}{R} + n^2 \frac{e_z}{R}\right)\left(\frac{EI_z}{aD}\right) - n^2\left(\frac{a}{R} + \frac{e_z}{R}\right)\left(\frac{GJ}{aD}\right)$$

$$\bar{D}_{75} = n^2\left[\frac{EI_{xz}}{aD} - 16 \frac{e_x}{R}\left(\frac{EI_x}{aD}\right)\right] - 4 \frac{e_x}{R}\left(\frac{a^2EA}{aD}\right)$$

$$\bar{D}_{76} = 4n^2\left(\frac{EI_x}{aD}\right) + \frac{a^2EA}{aD}$$

$$\bar{D}_{77} = 4n^2 \left[\frac{EI_z}{aD} - \frac{e_x}{R}\left(\frac{EI_{xz}}{aD}\right)\right] + \frac{GJ}{aD}$$

$$\bar{D}_{78} = \frac{EI_{xz}}{aD}$$

$$\bar{D}_{79} = \frac{GJ}{aD}$$

$$\bar{D}_{87} = \frac{e_x}{R}\left(\frac{EI_x}{aD} + \frac{a^2EA}{aD}\right) - \left(\frac{EI_{xz}}{aD}\right)$$

$$\bar{D}_{88} = \frac{EI_x}{aD} + \frac{a^2EA}{aD}$$

and the load parameters are

$$\lambda = N_0 \frac{cR}{Et^2} \quad \text{and} \quad \bar{q} = 4c \frac{R}{t^2} q \tag{F.17}$$

Equations (F.2a) and (F.2b) must hold for all values of $\theta = y/R$, which implies that the expressions in the brackets must be equal to zero. This yields then the following 8 boundary conditions at $x = 0$

$$\begin{aligned} &a_{11}^1 u_1 + a_{12}^1 v_1 + a_{13}^1 w_1 + a_{14}^1 w_{1,\bar{x}} - N_{x_1} = 0 \\ &a_{11}^2 u_2 + a_{12}^2 v_2 - N_{x_2} = 0 \\ &a_{11}^3 u_3 + a_{12}^3 v_3 + a_{13}^3 w_2 + a_{14}^3 w_{2,\bar{x}} - N_{x_3} = 0 \\ &a_{11}^4 u_4 + a_{12}^4 v_4 - N_{x_4} = 0 \end{aligned} \tag{F.18}$$

and

$$\begin{aligned} &a_{21}^1 u_1 + a_{22}^1 v_1 + a_{23}^1 w_1 + a_{24}^1 w_{1,\bar{x}} - N_{xy_1} = 0 \\ &a_{21}^2 u_2 + a_{22}^2 v_2 - N_{xy_2} = 0 \\ &a_{21}^3 u_3 + a_{22}^3 v_3 + a_{23}^3 w_2 + a_{24}^3 w_{2,\bar{x}} - N_{xy_3} = 0 \\ &a_{21}^4 u_4 + a_{22}^4 v_4 - N_{xy_4} = 0 \end{aligned} \tag{F.19}$$

For equations (F.2c) and (F.2d) the $\theta = y/R$ dependence is eliminated by using the following 3 Galerkin integrals

$$\int_0^{2\pi} \varepsilon_i d\theta = 0$$

$$\int_0^{2\pi} \varepsilon_i \cos n\theta d\theta = 0 \qquad \text{(F.20)}$$

$$\int_0^{2\pi} \varepsilon_i \sin n\theta d\theta = 0$$

where for i = 1, $\varepsilon_i = \varepsilon_1 = \varepsilon_H$ and for i = 2, $\varepsilon_i = \varepsilon_2 = \varepsilon_M$. Notice that the above are the same Galerkin integrals that have been used earlier to eliminate the y-displacement from the out-of-plane equilibrium equation (3). Substituting, in turn, for ε_H and ε_M and carrying out the integrals involved one obtains the following 6 boundary conditions at x = 0

$$a_{11}^{o}(W_v + W_{p_e} + W_t + w_o) + a_{12}^{o} w_{o,\bar{x}} - H_o = 0$$

$$a_{31}^{1} u_1 + a_{32}^{1} v_1 + a_{33}^{1} w_1 + a_{34}^{1} w_{1,\bar{x}} - H_1 = 0 \qquad \text{(F.21)}$$

$$a_{31}^{3} u_3 + a_{32}^{3} v_3 + a_{33}^{3} w_2 + a_{34}^{3} w_{2,\bar{x}} - H_2 = 0$$

$$a_{21}^{o}(W_v + W_{p_e} + W_t + w_o) + a_{22}^{o} w_{o,\bar{x}} - (M_{x_o} + \lambda\bar{q}) = 0$$

$$a_{41}^{1} u_1 + a_{42}^{1} v_1 + a_{43}^{1} w_1 + a_{44}^{1} w_{1,\bar{x}} - M_{x_1} = 0 \qquad \text{(F.22)}$$

$$a_{41}^{3} u_3 + a_{42}^{3} v_3 + a_{43}^{3} w_2 + a_{44}^{3} w_{2,\bar{x}} - M_{x_2} = 0$$

Notice that in the first of equations (F.22) q/R is neglected as compared to one. This same approximation has been used earlier in equations (46).

The boundary conditions at the upper edge (at x = L) have the same form as the ones at the lower edge (at x = 0), except that all boundary stiffness coefficients a_{ij}^{k} at x = L involve a minus sign in their definition as per Eqs. ((F.3) - (F.15)). Notice that these minus signs are due to the definition of the forces and moments acting at the ring centroid, which according to Eqs. (39) and (40) involve a minus sign, depending upon wether the ring is located at x = 0 or at x = L.

Finally, it must be mentioned that the elastic boundary conditions given by Eqs. (F.18), (F.19), (F.21) and (F.22) are valid in the limit as $E_r \to 0$ (free edges).

Another set of elastic boundary conditions valid in the limit as $E_r \to \infty$(fully clamped edges) can be derived by inverting the boundary stiffness matrix to yield the boundary flexibility matrix. To facilitate this operation initially the elastic boundary condition obtained earlier are regrouped into the following matrix notation

$$\begin{bmatrix} a^o_{11} & a^o_{12} \\ a^o_{21} & a^o_{22} \end{bmatrix} \begin{Bmatrix} W_v + W_{p_e} + W_t + w_o \\ w_{o,\bar{x}} \end{Bmatrix} = \begin{Bmatrix} H_o \\ M_{x_o} + \lambda\bar{q} \end{Bmatrix} \tag{F.23}$$

$$\begin{bmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{14} \\ a^1_{21} & a^1_{22} & a^1_{23} & a^1_{24} \\ a^1_{31} & a^1_{32} & a^1_{33} & a^1_{34} \\ a^1_{41} & a^1_{42} & a^1_{43} & a^1_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ w_{1,\bar{x}} \end{Bmatrix} = \begin{Bmatrix} N_{x_1} \\ N_{xy_1} \\ H_1 \\ M_{x_1} \end{Bmatrix} \tag{F.24}$$

$$\begin{bmatrix} a^2_{11} & a^2_{12} \\ a^2_{21} & a^2_{22} \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} N_{x_2} \\ N_{xy_2} \end{Bmatrix} \tag{F.25}$$

$$\begin{bmatrix} a^3_{11} & a^3_{12} & a^3_{13} & a^3_{14} \\ a^3_{21} & a^3_{22} & a^3_{23} & a^3_{24} \\ a^3_{31} & a^3_{32} & a^3_{33} & a^3_{34} \\ a^3_{41} & a^3_{42} & a^3_{43} & a^3_{44} \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ w_2 \\ w_{2,\bar{x}} \end{Bmatrix} = \begin{Bmatrix} N_{x_3} \\ N_{xy_3} \\ H_2 \\ M_{x_2} \end{Bmatrix} \tag{F.26}$$

$$\begin{bmatrix} a^4_{11} & a^4_{12} \\ a^4_{21} & a^4_{22} \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} N_{x_4} \\ N_{xy_4} \end{Bmatrix} \tag{F.27}$$

Inversion of these (2x2) and (4x4) boundary stiffness matrices yields the following (alternate) elastic boundary conditions valid in the limit as $E_r \to \infty$ (fully clamped edges)

$$b_{11}^{o}H_o + b_{12}^{o}(M_{x_o} + \lambda\bar{q}) = W_v + W_{p_e} + W_t + w_o$$

$$b_{21}^{o}H_o + b_{22}^{o}(M_{x_o} + \lambda\bar{q}) = w_{o,\bar{x}} \tag{F.28}$$

$$b_{11}^{1}N_{x_1} + b_{12}^{1}N_{xy_1} + b_{13}^{1}H_1 + b_{14}^{1}M_{x_1} = u_1$$

$$b_{21}^{1}N_{x_1} + b_{22}^{1}N_{xy_1} + b_{23}^{1}H_1 + b_{24}^{1}M_{x_1} = v_1$$

$$b_{31}^{1}N_{x_1} + b_{32}^{1}N_{xy_1} + b_{33}^{1}H_1 + b_{34}^{1}M_{x_1} = w_1 \tag{F.29}$$

$$b_{41}^{1}N_{x_1} + b_{42}^{1}N_{xy_1} + b_{43}^{1}H_1 + b_{44}^{1}M_{x_1} = w_{1,\bar{x}}$$

$$b_{11}^{2}N_{x_2} + b_{12}^{2}N_{xy_2} = u_2$$

$$b_{21}^{2}N_{x_2} + b_{22}^{2}N_{xy_2} = v_2 \tag{F.30}$$

$$b_{11}^{3}N_{x_3} + b_{12}^{3}N_{xy_3} + b_{13}^{3}H_2 + b_{14}^{3}M_{x_2} = u_3$$

$$b_{21}^{3}N_{x_3} + b_{22}^{3}N_{xy_3} + b_{23}^{3}H_2 + b_{24}^{3}M_{x_2} = v_3$$

$$b_{31}^{3}N_{x_3} + b_{32}^{3}N_{xy_3} + b_{33}^{3}H_2 + b_{34}^{3}M_{x_2} = w_2 \tag{F.31}$$

$$b_{41}^{3}N_{x_3} + b_{42}^{3}N_{xy_3} + b_{43}^{3}H_2 + b_{44}^{3}M_{x_2} = w_{2,\bar{x}}$$

$$b_{11}^{4}N_{x_4} + b_{12}^{4}N_{xy_4} = u_4$$

$$b_{21}^{4}N_{x_4} + b_{22}^{4}N_{xy_4} = v_4 \tag{F.32}$$

Notice that once again the boundary conditions at the upper edge (at x = L) have the same form as the ones at the lower edge (x = 0), except that all boundary flexibility coefficients b_{ij}^{k} at x = L involve a minus sign in their definition as per Eqs. ((F.3) - (F.15)).

APPENDIX G: Components of Jacobians J' (Eq. (180)) and $\hat{J}'$ (Eq. (208))

Using the 28-dimensional vector variable $\underset{\sim}{Y}$ defined on p. 31 the nonlinear governing differential equations (Eqs. (21)-(27)) can be put into the form given by Eq. (146), where the 28-dimensional vector function $\underset{\sim}{f}$ has the following components

$$\begin{array}{lll} f_1 = Y_8 & f_8 = Y_{15} & f_{15} = Y_{22} \\ f_2 = Y_9 & f_9 = Y_{16} & f_{16} = Y_{23} \\ f_3 = Y_{10} & f_{10} = Y_{17} & f_{17} = Y_{24} \\ f_4 = Y_{11} & f_{11} = Y_{18} & f_{18} = Y_{25} \\ f_5 = Y_{12} & f_{12} = Y_{19} & f_{19} = Y_{26} \\ f_6 = Y_{13} & f_{13} = Y_{20} & f_{20} = Y_{27} \\ f_7 = Y_{14} & f_{14} = Y_{21} & f_{21} = Y_{28} \end{array} \tag{G.1}$$

and

$$\begin{aligned} f_{22} = {} & C_1Y_{15} - C_2Y_1 - C_3Y_{20} - C_4\lambda(Y_{20} + A_1'') + C_5Y_6 + C_{201}Y_{24} + C_{202}Y_{10} + C_{203}Y_{28} \\ & + C_{204}Y_{14} + C_{211}\bar{p}_e(Y_6 + A_1) + C_{212}\bar{\tau}(Y_{14} + A_2') + C_6[Y_{19}(Y_6 + 2A_1) + Y_6(Y_{19} + 2A_o'')] \\ & - C_7Y_{19}(Y_6 + A_1) + C_8Y_5(Y_6 + A_1) - 2C_{10}Y_1(Y_{19} + A_o'') \\ & - C_9[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)](Y_6 + A_1) - C_{10}\{Y_{16}(Y_6 + A_1) + 4Y_9(Y_{13} + A_1') \\ & + 4Y_2(Y_{20} + A_1'') + Y_{18}(Y_7 + A_2) + 4Y_{11}(Y_{14} + A_2') + 4Y_4(Y_{21} + a_2'')\} \end{aligned} \tag{G.2}$$

$$\begin{aligned} f_{23} = {} & C_{11}Y_{16} - C_{12}Y_2 + C_{205}Y_{25} + C_{206}Y_{11} + C_{13}\{Y_{20}(Y_6 + 2A_1) - 2Y_{13}(Y_{13} + 2A_1') \\ & + Y_6(Y_{20} + 2A_1'') - [Y_{21}(Y_7 + 2A_2) - 2Y_{14}(Y_{14} + 2A_2') + Y_7(Y_{21} + 2A_2'')]\} \end{aligned} \tag{G.3}$$

$$\begin{aligned} f_{24} = {} & C_1Y_{17} - C_2Y_3 - C_3Y_{21} - C_4\lambda(Y_{21} + A_2'') + C_5Y_7 - C_{201}Y_{22} - C_{202}Y_8 \\ & - C_{203}Y_{27} - C_{204}Y_{13} + C_{211}\bar{p}_e(Y_7 + A_2) - C_{212}\bar{\tau}(Y_{13} + A_1') \\ & + C_6[Y_{19}(Y_7 + 2A_2) + Y_7(Y_{19} + 2A_o'')] - C_7Y_{19}(Y_7 + A_2) + C_8Y_5(Y_7 + A_2) \\ & - 2C_{10}Y_3(Y_{19} + A_o'') - C_9[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)](Y_7 + A_2) - C_{10}\{Y_{18}(Y_6 + A_1) \\ & + 4Y_{11}(Y_{13} + A_1') + 4Y_4(Y_{20} + A_1'') - [Y_{16}(Y_7 + A_2) + 4Y_9(Y_{14} + A_2') + 4Y_2(Y_{21} + A_2'')]\} \end{aligned} \tag{G.4}$$

$$\begin{aligned} f_{25} = {} & C_{11}Y_{18} - C_{12}Y_4 - C_{205}Y_{23} - C_{206}Y_9 + C_{13}\{Y_{21}(Y_6 + 2A_1) - 2Y_{14}(Y_{13} + 2A_1') \\ & + Y_7(Y_{20} + 2A_1'') + Y_{20}(Y_7 + 2A_2) - 2Y_{13}(Y_{14} + 2A_2') + Y_6(Y_{21} + 2A_2'')\} \end{aligned} \tag{G.5}$$

$$
\begin{aligned}
f_{26} = {} & C_{14}Y_{19} - C_{15}Y_5 - C_{19}\lambda(Y_{19} + A''_o) + C_{17}[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)] \\
& - C_{16}\{Y_{20}(Y_6 + 2A_1) + 2Y_{13}(Y_{13} + 2A'_1) + Y_6(Y_{20} + 2A''_1) + Y_{21}(Y_7 + 2A_2) \\
& + 2Y_{14}(Y_{14} + 2A'_2) + Y_7(Y_{21} + 2A''_2)\} - C_{18}\{Y_{15}(Y_6 + A_1) + 2Y_8(Y_{13} + A'_1) \\
& + Y_1(Y_{20} + A''_1) + Y_{17}(Y_7 + A_2) + 2Y_{10}(Y_{14} + A'_2) + Y_3(Y_{21} + A''_2)\}
\end{aligned}
\qquad \text{(G.6)}
$$

$$
\begin{aligned}
f_{27} = {} & C_{20}Y_{20} - C_{21}Y_6 + C_{22}Y_{15} - C_{24}Y_1 - C_{23}\lambda(Y_{20} + A''_1) + C_{209}Y_{28} + C_{210}Y_{14} + C_{207}Y_{24} \\
& + C_{208}Y_{10} + C_{213}\bar{p}_e(Y_6 + A_1) + C_{214}\bar{\tau}(Y_{14} + A'_2) - C_{25}[Y_{19}(Y_6 + 2A_1) \\
& + Y_6(Y_{19} + 2A''_o)] - C_{26}Y_{19}(Y_6 + A_1) + C_{27}Y_5(Y_6 + A_1) - C_{28}[Y_6(Y_6 + 2A_1) \\
& + Y_7(Y_7 + 2A_2)](Y_6 + A_1) - 2C_{29}Y_1(Y_{19} + A''_o) - C_{29}\{Y_{16}(Y_6 + A_1) + 4Y_9(Y_{13} + A'_1) \\
& + 4Y_2(Y_{20} + A''_1) + Y_{18}(Y_7 + A_2) + 4Y_{11}(Y_{14} + A'_2) + 4Y_4(Y_{21} + A''_2)\}
\end{aligned}
\qquad \text{(G.7)}
$$

$$
\begin{aligned}
f_{28} = {} & C_{20}Y_{21} - C_{21}Y_7 + C_{22}Y_{17} - C_{24}Y_3 - C_{23}\lambda(Y_{21} + A''_2) - C_{209}Y_{27} - C_{210}Y_{13} \\
& - C_{207}Y_{22} - C_{208}Y_8 + C_{213}\bar{p}_e(Y_7 + A_2) - C_{214}\bar{\tau}(Y_{13} + A'_1) - C_{25}[Y_{19}(Y_7 + 2A_2) \\
& + Y_7(Y_{19} + 2A''_o)] - C_{26}Y_{19}(Y_7 + A_2) + C_{27}Y_5(Y_7 + A_2) - C_{28}[Y_6(Y_6 + 2A_1) \\
& + Y_7(Y_7 + 2A_2)](Y_7 + A_2) - 2C_{29}Y_3(Y_{19} + A''_o) - C_{29}\{Y_{18}(Y_6 + A_1) + 4Y_{11}(Y_{13} + A'_1) \\
& + 4Y_4(Y_{20} + A''_1) \;\; - [Y_{16}(Y_7 + A_2) + 4Y_9(Y_{14} + A'_2) + 4Y_2(Y_{21} + A''_2]\}
\end{aligned}
\qquad \text{(G.8)}
$$

Next one must evaluate analytically the components of the Jacobian matrix

$$
J' = \frac{\partial}{\partial \underset{\sim}{Y}} \underset{\sim}{f} \, (\underset{\sim}{\bar{x}}, \underset{\sim}{Y} \, ; \lambda, \bar{p}_e, \bar{\tau}) = \frac{\partial}{\partial \underset{\sim}{U}} \underset{\sim}{f} = \frac{\partial}{\partial \underset{\sim}{V}} \underset{\sim}{f} \qquad \text{(G.9)}
$$

Introducing the short-hand notation

$$
J'_{i,j} = \frac{\partial f_i}{\partial Y_j} \qquad \text{(G.10)}
$$

one obtains the following values

$$
J'_{1,8} = J'_{2,9} + J'_{3,10} = J'_{4,11} = J'_{5,12} = J'_{6,13} = J'_{7,14} = J'_{8,15} = J'_{9,16} = J'_{10,17} = 1 \qquad \text{(G.11)}
$$

$$
J'_{11,18} = J'_{12,19} + J'_{13,20} = J'_{14,21} = J'_{15,22} = J'_{16,23} = J'_{17,24} = J'_{18,25} = J'_{19,26} = J'_{20,27} = J'_{21,28} = 1
$$

Further

$$
J'_{22,1} = - C_2 - 2C_{10}(Y_{19} + A''_o) \qquad J'_{22,4} = - 4C_{10}(Y_{21} + A''_2) \qquad \text{(G.12)}
$$

$J'_{22,2} = -4C_{10}(Y_{20} + A''_1)$ $\quad J'_{22,5} = C_8(Y_6 + A_1)$

$$J'_{22,6} = C_5 + C_{211}\bar{p}_e + 2C_6(Y_{19} + A''_o) - C_7Y_{19} + C_8Y_5 - C_9[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)] - 2C_9(Y_6 + A_1)(Y_6 + A_1) - C_{10}Y_{16}$$

$$J'_{22,7} = -2C_9(Y_6 + A_1)(Y_7 + A_2) - C_{10}Y_{18}$$

$J'_{22,9} = -4C_{10}(Y_{13} + A'_1)$ $\quad J'_{22,18} = -C_{10}(Y_7 + A_2)$

$J'_{22,10} = C_{202}$ $\quad J'_{22,19} = 2C_6(Y_6 + A_1) - C_7(Y_6 + A_1) - 2C_{10}Y_1$ (G.12)

$J'_{22,11} = -4C_{10}(Y_{14} + A'_2)$ $\quad J'_{22,20} = -C_3 - C_4\lambda - 4C_{10}Y_2$

$J'_{22,13} = -4C_{10}Y_9$ $\quad J'_{22,21} = -4C_{10}Y_4$

$J'_{22,14} = C_{204} + C_{212}\bar{\tau} - 4C_{10}Y_{11}$ $\quad J'_{22,24} = C_{201}$

$J'_{22,15} = C_1$ $\quad J'_{22,28} = C_{203}$

$J'_{22,16} = -C_{10}(Y_6 + A_1)$

Further

$J'_{23,2} = -C_{12}$ $\quad J'_{23,14} = 4C_{13}(Y_{14} + A'_2)$

$J'_{23,6} = 2C_{13}(Y_{20} + A''_1)$ $\quad J'_{23,16} = C_{11}$

$J'_{23,7} = -2C_{13}(Y_{21} + A''_2)$ $\quad J'_{23,20} = 2C_{13}(Y_6 + A_1)$ (G.13)

$J'_{23,11} = C_{206}$ $\quad J'_{23,21} = -2C_{13}(Y_7 + A_2)$

$J'_{23,13} = -4C_{13}(Y_{13} + A'_1)$ $\quad J'_{23,25} = C_{205}$

Further

$J'_{24,2} = 4C_{10}(Y_{21} + A''_2)$ $\quad J'_{24,16} = C_{10}(Y_7 + A_2)$ (G.14)

$$J'_{24,3} = -C_2 - 2C_{10}(Y_{19} + A''_o) \qquad J'_{24,17} = C_1$$

$$J'_{24,4} = -4C_{10}(Y_{20} + A''_1) \qquad J'_{24,18} = -C_{10}(Y_6 + A_1)$$

$$J'_{24,5} = C_8(Y_7 + A_2) \qquad J'_{24,19} = 2C_6(Y_7 + A_2) - C_7(Y_7 + A_2) - 2C_{10}Y_3$$

$$J'_{24,6} = -2C_9(Y_6 + A_1)(Y_7 + A_2) - C_{10}Y_{18} \qquad J'_{24,20} = -4C_{10}Y_4$$

$$J'_{24,7} = C_5 + C_{211}\bar{p}_e + 2C_6(Y_{19} + A''_o) - C_7Y_{19} + C_8Y_5$$
$$- C_9[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)] - 2C_9(Y_7 + A_2)(Y_7 + A_2) + C_{10}Y_{16} \qquad \text{(G.14)}$$

$$J'_{24,8} = -C_{202}$$

$$J'_{24,9} = 4C_{10}(Y_{14} + A'_2) \qquad J'_{24,21} = -C_3 - C_4\lambda + 4C_{10}Y_2$$

$$J'_{24,11} = -4C_{10}(Y_{13} + A'_1) \qquad J'_{24,22} = -C_{201}$$

$$J'_{24,13} = -C_{204} - C_{212}\bar{\tau} - 4C_{10}Y_{11} \qquad J'_{24,27} = -C_{203}$$

$$J'_{24,14} = 4C_{10}Y_9$$

Further

$$J'_{25,4} = -C_{12} \qquad J'_{25,14} = -4C_{13}(Y_{13} + A'_1)$$

$$J'_{25,6} = 2C_{13}(Y_{21} + A''_2) \qquad J'_{25,18} = C_{11}$$

$$J'_{25,7} = 2C_{13}(Y_{20} + A''_1) \qquad J'_{25,20} = 2C_{13}(Y_7 + A_2) \qquad \text{(G.15)}$$

$$J'_{25,9} = -C_{206} \qquad J'_{25,21} = 2C_{13}(Y_6 + A_1)$$

$$J'_{25,13} = -4C_{13}(Y_{14} + A''_2) \qquad J'_{25,23} = -C_{205}$$

Further

$$J'_{26,1} = -C_{18}(Y_{20} + A''_1) \qquad J'_{26,13} = -4C_{16}(Y_{13} + A'_1) - 2C_{18}Y_8 \qquad \text{(G.16)}$$

$$J'_{26,3} = -C_{18}(Y_{21} + A''_2) \qquad J'_{26,14} = -4C_{16}(Y_{14} + A'_2) - 2C_{18}Y_{10}$$

$$J'_{26,5} = -C_{15} \qquad J'_{26,15} = -C_{18}(Y_6 + A_1)$$

$$J'_{26,6} = 2C_{17}(Y_6 + A_1) - 2C_{16}(Y_{20} + A''_1) - C_{18}Y_{15} \qquad J'_{26,17} = -C_{18}(Y_7 + A_2) \tag{G.16}$$

$$J'_{26,7} = 2C_{17}(Y_7 + A_2) - 2C_{16}(Y_{21} + A''_2) - C_{18}Y_{17} \qquad J'_{26,19} = C_{14} - C_{19}\lambda$$

$$J'_{26,8} = -2C_{18}(Y_{13} + A'_1) \qquad J'_{26,20} = -2C_{16}(Y_6 + A_1) - C_{18}Y_1$$

$$J'_{26,10} = -2C_{18}(Y_{14} + A'_2) \qquad J'_{26,21} = -2C_{16}(Y_7 + A_2) - C_{18}Y_3$$

Further

$$J'_{27,1} = -C_{24} - 2C_{29}(Y_{19} + A''_o) \qquad J'_{27,14} = C_{210} + C_{214}\bar{\tau} - 4C_{29}Y_{11}$$

$$J'_{27,2} = -4C_{29}(Y_{20} + A''_1) \qquad J'_{27,15} = C_{22}$$

$$J'_{27,4} = -4C_{29}(Y_{21} + A''_2) \qquad J'_{27,16} = -C_{29}(Y_6 + A_1)$$

$$J'_{27,5} = C_{27}(Y_6 + A_1) \qquad J'_{27,18} = -C_{29}(Y_7 + A_2)$$

$$J'_{27,6} = -C_{21} + C_{213}\bar{p}_e - 2C_{25}(Y_{19} + A''_o) - C_{26}Y_{19} + C_{27}Y_5 - C_{18}[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)] - 2C_{28}(Y_6 + A_1)(Y_6 + A_1) - C_{29}Y_{16} \tag{G.17}$$

$$J'_{22,7} = -2C_{28}(Y_7 + A_2)(Y_6 + A_1) - C_{29}Y_{18} \qquad J'_{27,19} = -2C_{25}((Y_6 + A_1) - C_{26}(Y_6 + A_1) - 2C_{29}Y_1$$

$$J'_{27,9} = -4C_{29}(Y_{13} + A'_1) \qquad J'_{27,20} = C_{20} - C_{23}\lambda - 4C_{29}Y_2$$

$$J'_{27,10} = C_{208} \qquad J'_{27,21} = -4C_{29}Y_4$$

$$J'_{27,11} = -4C_{29}(Y_{14} + A'_2) \qquad J'_{27,24} = C_{207}$$

$$J'_{27,13} = -4C_{29}Y_9 \qquad J'_{27,28} = C_{209}$$

Further

$$J'_{28,2} = 4C_{29}(Y_{21} + A''_1) \qquad J'_{28,14} = 4C_{29}Y_9$$

$$J'_{28,3} = -C_{24} - 2C_{29}(Y_{19} + A''_0) \qquad J'_{28,16} = C_{29}(Y_7 + A_2)$$

$$J'_{28,4} = -4C_{29}(Y_{20} + A''_1) \qquad J'_{28,17} = C_{22}$$

$$J'_{28,5} = C_{27}(Y_7 + A_2) \qquad J'_{28,18} = -C_{29}(Y_6 + A_1) \tag{G.18}$$

$$J'_{28,6} = -2C_{28}(Y_6+A_1)(Y_7+A_2) - C_{29}Y_{18} \qquad J'_{28,19} = -2C_{25}(Y_7+A_2) - C_{26}(Y_7+A_2) - 2C_{29}Y_3$$

$$J'_{28,7} = -C_{21} + C_{213}\bar{p}_e - 2C_{25}(Y_{19} + A''_0) - C_{26}Y_{19} + C_{27}Y_5$$
$$- C_{28}[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)] - 2C_{28}(Y_7 + A_2)(Y_7 + A_2) + C_{29}Y_{16}$$

$$J'_{28,8} = -C_{208} \qquad J'_{28,20} = -4C_{29}Y_4$$

$$J'_{28,9} = 4C_{29}(Y_{14} + A'_2) \qquad J'_{28,21} = C_{20} - C_{23}\lambda + 4C_{29}Y_2$$

$$J'_{28,11} = -4C_{29}(Y_{13} + A'_1) \qquad J'_{28,22} = -C_{207}$$

$$J'_{28,13} = -C_{210} - C_{214}\bar{\tau} - 4C_{29}Y_{11} \qquad J'_{28,27} = -C_{209}$$

All other components of the Jacobian matrix J' are equal to zero. Notice that since the Jacobian J' is a function of $\underset{\sim}{Y}$ (or $\underset{\sim}{U}$ or $\underset{\sim}{V}$) therefore the variational equations (176)-(179) depend step-by-step on the results of the associated initial value problems. Thus the variational equations depend on the initial guesses $\underset{\sim}{S}^v$. Hence the variational equations must be integrated together with the corresponding associated initial value problems.

In shell buckling analysis the use of load increments to locate the limit point of the prebuckling equilibrium states leads to a singular problem. It has been shown by several authors[28,16] that with the introduction of the appropriate generalized displacements as an auxiliary constraint this singular behavior can be removed. It has been explained earlier (see Eqs. (152)-(153) and Eqs. (162)-(163)) that this auxiliary constraint condition can be written in the following form

$$\frac{d\Delta}{d\bar{x}} = f_\Delta(\bar{x}, \underset{\sim}{Y}\,;\, \lambda, \bar{p}_e, \bar{\tau}) \tag{G.19}$$

$$\Delta(0) = 0$$
$$\Delta(\tfrac{L}{R}) = \Delta_o \tag{G.20}$$

where, if the variable load $\Lambda = \lambda$, then $\Delta = \delta^{nl}$ and

$$f_\Delta = C_{50}Y_{19} + C_{51}Y_5 - C_{52}[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)] + C_{53}\{Y_{12}(Y_{12} + 2A_o') + \frac{1}{2}[Y_{13}(Y_{13} + 2A_1') + Y_{14}(Y_{14} + 2A_2')]\} \tag{G.21}$$

whereas if the variable load $\Lambda = \bar{p}_e$, then $\Delta = \bar{W}_{ave}$ and

$$f_\Delta = \frac{cR}{L} Y_5 \tag{G.22}$$

and finally, if the variable load $\Lambda = \bar{\tau}$, then $\Delta = \bar{\gamma}^{nl}$ and

$$f_\Delta = C_{550}Y_{19} - C_{551}Y_5 + C_{552}[Y_6(Y_6 + 2A_1) + Y_7(Y_7 + 2A_2)] - C_{553}\{Y_{13}(Y_7 + A_2) + Y_7A_1' - [Y_{14}(Y_6 + A_1) + Y_6A_2']\} \tag{G.23}$$

Considering the extended (29x29) dimensional Jacobian of Eq. (208)

$$\hat{J}' = \frac{\partial}{\partial \hat{\underset{\sim}{Y}}} \hat{\underset{\sim}{f}} (\bar{x}, \hat{\underset{\sim}{Y}} ; \lambda, \bar{p}_e, \bar{\tau})$$

then it can easily be seen that for the 29th column

$$\hat{J}'_{i,29} = \frac{\partial f_i}{\partial \Delta} = 0 \tag{G.24}$$

whereas for the 29th row the values of

$$\hat{J}'_{29,j} = \frac{\partial f_\Delta}{\partial Y_j} \tag{G.25}$$

depend on the generalized displacement used.
Thus for $\Delta = \delta^{nl}$

$$\begin{aligned} \hat{J}'_{29,5} &= C_{51} & \hat{J}'_{29,13} &= C_{53}(Y_{13} + A_1') \\ \hat{J}'_{29,6} &= -2C_{52}(Y_6 + A_1) & \hat{J}'_{29,14} &= C_{53}(Y_{14} + A_2') \\ \hat{J}'_{29,7} &= -2C_{52}(Y_7 + A_2) & \hat{J}'_{29,19} &= C_{50} \\ \hat{J}'_{29,12} &= 2C_{53}(Y_{12} + A_o') \end{aligned} \tag{G.26}$$

whereas for $\Delta = \bar{w}_{ave}$

$$\hat{J}'_{29,5} = \frac{cR}{L} \qquad \text{(G.27)}$$

and finally for $\Delta = \overline{\gamma}^{nl}$

$$\hat{J}'_{29,5} = -C_{551} \qquad \hat{J}'_{29,13} = -C_{553}(Y_7 + A_2)$$

$$\hat{J}'_{29,6} = 2C_{552}(Y_6 + A_1) + C_{553}(Y_{14} + A_2') \qquad \hat{J}'_{29,14} = C_{553}(Y_6 + A_1) \qquad \text{(G.28)}$$

$$\hat{J}'_{29,7} = 2C_{552}(Y_7 + A_2) - C_{553}(Y_{13} + A_1') \qquad \hat{J}'_{29,19} = C_{550}$$

All other $\hat{J}'_{29,j} = 0$.

APPENDIX H: The Forcing Function $\frac{\partial}{\partial\Lambda}\hat{\underset{\sim}{f}}$ and the Generalized Unit Vectors $\hat{\underset{\sim}{1}}_{\Lambda}$

The form of the forcing function $\partial\hat{\underset{\sim}{f}}/\partial\Lambda$ of the inhomogeneous variational equations (204)-(207) depends on the choice of the variable load parameter Λ. Thus if $\Lambda = \lambda$, then from Eqs. (G.2)-(G.8) and from Eq. (G.21)

$$\frac{\partial\hat{\underset{\sim}{f}}}{\partial\Lambda} = \frac{\partial\hat{\underset{\sim}{f}}}{\partial\lambda}$$
$$= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathring{f}_{22}, 0, \mathring{f}_{24}, 0, \mathring{f}_{26}, \mathring{f}_{27}, \mathring{f}_{28}, 0\}^T \tag{H.1}$$

where

$$\mathring{f}_{22} = \frac{\partial f_{22}}{\partial\lambda} = -\,C_4(Y_{20} + A_1'') \qquad \mathring{f}_{27} = \frac{\partial f_{27}}{\partial\lambda} = -C_{23}(Y_{20} + A_1'')$$
$$\mathring{f}_{24} = \frac{\partial f_{24}}{\partial\lambda} = -\,C_4(Y_{21} + A_2'') \qquad \mathring{f}_{28} = \frac{\partial f_{28}}{\partial\lambda} = -\,C_{23}(Y_{21} + A_2'') \tag{H.2}$$
$$\mathring{f}_{26} = \frac{\partial f_{26}}{\partial\lambda} = -\,C_{19}(Y_{19} + A_o'')$$

whereas if $\Lambda = \bar{p}_e$, then from Eqs. (G.2)-(G.8) and from Eq. (G.22), assuming that $\lambda = \bar{p}_e\hat{R}$

$$\frac{\partial\hat{\underset{\sim}{f}}}{\partial\Lambda} = \frac{\partial\hat{\underset{\sim}{f}}}{\partial\bar{p}_e}$$
$$= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathring{f}_{22}, 0, \mathring{f}_{24}, 0, \mathring{f}_{26}, \mathring{f}_{27}, \mathring{f}_{28}, 0\}^T \tag{H.3}$$

where

$$\mathring{f}_{22} = \frac{\partial f_{22}}{\partial\bar{p}_e} = -\,C_4(Y_{20} + A_1'')\hat{R} + C_{211}(Y_6 + A_1) \quad ; \mathring{f}_{27} = \frac{\partial f_{27}}{\partial\bar{p}_e} = -\,C_{23}(Y_{20} + A_1'')\hat{R} + C_{213}(Y_6 + A_1)$$
$$\mathring{f}_{24} = \frac{\partial f_{24}}{\partial\bar{p}_e} = -\,C_4(Y_{21} + A_2'')\hat{R} + C_{211}(Y_7 + A_2) \quad ; \mathring{f}_{28} = \frac{\partial_{28}}{\partial\bar{p}_e} = -\,C_{23}(Y_{21} + A_2'')\hat{R} + C_{213}(Y_7 + A_2) \tag{H.4}$$
$$\mathring{f}_{26} = \frac{\partial_{26}}{\partial\bar{p}_e} = -\,C_{19}(Y_{19} + A_o'')\hat{R}$$

Finally, if $\Lambda = \bar{\tau}$ then from Eqs. (G.2)-(G.8) and from Eq. (G.23)

$$\frac{\partial\hat{\underset{\sim}{f}}}{\partial\Lambda} = \frac{\partial\hat{\underset{\sim}{f}}}{\partial\bar{\tau}}$$
$$= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathring{f}_{22}, 0, \mathring{f}_{24}, 0, 0, \mathring{f}_{27}, \mathring{f}_{28}, 0\}^T \tag{H.5}$$

where

$$\mathring{f}_{22} = \frac{\partial f_{22}}{\partial\bar{\tau}} = C_{212}(Y_{14} + A_2') \qquad \mathring{f}_{27} = \frac{\partial f_{27}}{\partial\bar{\tau}} = C_{214}(Y_{14} + A_2')$$
$$\mathring{f}_{24} = \frac{\partial f_{24}}{\partial\bar{\tau}} = -\,C_{212}(Y_{13} + A_1') \qquad \mathring{f}_{28} = \frac{\partial f_{28}}{\partial\bar{\tau}} = -\,C_{214}(Y_{13} + A_1') \tag{H.6}$$

The form of the generalized unit vectors $\hat{\underset{\sim}{1}}_{\Lambda}$, used as the initial condition for the inhomogeneous variational equations (204) and (207), depends on both the choice of the variable load parameter Λ and on the boundary condition one wants to enforce. Introducing the following notation

$\hat{\underset{\sim}{1}}_{BC}$ - generalized unit vector for the specified boundary condition (for instance, BC = SS-1 etc)

$$\hat{\underset{\sim}{1}}_{\lambda} = \frac{\partial}{\partial \lambda}\hat{\underset{\sim}{1}}_{BC} \qquad \hat{\underset{\sim}{1}}_{p} = \frac{\partial}{\partial \bar{p}_e}\hat{\underset{\sim}{1}}_{BC} \qquad \hat{\underset{\sim}{1}}_{\tau} = \frac{\partial}{\partial \bar{\tau}}\hat{\underset{\sim}{1}}_{BC} \tag{H.7}$$

one has the following options.

SS-1 Boundary Condition

$$\hat{\underset{\sim}{1}}_{SS-1} = \{0,\ 0,\ 0,\ 0,\ f_u^1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6,\ s_7,\ f_u^2,\ f_u^3,\ f_u^4,\ s_8,\ s_9,\ s_{10},\ s_{11},\ s_{12},\ s_{13},\ s_{14},\ 0\}^T$$

$$\hat{\underset{\sim}{1}}_{\lambda} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_v,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{1}}_{p} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_c,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41}\hat{R},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{1}}_{\tau} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_t,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{220},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

where

$$f_u^1 = -\,(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \qquad f_U^3 = -\,B_1 s_4 - B_2 s_3$$

$$f_U^2 = C_{41}\lambda + C_{220}\bar{\tau} \qquad f_U^4 = -\,B_1 s_6 + B_2 s_2 \tag{H.8}$$

and

$$\hat{W}_v = \frac{1}{c}\bar{A}_{12}^* \quad ; \quad \hat{W}_p = \frac{1}{c}\bar{A}_{22}^* \quad ; \quad \hat{W}_t = -\frac{1}{c}\bar{A}_{26}^* \quad ; \quad \hat{W}_c = \hat{W}_v\hat{R} + \hat{W}_p \tag{H.9}$$

SS-2 Boundary Condition

$$\hat{\underset{\sim}{1}}_{SS-2} = \{s_1,\ s_2,\ s_3,\ s_4,\ f_u^1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ s_5,\ s_6,\ s_7,\ s_8,\ s_9,\ s_{10},\ s_{11},\ f_u^2,\ f_u^3,\ f_u^4,\ f_u^5,\ f_u^6,\ f_u^7,\ f_u^8,\ s_{12},\ s_{13},\ s_{14},\ 0\}^T$$

$$\hat{\underset{\sim}{1}}_{\lambda} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_v,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{1}}_{p} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_c,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41}\hat{R},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{1}}_{\tau} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_t,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{220},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

where

$$f_u^1 = -(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \qquad f_U^5 = B_5s_{10} - B_6s_3 - B_7s_6 + B_8s_{13} + 2B_9A_1s_5$$

$$f_u^2 = C_{41}\lambda + C_{220}\bar{\tau} \qquad f_U^6 = 2B_{10}s_{11} - 8B_{11}s_4 + B_9A_1s_6 - B_9A_2s_7$$

$$f_U^3 = -B_1s_8 + B_3s_1 - B_4s_7 \qquad f_U^7 = -B_5s_8 + B_6s_1 - B_7s_7 + B_8s_{14} + 2B_9A_2s_5 \tag{H.10}$$

$$f_U^4 = -B_1s_{10} + B_3s_3 + B_4s_6 \qquad f_U^8 = -2B_{10}s_9 + 8B_{11}s_2 + B_9A_2s_6 + B_9A_1s_7$$

SS-3 Boundary Condition

$$\hat{I}_{\sim SS-3} = \{0,\ 0,\ 0,\ 0,\ f_u^1,\ 0,\ 0,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6,\ s_7,\ f_u^2,\ f_u^3,\ f_u^4,\ f_u^5,\ f_u^6,\ f_u^7,\ f_u^8,\ s_8,\ s_9,\ s_{10},\ s_{11},\ s_{12},\ s_{13},\ s_{14},\ 0\}^T$$

$$\hat{I}_{\sim\lambda} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_v,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{I}_{\sim p} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_c,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41}\hat{R},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{I}_{\sim\tau} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_t,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{220},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

where

$$f_u^1 = -(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \qquad f_U^5 = -2B_{17}s_2$$

$$f_u^2 = B_{15}s_3 + B_{16}s_7 \qquad f_U^6 = C_{41}\lambda + C_{220}\bar{\tau}$$

$$f_U^3 = 2B_{17}s_4 \qquad f_U^7 = B_{18}s_3 - B_{19}s_7 \tag{H.11}$$

$$f_U^4 = -B_{15}s_1 - B_{16}s_6 \qquad f_U^8 = -B_{18}s_1 + B_{19}s_6$$

SS-4 Boundary Condition

$$\hat{I}_{\sim SS-4} = \{s_1,\ s_2,\ s_3,\ s_4,\ f_u^1,\ 0,\ 0,\ s_5,\ s_6,\ s_7,\ s_8,\ s_9,\ s_{10},\ s_{11},\ f_u^2,\ f_u^3,\ f_u^4,\ f_u^5,\ f_u^6,\ f_u^7,\ f_u^8,\ f_u^9,\ f_u^{10},\ f_u^{11},\ f_u^{12},\ s_{12},\ s_{13},\ s_{14},\ 0\}^T$$

$$\hat{I}_{\sim\lambda} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_v,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{I}_{\sim p} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_c,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{41}\hat{R},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{I}_{\sim\tau} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_t,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ C_{220},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

where

$$f_U^1 = -(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \qquad f_U^7 = B_{25}s_1 + B_{26}s_7 - B_{27}s_{11}$$

$$f_U^2 = B_{20}s_1 + B_{21}s_7 - B_{22}s_{11} \qquad f_U^8 = B_{25}s_3 - B_{26}s_5 + B_{27}s_{10} \tag{H.12}$$

$$f_U^3 = 4B_{24}s_2 + 2B_{17}s_8 \qquad f_U^9 = B_{28}s_5 - B_{29}s_3 + B_8s_{13} + B_{30}s_{10} + 2B_9A_1s_9$$

$$f_U^4 = B_{20}S_3 - B_{21}s_5 + B_{22}s_{10} \qquad f_U^{10} = 4B_{31}s_6 - 8B_{32}s_4 + B_9A_1s_{10} - B_9A_2s_{11}$$

$$f_u^5 = 4B_{24}s_4 - 2B_{17}s_6 \qquad f_U^{11} = B_{28}s_7 + B_{29}s_1 + B_8s_{14} + B_{30}s_{11} + 2B_9A_2s_9$$

$$f_U^6 = C_{41}\lambda + C_{220}\bar{\tau} \qquad f_U^{12} = 4B_{31}s_8 + 8B_{32}s_2 + B_9A_2s_{10} + B_9A_1s_{11}$$

C-1 Boundary Condition

$$\hat{\underset{\sim}{l}}_{C-1} = \{0, 0, 0, 0, f_u^1, 0, 0, 0, 0, 0, 0, 0, 0, 0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\lambda} = \{0, 0, 0, 0, -\hat{W}_v, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

$$\hat{\underset{\sim}{l}}_{p} = \{0, 0, 0, 0, -\hat{W}_c, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\tau} = \{0, 0, 0, 0, -\hat{W}_t, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

where

$$f_U^1 = -(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \tag{H.13}$$

C-2 Boundary Condition

$$\hat{\underset{\sim}{l}}_{C-2} = \{s_1, s_2, s_3, s_4, f_u^1, 0, 0, 0, 0, 0, 0, 0, 0, 0, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, f_u^2, f_u^3, f_u^4, f_u^5, s_{12}, s_{13}, s_{14}, 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\lambda} = \{0, 0, 0, 0, -\hat{W}_v, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

$$\hat{\underset{\sim}{l}}_{p} = \{0, 0, 0, 0, -\hat{W}_c, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\tau} = \{0, 0, 0, 0, -\hat{W}_t, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

where

$$f_U^1 = -(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \qquad f_U^4 = -B_{10}s_5 + B_{11}s_1 + B_8 s_{14} - B_{13}s_{10}$$

$$f_U^2 = B_{10}s_7 - B_{11}s_3 + B_8 s_{13} + B_{13}s_{11} \qquad f_U^5 = -2B_{10}s_6 + 8B_{11}s_2 \tag{H.14}$$

$$f_U^3 = 2B_{10}s_8 - 8B_{11}s_4$$

C-3 Boundary Condition

$$\hat{\underset{\sim}{l}}_{C-3} = \{0,\ 0,\ 0,\ 0,\ f_u^1,\ 0,\ 0,\ s_1,\ s_2,\ s_3,\ s_4,\ 0,\ 0,\ 0,\ f_u^2,\ f_u^3,\ f_u^4,\ f_u^5,\ s_5,\ s_6,\ s_7,\ s_8,\ s_9,\ s_{10},\ s_{11},\ s_{12},\ s_{13},\ s_{14},\ 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\lambda} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_v,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{l}}_{p} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_c,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\tau} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_t,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

where

$$f_U^1 = -(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \qquad f_U^4 = -B_{17}s_1 + B_8 s_7$$

$$f_U^2 = B_{17}s_3 + B_8 s_6 \qquad f_U^5 = -2B_{17}s_2 \tag{H.15}$$

$$f_U^3 = 2B_{17}s_4$$

C-4 Boundary Condition

$$\hat{\underset{\sim}{l}}_{C-4} = \{s_1,\ s_2,\ s_3,\ s_4,\ f_u^1,\ 0,\ 0,\ s_5,\ s_6,\ s_7,\ s_8,\ 0,\ 0,\ 0,\ f_u^2,\ f_u^3,\ f_u^4,\ f_u^5,\ s_9,\ s_{10},\ s_{11},\ f_u^6,\ f_u^7,\ f_u^8,\ f_u^9,\ s_{12},\ s_{13},\ s_{14},\ 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\lambda} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_v,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{l}}_{p} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_c,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

$$\hat{\underset{\sim}{l}}_{\tau} = \{0,\ 0,\ 0,\ 0,\ -\hat{W}_t,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}^T$$

where

$$f_U^1 = -(\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) \qquad f_U^6 = B_{31}s_5 - B_{32}s_3 + B_8 s_{13} + B_{14}s_{11}$$

$f_U^2 = B_{24}s_1 + B_{17}s_7 + B_8 s_{10}$

$f_U^3 = 4B_{24}s_2 + 2B_{17}s_8$

$f_U^4 = B_{24}s_3 - B_{17}s_5 + B_8 s_{11}$

$f_U^5 = 4B_{24}s_4 - 2B_{17}s_6$

$f_U^7 = 4B_{31}s_6 - 8B_{32}s_4$

$f_U^8 = B_{31}s_7 + B_{32}s_1 + B_8 s_{14} - B_{14}s_{10}$ (H.16)

$f_U^9 = 4B_{31}s_8 + 8B_{32}s_2$

Symmetry Condition at $\bar{x} = \frac{1}{2}\frac{L}{R}$

$$\hat{\underset{\sim sym}{1}} = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, 0, 0, 0, 0, 0, 0, 0, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, f_u^1, f_u^2, f_u^3, f_u^4, 0, f_u^5, f_u^6, 0\}^T$$

$$\hat{\underset{\sim\lambda}{1}} = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

$$\hat{\underset{\sim p}{1}} = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$$

$$\hat{\underset{\sim\tau}{1}} = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \hat{B}_1, 0, \hat{B}_2, 0, 0, \hat{B}_3, \hat{B}_4, 0\}^T$$

where

$f_U^1 = B_{33}s_{10} + B_{34}s_3 + B_{35}s_{14} + B_{36}s_7 + B_{37}A_2\bar{\tau}$

$f_U^2 = 2B_{10}s_{11} - 8B_{11}s_4$

$f_U^3 = -B_{33}s_8 - B_{34}s_1 - B_{35}s_{13} - B_{36}s_6 - B_{37}A_1\bar{\tau}$

$f_U^4 = -2B_{10}s_9 + 8B_{11}s_2$ (H.17)

$f_U^5 = -B_{38}s_{10} + B_{39}s_3 - B_{40}s_{14} + (B_{41} + B_{23}\bar{\tau})s_7 + B_{23}A_2\bar{\tau}$

$f_U^6 = B_{38}s_8 - B_{39}s_1 + B_{40}s_{13} - (B_{41} + B_{23}\bar{\tau})s_6 - B_{23}A_1\bar{\tau}$

and

$\hat{B}_1 = B_{37}A_2$

$\hat{B}_2 = -B_{37}A_1$

$\hat{B}_3 = B_{23}(s_7 + A_2)$ (H.18)

$\hat{B}_4 = -B_{23}(s_6 + A_1)$

Free Edge Condition

$$\hat{\underset{\sim}{I}}_{free} = \{0,\ 0,\ 0,\ 0,\ s_1,\ s_2,\ s_3,\ 0,\ 0,\ 0,\ 0,\ s_4,\ s_5,\ s_6,\ s_7,\ s_8,\ s_9\ s_{10},\ f_u^1,\ f_u^2,\ f_u^3,\ s_{11},\ s_{12},\ s_{13},\ s_{14},\ f_u^4,\ f_u^5,\ f_u^6,\ 0\}^T$$

$$\hat{\underset{\sim}{I}}_{\lambda} = \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ B_{50},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \hat{B}_1,\ \hat{B}_2,\ \hat{B}_3,\ 0\}^T$$

$$\hat{\underset{\sim}{I}}_{p} = \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \hat{B}_4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \hat{B}_5,\ \hat{B}_6,\ \hat{B}_7,\ 0\}^T$$

$$\hat{\underset{\sim}{I}}_{\tau} = \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -B_{52},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \hat{B}_8,\ \hat{B}_9,\ 0\}^T$$

where

$$\begin{aligned}
f_u^1 &= B_{42}s_1 + B_{50}\lambda + B_{51}\bar{p}_e - B_{52}\bar{\tau} - B_{53}[s_2(s_2 + 2A_1) + s_3(s_3 + 2A_2)] \\
f_u^2 &= B_{12}s_2 - B_2 s_6 - B_1 s_7 \\
f_u^3 &= B_{12}s_3 + B_2 s_5 - B_1 s_9 \\
f_u^4 &= (B_{42} - B_{43}\lambda)s_4 - B_{43}A_0'\lambda - B_{44}[s_5(s_2 + A_1) + s_2 A_1' + s_6(s_3 + A_2) + s_3 A_2'] \\
f_u^5 &= - B_1 s_{11} + B_{45}s_9 + (B_{46} - B_{47}\lambda)s_5 - B_{47}A_1'\lambda + (B_{48} + B_{49}\bar{\tau})s_3 + B_{49}A_2\bar{\tau} \\
f_u^6 &= - B_1 s_{13} - B_{45}s_7 + (B_{46} - B_{47}\lambda)s_6 - B_{47}A_2'\lambda - (B_{48} + B_{49}\bar{\tau})s_2 - B_{49}A_1\bar{\tau}
\end{aligned} \tag{H.19}$$

and

$$\begin{aligned}
\hat{B}_1 &= - B_{43}(s_4 + A_0') & \hat{B}_4 &= B_{50}\hat{R} + B_{51} & \hat{B}_7 &= - B_{47}(s_6 + A_2')\hat{R} \\
\hat{B}_2 &= - B_{47}(s_5 + A_1') & \hat{B}_5 &= - B_{43}(s_4 + A_0')\hat{R} & \hat{B}_8 &= B_{49}(s_3 + A_2) \\
\hat{B}_3 &= - B_{47}(s_6 + A_2') & \hat{B}_6 &= - B_{47}(s_5 + A_1')\hat{R} & \hat{B}_9 &= - B_{49}(s_2 + A_1)
\end{aligned} \tag{H.20}$$

Mixed Symmetry or Anti-symmetry Condition at $x = \frac{1}{2}\frac{L}{R}$

$$\hat{\underset{\sim}{I}}_{mixed} = \{s_1,\ s_2,\ 0,\ 0,\ s_3,\ s_4,\ 0,\ 0,\ 0,\ s_5,\ s_6,\ 0,\ 0,\ s_7,\ s_8,\ s_9,\ 0,\ 0,\ s_{10},\ s_{11},\ 0,\ 0,\ 0,\ s_{12},\ s_{13},\ 0,\ 0,\ s_{14},\ 0\}^T$$

whereas

$$\hat{\underset{\sim}{I}}_{\lambda} = \hat{\underset{\sim}{I}}_{p} = \hat{\underset{\sim}{I}}_{\tau} = \{\ \underset{\sim}{o}\ \}^T$$

APPENDIX I: The Extended Jacobian $\hat{J}(\hat{\underset{\sim}{S}}^{v}) = \dfrac{\partial}{\partial \hat{\underset{\sim}{S}}} \hat{\underset{\sim}{\phi}} (\hat{\underset{\sim}{S}}^{v})$

Using elastic end rings to impose the boudary conditions at the shell edges and employing Newton's Method with increments in deformation to obtain the location of the limit point involves the use of the following extended Jacobian $\hat{J}(\hat{\underset{\sim}{S}}^{v})$ given by Eq. (196)

$$
\hat{J} = \frac{\partial}{\partial \underset{\sim}{S}} \hat{\underset{\sim}{\phi}}(\hat{\underset{\sim}{S}}^{v}) =
\left[\begin{array}{cccccccccc}
\frac{\partial g_1}{\partial s_1} & \cdots & \frac{\partial g_1}{\partial s_{28}} & & & & & & & \frac{\partial g_1}{\partial \Lambda} \\
\cdots & \cdots & \cdots & & 0 & & & 0 & & \cdots \\
\frac{\partial g_{14}}{\partial s_1} & \cdots & \frac{\partial g_{14}}{\partial s_{28}} & & & & & & & \frac{\partial g_{14}}{\partial \Lambda} \\
\frac{\partial \varphi_1}{\partial s_1} & \cdots & \frac{\partial \varphi_1}{\partial s_{28}} & \frac{\partial \varphi_1}{\partial p_1} & \cdots & \frac{\partial \varphi_1}{\partial p_{28}} & & & & \frac{\partial \varphi_1}{\partial \Lambda} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \cdots \\
\cdots & & \cdots & \cdots & & \cdots & & 0 & & \cdots \\
\cdots & & \cdots & \cdots & & \cdots & & & & \cdots \\
\frac{\partial \varphi_{28}}{\partial s_1} & \cdots & \frac{\partial \varphi_{28}}{\partial \varphi_{28}} & \frac{\partial \varphi_{28}}{\partial p_1} & \cdots & \frac{\partial \varphi_{28}}{\partial p_{28}} & & & & \frac{\partial \varphi_{28}}{\partial \Lambda} \\
& & & \frac{\partial \varphi_{29}}{\partial p_1} & \cdots & \frac{\partial \varphi_{29}}{\partial p_{28}} & \frac{\partial \varphi_{29}}{\partial t_1} & \cdots & \frac{\partial \varphi_{29}}{\partial t_{28}} & \frac{\partial \varphi_{29}}{\partial \Lambda} \\
& & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& 0 & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & \cdots & \cdots & & \cdots & \cdots \\
& & & \frac{\partial \varphi_{56}}{\partial p_1} & \cdots & \frac{\partial \varphi_{56}}{\partial p_{28}} & \frac{\partial \varphi_{56}}{\partial t_1} & \cdots & \frac{\partial \varphi_{56}}{\partial t_{28}} & \frac{\partial \varphi_{56}}{\partial \Lambda} \\
& & & & & & \frac{\partial h_1}{\partial t_1} & \cdots & \frac{\partial h_1}{\partial t_{28}} & \frac{\partial h_1}{\partial \Lambda} \\
& 0 & & & 0 & & \cdots & \cdots & \cdots & \cdots \\
& & & & & & \frac{\partial h_{14}}{\partial t_1} & \cdots & \frac{\partial h_{14}}{\partial t_{28}} & \frac{\partial h_{14}}{\partial \Lambda} \\
\frac{\partial \Delta}{\partial s_1} & \cdots & \frac{\partial \Delta}{\partial s_{28}} & \frac{\partial \Delta}{\partial p_1} & \cdots & \frac{\partial \Delta}{\partial p_{28}} & \frac{\partial \Delta}{\partial t_1} & \cdots & \frac{\partial \Delta}{\partial t_{28}} & \frac{\partial \Delta}{\partial \Lambda}
\end{array}\right] \quad (I.1)
$$

A careful look at the structure of this block-diagonal matrix reveals that it is assembled out of the results of different computations. The components involving the derivatives of the matching and constraint conditions, the $\partial\varphi_i/\partial S_j$ terms, are obtained by solving the appropriate 29-dimensional variational equations (see Eqs. (200)-(203), whereby the 29th components are used to fill the last horizontal row. The components of the last column are assembled out of the solutions of the corresponding inhomogeneous variational equations (see Eqs. (204)-(207)). Finally those components of this Jacobian which involve derivatives of the specified boundary vectors $\underset{\sim}{g}$ and $\underset{\sim}{h}$ can be calculated analyticaly, as will be described in the following.

Limiting Case as $E_r \to 0$

In this case one must satisfy the 14-boundary conditions specified by Eq. (143), the first of which reads

$$g_1 = b_{11}^o H_o^s + b_{12}^o (M_{x_o}^s + \lambda\bar{q}) - (w_o^s + \hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}_t\bar{\tau}) = 0 \tag{I.2a}$$

Replacing the shell variables involved H_o^s and $M_{x_o}^s$ by their equivalent expressions in terms of the variables used in the anisotropic shell analysis $\bar{W}$, W and F derived earlier (see Eqs. (99) and (107), respectively), one obtains after some regrouping the following nonlinear boundary condition

$$\begin{aligned} g_1 = &- 2c\frac{R}{t} n^2 b_{11}^o[(Y_{13}+ A_1')Y_1 + (Y_{14}+ A_2')Y_3] + (2c\frac{R}{t} b_{12}^o \frac{\bar{B}_{21}^*}{\bar{A}_{22}^*} - 1)Y_5 \\ &- cn^2 \frac{\bar{B}_{21}^*}{\bar{A}_{22}^*} [b_{11}^o A_1' Y_6 + \frac{1}{2} b_{12}^o (Y_6 + 2A_1)Y_6] - cn^2 \frac{\bar{B}_{21}^*}{\bar{A}_{22}^*} [b_{11}^o A_2' Y_7 + \frac{1}{2} b_{12}^o (Y_7 + 2A_2)Y_7] \\ &- 2c\frac{R}{t} n^2 b_{11}^o [Y_6 + A_1)Y_8 + (Y_7 + A_2)Y_{10} - \frac{1}{n^2}(\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*} - 2\lambda)Y_{12}] \\ &- cn^2 b_{11}^o \frac{\bar{B}_{21}^*}{\bar{A}_{22}^*} [(Y_6 + A_1)Y_{13} + (Y_7 + A_2)Y_{14}] - (\bar{D}_{11}^* + \frac{\bar{B}_{21}^*\bar{B}_{21}^*}{\bar{A}_{22}^*})(b_{12}^o Y_{19} + b_{11}^o Y_{26}) \\ &- 4c\frac{R}{t} b_{11}^o \lambda A_o' + 2\frac{R}{t} b_{12}^o (\bar{B}_{11}^*\lambda + \bar{B}_{21}^*\bar{p}_e - \bar{B}_{61}^*\bar{\tau}) + b_{12}^o \lambda\bar{q} - (\hat{W}_v\lambda + \hat{W}_p\bar{p}_e + \hat{W}\bar{\tau}) = 0 \end{aligned} \tag{I.2b}$$

Next one can proceed to calculate the partial derivatives $\partial g_1/\partial Y_i$ analytically yielding the following nonzero terms

$$\frac{\partial g_1}{\partial Y_1} = - 2c\frac{R}{t} n^2 b_{11}^o (Y_{13} + A_1') \qquad \frac{\partial g_1}{\partial Y_{14}} = - cn^2 b_{11}^o [2\frac{R}{t} Y_3 + \frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}(Y_7 + A_2)]$$

$$\frac{\partial g_1}{\partial Y_3} = - 2c\frac{R}{t} n^2 b_{11}^o (Y_{14} + A_2') \qquad \frac{\partial g_1}{\partial Y_{19}} = - b_{12}^o (\bar{D}_{11}^* + \frac{\bar{B}_{21}^*\bar{B}_{21}^*}{\bar{A}_{22}^*})$$

$$\frac{\partial g_1}{\partial Y_5} = 2c\,\frac{R}{t}\,b^o_{12}\,\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}} - 1 \qquad \frac{\partial g_1}{\partial Y_{26}} = -\,b^o_{11}\Big(\bar{D}^*_{11} + \frac{\bar{B}^*_{21}\bar{B}^*_{21}}{\bar{A}^*_{22}}\Big)$$

$$\frac{\partial g_1}{\partial Y_6} = -\,2c\,\frac{R}{t}\,n^2 b^o_{11} Y_8 - cn^2\,\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}\,[b^o_{11}(Y_{13} + A_1') + b^o_{12}(Y_6 + A_1)]$$

$$\frac{\partial g_1}{\partial Y_7} = -\,2c\,\frac{R}{t}\,n^2 b^o_{11} Y_{10} - cn^2\,\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}\,[b^o_{11}(Y_{14} + A_2') + b^o_{12}(Y_7 + A_2)]$$

$$\frac{\partial g_1}{\partial Y_8} = -\,2c\,\frac{R}{t}\,n^2 b^o_{11}(Y_6 + A_1) \qquad \frac{\partial g_1}{\partial \lambda} = -\,4c\,\frac{R}{t}\,b^o_{11}(Y_{12} + A_o') + b^o_{12}\Big(2\,\frac{R}{t}\,\bar{B}^*_{11} + \bar{q}\Big) - \hat{W}_v$$

$$\frac{\partial g_1}{\partial Y_{10}} = -\,2c\,\frac{R}{t}\,n^2 b^o_{11}(Y_7 + A_2) \qquad \frac{\partial g_1}{\partial \bar{p}_e} = \Big(\frac{\partial g_1}{\partial \lambda}\Big)\hat{R} + 2\,\frac{R}{t}\,b^o_{12}\bar{B}^*_{21} - \hat{W}_p$$

$$\frac{\partial g_1}{\partial Y_{12}} = 2c\,\frac{R}{t}\,b^o_{11}\Big(\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}} - 2\lambda\Big) \qquad \frac{\partial g_1}{\partial \bar{\tau}} = -\,2\,\frac{R}{t}\,b^o_{12}\bar{B}^*_{61} - \hat{W}_t$$

$$\frac{\partial g_1}{\partial Y_{13}} = -\,cn^2 b^o_{11}\Big[2\,\frac{R}{t}\,Y_1 + \frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}(Y_6 + A_1)\Big]$$

Proceeding similarly with the other boundary conditions specified by Eq. (143) one obtains for

$$g_2 = b^o_{21}H^s_o + b^o_{22}(M^s_{x_o} + \lambda\bar{q}) - w^s_{o,\bar{x}} = 0 \tag{I.4}$$

the following nonzero partial derivatives

$$\frac{\partial g_2}{\partial Y_1} = -2c\,\frac{R}{t}\,n^2 b^o_{21}(Y_{13} + A_1') \qquad \frac{\partial g_2}{\partial Y_{14}} = -\,cn^2 b^o_{21}\Big[2\,\frac{R}{t}\,Y_3 + \frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}(Y_7 + A_2)\Big]$$

$$\frac{\partial g_2}{\partial Y_3} = -\,2c\,\frac{R}{t}\,n^2 b^o_{21}(Y_{14} + A_2') \qquad \frac{\partial g_2}{\partial Y_{19}} = -\,b^o_{22}\Big(\bar{D}^*_{11} + \frac{\bar{B}^*_{21}\bar{B}^*_{21}}{\bar{A}^*_{22}}\Big) \tag{I.5}$$

$$\frac{\partial g_2}{\partial Y_5} = 2c\,\frac{R}{t}\,b^o_{22}\,\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}} \qquad \frac{\partial g_2}{\partial Y_{26}} = -\,b^o_{21}\Big(\bar{D}^*_{11} + \frac{\bar{B}^*_{21}\bar{B}^*_{21}}{\bar{A}^*_{22}}\Big)$$

$$\frac{\partial g_2}{\partial Y_6} = -2c\frac{R}{t}n^2 b^o_{21} Y_8 - cn^2 \frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}[b^o_{21}(Y_{13}+A'_1) + b^o_{22}(Y_6+A_1)]$$

$$\frac{\partial g_2}{\partial Y_7} = -2c\frac{R}{t}n^2 b^o_{21} Y_{10} - cn^2 \frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}[b^o_{21}(Y_{14}+A'_2) + b^o_{22}(Y_7+A_2)]$$

$$\frac{\partial g_2}{\partial Y_8} = -2c\frac{R}{t}n^2 b^o_{21}(Y_6+A_1) \qquad \frac{\partial g_2}{\partial \lambda} = -4c\frac{R}{t}b^o_{21}(Y_{12}+A'_o) + b^o_{22}(2\frac{R}{t}\bar{B}^*_{11}+\bar{q}) \qquad (I.5)$$

$$\frac{\partial g_2}{\partial Y_{10}} = -2c\frac{R}{t}n^2 b^o_{21}(Y_7+A_2) \qquad \frac{\partial g_2}{\partial \bar{p}_e} = (\frac{\partial g_2}{\partial \lambda})\hat{R} + 2\frac{R}{t}b^o_{22}\bar{B}^*_{21}$$

$$\frac{\partial g_2}{\partial Y_{12}} = 2c\frac{R}{t}b^o_{21}(\frac{\bar{B}^*_{21}}{\bar{A}^*_{22}} - 2\lambda) - 1 \qquad \frac{\partial g_2}{\partial \bar{\tau}} = -2\frac{R}{t}b^o_{22}\bar{B}^*_{61}$$

$$\frac{\partial g_2}{\partial Y_{13}} = -cn^2 b^o_{21}[2\frac{R}{t}Y_1 + \frac{\bar{B}^*_{21}}{\bar{A}^*_{22}}(Y_6+A_1)]$$

the third condition

$$g_3 = b^1_{11}N^s_{x_1} + b^1_{12}N^s_{xy_1} + b^1_{13}H^s_1 + b^1_{14}M^s_{x_1} - u^s_1 = 0 \qquad (I.6)$$

yields the following nonzero partial derivatives

$$\frac{\partial g_3}{\partial Y_1} = -n^2 b^1_{11} - 4c\frac{R}{t}n^2 b^1_{13}(Y_{12}+A'_o) + 2\frac{R}{t}n^2 b^1_{14}\bar{B}^*_{11}$$

$$\frac{\partial g_3}{\partial Y_2} = -8c\frac{R}{t}n^2 b^1_{13}(Y_{13}+A'_1) \qquad \frac{\partial g_3}{\partial Y_{15}} = -2\frac{R}{t}b^1_{14}\bar{B}^*_{21}$$

$$\frac{\partial g_3}{\partial Y_3} = 4\frac{R}{t}n^3 b^1_{13}\bar{B}^*_{16} - \frac{n}{c}\bar{A}^*_{16} \qquad \frac{\partial g_3}{\partial Y_{17}} = -2\frac{R}{t}nb^1_{13}(2\bar{B}^*_{26} - \bar{B}^*_{61}) + \frac{2}{cn}\bar{A}^*_{26}$$

$$\frac{\partial g_3}{\partial Y_4} = -8c\frac{R}{t}n^2 b^1_{13}(Y_{14}+A'_2) \qquad \frac{\partial g_3}{\partial Y_{20}} = -b^1_{14}\bar{D}^*_{11}$$

$$\frac{\partial g_3}{\partial Y_6} = -4c\frac{R}{t}n^2 b^1_{13}Y_9 + n^2 b^1_{14}\bar{D}^*_{12} + \frac{t}{R}(Y_{12}+A'_o) \qquad \frac{\partial g_3}{\partial Y_{21}} = -4nb^1_{13}\bar{D}^*_{16} + \frac{1}{2}\frac{t}{R}\frac{1}{cn}(2\bar{B}^*_{26} - \bar{B}^*_{61})$$

$$\frac{\partial g_3}{\partial Y_7} = b^1_{13}(2n^3\bar{D}^*_{26} - 4c\frac{R}{t}n^2Y_{11} + 4c\frac{R}{t}n\bar{\tau}) + \frac{1}{2}\frac{t}{R}\frac{n}{c}\bar{B}^*_{62}$$

$$\frac{\partial g_3}{\partial Y_8} = nb^1_{12} + 2\frac{R}{t}n^2b^1_{13}(\bar{B}^*_{11} - 2\bar{B}^*_{66}) + \frac{1}{c}(\bar{A}^*_{12} + \bar{A}^*_{66})$$

$$\frac{\partial g_3}{\partial Y_9} = -4c\frac{R}{t}n^2b^1_{13}(Y_6 + A_1) \qquad \frac{\partial g_3}{\partial Y_{22}} = -2\frac{R}{t}b^1_{13}\bar{B}^*_{21} - \frac{1}{cn^2}\bar{A}^*_{22}$$

$$\frac{\partial g_3}{\partial Y_{10}} = 2\frac{R}{t}nb^1_{14}\bar{B}^*_{61} \qquad \frac{\partial g_3}{\partial Y_{27}} = -b^1_{13}\bar{D}^*_{11} + \frac{1}{2}\frac{t}{R}\frac{1}{cn^2}\bar{B}^*_{21} \tag{I.7}$$

$$\frac{\partial g_3}{\partial Y_{11}} = -4c\frac{R}{t}n^2b^1_{13}(Y_7 + A_2) \qquad \frac{\partial g_3}{\partial \lambda} = -4c\frac{R}{t}b^1_{13}(Y_{13} + A_1')$$

$$\frac{\partial g_3}{\partial Y_{12}} = -4c\frac{R}{t}n^2b^1_{13}Y_1 + \frac{t}{R}(Y_6 + A_1) \qquad \frac{\partial g_3}{\partial \bar{p}_e} = (\frac{\partial g_3}{\partial \lambda})\hat{R}$$

$$\frac{\partial g_3}{\partial Y_{13}} = -n^2b^1_{13}[8c\frac{R}{t}Y_2 - (\bar{D}^*_{12} + 4\bar{D}^*_{66}) + 4c\frac{R}{t}\frac{\lambda}{n^2}] - \frac{1}{n^2} - \frac{1}{2c}\frac{t}{R}(\bar{B}^*_{22} - 2\bar{B}^*_{66})$$

$$\frac{\partial g_3}{\partial Y_{14}} = -8c\frac{R}{t}n^2b^1_{13}Y_4 - 2n\,b^1_{14}\bar{D}^*_{16} \qquad \frac{\partial g_3}{\partial \bar{\tau}} = 4c\frac{R}{t}nb^1_{13}(Y_7 + A_2)$$

Further, the fourth condition

$$g_4 = b^1_{21}N^s_{x_1} + b^1_{22}N^s_{xy_1} + b^1_{23}H^s_1 + b^1_{24}M^s_{x_1} - v^s_1 = 0 \tag{I.8}$$

yields the following nonzero derivatives

$$\frac{\partial g_4}{\partial Y_1} = -n^2b^1_{21} - 4c\frac{R}{t}n^2b^1_{23}(Y_{12} + A_o') + 2\frac{R}{t}n^2b^1_{24}\bar{B}^*_{11} + \frac{n}{c}\bar{A}^*_{12}$$

$$\frac{\partial g_4}{\partial Y_2} = -8c\frac{R}{t}n^2b^1_{23}(Y_{13} + A_1') \qquad \frac{\partial g_4}{\partial Y_{15}} = -2\frac{R}{t}b^1_{24}\bar{B}^*_{21} - \frac{1}{cn}\bar{A}^*_{22}$$

$$\frac{\partial g_4}{\partial Y_3} = 4\,\frac{R}{t}\,n^3 b^1_{23}\bar{B}^*_{16}$$

$$\frac{\partial g_4}{\partial Y_{17}} = -\,2\,\frac{R}{t}\,n b^1_{23}(2\bar{B}^*_{26} - \bar{B}^*_{61})$$

$$\frac{\partial g_4}{\partial Y_4} = -\,8c\,\frac{R}{t}\,n^2 b^1_{23}(Y_{14} + A_2')$$

$$\frac{\partial g_4}{\partial Y_{20}} = -\,b^1_{24}\bar{D}^*_{11} + \frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\,\bar{B}^*_{21}$$

$$\frac{\partial g_4}{\partial Y_6} = -\,4c\,\frac{R}{t}\,n^2 b^1_{23}Y_9 + n^2 b^1_{24}\bar{D}^*_{12} - \frac{1}{n} - \frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\,\bar{B}^*_{22}$$

$$\frac{\partial g_4}{\partial Y_7} = b^1_{23}(2n^3\bar{D}^*_{26} - 4c\,\frac{R}{t}\,n^2 Y_{11} + 4c\,\frac{R}{t}\,n\bar{\tau})$$

$$\frac{\partial g_4}{\partial Y_{21}} = -\,4nb^1_{23}\bar{D}^*_{16}$$

$$\frac{\partial g_4}{\partial Y_8} = nb^1_{22} + 2\,\frac{R}{t}\,n^2 b^1_{23}(\bar{B}^*_{11} - 2\bar{B}^*_{66})$$

$$\frac{\partial g_4}{\partial Y_{22}} = -\,2\,\frac{R}{t}\,b^1_{23}\bar{B}^*_{21}$$

$$\frac{\partial g_4}{\partial Y_9} = -\,4c\,\frac{R}{t}\,n^2 b^1_{23}(Y_6 + A_1)$$

$$\frac{\partial g_4}{\partial Y_{27}} = -\,b^1_{23}\bar{D}^*_{11}$$

$$\frac{\partial g_4}{\partial Y_{10}} = 2\,\frac{R}{t}\,nb^1_{24}\bar{B}^*_{61} + \frac{1}{c}\,\bar{A}^*_{26}$$

$$\frac{\partial g_4}{\partial \lambda} = -\,4c\,\frac{R}{t}\,b^1_{23}(Y_{13} + A_1')$$

$$\frac{\partial g_4}{\partial Y_{11}} = -\,4c\,\frac{R}{t}\,n^2 b^1_{23}(Y_7 + A_2)$$

$$\frac{\partial g_4}{\partial \bar{p}_e} = \left(\frac{\partial g_4}{\partial \lambda}\right)\hat{R}$$

$$\frac{\partial g_4}{\partial Y_{12}} = -\,4c\,\frac{R}{t}\,n^2 b^1_{23}Y_1$$

$$\frac{\partial g_4}{\partial \bar{\tau}} = 4c\,\frac{R}{t}\,nb^1_{23}(Y_7 + A_2)$$

$$\frac{\partial g_4}{\partial Y_{13}} = -\,n^2 b^1_{23}\left[8c\,\frac{R}{t}\,Y_2 - (\bar{D}^*_{12} + 4\bar{D}^*_{66}) + 4c\,\frac{R}{t}\,\frac{\lambda}{n^2}\right]$$

$$\frac{\partial g_4}{\partial Y_{14}} = -\,8c\,\frac{R}{t}\,n^2 b^1_{23}Y_4 - 2nb^1_{24}\bar{D}^*_{16} + \frac{1}{c}\,\frac{t}{R}\,\bar{B}^*_{26}$$

Further, the fifth condition

$$g_5 = b^1_{31}N^s_{x_1} + b^1_{32}N^s_{xy_1} + b^1_{33}H^s_1 + b^1_{34}M^s_{x_1} - w^s_1 = 0 \qquad (I.10)$$

yields the following nonzero partial derivatives

$$\frac{\partial g_5}{\partial Y_1} = - n^2 b^1_{31} - 4c \frac{R}{t} n^2 b^1_{33}(Y_{12} + A'_o) + 2 \frac{R}{t} n^2 b^1_{34} \bar{B}^*_{11}$$

$$\frac{\partial g_5}{\partial Y_2} = - 8c \frac{R}{t} n^2 b^1_{33}(Y_{13} + A'_1)$$

$$\frac{\partial g_5}{\partial Y_3} = 4 \frac{R}{t} n^3 b^1_{33} \bar{B}^*_{16}$$

$$\frac{\partial g_5}{\partial Y_4} = - 8c \frac{R}{t} n^2 b^1_{33}(Y_{14} + A'_2)$$

$$\frac{\partial g_5}{\partial Y_6} = - 4c \frac{R}{t} n^2 b^1_{33} Y_9 + n^2 b^1_{34} \bar{D}^*_{12} - 1$$

$$\frac{\partial g_5}{\partial Y_7} = b^1_{33}(2n^3 \bar{D}^*_{26} - 4c \frac{R}{t} n^2 Y_{11} + 4c \frac{R}{t} n\bar{\tau})$$

$$\frac{\partial g_5}{\partial Y_8} = nb^1_{32} + 2 \frac{R}{t} n^2 b^1_{33}(\bar{B}^*_{11} - 2\bar{B}^*_{66})$$

$$\frac{\partial g_5}{\partial Y_9} = - 4c \frac{R}{t} n^2 b^1_{33}(Y_6 + A_1)$$

$$\frac{\partial g_5}{\partial Y_{10}} = 2 \frac{R}{t} nb^1_{34} \bar{B}^*_{61}$$

$$\frac{\partial g_5}{\partial Y_{11}} = - 4c \frac{R}{t} n^2 b^1_{33}(Y_7 + A_2)$$

$$\frac{\partial g_5}{\partial Y_{12}} = - 4c \frac{R}{t} n^2 b^1_{33} Y_1$$

$$\frac{\partial g_5}{\partial Y_{13}} = - n^2 b^1_{33}[8c \frac{R}{t} Y_2 - (\bar{D}^*_{12} + 4\bar{D}^*_{66}) + 4c \frac{R}{t} \frac{\lambda}{n^2}]$$

$$\frac{\partial g_5}{\partial Y_{14}} = - 8c \frac{R}{t} n^2 b^1_{33} Y_4 - 2nb^1_{34} \bar{D}^*_{16}$$

$$\frac{\partial g_5}{\partial Y_{15}} = - 2 \frac{R}{t} b^1_{34} \bar{B}^*_{21}$$

$$\frac{\partial g_5}{\partial Y_{17}} = - 2 \frac{R}{t} nb^1_{33}(2\bar{B}^*_{26} - \bar{B}^*_{61})$$

$$\frac{\partial g_5}{\partial Y_{20}} = - b^1_{34} \bar{D}^*_{11}$$

$$\frac{\partial g_5}{\partial Y_{21}} = - 4nb^1_{33} \bar{D}^*_{16} \qquad \text{(I.11)}$$

$$\frac{\partial g_5}{\partial Y_{22}} = - 2 \frac{R}{t} b^1_{33} \bar{B}^*_{21}$$

$$\frac{\partial g_5}{\partial Y_{27}} = - b^1_{33} \bar{D}^*_{11}$$

$$\frac{\partial g_5}{\partial \lambda} = - 4c \frac{R}{t} b^1_{33}(Y_{13} + A'_1)$$

$$\frac{\partial g_5}{\partial \bar{p}_e} = (\frac{\partial g_5}{\partial \lambda}) \hat{R}$$

$$\frac{\partial g_5}{\partial \bar{\tau}} = 4c \frac{R}{t} nb^1_{33}(Y_7 + A_2)$$

Further, the sixth condition

$$g_6 = b_{41}^1 N_{x_1}^s + b_{42}^1 N_{xy_1}^s + b_{43}^1 H_1^s + b_{44}^1 M_{x_1}^s - w_{1,\bar{x}}^s = 0 \tag{I.12}$$

yields the following nonzero partial derivatives

$$\frac{\partial g_6}{\partial Y_1} = -n^2 b_{41}^1 - 4c\frac{R}{t} n^2 b_{43}^1 (Y_{12} + A_0') + 2\frac{R}{t} n^2 b_{44}^1 \bar{B}_{11}^*$$

$$\frac{\partial g_6}{\partial Y_2} = -8c\frac{R}{t} n^2 b_{43}^1 (Y_{13} + A_1') \qquad \frac{\partial g_6}{\partial Y_{14}} = -8c\frac{R}{t} n^2 b_{43}^1 Y_4 - 2nb_{44}^1 \bar{D}_{16}^*$$

$$\frac{\partial g_6}{\partial Y_3} = 4\frac{R}{t} n^3 b_{43}^1 \bar{B}_{16}^* \qquad \frac{\partial g_6}{\partial Y_{15}} = -2\frac{R}{t} b_{44}^1 \bar{B}_{21}^*$$

$$\frac{\partial g_6}{\partial Y_4} = -8c\frac{R}{t} n^2 b_{43}^1 (Y_{14} + A_2') \qquad \frac{\partial g_6}{\partial Y_{17}} = -2\frac{R}{t} nb_{43}^1 (2\bar{B}_{26}^* - \bar{B}_{61}^*)$$

$$\frac{\partial g_6}{\partial Y_6} = -4c\frac{R}{t} n^2 b_{43}^1 Y_9 + n^2 b_{44}^1 \bar{D}_{12}^* \qquad \frac{\partial g_6}{\partial Y_{20}} = -b_{44}^1 \bar{D}_{11}^*$$

$$\frac{\partial g_6}{\partial Y_7} = b_{43}^1 (2n^3 \bar{D}_{26}^* - 4c\frac{R}{t} n^2 Y_{11} + 4c\frac{R}{t} n\bar{\tau}) \qquad \frac{\partial g_6}{\partial Y_{21}} = -4nb_{43}^1 \bar{D}_{16}^* \tag{I.13}$$

$$\frac{\partial g_6}{\partial Y_8} = nb_{42}^1 + 2\frac{R}{t} n^2 b_{43}^1 (\bar{B}_{11}^* - 2\bar{B}_{66}^*) \qquad \frac{\partial g_6}{\partial Y_{22}} = -2\frac{R}{t} b_{43}^1 \bar{B}_{21}^*$$

$$\frac{\partial g_6}{\partial Y_9} = -4c\frac{R}{t} n^2 b_{43}^1 (Y_6 + A_1) \qquad \frac{\partial g_6}{\partial Y_{27}} = -b_{43}^1 \bar{D}_{11}^*$$

$$\frac{\partial g_6}{\partial Y_{10}} = 2\frac{R}{t} nb_{44}^1 \bar{B}_{61}^* \qquad \frac{\partial g_6}{\partial \lambda} = -4c\frac{R}{t} b_{43}^1 (Y_{13} + A_1')$$

$$\frac{\partial g_6}{\partial Y_{11}} = -4c\frac{R}{t} n^2 b_{43}^1 (Y_7 + A_2) \qquad \frac{\partial g_6}{\partial \bar{p}_e} = \left(\frac{\partial g_6}{\partial \lambda}\right) \hat{R}$$

$$\frac{\partial g_6}{\partial Y_{12}} = -4c\frac{R}{t} n^2 b_{43}^1 Y_1 \qquad \frac{\partial g_6}{\partial \bar{\tau}} = 4c\frac{R}{t} nb_{43}^1 (Y_7 + A_2)$$

$$\frac{\partial g_6}{\partial Y_{13}} = - n^2 b_{43}^1 [8c \frac{R}{t} Y_2 - (\bar{D}_{12}^* + 4\bar{D}_{66}^*) + 4c \frac{R}{t} \frac{\lambda}{n^2}] - 1$$

Further, the seventh condition

$$g_7 = b_{11}^2 N_{x_2}^s + b_{12}^2 N_{xy_2}^s - u_2^s = 0 \tag{I.14}$$

yields the following nonzero partial derivatives

$$\frac{\partial g_7}{\partial Y_2} = - 4n^2 b_{11}^2 \qquad \frac{\partial g_7}{\partial Y_{13}} = \frac{1}{8} \frac{t}{R} (Y_6 + A_1)$$

$$\frac{\partial g_7}{\partial Y_4} = - \frac{2n}{c} \bar{A}_{16}^* \qquad \frac{\partial g_7}{\partial Y_{14}} = - \frac{1}{8} \frac{t}{R} (Y_7 + A_2)$$

$$\frac{\partial g_7}{\partial Y_6} = \frac{1}{8} \frac{t}{R} (Y_{13} + A_1') \qquad \frac{\partial g_7}{\partial Y_{18}} = \frac{1}{cn} \bar{A}_{26}^* \tag{I.15}$$

$$\frac{\partial g_7}{\partial Y_7} = - \frac{1}{8} \frac{t}{R} (Y_{14} + A_2') \qquad \frac{\partial g_7}{\partial Y_{23}} = - \frac{1}{4cn^2} \bar{A}_{22}^*$$

$$\frac{\partial g_7}{\partial Y_9} = 2nb_{12}^2 + \frac{1}{c} (\bar{A}_{12}^* + \bar{A}_{66}^*)$$

Further, the eighth condition

$$g_8 = b_{21}^2 N_{x_2}^s + b_{22}^2 N_{xy_2}^s - v_2^s = 0 \tag{I.16}$$

yields the following nonzero partial derivatives

$$\frac{\partial g_8}{\partial Y_2} = - 4n^2 b_{21}^2 + 2 \frac{n}{c} \bar{A}_{12}^* \qquad \frac{\partial g_8}{\partial Y_9} = 2nb_{22}^2$$

$$\frac{\partial g_8}{\partial Y_6} = - \frac{1}{4} \frac{t}{R} n(Y_6 + A_1) \qquad \frac{\partial g_8}{\partial Y_{11}} = \frac{1}{c} \bar{A}_{26}^* \tag{I.17}$$

$$\frac{\partial g_8}{\partial Y_7} = \frac{1}{4} \frac{t}{R} n(Y_7 + A_2) \qquad \frac{\partial g_8}{\partial Y_{16}} = - \frac{1}{2cn} \bar{A}_{22}^*$$

Further, the nineth condition

$$g_9 = b^3_{11}N^s_{x_3} + b^3_{12}N^s_{xy_3} + b^3_{13}H^s_2 + b^3_{14}M^s_{x_2} - u^s_3 = 0 \qquad (I.18)$$

yields the following nonzero partial derivatives

$$\frac{\partial g_9}{\partial Y_1} = -4\frac{R}{t}n^3 b^3_{13}\bar{B}^*_{16} + \frac{n}{c}\bar{A}^*_{16}$$

$$\frac{\partial g_9}{\partial Y_{15}} = 2\frac{R}{t}nb^3_{13}(2\bar{B}^*_{26} - \bar{B}^*_{61}) - \frac{2}{cn}\bar{A}^*_{26}$$

$$\frac{\partial g_9}{\partial Y_2} = 8c\frac{R}{t}n^2 b^3_{13}(Y_{14} + A'_2)$$

$$\frac{\partial g_9}{\partial Y_{17}} = -2\frac{R}{t}b^3_{14}\bar{B}^*_{21}$$

$$\frac{\partial g_9}{\partial Y_3} = -n^2 b^3_{11} - 4c\frac{R}{t}n^2 b^3_{13}(Y_{12} + A'_o) + 2\frac{R}{t}n^2 b^3_{14}\bar{B}^*_{11}$$

$$\frac{\partial g_9}{\partial Y_4} = -8c\frac{R}{t}n^2 b^3_{13}(Y_{13} + A'_1)$$

$$\frac{\partial g_9}{\partial Y_{20}} = 4nb^3_{13}\bar{D}^*_{16} - \frac{1}{2}\frac{t}{R}\frac{1}{cn}(2\bar{B}^*_{26} - \bar{B}^*_{61})$$

$$\frac{\partial g_9}{\partial Y_6} = -b^3_{13}(2n^3\bar{D}^*_{26} + 4c\frac{R}{t}n^2 Y_{11} + 4c\frac{R}{t}n\bar{\tau}) - \frac{1}{2}\frac{t}{R}\frac{n}{c}\bar{B}^*_{62} \qquad (I.19)$$

$$\frac{\partial g_9}{\partial Y_7} = 4c\frac{R}{t}n^2 b^3_{13}Y_9 + n^2 b^3_{14}\bar{D}^*_{12} + \frac{t}{R}(Y_{12} + A'_o)$$

$$\frac{\partial g_9}{\partial Y_{21}} = -b^3_{14}\bar{D}^*_{11}$$

$$\frac{\partial g_9}{\partial Y_8} = -2\frac{R}{t}nb^3_{14}\bar{B}^*_{61}$$

$$\frac{\partial g_9}{\partial Y_{24}} = -2\frac{R}{t}b^3_{13}\bar{B}^*_{21} - \frac{1}{cn^2}\bar{A}^*_{22}$$

$$\frac{\partial g_9}{\partial Y_9} = 4c\frac{R}{t}n^2 b^3_{13}(Y_7 + A_2)$$

$$\frac{\partial g_9}{\partial Y_{28}} = -b^3_{13}\bar{D}^*_{11} + \frac{1}{2}\frac{t}{R}\frac{1}{cn^2}\bar{B}^*_{21}$$

$$\frac{\partial g_9}{\partial Y_{10}} = -nb^3_{12} + 2\frac{R}{t}n^2 b^3_{13}(\bar{B}^*_{11} - 2\bar{B}^*_{66}) + \frac{1}{c}(\bar{A}^*_{12} + \bar{A}^*_{66})$$

$$\frac{\partial g_9}{\partial Y_{11}} = -4c\frac{R}{t}n^2 b^3_{13}(Y_6 + A_1)$$

$$\frac{\partial g_9}{\partial \lambda} = -4c\frac{R}{t}b^3_{13}(Y_{14} + A'_2)$$

$$\frac{\partial g_9}{\partial Y_{12}} = -4c\frac{R}{t}n^2 b^3_{13}Y_3 + \frac{t}{R}(Y_7 + A_2)$$

$$\frac{\partial g_9}{\partial \bar{p}_e} = \left(\frac{\partial g_9}{\partial \lambda}\right)\hat{R}$$

$$\frac{\partial g_9}{\partial Y_{13}} = -8c\,\frac{R}{t}\,n^2 b_{13}^3 Y_4 + 2nb_{14}^3 \bar{D}_{16}^* \qquad \frac{\partial g_9}{\partial \bar{\tau}} = -4c\,\frac{R}{t}\,nb_{13}^3(Y_6 + A_1)$$

$$\frac{\partial g_9}{\partial Y_{14}} = n^2 b_{13}^3[8c\,\frac{R}{t}\,Y_2 + (\bar{D}_{12}^* + 4\bar{D}_{66}^*) - 4c\,\frac{R}{t}\,\frac{\lambda}{n^2}] - \frac{1}{n^2} - \frac{1}{2c}\,\frac{t}{R}\,(\bar{B}_{22}^* - 2\bar{B}_{66}^*)$$

Further, the tenth condition

$$g_{10} = b_{21}^3 N_{x_3}^s + b_{22}^3 N_{xy_3}^s + b_{23}^3 H_2^s + b_{24}^3 M_{x_2}^s - v_3^s = 0 \tag{I.20}$$

yields the following nonzero partial deviatives

$$\frac{\partial g_{10}}{\partial Y_1} = -4\,\frac{R}{t}\,n^3 b_{23}^3 \bar{B}_{16}^* \qquad \frac{\partial g_{10}}{\partial Y_{14}} = n^2 b_{23}^3[8c\,\frac{R}{t}\,Y_2 + (\bar{D}_{12}^* + 4\bar{D}_{66}^*) - 4c\,\frac{R}{t}\,\frac{\lambda}{n^2}]$$

$$\frac{\partial g_{10}}{\partial Y_2} = 8c\,\frac{R}{t}\,n^2 b_{23}^3(Y_{14} + A_2') \qquad \frac{\partial g_{10}}{\partial Y_{15}} = 2\,\frac{R}{t}\,nb_{23}^3(2\bar{B}_{26}^* - \bar{B}_{61}^*)$$

$$\frac{\partial g_{10}}{\partial Y_3} = -n^2 b_{21}^3 - 4c\,\frac{R}{t}\,n^2 b_{23}^3(Y_{12} + A_o') + 2\,\frac{R}{t}\,n^2 b_{24}^3 \bar{B}_{11}^* - \frac{n}{c}\,\bar{A}_{12}^*$$

$$\frac{\partial g_{10}}{\partial Y_4} = -8c\,\frac{R}{t}\,n^2(Y_{13} + A_1') \qquad \frac{\partial g_{10}}{\partial Y_{17}} = -2\,\frac{R}{t}\,b_{24}^3 \bar{B}_{21}^* + \frac{1}{cn}\,\bar{A}_{22}^*$$

$$\frac{\partial g_{10}}{\partial Y_6} = -b_{23}^3(2n^3\bar{D}_{26}^* + 4c\,\frac{R}{t}\,n^2 Y_{11} + 4c\,\frac{R}{t}\,n\bar{\tau}) \qquad \frac{\partial g_{10}}{\partial Y_{20}} = 4nb_{23}^3 \bar{D}_{16}^*$$

$$\frac{\partial g_{10}}{\partial Y_7} = 4c\,\frac{R}{t}\,n^2 b_{23}^3 Y_9 + n^2 b_{24}^3 \bar{D}_{12}^* + \frac{1}{n} + \frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\,\bar{B}_{22}^* \tag{I.21}$$

$$\frac{\partial g_{10}}{\partial Y_8} = -2\,\frac{R}{t}\,nb_{24}^3 \bar{B}_{61}^* + \frac{1}{c}\,\bar{A}_{26}^* \qquad \frac{\partial g_{10}}{\partial Y_{21}} = -b_{24}^3 \bar{D}_{11}^* - \frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\,\bar{B}_{21}^*$$

$$\frac{\partial g_{10}}{\partial Y_9} = 4c\,\frac{R}{t}\,n^2 b_{23}^3(Y_7 + A_2) \qquad \frac{\partial g_{10}}{\partial Y_{24}} = -2\,\frac{R}{t}\,b_{23}^3 \bar{B}_{21}^*$$

$$\frac{\partial g_{10}}{\partial Y_{10}} = -nb_{22}^3 + 2\,\frac{R}{t}\,n^2 b_{23}^3(\bar{B}_{11}^* - 2\bar{B}_{66}^*) \qquad \frac{\partial g_{10}}{\partial Y_{28}} = -b_{23}^3 \bar{D}_{11}^*$$

$$\frac{\partial g_{10}}{\partial Y_{11}} = -4c\frac{R}{t}n^2b^3_{23}(Y_6+A_1) \qquad \frac{\partial g_{10}}{\partial \lambda} = -4c\frac{R}{t}b^3_{23}(Y_{14}+A_2')$$

$$\frac{\partial g_{10}}{\partial Y_{12}} = -4c\frac{R}{t}n^2b^3_{23}Y_3 \qquad \frac{\partial g_{10}}{\partial \bar{p}_e} = (\frac{\partial g_{10}}{\partial \lambda})\hat{R}$$

$$\frac{\partial g_{10}}{\partial Y_{13}} = -8c\frac{R}{t}n^2b^3_{23}Y_4 + 2nb^3_{24}\bar{D}^*_{16} + \frac{1}{c}\frac{t}{R}\bar{B}^*_{26} \qquad \frac{\partial g_{10}}{\partial \bar{\tau}} = -4c\frac{R}{t}nb^3_{23}(Y_6+A_1)$$

Further, the eleventh condition

$$g_{11} = b^3_{31}N^s_{x_3} + b^3_{32}N^s_{xy_3} + b^3_{33}H^s_2 + b^3_{34}M^s_{x_2} - w^s_2 = 0 \tag{I.22}$$

yields the following nonzero partial derivatives

$$\frac{\partial g_{11}}{\partial Y_1} = -4\frac{R}{t}n^3b^3_{33}\bar{B}^*_{16} \qquad \frac{\partial g_{11}}{\partial Y_{14}} = n^2b^3_{33}[8c\frac{R}{t}Y_2 + (\bar{D}^*_{12}+4\bar{D}^*_{66}) - 4c\frac{R}{t}\frac{\lambda}{n^2}]$$

$$\frac{\partial g_{11}}{\partial Y_2} = 8c\frac{R}{t}n^2b^3_{33}(Y_{14}+A_2') \qquad \frac{\partial g_{11}}{\partial Y_{15}} = 2\frac{R}{t}nb^3_{33}(2\bar{B}^*_{26}-\bar{B}^*_{61})$$

$$\frac{\partial g_{11}}{\partial Y_3} = -n^2b^3_{31} - 4c\frac{R}{t}n^2b^3_{33}(Y_{12}+A_o') + 2\frac{R}{t}n^2b^3_{34}\bar{B}^*_{11} \tag{I.23}$$

$$\frac{\partial g_{11}}{\partial Y_4} = -8c\frac{R}{t}n^2b^3_{33}(Y_{13}+A_1') \qquad \frac{\partial g_{11}}{\partial Y_{17}} = -2\frac{R}{t}b^3_{34}\bar{B}^*_{21}$$

$$\frac{\partial g_{11}}{\partial Y_6} = -b^3_{33}(2n^3\bar{D}^*_{26} + 4c\frac{R}{t}n^2Y_{11} + 4c\frac{R}{t}n\bar{\tau}) \qquad \frac{\partial g_{11}}{\partial Y_{20}} = 4nb^3_{33}\bar{D}^*_{16}$$

$$\frac{\partial g_{11}}{\partial Y_7} = 4c\frac{R}{t}n^2b^3_{33}Y_9 + n^2b^3_{34}\bar{D}^*_{12} - 1 \qquad \frac{\partial g_{11}}{\partial Y_{21}} = -b^3_{34}\bar{D}^*_{11}$$

$$\frac{\partial g_{11}}{\partial Y_8} = -2\frac{R}{t}nb^3_{34}\bar{B}^*_{61} \qquad \frac{\partial g_{11}}{\partial Y_{24}} = -2\frac{R}{t}b^3_{33}\bar{B}^*_{21}$$

$$\frac{\partial g_{11}}{\partial Y_9} = 4c\frac{R}{t}n^2b^3_{33}(Y_7+A_2) \qquad \frac{\partial g_{11}}{\partial Y_{28}} = -b^3_{33}\bar{D}^*_{11}$$

$$\frac{\partial g_{11}}{\partial Y_{10}} = -nb_{32}^{3} + 2\frac{R}{t}n^{2}b_{33}^{3}(\bar{B}_{11}^{*} - 2\bar{B}_{66}^{*}) \qquad \frac{\partial g_{11}}{\partial \lambda} = -4c\frac{R}{t}b_{33}^{3}(Y_{14} + A_{2}^{\prime}) \tag{I.23}$$

$$\frac{\partial g_{11}}{\partial Y_{11}} = -4c\frac{R}{t}n^{2}b_{33}^{3}(Y_{6} + A_{1}) \qquad \frac{\partial g_{11}}{\partial \bar{p}_{e}} = (\frac{\partial g_{11}}{\partial \lambda})\hat{R}$$

$$\frac{\partial g_{11}}{\partial Y_{12}} = -4c\frac{R}{t}n^{2}b_{33}^{3}Y_{3} \qquad \frac{\partial g_{11}}{\partial \bar{\tau}} = -4c\frac{R}{t}nb_{33}^{3}(Y_{6} + A_{1})$$

$$\frac{\partial g_{11}}{\partial Y_{13}} = -8c\frac{R}{t}n^{2}b_{33}^{3}Y_{4} + 2nb_{34}^{3}\bar{D}_{16}^{*}$$

Further, the twelveth condition

$$g_{12} = b_{41}^{3}N_{x_3}^{s} + b_{42}^{3}N_{xy_3}^{s} + b_{43}^{3}H_{2}^{s} + b_{44}^{3}M_{x_2}^{s} - w_{2,\bar{x}}^{s} = 0 \tag{I.24}$$

yields the following nonzero partial derivatives

$$\frac{\partial g_{12}}{\partial Y_{1}} = -4\frac{R}{t}n^{3}b_{43}^{3}\bar{B}_{16}^{*} \qquad \frac{\partial g_{12}}{\partial Y_{14}} = n^{2}b_{43}^{3}[8c\frac{R}{t}Y_{2} + (\bar{D}_{12}^{*} + 4\bar{D}_{66}^{*}) - 4c\frac{R}{t}\frac{\lambda}{n^{2}}] - 1$$

$$\frac{\partial g_{12}}{\partial Y_{2}} = 8c\frac{R}{t}n^{2}b_{43}^{3}(Y_{14} + A_{2}^{\prime}) \qquad \frac{\partial g_{12}}{\partial Y_{15}} = 2\frac{R}{t}nb_{43}^{3}(2\bar{B}_{26}^{*} - \bar{B}_{61}^{*})$$

$$\frac{\partial g_{12}}{\partial Y_{3}} = -n^{2}b_{41}^{3} - 4c\frac{R}{t}n^{2}b_{43}^{3}(Y_{12} + A_{o}^{\prime}) + 2\frac{R}{t}n^{2}b_{44}^{3}\bar{B}_{11}^{*} \tag{I.25}$$

$$\frac{\partial g_{12}}{\partial Y_{4}} = -8c\frac{R}{t}n^{2}b_{43}^{3}(Y_{13} + A_{1}^{\prime}) \qquad \frac{\partial g_{12}}{\partial Y_{17}} = -2\frac{R}{t}b_{44}^{3}\bar{B}_{21}^{*}$$

$$\frac{\partial g_{12}}{\partial Y_{6}} = -b_{43}^{3}(2n^{3}\bar{D}_{26}^{*} + 4c\frac{R}{t}n^{2}Y_{11} + 4c\frac{R}{t}n\bar{\tau}) \qquad \frac{\partial g_{12}}{\partial Y_{20}} = 4nb_{43}^{3}\bar{D}_{16}^{*}$$

$$\frac{\partial g_{12}}{\partial Y_{7}} = 4c\frac{R}{t}n^{2}b_{43}^{3}Y_{9} + n^{2}b_{44}^{3}\bar{D}_{12}^{*} \qquad \frac{\partial g_{12}}{\partial Y_{21}} = -b_{44}^{3}\bar{D}_{11}^{*}$$

$$\frac{\partial g_{12}}{\partial Y_{8}} = -2\frac{R}{t}nb_{44}^{3}\bar{B}_{61}^{*} \qquad \frac{\partial g_{12}}{\partial Y_{24}} = -2\frac{R}{t}b_{43}^{3}\bar{B}_{21}^{*}$$

$$\frac{\partial g_{12}}{\partial Y_9} = 4c\,\frac{R}{t}\,n^2 b_{43}^3 (Y_7 + A_2) \qquad \frac{\partial g_{12}}{\partial Y_{28}} = -\,b_{43}^3 \bar{D}_{11}^*$$

$$\frac{\partial g_{12}}{\partial Y_{10}} = -\,nb_{42}^3 + 2\,\frac{R}{t}\,n^2 b_{43}^3 (\bar{B}_{11}^* - 2\bar{B}_{66}^*) \qquad \frac{\partial g_{12}}{\partial \lambda} = -\,4c\,\frac{R}{t}\,b_{43}^3 (Y_{14} + A_2') \tag{I.25}$$

$$\frac{\partial g_{12}}{\partial Y_{11}} = -\,4c\,\frac{R}{t}\,n^2 b_{43}^3 (Y_6 + A_1) \qquad \frac{\partial g_{12}}{\partial \bar{p}_e} = \left(\frac{\partial g_{12}}{\partial \lambda}\right) \hat{R}$$

$$\frac{\partial g_{12}}{\partial Y_{12}} = -\,4c\,\frac{R}{t}\,n^2 b_{43}^3 Y_3 \qquad \frac{\partial g_{12}}{\partial \bar{\tau}} = -\,4c\,\frac{R}{t}\,nb_{43}^3 (Y_6 + A_1)$$

$$\frac{\partial g_{12}}{\partial Y_{13}} = -\,8c\,\frac{R}{t}\,n^2 b_{43}^3 Y_4 + 2nb_{44}^3 \bar{D}_{16}^*$$

Further, the thirteenth condition

$$g_{13} = b_{11}^4 N_{x_4}^s + b_{12}^4 N_{xy_4}^s - u_4^s = 0 \tag{I.26}$$

yields the following nonzero partial derivatives

$$\frac{\partial g_{13}}{\partial Y_2} = \frac{2n}{c}\,\bar{A}_{16}^* \qquad \frac{\partial g_{13}}{\partial Y_{13}} = \frac{1}{8}\,\frac{t}{R}\,(Y_7 + A_2)$$

$$\frac{\partial g_{13}}{\partial Y_4} = -\,4n^2 b_{11}^4 \qquad \frac{\partial g_{13}}{\partial Y_{14}} = \frac{1}{8}\,\frac{t}{R}\,(Y_6 + A_1)$$

$$\frac{\partial g_{13}}{\partial Y_6} = \frac{1}{8}\,\frac{t}{R}\,(Y_{14} + A_2') \qquad \frac{\partial g_{13}}{\partial Y_{16}} = -\,\frac{1}{cn}\,\bar{A}_{26}^* \tag{I.27}$$

$$\frac{\partial g_{13}}{\partial Y_7} = \frac{1}{8}\,\frac{t}{R}\,(Y_{13} + A_1') \qquad \frac{\partial g_{13}}{\partial Y_{25}} = -\,\frac{1}{4cn^2}\,\bar{A}_{22}^*$$

$$\frac{\partial g_{13}}{\partial Y_{11}} = -\,2nb_{12}^4 + \frac{1}{c}\,(\bar{A}_{12}^* + \bar{A}_{66}^*)$$

Finally, the fourteenth condition

$$g_{14} = b_{21}^{4} N_{x_4}^{s} + b_{22}^{4} N_{xy_4}^{s} - v_{4}^{s} = 0 \tag{I.28}$$

yields the following nonzero partial derivatives

$$\frac{\partial g_{14}}{\partial Y_4} = -4n^2 b_{41}^{4} - \frac{2n}{c} \bar{A}_{12}^{*} \qquad \frac{\partial g_{14}}{\partial Y_9} = \frac{1}{c} \bar{A}_{26}^{*}$$

$$\frac{\partial g_{14}}{\partial Y_6} = \frac{1}{4} \frac{t}{R} n(Y_7 + A_2) \qquad \frac{\partial g_{14}}{\partial Y_{11}} = -2nb_{22}^{4} \tag{I.29}$$

$$\frac{\partial g_{14}}{\partial Y_7} = \frac{1}{4} \frac{t}{R} n(Y_6 + A_1) \qquad \frac{\partial g_{14}}{\partial Y_{18}} = \frac{1}{2nc} \bar{A}_{22}^{*}$$

Limiting Case as $E_r \rightarrow \infty$

In this case one must satisfy the 14-boundary conditions specified by Eq. (141), the first of which reads

$$h_1 = a_{11}^{o}(w_o^{s} + W_v + W_p + W_t) + a_{12}^{o} w_{o,\bar{x}}^{s} - H_o^{s} = 0 \tag{I.30a}$$

Replacing the shell variable H_o^s by its equivalent expression in terms of the variables used in the anisotropic shell analysis $\bar{W}$, W and F derived earlier (see Eq. (99)), one obtains after some regrouping the following nonlinear boundary condition

$$h_1 = a_{11}^{o} Y_5 + \left(a_{12}^{o} - 2c \frac{R}{t} \frac{\bar{B}_{21}^{*}}{\bar{A}_{22}^{*}}\right) Y_{12} + 2c \frac{R}{t} n^2 [(Y_{13} + A_1')Y_1 + (Y_{14} + A_2')Y_3 \tag{I.30b}$$

$$+ (Y_6 + A_1)Y_8 + (Y_7 + A_2)Y_{10}] + cn^2 \frac{\bar{B}_{21}^{*}}{\bar{A}_{22}^{*}} [(Y_{13} + A_1')Y_6 + (Y_{14} + A_2')Y_7 + A_1 Y_{13} + A_2 Y_{14}]$$

$$+ \left(\bar{D}_{11}^{*} + \frac{\bar{B}_{21}^{*}\bar{B}_{21}^{*}}{\bar{A}_{22}^{*}}\right) Y_{26} + 4c \frac{R}{t} \lambda (Y_{12} + A_o') + a_{11}^{o}(\hat{W}_v \lambda + \hat{W}_p \bar{p}_e + \hat{W}_t \bar{\tau}) = 0$$

Next one can proceed to calculate the partial derivatives $\partial h_1/\partial Y_i$ analytically yielding the following nonzero terms

$$\frac{\partial h_1}{\partial Y_1} = 2c \frac{R}{t} n^2 (Y_{13} + A_1') \qquad \frac{\partial h_1}{\partial Y_{12}} = a_{12}^{o} + 2c \frac{R}{t} \left(2\lambda - \frac{\bar{B}_{21}^{*}}{\bar{A}_{22}^{*}}\right)$$

$$\frac{\partial h_1}{\partial Y_3} = 2c\,\frac{R}{t}\,n^2(Y_{14} + A_2') \qquad \frac{\partial h_1}{\partial Y_{13}} = 2c\,\frac{R}{t}\,n^2 Y_1 + cn^2\,\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}\,(Y_6 + A_1)$$

$$\frac{\partial h_1}{\partial Y_5} = a_{11}^o \qquad \frac{\partial h_1}{\partial Y_{14}} = 2c\,\frac{R}{t}\,n^2 Y_3 + cn^2\,\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}\,(Y_7 + A_2)$$

$$\frac{\partial h_1}{\partial Y_6} = 2c\,\frac{R}{t}\,n^2 Y_8 + cn^2\,\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}\,(Y_{13} + A_1') \qquad \frac{\partial h_1}{\partial Y_{26}} = \bar{D}_{11}^* + \frac{\bar{B}_{21}^*\bar{B}_{21}^*}{\bar{A}_{22}^*} \tag{I.31}$$

$$\frac{\partial h_1}{\partial Y_7} = 2c\,\frac{R}{t}\,n^2 Y_{10} + cn^2\,\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}\,(Y_{14} + A_2') \qquad \frac{\partial h_1}{\partial \lambda} = a_{11}^o \hat{W}_v + 4c\,\frac{R}{t}\,(Y_{12} + A_o')$$

$$\frac{\partial h_1}{\partial Y_8} = 2c\,\frac{R}{t}\,n^2(Y_6 + A_1) \qquad \frac{\partial h_1}{\partial \bar{p}_e} = \left(\frac{\partial h_1}{\partial \lambda}\right)\hat{R} + a_{11}^o \hat{W}_p$$

$$\frac{\partial h_1}{\partial Y_{10}} = 2c\,\frac{R}{t}\,n^2(Y_7 + A_2) \qquad \frac{\partial h_1}{\partial \bar{\tau}} = a_{11}^o \hat{W}_t$$

Proceeding similarly with the other boundary conditions specified by Eq. (141) one obtains for

$$h_2 = a_{21}^o(w_o^s + W_v + W_p + W_t) + a_{22}^o w_{o,\bar{x}}^s - (M_{x_o}^s + \lambda\bar{q}) = 0 \tag{I.32}$$

the following nonzero partial derivatives

$$\frac{\partial h_2}{\partial Y_5} = a_{21}^o - 2c\,\frac{R}{t}\,\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*} \qquad \frac{\partial h_2}{\partial Y_{19}} = \bar{D}_{11}^* + \frac{\bar{B}_{21}^*\bar{B}_{21}^*}{\bar{A}_{22}^*}$$

$$\frac{\partial h_2}{\partial Y_6} = cn^2\,\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}\,(Y_6 + A_1) \qquad \frac{\partial h_2}{\partial \lambda} = a_{21}^o \hat{W}_v - 2\,\frac{R}{t}\,\bar{B}_{11}^* - \bar{q}$$

$$\frac{\partial h_2}{\partial Y_7} = cn^2\,\frac{\bar{B}_{21}^*}{\bar{A}_{22}^*}\,(Y_7 + A_2) \qquad \frac{\partial h_2}{\partial \bar{p}_e} = \left(\frac{\partial h_2}{\partial \lambda}\right)\hat{R} + a_{21}^o \hat{W}_p - 2\,\frac{R}{t}\,\bar{B}_{21}^* \tag{I.33}$$

$$\frac{\partial h_2}{\partial Y_{12}} = a_{22}^o \qquad \frac{\partial h_2}{\partial \bar{\tau}} = a_{21}^o \hat{W}_t + 2\,\frac{R}{t}\,\bar{B}_{61}^*$$

The third condition

$$h_3 = a^1_{11}u^s_1 + a^1_{12}v^s_1 + a^1_{13}w^s_1 + a^1_{14}w^s_{1,\bar{x}} - N^s_{x_1} = 0 \tag{I.34}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_3}{\partial Y_1} = -\frac{n}{c}\, a^1_{12}\bar{A}^*_{12} + n^2 \qquad \frac{\partial h_3}{\partial Y_{14}} = -\frac{1}{c}\frac{t}{R}\, a^1_{12}\bar{B}^*_{26}$$

$$\frac{\partial h_3}{\partial Y_3} = \frac{n}{c}\, a^1_{11}\bar{A}^*_{16} \qquad \frac{\partial h_3}{\partial Y_{15}} = \frac{1}{cn}\, a^1_{12}\bar{A}^*_{22}$$

$$\frac{\partial h_3}{\partial Y_6} = -\frac{t}{R}\, a^1_{11}(Y_{12} + A'_o) + a^1_{12}\left(\frac{1}{n} + \frac{1}{2}\frac{t}{R}\frac{n}{c}\bar{B}^*_{22}\right) + a^1_{13}$$

$$\frac{\partial h_3}{\partial Y_7} = -\frac{1}{2}\frac{t}{R}\frac{n}{c}\, a^1_{11}\bar{B}^*_{62} \qquad \frac{\partial h_3}{\partial Y_{17}} = -\frac{2}{cn}\, a^1_{11}\bar{A}^*_{26} \tag{I.35}$$

$$\frac{\partial h_3}{\partial Y_8} = -\frac{1}{c}\, a^1_{11}(\bar{A}^*_{12} + \bar{A}^*_{66}) \qquad \frac{\partial h_3}{\partial Y_{20}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn}\, a^1_{12}\bar{B}^*_{21}$$

$$\frac{\partial h_3}{\partial Y_{10}} = -\frac{1}{c}\, a^1_{12}\bar{A}^*_{26} \qquad \frac{\partial h_3}{\partial Y_{21}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn}\, a^1_{11}(2\bar{B}^*_{26} - \bar{B}^*_{61})$$

$$\frac{\partial h_3}{\partial Y_{12}} = -\frac{t}{R}\, a^1_{11}(Y_6 + A_1) \qquad \frac{\partial h_3}{\partial Y_{22}} = \frac{1}{cn^2}\, a^1_{11}\bar{A}^*_{22}$$

$$\frac{\partial h_3}{\partial Y_{13}} = a^1_{11}\left[\frac{1}{n^2} + \frac{1}{2c}\frac{t}{R}(\bar{B}^*_{22} - 2\bar{B}^*_{66})\right] + a^1_{14} \qquad \frac{\partial h_3}{\partial Y_{27}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn^2}\, a^1_{11}\bar{B}^*_{21}$$

The fourth condition

$$h_4 = a^1_{21}u^s_1 + a^1_{22}v^s_1 + a^1_{23}w^s_1 + a^1_{24}w^s_{1,\bar{x}} - N^s_{xy_1} \tag{I.36}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_4}{\partial Y_1} = -\frac{n}{c}\, a^1_{22}\bar{A}^*_{12} \qquad \frac{\partial h_4}{\partial Y_{14}} = -\frac{1}{c}\frac{t}{R}\, a^1_{22}\bar{B}^*_{26}$$

$$\frac{\partial h_4}{\partial Y_3} = \frac{n}{c}\, a^1_{21}\bar{A}^*_{16} \qquad \frac{\partial h_4}{\partial Y_{15}} = \frac{1}{cn}\, a^1_{22}\bar{A}^*_{22}$$

$$\frac{\partial h_4}{\partial Y_6} = -\frac{t}{R}\, a^1_{21}(Y_{12}+ A'_o) + a^1_{22}\left(\frac{1}{n} + \frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\,\bar{B}^*_{22}\right) + a^1_{23}$$

$$\frac{\partial h_4}{\partial Y_7} = -\frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\, a^1_{21}\bar{B}^*_{62} \qquad \frac{\partial h_4}{\partial Y_{17}} = -\frac{2}{cn}\, a^1_{21}\bar{A}^*_{26}$$

$$\frac{\partial h_4}{\partial Y_8} = -\frac{1}{c}\, a^1_{21}(\bar{A}^*_{12}+ \bar{A}^*_{66}) - n \qquad \frac{\partial h_4}{\partial Y_{20}} = -\frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\, a^1_{22}\bar{B}^*_{21} \tag{I.37}$$

$$\frac{\partial h_4}{\partial Y_{10}} = -\frac{1}{c}\, a^1_{22}\bar{A}^*_{26} \qquad \frac{\partial h_4}{\partial Y_{21}} = -\frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\, a^1_{21}(2\bar{B}^*_{26}- \bar{B}^*_{61})$$

$$\frac{\partial h_4}{\partial Y_{12}} = -\frac{t}{R}\, a^1_{21}(Y_6+ A_1) \qquad \frac{\partial h_4}{\partial Y_{22}} = \frac{1}{cn^2}\, a^1_{21}\bar{A}^*_{22}$$

$$\frac{\partial h_4}{\partial Y_{13}} = a^1_{21}\left[\frac{1}{n^2} + \frac{1}{2c}\,\frac{t}{R}\,(\bar{B}^*_{22}- 2\bar{B}^*_{66})\right] + a^1_{24} \qquad \frac{\partial h_4}{\partial Y_{27}} = -\frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn^2}\, a^1_{21}\bar{B}^*_{21}$$

The fifth condition

$$h_5 = a^1_{31}u^s_1 + a^1_{32}v^s_1 + a^1_{33}w^s_1 + a^1_{34}w^s_{1,\bar{x}} - H^s_1 = 0 \tag{I.38}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_5}{\partial Y_1} = -\frac{n}{c}\, a^1_{32}\bar{A}^*_{12}+ 4c\,\frac{R}{t}\, n^2(Y_{12}+ A'_o) \qquad \frac{\partial h_5}{\partial Y_{15}} = \frac{1}{cn}\, a^1_{32}\bar{A}^*_{22}$$

$$\frac{\partial h_5}{\partial Y_2} = 8c\,\frac{R}{t}\, n^2(Y_{13}+ A'_1) \qquad \frac{\partial h_5}{\partial Y_{17}} = -\frac{2}{cn}\, a^1_{31}\bar{A}^*_{26}+ 2\,\frac{R}{t}\, n\,(2\bar{B}^*_{26}- \bar{B}^*_{61})$$

$$\frac{\partial h_5}{\partial Y_3} = \frac{n}{c}\, a^1_{31}\bar{A}^*_{16}- 4\,\frac{R}{t}\, n^3\bar{B}^*_{16} \qquad \frac{\partial h_5}{\partial Y_{20}} = -\frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\, a^1_{32}\bar{B}^*_{21} \tag{I.39}$$

$$\frac{\partial h_5}{\partial Y_4} = 8c\,\frac{R}{t}\, n^2\,(Y_{14}+ A'_2)$$

$$\frac{\partial h_5}{\partial Y_6} = -\frac{t}{R}\, a^1_{31}(Y_{12}+A_o') + a^1_{32}\left(\frac{1}{n} + \frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\,\bar{B}^*_{22}\right) + a^1_{33} + 4c\,\frac{R}{t}\,n^2 Y_9$$

$$\frac{\partial h_5}{\partial Y_7} = -\frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\,a^1_{31}\bar{B}^*_{62} + 4c\,\frac{R}{t}\,n^2 Y_{11} - 2n^3\bar{D}^*_{26} - 4c\,\frac{R}{t}\,n\bar{\tau}$$

$$\frac{\partial h_5}{\partial Y_8} = -\frac{1}{c}\,a^1_{31}(\bar{A}^*_{12}+\bar{A}^*_{66}) - 2\,\frac{R}{t}\,n^2(\bar{B}^*_{11} - 2\bar{B}^*_{66}) \qquad \frac{\partial h_5}{\partial Y_{21}} = -\frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\,a^1_{31}(2\bar{B}^*_{26} - \bar{B}^*_{61}) + 4n\bar{D}^*_{16}$$

$$\frac{\partial h_5}{\partial Y_9} = 4c\,\frac{R}{t}\,n^2(Y_6+A_1) \qquad \frac{\partial h_5}{\partial Y_{22}} = \frac{1}{cn^2}\,a^1_{31}\bar{A}^*_{22} + 2\,\frac{R}{t}\,\bar{B}^*_{21}$$

$$\frac{\partial h_5}{\partial Y_{10}} = -\frac{1}{c}\,a^1_{32}\bar{A}^*_{26} \qquad \frac{\partial h_5}{\partial Y_{27}} = -\frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn^2}\,a^1_{31}\bar{B}^*_{21} + \bar{D}^*_{11} \qquad (I.39)$$

$$\frac{\partial h_5}{\partial Y_{11}} = 4c\,\frac{R}{t}\,n^2(Y_7+A_2) \qquad \frac{\partial h_5}{\partial \lambda} = 4c\,\frac{R}{t}\,(Y_{13}+A_1')$$

$$\frac{\partial h_5}{\partial Y_{12}} = -\frac{t}{R}\,a^1_{31}(Y_6+A_1) + 4c\,\frac{R}{t}\,n^2 Y_1 \qquad \frac{\partial h_5}{\partial \bar{p}_e} = \left(\frac{\partial h_5}{\partial \lambda}\right)\hat{R}$$

$$\frac{\partial h_5}{\partial Y_{13}} = a^1_{31}\left[\frac{1}{n^2} + \frac{1}{2c}\,\frac{t}{R}\,(\bar{B}^*_{22} - 2\bar{B}^*_{66})\right] + a^1_{34} + 4c\,\frac{R}{t}\,(2n^2 Y_2 + \lambda) - n^2(\bar{D}^*_{12} + 4\bar{D}^*_{66})$$

$$\frac{\partial h_5}{\partial Y_{14}} = -\frac{1}{c}\,\frac{t}{R}\,a^1_{32}\bar{B}^*_{26} + 8c\,\frac{R}{t}\,n^2 Y_4 \qquad \frac{\partial h_5}{\partial \bar{\tau}} = -4c\,\frac{R}{t}\,n\,(Y_7+A_2)$$

The sixth condition

$$h_6 = a^1_{41}u^s_1 + a^1_{42}v^s_1 + a^1_{43}w^s_1 + a^1_{44}w^s_{1,\bar{x}} - M^s_{x_1} = 0 \qquad (I.40)$$

yields the following nonzero partial derivatives

$$\frac{\partial h_6}{\partial Y_1} = -\frac{n}{c}\,a^1_{42}\bar{A}^*_{12} - 2\,\frac{R}{t}\,n^2\bar{B}^*_{11} \qquad \frac{\partial h_6}{\partial Y_{14}} = -\frac{1}{c}\,\frac{t}{R}\,a^1_{42}\bar{B}^*_{26} + 2n\bar{D}^*_{16}$$

$$(I.41)$$

$$\frac{\partial h_6}{\partial Y_3} = \frac{n}{c}\,a^1_{41}\bar{A}^*_{16} \qquad \frac{\partial h_6}{\partial Y_{15}} = \frac{1}{cn}\,a^1_{42}\bar{A}^*_{22} + 2\,\frac{R}{t}\,\bar{B}^*_{21}$$

$$\frac{\partial h_6}{\partial Y_6} = -\frac{t}{R} a^1_{41}(Y_{12}+ A_o') + a^1_{42}(\frac{1}{n} + \frac{1}{2}\frac{t}{R}\frac{n}{c}\bar{B}^*_{22}) + a^1_{43} - n^2\bar{D}^*_{12}$$

$$\frac{\partial h_6}{\partial Y_7} = -\frac{1}{2}\frac{t}{R}\frac{n}{c} a^1_{41}\bar{B}^*_{62} \qquad \frac{\partial h_6}{\partial Y_{17}} = -\frac{2}{cn} a^1_{41}\bar{A}^*_{26}$$

$$\frac{\partial h_6}{\partial Y_8} = -\frac{1}{c} a^1_{41}(\bar{A}^*_{12}+ \bar{A}^*_{66}) \qquad \frac{\partial h_6}{\partial Y_{20}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn} a^1_{42}\bar{B}^*_{21}+ \bar{D}^*_{11} \tag{I.41}$$

$$\frac{\partial h_6}{\partial Y_{10}} = -\frac{1}{c} a^1_{42}\bar{A}^*_{26} - 2\frac{R}{t} n\,\bar{B}^*_{61} \qquad \frac{\partial h_6}{\partial Y_{21}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn} a^1_{41}(2\bar{B}^*_{26} - \bar{B}^*_{61})$$

$$\frac{\partial h_6}{\partial Y_{12}} = -\frac{t}{R} a^1_{41}(Y_6+ A_1) \qquad \frac{\partial h_6}{\partial Y_{22}} = \frac{1}{cn^2} a^1_{41}\bar{A}^*_{22}$$

$$\frac{\partial h_6}{\partial Y_{13}} = a^1_{41}[\frac{1}{n^2}+ \frac{1}{2c}\frac{t}{R}(\bar{B}^*_{22} - 2\bar{B}^*_{66})] + a^1_{44} \qquad \frac{\partial h_6}{\partial Y_{27}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn^2} a^1_{41}\bar{B}^*_{21}$$

The seventh condition

$$h_7 = a^2_{11}u^s_2 + a^2_{12}v^s_2 - N^s_{x_2} = 0 \tag{I.42}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_7}{\partial Y_2} = -2\frac{n}{c} a^2_{12}\bar{A}^*_{12}+ 4n^2 \qquad \frac{\partial h_7}{\partial Y_{13}} = -\frac{1}{8}\frac{t}{R} a^2_{11}(Y_6+ A_1)$$

$$\frac{\partial h_7}{\partial Y_4} = 2\frac{n}{c} a^2_{11}\bar{A}^*_{16} \qquad \frac{\partial h_7}{\partial Y_{14}} = \frac{1}{8}\frac{t}{R} a^2_{11}(Y_7+ A_2)$$

$$\frac{\partial h_7}{\partial Y_6} = -\frac{1}{8}\frac{t}{R} a^2_{11}(Y_{13}+ A_1') + \frac{1}{4}\frac{t}{R} na^2_{12}(Y_6+ A_1) \qquad \frac{\partial h_7}{\partial Y_{16}} = \frac{1}{2cn} a^2_{12}\bar{A}^*_{22} \tag{I.43}$$

$$\frac{\partial h_7}{\partial Y_7} = \frac{1}{8}\frac{t}{R} a^2_{11}(Y_{14}+ A_2') - \frac{1}{4}\frac{t}{R} na^2_{12}(Y_7+ A_2) \qquad \frac{\partial h_7}{\partial Y_{18}} = -\frac{1}{cn} a^2_{11}\bar{A}^*_{26}$$

$$\frac{\partial h_7}{\partial Y_9} = -\frac{1}{c} a^2_{11}(\bar{A}^*_{12}+ \bar{A}^*_{66}) \qquad \frac{\partial h_7}{\partial Y_{23}} = \frac{1}{4cn^2} a^2_{11}\bar{A}^*_{22}$$

$$\frac{\partial h_7}{\partial Y_{11}} = -\frac{1}{c}\, a_{12}^2 \bar{A}_{26}^* \tag{I.43}$$

The eighth condition

$$h_8 = a_{21}^2 u_2^s + a_{22}^2 v_2^s - N_{xy_2}^s = 0 \tag{I.44}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_8}{\partial Y_2} = -2\,\frac{n}{c}\, a_{22}^2 \bar{A}_{12}^*$$

$$\frac{\partial h_8}{\partial Y_4} = 2\,\frac{n}{c}\, a_{21}^2 \bar{A}_{16}^*$$

$$\frac{\partial h_8}{\partial Y_6} = -\frac{1}{8}\,\frac{t}{R}\, a_{21}^2 (Y_{13} + A_1') + \frac{1}{4}\,\frac{t}{R}\, n a_{22}^2 (Y_6 + A_1)$$

$$\frac{\partial h_8}{\partial Y_7} = \frac{1}{8}\,\frac{t}{R}\, a_{21}^2 (Y_{14} + A_2') - \frac{1}{4}\,\frac{t}{R}\, n a_{22}^2 (Y_7 + A_2)$$

$$\frac{\partial h_8}{\partial Y_9} = -\frac{1}{c}\, a_{21}^2 (\bar{A}_{12}^* + \bar{A}_{66}^*) - 2n$$

$$\frac{\partial h_8}{\partial Y_{11}} = -\frac{1}{c}\, a_{22}^2 \bar{A}_{26}^*$$

$$\frac{\partial h_8}{\partial Y_{13}} = -\frac{1}{8}\,\frac{t}{R}\, a_{21}^2 (Y_6 + A_1)$$

$$\frac{\partial h_8}{\partial Y_{14}} = \frac{1}{8}\,\frac{t}{R}\, a_{21}^2 (Y_7 + A_2)$$

$$\frac{\partial h_8}{\partial Y_{16}} = \frac{1}{2cn}\, a_{22}^2 \bar{A}_{22}^*$$

$$\frac{\partial h_8}{\partial Y_{18}} = -\frac{1}{cn}\, a_{21}^2 \bar{A}_{26}^* \tag{I.45}$$

$$\frac{\partial h_8}{\partial Y_{23}} = \frac{1}{4cn^2}\, a_{21}^2 \bar{A}_{22}^*$$

The nineth condition

$$h_9 = a_{11}^3 u_3^s + a_{12}^3 v_3^s + a_{13}^3 w_2^s + a_{14}^3 w_{2,\bar{x}}^s - N_{x_3}^s = 0 \tag{I.46}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_9}{\partial Y_1} = -\frac{n}{c}\, a_{11}^3 \bar{A}_{16}^*$$

$$\frac{\partial h_9}{\partial Y_3} = \frac{n}{c}\, a_{12}^3 \bar{A}_{12}^* + n^2$$

$$\frac{\partial h_9}{\partial Y_{14}} = a_{11}^3 \left[\frac{1}{n^2} + \frac{1}{2c}\,\frac{t}{R}\,(\bar{B}_{22}^* - 2\bar{B}_{66}^*)\right] + a_{14}^3 \tag{I.47}$$

$$\frac{\partial h_9}{\partial Y_{15}} = \frac{2}{cn}\, a_{11}^3 \bar{A}_{26}^*$$

$$\frac{\partial h_9}{\partial Y_6} = \frac{1}{2}\frac{t}{R}\frac{n}{c}a_{11}^3\bar{B}_{62}^* \qquad \frac{\partial h_9}{\partial Y_{17}} = -\frac{1}{cn}a_{12}^3\bar{A}_{22}^*$$

$$\frac{\partial h_9}{\partial Y_7} = -\frac{t}{R}a_{11}^3(Y_{12}+A_o') - a_{12}^3\left(\frac{1}{n}+\frac{1}{2}\frac{t}{R}\frac{n}{c}\bar{B}_{22}^*\right) + a_{13}^3$$

$$\frac{\partial h_9}{\partial Y_8} = -\frac{1}{c}a_{12}^3\bar{A}_{26}^* \qquad \frac{\partial h_9}{\partial Y_{20}} = \frac{1}{2}\frac{t}{R}\frac{1}{cn}a_{11}^3(2\bar{B}_{26}^* - \bar{B}_{61}^*)$$

(I.47)

$$\frac{\partial h_9}{\partial Y_{10}} = -\frac{1}{c}a_{11}^3(\bar{A}_{12}^* + \bar{A}_{66}^*) \qquad \frac{\partial h_9}{\partial Y_{21}} = \frac{1}{2}\frac{t}{R}\frac{1}{cn}a_{12}^3\bar{B}_{21}^*$$

$$\frac{\partial h_9}{\partial Y_{12}} = -\frac{t}{R}a_{11}^3(Y_7 + A_2) \qquad \frac{\partial h_9}{\partial Y_{24}} = \frac{1}{cn^2}a_{11}^3\bar{A}_{22}^*$$

$$\frac{\partial h_9}{\partial Y_{13}} = -\frac{1}{c}\frac{t}{R}a_{12}^3\bar{B}_{26}^* \qquad \frac{\partial h_9}{\partial Y_{28}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn^2}a_{11}^3\bar{B}_{21}^*$$

The tenth condition

$$h_{10} = a_{21}^3 u_3^s + a_{22}^3 v_3^s + a_{23}^3 w_2^s + a_{24}^3 w_{2,\bar{x}}^s - N_{xy_3}^s = 0 \tag{I.48}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_{10}}{\partial Y_1} = -\frac{n}{c}a_{21}^3\bar{A}_{16}^* \qquad \frac{\partial h_{10}}{\partial Y_{14}} = a_{21}^3\left[\frac{1}{n^2} + \frac{1}{2c}\frac{t}{R}(\bar{B}_{22}^* - 2\bar{B}_{66}^*)\right] + a_{24}^3$$

$$\frac{\partial h_{10}}{\partial Y_3} = \frac{n}{c}a_{22}^3\bar{A}_{12}^* \qquad \frac{\partial h_{10}}{\partial Y_{15}} = \frac{2}{cn}a_{21}^3\bar{A}_{26}^*$$

$$\frac{\partial h_{10}}{\partial Y_6} = \frac{1}{2}\frac{t}{R}\frac{n}{c}a_{21}^3\bar{B}_{62}^* \qquad \frac{\partial h_{10}}{\partial Y_{17}} = -\frac{1}{cn}a_{22}^3\bar{A}_{22}^*$$

(I.49)

$$\frac{\partial h_{10}}{\partial Y_7} = -\frac{t}{R}a_{21}^3(Y_{12}+A_o') - a_{22}^3\left(\frac{1}{n}+\frac{1}{2}\frac{t}{R}\frac{n}{c}\bar{B}_{22}^*\right) + a_{23}^3$$

$$\frac{\partial h_{10}}{\partial Y_8} = -\frac{1}{c}a_{22}^3\bar{A}_{26}^* \qquad \frac{\partial h_{10}}{\partial Y_{20}} = \frac{1}{2}\frac{t}{R}\frac{1}{cn}a_{21}^3(2\bar{B}_{26}^* - \bar{B}_{61}^*)$$

$$\frac{\partial h_{10}}{\partial Y_{10}} = -\frac{1}{c}\, a_{21}^3(\bar{A}_{12}^* + \bar{A}_{66}^*) + n \qquad \frac{\partial h_{10}}{\partial Y_{21}} = \frac{1}{2}\frac{t}{R}\frac{1}{cn}\, a_{22}^3\bar{B}_{21}^*$$

$$\frac{\partial h_{10}}{\partial Y_{12}} = -\frac{t}{R}\, a_{21}^3(Y_7 + A_2) \qquad \frac{\partial h_{10}}{\partial Y_{24}} = \frac{1}{cn^2}\, a_{21}^3\bar{A}_{22}^* \tag{I.49}$$

$$\frac{\partial h_{10}}{\partial Y_{13}} = -\frac{1}{c}\frac{t}{R}\, a_{22}^3\bar{B}_{26}^* \qquad \frac{\partial h_{10}}{\partial Y_{28}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn^2}\, a_{21}^3\bar{B}_{21}^*$$

The eleventh condition

$$h_{11} = a_{31}^3 u_3^s + a_{32}^3 v_3^s + a_{33}^3 w_2^s + a_{34}^3 w_{2,\bar{x}}^s - H_2^s = 0 \tag{I.50}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_{11}}{\partial Y_1} = -\frac{n}{c}\, a_{31}^3\bar{A}_{16}^* + 4\frac{R}{t}\, n^3\bar{B}_{16}^* \qquad \frac{\partial h_{11}}{\partial Y_{15}} = \frac{2}{cn}\, a_{31}^3\bar{A}_{26}^* - 2\,\frac{R}{t}\, n(2\bar{B}_{26}^* - \bar{B}_{61}^*)$$

$$\frac{\partial h_{11}}{\partial Y_2} = -8c\,\frac{R}{t}\, n^2\,(Y_{14} + A_2') \qquad \frac{\partial h_{11}}{\partial Y_{17}} = -\frac{1}{cn}\, a_{32}^3\bar{A}_{22}^*$$

$$\frac{\partial h_{11}}{\partial Y_3} = \frac{n}{c}\, a_{32}^3\bar{A}_{12}^* + 4c\,\frac{R}{t}\, n^2(Y_{12} + A_o') \qquad \frac{\partial h_{11}}{\partial Y_{20}} = \frac{1}{2}\frac{t}{R}\frac{1}{cn}\, a_{31}^3(2\bar{B}_{26}^* - \bar{B}_{61}^*) - 4n\bar{D}_{16}^*$$

$$\frac{\partial h_{11}}{\partial Y_4} = 8c\,\frac{R}{t}\, n^2(Y_{13} + A_1') \qquad \frac{\partial h_{11}}{\partial Y_{21}} = \frac{1}{2}\frac{t}{R}\frac{1}{cn}\, a_{32}^3\bar{B}_{21}^* \tag{I.51}$$

$$\frac{\partial h_{11}}{\partial Y_6} = \frac{1}{2}\frac{t}{R}\frac{n}{c}\, a_{31}^3\bar{B}_{62}^* + 4c\,\frac{R}{t}\, n^2 Y_{11} + 2n^3\bar{D}_{26}^* + 4c\,\frac{R}{t}\, n\bar{\tau}$$

$$\frac{\partial h_{11}}{\partial Y_7} = -\frac{t}{R}\, a_{31}^3(Y_{12} + A_o') - a_{32}^3\left(\frac{1}{n} + \frac{1}{2}\frac{t}{R}\frac{n}{c}\,\bar{B}_{22}^*\right) + a_{33}^3 - 4c\,\frac{R}{t}\, n^2 Y_9$$

$$\frac{\partial h_{11}}{\partial Y_8} = -\frac{1}{c}\, a_{32}^3\bar{A}_{26}^* \qquad \frac{\partial h_{11}}{\partial Y_{24}} = \frac{1}{cn^2}\, a_{31}^3\bar{A}_{22}^* + 2\,\frac{R}{t}\,\bar{B}_{21}^*$$

$$\frac{\partial h_{11}}{\partial Y_9} = -4c\,\frac{R}{t}\, n^2(Y_7 + A_2) \qquad \frac{\partial h_{11}}{\partial Y_{28}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn^2}\, a_{31}^3\bar{B}_{21}^* + \bar{D}_{11}^*$$

$$\frac{\partial h_{11}}{\partial Y_{10}} = -\frac{1}{c}\, a_{31}^{3}(\bar{A}_{12}^{*} + \bar{A}_{66}^{*}) - 2\,\frac{R}{t}\, n^{2}(\bar{B}_{11}^{*} - 2\bar{B}_{66}^{*}) \qquad \frac{\partial h_{11}}{\partial \lambda} = 4c\,\frac{R}{t}\,(Y_{14} + A_{2}')$$

$$\frac{\partial h_{11}}{\partial Y_{11}} = 4c\,\frac{R}{t}\, n^{2}(Y_{6} + A_{1}) \qquad \frac{\partial h_{11}}{\partial \bar{p}_{e}} = \left(\frac{\partial h_{11}}{\partial \lambda}\right)\hat{R}$$

$$\frac{\partial h_{11}}{\partial Y_{12}} = -\frac{t}{R}\, a_{31}^{3}(Y_{7} + A_{2}) + 4c\,\frac{R}{t}\, n^{2} Y_{3} \qquad \frac{\partial h_{11}}{\partial \bar{\tau}} = 4c\,\frac{R}{t}\, n\,(Y_{6} + A_{1}) \tag{I.51}$$

$$\frac{\partial h_{11}}{\partial Y_{13}} = -\frac{1}{c}\,\frac{t}{R}\, a_{32}^{3}\bar{B}_{26}^{*} + 8c\,\frac{R}{t}\, n^{2} Y_{4}$$

$$\frac{\partial h_{11}}{\partial Y_{14}} = a_{31}^{3}\left[\frac{1}{n^{2}} + \frac{1}{2c}\,\frac{t}{R}\,(\bar{B}_{22}^{*} - 2\bar{B}_{66}^{*})\right] + a_{34}^{3} - 4c\,\frac{R}{t}\,(2n^{2}Y_{2} - \lambda) - n^{2}(\bar{D}_{12}^{*} + 4\bar{D}_{66}^{*})$$

The twelfth condition

$$h_{12} = a_{41}^{3}u_{3}^{s} + a_{42}^{3}v_{3}^{s} + a_{43}^{3}w_{2}^{s} + a_{44}^{3}w_{2,\bar{x}}^{s} - M_{x_2}^{s} = 0 \tag{I.52}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_{12}}{\partial Y_{1}} = -\frac{n}{c}\, a_{41}^{3}\bar{A}_{16}^{*} \qquad \frac{\partial h_{12}}{\partial Y_{14}} = a_{41}^{3}\left[\frac{1}{n^{2}} + \frac{1}{2c}\,\frac{t}{R}\,(\bar{B}_{22}^{*} - 2\bar{B}_{66}^{*})\right] + a_{44}^{3}$$

$$\frac{\partial h_{12}}{\partial Y_{3}} = \frac{n}{c}\, a_{42}^{3}\bar{A}_{12}^{*} - 2\,\frac{R}{t}\, n^{2}\bar{B}_{11}^{*} \qquad \frac{\partial h_{12}}{\partial Y_{15}} = \frac{2}{cn}\, a_{41}^{3}\bar{A}_{26}^{*}$$

$$\frac{\partial h_{12}}{\partial Y_{6}} = \frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\, a_{41}^{3}\bar{B}_{62}^{*} \qquad \frac{\partial h_{12}}{\partial Y_{17}} = -\frac{1}{cn}\, a_{42}^{3}\bar{A}_{22}^{*} + 2\,\frac{R}{t}\,\bar{B}_{21}^{*} \tag{I.53}$$

$$\frac{\partial h_{12}}{\partial Y_{7}} = -\frac{t}{R}\, a_{41}^{3}(Y_{12} + A_{o}') - a_{42}^{3}\left(\frac{1}{n} + \frac{1}{2}\,\frac{t}{R}\,\frac{n}{c}\,\bar{B}_{22}^{*}\right) + a_{43}^{3} - n^{2}\bar{D}_{12}^{*}$$

$$\frac{\partial h_{12}}{\partial Y_{8}} = -\frac{1}{c}\, a_{42}^{3}\bar{A}_{26}^{*} + 2\,\frac{R}{t}\, n\,\bar{B}_{61}^{*} \qquad \frac{\partial h_{12}}{\partial Y_{20}} = \frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\, a_{41}^{3}(2\bar{B}_{26}^{*} - \bar{B}_{61}^{*})$$

$$\frac{\partial h_{12}}{\partial Y_{10}} = -\frac{1}{c}\, a_{41}^{3}(\bar{A}_{12}^{*} + \bar{A}_{66}^{*}) \qquad \frac{\partial h_{12}}{\partial Y_{21}} = \frac{1}{2}\,\frac{t}{R}\,\frac{1}{cn}\, a_{42}^{3}\bar{B}_{21}^{*} + \bar{D}_{11}^{*}$$

$$\frac{\partial h_{12}}{\partial Y_{12}} = -\frac{t}{R} a_{41}^{3}(Y_7 + A_2) \qquad \frac{\partial h_{12}}{\partial Y_{24}} = \frac{1}{cn^2} a_{41}^{3}\bar{A}_{22}^{*}$$

$$\frac{\partial h_{12}}{\partial Y_{13}} = -\frac{1}{c}\frac{t}{R} a_{42}^{3}\bar{B}_{26}^{*} - 2n\bar{D}_{16}^{*} \qquad \frac{\partial h_{12}}{\partial Y_{28}} = -\frac{1}{2}\frac{t}{R}\frac{1}{cn^2} a_{41}^{3}\bar{B}_{21}^{*} \tag{I.53}$$

The thirteenth condition

$$h_{13} = a_{11}^{4}u_4^{s} + a_{12}^{4}v_4^{s} - N_{x_4}^{s} = 0 \tag{I.54}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_{13}}{\partial Y_2} = -2\frac{n}{c} a_{11}^{4}\bar{A}_{16}^{*} \qquad \frac{\partial h_{13}}{\partial Y_{13}} = -\frac{1}{8}\frac{t}{R} a_{11}^{4}(Y_7 + A_2)$$

$$\frac{\partial h_{13}}{\partial Y_4} = 2\frac{n}{c} a_{12}^{4}\bar{A}_{12}^{*} + 4n^2 \qquad \frac{\partial h_{13}}{\partial Y_{14}} = -\frac{1}{8}\frac{t}{R} a_{11}^{4}(Y_6 + A_1)$$

$$\frac{\partial h_{13}}{\partial Y_6} = -\frac{1}{8}\frac{t}{R} a_{11}^{4}(Y_{14} + A_2') - \frac{1}{4}\frac{t}{R} na_{12}^{4}(Y_7 + A_2) \qquad \frac{\partial h_{13}}{\partial Y_{16}} = \frac{1}{cn} a_{11}^{4}\bar{A}_{26}^{*} \tag{I.55}$$

$$\frac{\partial h_{13}}{\partial Y_7} = -\frac{1}{8}\frac{t}{R} a_{11}^{4}(Y_{13} + A_1') - \frac{1}{4}\frac{t}{R} na_{12}^{4}(Y_6 + A_1) \qquad \frac{\partial h_{13}}{\partial Y_{18}} = -\frac{1}{2cn} a_{12}^{4}\bar{A}_{22}^{*}$$

$$\frac{\partial h_{13}}{\partial Y_9} = -\frac{1}{c} a_{12}^{4}\bar{A}_{26}^{*} \qquad \frac{\partial h_{13}}{\partial Y_{25}} = \frac{1}{4cn^2} a_{11}^{4}\bar{A}_{22}^{*}$$

$$\frac{\partial h_{13}}{\partial Y_{11}} = -\frac{1}{c} a_{11}^{4}(\bar{A}_{12}^{*} + \bar{A}_{66}^{*})$$

Finally, the fourteenth condition

$$h_{14} = a_{21}^{4}u_4^{s} + a_{22}^{4}v_4^{s} - N_{xy_4}^{s} = 0 \tag{I.56}$$

yields the following nonzero partial derivatives

$$\frac{\partial h_{14}}{\partial Y_2} = -2\frac{n}{c} a_{21}^{4}\bar{A}_{16}^{*} \qquad \frac{\partial h_{14}}{\partial Y_{13}} = -\frac{1}{8}\frac{t}{R} a_{21}^{4}(Y_7 + A_2)$$

$$\frac{\partial h_{14}}{\partial Y_4} = 2\,\frac{n}{c}\,a_{22}^4\bar{A}_{12}^*$$

$$\frac{\partial h_{14}}{\partial Y_{14}} = -\,\frac{1}{8}\,\frac{t}{R}\,a_{21}^4(Y_6 + A_1)$$

$$\frac{\partial h_{14}}{\partial Y_6} = -\,\frac{1}{8}\,\frac{t}{R}\,a_{21}^4(Y_{14} + A_2') - \frac{1}{4}\,\frac{t}{R}\,na_{22}^4(Y_7 + A_2)$$

$$\frac{\partial h_{14}}{\partial Y_{16}} = \frac{1}{cn}\,a_{21}^4\bar{A}_{26}^* \qquad (I.57)$$

$$\frac{\partial h_{14}}{\partial Y_7} = -\,\frac{1}{8}\,\frac{t}{R}\,a_{21}^4(Y_{13} + A_1') - \frac{1}{4}\,\frac{t}{R}\,na_{22}^4(Y_6 + A_1)$$

$$\frac{\partial h_{14}}{\partial Y_{18}} = -\,\frac{1}{2cn}\,a_{22}^4\bar{A}_{22}^*$$

$$\frac{\partial h_{14}}{\partial Y_9} = -\,\frac{1}{c}\,a_{22}^4\bar{A}_{26}^*$$

$$\frac{\partial h_{14}}{\partial Y_{25}} = \frac{1}{4cn^2}\,a_{21}^4\bar{A}_{22}^*$$

$$\frac{\partial h_{14}}{\partial Y_{11}} = -\,\frac{1}{c}\,a_{21}^4(\bar{A}_{12}^* + \bar{A}_{66}^*) + 2n$$

Table 1 Comparison of calculated buckling loads – Khot's glass-epoxy shell [26]
(-40, 40, 0) - $N_{c\ell} = -965.9814$ lb/in, R = 6.0 in, L = 12.5 in, t = 0.036 in

	ANILISA	COLLAPSE		ANILISA	COLLAPSE
SS-1	-282.324 (n=2)	-282.323 (n=2)	C-1	-519.656 (n=12)	-519.656 (n=12)
SS-2	-282.702 (n=2)	-283. (n=2)	C-2	-521.455 (n=12)	-521.454 (n=12)
SS-3	-518.116 (n=12)	-518.115 (n=12)	C-3	-520.333 (n=12)	-520.332 (n=12)
SS-4	-520.727 (n=12)	-520.663 (n=12)	C-4	-521.597 (n=12)	-521.597 (n=12)

Note: Buckling loads are given in lb/in

SS-1 : $N_x = -N_o$; $N_{xy} = w = 0$; $M_x = 0$

SS-2 : $u = u_o$; $N_{xy} = w = 0$; $M_x = 0$

SS-3 : $N_x = -N_o$; $v = w = 0$; $M_x = 0$

SS-4 : $u = u_o$; $v = w = 0$; $M_x = 0$

C-1 : $N_x = -N_o$; $N_{xy} = w = 0$; $w_{,x} = 0$

C-2 : $u = u_o$; $N_{xy} = w = 0$; $w_{,x} = 0$

C-3 : $N_x = -N_o$; $v = w = 0$; $w_{,x} = 0$

C-4 : $u = u_o$; $v = w = 0$; $w_{,x} = 0$

Table 2 Summary of Imperfection Sensitivity Calculations using ANILISA [29]
Khot's glass-epoxy shell (-40, 40, 0) - $N_{c\ell} = -965.9814$ lb/in

	λ_c^{nl}	n	b	α	β	$\tilde{\theta}_c$	$\tilde{\theta}_c^*$	$\bar{\xi}_2 = 0.1$	$\bar{\xi}_2 = 0.5$	$\bar{\xi}_2 = 1.0$
SS-1	0.292266	2	-0.0436	1.0042	1.0179	44.97	-59.08	0.869	0.677	0.553
SS-2	0.292659	2	+0.0328	1.0015	1.0139	44.97	+17.67	-	-	-
SS-3	0.536362	12	-0.3893	0.8975	0.6644	44.12	-122.26	0.757	0.436	0.251
SS-4	0.539065	12	-0.1726	0.9444	0.8083	44.09	-93.68	0.807	0.542	0.385
C-1	0.537958	12	-0.2654	0.9468	0.8267	44.56	-110.73	0.781	0.498	0.340
C-2	0.539820	12	-0.1115	0.9689	0.8939	44.55	-73.98	0.829	0.591	0.447
C-3	0.538658	12	-0.2176	0.9525	0.8403	44.56	-104.68	0.793	0.520	0.364
C-4	0.539966	12	-0.1101	0.9690	0.8934	44.55	-74.24	0.830	0.592	0.448

Note: $\tilde{\theta}_c$ and $\tilde{\theta}_c^*$ are given in degrees.

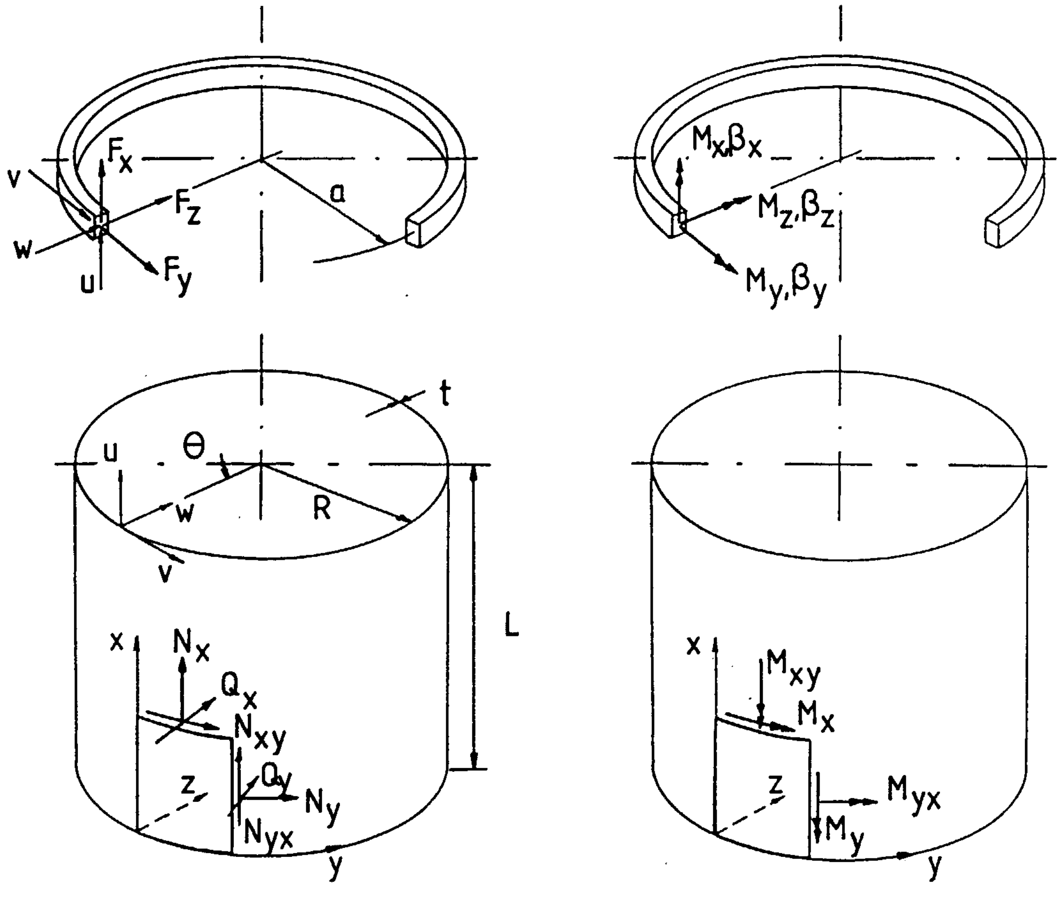


Fig. 1 Sign convention used for shell and ring analysis

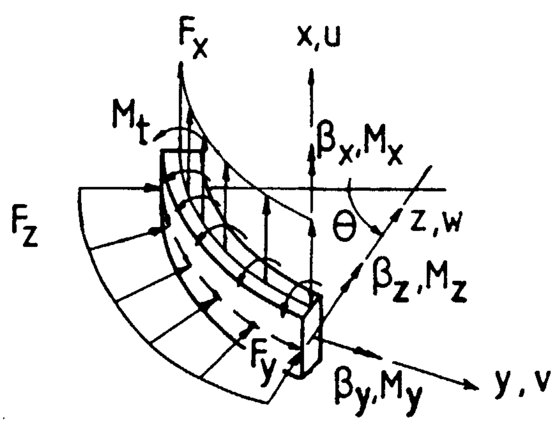


Fig. 2 Forces on a ring segment

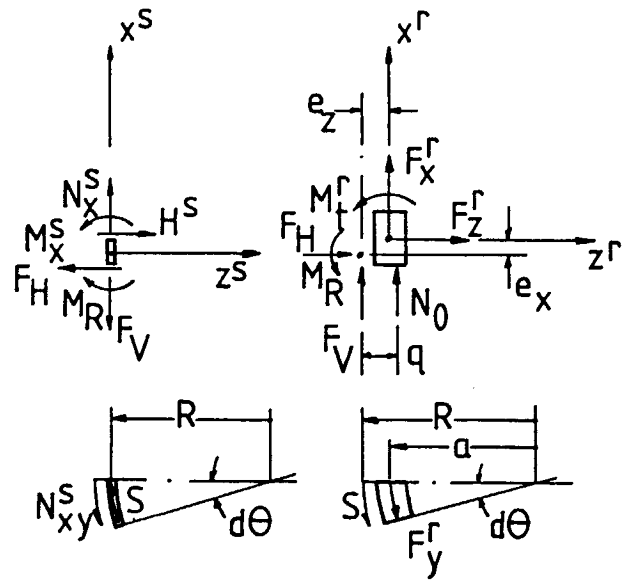


Fig. 3 Determination of forces and moments at the ring centroid - at x = 0 (lower edge)

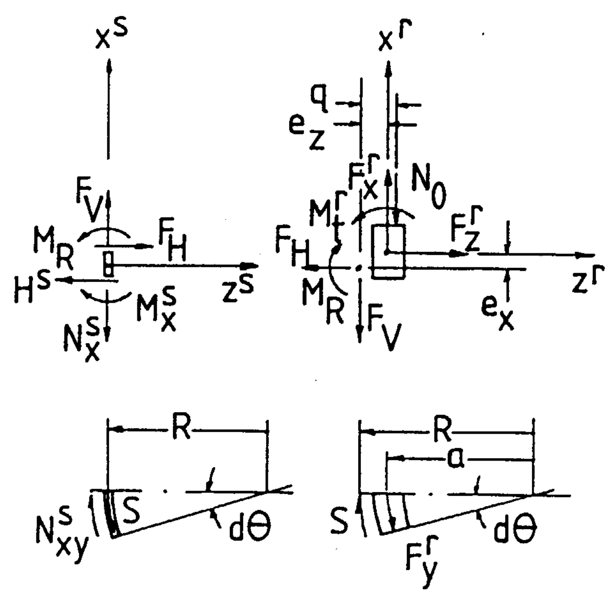


Fig. 4 Determination of forces and moments at the ring centroid - at x = L (upper edge)

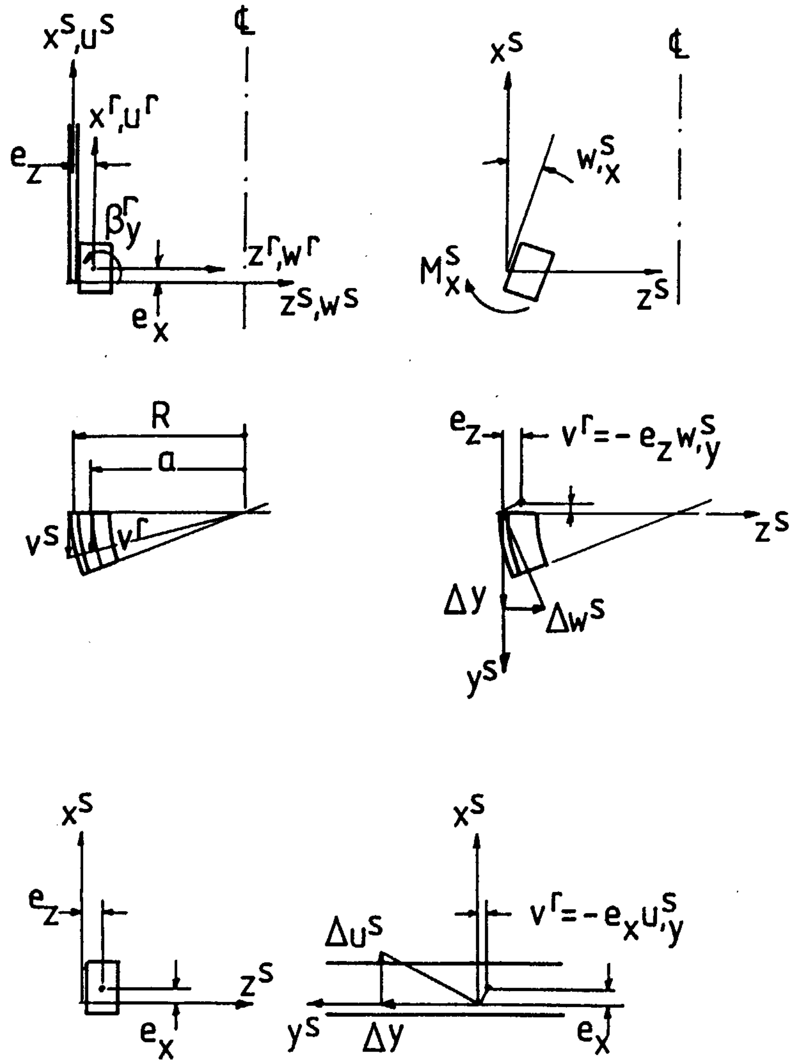


Fig. 5a Compatibility of displacements and rotations at x = 0 (lower edge)

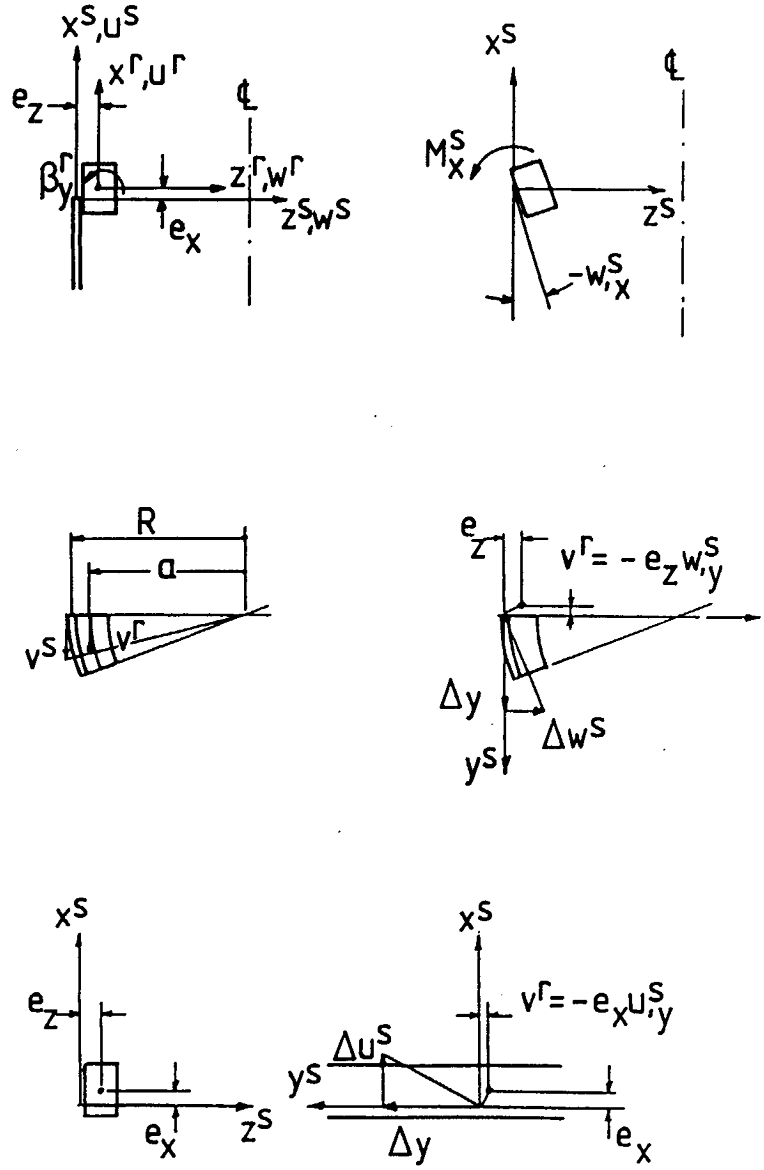


Fig. 5b Compatibility of displacements and rotations at x = L (upper edge)

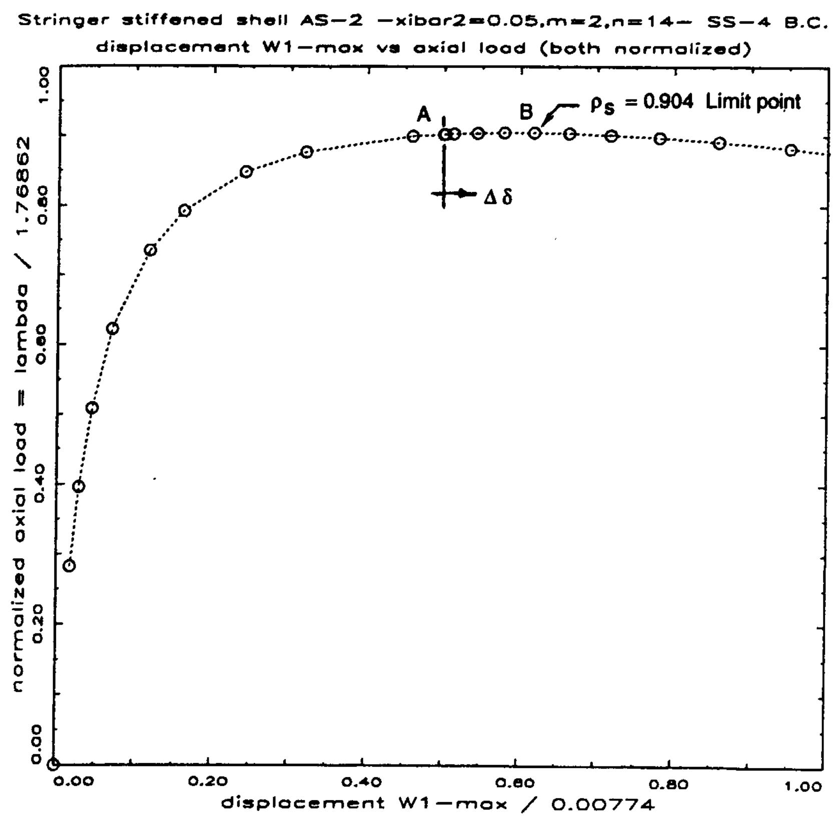


Fig. 6 Variation of $|W_{,max}|$ with axial load

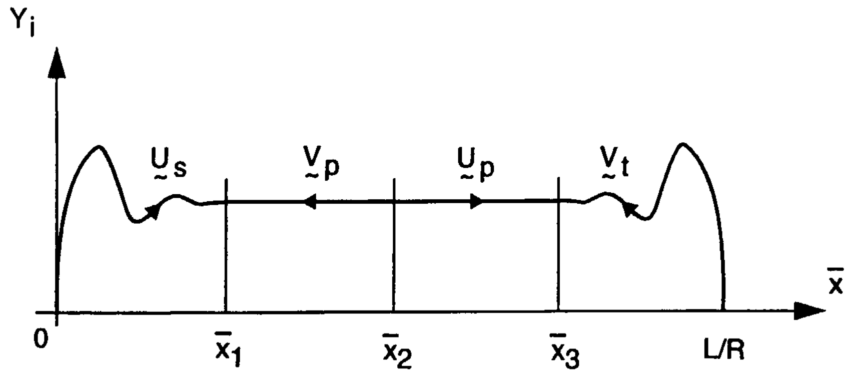


Fig. 7 Definition of the matching conditions

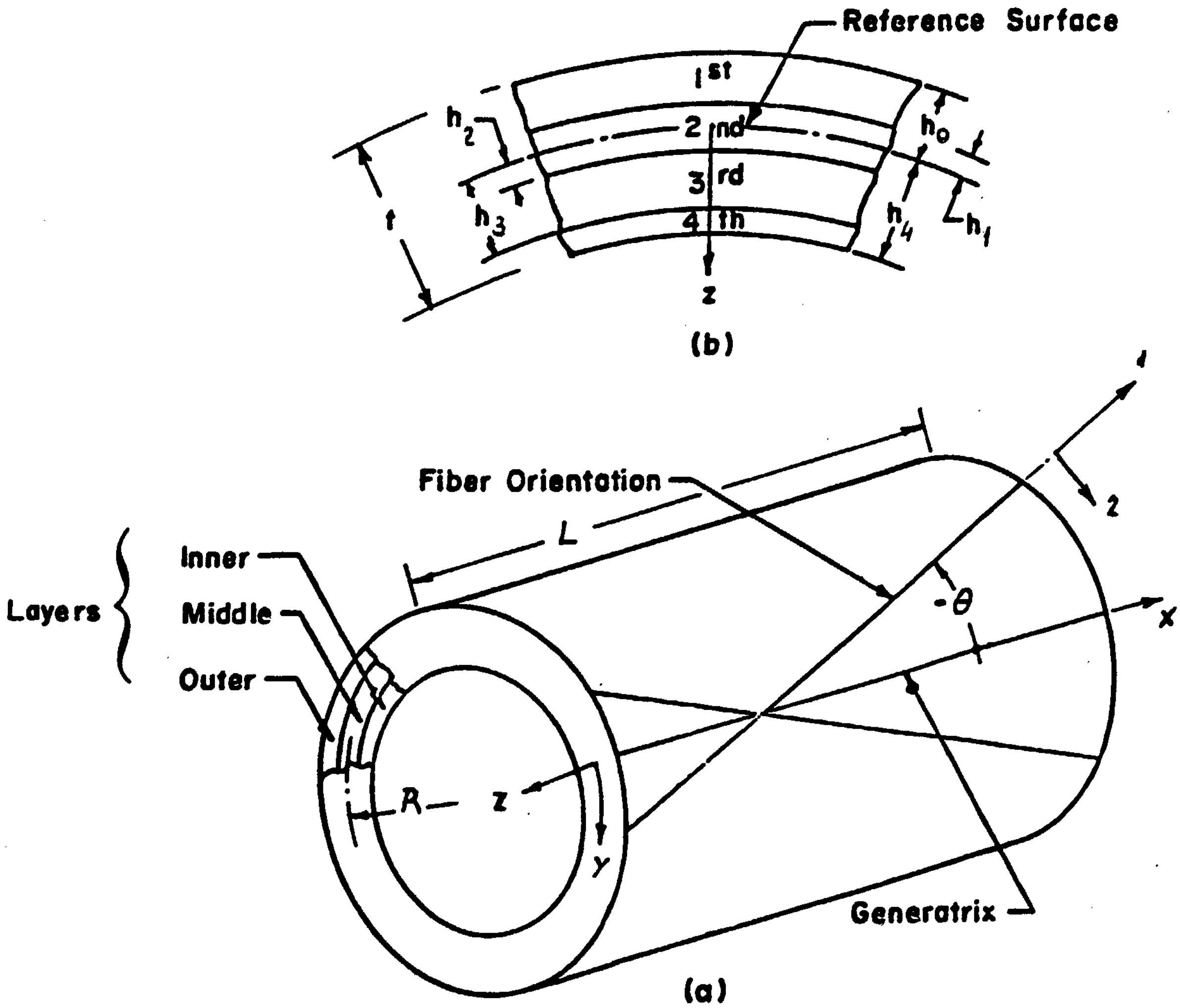


Fig. 8 Notation and sign convention for layered composite shell

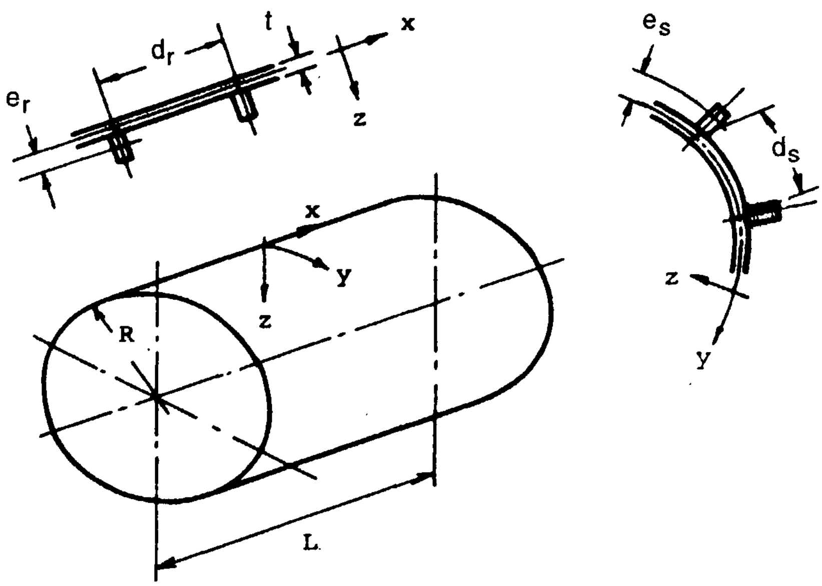
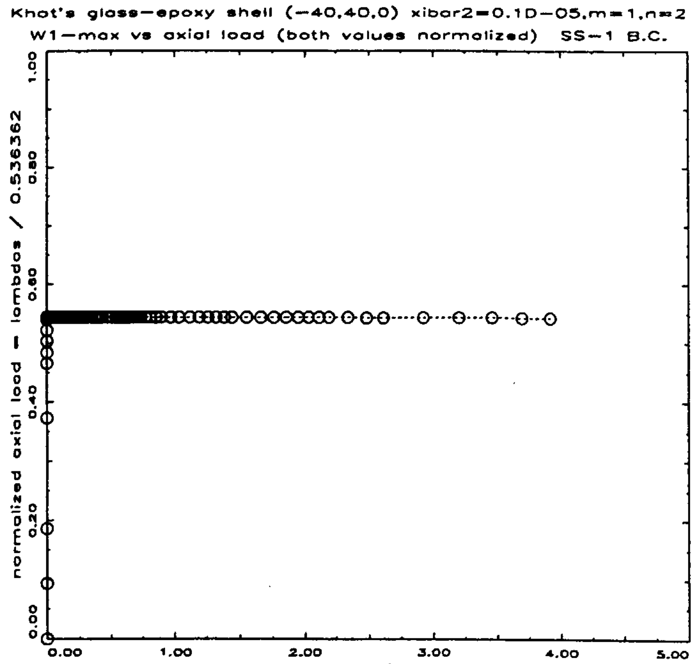
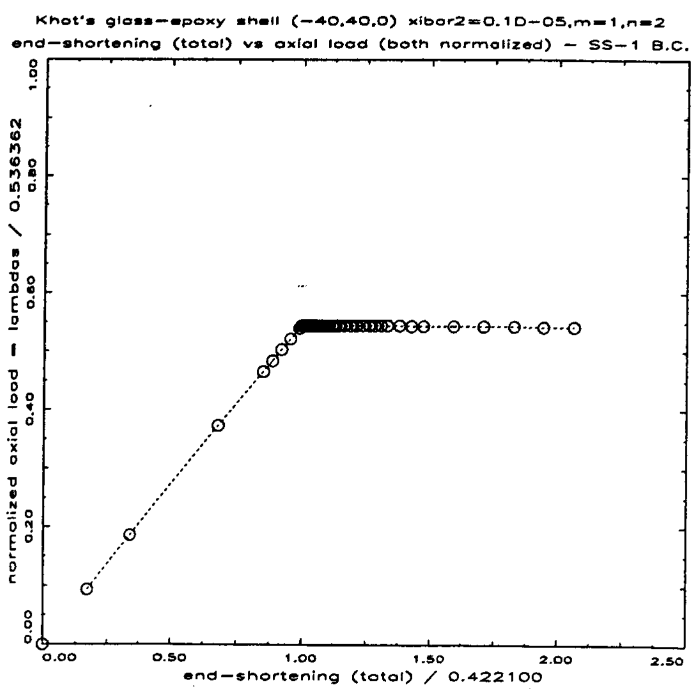


Fig. 9 Notation and sign convention for orthotropic stiffeners



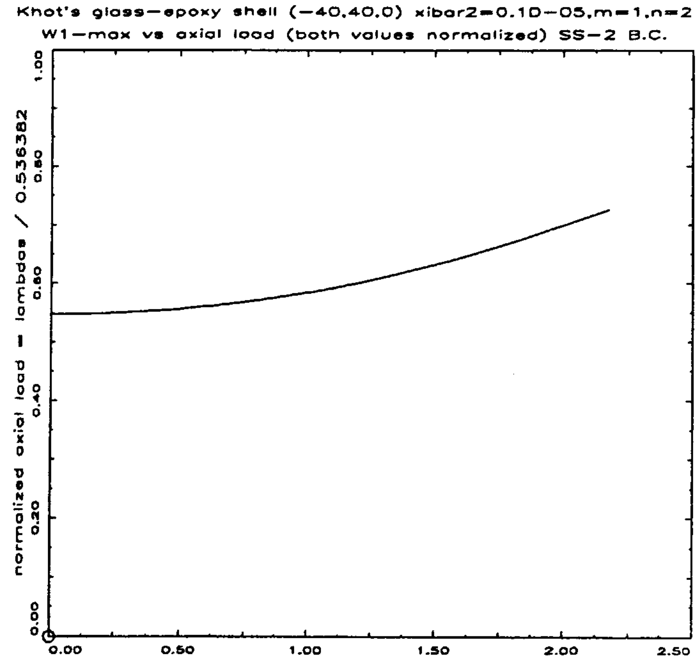
a. $w_{1,max}$ vs normalized axial load $\rho_s = \lambda_s/\lambda_c$



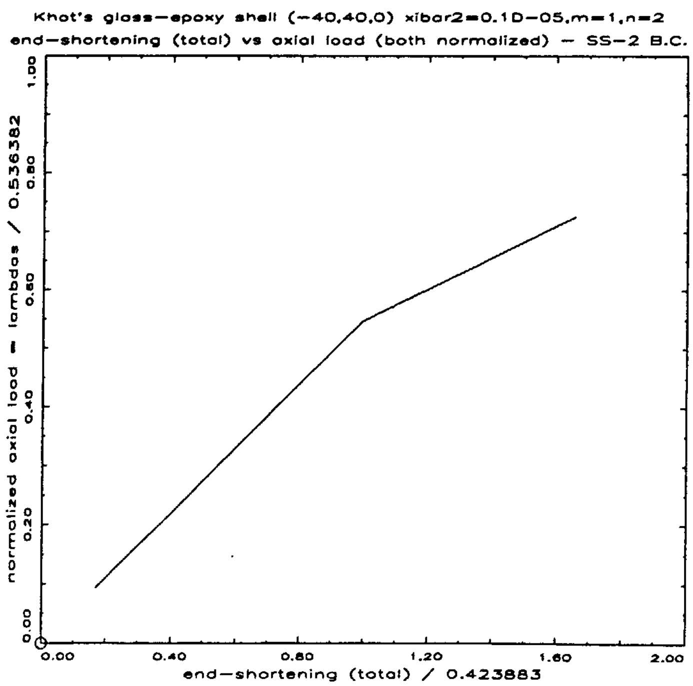
b. end-shortening vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

Fig. 10 Load vs deformation plots for SS-1 boundary conditions

$(N_x = -N_o,\ N_{xy} = w = M_x = 0\ ;\ N_{c\ell} = -965.9814$ lb/in$)$

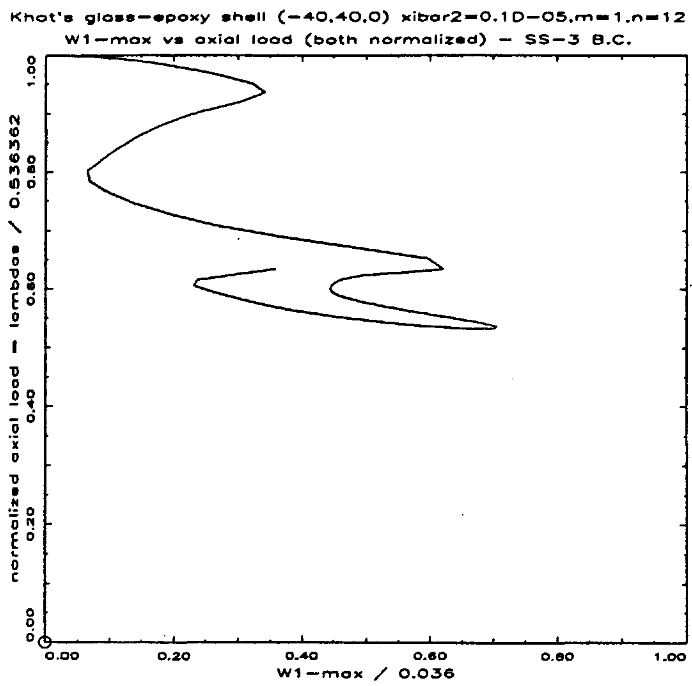


a. $w_{1,max}$ vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

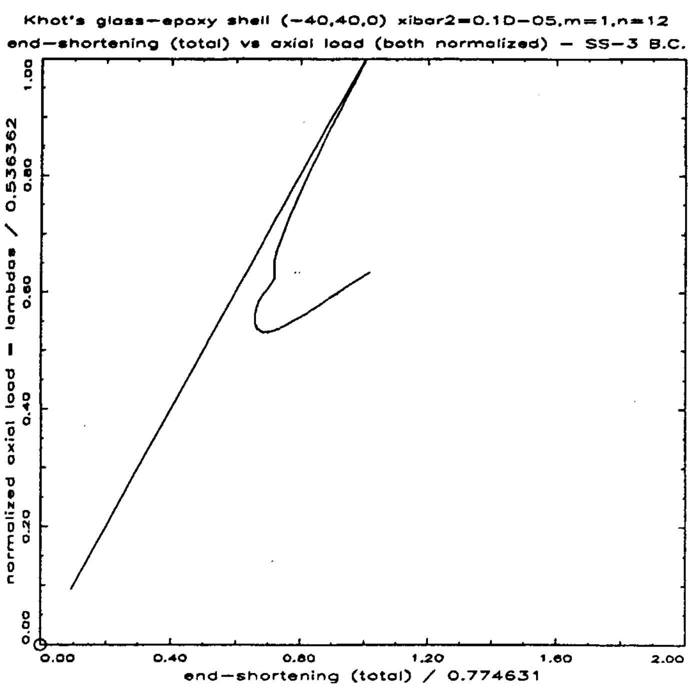


b. end-shortening vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

Fig. 11 Load vs deformation plots for SS-2 boundary conditions
($u = u_0$, $N_{xy} = w = M_x = 0$; $N_{c\ell} = -965.9814$ lb/in)



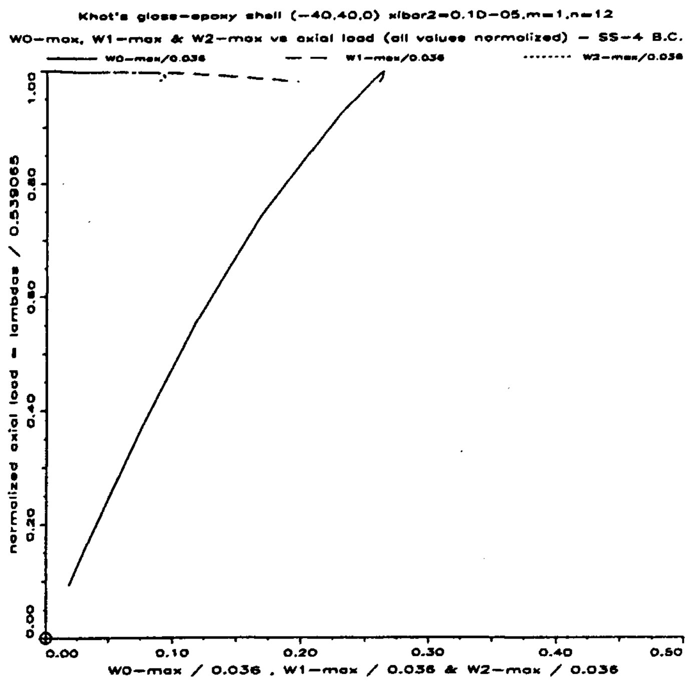
a. $w_{1,max}$ vs normalized axial load $\rho_s = \lambda_s/\lambda_c$



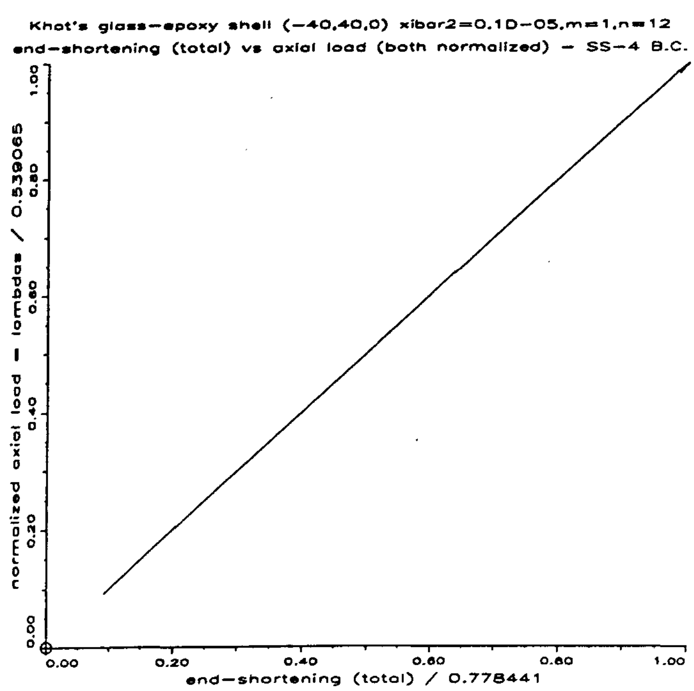
b. end-shortening vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

Fig. 12 Load vs deformation plots for SS-3 boundary conditions

($N_x = -N_o$, $v = w = M_x = 0$; $N_{c\ell} = -965.9814$ lb/in)



a. $w_{o,max}$, $w_{1,max}$, $w_{2,max}$ vs normalized axial load $\rho_s=\lambda_s/\lambda_c$



b. end-shortening vs normalized axial load $\rho_s= \lambda_s/\lambda_c$

Fig. 13 Load vs deformation plots for SS-4 boundary conditions

($u=u_o$, $v=w=M_x=0$; $N_{c\ell}=-965.9814$ lb/in)

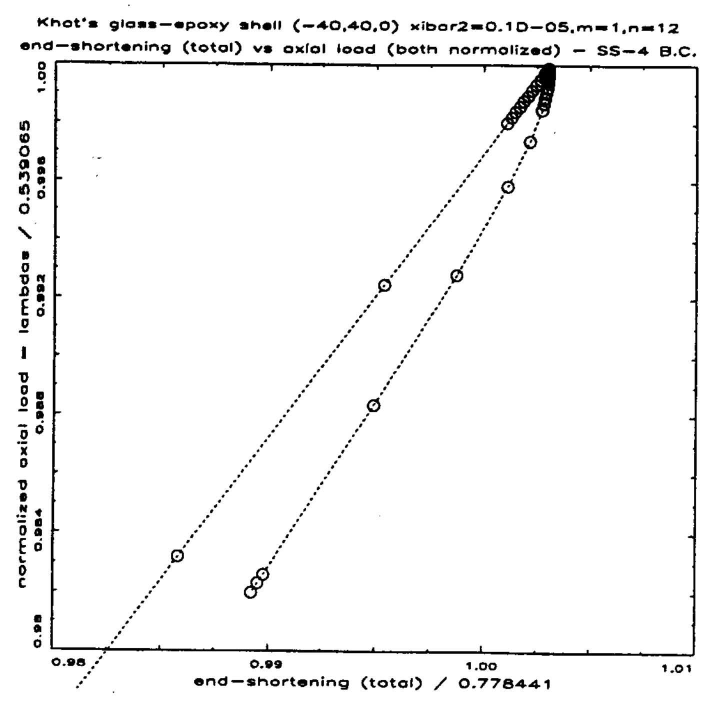
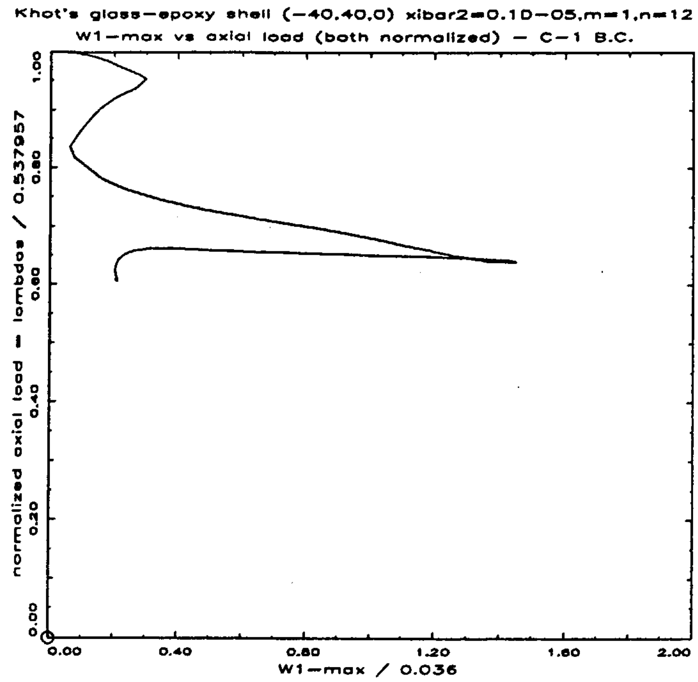
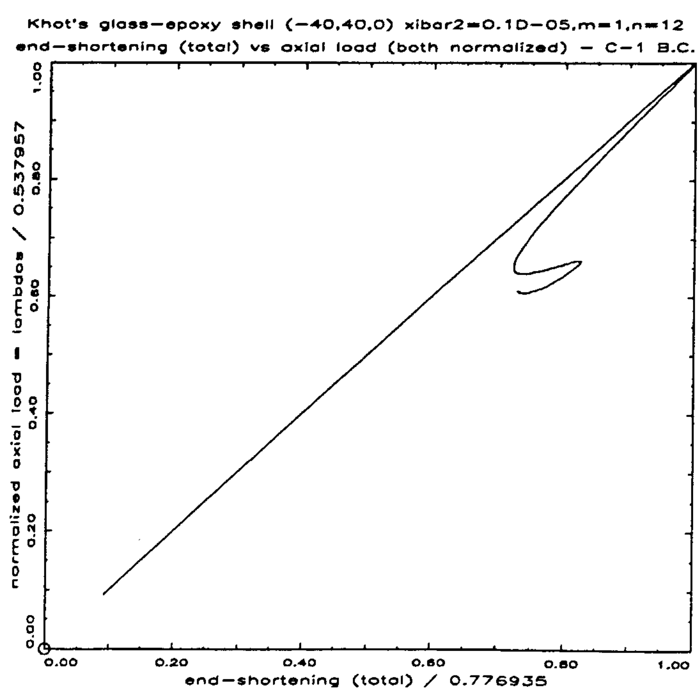


Fig. 14 Enlarged view of the limit point for SS-4 boundary conditions

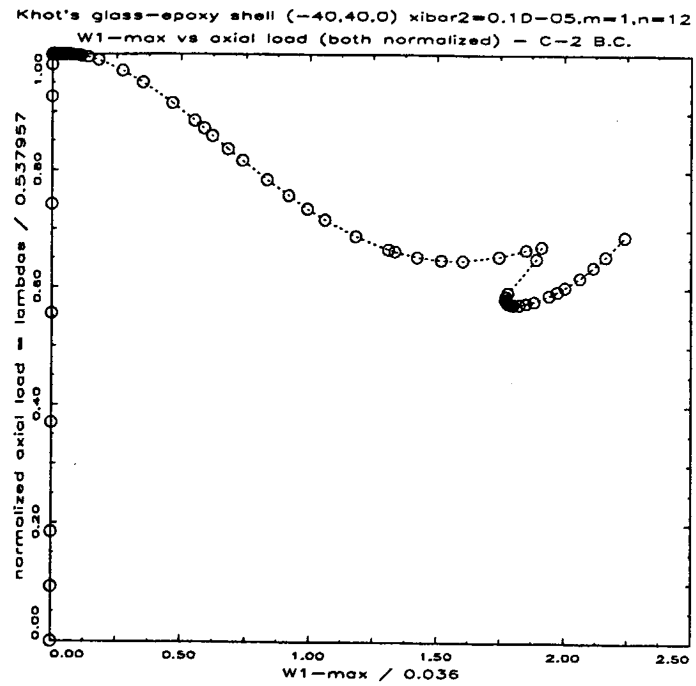


a. $w_{1,max}$ vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

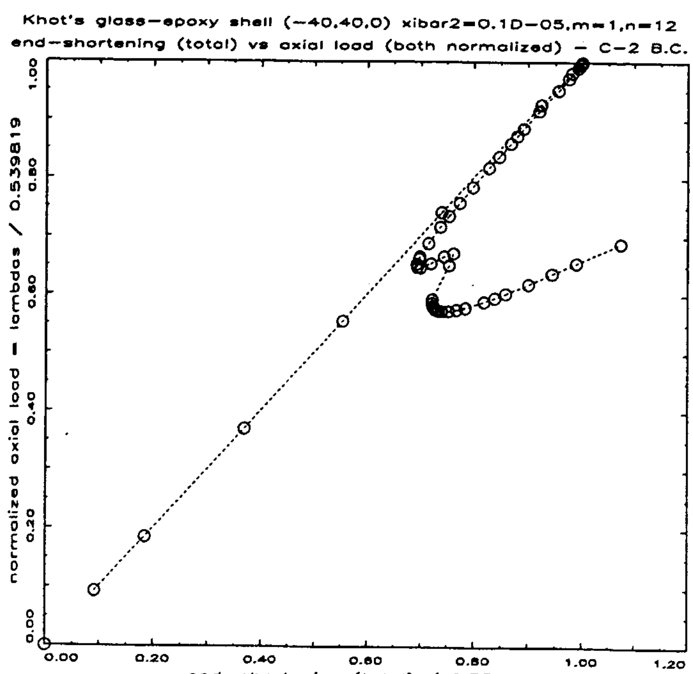


b. end-shortening vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

Fig. 15 Load vs deformation plots for C-1 boundary conditions ($N_x=-N_o$, $N_{xy}=w=w_{,x}=0$; $N_{c\ell}=-965.9814$ lb/in)

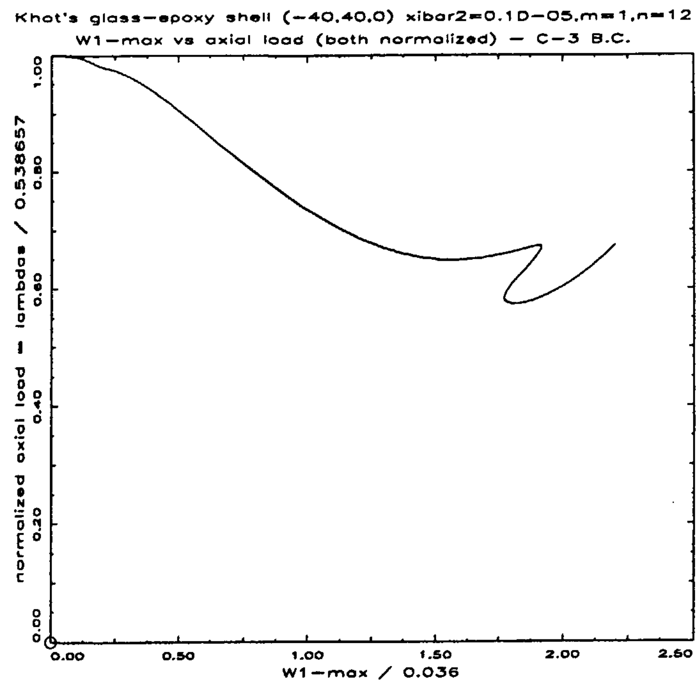


a. $w_{1,max}$ vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

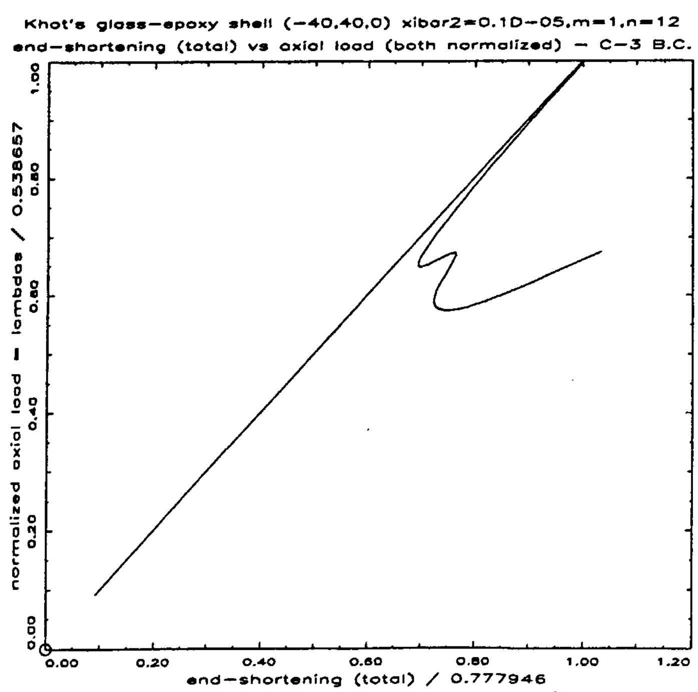


b. end-shortening vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

Fig. 16 Load vs deformation plots for C-2 boundary conditions
($u = u_o$, $N_{xy} = W = W_{,x} = 0$; $N_{c\ell} = -965.9814$ lb/in)

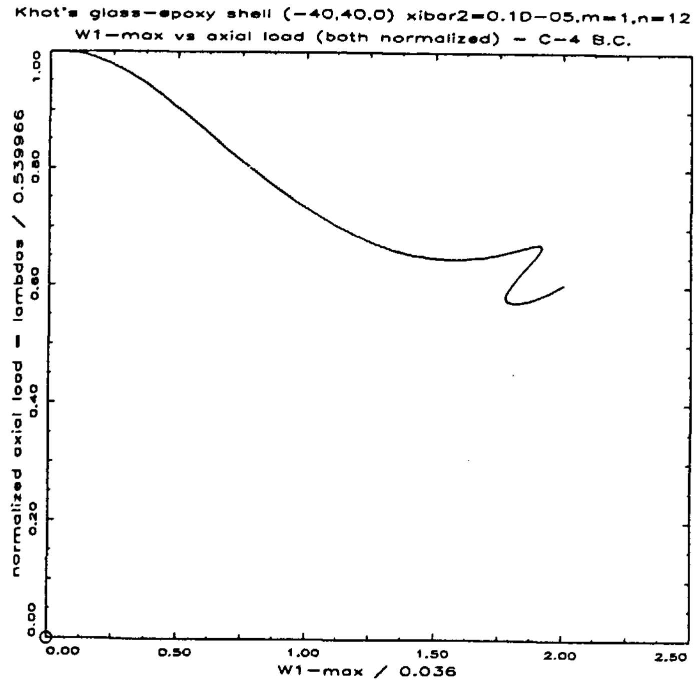


a. $w_{1,max}$ vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

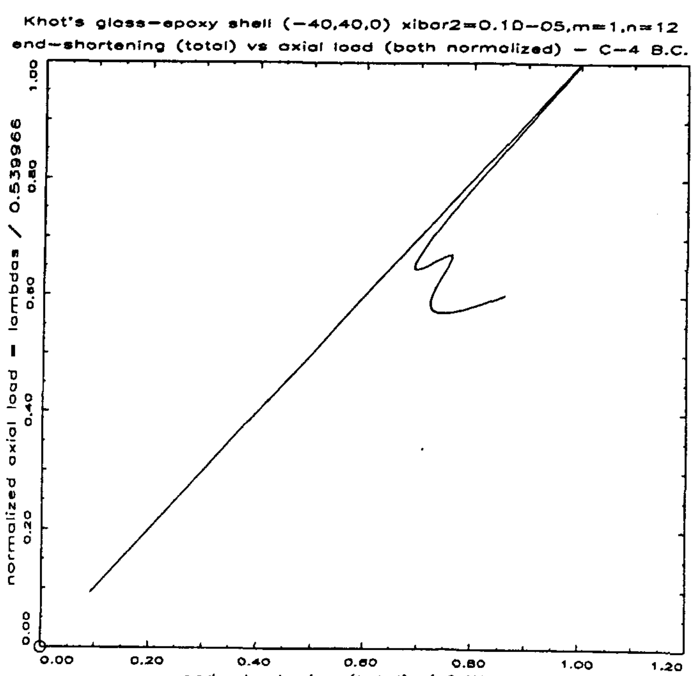


b. end-shortening vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

Fig. 17 Load vs deformation plots for C-3 boundary conditions
($N_x = -N_o$, $v = w = W_{,x} = 0$; $N_{c\ell} = -965.9814$ lb/in)



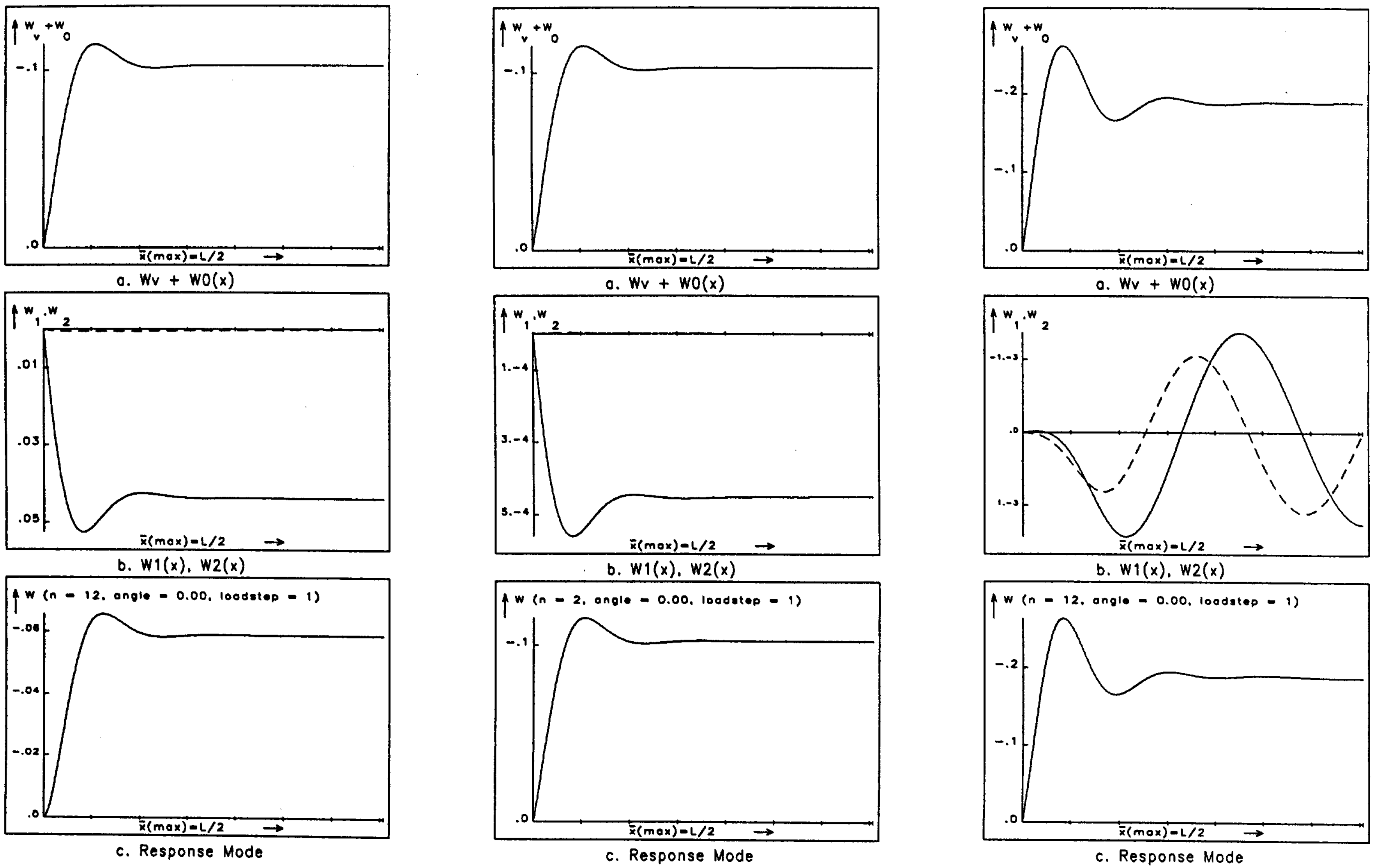
a. $w_{1,max}$ vs normalized axial load $\rho_s = \lambda_s/\lambda_c$



b. end-shortening vs normalized axial load $\rho_s = \lambda_s/\lambda_c$

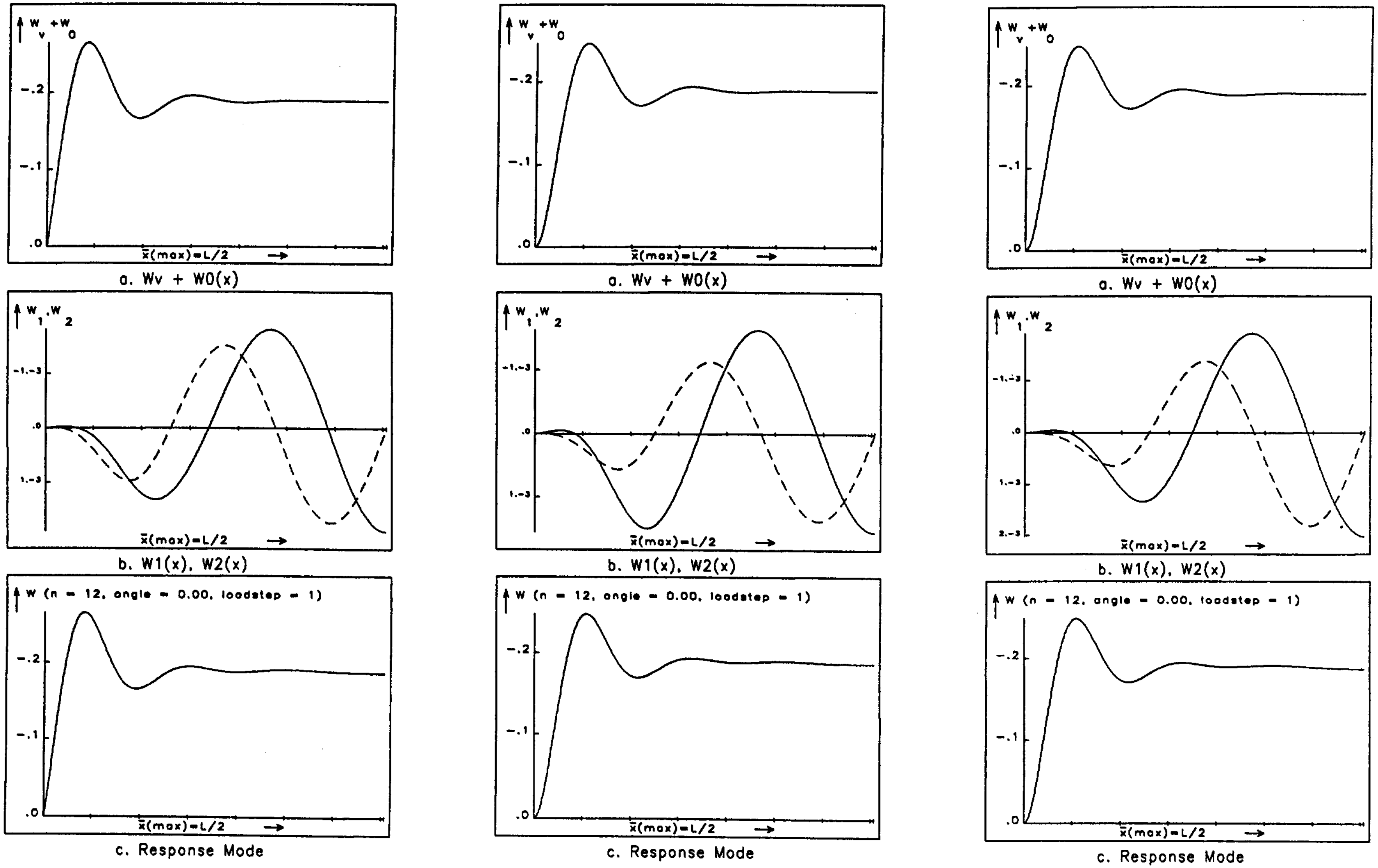
Fig. 18 Load vs deformation plots for C-4 boundary conditions

$(u = u_o,\ v = w = W_{,x} = 0\ ;\ N_{c\ell} = -965.9814$ lb/in$)$



a. SS-1 B.C. $\lambda_S = 0.292265$ (n=2)   b. SS-2 B.C. $\lambda_S \approx 0.293$ (n=2)   c. SS-3 B.C. $\lambda_S = 0.536361$ (n=12)

Fig. 19 Response mode shapes at the limit points for various boundary conditions

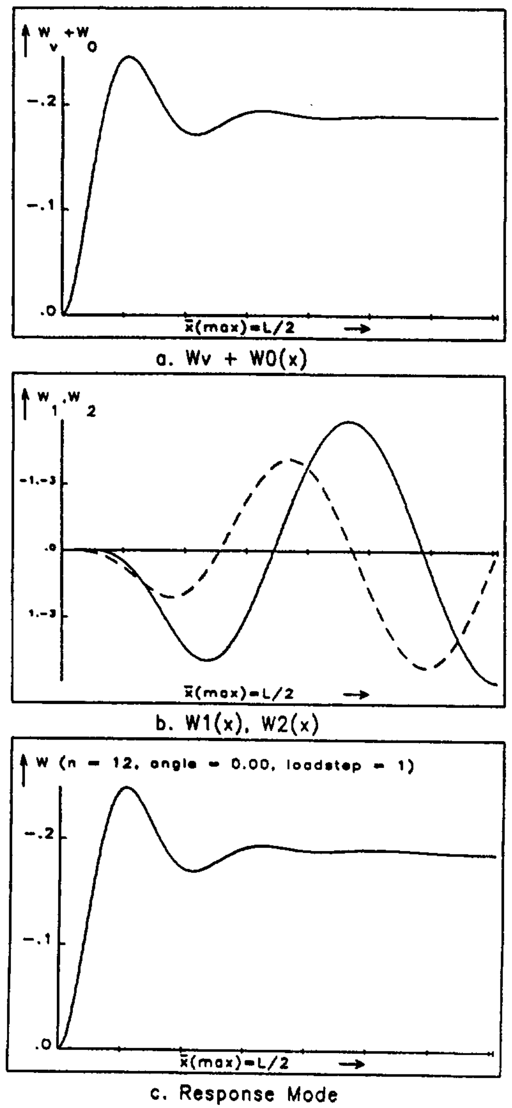


d. SS-4 B.C. $\lambda_S = 0.538999$ (n=12)

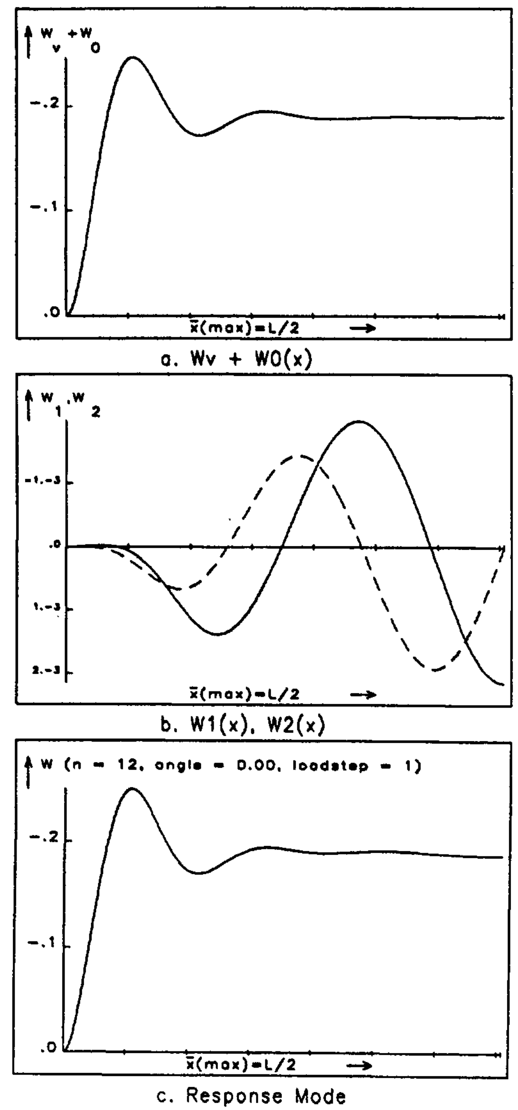
e. C-1 B.C. $\lambda_S = 0.537957$ (n=12)

f. C-2 B.C. $\lambda_S = 0.539819$ (n=12)

Fig. 19 Response mode shapes at the limit points for various boundary conditions (continuation)

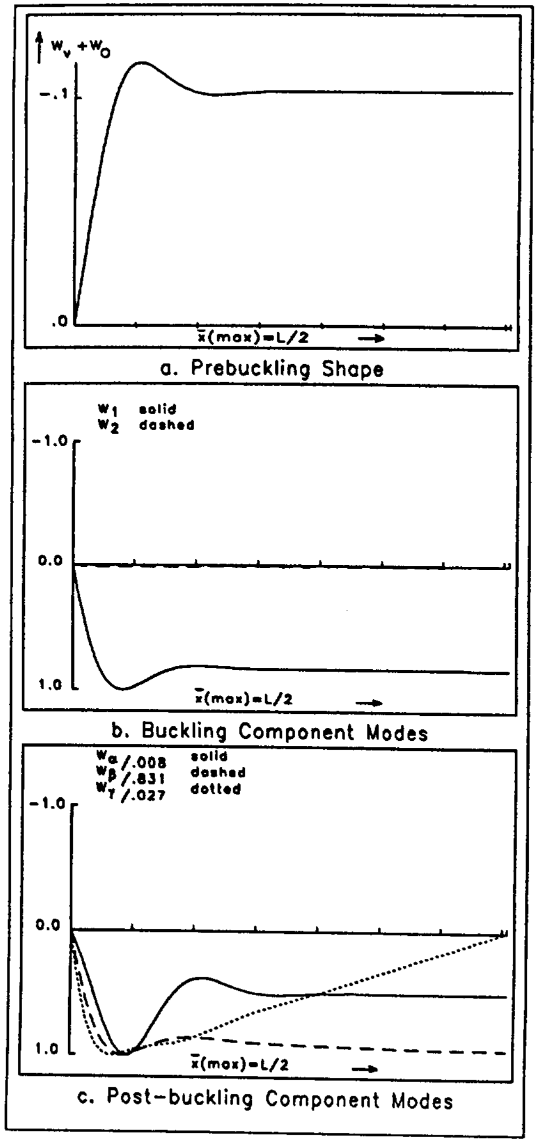


g. C-3 B.C. λ_S = 0.538657 (n=12)

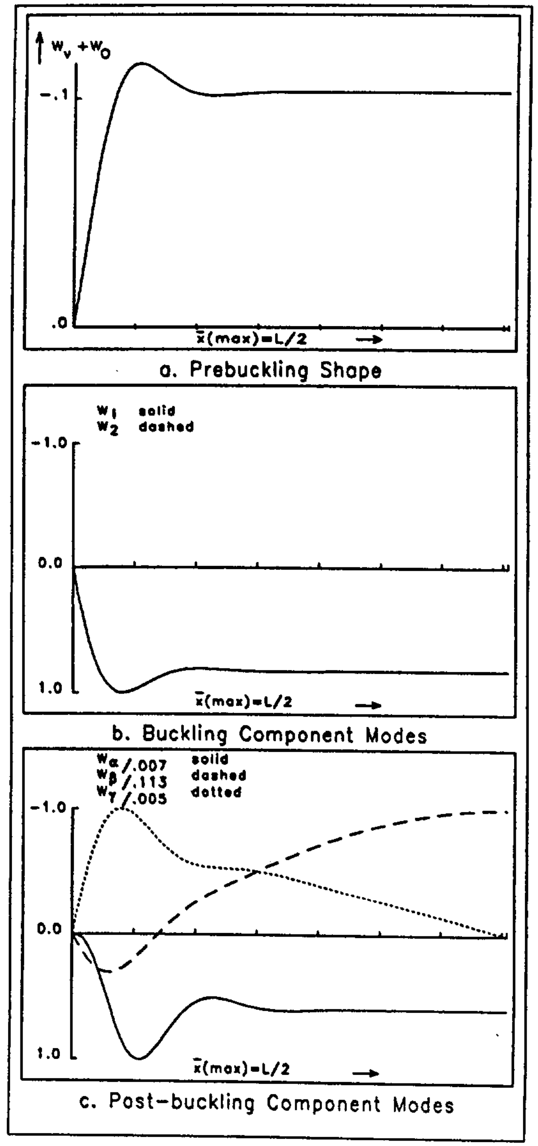


h. C-4 B.C. λ_S = 0.539966 (n=12)

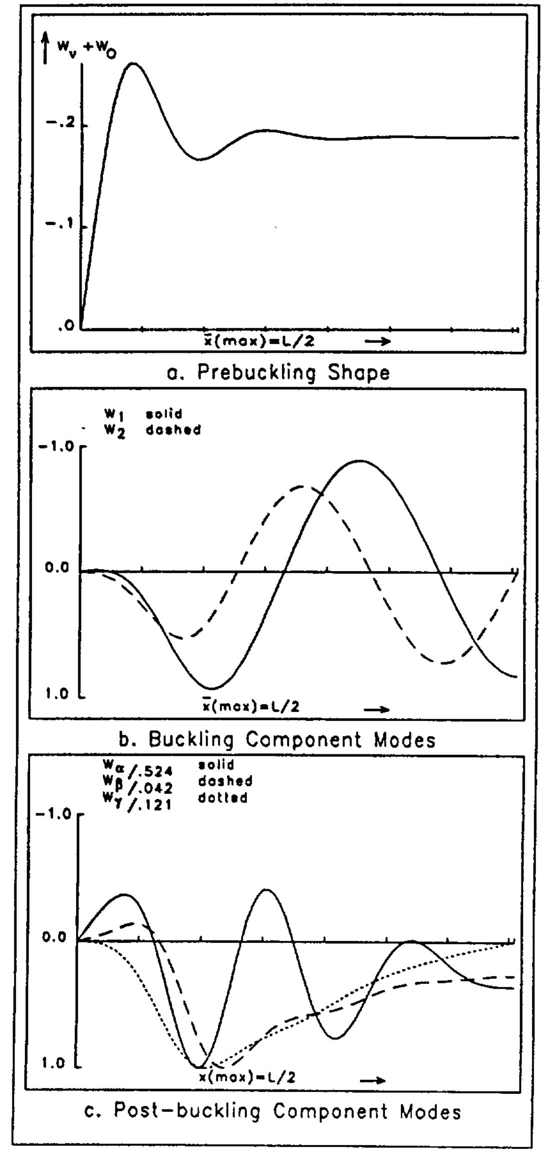
Fig. 19 Response mode shapes at the limit points for various boundary conditions (continuation)



a. SS-1 B.C. $\lambda_c^{nl} = 0.292266$ (n=2)

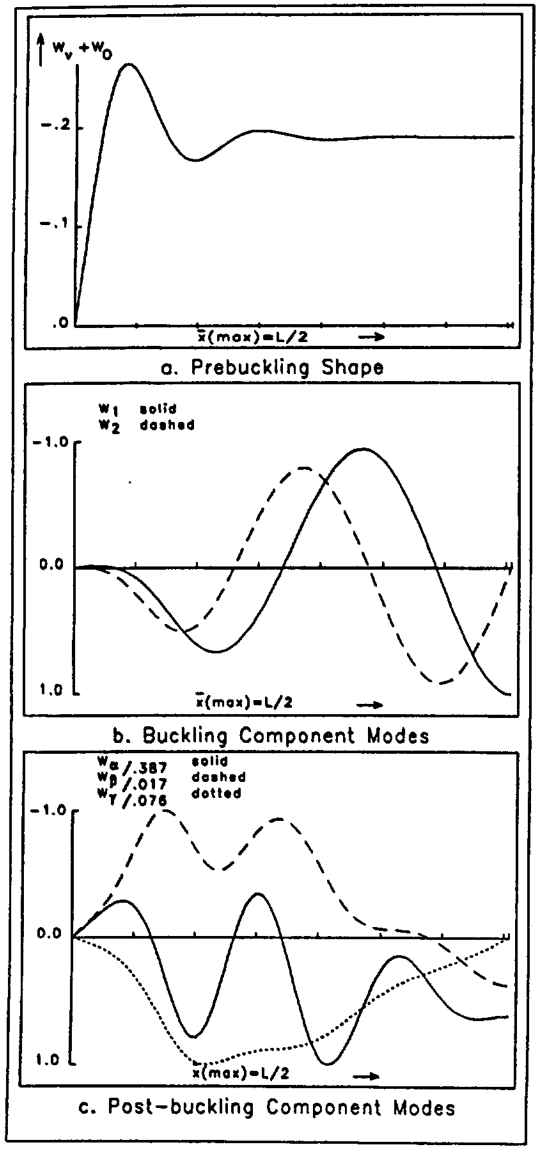


b. SS-2 B.C. $\lambda_c^{nl} = 0.292659$ (n=2)

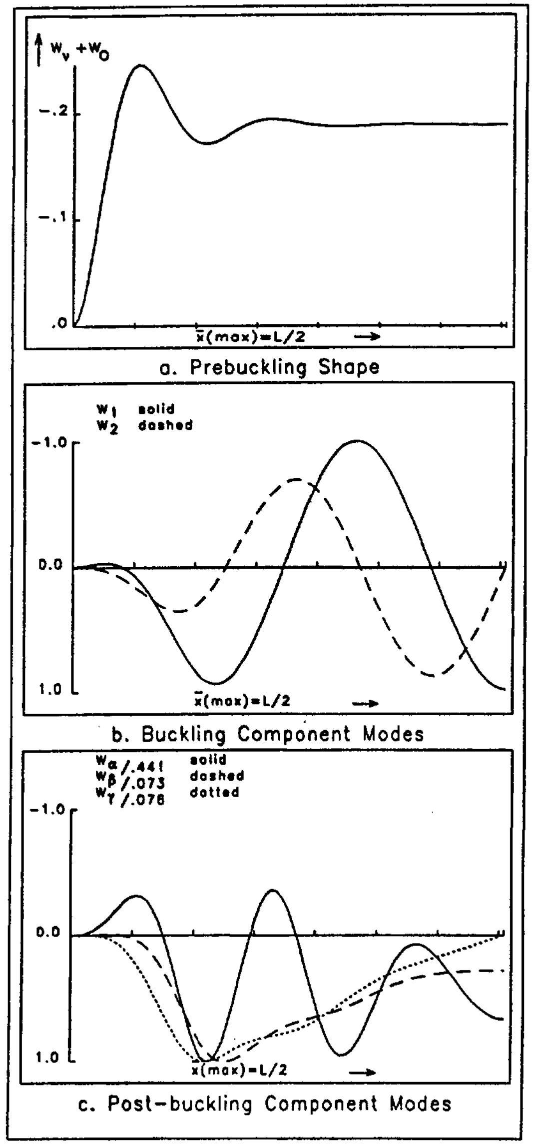


c. SS-3 B.C. $\lambda_c^{nl} = 0.536362$ (n=12)

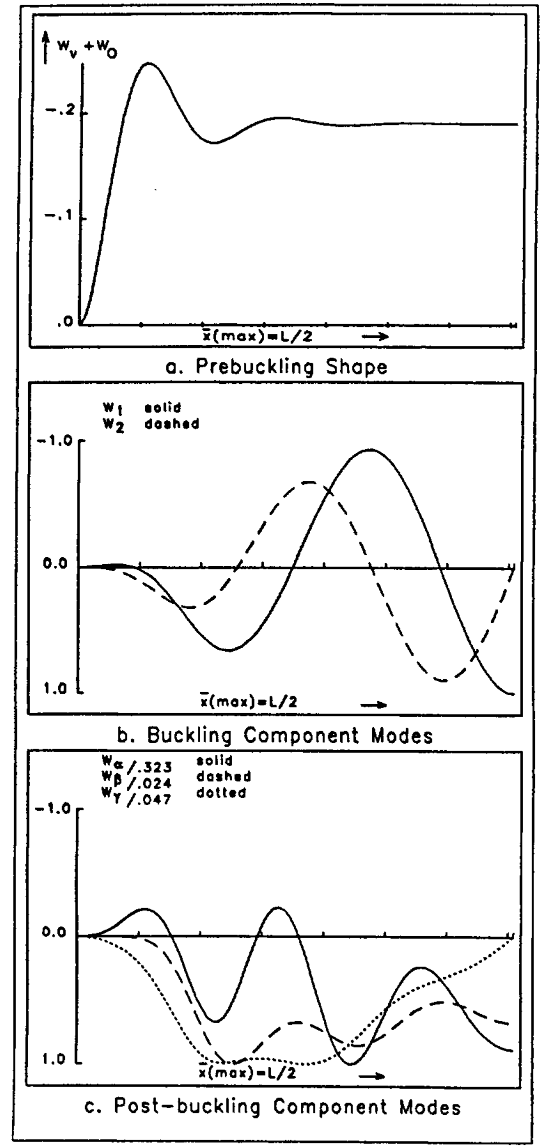
Fig. 20 Bifurcation buckling mode for various boundary conditions



d. SS-4 B.C. $\lambda_c^{nl} = 0.539065$ (n=12)

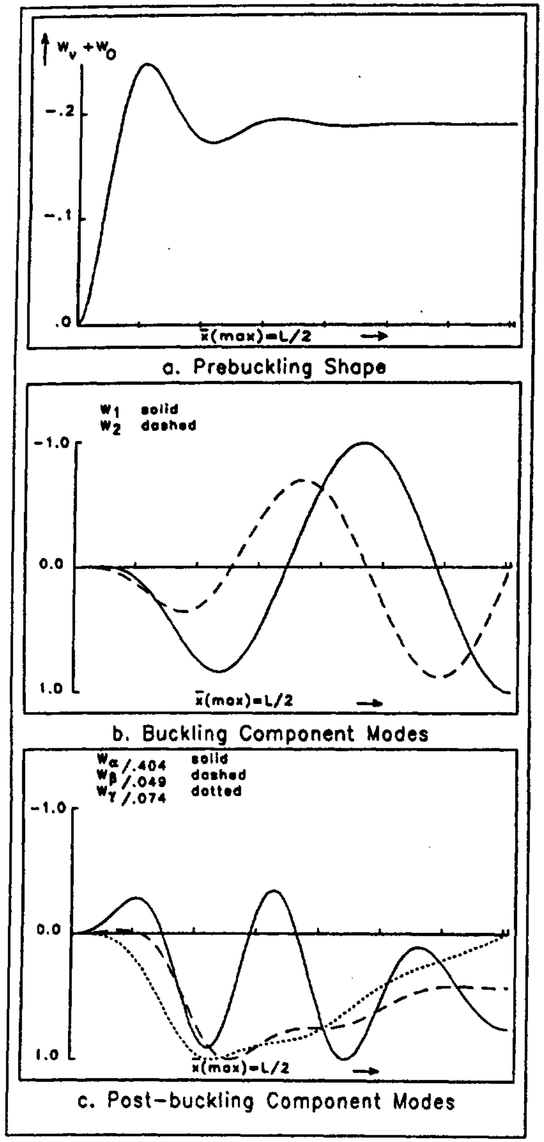


e. C-1 B.C. $\lambda_c^{nl} = 0.537958$ (n=12)

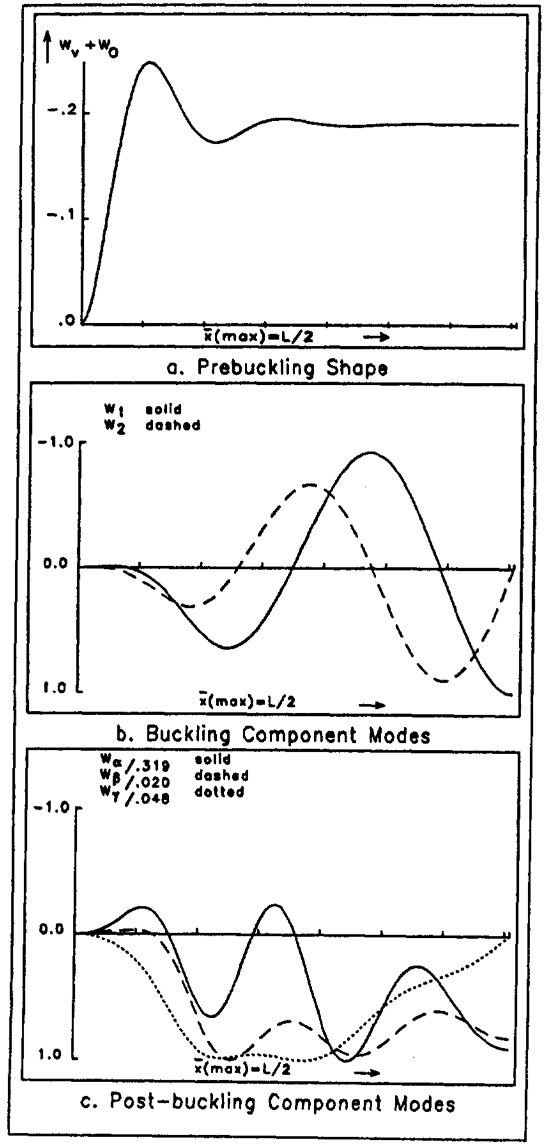


f. C-2 B.C. $\lambda_c^{nl} = 0.539820$ (n=12)

Fig. 20 Bifurcation buckling mode for various boundary conditions (continuation)



g. C-3 B.C. $\lambda_c^{nl} = 0.538658$ (n=12)



h. C-4 B.C. $\lambda_c^{nl} = 0.539966$ (n=12)

Fig. 20 Bifurcation buckling mode for various boundary conditions (continuation)

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