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ON THE COMPUTATION OF THE MOMENTS OF A POLYGON, WITH SOME APPLICATIONS

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A general formula for moments of a polygon is derived. The concept of 'moment' is a generalization of the statical, inertial and centrifugal moment respectively of a polygon, which as such are special moments of first and second order. Moments of higher order, however, also have applications in engineering mechanics. Some examples of these are given.

0 Introduction

Consider an area F which is given with reference to a Cartesian co-ordinate system (O, x, y). By the moments of this area are understood surface integrals of a term formed by the product of powers of the co-ordinates x and y whose exponents are non-negative integers. Expressed as a formula:

$$M_{p,q} = \iint_F x^p y^q \mathrm{d}x\mathrm{d}y \quad \ldots\ldots \quad (1)$$

The statical moments, the moments of inertia and the centrifugal moment of a section come within this definition, namely, for those values of p and q for which $(p+q) \leqslant 2$. Moments of higher order $[(p+q) > 2]$ may likewise have applications in engineering mechanics, and will be reverted to in Section 2.2.

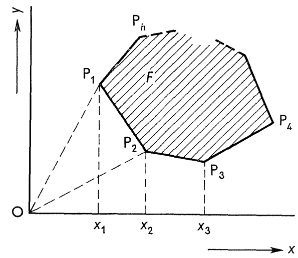
For working out the integral (1), GREEN's theorem in the plane, will be used, by means of which a surface integral (i.e., a double integral) can be transformed into a line integral (i.e., a single integral). In this way, partly by an analytical and partly by an inductive procedure, a general formula is obtained for moments of any order for a polygon. This formula is suitable for direct use in a computer. Dr. J. H. J. ALMERING subsequently found a proof for this formula. This proof, in an abridged form, is included in the present article.

In the further treatment of the subject, attention will in general be focused on a visual interpretation rather than on a rigorously mathematical approach.

1 The moments of a polygon

Consider a polygon, with corners $P_1, P_2, \ldots, P_h$, located in the co-ordinate plane Oxy (Fig. 1). Calculate the moments as defined by the surface integral (1). This surface integral will now be worked out by suitably dividing the integration area F. Two obvious methods suggest themselves.

Fig. 1. Given polygon F.



In the first place, the integration area F may be conceived as being composed of the triangles OP_1P_2, OP_2P_3, ..., OP_hP_1. The expression (1) can therefore be written as follows:

$$M_{p,q} = \sum_{i=1}^{i=h} \iint_{\Delta OP_iP_{i+1}} x^p y^q \mathrm{d}x\mathrm{d}y = \sum_{i=1}^{i=h} D_{i,i+1} \quad \ldots \quad (2)$$

The summation should of course be so interpreted that for $i = h$ must be substituted the contribution of the integral over the triangle OP_hP_1, that is to say, $D_{h,h+1} = D_{h,1}$. Without sacrifice of generality, it will suffice to determine the portion due to the triangle:

$$D_{1,2} = \iint_{\Delta OP_1P_2} x^p y^q \mathrm{d}x\mathrm{d}y \quad \ldots \quad (3)$$

whence the general term $D_{i,i+1}$ is obtained by simple substitution.

Secondly, the integration area F may be conceived as being composed of the trapeziums $x_1P_1P_2x_2$, $x_2P_2P_3x_3$, ..., $x_hP_hP_1x_1$, so that:

$$M_{p,q} = \sum_{i=1}^{i=h} \iint_{\substack{\text{trapezium} \\ x_iP_iP_{i+1}x_{i+1}}} x^p y^q \mathrm{d}x\mathrm{d}y = \sum_{i=1}^{i=h} T_{i,i+1} \quad \ldots \quad (4)$$

Here again the convention with regard to the summation must be that for $i = h$ the contribution of the integral over the trapezium $x_hP_hP_1x_1$ must be adopted, that is to say, $T_{h,h+1} = T_{h,1}$. Determination of the portion due to the trapezium:

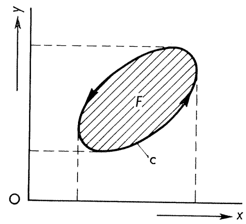
$$T_{1,2} = \iint_{\substack{\text{trapezium} \\ x_1P_1P_2x_2}} x^p y^q \mathrm{d}x\mathrm{d}y \quad \ldots \quad (5)$$

is here likewise sufficient for determining the general term $T_{i,i+1}$. In that case the problem has been reduced to a surface integral over the trapezium $x_1P_1P_2x_2$.

The next step consists in working out the surface integral (3) or (5).

One method of doing this is provided by GREEN's theorem in the plane, whereby a surface integral can be transformed into a contour integral [1]. This theorem appears in the literature in various forms, depending on the field of application concerned. The form best suited to the present purpose is given in the following.

Fig. 2. Diagram relating to GREEN's theorem.



1.1 *Green's theorem in the plane*

Consider an area F in the xy plane; this area is assumed to be bounded by a simple closed curve c (see Fig. 2). Let f and g denote functions of x and y which conform to certain requirements of regularity; then:

$$\oint_c (f\mathrm{d}x + g\mathrm{d}y) = \iint_F \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x\mathrm{d}y \quad \ldots \quad (6)$$

where the contour c is traversed in the positive direction (counterclockwise).

With the aid of the theorem a contour integral can be transformed into a surface integral, and vice versa. In the last-mentioned case the integrand v of the surface integral will be written as a derivative with respect to x of y:

$$v = \frac{\partial g}{\partial x}; \quad \iint_F v\,\mathrm{d}x\mathrm{d}y = \oint_c g\,\mathrm{d}y \quad \ldots \quad (6a)$$

$$v = -\frac{\partial f}{\partial y}; \quad \iint_F v\,\mathrm{d}x\mathrm{d}y = \oint_c f\,\mathrm{d}x \quad \ldots \quad (6b)$$

1.2 *Derivation of the general formula with the aid of Green's theorem in the plane*

In the present case a surface integral is required. In solving the problem the procedure embodied in the formula (6b) will be adopted, and the required quantity will be expressed in a contour integral. Comparison of the contours of the integration areas, the triangle OP_1P_2 and the trapezium $x_1P_1P_2x_2$, leads to adopting the latter as first choice, since it is simpler. The integrand v of the desired surface integral can be obtained by substitution of a suitably chosen function f into the formula (6b). In this case the functions are:

$$\left.\begin{aligned} v &= x^p y^q \\ f &= -\frac{1}{q+1} x^p y^{q+1} \end{aligned}\right\} \quad \ldots \quad (7)$$

The substitution yields the contour integral:

$$T_{1,2} = \oint_{x_1P_1P_2x_2} -\frac{1}{q+1} x^p y^{q+1} \mathrm{d}x \quad \ldots \quad (8)$$

However, of the trapezium contour $x_1P_1P_2x_2$ only the line segment P_1P_2 makes an effective contribution to the integral, since $y = 0$ along x_2x_1, while over the portions x_1P_1 and P_2x_2 the integration interval is of zero length. For the expression (8) it is therefore possible to write:

$$T_{1,2} = -\frac{1}{q+1}\int_{x_1}^{x_2} x^p y^{q+1}\mathrm{d}x \quad \ldots \quad (9)$$

The equation of the side P_1P_2 is:

$$\left.\begin{aligned} y &= Ax+B, \quad \text{where} \\ A &= \frac{y_2-y_1}{x_2-x_1} \text{ and } B = \frac{x_2y_1-x_1y_2}{x_2-x_1} \end{aligned}\right\} \quad \ldots \quad (10)$$

According to NEWTON's binomial theorem:

$$y^{q+1} = (Ax+B)^{q+1} = \sum_{k=0}^{k=q+1} \binom{q+1}{k} (Ax)^k B^{q+1-k}$$

so that (9) becomes:

$$T_{1,2} = -\frac{1}{q+1} \sum_{k=0}^{k=q+1} \binom{q+1}{k} A^k B^{q+1-k} \int_{x_1}^{x_2} x^{p+k}\mathrm{d}x \quad \ldots \quad (11)$$

Substitution of A and B from (10) into the integral (11), which is easy to work out, finally yields:

$$T_{1,2} = \frac{1}{(q+1)(x_1-x_2)^{q+1}} \sum_{k=0}^{k=q+1} \binom{q+1}{k} (y_1-y_2)^k (x_1y_2-x_2y_1)^{q+1-k} \times$$

$$\times \frac{1}{p+k+1}(x_1^{p+k+1}-x_2^{p+k+1}) \quad \ldots \quad (12)$$

From the form thus obtained one might suppose that the function is singular for $x_1 = x_2$. However, it is found that the division always terminates, resulting in a polynomial expression in the variables x_1, x_2, y_1 and y_2.This follows from the fact that if x_1 tends towards x_2, the trapezium portion remains finite; in fact, it tends to zero. This will be given further consideration a little later on.

With the aid of the formula for the trapezium portion $T_{1,2}$ (12) the triangle portion $D_{1,2}$ (3) can readily be calculated. To begin with, formula (12) is used in order to obtain trapezium portions for the connecting lines to the origin. In the summation only the term with $k = n$ is then retained. For the 'trapeziums' OP_1x_1 and Ox_2P_2 the following expressions are successively obtained:

$$\left.\begin{aligned} T_{0,1} &= -\frac{1}{(p+q+2)(q+1)}x_1^{p+1}y_1^{q+1} \\ T_{2,0} &= +\frac{1}{(p+q+2)(q+1)}x_2^{p+1}y_2^{q+1} \\ \text{Then:} \\ D_{1,2} &= T_{0,1}+T_{1,2}+T_{2,0} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (13)$$

We shall now revert to the fact $T_{1,2}$ (12) is in fact an integral function. $D_{1,2}$ is therefore a polynomial in the variables x_1, x_2, y_1 and y_2. It is, however, no easy matter to derive a general expression for this polynomial from (12) and (13) by an analytical procedure. Yet it has proved possible, after determining the coefficients of the said polynomial for a number of values of p and q by means of electronic computations, to establish this formula by an inductive procedure:

$$D_{1,2} = \frac{p!q!}{(p+q+2)!}(x_1y_2-x_2y_1)\sum_{i=0}^{i=p}\sum_{j=0}^{j=q}\binom{i+j}{i}\binom{p+q-i-j}{p-i}x_1^ix_2^{p-i}y_1^jy_2^{q-j} \qquad (14)$$

From (13) and (14) a form for the expression (12) as an integral function can also readily be found:

$$T_{1,2} = \frac{p!q!}{(p+q+2)!}(x_1-x_2)\sum_{i=0}^{i=p}\sum_{j=0}^{j=q+1}\binom{i+j}{i}\binom{p+q+1-i-j}{p-i}x_1^ix_2^{p-i}y_1^jy_2^{q+1-j} \qquad (15)$$

For the formula (14) dr. J. H. J. Almering had given a proof by working out the surface integral [2]. This proof will be given here in a somewhat abridged form.

1.3 *Another derivation and also proof of the general formula with the aid of gamma and beta functions*

In the previous section of this paper the desired surface integral was worked out by an indirect procedure by transforming it into a contour integral. The following direct method is due to dr. J. H. J. Almering, who conceived the idea of applying a transformation *before* working out the integral [2].

Consider, in the Oxy plane, the triangle OP_1P_2 with the corners $O(0,0)$, $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, as represented in Fig. 3a.

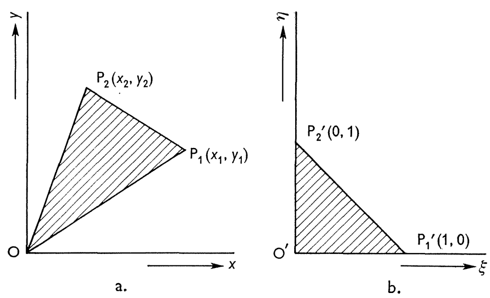
The moment $D_{1,2}$ of arbitrary order (p,q) for the triangle OP_1P_2 is, by definition:

$$D_{1,2} = \iint_{\Delta OP_1P_2} x^py^q\,\mathrm{d}x\mathrm{d}y$$

What we are required to prove is:

$$D_{1,2} = \frac{p!q!}{(p+q+2)!}(x_1y_2-x_2y_1)\sum_{i=0}^{i=p}\sum_{j=0}^{j=q}\binom{i+j}{i}\binom{p+q-i-j}{p-i}x_1^ix_2^{p-i}y_1^jy_2^{q-j}$$

Fig. 3. Transformation of the triangle OP_1P_2 in the Oxy plane into triangle $O'P_1'P_2'$ in the $O'\xi\eta$ plane.



The linear transformation written in matrix form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (16)$$

is a single-valued mapping of the Oxy plane in the $O'\xi\eta$ plane. The triangle OP_1P_2 is transformed into the triangle $O'P_1'P_2'$ with corners $O'(0,0)$, $P_1'(1,0)$ and $P_2'(0,1)$, as represented in Fig. 3b. The desired surface integral will then be:

$$D_{1,2} = \iint_{\Delta OP_1P_2} x^p y^q \mathrm{d}x\mathrm{d}y = \iint_{\Delta O'P_1'P_2'} (x_1\xi + x_2\eta)^p (y_1\xi + y_2\eta)^q \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathrm{d}\xi\mathrm{d}\eta \quad \ldots \quad (17)$$

where $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$ denotes the JACOBI functional determinant associated with the transformation (16). On writing out the integrand in the form represented by NEWTON's binomial theorem we obtain:

$$D_{1,2} = \iint_{\Delta O'P_1'P_2'} \sum_{i=0}^{i=p} \binom{p}{i} (x_1\xi)^i (x_2\eta)^{p-i} \sum_{j=0}^{j=q} \binom{q}{j} (y_1\xi)^j (y_2\eta)^{q-j} (x_1y_2 - x_2y_1) \mathrm{d}\xi\mathrm{d}\eta =$$

$$= (x_1y_2 - x_2y_1) \sum_{i=0}^{i=p} \sum_{j=0}^{j=q} \binom{p}{i} \binom{q}{j} x_1^i x_2^{p-i} y_1^j y_2^{q-j} \iint_{\Delta O'P_1'P_2'} \xi^{i+j} \eta^{p+q-i-j} \mathrm{d}\xi\mathrm{d}\eta \quad (18)$$

On further working out the integral we find:

$$\iint_{\Delta O'P_1'P_2'} \xi^{i+j} \eta^{p+q-i-j} \mathrm{d}\xi\mathrm{d}\eta =$$

$$= \int_0^1 \xi^{i+j} \left\{ \int_0^{1-\xi} \eta^{p+q-i-j} \mathrm{d}\eta \right\} \mathrm{d}\xi =$$

$$= \frac{1}{p+q-i-j+1} \int_0^1 \xi^{i+j} (1-\xi)^{p+q-i-j+1} \mathrm{d}\xi \quad \ldots\ldots\ldots \quad (19)$$

which is found to be the beta function. The expression (19) now becomes:

$$\frac{1}{p+q-i-j+1}\,B(i+j+1,p+q-i-j+2) \quad \ldots\ldots\ldots\ldots \quad (20)$$

which, as is known, can also be written with the aid of gamma functions:

$$\frac{1}{p+q-i-j+1}\cdot\frac{\Gamma(i+j+1)\Gamma(p+q-i-j+2)}{\Gamma(p+q+3)} \quad \ldots\ldots\ldots\ldots \quad (21)$$

Since p and q are integers and $\geqslant 0$, the expression (21) becomes:

$$\frac{1}{p+q-i-j+1}\cdot\frac{(i+j)!\;(p+q-i-j+1)!}{(p+q+2)!} \quad \ldots\ldots\ldots\ldots \quad (22)$$

Substitution of (22) into (18), followed by some rearrangement, yields the expression which it was desired to prove.

2 Some applications in engineering mechanics

2.1 *Section properties for a polygonal section*

Application of the method, described in the foregoing, to the section properties of a polygonal section yields results as given in Table 1. The first column indicates the usual notation for the quantities considered, the second column indicates the moment notation, and the third column indicates the 'triangle portion' for the side P_1P_2. The factor $(x_1y_2-x_2y_1)$ which occurs in all the formulae is designated by the letter d (of 'determinant'). For each quantity the following expression is again valid:

$$M_{p,q} = \iint_F x^p y^q \,\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{i=h} D_{i,i+1}$$

(the point P_{h+1} is identical with the point P_1)

Table 1

F	$M_{0,0}$	$^1/_2(x_1y_2-x_2y_1) = {}^1/_2d$
S_x	$M_{0,1}$	$^1/_6d(y_1+y_2)$
S_y	$M_{1,0}$	$^1/_6d(x_1+x_2)$
I_x	$M_{0,2}$	$^1/_{12}d(y_1{}^2+y_1y_2+y_2{}^2)$
I_y	$M_{2,0}$	$^1/_{12}d(x_1{}^2+x_1x_2+x_2{}^2)$
C_{xy}	$M_{1,1}$	$^1/_{24}d\{x_1(2y_1+y_2)+x_2(y_1+2y_2)\}$

As usual, S_y denotes the statical moment about the y-axis, i.e., the integral of $x\mathrm{d}x\mathrm{d}y$.

In technical applications it is generally desired to determine the moments of inertia with respect to axes through the centroid and to determine the position of the principal axes of inertia and the magnitude of the relevant principal moments of inertia. The transformations needed for this are elementary ones.

The formulae may also be used for multiply-connected sections. In that case an imaginary cut is applied from a corner on an external perimeter to a corner on an internal perimeter; the section can then be considered as singly-connected.

2.2 *Possible applications to moments of higher order*

In the following it will be shown, with reference to some examples, that the calculation procedure also presents possibilities of application in engineering mechanics with regard to moments of higher than second order. (The second order suffices for establishing the various section properties envisaged above.)

2.2.1 Analysis of concrete cross-section at failure

In the Netherlands Code of Practice for Reinforced Concrete it is specified that the analysis of sections in flexure in the stage of failure must be based (among other things) on the assumption that the compressive stress/strain-diagram is the symmetric half of a second-degree parabola with the maximum compressive concrete stress σ_u' as the maximum value (at the vertex). It is furthermore assumed that the concrete does not resist tensile stresses.

If the section is of a complex shape, or if bending occurs about two axes, the calculation of the resultant compressive force in the concrete and of the bending moments as integrals of stresses is quite laborious.

By way of example a rectangular section whose axes of symmetry are intersected obliquely by the neutral axis P_2P_3 will be considered (Fig. 4). Let the equation of P_2P_3 (with regard to the axes of symmetry of the section) be:

$$x - 2y - 30 = 0$$

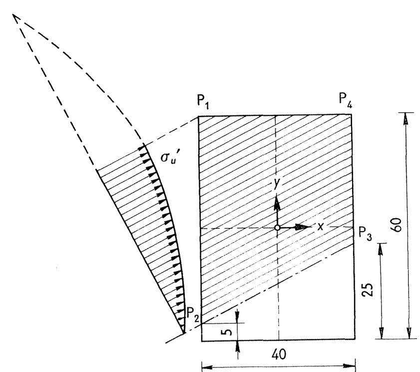


Fig. 4. Concrete section at failure (biaxial bending).

At the point $P_1(-20, 30)$ the compressive concrete stress attains the extreme value σ_u'. The stress distribution can be represented by a parabolic cylinder for which the line P_2P_3 constitutes one intersection with the zero plane. The other zero intersection (fictitious) is formed by a line which is symmetrically located with respect to the point P_1 and therefore extends parallel to the first-mentioned intersection line; its equation is:

$$x-2y+190 = 0$$

The compressive stress distribution can now be written as:

$$\begin{aligned}\sigma' &= c(x-2y-30)\,(x-2y+190)\\ &= c(x^2-4xy+4y^2+160x-320y-5700)\end{aligned}$$

Hence the stress at P_1 ($x = -20$, $y = +30$) is:

$$\sigma' = \sigma_u' = -12100c, \text{ so that } c = -\sigma_u'/12100$$

In general: $\sigma' = cL_1L_2$, where $L_1 = 0$ and $L_2 = 0$ are the equations of the zero intersection lines of the parabolic cylinder representing the stress distribution.

The magnitude and position of the compression resultant are determined from:

$$N' = \iint_F \sigma' \mathrm{d}x\mathrm{d}y$$

$$e_{0,x} \cdot N' = \iint_F x\sigma' \mathrm{d}x\mathrm{d}y$$

$$e_{0,y} \cdot N' = \iint_F y\sigma' \mathrm{d}x\mathrm{d}y$$

In the present example:

$$N' = c(M_{2,0}-4M_{1,1}+4M_{0,2}+160M_{1,0}-320M_{0,1}-5700M_{0,0}) = 1075\sigma_u'$$

$$e_{0,x} \cdot N' = c(M_{3,0}-4M_{2,1}+4M_{1,2}+160M_{2,0}-320M_{1,1}-5700M_{1,0}) = -2.4N'$$

$$e_{0,y} \cdot N' = c(M_{2,1}-4M_{1,2}+4M_{0,3}+160M_{1,1}-320M_{0,2}-5700M_{0,1}) = 13.1N'$$

As appears from the above, the ultimate load analysis of concrete sections involves third-order moments. For a compressive zone of polygonal shape these moments can be determined by means of the method described in Section 1 of this article.

2.2.2 The membrane analogy

PRANDTL's membrane analogy can be applied to the problem of determining the shear stress distribution in a section of a beam in the case of pure SAINT VENANT torsion [4]. This analogy is of great value, not only because of the possibility of experimental determination with the aid of a soap film, but also as a helpful mental concept.

Consider a bar of arbitrary quadrangular cross-sectional shape subjected to torsion (Fig. 5). The analogous membrane is attached to a perimeter of that same shape. When it is deflected, the membrane has intersections with the zero plane which coincide with the sides of the quadrangle. On the assumption that the deflection can be represented as a power series in x and y, it must have the following form:

$$w = L_1L_2L_3L_4(c_1+c_2x+c_3y+\ldots),$$

if the sides conform to the equations $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$. At the sides the deflections will then automatically be zero.

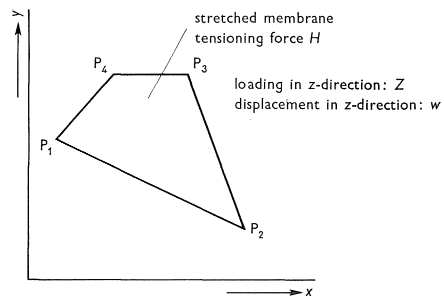


Fig. 5. Membrane as analogy for a torsionally loaded beam of arbitrary quadrangular cross-section.

Determining the deflection surface consists, in effect, in solving the potential equation. In one of the approximate methods suitable for the purpose a deflection surface is assumed conforming to a finite power series of the above-mentioned form. The coefficients c_1, c_2, etc. are so determined that the potential energy becomes a minimum [6]. This method is due to RITZ. The potential energy $\mathbf{P}$ of a deflected membrane with tensile prestress H in all directions and subjected to a loading Z is given by the expression:

$$\mathbf{P} = \iint_F \left[\tfrac{1}{2}H\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} - Zw\right]\mathrm{d}x\mathrm{d}y$$

If the deflection is assumed to conform to a power series, the integral will consist of a number of moments of higher order. We next determine the derivatives with respect to the various coefficients, needed for writing down the minimum condition.

After the coefficients have been determined by solving the resulting set of linear equations, it is necessary to perform one more integration if the so-called torsional moment of inertia I_w is required, which is proportional to the volume comprised between the deflected membrane and its initial plane:

$$I_w = \frac{4H}{Z} \iint_F w \, dx dy$$

By way of example the torsion of a bar of rectangular cross-section will be considered (Fig. 6). As an assumption for the membrane deflection a simple series with only one free constant will first be introduced:

$$w = c_1 x(x-a)\, y\, (y-b)$$

The operation of working out the condition of minimum potential energy nevertheless involves determining moments of sixth order.

The final result is found to be:

$$I_w = \frac{5}{18} \cdot \frac{a^3 b^3}{a^2+b^2}$$

In Fig. 6 this function has been plotted as a graph and compared with almost exact results obtained by other methods and published in the literature [5]. The results are relatively of the greatest accuracy if the lengths of the sides of the rectangle do not differ too much from one another; in the range of dimensional ratios indicated, the relative error is not more than 3%.

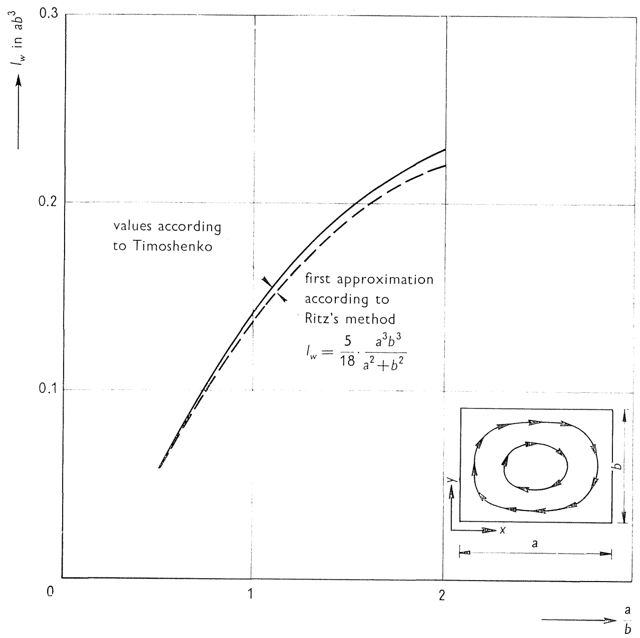


Fig. 6. Torsional stiffness of a beam of rectangular cross-section.

The accuracy can be fairly rapidly increased by the use of a more elaborately developed series as the assumption for the deflection. Thus with:

$$w = x(x-a)y(y-b)\{c_1+c_2(x-\tfrac{1}{2}a)^2+c_3(y-\tfrac{1}{2}b)^2\}$$

the result obtained is:

$$I_w = \frac{14}{9}\cdot\frac{a^3b^3}{a^2+b^2}\cdot\frac{9a^4+82a^2b^2+9b^4}{45a^4+464a^2b^2+45b^4}$$

In calculating this, moments up to and including the tenth order were worked out. In this case the relative error is found to be about 0.1% in the range indicated in Fig. 6, the discrepancy now being so small that it is not possible to show it in the graph.

It thus appears that RITZ's method yields sufficiently accurate results, provided, that one is prepared to calculate moments of high order. So here, too, there is scope for applying the formulae given in Section 1, which enable sections of more complex shape also to be dealt with. There is, however, a general restriction in that RITZ's method, in the form discussed, is applicable only to *convex singly-connected* sections.

2.2.3 Deflection of a clamped plate

Besides the potential equation the bipotential equation also (and even to a greater extent) plays a part in engineering mechanics, e.g., in the theory of flexurally rigid plates.

Problems in this field of application may, inter alia, be treated by means of RITZ's method [7, 10], for which purpose the following expression for the potential energy is adopted:

$$\mathbf{P} = \iint_F \left[\tfrac{1}{2}K\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} - Zw\right] \mathrm{d}x\mathrm{d}y$$

where $K = Et^3/12(1-\nu^2)$ is the stiffness of the plate and Z and w are the loading and the displacement in the z-direction respectively. If forces and moments, acting at the edge, do work during the deformation, then this work must be deducted when calculating the potential energy. The geometric boundary conditions are taken into account by choosing the assumption as to the deflection surface in such a manner as to correspond to them.

By way of example a rectangular plate will be considered. If the plate is clamped (rigidly restrained) along an edge that coincides with the y-axis, this fact can be taken into account by including a factor x^2 in the expression for the deflection (then at the edge both w and $\partial w/\partial x$ will be zero). If the plate is clamped on all four sides, the assumption will be at least:

$$w = c_1x^2(x-a)^2y^2(y-b)^2$$

or an expression of higher degree if, in order to obtain greater accuracy, the factor c_1

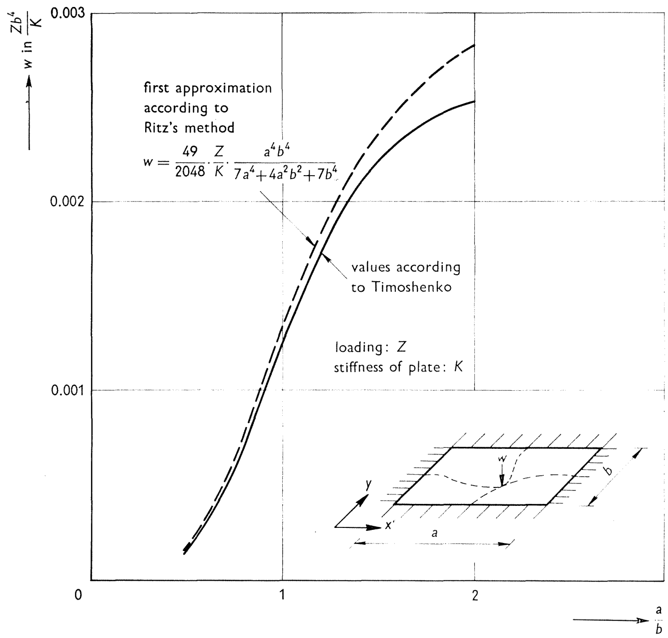


Fig. 7. Deflection at the centre of a rectangular plate, clamped on all four sides, under uniformly distributed loading.

is replaced by a polynomial with several constants. If the above assumption is adopted, twelfth-order moments will have to be determined; the final result will include the following value for the deflection at the centre:

$$w = \frac{49}{2048} \cdot \frac{Z}{K} \cdot \frac{a^4 b^4}{7a^4 + 4a^2 b^2 + 7b^4}$$

In Fig. 7 this result is compared with results obtained by different methods [9]. In the range represented in the diagram the relative error may be up to 12%. It can, however, be reduced to 0.4% by extending the assumed expression for the deflection [8, 11] and performing the necessary calculation of moments up to the 20th order.

2.2.4 Deformation of a square plate element in its own plane

A square element (with sides l, thickness t, and modulus of elasticity E) is elastically deformed in such a manner that the edges undergo the following displacements (see Fig. 8):

$$x = 0: \quad u = 0 \quad \text{and} \quad v = 0$$

$$x = l: \quad u = \frac{h}{l} y \quad \text{and} \quad v = 0$$

$$y = 0: \quad u = 0 \quad \text{and} \quad v = 0$$

$$y = l: \quad u = \frac{h}{l} x \quad \text{and} \quad v = 0$$

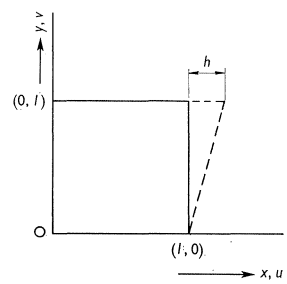


Fig. 8. The deformation of a square plate-element as considered.

Determine the energy needed for this, calculating it by RITZ's method.

Without detriment to the generality of the problem, we can base ourselves on the case $l = 1$, $h = 1$. The result obtained will afterwards merely have to be multiplied by h^2 to give the desired value for the energy. The specified displacements of *all the edges* can now be described by $u = xy$ and $v = 0$.

The following assumption for the displacement field of the plate will automatically satisfy the boundary conditions:

$$u = xy + x(x-1)y(y-1)\sum_{i=0}^{i=m}\sum_{j=0}^{j=m-i} c_{ij}x^i y^j$$

$$v = \quad x(x-1)y(y-1)\sum_{i=0}^{i=m}\sum_{j=0}^{j=m-i} d_{ij}x^i y^j$$

where c_{ij} and d_{ij} are constants to be further determined, and m is the degree chosen for the polynomial which is given by the summation form. By differentiation we obtain from this the strains $\varepsilon_x = \partial u/\partial x$, $\varepsilon_y = \partial v/\partial y$ as well as the angular rotation $\gamma = \partial u/\partial y + \partial v/\partial x$, which must be substituted into the expression for the strain energy, which in this case is identical with the potential energy. On the assumption of a plane state of stress, the expression for the energy is:

$$\mathbf{P} = \iint_F \frac{Et}{1-\nu^2}\left[\tfrac{1}{2}(\varepsilon_x+\varepsilon_y)^2 - (1-\nu)(\varepsilon_x\varepsilon_y - \tfrac{1}{4}\gamma^2)\right]\mathrm{d}x\mathrm{d}y$$

where E is the modulus of elasticity, F is the area of the element, and ν is POISSON's ratio. By imposing minimum conditions upon this expression for the energy a set of linear equations is obtained, from which the unknown constants c_{ij} and d_{ij} can be solved. The coefficients of this set of linear equations are moments of higher order.

Substitution of the values found for c_{ij} and d_{ij}, and once more carrying out the integration, yields the energy that we wished to determine. This, too, entails the calculation of moments of higher order.

The results of the above calculation are given in Table 2. The degree of the polynomial employed is at least 2 (no free constants) and is otherwise equal to $(m+4)$. A polynomial of odd degree provides no improvement over the polynomial of the next lower even degree. This is bound up with the symmetry of the element.

It should be noted that the approximation with a second-degree polynomial is quite usual in the finite element method. The error thus committed is of the same order of magnitude as other errors which are unavoidably associated with the method and which it is endeavoured to keep within acceptable limits by choosing small elements. Few numerical data concerning the magnitude of the error are available. For $\nu = 0$ one value is to be found in the literature, namely, the same as was found here [3]. The error increases with increasing value of ν, but need not give rise to anxiety, as is apparent from Fig. 9.

Table 2

degree of the polynomial	number of the Ritz constants	P/Eth^2			
		$\nu = 0$	$\nu = \frac{1}{6}$	$\nu = \frac{1}{3}$	$\nu = \frac{1}{2}$
2	0	0.25	0.242857	0.25	0.277778
4	1	0.244213	0.234279	0.236979	0.256944
6	6	0.244068	0.233998	0.236414	0.255727
8	12	0.244053	0.233968	0.236351	0.255583
extrapolation		0.244049	0.233961	0.236336	0.255547

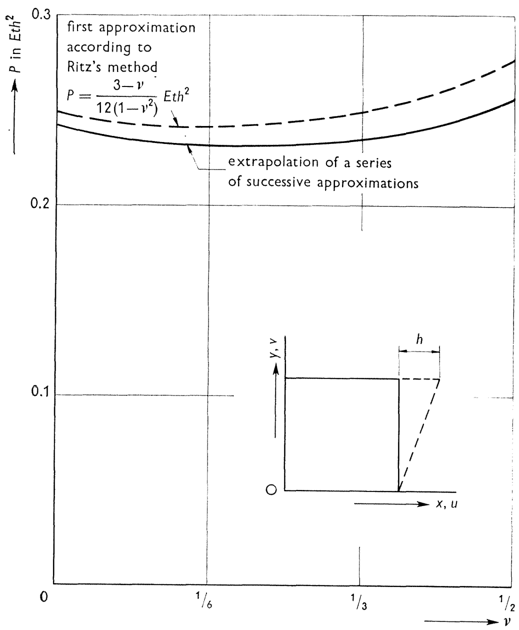


Fig. 9. Strain energy of a square plate-element.

3 Concluding remarks

As appears from the last few examples, the generalization of statical, inertial and centrifugal moments to moments of arbitrary order can yield fruitful results.

The formula that has been found is suitable for direct programming for processing by electronic computers. A standard programme for the computation of section properties according to the method described here was prepared, and favourable experience was gained with it.

In the application of RITZ's method to cases such as those discussed in the foregoing there are in general more problems than merely the calculation of moments of arbitrary order. In the case of a polygon with a large number of sides the degree of the polynomials to be dealt with is high, and a major proportion of the computation work consists in the manipulation of these polynomials. This is more particularly so if the section is not convex or is indeed multiply-connected, when the subdivision of the polygon into portions and the satisfying of continuity conditions constitutes a problem in itself.

Nevertheless, the automatic performance of the integrations is an important contribution to the solution of the overall problem, and the general formula has in this context, too, proved very useful.

4 Acknowledgment

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