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Hytönen, Tuomas; van Neerven, Jan; Veraar, Mark; Weis, Lutz

DOI
10.1007/978-3-031-46598-7_4

Publication date
2023

Document Version
Final published version

Published in
Ergebnisse der Mathematik und ihrer Grenzgebiete

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Function spaces

This chapter presents an in-depth study of several classes of vector-valued function spaces defined by smoothness conditions. In Volume I we have already encountered two such classes: the Sobolev spaces $W^{s,p}(\mathbb{R}^d;X)$ for $s \in \mathbb{N}$ and $s \in (0,1)$ (Chapter 2) and the Bessel potential spaces $H^{s,p}(\mathbb{R}^d;X)$ for $s \in \mathbb{R}$ (Chapter 5). Both classes are parametrised by an integrability parameter p and smoothness parameter s. The present chapter introduces two related classes of function spaces, the Besov spaces $B^s_{p,q}(\mathbb{R}^d;X)$ and the Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d;X)$. From the point of view of applications these spaces play an important role in the theory of partial differential equations, where they typically occur as trace spaces associated with initial value problems. What makes these spaces interesting from a mathematical point of view is the wealth of different characterisations of these classes: they can equivalently be introduced via Littlewood–Paley decompositions, difference norms, and interpolation.

In line with earlier developments, we introduce both Besov spaces and Triebel–Lizorkin spaces via their Littlewood–Paley decompositions. These involve a so-called inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$ of Schwartz functions on $\mathbb{R}^d$ whose Fourier transforms behave, informally speaking, as a dyadic partition of unity radially. In terms of such sequences, the Besov and Triebel–Lizorkin norms are defined by

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} = \big\|(2^k\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}$$

and

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} := \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))},$$

in the sense that a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^d;X)$ belongs to either one space if and only if the respective expression is well defined and finite. The third parameter q featuring in these definitions is often referred to as the *microscopic parameter*.

In both cases, the norms are independent of the Littlewood–Paley sequence up to a multiplicative constant independent of f. Accordingly, it will

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_4

be a standing assumption that *throughout the chapter we fix a Littlewood–Paley sequence* $(\varphi_k)_{k\geqslant 0}$ *once and for all* (Convention 14.2.8). Dependence of constants on this sequence will never be tracked.

Interestingly, the Bessel potential spaces studied in Chapter 5, and whose study is continued in the present chapter, admit a similar decomposition replacing ℓ^q-norms by Rademacher norms (Theorem 14.7.5) in case X has UMD:

$$\begin{aligned}\|f\|_{H^{s,p}(\mathbb{R}^d;X)} &\eqsim \big\|(\varepsilon_k 2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\varepsilon^p(L^p(\mathbb{R}^d;X))}\\ &= \big\|(\varepsilon_k 2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\varepsilon^p(X))},\end{aligned}$$

using the notation for Rademacher spaces introduced Section 6.3; the equality of the latter two norms is obtained by repeating the proof of Theorem 9.4.8 for Rademacher sums. Comparing these norms with the previous two, it is also of interest to note that equivalent norms are obtained if the ε^p-norm is replaced by an ε^q-norm, by the Kahane–Khintchine inequalities.

In view of their very similar definitions, it comes as no surprise that the theories of Besov and Triebel–Lizorkin spaces largely parallel each other and resemble the theory of Bessel potential spaces to some extent. There are some notable differences however, due to the different orders in which the L^p-norm and ℓ^q-norm are taken; as we have already pointed out, the Triebel–Lizorkin norm is generally speaking more difficult to handle. The main advantage of the Besov and Triebel–Lizorkin over the Bessel potential spaces is that they are often easier to work with, and indeed many basic results for these spaces in the vector-valued setting do not rely on the geometry of the Banach space X. This is in stark contrast with the theory of Bessel potential spaces, where the corresponding results often require geometrical properties such as the UMD property of X or the Radon–Nikodým property of X^*, as we have seen in Chapter 5.

After establishing notation and proving some preliminary results in Section 14.1, the class of Besov spaces is introduced in Section 14.4 via their Littlewood–Paley decompositions. Several basic aspects of these spaces are discussed, such as their independence of the inhomogeneous Littlewood–Paley sequence used in the definition, the density of smooth functions, and Sobolev type embeddings. We continue with several more advanced results, including a difference norm characterisation, identification the complex and real interpolation spaces, and identification of the dual spaces. In Section 14.5 these results are used to prove embedding theorems for the spaces $\gamma(L^2(\mathbb{R}^d), X)$ introduced in Chapter 9 and to prove R-boundedness of the ranges of smooth operator-valued functions under type and cotype assumptions. In the same section we discuss Fourier multiplier results for Besov spaces under (co)type and Fourier type assumptions.

In Section 14.6 the Triebel–Lizorkin spaces are introduced. Proving the same basic properties as before is more complicated, especially for the important endpoint exponent $q = 1$, and requires the boundedness of the so-called Peetre maximal function and the boundedness of Fourier multiplier operators for functions with compact Fourier support in an $L^p(\mathbb{R}^d;\ell^q(X))$-setting. Most of the elementary and more advanced results discussed for Besov spaces have a counterpart for Triebel–Lizorkin spaces and indeed our treatment mirrors that of the Besov spaces. Some results, however, have a different flavour, such as the Sobolev embedding theorem (Theorem 14.6.14), the Gagliardo–Nirenberg inequalities (Proposition 14.6.15), and the embedding theorem of Franke and Jawerth (Theorem 14.6.26), all of which have an improvement in the microscopic parameter q. In some situations this improvement makes it possible to derive results for general Banach spaces X in an effective way. For instance, for any Banach space X one has continuous embeddings (here and below denoted by "$\hookrightarrow$")

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X) \tag{14.1}$$

for $p \in (1,\infty)$ and $s \in \mathbb{R}$. For Hilbert spaces X this can be improved to

$$H^{s,p}(\mathbb{R}^d;X) = F^s_{p,2}(\mathbb{R}^d;X)$$

with equivalent norms for all $p \in (1,\infty)$ and $s \in \mathbb{R}$; this identity characterises Hilbert spaces up to isomorphism (Theorem 14.7.9). The "sandwich result" (14.1) often makes it possible to prove results about $H^{s,p}(\mathbb{R}^d;X)$ without conditions on X by factoring through a Triebel–Lizorkin space. At the end of the section apply some of the obtained result to prove boundedness of pointwise multiplication by the function $\mathbf{1}_{\mathbb{R}_+}$ in Triebel–Lizorkin spaces and Besov spaces. Such results are non-trivial due to the non-smoothness of $\mathbf{1}_{\mathbb{R}_+}$, and are important in applications to interpolation with boundary conditions of vector-valued function spaces used for evolution equations.

In Section 14.7 we return to the study of Bessel potential spaces and discuss some basic properties not covered in the earlier volumes. These include improvements of (14.1) for UMD spaces X under type and cotype assumptions, as well as some advanced results on complex interpolation of Bessel potential spaces (Corollary 14.7.13). At the end of the section we prove the boundedness of pointwise multiplication by the function $\mathbf{1}_{\mathbb{R}_+}$ in Bessel potential spaces again for UMD spaces.

As we will be using Fourier techniques practically everywhere, it will be a further standing assumption that *throughout the chapter we work over the complex scalar field.* As usually is the case, the case of real Banach spaces can be treated by standard complexification arguments. In some cases one can argue directly on real Banach spaces (see Remark 14.2.6). Unless stated otherwise, X will always denote an arbitrary complex Banach space.

14.1 Summary of the main results

Scattered over this section a wealth of inclusion and interpolation results are developed. For the convenience of the reader, we include a concise overview of them here, with pointers to their location.

In all identities, unless otherwise no geometric restrictions apply to Banach spaces and the occurring indices are taken in the ranges

$$p_0, p_1, p \in [1, \infty], \quad q_0, q_1, q \in [1, \infty], \quad s_0, s_1, s \in \mathbb{R}, \quad k_0, k_1 \in \mathbb{N},$$

or subsets thereof. The interpolation results assume that $\theta \in (0, 1)$ and where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$s_\theta = (1-\theta)s_0 + \theta s_1, \quad k_\theta = (1-\theta)k_0 + \theta k_1.$$

The complex and real interpolation spaces of an interpolation couple (X_0, X_1) of Banach spaces are denoted by

$$X_\theta = [X_0, X_1]_\theta, \quad X_{\theta,p} = (X_0, X_1)_{\theta,p}$$

respectively.

Identities. Up to equivalent norms we have the following identifications. If $p \in [1, \infty)$, $s \in (0, 1)$, then

$$W^{s,p}(\mathbb{R}^d; X) = B^s_{p,p}(\mathbb{R}^d; X) \qquad \text{(Corollary 14.4.25)}$$

and, if $s \in (0, \infty) \setminus \mathbb{N}$,

$$C^s_{\mathrm{ub}}(\mathbb{R}^d; X) = B^s_{\infty,\infty}(\mathbb{R}^d; X). \qquad \text{(Corollary 14.4.26)}$$

If H is a Hilbert space and $p \in (1, \infty)$, $s \in \mathbb{R}$, then

$$H^{s,p}(\mathbb{R}^d; H) = F^s_{p,2}(\mathbb{R}^d; H) \qquad \text{(Theorem 14.7.9)}$$

and, if $p \in (1, \infty)$ and $k \in \mathbb{N}$,

$$W^{k,p}(\mathbb{R}^d; H) = F^k_{p,2}(\mathbb{R}^d; H). \qquad \text{(Theorem 14.7.9)}$$

If X is a UMD space and $p \in (1, \infty)$, $k \in \mathbb{N}$, then

$$W^{k,r}(\mathbb{R}^d; X) = H^{k,r}(\mathbb{R}^d; X). \qquad \text{(Theorem 5.6.11)}$$

Embeddings. We have the following continuous embeddings:

$$\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X) \qquad \text{(Proposition 14.4.3)}$$
$$B^s_{p,1}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d; X) \qquad \text{(Proposition 14.4.18)}$$
$$\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X) \qquad \text{(Proposition 14.6.8)}$$

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.6.13)}$$
$$F^k_{p,1}(\mathbb{R}^d;X) \hookrightarrow W^{k,p}(\mathbb{R}^d;X) \hookrightarrow F^k_{p,\infty}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.6.13)}$$
$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.6.13)}$$

and, if $p \in [1,\infty)$,

$$B^s_{p,p\wedge q}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p\vee q}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.6.8)}$$

Sobolev embedding theorem I: If (and only if) either one of the following three conditions holds: $p_0 = p_1$ and $s_0 > s_1$; $p_0 = p_1$ and $s_0 = s_1$ and $q_0 \leqslant q_1$; $p_0 < p_1$ and $q_0 \leqslant q_1$ and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$; then

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.4.19)}$$

Sobolev embedding theorem II: Let $p_0, p_1 \in [1,\infty)$. If (and only if) either one of the following three conditions holds: $p_0 = p_1$ and $s_0 > s_1$; $p_0 = p_1$ and $s_0 = s_1$ and $q_0 \leqslant q_1$; $p_0 < p_1$ and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$ (no condition on q_0, q_1); then

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.6.14)}$$

Sobolev embedding theorem III: Let $p_0, p_1 \in (1,\infty)$. If (and only if) either one of the following three conditions holds: $p_0 = p_1$ and $s_0 \geqslant s_1$; $p_0 < p_1$ and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$; then

$$H^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^d;X) \qquad \text{(Theorem 14.7.1)}$$

and, if in addition $s_0, s_1 \in \mathbb{N}$, then the same necessary and sufficient conditions give

$$W^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow W^{s_1,p_1}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.7.1)}$$

For $k \in \mathbb{N}$,

$$B^k_{\infty,1}(\mathbb{R}^d;X) \hookrightarrow C^k_{\mathrm{ub}}(\mathbb{R}^d;X) \hookrightarrow B^k_{\infty,\infty}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.4.18)}$$

If $p_0 \in [1,\infty]$ and $s_0, s_1 \geqslant 0$ satisfy $s_0 - \frac{d}{p_0} \geqslant s_1$, then

$$B^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.4.27)}$$

and, if in addition $q \in [1,\infty]$ and $s_1 \notin \mathbb{N}$,

$$B^{s_0}_{p_0,q}(\mathbb{R}^d;X) \hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.4.27)}$$

Jawerth–Franke theorem: If $p_0 < p_1$, and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$, then

$$F^{s_0}_{p_0,q}(\mathbb{R}^d; X) \hookrightarrow B^{s_1}_{p_1,p_0}(\mathbb{R}^d; X) \qquad \text{(Theorem 14.6.26)}$$

and, if $p_1 < \infty$,

$$B^{s_0}_{p_0,p_1}(\mathbb{R}^d; X) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^d; X). \qquad \text{(Theorem 14.6.26)}$$

If $k \geqslant d$, then

$$F^{k}_{1,\infty}(\mathbb{R}^d; X) \hookrightarrow C^{k-d}_{\text{ub}}(\mathbb{R}^d; X). \qquad \text{(Corollary 14.6.27)}$$

Embeddings under (co)type assumptions: If (and only if) X has type $p \in [1, 2]$,

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X). \qquad \text{(Theorem 14.5.1)}$$

If (and only if) X has cotype $q \in [2, \infty]$,

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d; X). \qquad \text{(Theorem 14.5.1)}$$

If X has type p_0, then for all $p \in [1, p_0)$ we have

$$H^{(\frac{1}{p}-\frac{1}{2})d,p}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X). \qquad \text{(Corollary 14.7.7)}$$

If X has cotype q_0, then for all $q \in (q_0, \infty)$ we have

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow H^{(\frac{1}{q}-\frac{1}{2})d,q}(\mathbb{R}^d; X). \qquad \text{(Corollary 14.7.7)}$$

If X is a UMD Banach space with type $p_0 \in [1, 2]$ and cotype $q_0 \in [2, \infty]$, and if $p \in (1, \infty)$, $s \in \mathbb{R}$, then

$$F^{s}_{p,p_0}(\mathbb{R}^d; X) \hookrightarrow H^{s,p}(\mathbb{R}^d; X) \hookrightarrow F^{s}_{p,q_0}(\mathbb{R}^d; X). \qquad \text{(Proposition 14.7.6)}$$

Complex interpolation. Let (X_0, X_1) be an interpolation couple of Banach spaces. Let $p_0, p_1 \in [1, \infty]$ with $\min\{p_0, p_1\} < \infty$, $q_0, q_1 \in [1, \infty]$ with $\min\{q_0, q_1\} < \infty$, and $s_0, s_1 \in \mathbb{R}$ or $k_0, k_1 \in \mathbb{N}$. Under these assumptions:

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d; X_0), B^{s_1}_{p_0,q_0}(\mathbb{R}^d; X_1)]_\theta = B^{s_\theta}_{p_\theta,q_\theta}(\mathbb{R}^d; X_\theta). \qquad \text{(Theorem 14.4.30)}$$

If $p_0, p_1 \in (1, \infty)$ and $q_0, q_1 \in (1, \infty]$,

$$[F^{s_0}_{p,q}(\mathbb{R}^d; X_0), F^{s_1}_{p,q}(\mathbb{R}^d; X_1)]_\theta = F^{s_\theta}_{p,q}(\mathbb{R}^d; X_\theta) \qquad \text{(Theorem 14.6.23)}$$

and, if in addition X is a UMD space, then

$$[W^{k_0,p_0}(\mathbb{R}^d;X_0),W^{k_1,p_1}(\mathbb{R}^d;X_1)]_\theta = H^{k_\theta,p_\theta}(\mathbb{R}^d;X_\theta) \qquad \text{(Corollary 14.7.13)}$$
$$[H^{s_0,p_0}(\mathbb{R}^d;X_0),H^{s_1,p_1}(\mathbb{R}^d;X_1)]_\theta = H^{s_\theta,p_\theta}(\mathbb{R}^d;X_\theta). \qquad \text{(Theorem 14.7.12)}$$

Real interpolation. Let (X_0, X_1) be an interpolation couple of Banach spaces and X be a Banach space. Let $p_0, p_1 \in [1,\infty]$ with $\min\{p_0,p_1\} < \infty$, $q_0, q_1 \in [1,\infty]$ with $\min\{q_0,q_1\} < \infty$, $s_0, s_1 \in \mathbb{R}$, and $k_0, k_1 \in \mathbb{N}$. Under these assumptions:

If $s_0 \neq s_1$, then

$$(B^{s_0}_{p,q_0}(\mathbb{R}^d;X),B^{s_1}_{p,q_1}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X) \qquad \text{(Theorem 14.4.31)}$$
$$(H^{s_0,p}(\mathbb{R}^d;X),H^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.4.31)}$$

In addition, if $s_0, s_1 \in \mathbb{N}$ with $s_0 \neq s_1$, then

$$(W^{s_0,p}(\mathbb{R}^d;X),W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X) \qquad \text{(Theorem 14.4.31)}$$

and if $s_0, s_1 \in (0,1)$ with $s_0 \neq s_1$ and $p \in [1,\infty)$, then

$$(W^{s_0,p}(\mathbb{R}^d;X),W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.4.31)}$$

If $s_0, s_1 \in [0,\infty)$ satisfy $s_0 \neq s_1$, then

$$(C^{s_0}_{\rm ub}(\mathbb{R}^d;X),C^{s_1}_{\rm ub}(\mathbb{R}^d;X))_{\theta,\infty} = B^{s_\theta}_{\infty,\infty}(\mathbb{R}^d;X). \qquad \text{(Corollary 14.4.32)}$$

If $p \in [1,\infty)$ and $s_0 \neq s_1$, then

$$(F^{s_0}_{p,q_0}(\mathbb{R}^d;X),F^{s_1}_{p,q_1}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.6.24)}$$

Duality. With respect to the natural duality pairing of $L^2(\mathbb{R}^d;X)$ and $L^2(\mathbb{R}^d;X^*)$, for $p,q \in [1,\infty)$ and $s \in \mathbb{R}$ we have, up to equivalent norms,

$$B^s_{p,q}(\mathbb{R}^d;X)^* = B^{-s}_{p',q'}(\mathbb{R}^d;X^*) \qquad \text{(Theorem 14.4.34)}$$

and, for $p,q \in (1,\infty)$ and $s \in \mathbb{R}$,

$$F^s_{p,q}(\mathbb{R}^d;X)^* = F^{-s}_{p',q'}(\mathbb{R}^d;X^*). \qquad \text{(Theorem 14.6.28)}$$

If X^* has the Radon-Nikodým property, $p \in [1,\infty)$, and $s \in \mathbb{R}$, then

$$H^{s,p}(\mathbb{R}^d;X)^* = H^{-s,p'}(\mathbb{R}^d;X^*). \qquad \text{(Proposition 5.6.7)}$$

14.2 Preliminaries

In this section we prepare some, mostly technical, results that will be of use in our treatments of both Besov and Triebel–Lizorkin spaces.

14.2.a Notation

We start by reviewing some notation that has been introduced in the two earlier volumes. We use the standard multi-index notation explained in Section 2.5. For the details we refer to the relevant sections (Section 2.4.c for Schwartz functions, 2.4.d for tempered distributions, 2.5.b and 2.5.d for Sobolev spaces, and 5.6.a for Bessel potential spaces).

Let X be a Banach space and let $d \geqslant 1$ be an integer. The *Schwartz space* $\mathscr{S}(\mathbb{R}^d; X)$ is the space of all $f \in C^\infty(\mathbb{R}^d; X)$ for which the seminorms

$$[f]_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \|x^\beta \partial^\alpha f(x)\| \tag{14.2}$$

are finite for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. These seminorms define a locally convex topology $\mathscr{S}(\mathbb{R}^d; X)$ in which sequential convergence $f_n \to f$ is equivalent to the convergence $[f - f_n]_{\alpha,\beta} \to 0$ for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. This topology is metrisable by the metric

$$d(f,g) := \sum_{\alpha,\beta\in\mathbb{N}^d} 2^{-|\alpha|-|\beta|} \frac{[f-g]_{\alpha,\beta}}{1+[f-g]_{\alpha,\beta}}$$

which turns $\mathscr{S}(\mathbb{R}^d; X)$ into a complete metric space. Thus $\mathscr{S}(\mathbb{R}^d; X)$ has the structure of a Fréchet space. As a consequence of Lemma 1.2.19 or Lemma 14.2.1, the space $C_c^\infty(\mathbb{R}^d) \otimes X$ is dense in $L^p(\mathbb{R}^d; X)$ for $1 \leqslant p < \infty$. We will prove in Lemma 14.2.1 that $C_c^\infty(\mathbb{R}^d) \otimes X$ is sequentially dense in both $C_c^\infty(\mathbb{R}^d; X)$ and $\mathscr{S}(\mathbb{R}^d; X)$.

The space of continuous linear operators

$$\mathscr{S}'(\mathbb{R}^d; X) := \mathscr{L}(\mathscr{S}(\mathbb{R}^d), X)$$

is called the space of *tempered distributions* with values in X.

Let D be an open subset of $\mathbb{R}^d$. For $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}$ the *Sobolev space* $W^{k,p}(D; X)$ is the space of functions $f \in L^p(D; X)$ whose weak derivatives $\partial^\alpha f$ of order $|\alpha| \leqslant k$ exist and belong to $L^p(D; X)$. Recall that a function $g \in L^1_{\rm loc}(D)$ is said to be the *weak derivative* of order α of f if

$$\int_D f(x)\partial^\alpha \phi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_D g(x)\phi(x)\,\mathrm{d}x \quad \text{for all } \phi \in C_c^\infty(D).$$

Such a function g, if it exists, is unique. With respect to the norm

$$\|f\|_{W^{k,p}(D;X)} := \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_p,$$

the space $W^{k,p}(D;X)$ is a Banach space. For $1 \leqslant p < \infty$ and $0 < s < 1$, the *Sobolev–Slobodetskii space* $W^{s,p}(\mathbb{R}^d;X)$ is the space of all functions $f \in L^p(\mathbb{R}^d;X)$ for which the seminorm

$$[f]_{W^{s,p}(D;X)} := \Big(\int_D\int_D \frac{\|f(x)-f(y)\|^p}{|x-y|^{sp+d}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}$$

is finite. With respect to the norm

$$\|f\|_{W^{s,p}(D;X)} := \|f\|_p + [f]_{W^{s,p}(D;X)},$$

the space $W^{s,p}(\mathbb{R}^d;X)$ is a Banach space. By Theorem 2.5.17, for $1 \leqslant p <\in \infty$ and $0 < s < 1$ the real interpolation method gives

$$(L^p(\mathbb{R}^d;X), W^{1,p}(\mathbb{R}^d;X))_{\theta,p} = W^{\theta,p}(\mathbb{R}^d;X)$$

with equivalent norms.

For $1 \leqslant p \leqslant \infty$ and $s \in \mathbb{R}$ the *Bessel potential space* $H^{s,p}(\mathbb{R}^d;X)$ consists of all $u \in \mathscr{S}'(\mathbb{R}^d;X)$ for which the tempered distribution $J_s u \in \mathscr{S}'(\mathbb{R}^d;X)$ defined by

$$J_s u := ((1+4\pi^2|\cdot|^2)^{s/2}\widehat{u})^{\smallsmile}$$

belongs to $L^p(\mathbb{R}^d;X)$. Recall that the Fourier transform of u is defined by $\widehat{u}(f) = u(\widehat{f})$ for $f \in \mathscr{S}(\mathbb{R}^d;X)$, where the Fourier transform of a function $f \in L^1(\mathbb{R}^d;X)$ is defined as

$$\widehat{f}(\xi) = \mathscr{F}f(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

The inverse Fourier transform of a tempered distribution is defined similarly. With respect to the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^d;X)} := \|J_s u\|_{L^p(\mathbb{R}^d;X)},$$

$H^{s,p}(\mathbb{R}^d;X)$ is a Banach space. The following continuous embeddings hold, the first being dense if $1 \leqslant p < \infty$:

$$\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X).$$

By Theorem 5.6.1, complex interpolation gives

$$[L^p(\mathbb{R}^d;X), W^{k,p}(\mathbb{R}^d;X)]_\theta = H^{\theta k,p}(\mathbb{R}^d;X)$$

with equivalent norms, provided X is a UMD space, $1 < p < \infty$, and $k \geqslant 1$ is an integer. Under the same assumptions, Theorem 5.6.9 gives

$$[H^{s_0,p}(\mathbb{R}^d;X), H^{s_1,p}(\mathbb{R}^d;X)]_\theta = H^{s_\theta,p}(\mathbb{R}^d;X)$$

with equivalent norms, for $s_0, s_1 \in \mathbb{R}$ satisfying $s_0 < s_1$ and with $s_\theta = (1-\theta)s_0 + \theta s_1$. Still for UMD spaces X and $1 < p < \infty$, by Theorem 5.6.11 for all integers $k \geqslant 1$ we have

$$W^{k,p}(\mathbb{R}^d; X) = H^{k,p}(\mathbb{R}^d; X)$$

with equivalent norms. For $k = 0$ we have the trivial identities

$$W^{0,p}(\mathbb{R}^d; X) = H^{0,p}(\mathbb{R}^d; X) = L^p(\mathbb{R}^d; X),$$

valid for all Banach spaces X and $1 \leqslant p < \infty$.

For $k \in \mathbb{N}$ the space $C^k_{\rm b}(\mathbb{R}^d; X)$ consists of all k-times continuously differentiable functions $f : \mathbb{R}^d \to X$ whose partial derivatives $\partial^\alpha f$ are bounded for all multi-indices $\alpha \in \mathbb{N}^d$ satisfying $|\alpha| \leqslant k$. With respect to the norm

$$\|f\|_{C^k_{\rm b}(\mathbb{R}^d;X)} := \sup_{|\alpha| \leqslant k} \|\partial^\alpha f\|_\infty,$$

the space $C^k_{\rm b}(\mathbb{R}^d; X)$ is a Banach space. We denote by $C^k_{\rm ub}(\mathbb{R}^d; X)$ its closed subspace of functions for which $\partial^\alpha f$ is uniformly continuous for all $|\alpha| \leqslant k$.

For $\theta \in (0, 1)$ the space of *Hölder continuous* functions $C^\theta_{\rm b}(\mathbb{R}^d; X)$ consists of all bounded continuous $f : \mathbb{R}^d \to X$ for which the seminorm

$$[f]_{C^\theta_{\rm b}(\mathbb{R}^d;X)} := \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{\|f(x) - f(y)\|}{|x - y|^\theta}$$

is finite. With respect to the norm

$$\|f\|_{C^\theta_{\rm b}(\mathbb{R}^d;X)} := \|f\|_\infty + [f]_{C^\theta_{\rm b}(\mathbb{R}^d;X)}$$

the space $C^\theta_{\rm b}(\mathbb{R}^d; X)$ is a Banach space. The Banach space obtained by taking $\theta = 1$ in these expressions is called the space of *Lipschitz continuous* functions and is denoted by $\mathrm{Lip}(\mathbb{R}^d; X)$.

For $s = k + \theta$, with $k \in \mathbb{N}$ and $\theta \in (0, 1)$, the space $C^s_{\rm b}(\mathbb{R}^d; X)$ is defined as the space of all $f \in C^k_{\rm b}(\mathbb{R}^d; X)$ for which $\partial^\alpha f \in C^\theta_{\rm b}(\mathbb{R}^d; X)$ for all multi-indices satisfying $|\alpha| \leqslant k$. With the norm

$$\|f\|_{C^s_{\rm b}(\mathbb{R}^d;X)} := \sup_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{C^\theta_{\rm b}(\mathbb{R}^d;X)},$$

this space is a Banach space. For all $s \in [0, \infty)$ we have continuous embeddings

$$\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow C^s_{\rm b}(\mathbb{R}^d; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X).$$

The first embedding is not dense, as non-zero constant functions cannot be approximated by Schwartz functions. For non-integers $s > 0$ we will use the notation

$$C^s_{\rm ub}(\mathbb{R}^d; X) = C^s_{\rm b}(\mathbb{R}^d; X).$$

14.2.b A density lemma and Young's inequality

Let $U \subseteq \mathbb{R}^d$ be an open set. The elements of the space $C_c^\infty(U;X)$ will be referred to as X-valued *test functions*. Sequential convergence in $C_c^\infty(U;X)$ is defined by insisting that $f_n \to f$ in $C_c^\infty(U;X)$ if there exists a compact set K of U containing the support of all f_n and $\|\partial^\alpha f - \partial^\alpha f\|_\infty \to 0$ for all $\alpha \in \mathbb{N}^d$. Related sequential notions, such as Cauchy sequences, are defined similarly. Note that if $f_n \to f$ in $C_c^\infty(U;X)$, then also $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d;X)$, provided we extend the functions identically zero outside U.

Lemma 14.2.1. *The space $C_c^\infty(\mathbb{R}^d)\otimes X$ is sequentially dense in $C_c^\infty(\mathbb{R}^d;X)$ and $\mathscr{S}(\mathbb{R}^d;X)$.*

Proof. We prove the lemma in two steps.

Step 1 – We first prove that $C_c^\infty(\mathbb{R}^d;X)$ is sequentially dense in $\mathscr{S}(\mathbb{R}^d;X)$.

Let $f \in \mathscr{S}(\mathbb{R}^d;X)$. Let $\zeta \in C^\infty(\mathbb{R}^d)$ satisfy $\zeta \equiv 1$ on $\{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$ and $\zeta \equiv 0$ on $\{\xi \in \mathbb{R}^d : |\xi| \geqslant 2\}$, and put $f_n(x) := \zeta(x/n)f(x)$. Then $f_n \in C_c^\infty(\mathbb{R}^d;X)$. To prove that $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d;X)$ it suffices to check that for all multi-indices $\alpha,\beta \in \mathbb{N}^d$ we have

$$\lim_{n\to\infty}\big\|(\cdot)^\beta\partial^\alpha[(1-\zeta(\cdot/n))f]\big\|_\infty = 0.$$

The elementary verification is left to the reader.

Step 2 – Let $f \in C_c^\infty(\mathbb{R}^d;X)$. Choose bounded open sets $O,U,V \subseteq \mathbb{R}^d$ such that $\mathrm{supp}(f) \subseteq U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq O$. We first claim that for every $\varepsilon > 0$ there exists a $g \in C_c^\infty(V)\otimes X$ such that $\|f-g\|_\infty \leqslant \varepsilon$. Fix $\varepsilon > 0$. Since $f(\overline{U}) \subseteq X$ is compact, it follows that there exist $x_1,\dots,x_n \in X$ such that $f(\overline{U}) \subseteq B(x_1,\varepsilon)\cup\dots B(x_n,\varepsilon)$. The sets $U_0 = O\backslash\overline{U}$ and $U_j = f^{-1}(B(x_j,\varepsilon))\cap V$ for $j = 1,\dots,n$ define an open cover $(U_j)_{j=0}^n$ of $\overline{V}$. Let $(\psi_j)_{j=0}^n$ be a smooth partition of unity subordinate to this cover, i.e., $\psi_j \in C_c^\infty(U_j)$, $0 \leqslant \psi_j \leqslant 1$, and $\sum_{j=0}^n \psi_j \equiv 1$ on $\overline{V}$. Letting $g := \sum_{j=0}^n \psi_j \otimes x_j$ with $x_0 = 0$, for all $u \in \mathbb{R}^d$ we have

$$\|f(u)-g(u)\| \leqslant \sum_{j=0}^n \psi_j(u)\|f(u)-x_j\| < \varepsilon.$$

which proves the claim.

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d}\phi(u)\,\mathrm{d}u = 1$ and put $\phi_j(u) := j^d\phi(ju)$. By compactness, the exists an index $j_0 \in \mathbb{N}$ such that for all $j \geqslant j_0$ and all $g \in C_c^\infty(V;X)$ we have $\phi_j * g \in C_c^\infty(O;X)$ and, for all multi-indices $\alpha,\beta \in \mathbb{N}^d$,

$$[\phi_j * g - g]_{\alpha,\beta} \leqslant C_{O,\beta}\|\phi_j * \partial^\alpha g - \partial^\alpha g\|_\infty \to 0$$

as $j \to \infty$, by the uniform continuity of $\partial^\alpha g$. We conclude that for all such g and $j \geqslant 0$ we have $\phi_j * g \to g$ in $\mathscr{S}(\mathbb{R}^d;X)$. In particular, this holds with $g = f$. By the claim, we can find a sequence $(g_k)_{k\geqslant 1}$ in $C_c^\infty(V)\otimes X$ such that $\|f-g_k\|_\infty \to 0$. Now for each $j \geqslant j_0$ the functions $g_{kj} := \psi_j * g_k$ belong to

$C_c^\infty(O) \otimes X$, and by the above we have $g_{kj} \to g_k$ in $\mathscr{S}(\mathbb{R}^d; X)$. For appropriate $j_k \geqslant j_0$ we find that $g_{kj_k} \to g$ in $\mathscr{S}(\mathbb{R}^d; X)$. Since $g_{kj_k} \in C_c^\infty(O) \otimes X$, this proves density in $C_c^\infty(\mathbb{R}^d; X)$.

To prove density in $\mathscr{S}(\mathbb{R}^d; X)$ let $f \in \mathscr{S}(\mathbb{R}^d; X)$. By Step 1 there exists a sequence $(f_n)_{n\geqslant 1}$ in $C_c^\infty(\mathbb{R}^d; X)$ such that $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d; X)$. Using Step 2, for every $n \geqslant 1$ choose a sequence $(f_{n,k})_{k\geqslant 1}$ in $C_c^\infty(\mathbb{R}^d) \otimes X$ such that $f_{n,k} \to f_n$ in $C_c^\infty(\mathbb{R}^d; X)$. Then in particular, $f_{n,k} \to f_n$ in $\mathscr{S}(\mathbb{R}^d; X)$. Since convergence in $\mathscr{S}(\mathbb{R}^d; X)$ is governed by countably many seminorms, a standard diagonal argument allows us to find a subsequence such that $f_{n,k_n} \to f$ in $\mathscr{S}(\mathbb{R}^d; X)$. □

As a corollary to the above lemma we record:

Proposition 14.2.2. *For all $p \in [1, \infty)$ and $s \in \mathbb{R}$ the space $C_c^\infty(\mathbb{R}^d) \otimes X$ is dense in $H^{s,p}(\mathbb{R}^d; X)$.*

Proof. By Proposition 5.6.4, for all $p \in [1, \infty)$ and $s \in \mathbb{R}$ we have a dense embedding $\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow H^{s,p}(\mathbb{R}^d; X)$. □

We will often make use of the following version of Young's inequality, which extends a special case already proven in Lemma 1.2.30.

Lemma 14.2.3 (Young's inequality). *Let $p, q, r \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^d; \mathscr{L}(X, Y))$ and $g \in L^q(\mathbb{R}^d; X)$, then $f * g \in L^r(\mathbb{R}^d; Y)$ and*

$$\|f * g\|_{L^r(\mathbb{R}^d;Y)} \leqslant \|f\|_{L^p(\mathbb{R}^d;\mathscr{L}(X,Y))} \|g\|_{L^q(\mathbb{R}^d;X)}.$$

Proof. For $1 \leqslant q < \infty$, by density it suffices to prove the estimate for $g \in C_c^\infty(\mathbb{R}^d) \otimes X$, and if $q = \infty$, then $p = 1$ and $r = \infty$ and it suffices to prove the required estimate for $f \in C_c^\infty(\mathbb{R}^d) \otimes \mathscr{L}(X, Y)$. In either case, $f * g$ is strongly measurable and we have the bound $\|f * g\| \leqslant \|f\| * \|g\|$. The estimate then follows from the scalar version of Young's inequality. □

We recall from Section 1.3 that the *variation* of an operator-valued measure $\Phi : \mathscr{A} \to \mathscr{L}(X, Y)$, where $(S, \mathscr{A})$ is a measurable space, is the measure $\|\Phi\| : \mathscr{A} \to [0, \infty]$ given by

$$\|\Phi\|(A) = \sup_{\pi} \sum_{B \in \pi} \|\Phi(B)\|,$$

the supremum being taken over all finite disjoint partitions π of the set $A \in \mathscr{A}$; the is taken in $\mathscr{L}(X, Y)$. We say that Φ has *bounded variation* if $\|\Phi\|(S) < \infty$. For a strongly measurable function $f : S \to X$ such that

$$\int_S \|f(s)\| \, \mathrm{d}\|\Phi\|(s) < \infty,$$

the construction of the Bochner integral (see Section 1.2.a) can be repeated to define $\int_S f \,\mathrm{d}\Phi$ as an element of Y satisfying

$$\Big\| \int_S f \,\mathrm{d}\Phi \Big\| \leqslant \int_S \|f\| \,\mathrm{d}\|\Phi\|.$$

When $(S, \mathscr{A}, \mu)$ is a measure space, a simple example of an operator-valued measure with bounded variation is obtained by taking $\Phi(A) := \int_A \phi \,\mathrm{d}\mu$ with $\phi \in L^1(S, \mu; \mathscr{L}(X, Y))$. The total variation of this measure satisfies

$$\|\Phi\|(S) \leqslant \|\phi\|_{L^1(S,\mu;\mathscr{L}(X,Y))}.$$

Standard arguments show that $\int_S \|f\|_X \,\mathrm{d}\|\Phi\| < \infty$ if and only if $\phi f \in L^1(S; Y)$ and in that case

$$\int_S f \,\mathrm{d}\Phi = \int_S \phi f \,\mathrm{d}\mu.$$

Lemma 14.2.4 (Convolutions with measures). *Let $\Phi : \mathbb{R}^d \to \mathscr{L}(X, Y)$ be an operator-valued measure of bounded variation, and let $f \in L^p(\mathbb{R}^d; X)$. For almost all $x \in \mathbb{R}^d$ the integral $\int_{\mathbb{R}^d} f(x - y) \,\mathrm{d}\Phi(y)$ is well defined in the above sense, and the convolution*

$$\Phi * f(x) := \int_{\mathbb{R}^d} f(x - y) \,\mathrm{d}\Phi(y)$$

defines a function $\Phi * f \in L^p(\mathbb{R}^d; Y)$ which satisfies

$$\|\Phi * f\|_{L^p(\mathbb{R}^d;Y)} \leqslant \|\Phi\|(\mathbb{R}^d)\|f\|_{L^p(\mathbb{R}^d;X)}.$$

Proof. For $1 \leqslant p < \infty$, Minkowski's inequality (Proposition 1.2.22) implies

$$\begin{aligned} \Big\| x \mapsto \int_{\mathbb{R}^d} \|f(x - y)\| \,\mathrm{d}\|\Phi\|(y) \Big\|_{L^p(\mathbb{R}^d)} &\leqslant \int_{\mathbb{R}^d} \big\| x \mapsto f(x - y) \big\|_{L^p(\mathbb{R}^d;X)} \,\mathrm{d}\|\Phi\|(y) \\ &= \|f\|_{L^p(\mathbb{R}^d;X)} \|\Phi\|(\mathbb{R}^d). \end{aligned}$$

For $p = \infty$, the same holds for trivial reasons. It follows that for almost all $x \in \mathbb{R}^d$ the integral $\Phi * f(x) = \int_{\mathbb{R}^d} f(x - y) \,\mathrm{d}\Phi(y)$ is well defined in Y. By approximation with simple functions it is seen that $\Phi * f$ is strongly measurable, and since

$$\|\Phi * f(x)\| \leqslant \int_{\mathbb{R}^d} \|f(x - y)\| \,\mathrm{d}\|\Phi\|(y),$$

the required estimate also follows. □

14.2.c Inhomogeneous Littlewood–Paley sequences

We now introduce one of our main technical tools, which allows us to break up a function spectrally into pieces with control on their frequencies.

Let Φ denote the set of all Schwartz functions $\varphi \in \mathscr{S}(\mathbb{R}^d)$ with the following properties:

(i) $0 \leqslant \widehat{\varphi}(\xi) \leqslant 1, \quad \xi \in \mathbb{R}^d$,
(ii) $\widehat{\varphi}(\xi) = 1$ if $|\xi| \leqslant 1$,
(iii) $\widehat{\varphi}(\xi) = 0$ if $|\xi| \geqslant \frac{3}{2}$.

Such functions can be constructed in a similar way as in Lemma 5.5.21.

Remark 14.2.5. If $\phi \in \Phi$, the function $\psi \in \mathscr{S}(\mathbb{R}^d)$ given by

$$\widehat{\psi}(\xi) := \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)$$

is a smooth Littlewood–Paley function in the sense of Definition 5.5.20, i.e.,

(i) $\widehat{\psi}$ is smooth, non-negative, and supported in $\{\xi \in \mathbb{R}^d : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$;
(ii) $\sum_{k\in\mathbb{Z}} \widehat{\psi}(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Remark 14.2.6. It is possible to choose the function φ is real and even (or equivalently $\widehat{\varphi}$ real and even). In that case it would be possible to use real Banach spaces in several of the definitions and results of this chapter. For instance if $f \in L^p(\mathbb{R}^d; X)$ or even $\mathscr{S}'(\mathbb{R}^d; X)$, then $\varphi * f$ can be defined without using any complex structure.

Definition 14.2.7 (Inhomogeneous Littlewood–Paley sequence). *The* inhomogeneous Littlewood–Paley sequence *associated with a function $\varphi \in \Phi$ is the sequence $(\varphi_k)_{k\geqslant 0}$ in $\mathscr{S}(\mathbb{R}^d)$ given by*

$$\begin{aligned} \widehat{\varphi}_0(\xi) &:= \widehat{\varphi}(\xi), & k = 0,\ \xi \in \mathbb{R}^d, \\ \widehat{\varphi}_k(\xi) &:= \widehat{\varphi}(2^{-k}\xi) - \widehat{\varphi}(2^{-k+1}\xi), & k \geqslant 1,\ \xi \in \mathbb{R}^d. \end{aligned} \tag{14.3}$$

Note the scaling property

$$\widehat{\varphi}_k(\xi) = \widehat{\varphi}_1(2^{-k+1}\xi), \quad k \geqslant 1, \tag{14.4}$$

and the telescoping properties

$$\sum_{k=0}^{n} \widehat{\varphi}_k(\xi) = \widehat{\varphi}_0(2^{-n}\xi), \quad \sum_{k\geqslant 0} \widehat{\varphi}_k(\xi) = 1. \tag{14.5}$$

We will often use the simple L^1-norm identity

$$\Big\| \sum_{k=0}^{n} \varphi_k \Big\|_1 = \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \widehat{\varphi}_0(2^{-n}\xi)\,\mathrm{d}\xi \Big|\,\mathrm{d}x = 2^n \int_{\mathbb{R}^d} |\varphi_0(2^n x)|\,\mathrm{d}x = \|\varphi_0\|_1, \tag{14.6}$$

which implies

$$\|\varphi_k\|_1 = \Big\| \sum_{k=0}^{n} \varphi_k - \sum_{k=0}^{n-1} \varphi_k \Big\|_1 \leqslant 2\|\varphi_0\|_1, \quad k \geqslant 1. \tag{14.7}$$

The adjective ‘inhomogeneous’ refers to the special role played by the function φ_0 whose support contains an open neighbourhood of the origin.

Inhomogeneous Littlewood–Paley sequences will be used to define the classes of Besov spaces and Triebel–Lizorkin spaces. Up to equivalent norms, the definitions of these spaces turn out to be independent of the particular inhomogeneous Littlewood–Paley sequence chosen. This allows us to fix an arbitrary such sequence once and for all and always work with that given sequence. In order to avoid endless repetitions we therefore make the following convention.

Convention 14.2.8. *Throughout this entire chapter, $(\varphi_k)_{k\in\mathbb{N}}$ denotes the inhomogeneous Littlewood–Paley sequence associated with a function $\varphi \in \Phi$ which we fix once and for all. Whenever this is useful, we extend the index set of the sequence to include the negative integers by setting*

$$\phi_k \equiv 0, \quad k = -1, -2, \dots$$

Constants in estimates involving a Littlewood–Paley sequences or spaces defined by using them will often also depend on the generating function $\varphi \in \Phi$, but since it is considered to be fixed we will not express these dependencies in our estimates.

Let us collect some easy properties of inhomogeneous Littlewood–Paley sequences. It is immediate to check the Fourier support property

$$\widehat{\varphi}_k(\xi) \equiv 1 \;\text{ for }\; \frac{3}{4}\cdot 2^k \leqslant |\xi| \leqslant 2^k, \quad k \geqslant 1, \tag{14.8}$$

and

$$\operatorname{supp} \widehat{\varphi}_k \subseteq \{\xi \in \mathbb{R}^d :\, 2^{k-1} \leqslant |\xi| \leqslant 3\cdot 2^{k-1}\}, \qquad k \geqslant 1. \tag{14.9}$$

In particular we have the disjointness property

$$\operatorname{supp} \widehat{\varphi}_j \cap \operatorname{supp} \widehat{\varphi}_k = \varnothing, \quad |j-k| \geqslant 2, \tag{14.10}$$

which implies the orthogonality properties

$$\widehat{\varphi}_j \widehat{\varphi}_k = 0 \;\text{ and }\; \varphi_j * \varphi_k = 0, \quad |j-k| \geqslant 2. \tag{14.11}$$

From (14.5) and (14.11) we infer

$$\sum_{j=-1}^{1} \widehat{\varphi}_{k+j} \equiv 1 \;\text{ on }\; \operatorname{supp}(\widehat{\varphi}_k), \quad k \geqslant 0, \tag{14.12}$$

using the convention $\varphi_{-1} = 0$ for the case $k = 0$.

By Proposition 2.4.32, for $\psi \in \mathscr{S}(\mathbb{R}^d)$ and $u \in \mathscr{S}'(\mathbb{R}^d; X)$ the convolution

$$\psi * u = \mathscr{F}^{-1}(\widehat{\psi}\widehat{f}) \tag{14.13}$$

is well defined as element of $C^\infty(\mathbb{R}^d; X)$ and as such it has at most polynomial growth. For later use we record the following useful consequence:

Lemma 14.2.9. *Every $f \in \mathscr{S}'(\mathbb{R}^d; X)$ with compact Fourier support belongs to $C^\infty(\mathbb{R}^d; X)$ and has at most polynomial growth.*

Proof. This follows from Proposition 2.4.32 by writing $f = f * g$ with $g \in \mathscr{S}(\mathbb{R}^d)$ satisfying $\widehat{g} \equiv 1$ on $\mathrm{supp}(f)$. □

Returning to the main line of development, by applying (14.13) to the convolutions $\varphi_k * u$, the latter can be identified with distributions in $\mathscr{S}'(\mathbb{R}^d; X)$ and we have the following result:

Lemma 14.2.10. *Let $E = \mathscr{S}(\mathbb{R}^d; X)$ or $E = \mathscr{S}'(\mathbb{R}^d; X)$. For all $f \in E$ we have*

$$f = \sum_{k\geqslant 0} \varphi_k * f = \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} \varphi_{k+\ell} * \varphi_k * f$$

with convergence of the sums in E.

Proof. The second identity follows by applying the first twice and (14.11). It thus remains to prove the first identity.

By the second identity in (14.5), (14.13), and the continuity of the Fourier transform on E proved in Proposition 2.4.22, it suffices to show that $\sum_{k\geqslant 0} \widehat{\varphi}_k g = g$ for all $g \in E$, with convergence of the sum in E.

First suppose that $g \in \mathscr{S}(\mathbb{R}^d; X)$. In view of the first identity in (14.5) we must to show that, for arbitrary multi-indices α, β,

$$\lim_{n\to\infty} \big\| (\cdot)^\beta \partial^\alpha [(1 - \widehat{\varphi}(2^{-n}\cdot))g] \big\|_\infty = 0.$$

This is elementary and left to the reader.

Next suppose that $g \in \mathscr{S}'(\mathbb{R}^d; X)$. Fix a function $\psi \in \mathscr{S}(\mathbb{R}^d)$. We need to check that $\sum_{k\geqslant 0} g(\psi \widehat{\varphi}_k) = g(\psi)$. For this it suffices to check that $\sum_{k\geqslant 0} \psi \widehat{\varphi}_k = \psi$ in $\mathscr{S}(\mathbb{R}^d)$, which is the content of the previous case. □

As a first application of Littlewood–Paley sequence techniques we prove a lemma that will be useful for establishing Fourier multiplier results in later subsections. For its proof we recall from Volume I the space

$$\check{L}^1(\mathbb{R}^d; X) := \{f \in L^\infty(\mathbb{R}^d; X) : \mathscr{F}^{-1} f \in L^1(\mathbb{R}^d; X)\},$$

where the inverse Fourier transforms is viewed as an element of $\mathscr{S}'(\mathbb{R}^d; X)$. With respect to the norm

$$\|f\|_{\check{L}^1(\mathbb{R}^d;X)} = \|\widehat{f}\|_{L^1(\mathbb{R}^d;X)},$$

$\check{L}^1(\mathbb{R}^d; X)$ is a Banach space. It enjoys the scaling invariance property

$$\|f(\lambda\cdot)\|_{\check{L}^1(\mathbb{R}^d;X)} = \|f\|_{\check{L}^1(\mathbb{R}^d;X)}, \quad \lambda > 0, \tag{14.14}$$

which is proved by a simple change of variables.

Lemma 14.2.11 (Integrability of Fourier transforms – I). *Let* $f \in C^{d+1}(\mathbb{R}^d; X)$*, and suppose that there exists an* $\varepsilon > 0$ *such that*

$$C_{f,d,\varepsilon} := \max_{|\alpha| \leqslant d+1} \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^{|\alpha|+\varepsilon}) \|\partial^\alpha f(\xi)\| < \infty.$$

Then $\widehat{f} \in L^1(\mathbb{R}^d; X)$ *and* $\|\widehat{f}\|_{L^1(\mathbb{R}^d;X)} \lesssim_{d,\varepsilon} C_{f,d,\varepsilon}$.

Note that $C_{f,d,\varepsilon}$ is trivially finite (for all $\varepsilon > 0$) if $f \in C^{d+1}(\mathbb{R}^d; X)$ has compact support.

Proof. In view of (14.5) we have $\|f\|_{\check{L}^1(\mathbb{R}^d;X)} \leqslant \sum_{j \geqslant 0} \|\widehat{\varphi}_j f\|_{\check{L}^1(\mathbb{R}^d;X)}$, and therefore it is enough to show that for all $j \geqslant 0$ we have

$$\|\widehat{\varphi}_j f\|_{\check{L}^1(\mathbb{R}^d;X)} \lesssim_d 2^{-(j-1)\varepsilon} C_{f,d,\varepsilon}. \tag{14.15}$$

First we consider indices $j \geqslant 1$. Setting $B := \{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$, by (14.4) and (14.14) we obtain

$$\begin{aligned} \|\widehat{\varphi}_j f\|_{\check{L}^1(\mathbb{R}^d;X)} &= \|\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot)\|_{\check{L}^1(\mathbb{R}^d;X)} \\ &= \|\mathscr{F}(\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot))\|_{L^1(B;X)} + \|\mathscr{F}(\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot))\|_{L^1(\mathbb{R}^d\setminus B;X)} \\ &=: T_1 + T_2. \end{aligned}$$

The first term is easy to handle. Indeed, since $\|\mathscr{F}\|_{L^1 \to L^\infty} \leqslant 1$ and $0 \leqslant \widehat{\varphi}_1 \leqslant 1$,

$$\begin{aligned} T_1 &\leqslant |B|\, \|\mathscr{F}(\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot))\|_\infty \\ &\leqslant |B| \|\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot)\|_{L^1(\mathbb{R}^d;X)} \leqslant |B| \|f(2^{j-1}\cdot)\|_{L^1(3B\setminus B;X)}, \end{aligned}$$

using that $\widehat{\varphi}_1$ is supported in $3B \setminus B$ in the last step. Together with the assumed bound for f with $\alpha = 0$, for $\xi \in 3B \setminus B$ we have

$$\|f(2^{j-1}\xi)\| \leqslant \frac{C_{f,d,\varepsilon}}{1 + 2^{(j-1)\varepsilon}|\xi|^\varepsilon} \leqslant 2^{-(j-1)\varepsilon} C_{f,d,\varepsilon}.$$

Combining this with the previous estimate, this gives the bound $T_1 \leqslant 2^{-(j-1)\varepsilon} C_{f,d,\varepsilon} |3B \setminus B||B|$.

For the second term we use the finiteness of $C_d := \int_{\mathbb{R}^d \setminus B} |x|^{-d-1} \, \mathrm{d}x$ to obtain

$$T_2 \leqslant C_d \big\| \xi \mapsto |\xi|^{d+1} \mathscr{F}(\widehat{\varphi}_1 f(2^{j-1}\cdot))(\xi) \big\|_\infty.$$

By the estimate $|\xi|^{d+1} \lesssim_d \sum_{|\alpha|=d+1} |\xi^\alpha|$ and the identity $(2\pi i)^{|\alpha|} \xi^\alpha \mathscr{F}(g)(\xi) = \mathscr{F}(\partial^\alpha g)(\xi)$, for each $\xi \in \mathbb{R}^d$ we can further estimate

$$\big\| |\xi|^{d+1} \mathscr{F}(\widehat{\varphi}_1 f(2^{j-1}\cdot))(\xi) \big\|_X \lesssim_d \sum_{|\alpha|=d+1} \big\| (2\pi\xi)^\alpha \mathscr{F}(\widehat{\varphi}_1 f(2^{j-1}\cdot))(\xi) \big\|_X$$

$$= \sum_{|\alpha|=d+1} \big\| \mathscr{F}(\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)])(\xi)\big\|_X.$$

Using that $\widehat{\varphi}_1$ is compactly supported we obtain

$$\big\|\mathscr{F}(\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)])\big\|_\infty \leqslant \big\|\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)]\big\|_1 \lesssim_d \big\|\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)]\big\|_\infty.$$

After an application of the Leibniz rule it remains to estimate terms of the form $\partial^\beta\widehat{\varphi}_1\partial^\gamma[f(2^{j-1}\cdot)]$ with $|\beta|+|\gamma| = |\alpha| = d+1$. By the assumptions and the fact that $\widehat{\varphi}_1$ is supported in $3B\setminus B$,

$$\|\partial^\beta\widehat{\varphi}_1\partial^\gamma[f(2^{j-1}\cdot)]\|_\infty \lesssim_d \sup_{1\leqslant|\xi|\leqslant 3} \|2^{(j-1)|\gamma|}\partial^\gamma f(2^{j-1}\xi)\| \leqslant 2^{-(j-1)\varepsilon}C_{f,d,\varepsilon}.$$

It follows that $T_2 \lesssim_d 2^{-(j-1)\varepsilon}C_{f,d,\varepsilon}$. This proves (14.15) for $j\geqslant 1$. The case $j=0$ can be shown in a similar way, skipping the dilation step. □

For later reference we state the following consequence of Lemma 14.2.11.

Lemma 14.2.12. *Let $\lambda\geqslant 0$ and suppose that $f\in C^{d+1+\lceil\lambda\rceil}(\mathbb{R}^d;X)$ has support in the ball $B_R = \{\xi\in\mathbb{R}^d : |\xi|\leqslant R\}$. Then $(1+|\cdot|)^\lambda\widehat{f}(\cdot)\in L^1(\mathbb{R}^d;X)$ and*

$$\|(1+|\cdot|)^\lambda \widehat{f}(\cdot)\|_{L^1(\mathbb{R}^d;X)} \leqslant C_{R,d}\|f\|_{C_{\rm b}^{d+1+\lceil\lambda\rceil}(\mathbb{R}^d;X)}.$$

Proof. Upon replacing λ by $\lceil\lambda\rceil$ we may assume that $\lambda\in\mathbb{N}$. By Lemma 14.2.11 we have $\widehat{f}\in L^1(\mathbb{R}^d;X)$. Therefore it suffices to prove the estimate with $(1+|\cdot|)^\lambda$ replaced by $|\cdot|^\lambda$.

Arguing as before, since $|x|^\lambda \lesssim_d \sum_{|\beta|=\lambda}|x^\beta|$,

$$\big\||\cdot|^\lambda\widehat{f}\big\|_{L^1(\mathbb{R}^d;X)} \lesssim_{d,R} \sum_{|\beta|=\lambda}\|\widehat{\partial^\beta f}\|_{L^1(\mathbb{R}^d;X)}.$$

Therefore, the required result follows from Lemma 14.2.11 applied to $\partial^\beta f$. □

14.3 Interpolation of L^p-spaces with change of weights

When $(S,\mathscr{A},\mu)$ is σ-finite measure space, we call a measurable function $w: S\to[0,\infty]$ a *weight* if $w(x)\in(0,\infty)$ for almost all $x\in S$. On earlier occasions (e.g., in Appendix J and Chapter 11) we have considered the weighted spaces

$$L^q(w;X) := \Big\{f: S\to X \text{ strongly measurable,} \quad \|f\|_{L^q(w;X)} := \Big(\int_S \|f(x)\|_X^q w(x)\,\mathrm{d}\mu(x)\Big)^{1/q} < \infty\Big\}.$$

For the present purposes, it is more convenient to introduce the variant

$$L^q_w(S;X) := \Big\{ f : S \to X \text{ strongly measurable}, \\ \|f\|_{L^q_w(S;X)} := \Big(\int_S \|f(x)w(x)\|_X^q \, \mathrm{d}\mu(x) \Big)^{1/q} < \infty \Big\}.$$

For $q < \infty$, this is just another way of expressing the same spaces with a different normalisation of the weight, namely $L^q_w(S;X) = L^q(w^q;X)$. However, using the usual modification for $q = \infty$, the first version reduces to just $L^\infty(w;X) = L^\infty(S;X)$ (since $\mathrm{d}\mu$ and $w\,\mathrm{d}\mu$ share the same zero sets), whereas $L^\infty_q(S;X)$ with norm $\|f\|_{L^\infty_w(S;X)} = \|fw\|_{L^\infty(S;X)}$ is a new space with non-trivial dependence on the weight w.

14.3.a Complex interpolation

Our first main result concerning these spaces is the following:

Theorem 14.3.1 (Stein–Weiss). *Let (Y_0, Y_1) be an interpolation couple of Banach spaces, let $q_0, q_1 \in [1,\infty]$ satisfy $\min\{q_0,q_1\} < \infty$. Let $(S, \mathscr{A}, \mu)$ be a σ-finite measure space, let w_0, w_1 be two weight functions on S, and let $\theta \in (0,1)$. Then*

$$[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta = L^q_w(S;[Y_0,Y_1]_\theta)$$

isometrically, where

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad w = w_0^{1-\theta} w_1^\theta.$$

We first record the simple:

Lemma 14.3.2. *In the setting of Theorem 14.3.1, if $f_n \to f$ in the norm of $L^{q_0}_{w_0}(S;Y_0) + L^{q_1}_{w_1}(S;Y_1)$, then a subsequence converges almost everywhere in the norm of $Y_0 + Y_1$ to the same limit function.*

Proof. We assume that $\|f_n - f\|_{L^{q_0}_{w_0}(S;Y_0)+L^{q_1}_{w_1}(S;Y_1)} \to 0$. Hence, for every n, there is a decomposition $f_n - f = f_n^0 + f_n^1$, where $\|f_n^j\|_{L^{q_j}_{w_j}(S;Y_j)} \to 0$ for $j = 0, 1$. By the well known version of the Lemma in just one L^p space, a subsequence of f_n^0 converges to 0 almost everywhere in the norm of Y_0. By the same result, a further subsequence of f_n^1 also converges to 0 almost everywhere in the norm if Y_1. Thus, along this last subsequence, $f_n - f = f_n^0 + f_n^1$ converges to 0 almost everywhere in the norm of $Y_0 + Y_1$. □

Proof of Theorem 14.3.1. The unweighted version ($w_0 = w_1 = w \equiv 1$) of this result is contained in Theorem 2.2.6. We will reduce the weighted version to this special case. Let us abbreviate $Y := [Y_0, Y_1]_\theta$. For $n \in \mathbb{Z}_+$, we denote $S_n := \{n^{-1} \leqslant w_0, w_1 \leqslant n\}$. Then $\bigcup_{n=1}^\infty S_n$ exhausts S, up to a set of measure zero, by definition of weights.

Step 1 – $L^q_w(S;[Y_0,Y_1]_\theta) \subseteq [L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$:
Let $f \in L^q_w(S;Y)$, and assume first assume that $\{f \neq 0\}$ is contained in S_n for some $n \in \mathbb{N}$. Thus

$$\phi := fw \in L^q(S;Y) = [L^{q_0}(S;Y_0), L^{q_1}(S;Y_1)]_\theta,$$

where the equality of space is Theorem 2.2.6, and hence $\phi = \Phi(\theta)$ for some $\Phi \in \mathscr{H}(L^{q_0}(S;Y_0), L^{q_1}(S;Y_1))$, where this notation of holomorphic functions on the unit strip with appropriate boundary behaviour is defined in Section C.2. The relation $\phi = \Phi(\theta)$ remains valid if we replace $\Phi(z)$ by $\Phi(z)\mathbf{1}_{E_n}$, and hence all the subsequent considerations can be restricted to E_n. In particular, multiplication by any power of w_0 or w_1 is then a bounded operation on any of the (weighted or not) L^p spaces appearing in this argument. Now

$$f = \phi w^{-1} = \Phi(\theta) w_0^{-(1-\theta)} w_1^{-\theta} = F(\theta),$$

where $F(z) := \Phi(z) w_0^{-(1-z)} w_1^{-z} \in \mathscr{H}(L^{q_0}(w_0;Y_0), L^{q_1}(w_1;Y_1))$. Qualitatively, the last inclusion is easy from the corresponding relation for Φ and the restriction of the supports on E_n, where all multiplications by powers of w_i are bounded. Quantitatively, we have

$$\begin{aligned} \|F(j+it)\|_{L^{q_j}(w_j;Y_j)} &= \|\Phi(j+it) w_0^{-(1-j)} w_1^{-j}\|_{L^{q_j}(w_j;Y_j)} \\ &= \|\Phi(j+it)\|_{L^{q_j}(S;Y_j)}, \qquad j = 0,1, \end{aligned}$$

thus, recalling that $\|F\|_{\mathscr{H}(X_0,X_1)} := \max_{j=0,1} \sup_{t\in\mathbb{R}} \|F(j+it)\|_{X_j}$,

$$\|F\|_{\mathscr{H}(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))} = \|\Phi\|_{\mathscr{H}(L^{q_0}(S;Y_0), L^{q_1}(S;Y_1))}, \tag{14.16}$$

and hence

$$\begin{aligned} \|f\|_{[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta} &\leqslant \|F\|_{\mathscr{H}(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))} \\ &= \|\Phi\|_{\mathscr{H}(L^{q_0}(S;Y_0), L^{q_1}(S;Y_1))}. \end{aligned}$$

Taking the infimum over all Φ in this space with $\phi = \Phi(\theta)$, we obtain

$$\begin{aligned} \|f\|_{[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta} &\leqslant \|\phi\|_{[L^{q_0}(S;Y_0), L^{q_1}(S;Y_1)]_\theta} \\ &= \|\phi\|_{L^q(S;Y)} = \|f\|_{L^q_w(S;Y)}. \end{aligned}$$

Recall that the previous estimate was obtained under the assumption that $f \in L^q_w(S;Y)$ satisfies $\{f \neq 0\} \subseteq S_n$. For a general $f \in L^q_w(S;Y)$, this bound holds with either $\mathbf{1}_{S_n} f$ of $\mathbf{1}_{S_n} f - \mathbf{1}_{S_m} f$ in place of f. Since $\mathbf{1}_{S_n} f \to f$ in $L^q_w(S;Y)$ by dominated convergence, it follows that $\mathbf{1}_{S_n} f$ is a Cauchy sequence, and hence convergent, in the interpolation space $[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$ and thus in the sum space $L^{q_0}_{w_0}(S;Y_0) + L^{q_1}_{w_1}(S;Y_1)$ by Lemma C.2.5. By Lemma 14.3.2, a subsequence converges almost everywhere to the same limit function. But it is clear that the a.e. limit is f, and hence

$$\begin{aligned}\|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1)]_\theta} &= \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1)]_\theta} \\ &\leqslant \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{L^q_w(S;Y)} = \|f\|_{L^q_w(S;Y)}.\end{aligned}$$

Step 2 – $[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta \subseteq L^q_w(S;[Y_0,Y_1]_\theta)$:
Let $f = F(\theta) \in [L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$, where

$$F \in \mathscr{H}(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)).$$

As before, we first assume that $\{f \neq 0\} \subseteq S_n$, and then without loss of generality (multiplying by $\mathbf{1}_{E_n}$ if necessary) that $F(z)$ has the same property for every z. We can then reverse the previous reasoning. Defining

$$\Phi(z) := F(z) w_0^{(1-z)} w_1^z,$$

we check the same relation (14.16), and hence

$$\begin{aligned}\|f\|_{L^q(w;Y)} &= \|F(\theta)w\|_{L^q(S;Y)} = \|\Phi(\theta)\|_{[L^{q_0}(S;Y_0),L^{q_1}(S;Y_0)]_\theta} \\ &\leqslant \|\Phi\|_{\mathscr{H}(L^{q_0}(S;Y_0),L^{q_1}(S;Y_0))} = \|F\|_{\mathscr{H}(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0))}.\end{aligned}$$

Taking the infimum over the relevant F with $F(\theta) = f$, we get

$$\|f\|_{L^q_w(S;Y)} \leqslant \|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta}, \quad \{f \neq 0\} \subseteq S_n. \tag{14.17}$$

Consider next a general $f \in [L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_0)]_\theta$. Multiplication by $\mathbf{1}_{S_n}$ contracts all L^p spaces, including weighted ones, and hence also the interpolation space $[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$ by Theorem C.2.6. Now (14.17) holds with $\mathbf{1}_{S_n} f$ in place of f, and hence

$$\|\mathbf{1}_{S_n} f\|_{L^q(w;Y)} \leqslant \|\mathbf{1}_{S_n} f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta} \leqslant \|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta}.$$

But then monotone convergence shows that

$$\|f\|_{L^q(w;Y)} = \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{L^q_w(S;Y)} \leqslant \|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta}.$$

This completes the proof. □

For easy reference later in this chapter, we state the special case of the previous result for sequence space with the weights $w_s(k) = 2^{ks}$ on the integers.

Proposition 14.3.3 (Complex interpolation of the spaces $\ell^q_{w_s}(Y)$). *Let (Y_0, Y_1) be an interpolation couple of Banach spaces, let $q_0, q_1 \in [1,\infty]$ satisfy* $\min\{q_0,q_1\} < \infty$*, and let $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0,1)$. Then*

$$[\ell^{q_0}_{w_{s_0}}(Y_0), \ell^{q_1}_{w_{s_1}}(Y_1)]_\theta = \ell^q_{w_s}([Y_0,Y_1]_\theta)$$

isometrically, where $s = (1-\theta)s_0 + \theta s_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. The condition $s = (1-\theta)s_0 + \theta s_1$ is equivalent to $w_s = w_{s_0}^{1-\theta} w_{s_1}^\theta$; whence the Proposition is a special case of Theorem 14.3.1. □

14.3.b Real interpolation

We next turn to the case of real interpolation. Recall that for a Banach couple (X_0, X_1), the real interpolation space $(X_0, X_1)_{\theta,p}$ with $p \in [1, \infty]$ and $\theta \in (0, 1)$, was introduced in Section C.3. Also recall from Theorem C.3.14 that if $p_0, p_1 \in [1, \infty]$ satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then $(X_0, X_1)_{\theta,p} = (X_0, X_1)_{\theta,p_0,p_1}$ with equivalent norms, where the latter denotes the Lions–Peetre interpolation of X_0 and X_1 (second mean method). The main result of this section is as follows.

Theorem 14.3.4 (Stein–Weiss, real version). *Let (Y_0, Y_1) be an interpolation couple of Banach spaces, let $q_0, q_1 \in [1, \infty]$ satisfy $\min\{q_0, q_1\} < \infty$. Let $(S, \mathscr{A}, \mu)$ be a σ-finite measure space, let w_0, w_1 be two weight functions on S, and let $\theta \in (0, 1)$. Then*

$$(L^{q_0}_{w_0}(S; Y_0), L^{q_1}_{w_1}(S; Y_1))_{\theta,q_0,q_1} = L^q_w(S; (Y_0, Y_1)_{\theta,q_0,q_1})$$

isometrically, where

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad w = w_0^{1-\theta} w_1^{\theta}.$$

In particular,

$$(L^{q_0}_{w_0}(S; Y_0), L^{q_1}_{w_1}(S; Y_1))_{\theta,q} = L^q_w(S; (Y_0, Y_1)_{\theta,q}),$$

with equivalent norms.

Proof. The unweighted version ($w_0 = w_1 = w \equiv 1$) of this result is contained in Theorem 2.2.10. We will reduce the weighted version to this special case. Let us abbreviate $Y := (Y_0, Y_1)_{\theta,q_0,q_1}$. As in the proof of Theorem 14.3.1 we denote $S_n := \{n^{-1} \leqslant w_0, w_1 \leqslant n\}$ for each $n \in \mathbb{Z}_+$, and observe that $\bigcup_{n=1}^{\infty} S_n$ exhausts S, up to a set of measure zero, by definition of weights.

Step 1 – $L^q_w(S; Y) \subseteq (L^{q_0}_{w_0}(S; Y_0), L^{q_1}_{w_1}(S; Y_1))_{\theta,q_0,q_1}$:
Let $f \in L^q(w; Y)$, and assume first that $\{f \neq 0\}$ is contained in S_n for some $n \in \mathbb{N}$. We also make the technical assumption that the weights w_j are discrete, in that they only take values of the form ρ^k, where $\rho > 1$ is a fixed number, and $k \in \mathbb{Z}$. This plays a role in the representation (14.18) below. Now

$$\phi := fw \in L^q(S; Y) = (L^{q_0}(S; Y_0), L^{q_1}(S; Y_1))_{\theta,q_0,q_1},$$

where the equality of spaces is Theorem 2.2.10. Hence, by Definition C.3.10 of the Lions–Peetre interpolation method $(\ ,\ )_{\theta,q_0,q_1}$, for some strongly measurable $\Phi : (0, \infty) \to L^{q_0}(S; Y_0) \cap L^{q_1}(S; Y_1)$, we have

$$\phi = \int_0^\infty \Phi(t) \frac{\mathrm{d}t}{t},$$

where $t^{j-\theta}\Phi(t) \in L^{q_j}(\,\mathrm{d}t/t; L^{q_j}(S;Y_j))$ for $j = 0,1$, and (as a consequence) the improper integral converges in $L^{q_0}(S;Y_0)+L^{q_1}(S;Y_1)$. Multiplying by $\mathbf{1}_{S_n}$ if necessary, we may assume that each $\Phi(t)$ is also supported on S_n.

Choosing the auxiliary weight $W := w_0^{-1}w_1$, we then have

$$f = \phi w^{-1} = \int_0^\infty \Phi(t)w^{-1}\frac{\mathrm{d}t}{t} = \int_0^\infty \Phi(tW)w^{-1}\frac{\mathrm{d}t}{t} =: \int_0^\infty F(t)\frac{\mathrm{d}t}{t}.$$

On S_n, both w_j are bounded from above and below. Due to the technical assumption on the discreteness of their ranges, both these weights, and hence W, only take finitely many possible value on S_n. Hence

$$F(t) = \Phi(tW)w^{-1} = \sum_{k=1}^K \mathbf{1}_{E_k}\Phi(t\alpha_k)\beta_k^{-1} \tag{14.18}$$

for some $\alpha_k, \beta_k \in (0,\infty)$ and sets $E_k \subseteq S_n$, from which it is immediate that also $F : (0,\infty) \to L^{q_0}(S;Y_0) \cap L^{q_1}(S;Y_1)$ is strongly measurable. This still remains true with each $L^{q_j}(S;Y_j)$ replaced by $L^{q_j}(w_j;Y_j)$ since the intersections of these spaces with functions supported on S_n coincide. With these qualitative issues out of the way, we make the quantitative observation

$$\begin{aligned}
&\int_0^\infty \|t^{j-\theta}F(t)\|^{q_j}_{L^{q_j}_{w_j}(S;Y_j)}\frac{\mathrm{d}t}{t}\\
&\quad= \int_0^\infty \|t^{j-\theta}\Phi(tW)w^{-1}w_j\|^{q_j}_{L^{q_j}(S;Y_j)}\frac{\mathrm{d}t}{t}\\
&\quad= \int_0^\infty \|W^{\theta-j}w^{-1}w_j t^{j-\theta}\Phi(t)\|^{q_j}_{L^{q_j}(S;Y_j)}\frac{\mathrm{d}t}{t}\\
&\quad= \int_0^\infty \|t^{j-\theta}\Phi(t)\|^{q_j}_{L^{q_j}(S;Y_j)}\frac{\mathrm{d}t}{t},
\end{aligned} \tag{14.19}$$

where in the last step our choice $W := w_0^{-1}w_1$ and the assumption $w = w_0^{1-\theta}w_1^{\theta}$ show that $W^{\theta-j}w^{-1}w_j \equiv 1$ for both $j = 0,1$ (and indeed having this identity dictates our choice of the auxiliary W).

Now, by the Lions–Peetre method, we have

$$\begin{aligned}
\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \sup_{j=0,1}\|t \mapsto t^{j-\theta}F(t)\|_{L^{q_j}(\,\mathrm{d}t/t;L^{q_j}_{w_j}(S;Y_j))}\\
&= \sup_{j=0,1}\|t \mapsto t^{j-\theta}\Phi(t)\|_{L^{q_j}(\,\mathrm{d}t/t;L^{q_j}(S;Y_j))},
\end{aligned}$$

and taking the infimum over all such Φ shows that

$$\begin{aligned}
\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \|\phi\|_{(L^{q_0}(S;Y_0),L^{q_1}(S;Y_1))_{\theta,q_0,q_1}}\\
&= \|\phi\|_{L^q(S;Y)} = \|f\|_{L^q_w(S;Y)}.
\end{aligned}$$

We proved this assuming that $\{f \neq 0\} \subseteq S_n$. For arbitrary $f \in L^q_w(S;Y)$, this is true with either $\mathbf{1}_{S_n}f$ or $\mathbf{1}_{S_n}f - \mathbf{1}_{S_m}f$ in place of f. It follows that $\mathbf{1}_{S_n}f$

is a Cauchy sequence, and hence convergent, in $(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}$, and thus in $L^{q_0}_{w_0}(S;Y_0) + L^{q_1}_{w_1}(S;Y_1)$ by the very Definition C.3.10 (recall that $f \in (X_0, X_1)_{\theta,q_0,q_1}$ is given by an integral that converges in $X_0 + X_1$). By Lemma 14.3.2, a subsequence converges almost everywhere to the same limit, and hence this limit must be f. Thus $f \in (L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}$, and

$$\begin{aligned}\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &= \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} \\ &\leqslant \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{L^q_w(S;Y)} = \|f\|_{L^q_w(S;Y)}.\end{aligned}$$

We still had the additional hypothesis on the discreteness of the ranges of both w_j. For arbitrary weights w_j and $\rho > 1$, we consider

$$w_{j,\rho} := \sup\{\rho^k : \rho^k \leqslant w_j, k \in \mathbb{Z}\},$$

which clearly satisfy the discreteness property, as well as $w_{j,\rho} \leqslant w_j \leqslant \rho w_{j,\rho}$. Hence

$$\|f\|_{L^{q_j}_{w_j}(S;Y_j)} \leqslant \rho \|f\|_{L^{q_j}_{w^\rho_j}(S;Y_j)}$$

and Theorem C.3.16 gives the first estimate in

$$\begin{aligned}\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \rho^{1-\theta}\rho^{\theta} \|f\|_{(L^{q_0}_{w_{0,\rho}}(S;Y_0),L^{q_1}_{w_{1,\rho}}(S;Y_1))_{\theta,q_0,q_1}} \\ &= \rho \|f\|_{L^q_{w^{1-\theta}_{0,\rho} w^{\theta}_{1,\rho}}(S;Y)} \\ &\leqslant \rho \|f\|_{L^q(w;Y)}.\end{aligned}$$

Taking the limit $\rho \to 1$, we finally deduce

$$\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} \leqslant \|f\|_{L^q_w(S;Y)}$$

unconditionally.

Step 2 – $(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1} \subseteq L^q_w(S;Y)$:
Let $f \in (L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}$. We make the same initial assumptions on both f and the weights w_j as in the previous part. By definition, we have $f = \int_0^\infty F(t) \frac{\mathrm{d}t}{t}$ with $t^{j-\theta} F(t) \in L^{q_j}(\mathrm{d}t/t; L^{q_j}_{w_j}(S;Y_j))$. Working the previous computations backwards, we find that

$$\phi := fw = \int_0^\infty F(t) w \frac{\mathrm{d}t}{t} = \int_0^\infty F(tW^{-1}) w \frac{\mathrm{d}t}{t} =: \int_0^\infty \Phi(t) \frac{\mathrm{d}t}{t},$$

where Φ satisfies the relevant measurability conditions (by the structural assumptions on the weights) and the quantitative relation (14.19). We conclude that

$$\begin{aligned}\|\phi\|_{(L^{q_0}(S;Y_0),L^{q_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \sup_{j=0,1} \|t^{j-\theta}\Phi(t)\|_{L^{q_j}(\mathrm{d}t/t;L^{q_j}(S;Y_j))} \\ &= \sup_{j=0,1} \|t^{j-\theta}F(t)\|_{L^{q_j}(\mathrm{d}t/t;L^{q_j}_{w_j}(S;Y_j))},\end{aligned}$$

and taking the infimum over all relevant F,

$$\begin{aligned}\|f\|_{L^q_w(S;Y)} = \|\phi\|_{L^q(S;Y)} &= \|\phi\|_{(L^{q_0}(S;Y_0),L^{q_1}(S;Y_1))_{\theta,q_0,q_1}} \\ &\leqslant \|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}}.\end{aligned}$$

For a general f in the interpolation space, applying the previous conclusion to $\mathbf{1}_{S_n}f$ in place of f, we have

$$\begin{aligned}\|\mathbf{1}_{S_n}f\|_{L^q_w(S;Y)} &\leqslant \|\mathbf{1}_{S_n}f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} \\ &\leqslant \|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}},\end{aligned}$$

since multiplication by $\mathbf{1}_{S_n}$ is clearly contractive on each $L^{q_j}(w_j;Y_j)$, and hence on the interpolation space by Theorem C.3.16. It then follows from monotone convergence that

$$\|f\|_{L^q_w(S;Y)} = \lim_{n\to\infty} \|\mathbf{1}_{S_n}f\|_{L^q_w(S;Y)} \leqslant \|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}}.$$

Finally, the discreteness assumption on the weights can be removed by the same considerations as in the previous part: For general weights w_j and the auxiliary discrete $w_{j,\rho}$ as in the previous part, we have

$$\begin{aligned}\|f\|_{L^q_w(S;Y)} = \|f\|_{L^q_{w_0^{1-\theta}w_1^{\theta}}(S;Y)} &\leqslant \rho^{(1-\theta)+\theta}\|f\|_{L^q_{w_{0,\rho}^{1-\theta}w_{1,\rho}^{\theta}}(S;Y)} \\ &\leqslant \rho\|f\|_{((L^{q_0}_{w_{0,\rho}}(S;Y_0),L^{q_1}_{w_{1,\rho}}(S;Y_1))_{\theta,q_0,q_1}} \\ &\leqslant \rho\|f\|_{((L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}},\end{aligned}$$

and taking the limit $\rho \to 1$ completes the proof. □

For applications of the real interpolation theorem to Besov spaces, it is useful to include a version that is genuine variant, rather than just a special case of the previous theorem. This version is concerned with the particular case of $S = \mathbb{N}$ or $S = \mathbb{Z}$ with the exponential weights $w_s(k) = 2^{ks}$, and restricting to just one range space $Y_0 = Y_1 = Y$. Remarkably, under these circumstances the condition $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ of Theorem 14.3.4 can be omitted:

Proposition 14.3.5 (Real interpolation of the spaces $\ell^q_{w_s}(Y)$). *Let $p, q_0, q_1 \in [1,\infty]$, let $s_0, s_1 \in \mathbb{R}$ satisfy $s_0 \neq s_1$, let $\theta \in (0,1)$, and let $s = (1-\theta)s_0 + \theta s_1$. Then*

$$(\ell^{q_0}_{w_{s_0}}(Y), \ell^{q_1}_{w_{s_1}}(Y))_{\theta,p} = \ell^p_{w_s}(Y) \quad \textit{with equivalent norms,}$$

with constants in the norm estimates only depending on θ, p, s_0, s_1. Moreover, for all $y \in \ell^{q_0}_{w_{s_0}}(Y) \cap \ell^{q_1}_{w_{s_1}}(Y)$ we have

$$\|y\|_{\ell^p_{w_s}(Y)} \leqslant C\|y\|^{1-\theta}_{\ell^{q_0}_{w_{s_0}}(Y)}\|y\|^{\theta}_{\ell^{q_1}_{w_{s_1}}(Y)},$$

where C only depends on s_0, s_1, θ.

Proof. We will present the details for $S = \mathbb{N}$, as the case $S = \mathbb{Z}$ is proved in the same way. By interchanging the roles of $\ell^{q_0}_{w_{s_0}}(Y)$ and $\ell^{q_1}_{w_{s_1}}(Y)$ if necessary, without loss of generality we may assume that $s_0 > s_1$.

Since $\ell^{q_0}_{w_{s_0}}(Y) \hookrightarrow \ell^{\infty}_{w_{s_0}}(Y)$ and $\ell^{q_1}_{w_{s_1}}(Y) \hookrightarrow \ell^{\infty}_{w_{s_1}}(Y)$ continuously, real interpolation (Theorem C.3.3) gives $(\ell^{q_0}_{w_{s_0}}(Y), \ell^{q_1}_{w_{s_1}}(Y))_{\theta,p} \hookrightarrow (\ell^{\infty}_{w_{s_0}}(Y), \ell^{\infty}_{w_{s_1}}(Y))_{\theta,p}$ continuously. Hence to show that $(\ell^{q_0}_{w_{s_0}}(Y), \ell^{q_1}_{w_{s_1}}(Y))_{\theta,p}$ embeds into $\ell^{p}_{w_s}(Y)$ it suffices to consider the case $q_0 = q_1 = \infty$. If $y = y^{(0)} + y^{(1)} \in \ell^{\infty}_{w_{s_0}}(Y) + \ell^{\infty}_{w_{s_1}}(Y)$, then

$$\|y_k\| \leqslant \|y_k^{(0)}\| + \|y_k^{(1)}\| \leqslant 2^{-ks_0}\|y^{(0)}\|_{\ell^{\infty}_{w_{s_0}}(Y)} + 2^{-ks_1}\|y^{(1)}\|_{\ell^{\infty}_{w_{s_1}}(Y)}.$$

Multiplying with 2^{ks_0} and taking the infimum over all admissible pairs $(y^{(0)}, y^{(1)})$, we find

$$2^{ks_0}\|y_k\| \leqslant K(2^{k(s_0-s_1)}, y)$$

using the notation of Section C.3. In combination with the identity $\theta(s_1 - s_0) = s - s_0$ and the fact that the K-functional is non-decreasing, this gives

$$\begin{aligned}
\|y\|_{\ell^p_{w_s}} &\leqslant \Big(\sum_{k\geqslant 0} |2^{k(s-s_0)} K(2^{k(s_0-s_1)}, y)|^p\Big)^{1/p} \\
&\leqslant C_0\Big(\sum_{k\geqslant 0} \int_{2^{k(s_0-s_1)}}^{2^{(k+1)(s_0-s_1)}} |t^{-\theta} K(2^{k(s_0-s_1)}, y)|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} \\
&\leqslant C_0\Big(\int_0^{\infty} |t^{-\theta} K(t, y)|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} = C_0\|y\|_{(\ell^{q_0}_{w_{s_0}}(Y), \ell^{q_1}_{w_{s_1}}(Y))_{\theta,p}},
\end{aligned}$$

where $C_0 = \frac{(\theta p)^{1/p}}{(1-2^{-(s_0-s)p})^{1/p}}$ if $p < \infty$. A simple modification of this argument gives the same result with $C_0 = 1$ if $p = \infty$.

To prove the reverse inequality it suffices to consider the case $q_0 = q_1 = 1$. Discretising as before, we find

$$\begin{aligned}
\|y\|_{(\ell^{1}_{w_{s_0}}(Y), \ell^{1}_{w_{s_1}}(Y))_{\theta,p}} &\leqslant \Big(\sum_{k\geqslant 0} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} |t^{-\theta} K(t, y)|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} \\
&\leqslant C_1\Big(\sum_{k\geqslant 0} |2^{-\theta k(s_0-s_1)} K(2^{k(s_0-s_1)}, y)|^p\Big)^{1/p},
\end{aligned}$$

where $C_1 = \frac{(2^{(s_0-s)p}-1)^{1/p}}{(\theta p)^{1/p}}$. If $p = \infty$ we consider the supremum norm in the above and take $C_1 = 2^{s_0-s}$. Splitting $y_m = y_m\mathbf{1}_{\{m\leqslant k\}} + y_m\mathbf{1}_{\{m>k\}}$, we estimate

$$K(2^{k(s_0-s_1)}, y) \leqslant \sum_{m=-\infty}^{k} 2^{ms_0}\|y_m\| + 2^{k(s_0-s_1)} \sum_{m=k+1}^{\infty} 2^{s_1 m}\|y_m\|.$$

Therefore, since $\theta(s_1 - s_0) = s - s_0$ and $(1-\theta)(s_1 - s_0) = s - s_1$,

$$2^{-\theta k(s_0-s_1)}K(2^{k(s_0-s_1)}, y) \leqslant \sum_{m=-\infty}^{k} 2^{(m-k)(s_0-s)}2^{ms}\|y_m\| + \sum_{m=k+1}^{\infty} 2^{-(m-k)(s-s_1)}2^{ms}\|y_m\|.$$

Taking ℓ^p-norms in k and using Young's inequality for convolutions we obtain

$$\Big(\sum_{k\geqslant 0} |2^{-\theta k(s_0-s_1)}K(2^{k(s_0-s_1)}, y)|^p\Big)^{1/p} \leqslant (C_2 + C_3)\|y\|_{\ell^p_{w_s}(Y)}.$$

where $C_2 = \sum_{k=0}^{\infty} 2^{-k(s_0-s)}$ and $C_3 = \sum_{k=1}^{\infty} 2^{-k(s-s_1)}$. This gives the inequality

$$\|y\|_{(\ell^1_{w_{s_0}}(Y),\ell^1_{w_{s_1}}(Y))_{\theta,p}} \leqslant C_1(C_2 + C_3)\|y\|_{\ell^p_{w_s}(Y)}.$$

The final assertion is immediate from the first assertion and the log-convexity inequality (L.2). □

14.4 Besov spaces

The various Littlewood–Paley decompositions encountered in Chapter 5 express the norm of a function $f \in L^p(\mathbb{R}^d; X)$ in terms of (sharp or smooth) dyadic cut-offs in the frequency domain. For instance, in Theorem 5.5.22 we have seen that if X is a UMD Banach space, $p \in (1, \infty)$, and ψ is a smooth Littlewood–Paley function,

$$\|f\|_{L^p(\mathbb{R}^d;X)} \eqsim \Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k \psi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}, \tag{14.20}$$

where $\psi_k(x) := 2^k\psi(2^k x)$ and $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a Rademacher sequence. With an eye toward the ensuing discussion we also remark that we have an equivalence of norms

$$\|f\|_{L^p(\mathbb{R}^d;X)} \eqsim \Big\| \sum_{k\in\mathbb{N}} \varepsilon_k \varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}, \tag{14.21}$$

where now $(\varphi_k)_{k\in\mathbb{N}}$ is an inhomogeneous Littlewood–Paley sequence as in (14.20). This follows from Theorem 14.7.5 below, but could already have been proved in Chapter 5 with the methods presented there.

The idea behind the Littlewood–Paley approach to Besov spaces is to take this representation as a starting point, introducing an additional smoothness parameter $s \in \mathbb{R}$, and trading the norm of the Rademacher sum for an $\ell^q_{w_s}$-sum. The possibility of having $p \neq q$ presents us with two possible definitions, utilising the spaces $\ell^q_{w_s}(L^p(\mathbb{R}^d; X))$ and $L^p(\mathbb{R}^d; \ell^q_{w_s}(X))$ respectively. For $p =$

q, these spaces are canonically isometric by Fubini's theorem. The two choices lead to the theory of Besov spaces and Triebel–Lizorkin spaces, respectively.

The choice $\ell^q_{w_s}(\mathbb{Z})$ with the (homogeneous) Littlewood–Paley sequence $(\psi_k)_{k\in\mathbb{Z}}$ as in (14.20) leads to the so-called *homogeneous* Besov and Triebel–Lizorkin spaces. Alternatively, the choice $\ell^q_{w_s}(\mathbb{N})$ and the use of Littlewood–Paley sequences $(\varphi_k)_{k\in\mathbb{N}}$ as introduced in Definition 14.2.7 leads to the inhomogeneous version of these spaces. In what follows we will only present in the inhomogeneous case. Both classes of spaces are used in applications to PDE. The advantage of inhomogeneous spaces is that, in the development of their theory, one can make effective use of Schwartz functions and tempered distributions. The theory of homogeneous spaces is technically more involved and requires the use of different classes of test functions and equivalence classes of tempered distributions modulo polynomials. Since we have already encountered Schwartz functions and tempered distributions in many places, we choose to only develop the theory of inhomogeneous spaces here. Homogeneous spaces have better scaling properties, and scaling often plays a crucial role in PDE, but for the purposes of the theory developed here homogeneous spaces are not essential.

The proofs of (14.20) and (14.21) require the Banach space X to be UMD. In contrast, in the theory of Besov and Triebel–Lizorkin spaces these norm equivalences *are promoted to definitions*, thus eliminating the need of imposing any conditions on X. By taking this approach, most of the fundamental results in the theory of Besov spaces and Triebel–Lizorkin spaces are true for arbitrary Banach spaces X. They come with their own versions of the Mihlin multiplier theorem which does not require the UMD property either, allowing multipliers without singularities at the origin in case of inhomogeneous spaces. The more general multipliers considered in Chapter 5 have corresponding versions for homogeneous Besov and Triebel–Lizorkin spaces. Perhaps more surprising is the fact that also for the duality theory of these spaces no geometrical conditions need to be imposed on X. This contrast the duality theory for the Bochner spaces, which requires that X^* have the Radon–Nikodým property.

14.4.a Definitions and basic properties

As anticipated in the above discussion, we now introduce scale of Besov spaces through a Littlewood–Paley decomposition.

Definition 14.4.1. *Let $p, q \in [1,\infty]$ and $s \in \mathbb{R}$. The* Besov space $B^s_{p,q}(\mathbb{R}^d;X)$ *is the space of all $f \in \mathscr{S}'(\mathbb{R}^d;X)$ for which $\varphi_k * f \in L^p(\mathbb{R}^d;X)$ for all $k \geqslant 0$ and for which the quantity*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} := \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}$$

is finite.

Here, $(\varphi_k)_{k\geqslant 0}$ is the inhomogeneous Littlewood–Paley sequence that has been fixed throughout the chapter (see Convention 14.2.8). By the discussion of (14.13), the tempered distribution $\varphi_k * f$ is a C^∞-function of polynomial growth, so that the L^p-norm in the above definition makes sense.

To see that $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^d;X)}$ is indeed a norm, suppose that $\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} = 0$. Then $\widehat{\varphi}_k \widehat{f} = \mathscr{F}(\varphi_k * f) = 0$ for all $k \geqslant 0$, so $\varphi_k * f = 0$ for all $k \geqslant 0$, and therefore $f = 0$ by Lemma 14.2.10. All other properties of a norm can be deduced from the fact that $\|\cdot\|_{\ell^q(L^p(\mathbb{R}^d;X))}$ is a norm.

It is immediate from Young's inequality, applied term-wise with respect to the ℓ^q-sum, that $\psi * f \in B^s_{p,q}(\mathbb{R}^d;X)$ whenever $\psi \in L^1(\mathbb{R}^d)$ and $f \in B^s_{p,q}(\mathbb{R}^d;X)$, and more generally the analogue of Proposition 14.2.3 is valid.

Up to an equivalent norm the above definition is independent on the choice of the sequence $(\varphi_k)_{k\geqslant 0}$, as will be shown in Proposition 14.4.2.

From the continuous embedding $\ell^{q_0} \hookrightarrow \ell^{q_1}$ for $1 \leqslant q_0 \leqslant q_1 \leqslant \infty$ we obtain the continuous embedding

$$B^s_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,q_1}(\mathbb{R}^d;X). \tag{14.22}$$

For $1 \leqslant q_0, q_1 \leqslant \infty$ and $s_0 > s_1$ we have the continuous embedding

$$B^{s_0}_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p,q_1}(\mathbb{R}^d;X). \tag{14.23}$$

Indeed, for $q_0 \leqslant q_1$ this follows from (14.22) and the inequality $2^{ks_0} \leqslant 2^{ks_1}$ for $k \geqslant 0$. For $q_0 > q_1$ this follows from Hölder's inequality with $\frac{1}{q_1} = \frac{1}{q_0} + \frac{1}{r}$ and using that $\sum_{k\geqslant 0} 2^{-k(s_0-s_1)r} < \infty$.

Proposition 14.4.2. *For all $p, q \in [1,\infty]$ and $s \in \mathbb{R}$, up to an equivalent norm the space $B^s_{p,q}(\mathbb{R}^d;X)$ is independent of the choice of inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$.*

The proof will give explicit constants depending only on s and φ_0 (in one direction), respectively s and ψ_0 (in the other direction).

Proof. Suppose $(\psi_k)_{k\geqslant 0}$ is another inhomogeneous Littlewood–Paley sequence. Then the analogues of (14.10) and (14.11) hold with φ_j and ψ_k; in particular for all $j, k \geqslant 0$ with $|j-k| \geqslant 2$ we have $\varphi_k * \psi_j = 0$. Using (14.12) for the sequence $(\psi_k)_{k\geqslant 0}$, the triangle inequality, Young's inequality, and (14.7), we obtain

$$\begin{aligned}
&\big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \\
&\qquad \leqslant \sum_{j=-1}^{1} \big\|(2^{ks}\varphi_k * \psi_{k+j} * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \\
&\qquad \leqslant \|\varphi_k\|_1 \sum_{j=-1}^{1} 2^{|s|} \big\|(2^{(k+j)s}\psi_{k+j} * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}
\end{aligned}$$

$$\leqslant 6\|\varphi_0\|_1 2^{|s|}\big\|(2^{ks}\psi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))},$$

where we used (14.7). This gives the required estimate in one direction. The reverse estimate is obtained by reversing the rôles of φ_k and ψ_k. □

Proposition 14.4.3. *For all $p,q \in [1,\infty]$ and $s \in \mathbb{R}$ we have continuous embeddings*

$$\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X).$$

Moreover, if $1 \leqslant p,q < \infty$, then $C^\infty_c(\mathbb{R}^d)\otimes X$ is dense in $B^s_{p,q}(\mathbb{R}^d;X)$.

Proof. We split the proof into three steps.

Step 1 – For the first embedding, by (14.22) it is enough to prove that $\mathscr{S}(\mathbb{R}^d;X)$ embeds into $B^s_{p,1}(\mathbb{R}^d;X)$. For $f \in \mathscr{S}(\mathbb{R}^d;X)$ and $L = L_{p,d} \in \mathbb{N}$ so large that $(1+|2\pi\cdot|^{2L})^{-1} \in L^p(\mathbb{R}^d)$ we find

$$\begin{aligned}\|f\|_{B^s_{p,1}(\mathbb{R}^d;X)} &= \sum_{k\geqslant 0} 2^{ks}\|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)}\\ &\lesssim_{d,p} \sum_{k\geqslant 0} 2^{ks}\big\|(1+|2\pi\cdot|^{2L})\varphi_k * f\big\|_{L^\infty(\mathbb{R}^d;X)}\\ &\leqslant \sum_{k\geqslant 0} 2^{ks}\big\|(1+(-\Delta)^L)(\widehat{\varphi}_k\widehat{f})\big\|_{L^1(\mathbb{R}^d;X)},\end{aligned}$$

where we used the fact that $\mathscr{F}^{-1}$ maps L^1 into L^∞. It remains to estimate $2^{ks}\|\partial^\alpha(\widehat{\varphi}_k\widehat{f})\|_{L^1(\mathbb{R}^d;X)}$ for multi-indices $|\alpha| \leqslant 2L$.

First consider $k \geqslant 1$. Then $\operatorname{supp}\varphi_k \subseteq B_k := \{\xi \in \mathbb{R}^d : 2^{k-1} \leqslant |\xi| \leqslant 3\cdot 2^k\}$ and $|B_k| \lesssim_d 2^{kd}$. By Leibniz' rule and the boundedness on B_k of the functions $\partial^\gamma\widehat{\varphi}_k$ with $|\gamma| \leqslant |\alpha| \leqslant 2L = 2L_{p,d}$,

$$\|\partial^\alpha(\widehat{\varphi}_k\widehat{f})\|_{L^1(\mathbb{R}^d;X)} \lesssim_{d,p} \sum_{|\beta|\leqslant|\alpha|} \|\mathbf{1}_{B_k}\partial^\beta\widehat{f}\|_{L^1(\mathbb{R}^d;X)}.$$

To estimate the terms on the right-hand side, fix an $M \in \mathbb{N}$ which is arbitrary for the moment. Then

$$\begin{aligned}\|\mathbf{1}_{B_k}\partial^\beta\widehat{f}\|_{L^1(\mathbb{R}^d;X)} &\leqslant \|\mathbf{1}_{B_k}(1+|\cdot|^{2M})^{-1}\|_{L^1(\mathbb{R}^d)}\|(1+|\cdot|^{2M})\partial^\beta\widehat{f}\|_{L^\infty(\mathbb{R}^d;X)}\\ &\leqslant |B_k|(1+2^{2M(k-1)})^{-1}\sum_{|\delta|\leqslant 2M}[\widehat{f}]_{\beta,\delta},\end{aligned}$$

using the notation (14.2) for the seminorms defining the Schwartz space. Keeping in mind that $|B_k| \lesssim_d 2^{kd}$ we now choose $M = M_{s,p,d} \in \mathbb{N}$ so large that $\sum_{k\geqslant 0} 2^{ks}2^{kd}(1+2^{2M(k-1)})^{-1} < \infty$. With this choice, we obtain the estimate

$$\|f\|_{B^s_{p,1}(\mathbb{R}^d;X)} \lesssim_{d,p,s} \sum_{|\delta|\leqslant 2M}[\widehat{f}]_{\beta,\delta}.$$

A similar estimate in the case $k = 0$ can be obtained in a similar, but simpler, way. Since $\mathscr{F}$ is continuous on $\mathscr{S}(\mathbb{R}^d; X)$ (see Proposition 2.4.22), this proves that we have a continuous embedding $\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d; X)$.

Step 2 – Next we prove that $B^s_{p,q}(\mathbb{R}^d; X)$ embeds into $\mathscr{S}'(\mathbb{R}^d; X)$. By (14.22) it is enough to prove that the inclusion mapping $B^s_{p,\infty}(\mathbb{R}^d; X) \subseteq \mathscr{S}'(\mathbb{R}^d; X)$ (by definition $B^s_{p,\infty}(\mathbb{R}^d; X)$ is contained in $\mathscr{S}'(\mathbb{R}^d; X)$) is continuous.

Fix $f \in B^s_{p,\infty}(\mathbb{R}^d; X)$ and $\psi \in \mathscr{S}(\mathbb{R}^d)$, and set $f_k := \varphi_k * f$ and $\psi_k := \varphi_k * \psi$. By Lemma 14.2.10 and (14.10) we have

$$f(\psi) = \sum_{k,\ell \geqslant 0} f_k(\psi_\ell) = \sum_{\ell=-1}^{1} \sum_{k \geqslant 0} f_k(\psi_{k+\ell}).$$

Thus, by (14.13),

$$\begin{aligned}
\|f(\psi)\| &\leqslant \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} \int_{\mathbb{R}^d} \|f_k(x)\| |\psi_{k+\ell}(x)| \,\mathrm{d}x \\
&\leqslant \sum_{\ell=-1}^{1} \big\|(2^{ks}\|f_k(\cdot)\|)_{k\geqslant 0}\big\|_{\ell^\infty(L^p(\mathbb{R}^d;X))} \big\|(2^{-ks}\psi_{k+\ell})_{k\geqslant 0}\big\|_{\ell^1(L^{p'}(\mathbb{R}^d))} \\
&\leqslant 3 \cdot 2^{|s|} \|f\|_{B^s_{p,\infty}(\mathbb{R}^d;X)} \|\psi\|_{B^{-s}_{p',1}(\mathbb{R}^d)}.
\end{aligned}$$

Since $\mathscr{S}(\mathbb{R}^d) \hookrightarrow B^{-s}_{p',1}(\mathbb{R}^d)$ continuously by the previous step, the result follows from this.

Step 3 – To prove density, by Lemma 14.2.1 it suffices to prove the density of $\mathscr{S}(\mathbb{R}^d; X)$ in $B^s_{p,q}(\mathbb{R}^d; X)$.

Fix $f \in B^s_{p,q}(\mathbb{R}^d; X)$ and set $\zeta_n := \sum_{k=0}^n \varphi_k$. By (14.6) we have $\|\zeta_n\|_1 = \|\varphi\|_1$.

We will first show that $\zeta_n * f \to f$ in $B^s_{p,q}(\mathbb{R}^d; X)$. Fix $\varepsilon > 0$ and choose $K \in \mathbb{N}$ such that

$$\sum_{k>K} 2^{ksq} \|\varphi_k * f\|^q_{L^p(\mathbb{R}^d;X)} < \varepsilon^q.$$

By Young's inequality combined with the identity $\|\zeta_n\|_1 = \|\varphi_0\|_1$ we have $\zeta_n * \varphi_k * f \in L^p(\mathbb{R}^d; X)$ and $\|\zeta_n * \varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \leqslant \|\varphi\|_1 \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)}$. From this we infer that $\zeta_n * f \in B^s_{p,q}(\mathbb{R}^d; X)$ and

$$\sum_{k>K} 2^{ksq} \|\zeta_n * \varphi_k * f\|^q_{L^p(\mathbb{R}^d;X)} < \varepsilon^q \|\varphi\|_1^q.$$

Hence by the triangle inequality in $\ell^q(L^p(\mathbb{R}^d; X))$,

$$\|f - \zeta_n * f\|_{B^s_{p,q}(\mathbb{R}^d;X)}$$

$$= \Big(\sum_{k\geqslant 0} 2^{ksq} \|\varphi_k * (f - \zeta_n * f)\|^q_{L^p(\mathbb{R}^d;X)} \Big)^{1/q}$$
$$\leqslant \Big(\sum_{k=0}^{K} 2^{ksq} \|\varphi_k * (f - \zeta_n * f)\|^q_{L^p(\mathbb{R}^d;X)} \Big)^{1/q} + \varepsilon(1 + \|\varphi\|_1).$$

The first term in the last expression tends to zero as $n \to \infty$ by Proposition 1.2.32; here we use that $\zeta_n = 2^{nd}\varphi(2^n\cdot)$ and $\int_{\mathbb{R}^d} \varphi \,\mathrm{d}x = \widehat{\varphi}(0) = 1$. This concludes the proof that $\zeta_n * f \to f$ in $B^s_{p,q}(\mathbb{R}^d;X)$.

It remains to approximate each of the functions $f_n = \zeta_n * f$ by elements in $\mathscr{S}(\mathbb{R}^d;X)$. Observe that $f_n \in L^p(\mathbb{R}^d;X)$ since the functions $\varphi_k * f$ belong to $L^p(\mathbb{R}^d;X)$. Let $\eta \in \mathscr{S}(\mathbb{R}^d)$ be a functions satisfying $\eta(0) = 1$ and $\operatorname{supp}(\widehat{\eta}) \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$. Since $\mathscr{F}(\eta(\delta\cdot)) = \delta^{-d}\widehat{\eta}(\delta^{-1}\cdot)$, for all $\delta \in (0,1)$ the support of $\mathscr{F}(\eta(\delta\cdot)f_n)$ is contained in a ball of radius $3 \cdot 2^{n-1} + 1 \leqslant 2^{n+1}$; here we use the definition of ζ_n and (14.9). Using (14.11), (14.7), and Young's inequality, it follows that

$$\|f_n - \eta(\delta\cdot)f_n\|_{B^s_{p,q}(\mathbb{R}^d;X)} = \Big(\sum_{k=0}^{n+2} 2^{ksq} \|\varphi_k * (f_n - \eta(\delta\cdot)f_n)\|^q_{L^p(\mathbb{R}^d;X)} \Big)^{1/q}$$
$$\leqslant C\|f_n - \eta(\delta\cdot)f_n\|_{L^p(\mathbb{R}^d;X)},$$

where $C = C_{n,s,q} = (\sum_{k=0}^n 2^{ksq})^{1/q}$. For each fixed n, the right-hand side tends to zero as $\delta \downarrow 0$ by the dominated convergence theorem. □

Next we will prove the completeness of the normed space $B^s_{p,q}(\mathbb{R}^d;X)$.

Proposition 14.4.4. *For $p, q \in [1,\infty]$ and $s \in \mathbb{R}$ the space $B^s_{p,q}(\mathbb{R}^d;X)$ is a Banach space.*

The proof requires some preparations. Recall that a sequence $(f_n)_{n\geqslant 1}$ is said to *converge* in $\mathscr{S}'(\mathbb{R}^d;X)$ if there exists an $f \in \mathscr{S}'(\mathbb{R}^d;X)$ such that $f_n(\phi) \to f(\phi)$ in X for all $\phi \in \mathscr{S}(\mathbb{R}^d)$. Likewise, it is said to be *Cauchy* in $\mathscr{S}'(\mathbb{R}^d;X)$ if $(f_n(\phi))_{n\geqslant 1}$ is a Cauchy sequence in X for all $\phi \in \mathscr{S}(\mathbb{R}^d)$.

Lemma 14.4.5. *The space $\mathscr{S}'(\mathbb{R}^d;X)$ is sequentially complete, i.e., every Cauchy sequence in $\mathscr{S}'(\mathbb{R}^d;X)$ is convergent in $\mathscr{S}'(\mathbb{R}^d;X)$.*

Proof. Let $(f_n)_{n\geqslant 1}$ be a Cauchy sequence in $\mathscr{S}'(\mathbb{R}^d;X)$. Since X is complete we may define a linear mapping $f : \mathscr{S}(\mathbb{R}^d) \to X$ by $f(\phi) := \lim_{n\to\infty} f_n(\phi)$. We claim that f is continuous. Indeed, for every $\phi \in \mathscr{S}(\mathbb{R}^d)$ the sequence $(f_n(\phi))_{n\geqslant 1}$ is bounded in X, and therefore the Banach–Steinhaus theorem for topological vector spaces implies that the sequence $(f_n)_{n\geqslant 1}$ is equicontinuous. Hence, given an $\varepsilon > 0$, we can find an open neighbourhood V of 0 in $\mathscr{S}(\mathbb{R}^d)$ such that $|f_n(\phi)| \leqslant \varepsilon$ for all $\phi \in V$ and $n \geqslant 1$. Taking limits, it follows that $|f(\phi)| \leqslant \varepsilon$ for all $\phi \in V$. This means that f is continuous at zero and hence continuous. □

A normed space $E \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$ is said to have the *Fatou property* if for all sequences $(f_n)_{n\geqslant 1}$ in E such that

$$f_n \to f \text{ in } \mathscr{S}'(\mathbb{R}^d; X) \quad \text{and} \quad \liminf_{n\to\infty} \|f_n\|_E < \infty$$

we have $f \in E$ and $\|f\|_E \leqslant \liminf_{n\to\infty} \|f_n\|_E$.

Lemma 14.4.6. *For all $p, q \in [1, \infty]$ and $s \in \mathbb{R}$ the space $B^s_{p,q}(\mathbb{R}^d; X)$ has the Fatou property.*

Proof. Choose a sequence $(f_n)_{n\geqslant 1}$ of functions from $B^s_{p,q}(\mathbb{R}^d; X)$ with

$$f_n \to f \text{ in } \mathscr{S}'(\mathbb{R}^d; X) \quad \text{and} \quad \liminf_{n\to\infty} \|f_n\|_{B^s_{p,q}(\mathbb{R}^d;X)} < \infty.$$

Then $\lim_{n\to\infty} \varphi_k * f_n = \varphi_k * f$ pointwise. In case $p < \infty$, Fatou's lemma gives

$$\|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \leqslant \liminf_{n\to\infty} \|\varphi_k * f_n\|_{L^p(\mathbb{R}^d;X)} < \infty.$$

Multiplying with 2^{ks} and taking ℓ^q-norms, it follows that we have $f \in B^s_{p,q}(\mathbb{R}^d; X)$ and $\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \leqslant \liminf_{n\to\infty} \|f_n\|_{B^s_{p,q}(\mathbb{R}^d;X)}$ (by Fatou's lemma if $q < \infty$ and directly if $q = \infty$). For $p = \infty$ the proof is similar. □

Lemma 14.4.7. *Every normed space $E \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$ with the Fatou property is complete.*

Proof. Let $(f_n)_{n\geqslant 1}$ be a Cauchy sequence in E. Since $\mathscr{S}'(\mathbb{R}^d; X)$ is sequentially complete by Lemma 14.4.5, and E is continuously embedded in $\mathscr{S}'(\mathbb{R}^d; X)$, it follows that there exists an $f \in \mathscr{S}'(\mathbb{R}^d; X)$ such that $f_n \to f$ in $\mathscr{S}'(\mathbb{R}^d; X)$. Since $(f_n)_{n\geqslant 1}$ is a Cauchy sequence in E it is bounded in E. By the Fatou property of E it follows that $f \in E$. To prove that $f_n \to f$ in E we fix an $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n, m \geqslant N$ we have $\|f_m - f_n\|_E < \varepsilon$. Using the Fatou property once more, we obtain

$$\|f - f_n\|_E \leqslant \liminf_{m\to\infty} \|f_m - f_n\|_E \leqslant \varepsilon$$

and the result follows. □

Proof of Proposition 14.4.4. Combine Lemmas 14.4.6 and 14.4.7 and Proposition 14.4.3. □

Coming back to the discussion on homogeneous verses inhomogeneous norms (see (14.20) and (14.21)), we have the following remark.

Remark 14.4.8. Let $p, q \in [1, \infty]$ and $s > 0$. For $f \in \mathscr{S}'(\mathbb{R}^d; X)$ one has

$$\big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \eqsim \big\|(2^{ks}\psi_k * f)_{k\in\mathbb{Z}}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} + \|f\|_{L^p(\mathbb{R}^d;X)},$$

where both expressions are infinite whenever one of them is. Here the $(\varphi_k)_{k\geqslant 0}$ are as in Definition 14.4.1, and thus the left-hand side of the above identity equals $\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}$. The $(\psi_k)_{k\in\mathbb{Z}}$ are as in (14.20). The first expression on the right-hand side is equal to the homogeneous Besov norm, which we will not discuss in detail.

To prove the norm equivalence first recall that $\psi_k = \varphi_k$ for $k \geqslant 1$. For "$\lesssim$" it suffices to observe that by Young's inequality

$$\big\|\varphi_0 * f\big\|_{L^p(\mathbb{R}^d;X)} \leqslant \|\varphi_0\|_1 \|f\|_{L^p(\mathbb{R}^d;X)}.$$

Conversely, assume that $f \in B^s_{p,q}(\mathbb{R}^d;X)$. Since $\widehat{\varphi}_0 = 1$ on $\operatorname{supp}(\widehat{\psi}_k)$ for $k \leqslant 0$, we can write

$$\begin{aligned}\|\psi_k * f\|_{L^p(\mathbb{R}^d;X)} &= \|\psi_k * \varphi_0 * f\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \|\psi_k\|_1 \|\varphi_0 * f\|_{L^p(\mathbb{R}^d;X)} = \|\psi_0\|_1 \|\varphi_0 * f\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

and thus using that $s > 0$ we obtain

$$\begin{aligned}\big\|(2^{ks}\psi_k * f)_{k\leqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} &\leqslant \big\|(2^{ks}\varphi_0 * f)_{k\leqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \\ &\leqslant C_s \|\varphi_0 * f\|_{L^p(\mathbb{R}^d;X)}.\end{aligned}$$

Moreover, since $s > 0$, from (14.23) $B^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^0_{p,1}(\mathbb{R}^d;X)$, and thus by Lemma 14.2.10

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^d;X)} = \Big\|\sum_{k\geqslant 0} \varphi_k * f\Big\|_{L^p(\mathbb{R}^d;X)} &\leqslant \sum_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \\ &= \|f\|_{B^0_{p,1}(\mathbb{R}^d;X)} \leqslant C_{s,q}\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}\end{aligned}$$

14.4.b Fourier multipliers

The goal of this section is to prove a version of the Mihlin multiplier theorem for operator-valued Fourier multipliers acting on vector-valued Besov spaces. In contrast to the situation in the L^p-setting (cf. Theorems 5.3.18 and 5.5.10), where we had to assume the UMD property, a variant of the Mihlin theorem for Besov spaces holds for arbitrary Banach spaces.

We wish to emphasise that the main result, Theorem 14.4.16 below, is not applicable to multipliers which are non-smooth or even singular near the origin. This is due to the presence of the term φ_0 in the definition of inhomogeneous Littlewood–Paley sequences, whose support contains the origin in its interior. For instance, the Fourier multiplier associated to the Hilbert transform does not satisfy the conditions of the theorem.

Unlike in other chapters, we also include the case $p = \infty$. In order to avoid density issues, we define $\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y)$ as the space of Fourier transforms of operator-valued measures of bounded variation:

Definition 14.4.9. *We define*

$$\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y) := \big\{\widehat{\Phi} : \textit{the operator-valued measure}\quad \Phi : \mathscr{B}(\mathbb{R}^d) \to \mathscr{L}(X,Y) \textit{ is of bounded variation}\big\}.$$

With the norm $\|\widehat{\Phi}\|_{\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y)} = \|\Phi\|(\mathbb{R}^d)$, the space $\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y)$ is a Banach space.

For $m \in \mathfrak{M}L^\infty(\mathbb{R}^d;X,Y)$ and $f \in L^\infty(\mathbb{R}^d;X)$ we define

$$T_m * f := \check{m} * f,$$

recalling that the convolutions with an operator-valued measure of bounded variation has been introduced in Lemma 14.2.4.

Remark 14.4.10. In the scalar case it can be shown that the space $\mathfrak{M}L^\infty(\mathbb{R}^d) = \mathfrak{M}L^\infty(\mathbb{R}^d;\mathbb{C},\mathbb{C})$ as defined in Definition 14.4.9 coincides with the space of all $m \in L^\infty(\mathbb{R}^d)$ for which the quantity

$$\sup\{\|T_m f\|_\infty : f \in \mathscr{S}(\mathbb{R}^d) \text{ with } \|f\|_\infty \leqslant 1\}$$

is finite, and that this quantity then equals the norm on $\mathfrak{M}L^\infty(\mathbb{R}^d)$ introduced above. This provides further motivation for Definition 14.4.9.

Various properties discussed in Section 5.3.a extend to $p = \infty$. Moreover, from the definition of the Fourier transform one sees that

$$\|\widehat{\Phi}\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))} \leqslant \|\Phi\|(\mathbb{R}^d).$$

This induces a contractive embedding

$$\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y) \hookrightarrow L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y)).$$

For $m \in \mathfrak{M}L^\infty(\mathbb{R}^d;X,Y)$ and $f \in \mathscr{S}(\mathbb{R}^d;X)$ one can check that $m\widehat{f} = \mathscr{F}(\check{m} * f)$, and by Lemma 14.2.4 for all $p \in [1,\infty]$ we have

$$\|\check{m} * f\|_{L^p(\mathbb{R}^d;Y)} \leqslant \|\check{m}\|(\mathbb{R}^d)\|f\|_{L^p(\mathbb{R}^d;Y)}.$$

This shows that for all $p \in [1,\infty]$ we have a contractive embedding

$$\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y) \hookrightarrow \mathfrak{M}L^p(\mathbb{R}^d;X,Y). \tag{14.24}$$

In the discussion preceding Lemma 14.2.4 it was observed that for any function $\phi \in L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$, an operator-valued measure $\Phi : \mathscr{B}(\mathbb{R}^d) \to \mathscr{L}(X,Y)$ of bounded variation is obtained by setting

$$\Phi(A) := \int_A \phi \,\mathrm{d}x,$$

and that its total variation satisfies $\|\Phi\|(\mathbb{R}^d) \leqslant \|\phi\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))}$. In this way we obtain contractive embeddings

$$\check{L}^1(\mathbb{R}^d;\mathscr{L}(X,Y)) \hookrightarrow \mathfrak{M}L^\infty(\mathbb{R}^d;X,Y) \hookrightarrow \mathfrak{M}L^p(\mathbb{R}^d;X,Y).$$

In combination with Lemma 14.2.11 we now obtain the following sufficient condition on m for membership of $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$.

Proposition 14.4.11. *If the multiplier $m \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ satisfies $\check{m} \in L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$, then for all $p \in [1,\infty]$ we have $m \in \mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ and*

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \|\check{m}\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

In particular, if $m \in C^{d+1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ and there exists an $\varepsilon > 0$ such that

$$C_{m,d,\varepsilon} := \max_{|\alpha|\leqslant d+1} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|+\varepsilon})\|\partial^\alpha m(\xi)\| < \infty,$$

then $m \in \mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ and $\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_{d,\varepsilon} C_{m,d,\varepsilon}$.

Remark 14.4.12. In applications it can be useful to apply Proposition 14.2.11 to a dilated multiplier $m(t\cdot)$ instead of $m(\cdot)$. The $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$-norm is invariant under dilations, but the expression for $C_{m,d\varepsilon}$ is not. A similar remark applies to Lemma 14.2.11.

Remark 14.4.13. If $m \in C_{\rm c}^{d+1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ is supported in the ball B_R around the origin, one easily checks that $C_{m,d,\varepsilon} \lesssim_R \|m\|_{C_{\rm b}^{d+1}(\mathbb{R}^d;\mathscr{L}(X,Y))}$. As a consequence we obtain that every $m \in C_{\rm c}^{d+1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ belongs to $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ and $\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_{d,\varepsilon,R} \|m\|_{C_{\rm b}^{d+1}(\mathbb{R}^d;\mathscr{L}(X,Y))}$.

Remark 14.4.14. Multipliers with singularities in the origin, such as the multiplier giving rise to the Hilbert transform, are not covered by Proposition 14.4.11.

Before moving to a Mihlin multiplier theorem for Besov space we present an important result on lifting operators. Recall from Subsection 5.6.a that the *Bessel potential operators* are the continuous operators J_σ, $\sigma \in \mathbb{R}$, acting on $\mathscr{S}'(\mathbb{R}^d;X)$ by

$$J_\sigma u := ((1+4\pi^2|\cdot|^2)^{\sigma/2}\widehat{u})^{\check{}}, \quad u \in \mathscr{S}'(\mathbb{R}^d;X).$$

They satisfy $J_0 = I$ and $J_{\sigma_1+\sigma_2} = J_{\sigma_1} \circ J_{\sigma_2}$.

Proposition 14.4.15 (Lifting). *Let $p,q \in [1,\infty]$ and $s \in \mathbb{R}$. For all $\sigma \in \mathbb{R}$ we have*

$$J_\sigma : B_{p,q}^s(\mathbb{R}^d;X) \simeq B_{p,q}^{s-\sigma}(\mathbb{R}^d;X) \quad \textit{isomorphically.}$$

Proof. Noting that J_σ is a bijection from $\mathscr{S}'(\mathbb{R}^d;X)$ to $\mathscr{S}'(\mathbb{R}^d;X)$, with inverse $J_\sigma^{-1} = J_{-\sigma}$, it suffices to prove that J_σ maps $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^{s-\sigma}_{p,q}(\mathbb{R}^d;X)$ and is bounded for each $\sigma \in \mathbb{R}$.

We claim that there exists a constant $C \geqslant 0$, independent of $k \geqslant 0$, such that for all $f \in \mathscr{S}'(\mathbb{R}^d;X)$,

$$\|\varphi_k * J_\sigma f\|_{L^p(\mathbb{R}^d;X)} \leqslant C 2^{k\sigma} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)}.$$

This will imply the result.

To prove the claim we use that $\sum_{\ell=-1}^{1} \widehat{\varphi}_{k+\ell} \equiv 1$ on the support of $\widehat{\varphi}_k$ to write

$$2^{-k\sigma} J_\sigma \varphi_k * f = \sum_{\ell=-1}^{1} \mathscr{F}^{-1}(\widehat{\varphi}_k m \widehat{\varphi}_{k+\ell} \widehat{f}),$$

where $m(\xi) = 2^{-k\sigma}(1+4\pi^2|\xi|^2)^{\sigma/2}$. Using a dilation, Proposition 14.4.11, and the Fourier support property (14.9), for $k \geqslant 1$ we obtain

$$\begin{aligned}
\|\varphi_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &= \|\varphi_1(2\cdot) m(2^k \cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\
&\lesssim_d \max_{|\alpha| \leqslant d+1} \sup_{\xi \in \mathbb{R}^d} (1+|\xi|^{|\alpha|+1}) \|\partial^\alpha [\varphi_1(2\cdot) m(2^k \xi)](\xi)\| \\
&\lesssim_d \max_{|\alpha| \leqslant d+1} \sup_{\frac12 \leqslant |\xi| \leqslant \frac32} \|\partial^\alpha [m(2^k \cdot)](\xi)\|,
\end{aligned}$$

where in the last step we applied the Leibniz rule as before and the Fourier support properties of φ_1 given by (14.8) and (14.9). Since $m(2^k\xi) = (2^{-2k} + |\xi|^2)^{\sigma/2}$, it is elementary to check that the latter expression is uniformly bounded in $k \geqslant 1$. A similar argument shows that $\varphi_0 m \in \mathfrak{M}L^p(\mathbb{R}^d;X)$. □

The simple multiplier result of Proposition 14.4.11 is already strong enough to prove the version of Mihlin's multiplier theorem for Besov spaces $B^s_{p,q}(\mathbb{R}^d;X)$ contained in Theorem 14.4.16 below, valid for arbitrary Banach spaces X and integrability exponents $p,q \in [1,\infty]$. In the statement of the theorem the end-points $p=\infty$ and $q=\infty$ create some technical difficulties, since we cannot use the density of the Schwartz functions to define T_m. It is for this reason that in the theorem we assume that the multiplier m is smooth and has derivatives of polynomial growth. Many interesting multipliers satisfy this condition, and to proceed with the development of the theory this version suffices for the time being. A version which avoids this restriction on m will be presented in Theorem 14.5.6.

When $m \in C^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ has derivatives of polynomial growth, one can define the Fourier multiplier T_m as an operator from $\mathscr{S}'(\mathbb{R}^d;X)$ into $\mathscr{S}'(\mathbb{R}^d;Y)$ by $T_m f := \mathscr{F}^{-1}(m\widehat{f})$. To see that this is well-defined it suffices to note that $m\widehat{f} \in \mathscr{S}'(\mathbb{R}^d;Y)$ for $f \in \mathscr{S}'(\mathbb{R};X)$. In the next theorem, T_m is understood to be the restriction of this operator to $B^s_{p,q}(\mathbb{R}^d;X)$. The theorem then asserts that, under Mihlin type conditions on m, it maps $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$.

Theorem 14.4.16 (Mihlin multiplier theorem for Besov spaces). *Let X and Y be Banach spaces and let $p,q \in [1,\infty]$ and $s \in \mathbb{R}$. Suppose that $m \in C^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ has derivatives of polynomial growth, and that*

$$\sup_{|\alpha|\leqslant d+1} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|})\|\partial^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} =: K_m < \infty. \tag{14.25}$$

Then the Fourier multiplier $T_m = \mathscr{F}^{-1} m \mathscr{F}$ restricts to a bounded operator from $B^s_{p,q}(\mathbb{R}^d;X)$ to $B^s_{p,q}(\mathbb{R}^d;Y)$ of norm $\|T_m\| \leqslant C_{s,d} K_m$.

The usual Mihlin condition involves a factor $|\xi|^{|\alpha|}$ instead of $1+|\xi|^{|\alpha|}$. A multiplier theorem involving the former can be shown to hold for the scale of homogeneous Besov spaces.

For finite p and q, the condition $m \in C^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ can be weakened to $m \in C^{d+1}(\mathbb{R}^d; \mathscr{L}(X,Y))$. This can be seen by taking f in the dense class $\mathscr{S}(\mathbb{R}^d)\otimes X$ in the proof below.

Proof. For $f \in B^s_{p,q}(\mathbb{R}^d;X)$ let $f_n := \varphi_n * f$ for $n \geqslant 0$. Since $\sum_{\ell=-1}^1 \widehat{\varphi}_{k+\ell} \equiv 1$ on the support of $\widehat{\varphi}_k$,

$$\begin{aligned}
\|T_m f\|_{B^s_{p,q}(\mathbb{R}^d;Y)} &= \big\|(2^{ks}\varphi_k * \mathscr{F}^{-1} m \widehat{f})_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;Y))} \\
&= \Big\|\Big(2^{ks}\mathscr{F}^{-1}\widehat{\varphi}_k m \sum_{\ell=-1}^{1} \widehat{\varphi}_{k+\ell}\widehat{f}\Big)_{k\geqslant 0}\Big\|_{\ell^q(L^p(\mathbb{R}^d;Y))} \\
&\leqslant \sum_{\ell=-1}^{1} \big\|2^{ks}\mathscr{F}^{-1}(\widehat{\varphi}_k m \widehat{f}_{k+\ell})_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;Y))} \\
&\leqslant \sup_{k\geqslant 0}\|\widehat{\varphi}_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \sum_{\ell=-1}^{1} \big\|(2^{ks} f_{k+\ell})_{n\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;Y))} \\
&\leqslant 2^{|s|}\sup_{k\geqslant 0}\|\widehat{\varphi}_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}.
\end{aligned}$$

To complete the proof we must show that $\sup_{k\geqslant 0}\|\widehat{\varphi}_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_d K_m$.

First consider the case $k \geqslant 1$. Since the multiplier norm is invariant under dilations by Proposition 5.3.8, it suffices to show that

$$\sup_{k\geqslant 1}\|\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_d K_m.$$

By Proposition 14.4.11, it even suffices to show that there exists an $\varepsilon > 0$ such that

$$\max_{|\alpha|\leqslant d+1}\sup_{\xi\in\mathbb{R}^d}(1+|\xi|^{|\alpha|+\varepsilon})\|\partial^\alpha[\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)](\xi)\| \lesssim_d K_m.$$

We will verify this bound for $\varepsilon = 1$. By the Fourier support properties of φ_1 implied by (14.8) and (14.9), for $\beta \leqslant \alpha$ with $|\alpha| \leqslant d+1$ we have

$$\sup_{\xi\in\mathbb{R}^d} |(1+|\xi|^{|\alpha|+1})\partial^\beta \widehat{\varphi}_1(\xi)| \leqslant C_{\beta,d}.$$

Hence, by Leibniz's rule the Mihlin condition on m, and the Fourier support property of φ_1 given by (14.9), for all $|\alpha| \leqslant d+1$ we have

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|+1})\|\partial^\alpha[\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)](\xi)\| \\
&\quad = \sup_{|\xi|\geqslant 1} (1+|\xi|^{|\alpha|+1})\|\partial^\alpha[\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)](\xi)\| \\
&\quad \leqslant \sup_{|\xi|\geqslant 1} (1+|\xi|^{|\alpha|+1}) \sum_{\beta\leqslant\alpha} C_{\alpha,\beta}|\partial^\beta\varphi_1(\xi)|\cdot 2^{(k-1)|\alpha-\beta|}|\partial^{\alpha-\beta}m(2^{k-1}\xi)| \\
&\quad \lesssim_d \sup_{|\xi|\geqslant 1} \sum_{\beta\leqslant\alpha} 2^{(k-1)|\alpha-\beta|}|\partial^{\alpha-\beta}m(2^{k-1}\xi)| \\
&\quad \leqslant \sup_{|\xi|\geqslant 1} \sum_{\beta\leqslant\alpha} 2^{(k-1)|\alpha-\beta|}\frac{K_m}{1+|2^{k-1}\xi|^{|\alpha-\beta|}} \\
&\quad \lesssim_d K_m.
\end{aligned} \tag{14.26}$$

The case $k=0$ is proved in similarly, omitting the dilation argument. □

As an application of Theorem 14.4.16, we obtain the following analogue of Theorem 5.6.11.

Proposition 14.4.17. *Let $p\in[1,\infty)$, $q\in[1,\infty]$, and $s\in\mathbb{R}$. For all $k\in\mathbb{N}$,*

$$|||f|||_{B^s_{p,q}(\mathbb{R}^d;X)} := \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \tag{14.27}$$

defines an equivalent norm on $B^s_{p,q}(\mathbb{R}^d;X)$

Proof. As a consequence of Proposition 14.4.15 it suffices to prove the equivalence of (14.27) with $\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}$. This can be deduced from Theorem 14.4.16 by an argument similar to the one in Theorem 5.6.11. In the present situation it is important to note that the multipliers in the proof of the proposition also satisfy the more restrictive condition (14.25). Below we present a simplification of the argument of Theorem 5.6.11 adapted to the Besov space case. Let $\langle\xi\rangle = (1+|2\pi\xi|^2)^{1/2}$.

First we prove the estimate

$$\|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \leqslant C\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.$$

Applying the Fourier transform, we have

$$\mathscr{F}[\partial^\alpha f](\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi) = \frac{(2\pi i\xi)^\alpha}{\langle\xi\rangle^k}\langle\xi\rangle^k\widehat{f}(\xi) =: m_\alpha(\xi)\langle\xi\rangle^k\widehat{f}(\xi).$$

One checks that m_α satisfies the conditions of Theorem 14.4.16, and thus

$$\begin{aligned}\|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} &\leqslant C_\alpha C_{d,p,q}\|\mathscr{F}^{-1}[\langle\cdot\rangle^k \widehat{f}]\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}\\ &= C_\alpha C_{d,p,q}\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

For the reverse estimate it suffices to show

$$\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \leqslant C\sum_{|\alpha|\leqslant k}\|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.$$

Again we apply Theorem 14.4.16. By induction on k,

$$\langle\xi\rangle^{2k} = (1+|2\pi\xi|^2)^k = \sum_{|\alpha|\leqslant k} c_{\alpha,k}(2\pi i\xi)^\alpha(2\pi i\xi)^\alpha,$$

and therefore

$$\begin{aligned}\langle\xi\rangle^k\widehat{f}(\xi) = \frac{\langle\xi\rangle^{2k}}{\langle\xi\rangle^k}\widehat{f}(\xi) &= \sum_{|\alpha|\leqslant k} c_{\alpha,k}m_\alpha(\xi)(2\pi i\xi)^\alpha\widehat{f}(\xi)\\ &= \sum_{|\alpha|\leqslant k} c_{\alpha,k}m_\alpha(\xi)\widehat{\partial^\alpha f}(\xi),\end{aligned}$$

where $m_\alpha(\xi) = \frac{(2\pi i\xi)^\alpha}{\langle\xi\rangle^k}$. Applying Theorem 14.4.16 to m_α now gives

$$\begin{aligned}\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} &= \|\mathscr{F}^{-1}[\langle\cdot\rangle^k\widehat{f}]\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}\\ &\leqslant \sum_{|\alpha|\leqslant k}|c_{\alpha,k}|\|T_{m_\alpha}\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}\\ &\leqslant C_{d,p,k}\sum_{|\alpha|\leqslant k}\|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

□

14.4.c Embedding theorems

We begin by showing that various classes of function spaces lie ‘sandwiched’ between Besov spaces.

Proposition 14.4.18 (Sandwiching with Besov spaces). *For all* $p \in [1,\infty]$, $s\in\mathbb{R}$, *and* $m\in\mathbb{N}$, *we have continuous embeddings*

$$B^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d;X), \tag{14.28}$$

$$B^m_{p,1}(\mathbb{R}^d;X) \hookrightarrow W^{m,p}(\mathbb{R}^d;X) \hookrightarrow B^m_{p,\infty}(\mathbb{R}^d;X), \tag{14.29}$$

$$B^m_{\infty,1}(\mathbb{R}^d;X) \hookrightarrow C^m_{\rm ub}(\mathbb{R}^d;X) \hookrightarrow B^m_{\infty,\infty}(\mathbb{R}^d;X). \tag{14.30}$$

An improvement for $p \in (1,\infty)$ will be given in Proposition 14.6.13.

Proof. In order to prove (14.28), by Propositions 5.6.3 and 14.4.15 it suffices to consider $s = m = 0$. Similarly, in order to prove (14.29) and (14.30), by Proposition 14.4.17 it suffices to consider $s = m = 0$. Therefore, (14.28) and (14.29) reduce to proving the continuous embeddings

$$B^0_{p,1}(\mathbb{R}^d;X) \hookrightarrow L^p(\mathbb{R}^d;X) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d;X). \tag{14.31}$$

Fix $f \in B^0_{p,1}(\mathbb{R}^d;X)$. By definition,

$$\|f\|_{B^0_{p,1}(\mathbb{R}^d;X)} = \sum_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)}.$$

In particular, the sum $\sum_{k\geqslant 0}\varphi_k * f$ converges absolutely in $L^p(\mathbb{R}^d;X)$, and the required result follows by Lemma 14.2.10 and the triangle inequality.

To prove the second embedding in (14.31), fix $f \in L^p(\mathbb{R}^d;X)$. By Young's inequality,

$$\begin{aligned}\|f\|_{B^0_{p,\infty}(\mathbb{R}^d;X)} &= \sup_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \sup_{k\geqslant 0} \|\varphi_k\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant 2\|\varphi_0\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

where the last step uses (14.7). This completes the proof of (14.31).

As we already noted, in order to prove the embeddings in (14.30) it suffices to consider the case $m = 0$. Fix $f \in B^0_{\infty,1}(\mathbb{R}^d;X)$. As before we see that the sum $\sum_{k=0}^\infty \varphi_k * f$ is absolutely convergent in $L^\infty(\mathbb{R}^d;X)$. By Lemma 14.2.10 its sum equals f and

$$\|f\|_\infty \leqslant \sum_{k=0}^\infty \|\varphi_k * f\|_\infty = \|f\|_{B^0_{\infty,1}(\mathbb{R}^d;X)}.$$

To see that f has a uniformly continuous version, we note that by Proposition 2.4.32 we have $\varphi_k * f \in C^\infty(\mathbb{R}^d;X)$ and

$$\|\partial_j(\varphi_k * f)\|_\infty = \|(\partial_j\varphi_k) * f\|_\infty \leqslant \|\partial_j\varphi_k\|_1\|f\|_\infty \leqslant \|\partial_j\varphi_k\|_1\|f\|_{B^0_{\infty,1}(\mathbb{R}^d;X)}.$$

In particular, each function $\varphi_k * f$ is Lipschitz continuous and hence uniformly continuous. Therefore $f \in C_{\rm ub}(\mathbb{R}^d;X)$ by uniform convergence.

The second embedding in (14.30) follows by combining the embedding $C^m_{\rm ub}(\mathbb{R}^d;X) \hookrightarrow W^{m,\infty}(\mathbb{R}^d;X)$ and (14.29). □

Theorem 14.4.19 (Sobolev embedding for Besov spaces). *For given p_0, p_1, $q_0, q_1 \in [1,\infty]$, and $s_0, s_1 \in \mathbb{R}$, we have a continuous embedding*

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)$$

if and only if one of the following three conditions holds:

(i) $p_0 = p_1$ *and* $[s_0 > s_1$ *or* $(s_0 = s_1$ *and* $q_0 \leqslant q_1)]$;
(ii) $p_0 < p_1$, $q_0 \leqslant q_1$, *and* $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$;
(iii) $p_0 < p_1$ *and* $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$.

The most interesting cases are (ii) and (iii), since they can be used to change the integrability parameter from p_0 into p_1.

For the proof of the sufficiency of the three conditions we need two lemmas. The first provides an L^p-estimate for the derivatives under suitable Fourier support assumptions. Recall from Lemma 14.2.9 that every $f \in \mathscr{S}'(\mathbb{R}^d; X)$ with compact Fourier support belongs to $C^\infty(\mathbb{R}^d; X)$ and has at most polynomial growth.

Lemma 14.4.20 (Bernstein–Nikolskii inequality). *Let* $p_0, p_1 \in [1, \infty]$ *satisfy* $p_0 \leqslant p_1$. *If* $f \in L^{p_0}(\mathbb{R}^d; X)$ *satisfies*

$$\operatorname{supp} \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| < t\}$$

for some $t > 0$, *then for any multi-index* $\alpha \in \mathbb{N}^d$ *there is a constant* $C = C_{\alpha,d,p_0,p_1}$ *such that*

$$\|\partial^\alpha f\|_{L^{p_1}(\mathbb{R}^d;X)} \leqslant C t^{|\alpha| + \frac{d}{p_0} - \frac{d}{p_1}} \|f\|_{L^{p_0}(\mathbb{R}^d;X)}.$$

An extension to exponents $0 < p_0 \leqslant p_1 \leqslant \infty$ will be given in Remark 14.6.4.

Proof. By a routine scaling argument it suffices to consider the case $t = 1$.

Let $\psi \in \mathscr{S}(\mathbb{R}^d)$ satisfy $\widehat{\psi} \equiv 1$ on $B_1 := \{x \in \mathbb{R}^d : |x| < 1\}$ and put $\psi_\alpha := \partial^\alpha \psi$. Then $f = \psi * f$, and by Young's inequality with $\frac{1}{p_1} + 1 = \frac{1}{p_0} + \frac{1}{q}$ we obtain

$$\begin{aligned} \|\partial^\alpha f\|_{L^{p_1}(\mathbb{R}^d;X)} &= \|\partial^\alpha (\psi * f)\|_{L^{p_1}(\mathbb{R}^d;X)} \\ &= \|\psi_\alpha * f\|_{L^{p_1}(\mathbb{R}^d;X)} \leqslant \|\psi_\alpha\|_{L^q(\mathbb{R}^d)} \|f\|_{L^{p_0}(\mathbb{R}^d;X)}. \end{aligned}$$

□

The next lemma provides shows how the L^p-norm of $\varphi_k * \varphi_{k+j}$ scales with k.

Lemma 14.4.21. *For all* $j \in \mathbb{Z}$ *there exists a constant* $C_{d,j,p} \geqslant 0$ *such that for all* $k \geqslant 0$ *and* $k + \ell \geqslant 0$ *we have*

$$\|\varphi_{k+\ell} * \varphi_k\|_{L^p(\mathbb{R}^d)} = C_{\ell,p,d} 2^{kd/p'}.$$

Proof. The identity $\widehat{\varphi}_k(\xi) = \widehat{\varphi}_1(2^{-k+1}\xi)$ implies $\varphi_k(x) = 2^{(k-1)d}\varphi_1(2^{k-1}x)$ and therefore, by a change of variables in x and y,

$$\begin{aligned} &\|\varphi_{k+j} * \varphi_k\|^p_{L^p(\mathbb{R}^d)} \\ &\quad = \int_{\mathbb{R}^d} \Big| 2^{(k-1)d} 2^{(k+j-1)d} \int_{\mathbb{R}^d} \varphi_1(2^j 2^{k-1}(x-y)) \varphi_1(2^{k-1}y) \,\mathrm{d}y \Big|^p \,\mathrm{d}x \end{aligned}$$

$$= 2^{kd(p-1)} \underbrace{2^{jdp-d(p-1)} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \varphi_1(2^j(x-y))\varphi_1(y)\,\mathrm{d}y \Big|^p \,\mathrm{d}x}_{=:C^p_{j,p,d}}$$

and the result follows. □

Proof of Theorem 14.4.19. For the sufficiency of (i), first consider the case $s_0 \geqslant s_1$ and $q_0 \leqslant q_1$. Then the result follows from the fact that for any scalar sequence $(a_k)_{k\geqslant 0}$,

$$\big\|(2^{ks_1}a_k)_{k\geqslant 0}\big\|_{\ell^{q_1}} \leqslant \big\|(2^{ks_0}a_k)_{k\geqslant 0}\big\|_{\ell^{q_0}}.$$

If $s_0 > s_1$, the result follows from (14.23).

If (ii) holds, then writing $f_k := \varphi_k * f$ for $k \geqslant 0$, from Lemma 14.4.20 we infer that

$$\|f_k\|_{L^{p_1}(\mathbb{R}^d;X)} \leqslant C 2^{k(\frac{d}{p_0}-\frac{d}{p_1})}\|f_k\|_{L^{p_0}(\mathbb{R}^d;X)} = C2^{k(s_0-s_1)}\|f_k\|_{L^{p_0}(\mathbb{R}^d;X)}.$$

It follows that

$$\begin{aligned}\|f\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)} &= \big\|(2^{ks_1}f_k)_{k\geqslant 0}\big\|_{\ell^{q_1}(L^{p_1}(\mathbb{R}^d;X))}\\ &\leqslant C\big\|(2^{ks_0}f_k)_{k\geqslant 0}\big\|_{\ell^{q_1}(L^{p_0}(\mathbb{R}^d;X))}\\ &= C\|f\|_{B^{s_0}_{p_0,q_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)},\end{aligned}$$

using (14.22) in the last step.

Suppose now that (iii) holds and let $t := s_0 - \frac{d}{p_0} + \frac{d}{p_1}$. Then $t - \frac{d}{p_1} = s_0 - \frac{d}{p_0}$ and therefore, by the previous step,

$$\|f\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{t}_{p_0,q_1}(\mathbb{R}^d;X)}.$$

Since $t > s_1$, it follows that the conditions (i) are satisfied, and thus

$$\|f\|_{B^{t}_{p_0,q_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}.$$

Next we move to the necessity of the conditions (i), (ii), and (iii). It suffices to consider the case $X = \mathbb{K}$.

Suppose that we have the continuous embedding stated in the theorem. By the closed graph theorem there is a constant $C = C_{d,p_0,p_1,q_0,q_1,s_0,s_1}$ such that for all $f \in B^{s_0}_{p_0,q_0}(\mathbb{R}^d)$,

$$\|f\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d)} \leqslant C\|f\|_{B^{s_0}_{p_0,q_0}(\mathbb{R}^d)}. \tag{14.32}$$

First we will derive

$$s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1} \;\text{ and }\; p_0 \leqslant p_1. \tag{14.33}$$

By (14.22), (14.32) also holds (with a possibly different constant) for $q_0 = 1$ and $q_1 = \infty$. The Fourier support properties (14.8) and (14.9) of φ_k then imply

$$2^{ks_1}\|\varphi_k * \varphi_k\|_{L^{p_1}(\mathbb{R}^d)} \leqslant \|\varphi_k\|_{B^{s_1}_{p_1,\infty}(\mathbb{R}^d)} \leqslant C\|\varphi_k\|_{B^{s_0}_{p_0,1}(\mathbb{R}^d)} \leqslant C2^{ks_0}\sum_{j=-1}^{1}\|\varphi_k * \varphi_{k+j}\|_{L^{p_0}(\mathbb{R}^d)}.$$

By Lemma 14.4.21 this implies

$$2^{ks_1}2^{kd/p_1'} \leqslant \tilde{C}2^{ks_0}2^{kd/p_0'}$$

for some possibly different constant $\widetilde{C}$ independent of k. Upon letting $k \to \infty$, this gives the inequality $s_1 - \frac{d}{p_1'} \leqslant s_0 - \frac{d}{p_0'}$, or equivalently, $s_1 - \frac{d}{p_1} \leqslant s_0 - \frac{d}{p_0}$.

Define $f_t : \mathbb{R}^d \to \mathbb{C}$ by $\widehat{f_t}(x) := \widehat{\varphi}_0(t^{-1}\cdot)$. Then $\widehat{\varphi}_0 = 1$ and $\varphi_k = 0$ for $k \geqslant 1$ on $\operatorname{supp}(\widehat{f_t})$ for $t > 0$ small enough. Therefore,

$$t^{-\frac{d}{p_j}}\|f_1\|_{L^{p_j}(\mathbb{R}^d)} = \|f_t\|_{L^{p_j}(\mathbb{R}^d)} = \|\varphi_0 * f_t\|_{L^{p_j}(\mathbb{R}^d)} = \|f_t\|_{B^{s_j}_{p_j,q_j}(\mathbb{R}^d)}$$

Combining this with (14.32) gives

$$t^{-\frac{d}{p_1}}\|f_1\|_{L^{p_1}(\mathbb{R}^d)} \leqslant Ct^{-\frac{d}{p_0}}\|f_1\|_{L^{p_0}(\mathbb{R}^d)}.$$

Upon letting $t \downarrow 0$, we find that $p_0 \leqslant p_1$. This completes the proof of (14.33).

Now there are two possibilities: (i) $p_0 < p_1$, or (ii) $p_0 = p_1$. First consider the case (i). If $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$, then (iii) follows. Still assuming (i), if $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$, then in order to deduce (ii) it suffices to show that $q_0 \leqslant q_1$. We claim that for any finite sequence of scalars $(a_k)_{k=1}^n$,

$$\|(a_k)_{k=1}^n\|_{\ell^{q_1}} \leqslant C\|(a_k)_{k=1}^n\|_{\ell^{q_0}}, \tag{14.34}$$

where C is a constant independent of $n \geqslant 1$ and the sequence $(a_k)_{k=1}^n$. Once established, this claim gives $q_0 \leqslant q_1$.

To prove the claim fix a scalar sequence $(a_k)_{k=1}^n$. Applying (14.32) to the function $f := \sum_{k=1}^n 2^{-3k(s_0+\frac{d}{p_0'})}a_k\varphi_{3k} = \sum_{k=1}^n 2^{-3k(s_1+\frac{d}{p_1'})}a_k\varphi_{3k}$ gives the inequality

$$\begin{aligned}\Big(\sum_{m\geqslant 0} 2^{ms_1q_1}\Big\|\sum_{k=1}^n 2^{-3k(s_1+\frac{d}{p_1'})}a_k\varphi_m * \varphi_{3k}\Big\|^{q_1}_{L^{p_1}(\mathbb{R}^d)}\Big)^{1/q_1} \\ \leqslant C\Big(\sum_{m\geqslant 0} 2^{ms_0q_0}\Big\|\sum_{k=1}^n 2^{-3k(s_0+\frac{d}{p_0'})}a_k\varphi_m * \varphi_{3k}\Big\|^{q_0}_{L^{p_0}(\mathbb{R}^d)}\Big)^{1/q_0}.\end{aligned} \tag{14.35}$$

Let us analyse the expressions on the left-hand and right-hand sides for general values of p, q, and s. We have $\varphi_m * \varphi_{3k} \neq 0$ only for $m = 3k + \ell$ with

$\ell \in \{-1,0,1\}$. This suggests splitting the sum over m into the sums over $m = 3j+\ell$ for $\ell \in \{-1,0,1\}$. Using the lemma, they evaluate as

$$\begin{aligned}
&\Big(\sum_{j\geqslant 0} 2^{(3j+\ell)sq}\Big\|\sum_{k=1}^{n} 2^{-3k(s+\frac{d}{p'})} a_k\varphi_{3j+\ell} * \varphi_{3k}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \\
&\quad= \Big(\sum_{j=1}^{n} 2^{(3j+\ell)sq}\Big\|2^{-3j(s+\frac{d}{p'})} a_j\varphi_{3j+\ell} * \varphi_{3j}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \\
&\quad= C_{\ell,p,d}\Big(\sum_{j=1}^{n} 2^{(3j+\ell)sq}2^{-3j(sq+\frac{dq}{p'})}\|a_j\|^q 2^{3jdq/p'}\Big)^{1/q} \\
&\quad= 2^{\ell s}C_{\ell,p,d}\Big(\sum_{j=1}^{n}\|a_j\|^q\Big)^{1/q}.
\end{aligned}$$

We thus find (using the triangle inequality in ℓ^q_3 for the upper estimate)

$$\Big(\sum_{m\geqslant 0} 2^{msq}\Big\|\sum_{k=1}^{n} 2^{-3k(s+\frac{d}{p'})} a_k\varphi_m * \varphi_{3k}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \eqsim_{d,p,s} \Big(\sum_{\ell=1}^{n}\|a_\ell\|^q\Big)^{1/q}.$$

Inserting this norm equivalence into (14.35) (taking $(p,q,s) = (p_0,q_0,s_0)$ on the left and $(p,q,s) = (p_1,q_1,s_1)$ on the right) we obtain (14.34).

Finally suppose that (ii) holds. Then from $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$ we see that $s_0 \geqslant s_1$. If $s_0 = s_1$, then by arguing as above it follows that $q_0 \leqslant q_1$ and (i) follows. □

14.4.d Difference norms

In this section we show that Besov spaces with smoothness parameter $s > 0$ admit a characterisation in terms of difference norms. This characterisation can be often used to effectively check whether a given concrete function belongs to a given Besov space. For example, we check in Corollary 14.4.26 that the Besov spaces $B^s_{\infty,\infty}(\mathbb{R}^d;X)$ coincide with certain spaces of s-Hölder continuous functions.

For functions $f : \mathbb{R}^d \to X$ and vectors $h \in \mathbb{R}^d$, the function $\Delta_h f : \mathbb{R}^d \to X$ is defined by

$$\Delta_h f(x) := f(x+h) - f(x).$$

Clearly, the *difference operator* Δ_h thus defined is bounded as an operator on $L^p(\mathbb{R}^d;X)$ for all $1 \leqslant p \leqslant \infty$, with norm at most 2. We have the following formula for the powers $\Delta_h^m = (\Delta_h)^m$.

Lemma 14.4.22. *For all $f \in L^1(\mathbb{R}^d;X)$ and $h,\xi \in \mathbb{R}^d$ we have*

$$\Delta_h^m f = \sum_{j=0}^{m}\binom{m}{j}(-1)^j f(\cdot + (m-j)h).$$

Proof. The identity $\mathscr{F}(f(\cdot+h))(\xi) = e^{2\pi i h\cdot\xi}\widehat{f}$ implies $\mathscr{F}(\Delta_h f)(\xi) = (e^{2\pi i h\cdot\xi} - 1)\widehat{f}(\xi)$, from which it follows that

$$\mathscr{F}(\Delta_h^m f)(\xi) = (e^{2\pi i h\cdot\xi} - 1)^m \widehat{f}(\xi) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j e^{2\pi i h\cdot\xi(m-j)} \widehat{f}(\xi).$$

Now apply the inverse Fourier transform. □

Definition 14.4.23 (Difference norm for Besov spaces). *Let $p, q, \tau \in [1,\infty]$, $s \in \mathbb{R}$, and $m \in \mathbb{N} \setminus \{0\}$. For functions $f \in L^p(\mathbb{R}^d; X)$ we define the* difference norm *by setting*

$$[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} := \Big(\int_0^\infty t^{-sq} \Big\| \Big(\fint_{\{|h|\leqslant t\}} \|\Delta_h^m f\|^\tau \,\mathrm{d}h \Big)^{1/\tau} \Big\|^q_{L^p(\mathbb{R}^d)} \frac{\mathrm{d}t}{t} \Big)^{1/q}$$

with obvious modifications for $q = \infty$ and/or $\tau = \infty$ where the integral with respect to $\mathrm{d}t/t$ and the average are replaced by essential suprema, and

$$|||f|||^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} := \|f\|_{L^p(\mathbb{R}^d;X)} + [f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)}.$$

Here we used the notation $\fint_F := \frac{1}{|F|}\int_F$ to denote the average over the set F. In typical applications one takes $\tau \in \{1, p, \infty\}$.

It is clear that $\tau_0 \leqslant \tau_1$ implies

$$[f]^{(m,\tau_0)}_{B^s_{p,q}(\mathbb{R}^d;X)} \leqslant [f]^{(m,\tau_1)}_{B^s_{p,q}(\mathbb{R}^d;X)}. \tag{14.36}$$

The next theorem implies that if $s > 0$, then each of the norms $|||\cdot|||^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)}$ with $m > s$ defines an equivalent norm on $B^s_{p,q}(\mathbb{R}^d; X)$.

Theorem 14.4.24 (Difference norms for Besov spaces). *Let $p, q \in [1,\infty]$, $s > 0$, $\tau \in [1,\infty]$, and let $m > s$ be an integer. A function $f \in L^p(\mathbb{R}^d; X)$ belongs to $B^s_{p,q}(\mathbb{R}^d; X)$ if and only if $[f]^{m,\tau}_{B^s_{p,q}(\mathbb{R}^d;X)} < \infty$, and the following equivalence of norms holds:*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,m,s} |||f|||^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)}.$$

Before turning to the details of the proof we give some simple applications. The first two identify the Sobolev–Slobodetskii spaces and the Hölder spaces (cf. Section 14.1 for the relevant notation) as Besov spaces.

Corollary 14.4.25 (Sobolev–Slobodetskii spaces). *Let $p \in [1,\infty)$ and $s \in (0,1)$. Then*

$$B^s_{p,p}(\mathbb{R}^d; X) = W^{s,p}(\mathbb{R}^d; X)$$

with equivalent norms. In fact,

$$[f]^{(1,p)}_{B^s_{p,p}(\mathbb{R}^d;X)} = \frac{1}{(sp+d)^{1/p}|B_1|}[f]_{W^{s,p}(\mathbb{R}^d;X)}. \tag{14.37}$$

Proof. By Theorem 14.4.24 it suffices to prove the identity (14.37) for the seminorms, which follows from Fubini's theorem and a change of variable:

$$\begin{aligned}|B_1|^p\big([f]^{(1,p)}_{B^s_{p,p}(\mathbb{R}^d;X)}\big)^p &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty \mathbf{1}_{\{|h|\leqslant t\}} t^{-sp-d-1}\|\Delta_h f(x)\|^p \,\mathrm{d}t\,\mathrm{d}h\,\mathrm{d}x \\ &= (sp+d)^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h|^{-sp-d}\|\Delta_h f(x)\|^p\,\mathrm{d}h\,\mathrm{d}x \\ &= (sp+d)^{-1}[f]^p_{W^{s,p}(\mathbb{R}^d;X)}.\end{aligned}$$

□

Corollary 14.4.26 (Hölder spaces). *Let X be a Banach space and let $s \in (0,\infty)\setminus\mathbb{N}$. Then*

$$B^s_{\infty,\infty}(\mathbb{R}^d;X) = C^s_{\mathrm{ub}}(\mathbb{R}^d;X)$$

with equivalent norms.

Proof. Let $s = k+\theta$, where $k\in\mathbb{N}$ and $\theta\in(0,1)$. It follows from Proposition 14.4.18 and Theorem 14.4.19 that we have continuous embeddings

$$B^s_{\infty,\infty}(\mathbb{R}^d;X)\hookrightarrow B^k_{\infty,1}(\mathbb{R}^d;X)\hookrightarrow C^k_{\mathrm{ub}}(\mathbb{R}^d;X).$$

Therefore there is no loss of generality in assuming that our functions are k-times continuously differentiable. For functions $f\in C^k_{\mathrm{ub}}(\mathbb{R}^d;X)$ and multi-indices $|\alpha|\leqslant k$, from Theorem 14.4.24 we infer the equivalences

$$\|\partial^\alpha f\|_{B^\theta_{\infty,\infty}(\mathbb{R}^d;X)} \eqsim_{d,\theta} \|\partial^\alpha f\|^{(1,\infty)}_{B^\theta_{\infty,\infty}(\mathbb{R}^d;X)} = \|\partial^\alpha f\|_{C^\theta_{\mathrm{ub}}(\mathbb{R}^d;X)},$$

where we used the continuous version of $\partial^\alpha f$ to replace the essential supremum by a supremum. Now the result follows after summation over all multi-indices $|\alpha|\leqslant k$ and an application of Proposition 14.4.17. □

Corollary 14.4.27 (Embeddings into Hölder spaces). *Let $p_0, q\in[1,\infty]$ and $s_0,s_1\geqslant 0$ satisfy $s_0-\frac{d}{p_0}\geqslant s_1$. Then we have the following continuous embeddings:*

(1) $B^{s_0}_{p_0,q}(\mathbb{R}^d;X)\hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X)$ *if* $s_1\notin\mathbb{N}$;
(2) $B^{s_0}_{p_0,1}(\mathbb{R}^d;X)\hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X)$.

Proof. (1): By Theorem 14.4.19 and Corollary 14.4.26,

$$B^{s_0}_{p_0,q}(\mathbb{R}^d;X)\hookrightarrow B^{s_1}_{\infty,\infty}(\mathbb{R}^d;X) = C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X).$$

(2): The case $s_1\notin\mathbb{N}$ follows from the previous case. If $s_1\in\mathbb{N}$, then by Theorem 14.4.19 and Proposition 14.4.18,

$$B^{s_0}_{p_0,1}(\mathbb{R}^d;X)\hookrightarrow B^{s_1}_{\infty,1}(\mathbb{R}^d;X)\hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X).$$

□

The proof of Theorem 14.4.24 makes use the following simple lemma. Recall the Fourier multiplier notation of Subsection 14.4.b.

Lemma 14.4.28. *For non-zero* $\xi, h \in \mathbb{R}^d$ *let*

$$m_h(\xi) := \frac{e^{2\pi i h\cdot\xi} - 1}{2\pi i h \cdot \xi}.$$

Then for all $p \in [1,\infty]$ *we have* $m_h \in \mathfrak{M}L^p(\mathbb{R}^d; X)$ *and* $\|m_h\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \leqslant 1$.

Proof. By an elementary computation, the associated Fourier multiplier is given by

$$T_{m_h} f(x) = \int_0^1 f(x - ht)\,\mathrm{d}t = \mu_h * f(x), \quad f \in L^p(\mathbb{R}^d; X),$$

where $\mu_h(A) = \int_0^1 \mathbf{1}_{th\in A}\,\mathrm{d}t$ defines a measure by monotone convergence. Hence the result follows from (14.24). For $p < \infty$, one can also use the direct estimate

$$\|T_{m_h} f\|_{L^p(\mathbb{R}^d;X)} \leqslant \int_0^1 \|f(\cdot - ht)\|_{L^p(\mathbb{R}^d;X)}\,\mathrm{d}t = \|f\|_{L^p(\mathbb{R}^d;X)}.$$

□

Proof of Theorem 14.4.24. Let

$$I_p^{m,\tau}(f,k) := \Big\|\Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-k}h} f\|^\tau\,\mathrm{d}h\Big)^{1/\tau}\Big\|_{L^p(\mathbb{R}^d)},$$

where the integral average has to be replaced by $\sup_{|h|\leqslant 1}$ if $\tau = \infty$. Discretising the integral over t in the definition of the difference norm (Definition 14.4.23) and noting that

$$\fint_{\{|h|\leqslant t\}} \leqslant \frac{1}{\omega_d 2^{-kd}} \int_{\{|h|\leqslant 2^{-k+1}\}} = 2^d \fint_{\{|h|\leqslant 2^{-k+1}\}},$$

we obtain

$$\begin{aligned}
[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} &= \Big(\sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} t^{-sq-1} \Big\|\Big(\fint_{\{|h|\leqslant t\}} \|\Delta_h^m f\|^\tau\,\mathrm{d}h\Big)^{1/\tau}\Big\|^q_{L^p(\mathbb{R}^d)}\,\mathrm{d}t\Big)^{1/q} \\
&\leqslant 2^{d/\tau}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big\|\Big(\fint_{\{|h|\leqslant 2^{-k+1}\}} \|\Delta_h^m f\|^\tau\,\mathrm{d}h\Big)^{1/\tau}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \\
&= 2^{d/\tau}\Big(\sum_{j\in\mathbb{Z}} 2^{(j+1)sq}\Big\|\Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-j}h} f\|^\tau\,\mathrm{d}h\Big)^{1/\tau}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \\
&= 2^{s+d/\tau}\Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\Big\|\Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-j}h} f\|^\tau\,\mathrm{d}h\Big)^{1/\tau}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q}.
\end{aligned}$$

Similarly,

$$[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} \geqslant 2^{-s-1-d/\tau}\Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\Big\|\Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-j}h} f\|^\tau \,\mathrm{d}h\Big)^{1/\tau}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q}.$$

Hence,

$$[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,s,\tau} \big\|(2^{ks} I^{m,\tau}_p(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})}. \tag{14.38}$$

In view of (14.36) and (14.38) it thus suffices to prove the two estimates

$$\|f\|_{L^p(\mathbb{R}^d;X)} + \big\|(2^{ks} I^{m,\infty}_p(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \lesssim_{d,m,s} \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}, \tag{14.39}$$

$$\|f\|_{L^p(\mathbb{R}^d;X)} + \big\|(2^{ks} I^{m,1}_p(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \gtrsim_{s,m,d} \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}. \tag{14.40}$$

Throughout the proof of (14.39) and (14.40) we will use the standard algebraic properties of L^p-multipliers discussed in Section 5.3.a.

Put $f_j := \varphi_j * f$ for $j \geqslant 0$. By Hölder's inequality,

$$\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant \sum_{j\geqslant 0} \|f_j\|_{L^p(\mathbb{R}^d;X)} \leqslant \big\|(2^{-js})_{j\geqslant 0}\big\|_{\ell^{q'}} \|f\|_{B^q_{p,q}(\mathbb{R}^d;X)},$$

where the assumption $s > 0$ implies the finiteness of the $\ell^{q'}$-norm. To prove (14.39) and (14.40) it therefore remains to estimate $I^{m,\infty}_p(f,k)$ from above and $I^{m,1}_p(f,k)$ from below.

Step 1 – We begin with the proof of (14.39). By Lemma 14.2.10 and the triangle inequality,

$$I^{m,\infty}_p(f,k) \leqslant \sum_{\ell=-1}^{1}\sum_{j\geqslant 0} I^{m,\infty}_p(\varphi_j * f_{j+\ell}, k),$$

observing the standing convention $\varphi_{-1} \equiv 0$ which implies that $f_{-1} \equiv 0$. Keeping in mind the operator norm inequality $\|\Delta_h\| \leqslant 2$ and (14.7), for $j \geqslant 1$ and arbitrary $g \in L^p(\mathbb{R}^d;X)$ we have

$$\begin{aligned} I^{m,\infty}_p(\varphi_j * g, k) &= \sup_{|h|\leqslant 1|} \big\|\Delta^m_{2^{-k}h}\varphi_j * g\big\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant 2^m\|\varphi_j * g\|_p \leqslant 2^{m+1}\|\varphi\|_1\|g\|_p. \end{aligned} \tag{14.41}$$

On the other hand, using that $\widehat{\varphi}_j(\xi) = \widehat{\varphi}_1(2^{-(j-1)}\xi)$, we find that

$$\begin{aligned} I^{m,\infty}_p(\varphi_j * g, k) &\leqslant \sup_{|h|\leqslant 1} \|\mathscr{F}(\Delta^m_{2^{-k}h}\varphi_j)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\|g\|_p \\ &\leqslant \sup_{|h|\leqslant 1} \|\xi \mapsto (e^{2\pi i 2^{-k}h\cdot\xi} - 1)^m \widehat{\varphi}_1(2^{-(j-1)}\cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\|g\|_p. \end{aligned} \tag{14.42}$$

By Lemma 14.4.28 and a dilation

$$\|\xi \mapsto (e^{2\pi i 2^{-k} h\cdot\xi} - 1)^m (h\cdot\xi)^{-m}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \leqslant (2\pi)^m 2^{-km}. \tag{14.43}$$

Moreover, since φ_1 is a Schwartz function, dilation, and $|h| \leqslant 1$,

$$\begin{aligned}
\|\xi \mapsto (h\cdot\xi)^m \widehat{\varphi}_1(2^{-(j-1)}\xi)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\
&= 2^{(j-1)m}\|\xi \mapsto (h\cdot\xi)^m \widehat{\varphi}_1(\xi)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\
&\leqslant 2^{(j-1)m} \sum_{|\alpha|=m} c_{\alpha,m}\|\xi \mapsto \xi^\alpha \widehat{\varphi}_1(\xi)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\
&\leqslant C_{m,d} 2^{(j-1)m},
\end{aligned} \tag{14.44}$$

where in the last step we used Proposition 14.4.11 with $\partial^\alpha \varphi_1 \in L^1(\mathbb{R}^d)$. Combining (14.41) with (14.42), estimating the latter using (14.43) and (14.44), we obtain the estimate

$$I_p^{m,\infty}(\varphi_j * g, k) \lesssim_{d,m} \min\{1, 2^{(j-k)m}\}\|g\|_p, \quad j \geqslant 1.$$

Similarly one checks that

$$I_p^{m,\infty}(\varphi_0 * g, k) \lesssim_{d,m} \min\{1, 2^{-km}\}\|g\|_p$$

Therefore, with $a_{j,m} = \min\{1, 2^{jm}\}$,

$$\begin{aligned}
\big\|(2^{ks} I_p^{m,\infty}(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \\
&\leqslant \sum_{\ell=-1}^{1} \Big\|\Big(2^{ks}\sum_{j\geqslant 0} I_p^{m,\infty}(\varphi_j * f_{j+\ell}, k)\Big)_{k\in\mathbb{Z}}\Big\|_{\ell^q(\mathbb{Z})} \\
&\lesssim_{d,m,s} \sum_{\ell=-1}^{1} \Big\|\Big(\sum_{j\geqslant 0} 2^{-(j-k)s} a_{j-k,m} 2^{(j+\ell)s}\|f_{j+\ell}\|_p\Big)_{k\in\mathbb{Z}}\Big\|_{\ell^q(\mathbb{Z})} \\
&\lesssim_s \big\|(2^{-js}a_{j,m})_{j\geqslant 0}\big\|_{\ell^1} \big\|(2^{(j+\ell)s}\|f_j\|_p)_{j\geqslant 0}\big\|_{\ell^q} \\
&\lesssim_s \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)},
\end{aligned}$$

where we applied the discrete version of Young's inequality and used the assumption $m > s$ for the finiteness of the ℓ^1 norm.

Step 2 – In this step we prove (14.40). For $k \geqslant 0$ let $T_k f := 2^{kd}\varphi(2^k\cdot) * f$ and $S_k f := \varphi_k * f$. By (14.3), for $k \geqslant 1$ we have $S_k = T_k - T_{k-1} = (I - T_{k-1}) - (I - T_k)$ and therefore

$$\begin{aligned}
\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} &= \big\|(2^{ks}\|S_k f\|_{L^p(\mathbb{R}^d;X)})_{k\geqslant 0}\big\|_{\ell^q} \\
&\leqslant \|S_0 f\|_{L^p(\mathbb{R}^d;X)} + 2\big\|(2^{ks}\|T_k f - f\|_{L^p(\mathbb{R}^d;X)})_{k\geqslant 0}\big\|_{\ell^q}.
\end{aligned} \tag{14.45}$$

By Young's inequality,

$$\|S_0 f\|_{L^p(\mathbb{R}^d;X)} \leqslant \|\varphi_0\|_1 \|f\|_{L^p(\mathbb{R}^d;X)}. \tag{14.46}$$

It remains to estimate the terms with $k \geqslant 0$ by the difference norm.

Choose $\psi \in \mathscr{S}(\mathbb{R}^d)$ such that $\widehat{\psi}(\xi) = 1$ if $|\xi| \leqslant 1$ and $\widehat{\psi}(\xi) = 0$ if $|\xi| \geqslant 3/2$. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ be given by

$$\widehat{\varphi}(\xi) = (-1)^{m+1} \sum_{j=0}^{m-1} \binom{m}{j} (-1)^j \widehat{\psi}(-(m-j)\xi)$$

and define the sequence $(\varphi_k)_{k\geqslant 0}$ as in (14.3). For $|\xi| \leqslant 1/m$ and $0 \leqslant j \leqslant m-1$ we have $\widehat{\psi}(-(m-j)\xi) = 1$ and therefore

$$\widehat{\varphi}(\xi) = (-1)^{m+1} \sum_{j=0}^{m-1} \binom{m}{j} (-1)^j = (-1)^{m+1} \Big(\sum_{j=0}^{m} \binom{m}{j} (-1)^j - (-1)^m \Big) = 1$$

by the binomial theorem, and for $|\xi| \geqslant 3/2$ we have $\widehat{\varphi}(\xi) = 0$. Furthermore the Fourier supports of φ_j and φ_k are disjoint for $|j-k| \geqslant N_m$, where $N_m \in \mathbb{N}$ only depends on m (rather than for $|j-k| \geqslant 2$ as in (14.10) in the case of an inhomogeneous Littlewood–Paley sequence). Thanks to these properties, the proof of Proposition 14.4.2 may be repeated to see that this system leads to an equivalent norm on $B^s_{p,q}(\mathbb{R}^d;X)$.

Let $f \in L^p(\mathbb{R}^d;X)$. We claim that

$$T_k f(x) - f(x) = (-1)^{m+1} \int_{\mathbb{R}^d} \Delta^m_{2^{-k}y} f(x) \psi(y) \,\mathrm{d}y \tag{14.47}$$

Indeed, taking Fourier transforms in the x-variable and using Lemma 14.4.22 and the fact that $\widehat{\psi}(0) = 1$, we have

$$\begin{aligned}
\widehat{T_k f}(\xi) - \widehat{f}(\xi) &= (\widehat{\varphi}(2^{-k}\xi) - 1)\widehat{f}(\xi) \\
&= \Big((-1)^{m+1} \sum_{j=0}^{m-1} \binom{m}{j} (-1)^j \widehat{\psi}(-(m-j)2^{-k}\xi) - 1 \Big) \widehat{f}(\xi) \\
&= (-1)^{m+1} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \widehat{\psi}(-(m-j)2^{-k}\xi) \\
&= (-1)^{m+1} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \int_{\mathbb{R}^d} e^{2\pi i (m-j) 2^{-k} y\cdot\xi} \psi(y) \,\mathrm{d}y \\
&= (-1)^{m+1} \int_{\mathbb{R}^d} (e^{2\pi i 2^{-k} y\cdot\xi} - 1)^m \widehat{f}(\xi) \psi(y) \,\mathrm{d}y \\
&= (-1)^{m+1} \int_{\mathbb{R}^d} \mathscr{F}(\Delta^m_{2^{-k}y} f)(\xi) \psi(y) \,\mathrm{d}y
\end{aligned}$$

and the claim follows.

Fix a real number $r > 0$, the numerical value of which will be fixed in a moment. Taking norms in (14.47), using that $\sup_{x\in\mathbb{R}^d}(1+|x|^r)|\psi(x)| < \infty$, and writing $B_R := \{\xi \in \mathbb{R}^d : |\xi| \leqslant R\}$, it follows that

$$\begin{aligned}
&\|f(x) - T_k f(x)\| \\
&\quad \leqslant \int_{\mathbb{R}^d} \|\Delta^m_{2^{-k}y} f(x)\psi(y)\| \,\mathrm{d}y \\
&\quad \lesssim_\psi \int_{B_1} \|\Delta^m_{2^{-k}y} f(x)\| \,\mathrm{d}y + \sum_{j\geqslant 0} 2^{-(j+1)r} \int_{B_{2^{j+1}}\setminus B_{2^j}} \|\Delta^m_{2^{-k}y} f(x)\| \,\mathrm{d}y \\
&\quad = \int_{B_1} \|\Delta^m_{2^{-k}y} f(x)\| \,\mathrm{d}y + \sum_{j\geqslant 0} 2^{-(j+1)(r-d)} \int_{B_1\setminus B_{\frac12}} \|\Delta^m_{2^{j+1-k}h} f(x)\| \,\mathrm{d}h \\
&\quad \leqslant \sum_{j\geqslant 0} 2^{-j(r-d)} \int_{B_1} \|\Delta^m_{2^{j-k}h} f(x)\| \,\mathrm{d}h.
\end{aligned}$$

Taking L^p-norms with respect to x, we obtain the estimate

$$\|T_k f - f\|_{L^p(\mathbb{R}^d;X)} \lesssim_{d,\psi} \sum_{j\geqslant 0} 2^{-j(r-d)} I^{m,1}_p(f, k-j).$$

Taking ℓ^q-norms with respect to $k \geqslant 0$ and choosing $r > d + s$, we obtain

$$\begin{aligned}
&\big\|(2^{ks}\|T_k f - f\|_{L^p(\mathbb{R}^d;X)})_{k\geqslant 0}\big\|_{\ell^q} \\
&\quad \lesssim_{d,\psi} \Big\|\Big(\sum_{j\geqslant 0} 2^{-j(r-d)} 2^{ks} I^{m,1}_p(f, k-j)\Big)_{k\geqslant 0}\Big\|_{\ell^q} \\
&\quad = \Big\|\Big(\sum_{j\geqslant 0} 2^{-j(r-d-s)} 2^{(k-j)s} I^{m,1}_p(f, k-j)\Big)_{k\geqslant 0}\Big\|_{\ell^q} \\
&\quad \leqslant \sum_{j\geqslant 0} 2^{-j(r-d-s)} \big\|(2^{(k-j)s} I^{m,1}_p(f, k-j))_{k\geqslant 0}\big\|_{\ell^q} \\
&\quad \leqslant \sum_{j\geqslant 0} 2^{-j(r-d-s)} \big\|(2^{ks} I^{m,1}_p(f, k))_{k\in\mathbb{Z}}\big\|_{\ell^q} \\
&\quad = \sum_{j\geqslant 0} 2^{-j(r-d-s)} \big\|(2^{ks} I^{m,1}_p(f, k))_{k\in\mathbb{Z}}\big\|_{\ell^q}.
\end{aligned}$$

In combination with (14.45) and (14.46) this proves estimate (14.40). □

14.4.e Interpolation

In order to consider interpolation for Besov spaces, we will now introduce the so-called retraction and co-retraction operators, which allow us to reduce questions about the interpolation of Besov spaces to the corresponding questions about the spaces $\ell^q_{w_s}(L^p(\mathbb{R}^d; X))$.

Lemma 14.4.29. *Let $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. For $k \geqslant 0$ set $\psi_k := \varphi_{k-1} + \varphi_k + \varphi_{k+1}$. Define the operators*

$$R : \ell^q_{w_s}(L^p(\mathbb{R}^d; X)) \to B^s_{p,q}(\mathbb{R}^d; X)$$
$$S : B^s_{p,q}(\mathbb{R}^d; X) \to \ell^q_{w_s}(L^p(\mathbb{R}^d; X))$$

by

$$R((f_k)_{k\geqslant 0}) = \sum_{k\geqslant 0} \psi_k * f_k, \quad Sf = (\varphi_k * f)_{k\geqslant 0}.$$

Then R is bounded of norm $\leqslant 60\|\varphi_0\|_1^2 4^{|s|}$, S is an isometry, and $RS = I$.

Proof. It is clear from the definitions of the spaces involved that S is an isometry. Next we turn to the proof that R is well defined and bounded. By (14.7) and Young's inequality, $\|\varphi_{k+\ell} * \psi_k\|_1 \leqslant 12\|\varphi_0\|_1^2$. Therefore, by another application of Young's inequality and (14.11),

$$\begin{aligned}
\Big\| \sum_{k\geqslant 0} \psi_k * f_k \Big\|_{B^s_{p,q}(\mathbb{R}^d;X)} &= \Big\| \Big(\varphi_j * \sum_{k\geqslant 0} \psi_k * f_k \Big)_{j\geqslant 0} \Big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X)} \\
&= \Big\| \Big(\varphi_j * \sum_{|\ell|\leqslant 2} \psi_{j+\ell} * f_{j+\ell} \Big)_{j\geqslant 0} \Big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X)} \\
&\leqslant \sum_{|\ell|\leqslant 2} \big\| (\varphi_j * \psi_{j+\ell} * f_{j+\ell})_{j\geqslant 0} \big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \\
&\leqslant 12\|\varphi_0\|_1^2 \sum_{|\ell|\leqslant 2} \big\| (f_{j+\ell})_{j\geqslant 0} \big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \\
&\leqslant 60\|\varphi_0\|_1^2 4^{|s|} \|(f_j)_{j\geqslant 0}\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))},
\end{aligned}$$

the convergence of the sum $\sum_{k\geqslant 0} \psi_k * f_k$ in $B^s_{p,q}(\mathbb{R}^d; X)$ being a consequence of the convergence of the sum $\sum_{j\geqslant 0} 2^{js} f_j$ in $L^p(\mathbb{R}^d; X)$, for this allows to first perform the same estimates for differences of partial sums.

The identity $RS = I$ follows from Lemma 14.2.10 and the fact that $\widehat{\psi}_k \equiv 1$ on $\mathrm{supp}(\widehat{\varphi}_k)$. □

Now we are ready identify the complex interpolation spaces of Besov spaces in a very general setting. In contrast to the complex interpolation results for Sobolev and Bessel potential spaces in Section 5.6, where it was necessary to impose UMD assumptions, no geometric restrictions on the interpolation couple (X_0, X_1) are needed.

Theorem 14.4.30 (Complex interpolation of Besov spaces). *Let (X_0, X_1) be an interpolation couple of Banach spaces, let $p_0, p_1, q_0, q_1 \in [1, \infty]$ satisfy $\min\{p_0, p_1\} < \infty$ and $\min\{q_0, q_1\} < \infty$, and let $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0, 1)$. Furthermore let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X_1)]_\theta = B^s_{p,q}(\mathbb{R}^d;[X_0,X_1]_\theta)$$

with equivalent norms.

Proof. Let $R : \ell^q_{w_s}(L^p(\mathbb{R}^d;X)) \to B^s_{p,q}(\mathbb{R}^d;X)$ and $S : B^s_{p,q}(\mathbb{R}^d;X) \to \ell^q_{w_s}(L^p(\mathbb{R}^d;X))$ be the retraction and co-retraction operators of Lemma 14.4.29. Set

$$E_j := \ell^{q_j}_{w_{s_j}}(L^{p_j}(\mathbb{R}^d;X_j)), \quad F_j := B^{s_j}_{p_j,q_j}(\mathbb{R}^d;X_j), \quad j \in \{0,1\},$$

and

$$E_\theta := (E_0,E_1)_\theta, \quad F_\theta := (F_0,F_1)_\theta, \quad X_\theta := [X_0,X_1]_\theta.$$

By Theorem 2.2.6 and Proposition 14.3.3, $E_\theta = \ell^q_{w_s}(L^p(\mathbb{R}^d;X_\theta))$ isometrically. Therefore,

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X_0), B^{s_1}_{p_0,q_0}(\mathbb{R}^d;X_1)]_\theta = F_\theta = RSF_\theta \subseteq RE_\theta \subseteq B^s_{p,q}(\mathbb{R}^d;X_\theta),$$

and for all $f \in F_\theta$ we have

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X_\theta)} = \|Sf\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X_\theta))} = \|Sf\|_{E_\theta} \leqslant \|f\|_{F_\theta}$$

using Theorem C.3.3. Conversely, by Theorem C.3.3,

$$B^s_{p,q}(\mathbb{R}^d;X_\theta) = RSB^s_{p,q}(\mathbb{R}^d;X_\theta) \subseteq RE_\theta \subseteq F_\theta,$$

and for all $f \in B^s_{p,q}(\mathbb{R}^d;X_\theta)$ we have

$$\|f\|_{F_\theta} = \|RSf\|_{F_\theta} \leqslant C\|Sf\|_{E_\theta} = C\|Sf\|_{\ell^r_{w_s}(L^p(\mathbb{R}^d;X_\theta))} = C\|f\|_{B^s_{p,q}(\mathbb{R}^d;X_\theta)},$$

where $C = 60\|\varphi_0\|_1^2 4^{|s|}$ is the constant of Lemma 14.4.29. □

In the next result we identify the Besov spaces as the real interpolation spaces of Besov spaces, Bessel potential spaces, and Sobolev spaces, allowing only non-negative integer values of s in the latter case. In contrast to the case of complex interpolation, the integrability exponent p as well as the range space X are fixed.

Theorem 14.4.31 (Real interpolation of Besov spaces). *Let X be a Banach space, let $p,q,q_0,q_1 \in [1,\infty]$, let $s_0,s_1 \in \mathbb{R}$ satisfy $s_0 \neq s_1$, and let $\theta \in (0,1)$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$(B^{s_0}_{p,q_0}(\mathbb{R}^d;X), B^{s_1}_{p,q_1}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X), \tag{14.48}$$

$$(H^{s_0,p}(\mathbb{R}^d;X), H^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X), \tag{14.49}$$

with equivalent norms. If we additionally assume that $s_0,s_1 \in \mathbb{N}$, then

$$(W^{s_0,p}(\mathbb{R}^d;X), W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X) \tag{14.50}$$

with equivalent norms. If instead we additionally assume that $p \in [1,\infty)$ *and* $s_0, s_1 \in (0,1)$*, then*

$$(W^{s_0,p}(\mathbb{R}^d;X), W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X) \tag{14.51}$$

with equivalent norms.

Proof. The identification (14.51) follows from (14.48) and Corollary 14.4.25. We will give the proof of the remaining identifications in two steps.

Step 1 – If we can prove that (14.48) holds for $q_0 = q_1 \in \{1,\infty\}$, then all remaining cases can be inferred as follows. Let $\mathcal{A}^{s_j,p}_{q_j} \in \{B^{s_j}_{p,q_j}, H^{s_j,p}, W^{s_j,p}\}$, where we assume that $s_j \in \mathbb{N}$ if $\mathcal{A}^{s_j,p}_{q_j} = W^{s_j,p}$. Then by (14.48), Theorem C.3.3, (14.22), and Theorem 14.4.18, we have continuous embeddings

$$\begin{aligned} B^s_{p,q}(\mathbb{R}^d;X) &= (B^{s_0}_{p,1}(\mathbb{R}^d;X), B^{s_1}_{p,1}(\mathbb{R}^d;X))_{\theta,q} \\ &\hookrightarrow (\mathcal{A}^{s_0,p}(\mathbb{R}^d;X), \mathcal{A}^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} \\ &\hookrightarrow (B^{s_0}_{p,\infty}(\mathbb{R}^d;X), B^{s_1}_{p,\infty}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X), \end{aligned}$$

and (14.48), (14.49), (14.50) follow.

Step 2 – It remains to prove (14.48) for $r := q_0 = q_1 \in \{1,\infty\}$. The argument is similar to that of Theorem 14.4.30.

Let R and S be the retraction and co-retraction operators considered in Lemma 14.4.29. Let

$$E_j := \ell^r_{w_{s_j}}(L^p(\mathbb{R}^d;X)), \quad F_j := B^{s_j}_{p,r}(\mathbb{R}^d;X), \quad j \in \{0,1\},$$

and

$$E_{\theta,q} := (E_0, E_1)_{\theta,q}, \quad F_{\theta,q} := (F_0, F_1)_{\theta,q}.$$

By Proposition 14.3.5, $E_{\theta,q} = \ell^q_{w_s}(L^p(\mathbb{R}^d;X))$ with equivalent norms, say with constants C_1, C_2 (depending on θ, p, q, s_0, s_1), i.e.,

$$C_1^{-1}\|g\|_{E_{\theta,q}} \leqslant \|g\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \leqslant C_2\|g\|_{E_{\theta,q}}.$$

From Theorem C.3.3 it follows that

$$(B^{s_0}_{p,r}(\mathbb{R}^d;X), B^{s_1}_{p,r}(\mathbb{R}^d;X))_{\theta,q} = F_{\theta,q} = RSF_{\theta,q} \subseteq RE_{\theta,q} \subseteq B^s_{p,q}(\mathbb{R}^d;X),$$

and for all $f \in F_{\theta,q}$ we have

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} = \|Sf\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \leqslant C_2\|Sf\|_{E_{\theta,q}} = C_2\|f\|_{F_{\theta,q}}.$$

In the converse direction, interpolation R and S by Theorem C.3.3,

$$B^s_{p,q}(\mathbb{R}^d;X) = RSB^s_{p,q}(\mathbb{R}^d;X) \subseteq RE_{\theta,q} \subseteq F_{\theta,q},$$

and for all $f \in B^s_{p,q}(\mathbb{R}^d;X)$ we have

$$\begin{aligned} \|f\|_{F_{\theta,q}} &= \|RSf\|_{F_{\theta,q}} \\ &\leqslant C\|Sf\|_{E_{\theta,q}} \lesssim C\|Sf\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} = C_3C_1\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}, \end{aligned}$$

where $C = 60\|\varphi_0\|_1^2 4^{|s|}$ is the constant of Lemma 14.4.29. □

Corollary 14.4.32. *Let $s_0, s_1 \in [0,\infty)$ satisfy $s_0 \neq s_1$, let $\theta \in [0,1]$, and put $s := (1-\theta)s_0 + \theta s_1$. Then*

$$(C^{s_0}_{\rm ub}(\mathbb{R}^d;X), C^{s_1}_{\rm ub}(\mathbb{R}^d;X))_{\theta,\infty} = B^s_{\infty,\infty}(\mathbb{R}^d;X)$$

with equivalent norms. Moreover, if $s \notin \mathbb{N}$, then $B^s_{\infty,\infty}(\mathbb{R}^d;X) = C^s_{\rm ub}(\mathbb{R}^d;X)$ with equivalent norms and therefore

$$(C^{s_0}_{\rm ub}(\mathbb{R}^d;X), C^{s_1}_{\rm ub}(\mathbb{R}^d;X))_{\theta,\infty} = (C^{s}_{\rm ub}(\mathbb{R}^d;X).$$

Proof. By Corollary 14.4.26 it suffices to prove the first identity. Since by Proposition 14.4.18 we have continuous embeddings $B^{s_j}_{\infty,1}(\mathbb{R}^d;X) \hookrightarrow C^{s_j}_{\rm ub}(\mathbb{R}^d;X) \hookrightarrow B^{s_j}_{\infty,\infty}(\mathbb{R}^d;X)$ we can straightforwardly adapt the proof of Theorem 14.4.31. □

As a simple application we show that multiplication by a smooth function leads to a bounded operator on Besov spaces.

Example 14.4.33 (Pointwise multiplication by smooth functions – I). Let $p,q \in [1,\infty]$ and $s > 0$, and let $k \in (s,\infty) \cap \mathbb{N}$. If $\zeta \in C^k_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))$, then pointwise multiplication

$$f \mapsto \zeta f$$

defines a bounded operator from $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$ of norm

$$\|f \mapsto \zeta f\|_{\mathscr{L}(B^s_{p,q}(\mathbb{R}^d;X),B^s_{p,q}(\mathbb{R}^d;Y))} \lesssim_{k,s} \|\zeta\|_{C^k_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

Indeed, $f \mapsto \zeta f$ is bounded as a mapping from $W^{j,p}(\mathbb{R}^d;X)$ into $W^{j,p}(\mathbb{R}^d;Y)$ for each $j \in \{0,\dots,k\}$. Interpolating between the cases $j=0$ and $j=k$ by the real method with parameters $(\frac{s}{k},q)$ and applying Theorems 14.4.31 and C.3.3, the desired result is obtained. Alternatively one can prove the boundedness as a consequence of Theorem 14.4.24.

14.4.f Duality

The main result of this section identifies the duals of Besov spaces $B^s_{p,q}(\mathbb{R}^d;X)$ for $p,q \in [1,\infty)$. It is interesting that no geometric assumptions are needed on X. This contrasts with the situation for vector-valued Bochner spaces: recall that, by Theorem 1.3.10, for σ-finite measures spaces one has $L^p(S;X) = L^{p'}(S;X^*)$ if and only if X^* has the Radon–Nikodým property.

We start with the preliminary observation that elements in the duals of Besov spaces can be naturally identified with tempered distributions. Indeed, if $g \in B^s_{p,q}(\mathbb{R}^d;X)^*$, then for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $x \in X$ we have

$$|\langle \varphi \otimes x, g\rangle| \leqslant \|\varphi \otimes x\|_{B^s_{p,q}(\mathbb{R}^d;X)}\|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*} = \|\varphi\|_{B^s_{p,q}(\mathbb{R}^d)}\|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}\|x\|,$$

where we used Proposition 14.4.3 to identify the Schwartz function φ with an element of $B^s_{p,q}(\mathbb{R}^d)$. Thus the mapping $x \mapsto \langle \varphi \otimes x, g\rangle$ defines an element

$g_\varphi \in X^*$, of norm $\|g_\varphi\| \leqslant \|\varphi\|_{B^s_{p,q}(\mathbb{R}^d)} \|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}$. By the continuity of the embedding $\mathscr{S}(\mathbb{R}^d) \hookrightarrow B^s_{p,q}(\mathbb{R}^d)$ (see Proposition 14.4.3), this implies that the mapping $\varphi \to g_\varphi$ defines an element in $\mathscr{S}'(\mathbb{R}^d; X^*)$.

In the converse direction, for $g \in \mathscr{S}'(\mathbb{R}^d; X^*)$ and elements $f = \sum_{n=1}^N \zeta_n \otimes x_n$ in $\mathscr{S}(\mathbb{R}^d) \otimes X$, we can define

$$g(f) := \sum_{n=1}^N \langle x_n, g(\zeta_n)\rangle. \tag{14.52}$$

In order to check whether the mapping $f \mapsto g(f)$ defines an element of $B^s_{p,q}(\mathbb{R}^d; X)^*$, with $p, q \in [1, \infty)$, by the density results contained in Lemma 14.2.1 and Proposition 14.6.8, it suffices to check that there is a constant $C \geqslant 0$ such that

$$|g(f)| \leqslant C\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}, \quad f \in \mathscr{S}(\mathbb{R}^d) \otimes X. \tag{14.53}$$

Theorem 14.4.34. *Let X be a Banach space and let $p, q \in [1, \infty)$ and $s \in \mathbb{R}$. Then every $g \in B^{-s}_{p',q'}(\mathbb{R}^d; X^*)$, when viewed as an element of $\mathscr{S}'(\mathbb{R}^d; X^*)$, determines a unique element of $B^s_{p,q}(\mathbb{R}^d; X)^*$, and this identification sets up a natural isomorphism of Banach spaces*

$$B^s_{p,q}(\mathbb{R}^d; X)^* \simeq B^{-s}_{p',q'}(\mathbb{R}^d; X^*).$$

Proof. The second assertion follows from the first, combined with Corollary 14.4.25.

As a preliminary observation to the proof of the first assertion, we recall Proposition 2.4.32, which asserts that if $g \in \mathscr{S}'(\mathbb{R}^d; X^*)$ and $\zeta \in \mathscr{S}(\mathbb{R}^d)$, then $\zeta * g$ is in $C^\infty(\mathbb{R}^d; X^*)$ and $\partial^\alpha g$ has polynomial growth for any $\alpha \in \mathbb{N}^d$. Moreover, by Lemma 14.2.10, and the support properties (14.11), (14.12), we have the identity

$$g(\zeta) = \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle \zeta(t), g_j(t)\rangle \,\mathrm{d}t = \sum_{\ell=-1}^{1} \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle \varphi_{j+\ell} * \zeta(t), g_j(t)\rangle \,\mathrm{d}t, \tag{14.54}$$

where $g_j := \varphi_j * g$.

We split the proof of the theorem into three steps.

Step 1 – First let $g \in B^{-s}_{p',q'}(\mathbb{R}^d; X^*)$. Identifying g with an element of $\mathscr{S}'(\mathbb{R}^d; X^*)$, in order to prove that g defines an element of $B^s_{p,q}(\mathbb{R}^d; X)^*$ we will check that the duality given by (14.52) satisfies the bound (14.53).

By (14.54), if $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$ is as in (14.52), then with $f_j := \varphi_j * f$ we have

$$g(f) = \sum_{\ell=-1}^{1} \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle f_{j+\ell}(t), g_j(t)\rangle \,\mathrm{d}t.$$

By Hölder's inequality,

$$\begin{aligned}|g(f)| &\leqslant \sum_{\ell=-1}^{1}\sum_{j\geqslant 0}\int_{\mathbb{R}^d} |\langle f_{j+\ell}(t), g_j(t)\rangle|\,\mathrm{d}t \\ &\leqslant \sum_{\ell=-1}^{1} 2^{-\ell s}\big\|(2^{(j+\ell)s} f_{j+\ell})_{j\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}\big\|(2^{-js}g_j)_{j\geqslant 0}\big\|_{\ell^{q'}(L^{p'}(\mathbb{R}^d;X^*))} \\ &\leqslant 3\cdot 2^{|s|}\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}\|g\|_{B^{-s}_{p',q'}(\mathbb{R}^d;X^*)}.\end{aligned}$$

This verifies the bound (14.53).

Step 2 – Suppose next that $g \in B^s_{p,q}(\mathbb{R}^d;X)^*$. As explained above, we can identify g with an element of $\mathscr{S}'(\mathbb{R}^d;X^*)$. Let $(f_j)_{j\geqslant 0}$ be any finitely non-zero sequence in $\mathscr{S}(\mathbb{R}^d)\otimes X$ such that $\|(2^{js}f_j)_{j\geqslant 0}\|_{\ell^q(L^p(\mathbb{R}^d;X))} \leqslant 1$. Put $f := R(f_j)_{j\geqslant 0}$, where $R : \ell^q_{w_s}(L^p(\mathbb{R}^d;X)) \to B^s_{p,q}(\mathbb{R}^d;X)$ is the operator considered in Lemma 14.4.29. Then by (14.54) and the fact that $\widehat{\psi}_j = \widehat{\varphi}_{j-1}+\widehat{\varphi}_j+\widehat{\varphi}_{j+1} = 1$ on $\mathrm{supp}(\widehat{\varphi}_j)$ we see that

$$g(f) = \sum_{j\geqslant 0}\int_{\mathbb{R}^d}\langle f(t), g_j(t)\rangle\,\mathrm{d}t = \sum_{j\geqslant 0}\int_{\mathbb{R}^d}\langle f_j(t), g_j(t)\rangle\,\mathrm{d}t.$$

Therefore,

$$\begin{aligned}\Big|\sum_{j\geqslant 0}\int_{\mathbb{R}^d}\langle 2^{js}f_j(t), 2^{-js}g_j(t)\rangle\,\mathrm{d}t\Big| = |g(f)| &\leqslant \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}\|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*} \\ &\leqslant \|R\|\,\|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}.\end{aligned}$$

Taking the supremum over all admissible finitely non-zero sequences $(f_j)_{j\geqslant 0}$, Propositions 1.3.1 and 1.3.3 imply that g belongs to $B^{-s}_{p',q'}(\mathbb{R}^d;X^*)$ and

$$\|g\|_{B^{-s}_{p',q'}(\mathbb{R}^d;X^*)} = \big\|(2^{-js}g_j)_{j\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X^*))} \leqslant \|R\|\,\|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}.$$

Step 3 – Since the identifications in Steps 2 and 3 are inverse to each other, they set up a bijective correspondence, and the estimates in the above proof show that this correspondence is bounded in both directions. □

Theorem 14.4.34 permits an extension of Example 14.4.33 to negative smoothness exponents.

Example 14.4.35 (Pointwise multiplication by smooth functions – II). Let X and Y be Banach spaces, let $p \in (1,\infty)$, $q \in [1,\infty]$, $s \leqslant 0$, and let $k \in (|s|,\infty)\cap\mathbb{N}$. For functions $\zeta \in C^k_{\mathrm{b}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, the pointwise multiplication

$$f \mapsto \zeta f$$

defines a bounded operator from $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$ of norm

$$\|f \mapsto \zeta f\|_{\mathscr{L}(B^s_{p,q}(\mathbb{R}^d;X),B^s_{p,q}(\mathbb{R}^d;Y))} \lesssim_{k,s} \|\zeta\|_{C^k_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))}. \tag{14.55}$$

To prove this, first assume that $q \in (1,\infty)$ and $s < 0$. From Example 14.4.33 we obtain the boundedness of $g \mapsto \zeta^* g$ from $B^{-s}_{p',q'}(\mathbb{R}^d;Y^*)$ into $B^{-s}_{p',q'}(\mathbb{R}^d;X^*)$. Therefore, by Theorem 14.4.34, the adjoint mapping $f \mapsto \zeta f$ is bounded from $B^s_{p,q}(\mathbb{R}^d;X^{**})$ into $B^s_{p,q}(\mathbb{R}^d;Y^{**})$. Restricting to $\mathscr{S}(\mathbb{R}^d;X)$ and using density (Proposition 14.4.3) we obtain boundedness from $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$.

Next let $q \in \{1,\infty\}$ and $s < 0$. Interpolating the inequality (14.55) for the cases $B^{s+\varepsilon}_{p,2}$ and $B^{s-\varepsilon}_{p,2}$ by the real method with parameters $(\frac{1}{2},q)$, and using Theorems 14.4.31 to the effect that $(B^{s+\varepsilon}_{p,2}, B^{s-\varepsilon}_{p,2})_{\frac{1}{2},q} = B^s_{p,q}$ we obtain boundedness in the endpoint cases $q \in \{1,\infty\}$ by Theorem C.3.3.

Finally, if $q \in [1,\infty]$ and $s = 0$, then by interpolating the cases $B^{\varepsilon}_{p,q}$ and $B^{-\varepsilon}_{p,q}$ by the real method with parameters $(\frac{1}{2},q)$ we obtain the boundedness also in this case.

As another application of interpolation and duality we present a density result, which at first sight looks a bit technical. It will be used to derive an analogues density result for Triebel–Lizorkin spaces (see Proposition 14.6.17) which will serve to show that several end-point results do not hold (see the text below Theorem 14.6.32 and Example 14.6.33). Moreover, some of these density results will be used to prove results on pointwise multiplication by the non-smooth function $\mathbf{1}_{\mathbb{R}_+}$ (see Sections 14.6.h and 14.7.d).

Let

$$\ddot{\mathbb{R}}^d := (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{d-1}.$$

Proposition 14.4.36 (Density of compactly supported functions). *Let $p,q \in [1,\infty)$ and $s \in \mathbb{R}$. Then $C^\infty_{\rm c}(\ddot{\mathbb{R}}^d) \otimes X$ is dense in $B^s_{p,q}(\mathbb{R}^d;X)$ in each of the following situations:*

(1) *$s < 1/p$;*
(2) *$p,q \in (1,\infty)$ and $s = 1/p$.*

Proof. By Proposition 14.4.3 it suffices to show that for every $f \in C^\infty_{\rm c}(\mathbb{R}^d)$ there exist $f_n \in C^\infty_{\rm c}(\ddot{\mathbb{R}}^d)$ such that $f_n \to f$ in $B^s_{p,q}(\mathbb{R}^d)$. Moreover, by the embedding (14.23) and Theorem 14.4.19 it suffices to prove (2).

In order to prove (2) let $f_n := \zeta_n f$, where $\zeta_n(x) = \zeta(nx_1, x_2, \dots, x_n)$ is multiplication by n in the first coordinate, and where $\zeta \in C^\infty(\mathbb{R}^d)$ satisfies $\zeta = 1$ if $|x_1| \geqslant 2$ and $\zeta = 0$ if $|x_1| \leqslant 1$. Then by Theorem 14.4.31 the following interpolation inequality holds:

$$\|f_n\|_{B^{1/p}_{p,q}(\mathbb{R}^d)} \leqslant C\|f_n\|^{1/p'}_{L^p(\mathbb{R}^d)}\|f_n\|^{1/p}_{W^{1,p}(\mathbb{R}^d)}.$$

Since

$$\|f_n\|_{L^p(\mathbb{R}^d)} \leqslant \|f\|_\infty \|\zeta_n\|_{L^p(\mathbb{R}^d)} \lesssim_\zeta n^{-1/p}\|f\|_\infty$$

and similarly

$$\|f_n\|_{W^{1,p}(\mathbb{R}^d)} \lesssim_\zeta n^{1/p'}(\|f\|_\infty + \|\nabla f\|_\infty),$$

the interpolation inequality implies that $(f_n)_{n\geqslant 1}$ is a bounded sequence in $B^{1/p}_{p,q}(\mathbb{R}^d)$. Using the reflexivity of $B^{1/p}_{p,q}(\mathbb{R}^d)$ (which follows Theorem 14.4.34) we find that $(f_n)_{n\geqslant 1}$ has a weakly convergent subsequence, say $f_{n_k} \to g$ weakly in $B^{1/p}_{p,q}(\mathbb{R}^d)$. Since also $f_n \to f$ in $\mathscr{S}'(\mathbb{R}^d)$, we find that $g = f$ and therefore $f_{n_k} \to f$ weakly in $B^{1/p}_{p,q}(\mathbb{R}^d)$. Therefore, $f \in \overline{C_c^\infty(\ddot{\mathbb{R}})}^{w} = \overline{C_c^\infty(\ddot{\mathbb{R}})}^{\|\cdot\|}$, where the closures are taken in the weak and norm topology of $B^{1/p}_{p,q}(\mathbb{R}^d)$, respectively. This completes the proof. □

14.5 Besov spaces, random sums, and multipliers

In the preceding subsections we have proved various results on embedding Besov spaces into other function spaces and vice versa. In the present subsection we take a look at the embeddability of Besov spaces into spaces of γ-radonifying operators. This question turns out to be intimately connected with the type and cotype properties of the space X.

The point of departure is provided by Theorems 9.2.10 and 9.7.3, by which we have the following natural continuous embeddings:

- $L^2(S;X) \hookrightarrow \gamma(L^2(S),X)$ if and only if X has type 2;
- $\gamma(L^2(S),X) \hookrightarrow L^2(S;X)$ if and only if X has cotype 2;
- $W^{\frac{1}{p}-\frac{1}{2},p}(\mathbb{R};X) \hookrightarrow \gamma(L^2(\mathbb{R}),X)$ if and only if X has type p.

In the first two embeddings $(S,\mathscr{A},\mu)$ is an arbitrary measure space.

The main result of this section is the following characterisation of type p and cotype q in terms of embedding properties:

Theorem 14.5.1 (γ-Sobolev embedding – I). *Let X be a Banach space and let $p \in [1,2]$ and $q \in [2,\infty]$.*

(1) X has type p if and only if the identity mapping on $C_c^\infty(\mathbb{R}^d)\otimes X$ extends to a continuous embedding

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X);$$

(2) X has cotype q if and only if the identity mapping on $C_c^\infty(\mathbb{R}^d)\otimes X$ extends to a continuous embedding

$$\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d;X).$$

In particular, for any Banach space X we have continuous embeddings

$$B^{\frac{1}{2}d}_{1,1}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X) \hookrightarrow B^{-\frac{1}{2}d}_{\infty,\infty}(\mathbb{R}^d;X).$$

The proof of Theorem 14.5.1 provides quantitative estimates for the norms of these embeddings. It relies on the following Gaussian version of the Bernstein–Nikolskii inequality (Lemma 14.4.20).

Lemma 14.5.2 (γ-Bernstein–Nikolskii inequality). *Let $p \in [1, 2]$ and $q \in [2, \infty]$.*

(1) *Let X have type p. If $f \in \mathscr{S}(\mathbb{R}^d; X)$ satisfies* $\operatorname{supp} \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| < t\}$*, then for all multi-indices $\alpha \in \mathbb{N}^d$ we have*

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d,X)} \leqslant \kappa_{2,p} \tau_{p,X}^\gamma \pi^{|\alpha|} t^{|\alpha| + \frac{d}{p} - \frac{d}{2}} \|f\|_{L^p(\mathbb{R}^d;X)}.$$

(2) *Let X have cotype q. If $f \in \mathscr{S}(\mathbb{R}^d; X)$ satisfies* $\operatorname{supp} \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| < t\}$*, then for all multi-indices $\alpha \in \mathbb{N}^d$ we have*

$$\|\partial^\alpha f\|_{L^q(\mathbb{R}^d;X)} \leqslant \kappa_{q,2} c_{q,X}^\gamma \pi^{|\alpha|} t^{|\alpha| + \frac{d}{2} - \frac{d}{q}} \|f\|_{\gamma(\mathbb{R}^d;X)}.$$

Here, $\kappa_{2,p}$ and $\kappa_{q,2}$ are the Kahane–Khintchine constants introduced in Section 6.2 and $\tau_{q,X}^\gamma$ and $c_{q,X}^\gamma$ are the Gaussian type and cotype constants of X, respectively, introduced in Section 7.1.d.

Proof. (1): By a scaling argument it suffices to consider the case $t = \frac{1}{2}$. By Example 9.6.5, $\partial^\alpha f \in \gamma(\mathbb{R}^d; X)$ if and only if $\xi \mapsto \xi^\alpha \widehat{f} \in \gamma(\mathbb{R}^d; X)$ and in this case

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} = (2\pi)^{|\alpha|} \|\xi \mapsto \xi^\alpha \widehat{f}(\xi)\|_{\gamma(\mathbb{R}^d;X)}.$$

In order to show that $\xi \mapsto \xi^\alpha \widehat{f}(\xi) \in \gamma(\mathbb{R}^d; X)$, by Examples 9.1.12 and 9.4.4 it suffices to check $\widehat{f} \in \gamma(Q; X)$, where $Q := [-\frac{1}{2}, \frac{1}{2}]^d$; in that case

$$(2\pi)^{|\alpha|} \|\xi \mapsto \xi^\alpha \widehat{f}\|_{\gamma(\mathbb{R}^d;X)} \leqslant (2\pi)^{|\alpha|} \|\xi \mapsto \xi^\alpha \widehat{f}\|_{\gamma(Q;X)} \leqslant \pi^{|\alpha|} \|\widehat{f}\|_{\gamma(Q;X)}.$$

The assertion $\widehat{f} \in \gamma(Q; X)$ is short-hand for the statement that the Pettis integral operator $\mathbb{I}_{\widehat{f}} : L^2(Q) \to X$ defined by

$$\mathbb{I}_{\widehat{f}} g := \int_Q \widehat{f}(\xi) g(\xi) \, \mathrm{d}\xi, \quad g \in L^2(Q),$$

belongs to $\gamma(L^2(Q), X)$ (see Section 9.2.a). We will prove the latter by testing against an orthonormal bases, making use of Theorem 9.1.17.

Let $e_n(\xi) := e^{2\pi i n \cdot \xi}$ for $n \in \mathbb{Z}^d$ and $\xi \in Q$. These functions define an orthonormal basis for $L^2(Q)$ and we have

$$\mathbb{I}_{\widehat{f}} e_n = \int_Q \widehat{f}(\xi) e^{2\pi i n \cdot \xi} \, \mathrm{d}\xi = f(n).$$

By the Kahane–Khintchine inequalities (Theorem 6.2.6) and the type p condition, for any finite subset $F \subseteq \mathbb{Z}^d$ we have

$$\begin{aligned}\Big\|\sum_{n\in F}\gamma_n \mathbb{I}_{\widehat{f}} e_n\Big\|_{L^2(\Omega;X)} &= \Big\|\sum_{n\in F}\gamma_n f(n)\Big\|_{L^2(\Omega;X)}\\ &\leqslant \kappa_{2,p}\Big\|\sum_{n\in F}\gamma_n f(n)\Big\|_{L^p(\Omega;X)}\\ &\leqslant \kappa_{2,p}\tau_{p,X}^{\gamma}\Big(\sum_{n\in F}\|f(n)\|^p\Big)^{1/p}.\end{aligned}$$

It follows from Theorem 9.1.17 that $\widehat{f}\in\gamma(Q,X)$ and, by the above observations,

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} \leqslant \pi^{|\alpha|}\|\widehat{f}\|_{\gamma(Q;X)} \leqslant \kappa_{2,p}\tau_{p,X}^{\gamma}\pi^{|\alpha|}\Big(\sum_{n\in\mathbb{Z}^d}\|f(n)\|^p\Big)^{1/p}.$$

To deduce the estimate in the statement of the theorem from it, for $h\in Q$ and $s\in\mathbb{R}^d$ put $f_h(s) := f(s+h)$. Then $\operatorname{supp}\widehat{f_h}\subseteq Q$ and

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} = \|\partial^\alpha f_h\|_{\gamma(\mathbb{R}^d;X)} \leqslant \kappa_{2,p}\tau_{p,X}^{\gamma}\pi^{|\alpha|}\Big(\sum_{n\in\mathbb{Z}^d}\|f_h(n)\|^p\Big)^{1/p}.$$

Raising both sides to the power p and integrating over $h\in Q$ we obtain

$$\begin{aligned}\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} &\leqslant \kappa_{2,p}\tau_{p,X}^{\gamma}\Big(\int_Q\sum_{n\in\mathbb{Z}^d}\|f_h(n)\|^p\,\mathrm{d}h\Big)^{1/p}\\ &= \kappa_{2,p}\tau_{p,X}^{\gamma}\Big(\int_{\mathbb{R}^d}\|f(s)\|^p\,\mathrm{d}s\Big)^{1/p}.\end{aligned}$$

(2): This is proved similarly. □

Proof of Theorem 14.5.1. (1): First we prove the 'only if' part and assume that X has type p. Let $f\in\mathscr{S}(\mathbb{R}^d;X)$, put $f_k := \varphi_k * f$, and note that $\operatorname{supp}\widehat{f_0}\subseteq\{\xi\in\mathbb{R}^d : |\xi|\leqslant\frac{3}{2}\}$ and

$$\operatorname{supp}\widehat{f_k}\subseteq S_k := \{\xi\in\mathbb{R}^d : 2^{k-1}\leqslant|\xi|\leqslant 2^{k+1}\},\quad k\geqslant 1.$$

By Lemma 14.5.2, $f_k\in\gamma(\mathbb{R}^d;X)$ and

$$\|f_k\|_{\gamma(\mathbb{R}^d;X)} \leqslant \kappa_{2,p}\tau_{p,X}^{\gamma}2^{k(\frac{1}{p}-\frac{1}{2})d}\|f_k\|_{L^p(\mathbb{R}^d;X)}.$$

By Proposition 9.4.13, applied to the decompositions $(S_{2k})_{k\geqslant0}$ and $(S_{2k+1})_{k\geqslant0}$ of $\mathbb{R}^d\setminus\{0\}$, for $n\geqslant m\geqslant 0$ we obtain

$$\Big\|\sum_{k=2m}^{2n}f_k\Big\|_{\gamma(\mathbb{R}^d;X)} \leqslant \kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}\Big(\sum_{j=m}^{n}2^{2j(\frac{1}{p}-\frac{1}{2})pd}\|f_{2j}\|_{L^p(\mathbb{R}^d;X)}^p\Big)^{1/p}$$

$$+ \kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}\Big(\sum_{j=m}^{n-1} 2^{(2j+1)(\frac{1}{p}-\frac{1}{2})pd}\|f_{2j+1}\|^p_{L^p(\mathbb{R}^d;X)}\Big)^{1/p}.$$

Sums of the form $\sum_{k=2m}^{2n+1}$, $\sum_{k=2m+1}^{2n}$, and $\sum_{k=2m+1}^{2n+1}$ can be estimated in a similar way. Since $f = \sum_{k\in\mathbb{Z}} \varphi_k * f = \sum_{k\in\mathbb{Z}} f_k$ in $\mathscr{S}(\mathbb{R}^d;X)$ (by Lemma 14.2.10) and hence in $\gamma(\mathbb{R}^d;X)$ (by the continuous embedding $\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow \gamma(\mathbb{R}^d;X)$), it follows that $f \in \gamma(\mathbb{R}^d;X)$ and

$$\begin{aligned}\|f\|_{\gamma(\mathbb{R}^d;X)} &\leqslant 2\kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}\Big(\sum_{j\in\mathbb{Z}} 2^{j(\frac{1}{p}-\frac{1}{2})pd}\|f_{2j}\|^p_{L^p(\mathbb{R}^d;X)}\Big)^{1/p} \\ &= 2\kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}\|f\|_{B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;X)}.\end{aligned}$$

Since $\mathscr{S}(\mathbb{R}^d;X)$ is dense in $B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;X)$ by Proposition 14.4.3, the identity mapping on $\mathscr{S}(\mathbb{R}^d;X)$ extends to a bounded operator from $B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;X)$ into $\gamma(\mathbb{R}^d;X)$ of norm at most $2\kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}$. The simple proof that this extension is injective is left to the reader.

Next we prove the 'if' part. Since every Banach space has type 1, the 'if' part is trivial for $p=1$. In the rest of the proof of (1) we may therefore assume that $p\in(1,2]$. We will prove the stronger statement that if for some $r\in(1,\infty]$ the identity operator on $\mathscr{S}(\mathbb{R}^d;X)$ extends to a bounded operator, say I, from $B_{p,r}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;X)$ into $\gamma(L^2(\mathbb{R}^d),X)$, X has type r (and then necessarily $r\in(1,2]$).

Let $\psi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$ and $\mathrm{supp}(\widehat{\psi}) \subseteq \{\xi\in\mathbb{R}^d : \widehat{\varphi}_1(\xi)=1\}$. For $n\geqslant 1$, let $\psi_n \in \mathcal{S}(\mathbb{R}^d)$ be defined by

$$\widehat{\psi}_n(\xi) := 2^{(-n+1)d/2}\widehat{\psi}(2^{-n+1}\xi).$$

Then $(\psi_n)_{n\geqslant 1}$ is an orthonormal system in $L^2(\mathbb{R}^d)$. By Proposition 9.1.3, for any finite sequence $(x_n)_{n=1}^N$ in X we then have, with $f := \sum_{n=1}^N \psi_n\otimes x_n$,

$$\|f\|^2_{\gamma(\mathbb{R}^d;X)} = \mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2.$$

On the other hand, since $\varphi_k * \psi_n = \delta_{kn}\psi_n$ (this is seen by taking Fourier transforms and using the Fourier support properties of φ_k),

$$\|f\|^q_{B_{p,r}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;X)} = \sum_{n=1}^N 2^{(\frac{1}{p}-\frac{1}{2})dr}\|\psi_n\|_p^r\|x_n\|^r = \|\psi\|_p^r\sum_{n=1}^N\|x_n\|^r.$$

By putting things together we see that X has type r, with Gaussian type r constant $\tau_{r,X}^{\gamma} \leqslant \|\psi\|_p\|I\|$.

(2): This is proved similarly. □

14.5.a The Fourier transform on Besov spaces

This section presents some mapping properties of the Fourier transform on spaces of functions taking values in a Banach space with (co)type or Fourier type properties. Recall from Section 2.4.b that a Banach space has *Fourier type* $p \in [1,2]$ if the Fourier transform, initially defined on $L^1(\mathbb{R}^d;X) \cap L^p(\mathbb{R}^d;X)$, extends to a bounded operator from $L^p(\mathbb{R}^d;X)$ into $L^{p'}(\mathbb{R}^d;X)$. If that is the case, the norm of this extension is denoted by $\varphi_{p,X}(\mathbb{R}^d)$.

Proposition 14.5.3 (Integrability of Fourier transforms – II). *Let $p \in [1,2]$, and suppose that one of the following two conditions holds:*

(i) *$q \in [p,\infty]$ and X has Fourier type p;*
(ii) *$q \in [2,\infty]$ and X has type p and cotype 2.*

Let $\mathscr{F}$ denote the Fourier transform on $\mathscr{S}'(\mathbb{R}^d;X)$ and let $s := (\frac{1}{p} - \frac{1}{q})d$.

(1) *$\mathscr{F}$ restricts to a bounded operator from $B^s_{p,q'}(\mathbb{R}^d;X)$ into $L^{q'}(\mathbb{R}^d;X)$;*
(2) *$\mathscr{F}$ restricts to a bounded operator from $W^{\lfloor s\rfloor+1,p}(\mathbb{R}^d;X)$ into $L^{q'}(\mathbb{R}^d;X)$.*

The case $q = \infty$ gives sufficient conditions for the Fourier transform to take values in $L^1(\mathbb{R}^d;X)$. Different conditions guaranteeing this have been discussed in Lemma 14.2.11, where growth assumptions on the functions and their derivatives where imposed.

Proof. We start with case (i). Accordingly, let $q \in [p,\infty]$ and let X have Fourier type p

(1): Let $f \in B^s_{p,q'}(\mathbb{R}^d;X)$. Put $f_k := \varphi_k * f$ for $k \geqslant 0$. Let $I_0 = \{\xi \in \mathbb{R}^d : |\xi| < 1\}$ and

$$I_n := \{\xi \in \mathbb{R}^d : 2^{n-1} \leqslant |\xi| < 2^n\}, \quad n \geqslant 1.$$

The sets I_n thus defined are pairwise disjoint, we have $\bigcup_{n\geqslant 0} I_n = \mathbb{R}^d$, and

$$\|\widehat{f}\|_{q'} = \Big(\sum_{n\geqslant 0} \|\mathbf{1}_{I_n}\widehat{f}\|_{q'}^{q'}\Big)^{1/q'} \leqslant \sum_{\ell=-1}^{1} \Big(\sum_{n\geqslant 0} \|\mathbf{1}_{I_n}\widehat{f}_{n+\ell}\|_{q'}^{q'}\Big)^{1/q'},$$

where we used that $\mathrm{supp}(\widehat{\varphi}_k) \cap I_n = \varnothing$ for $|n-k| \geqslant 2$ and that $\sum_{k\geqslant 0} \widehat{\varphi}_k = 1$. By Hölder's inequality with $\frac{1}{q'} = \frac{s}{d} + \frac{1}{p'}$ and the Fourier type p assumption, for $\ell \in \{-1,0,1\}$ we have

$$\begin{aligned}\|\mathbf{1}_{I_n}\widehat{f}_{n+\ell}\|_{q'} &\leqslant \|\mathbf{1}_{I_n}\|_{\frac{d}{s}}\|\widehat{f}_{n+\ell}\|_{p'} \\ &\leqslant \varphi_{p,X}(\mathbb{R}^d)2^{(n+1)s}\|f_{n+\ell}\|_{p'} \leqslant 2^{2s}\varphi_{p,X}(\mathbb{R}^d)2^{(n+\ell)s}\|f_{n+\ell}\|_{p'}.\end{aligned}$$

Taking $\ell^{q'}$-norms on both sides we obtain $\widehat{f} \in L^{q'}(\mathbb{R}^d;X)$ and

$$\|\widehat{f}\|_{q'} \leqslant 2^{2s}3\varphi_{p,X}(\mathbb{R}^d)\|f\|_{B^s_{p,q'}(\mathbb{R}^d;X)}.$$

(2): This follows from (1) since by Proposition 14.4.18 and Theorem 14.4.19 we have the embeddings $W^{\lfloor s\rfloor+1,p}(\mathbb{R}^d;X) \hookrightarrow B^{\lfloor s\rfloor+1}_{p,\infty}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,1}(\mathbb{R}^d;X)$.

Case (ii): Assume now that $q\in[2,\infty]$ and that X has type p and cotype 2. Using the same notation as in case (i), by Hölder's inequality with $\frac{1}{q'} = \frac{1}{r}+\frac{1}{2}$, Theorem 9.2.10, and Lemma 14.5.2 we have

$$\begin{aligned}\|\mathbf{1}_{I_n}\widehat{f}_{n+\ell}\|_{q'} &\leqslant \|\mathbf{1}_{I_n}\|_r\|\widehat{f}_{n+\ell}\|_2\\ &\leqslant c^\gamma_{2,X}2^{d/r(n+1)}\|f_{n+\ell}\|_{\gamma(\mathbb{R}^d;X)}\\ &\lesssim_{d,p} c_{2,X}\tau_{p,X}2^{(n+1)d/r}2^{(n+1)(\frac{1}{p}-\frac{1}{2})d}\|f_{n+\ell}\|_p\\ &= c_{2,X}\tau_{p,X}2^{(n+1)s}\|f_{n+\ell}\|_p.\end{aligned}$$

The proof can now be finished as in case (i). □

As an application of Proposition 14.5.3 using the Fourier type of X, we give an improvement of the Mihlin multiplier theorem for vector-valued Besov spaces presented in Theorem 14.4.16. Before we do that we derive an immediate consequence of Propositions 14.4.11.

Corollary 14.5.4 (Fourier multiplier theorem for L^p under Fourier type). *Let $p\in[1,\infty]$ and $s\in\mathbb{R}$, let X and Y be Banach spaces, and suppose that one of the following conditions holds:*

(i) *Y has Fourier type τ;*
(ii) *Y has type τ and cotype 2.*

Then we have a continuous embedding

$$B^{d/\tau}_{\tau,1}(\mathbb{R}^d;\mathscr{L}(X,Y)) \hookrightarrow \mathfrak{M}L^p(\mathbb{R}^d;X,Y),$$

i.e., every $m\in B^{d/\tau}_{\tau,1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ defines a bounded operator T_m from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$.

Proof. The result is immediate from the fact that $\check{m}\in L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$ by Proposition 14.5.3. □

Remark 14.5.5. It is possible to prove a result as in Corollary 14.5.4 under assumptions on m and m^* in the strong operator topology if X (equivalently X^*) has Fourier type τ_1 and Y has Fourier type τ_2. Indeed, assume there is a constant C_m such that

$$\|mx\|_{B^{d/\tau_2}_{\tau_2,1}(\mathbb{R}^d;Y)} \leqslant C_m\|x\|,\quad x\in X, \tag{14.56}$$

$$\|m^*y^*\|_{B^{d/\tau_1}_{\tau_1,1}(\mathbb{R}^d;X^*)} \leqslant C_m\|y^*\|,\quad y^*\in Y^*. \tag{14.57}$$

First observe that by (14.56), (14.57) and Proposition 14.5.3,

$$\begin{aligned}\|\check{m}x\|_{L^1(\mathbb{R}^d;Y)} &\leqslant C_{\tau_2,Y} C_m \|x\| \\ \|\check{m}^* y^*\|_{L^1(\mathbb{R}^d;X^*)} &\leqslant C_{\tau_1,X} C_m \|y^*\|.\end{aligned} \tag{14.58}$$

Here $\check{m}x := \mathscr{F}^{-1}(mx)$ and $\check{m}^* y^* := \mathscr{F}^{-1}(m^* y^*)$. Therefore, for $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$, by Fubini's theorem one can write

$$\begin{aligned}\|\check{m} * f\|_{L^1(\mathbb{R}^d;Y)} &\leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\check{m}(t-s) f(s)\| \, \mathrm{d}s \, \mathrm{d}t \\ &\leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\check{m}(r) f(s)\| \, \mathrm{d}r \, \mathrm{d}s \leqslant C_m \|f\|_{L^1(\mathbb{R}^d;Y)}.\end{aligned}$$

This proves that T_m extends uniquely to $T_m \in \mathscr{L}(L^1(\mathbb{R}^d;X), L^1(\mathbb{R}^d;Y))$. Since the second line of (14.58) trivially implies that the kernel $\check{m}$ satisfies the dual Hörmander's condition, it follows from the Calderón–Zygmund extrapolation theorem (Theorem 11.2.5) that T_m extends uniquely to $T_m \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ for all $p \in [1,\infty)$. By a duality argument a similar result can be derived for $p = \infty$.

It is clear from the above proof that we can replace the Fourier type conditions by the conditions that Y has type τ_2 and cotype 2, and X^* has type τ_1 and cotype 2.

We continue with an improvement of Theorem 14.4.16 using the Fourier type or type and cotype Y.

Theorem 14.5.6 (Mihlin multiplier theorem for $B^s_{p,q}(\mathbb{R}^d;X)$ under type conditions). *Let $p,q \in [1,\infty]$ and $s \in \mathbb{R}$ and X and Y be Banach spaces and suppose that one of the following conditions holds:*

(i) *Y has Fourier type τ;*
(ii) *Y has type τ and cotype 2.*

If $m \in C^{\lfloor \frac{d}{\tau} \rfloor + 1}(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfies

$$K_m := \sup_{|\alpha| \leqslant \lfloor \frac{d}{\tau} \rfloor + 1} \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^{|\alpha|}) \|\partial^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} < \infty,$$

then there is a bounded operator $T : B^s_{p,q}(\mathbb{R}^d;X) \to B^s_{p,q}(\mathbb{R}^d;Y)$ with $\|T\| \leqslant C_{d,s,X,Y} K_m$ such that $Tf = \mathscr{F}^{-1}(m\widehat{f})$ for all $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$.

Note that in the case $p,q < \infty$, one has that T is the unique bounded extension of $T_m : \mathscr{S}(\mathbb{R}^d) \otimes X \to \mathscr{S}'(\mathbb{R}^d;Y)$. In the end point case $p = \infty$ or $q = \infty$ this does not make sense since $\mathscr{S}(\mathbb{R}^d) \otimes X$ is not dense in $B^s_{p,q}(\mathbb{R}^d;X)$. This is the main reason for the unusual formulation in Theorem 14.5.6.

By a duality argument one can also formulate the (Fourier) (co)type conditions on X^*, but the end-point cases require some caution.

Proof. For $f \in B^s_{p,q}(\mathbb{R}^d;X)$ let $f_k = \varphi_k * f$ and $m_k = \widehat{\varphi}_k m$. Define

$$Tf = \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} T_{m_{k+\ell}} f_k. \tag{14.59}$$

We will check that the series converges in $\mathscr{S}'(\mathbb{R}^d;Y)$ and defines an element in $B^s_{p,q}(\mathbb{R}^d;Y)$.

The proof follows the lines of Theorem 14.4.16. First we show that m_k bound $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ with a uniform bound in $k \geqslant 0$. First let $k \geqslant 1$. By invariance under dilations (see Proposition 5.3.8), Corollary 14.5.4, and the embeddings (14.23) and (14.29), we have

$$\begin{aligned}
\|m_k\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} &= \|m_k(2^{k-1}\cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \\
&\leqslant C_{\tau,Y}\|m_k(2^{k-1}\cdot)\|_{B^{d/\tau}_{\tau,1}(\mathbb{R}^d;\mathscr{L}(X,Y))} \\
&\leqslant C_{\tau,Y}\|m_k(2^{k-1}\cdot)\|_{W^{\lfloor\frac{d}{\tau}\rfloor+1,\tau}(\mathbb{R}^d;\mathscr{L}(X,Y))}
\end{aligned}$$

Since $m_k(2^{k-1}\cdot) = \widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)$, by the support properties of $\widehat{\varphi}_1$ is suffices to bound $\partial^\alpha[\widehat{\varphi}_1(\xi)m(2^{k-1}\xi)]$ for $|\alpha| \leqslant \lfloor\frac{d}{\tau}\rfloor + 1$, uniformly in $k \geqslant 1$ and $1 \leqslant |\xi| \leqslant 3$. This can be done in the same way as in (14.26). The case $k = 0$ can be proved in the same way without the dilation argument. We can conclude that

$$\|T_{m_{k+\ell}} f_k\|_{L^p(\mathbb{R}^d;Y)} \leqslant C_{d,s,X,Y} K_m \|f_k\|_{L^p(\mathbb{R}^d;X)} \tag{14.60}$$

Next we check the convergence of the series in (14.59). For $\zeta \in \mathscr{S}(\mathbb{R}^d)$ one has $T_{m_{k+\ell}} f_k(\zeta) = \sum_{j=-1}^{1} T_{m_{k+\ell}} f_k(\zeta_{k+j})$, where $\zeta_k = \varphi_k * \zeta$, and thus

$$\begin{aligned}
\|T_{m_{k+\ell}} f_k(\zeta)\|_Y &\leqslant \|T_{m_{k+\ell}} f_k\|_{L^p(\mathbb{R}^d;Y)} \sum_{j=-1}^{1} \|\zeta_{k+j}\|_{L^{p'}(\mathbb{R}^d)} \\
&\leqslant C_{d,s,X,Y} K_m 2^{sk} \|f_k\|_{L^p(\mathbb{R}^d;Y)} \sum_{j=-1}^{1} 2^{|s|} 2^{-s(k+j)} \|\zeta_{k+j}\|_{L^{p'}(\mathbb{R}^d)}
\end{aligned}$$

Summing over k we see that

$$\begin{aligned}
\sum_{k\geqslant 0} \|T_{m_{k+\ell}} f_k(\zeta)\|_Y &\leqslant C_{d,s,X,Y} K_m \sum_{k\geqslant 0} 2^{sk} \|f_k\|_{L^p(\mathbb{R}^d;Y)} \sum_{j=-1}^{1} 2^{-sk} \|\zeta_{k+j}\|_{L^{p'}(\mathbb{R}^d)} \\
&\leqslant 3 \cdot 2^{|s|} C_{d,s,X,Y} K_m \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \|\zeta\|_{B^{-s}_{p',q'}(\mathbb{R}^d)},
\end{aligned}$$

which gives the required convergence.

By the properties of $(\varphi_n)_{n\geqslant 0}$ we can write

$$\mathscr{F}(\varphi_j * Tf) = \sum_{k=j-1}^{j+1} \widehat{\varphi}_j \sum_{\ell=-1}^{1} \widehat{\varphi}_{k+\ell} m \widehat{\varphi}_k f = \sum_{k=j-1}^{j+1} \widehat{\varphi}_j m \widehat{\varphi}_k f = \sum_{\ell-1}^{1} m_j f_{j+\ell}.$$

Therefore, the boundedness follows from

$$\begin{aligned}\|Tf\|_{B^s_{p,q}(\mathbb{R}^d;Y)} &\leqslant \sum_{\ell-1}^{1} \|(T_{m_j} f_{j+\ell})_{j\geqslant 0}\|_{\ell^q(L^p(\mathbb{R}^d;Y))}\\ &\leqslant C_{d,s,X,Y} K_m \sum_{\ell-1}^{1} \|(f_{j+\ell})_{j\geqslant 0}\|_{\ell^q(L^p(\mathbb{R}^d;X))}\\ &\leqslant C'_{d,s,X,Y} K_m \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

It remains to observe that for $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$, the following identities hold in $\mathscr{S}'(\mathbb{R}^d;X)$

$$\widehat{Tf} = \sum_{k\geqslant 0}\sum_{\ell=-1}^{1} \widehat{\varphi}_{k+\ell} m \widehat{\varphi}_k \widehat{f} = \sum_{k\geqslant 0} m\widehat{\varphi}_k \widehat{f} = m\widehat{f}.$$

□

A further consequence of Proposition 14.5.3 is a Fourier multiplier theorem of a very different nature, in which the multiplier is non-smooth but the domain and range spaces have different integrability and smoothness exponents.

Proposition 14.5.7. *Let X and Y be Banach spaces with Fourier type $p \in [1,2]$ and let $s := (\frac{1}{p} - \frac{1}{p'})d$. Let $m : \mathbb{R}^d \to \mathscr{L}(X,Y)$ be strongly measurable in the strong operator topology and uniformly bounded. Then the Fourier multiplier $T_m = \mathscr{F}^{-1} m \mathscr{F}$ is bounded as an operator from $B^s_{p,p}(\mathbb{R}^d;X)$ into $L^{p'}(\mathbb{R}^d;Y)$ with norm*

$$\|T_m\|_{\mathscr{L}(B^s_{p,p}(\mathbb{R}^d;X),L^{p'}(\mathbb{R}^d;Y))} \lesssim_p \varphi_{p,X}(\mathbb{R}^d)\varphi_{p,Y}(\mathbb{R}^d) \sup_{\xi\in\mathbb{R}^d} \|m(\xi)\|_{\mathscr{L}(X,Y)}.$$

Proof. By the Fourier type p of Y,

$$\begin{aligned}\|T_m f\|_{L^{p'}(\mathbb{R}^d;Y)} &\leqslant \varphi_{p,Y}(\mathbb{R}^d)\|m\widehat{f}\|_{L^p(\mathbb{R}^d;Y)}\\ &\leqslant \varphi_{p,Y}(\mathbb{R}^d) \sup_{\xi\in\mathbb{R}^d} \|m(\xi)\|_{\mathscr{L}(X,Y)} \|\widehat{f}\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

The Fourier type p of X and Proposition 14.5.3, applied with $q = p'$, give

$$\|\widehat{f}\|_{L^p(\mathbb{R}^d;X)} \lesssim_p \varphi_{p,X}(\mathbb{R}^d)\|f\|_{B^s_{p,p}(\mathbb{R}^d;X)},$$

and the result follows. □

14.5.b Smooth functions have *R*-bounded ranges

In Chapter 8 we have seen several instances of the general principle that sufficiently smooth operator-valued functions have R-bounded ranges. The amount of smoothness needed depends on the geometry of the underlying Banach spaces. For instance, it was shown in Theorem 8.5.21 that if X has cotype q and Y has type p, and if $T \in W^{s,r}(\mathbb{R}^d;\mathscr{L}(X,Y))$ with $(\frac{1}{p}-\frac{1}{q})d < \frac{d}{r} < s < 1$, then T has a continuous version whose range is R-bounded.

In the present section we will show that if the Besov scale is used instead of the Sobolev scale, the analogous result holds for the optimal smoothness exponent $s = (\frac{1}{p}-\frac{1}{q})d$ and the restriction $s<1$ can be omitted. The precise statement reads as follows.

Theorem 14.5.8 (Besov functions with R-bounded range – I). *Let X and Y be Banach spaces, X having cotype $q \in [2,\infty]$ and Y having type $p \in [1,2]$. If $r \in [1,\infty]$ satisfies $\frac{1}{r} \geqslant \frac{1}{p}-\frac{1}{q}$, then every $T \in B^{d/r}_{r,1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ has R-bounded range, with R-bound*

$$\mathscr{R}(T(t) : t \in \mathbb{R}^d) \leqslant C\|T\|_{B^{d/r}_{r,1}(\mathbb{R}^d;\mathscr{L}(X,Y))}, \tag{14.61}$$

where C is a constant depending on d,p,q,r,X,Y.

By Theorem 14.4.19, the spaces $B^{d/r}_{r,1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ increasing in the exponent $r \in [1,\infty]$ and we have continuous embeddings

$$B^{d/r}_{r,1}(\mathbb{R}^d;\mathscr{L}(X,Y)) \hookrightarrow B^{0}_{\infty,1}(\mathbb{R}^d;\mathscr{L}(X,Y)) \hookrightarrow C_{\rm ub}(\mathbb{R}^d;\mathscr{L}(X,Y)), \tag{14.62}$$

the second being a consequence of Proposition 14.4.18. The continuous version provided by (14.62) is used in the left-hand side of (14.61).

In the proof below, we will use the Lorentz space $L^{r',\sigma}(\mathbb{R}^d)$ with $\sigma = \frac{\min\{\frac{1}{p'},\frac{1}{q}\}}{\frac{1}{p'}+\frac{1}{q}} \in (0,1]$. Referring to Appendix F, we recall that the Lorentz space $L^{r',\sigma}(\mathbb{R}^d)$ is the space of all measurable functions $f:\mathbb{R}^d \to \mathbb{K}$ for which the (quasi-)norm

$$\|f\|_{L^{r',\sigma}(\mathbb{R}^d)} := \big\|\tau \mapsto \tau^{1/r'} f^*(\tau)\big\|_{L^\sigma(\mathbb{R}_+,\frac{\mathrm{d}\tau}{\tau})}$$

is finite, where

$$f^*(\tau) := \inf\big\{\lambda > 0 : |\{|f|>\lambda\}| \leqslant \tau\big\}, \quad \tau \in \mathbb{R}_+,$$

is the non-increasing rearrangement of f.

Proof. By the observation before (14.62) it suffices to prove the theorem in the case $\frac{1}{r} = \frac{1}{p}-\frac{1}{q}$. In the proof we will only consider $r \in (1,\infty]$; in Theorem 14.5.9 a stronger result is proved which covers the case $r=1$ of the present theorem.

Let us write

$$T = \sum_{k\geqslant 0} T_k = \sum_{\ell=-1}^{1}\sum_{k\geqslant 0} \varphi_{k+\ell} * T_k,$$

where $T_k = \varphi_k * T$, and we used (14.12) in the second identity. Since $T \in B^0_{\infty,1}(\mathbb{R}^d; \mathscr{L}(X,Y))$ (see (14.62)), the series $\sum_{k\geqslant 0} T_k$ converges uniformly on $\mathbb{R}^d$ with respect to the operator norm of $\mathscr{L}(X,Y)$. By Propositions 8.1.19 and 8.1.22,

$$\mathscr{R}(T(t) : t \in \mathbb{R}^d) \leqslant \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} \mathscr{R}(\varphi_{k+\ell} * T_k(t) : t \in \mathbb{R}^d), \tag{14.63}$$

provided of course that the operator families occurring in the sums are R-bounded and their R-bounds are summable. Proving this will occupy us in the remainder of the proof.

Fix an integer $n \geqslant 1$. Starting from the identity $\varphi_n(t) = 2^{(n-1)d}\varphi_1(2^{n-1}t)$ (see (14.4)), it is elementary to check that the non-increasing rearrangements satisfy $\varphi_n^*(\tau) = 2^{(n-1)d}\varphi_1^*(2^{n-1}\tau)$. Therefore,

$$\begin{aligned}\|\varphi_n\|_{L^{r',\sigma}(\mathbb{R}^d)} &= 2^{(n-1)d}\|\tau \mapsto \tau^{1/r'}\varphi_1^*(2^{n-1}\tau)\|_{L^\sigma(\mathbb{R}_+,\frac{\mathrm{d}\tau}{\tau})}\\ &= 2^{(n-1)d/r}\|\tau \mapsto \tau^{1/r'}\varphi_1^*(\tau)\|_{L^\sigma(\mathbb{R}_+,\frac{\mathrm{d}\tau}{\tau})} = 2^{(n-1)d/r}\|\varphi_1\|_{L^{r',\sigma}(\mathbb{R}^d)},\end{aligned}$$

the latter being finite since $\varphi_1 \in \mathscr{S}(\mathbb{R}^d)$. A similar calculation can be done for $n = 0$.

For $t \in \mathbb{R}^d$ define $\varphi_{n,t} \in \mathscr{S}(\mathbb{R}^d)$ by $\varphi_{n,t}(s) := \varphi_n(t-s)$. Then $\varphi_{n,t}$ is identically distributed with φ_n. Letting $T_{k,\varphi_{n,t}} \in \mathscr{L}(X,Y)$ be the integral operator from Proposition 8.5.16, i.e.,

$$T_{k,\varphi_{n,t}}x := \int_{\mathbb{R}^d} \varphi_{n,t}(s)T_k(s)x \,\mathrm{d}s,$$

it follows from Proposition 8.5.16 with $\sigma = r' \min\{\frac{1}{p'}, \frac{1}{q}\}$ and $\psi = \varphi_n$ that for all $n \geqslant 0$ and $k \geqslant 0$ the set $\{\varphi_n * T_k(t) : t \in \mathbb{R}^d\}$ is R-bounded, with R-bound

$$\mathscr{R}(\varphi_n * T_k(t) : t \in \mathbb{R}^d) = \mathscr{R}(T_{k,\varphi_{n,t}} : t \in \mathbb{R}^d) \leqslant C2^{nd/r}\|T_k\|_{L^r(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

With (14.63) we conclude that

$$\begin{aligned}\mathscr{R}(T(t) : t \in \mathbb{R}^d) &\leqslant C\sum_{\ell=-1}^{1}\sum_{k\geqslant 0} 2^{(k+\ell)d/r}\|T_k\|_{L^r(\mathbb{R}^d;\mathscr{L}(X,Y))}\\ &\leqslant 3\cdot 2^{\frac{d}{r}}C\|T\|_{B^{d/r}_{r,1}(\mathbb{R}^d;\mathscr{L}(X,Y))}.\end{aligned}$$

□

We have the following variation of this result for the strong operator topology:

Theorem 14.5.9 (Besov functions with R-bounded range – II). *Let X and Y be Banach spaces and assume that Y has type $p \in [1,2]$. Suppose that $T : \mathbb{R}^d \to \mathscr{L}(X,Y)$ satisfies $Tx \in B^{d/p}_{p,1}(\mathbb{R}^d;Y)$ for all $x \in X$ and*

$$\|Tx\|_{B^{d/p}_{p,1}(\mathbb{R}^d;Y)} \leqslant C_T\|x\|, \quad x \in X.$$

Then the family $\{T(t) \in \mathscr{L}(X,Y) : t \in \mathbb{R}^d\}$ *is R-bounded, with R-bound*

$$\mathscr{R}(T(t) \in \mathscr{L}(X,Y) : t \in \mathbb{R}^d) \leqslant CC_T,$$

where C *is a constant depending on* p *and* Y*.*

Proof. We begin with the case $p = 1$, which corresponds to the case where Y is an arbitrary Banach space. By Proposition 14.5.3 we have $\widehat{T}x \in L^1(\mathbb{R}^d;Y)$ and

$$\|\widehat{T}x\|_{L^1(\mathbb{R}^d;Y)} \lesssim_d \|Tx\|_{B^{d/p}_{p,1}(\mathbb{R}^d;Y)} \leqslant C_T\|x\|.$$

This implies that we have the integral representation

$$T(t)x = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot t}\widehat{T}(\xi)x \,\mathrm{d}\xi, \quad t \in \mathbb{R}^d,$$

where the operator-valued kernel is strongly in L^1. Now Theorem 8.5.4 implies that the family $\{T(t) : t \in \mathbb{R}^d\}$ is R-bounded, with R-bound $\mathscr{R}_p(T(t) : t \in \mathbb{R}^d) \lesssim_d C_T$.

Next assume that $p \in (1,2]$. For $k \geqslant 0$ and $x \in X$ set $T_k(t)x := \varphi_k * T(t)x$. By Theorem 14.5.1,

$$\|T_kx\|_{\gamma(L^2(\mathbb{R}^d),Y)} \leqslant C\|T_kx\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;Y)} \leqslant C_{d,p,s}2^{kd(\frac{1}{p}-\frac{1}{2})}\|T_kx\|_{L^p(\mathbb{R}^d;Y)}, \tag{14.64}$$

where (setting $s = d(\frac{1}{p}-\frac{1}{2})$ for brevity) the second inequality follows from

$$\begin{aligned}\|T_kx\|^p_{B^s_{p,p}(\mathbb{R}^d;Y)} &= \sum_{n\geqslant 0} 2^{nsp}\|\varphi_n * \varphi_k * Tx\|^p_{L^p(\mathbb{R}^d;Y)} \\ &= \sum_{\ell=-1}^{1} 2^{(k+\ell)sp}\|\varphi_{k+\ell} * \varphi_k * Tx\|^p_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \sum_{\ell=-1}^{1} 2^{(k+\ell)sp}\|\varphi_{k+\ell}\|_1^p\|\varphi_k * Tx\|^p_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant 3\cdot 2^{(k+1)sp}\cdot 2^p\|\varphi\|_1^p\|T_kx\|^p_{L^p(\mathbb{R}^d;Y)}\end{aligned}$$

using (14.11) and (14.7).

Choose arbitrary finite sequences $(t_m)_{m=1}^M$ in $\mathbb{R}^d$ and $(x_m)_{m=1}^M$ in X, and let $(\varepsilon_m)_{m=1}^M$ be a Rademacher sequence on a probability space $(\Omega,\mathbb{P})$. Since Y has type $p > 1$ it follows from Theorem 9.6.14 with constant $L_{p,Y}$ hat

$$\Big\|\sum_{m=1}^M \varepsilon_m T(t_m)x_m\Big\|_{L^2(\Omega;Y)}$$

$$
\begin{aligned}
&\leqslant \sum_{k\geqslant 0}\sum_{\ell=-1}^{1}\Big\|\sum_{m=1}^{M}\varepsilon_m\varphi_{k+\ell}*T_k(t_m)x_m\Big\|_{L^2(\Omega;Y)}\\
&=\sum_{k\geqslant 0}\sum_{\ell=-1}^{1}\Big\|\sum_{m=1}^{M}\varepsilon_m\int_{\mathbb{R}^d}T_k(u)x_m\,\varphi_{k+\ell}(t_m-u)\,\mathrm{d}u\Big\|_{L^2(\Omega;Y)}\\
&\leqslant L_{p,Y}\sum_{k\geqslant 0}\sum_{\ell=-1}^{1}\|\varphi_{k+\ell}\|_{L^2(\mathbb{R}^d)}\Big\|\sum_{m=1}^{M}\varepsilon_m T_k x_m\Big\|_{L^2(\Omega;\gamma(L^2(\mathbb{R}^d),Y))}\\
&\leqslant L_{p,Y}C_\varphi\sum_{k\geqslant 0}2^{kd/2}\Big\|T_k\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{L^2(\Omega;\gamma(L^2(\mathbb{R}^d),Y))},
\end{aligned}
$$

where we used that (14.9) implies $\|\varphi_{k+\ell}\|_{L^2(\mathbb{R}^d)} = \|\widehat{\varphi}_{k+\ell}\|_{L^2(\mathbb{R}^d)} \leqslant C_\varphi 2^{kd/2}$. Applying (14.64) pointwise in Ω, setting $C_0 := L_{p,Y}C_\varphi C_{d,p,s}$, and using the Kahane-Khintchine inequalities, we continue estimating

$$
\begin{aligned}
&\leqslant C_0\sum_{k\geqslant 0}2^{kd/2}2^{k(\frac{1}{p}-\frac{1}{2})d}\Big\|T_k\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{L^2(\Omega;L^p(\mathbb{R}^d;Y))}\\
&\leqslant C_0\kappa_{2,1}\int_\Omega\sum_{k\geqslant 0}2^{kd/p}\Big\|T_k\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{L^p(\mathbb{R}^d;Y)}\,\mathrm{d}\mathbb{P}\\
&= C_0\kappa_{2,1}\int_\Omega\Big\|T\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{B^{d/p}_{p,1}(\mathbb{R}^d;Y)}\,\mathrm{d}\mathbb{P}\\
&\leqslant C_0\kappa_{2,1}C_T\int_\Omega\Big\|\sum_{m=1}^{M}\varepsilon_m x_m\Big\|_X\,\mathrm{d}\mathbb{P}\\
&\leqslant C_0\kappa_{2,1}C_T\Big\|\sum_{m=1}^{M}\varepsilon_m x_m\Big\|_{L^2(\Omega;X)}.
\end{aligned}
$$

Putting things together gives the required R-boundedness estimate. □

Remark 14.5.10.

(1) The method of proof for $p = 1$ in Theorem 14.5.9 could be extended to $p \in (1, 2]$ if Y has Fourier type p. We have not done this, because Proposition 7.3.6 shows that having type p is weaker than having Fourier type p.

(2) In the case $p = 1$ and $d = 1$, a variation of the argument in Proposition 8.5.7 actually gives a stronger result than Theorem 14.5.9, namely that if $Tx \in W^{d,1}(\mathbb{R}^d; \mathscr{L}(X, Y))$ for all $x \in X$, then the range of T is R-bounded.

14.6 Triebel–Lizorkin spaces

As we have seen in the preceding sections, the study of Besov spaces is intimately connected with the space $\ell^q(L^p(\mathbb{R}^d;X))$ through the very definition, which features the norm

$$\|f\|_{B^p_{q,s}(\mathbb{R}^d;X)} = \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}.$$

The class of Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d;X)$ is obtained upon replacing $\ell^q(L^p(\mathbb{R}^d;X))$ by $L^p(\mathbb{R}^d;\ell^q(X))$, putting

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} = \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}.$$

The theory of Triebel–Lizorkin spaces is in many respect analogous to the theory of Besov spaces, but the occurrence of the ℓ^q-norm inside the L^p-norm precludes the use of Young's inequality to estimate the norm of term-wise convolutions, a technique that was critically used in our treatment of Besov spaces. This makes the norm of Triebel–Lizorkin spaces more difficult to deal with.

14.6.a The Peetre maximal function

The obstruction just noted already makes itself felt if one tries to adapt the proof that Besov spaces are independent up to an equivalent norm of the inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$ to Triebel–Lizorkin spaces. The encountered difficulty will be resolved by a variant on the Fefferman–Stein inequality due to Peetre, to which we turn in the present preliminary subsection.

Throughout this section, unless otherwise stated X is an arbitrary Banach space. For a strongly measurable function $f:\mathbb{R}^d\to X$ and $r\in(0,\infty)$ we let

$$M_r f(x) := (M(\|f\|^r)(x))^{1/r}, \quad x\in\mathbb{R}^d, \tag{14.65}$$

where M is the Hardy–Littlewood maximal operator introduced in Section 2.3,

$$Mf(x) := \sup_{B\ni x}\frac{1}{|B|}\int_B \|f(y)\|\,\mathrm{d}y,$$

the supremum being taken over all Euclidean balls B in $\mathbb{R}^d$ that contain x.

Lemma 14.6.1 (Peetre's maximal inequality). *Fix $r,t\in(0,\infty)$ and a multi-index $\alpha\in\mathbb{N}^d$, and let $f\in\mathscr{S}'(\mathbb{R}^d;X)$ satisfy*

$$\operatorname{supp}\widehat{f}\subseteq B_t := \{\xi\in\mathbb{R}^d : |\xi|\leqslant t\}.$$

Then $f\in C^\infty(\mathbb{R}^d;X)$ and there exist constants C_1 and C_2, depending only on $|\alpha|$, d, r such that for all $x\in\mathbb{R}^d$ we have

$$\sup_{z\in\mathbb{R}^d} t^{-|\alpha|}\frac{\|\partial^\alpha f(x-z)\|}{(1+t|z|)^{d/r}} \leqslant C_1 \sup_{z\in\mathbb{R}^d}\frac{\|f(x-z)\|}{(1+t|z|)^{d/r}} \leqslant C_2 M_r f(x)$$

In particular, taking $z=0$, for all $x\in\mathbb{R}^d$ we have

$$t^{-|\alpha|}\|\partial^\alpha f(x)\| \leqslant \|f(x)\| \leqslant C_2 M_r f(x).$$

Proof. That the tempered distribution f is represented by a function in $C^\infty(\mathbb{R}^d;X)$ has already been observed in Lemma 14.2.9. In the remainder of the proof we assume that this identification has been made.

By an iteration argument it suffices to consider multi-indices satisfying $|\alpha|=1$. The short-hand notation $\|\nabla f(x)\| = \sum_{j=1}^d \|\partial_j f(x)\|$ will be used throughout the proof. We first consider the case $f\in\mathscr{S}(\mathbb{R}^d;X)$. Replacing f by $f(t^{-1}\cdot)$, it suffices to prove the result for $t=1$.

Step 1 – Choose $\psi\in\mathscr{S}(\mathbb{R}^d)$ such that $\widehat{\psi}\equiv 1$ on B_1. Since $\widehat{f}$ is supported on B_1, we have $f=\psi * f$ and $\nabla f = (\nabla\psi)*f$. It follows that for $x,z\in\mathbb{R}^d$ and $\lambda>0$,

$$\begin{aligned}\|\partial_j f(x-z)\| &\leqslant \int_{\mathbb{R}^d}|\partial_j\psi(x-z-y)|\,\|f(y)\|\,\mathrm{d}y\\ &\leqslant c_\lambda\int_{\mathbb{R}^d}(1+|x-z-y|)^{-\lambda}\,\|f(y)\|\,\mathrm{d}y,\end{aligned}$$

where $c_\lambda = \sup_{y\in\mathbb{R}^d}(1+|y|)^\lambda|\partial_j\psi(y)|$. Clearly we have $1+|x-y|\leqslant(1+|x-z-y|)(1+|z|)$, and upon taking $\lambda = d+1+d/r$ we obtain

$$\begin{aligned}\frac{\|\partial_j f(x-z)\|}{(1+|z|)^{d/r}} &\leqslant c_\lambda\int_{\mathbb{R}^d}(1+|x-z-y|)^{-\lambda}(1+|z|)^{-d/r}\,\|f(y)\|\,\mathrm{d}y\\ &\leqslant c_\lambda\int_{\mathbb{R}^d}(1+|x-z-y|)^{-d-1}(1+|x-y|)^{-d/r}\,\|f(y)\|\,\mathrm{d}y\\ &\leqslant C_1\sup_{y\in\mathbb{R}^d}\frac{\|f(x-y)\|}{(1+|y|)^{d/r}},\end{aligned}$$

where $C_1 = c_\lambda\int_{\mathbb{R}^d}(1+|y|)^{-d-1}\,\mathrm{d}y$. This gives the first inequality in the statement of the lemma.

Step 2 – Fix $\varepsilon>0$ and let Q_ε be the closed cube centred at zero and of side-length ε. We claim that for all $g\in C^1(Q_\varepsilon;X)$,

$$\|g(0)\| \leqslant \frac{\varepsilon}{2}\sup_{y\in Q_\varepsilon}\|\nabla g(y)\| + \Big(\fint_{Q_\varepsilon}\|g(y)\|^r\,\mathrm{d}y\Big)^{1/r}, \tag{14.66}$$

where we write $\fint_Q = \frac{1}{|Q|}\int_Q$ for averages. By scaling it suffices prove (14.66) for $\varepsilon=1$.

Fix $g\in C^1(Q_1;X)$. For all $y\in Q_1$ we have $\|y\|\leqslant\frac{1}{2}$ and

$$g(0) = g(y) + \int_0^1 \nabla g(ty) \cdot y \, dt.$$

Therefore, $\|g(0)\| \leqslant \|g(y)\| + \frac{1}{2} \sup_{y \in Q_1} \|\nabla g(y)\|$. Taking L^r-average over Q_1 gives (14.66) for $\varepsilon = 1$.

Step 3 – By Step 2, applied to the function $f(x - z - \cdot)$,

$$\|f(x-z)\| \leqslant \frac{\varepsilon}{2} \sup_{y \in Q_\varepsilon} \|\nabla f(x-z-y)\| + \Big(\fint_{Q_\varepsilon} \|f(x-z-y)\|^r \, dy \Big)^{1/r}. \tag{14.67}$$

Now let $\varepsilon \in (0,1]$. It follows from $z - Q_\varepsilon \subseteq Q_{1+|z|}$ that

$$\begin{aligned} \fint_{Q_\varepsilon} \|f(x-z-y)\|^r \, dy &= \fint_{z-Q_\varepsilon} \|f(x-y)\|^r \, dy \\ &\leqslant \frac{|Q_{1+|z|}|}{|Q_\varepsilon|} \fint_{Q_{1+|z|}} \|f(x-y)\|^r \, dy \\ &\leqslant \varepsilon^{-d} (1+|z|)^d M(\|f\|^r)(x). \end{aligned}$$

Substituting this into (14.67) and dividing by $(1+|z|)^{d/r}$, it follows that

$$\begin{aligned} \sup_{z \in \mathbb{R}^d} \frac{\|f(x-z)\|}{(1+|z|)^{d/r}} &\leqslant \frac{\varepsilon}{2} \sup_{z \in \mathbb{R}^d} \sup_{y \in Q_\varepsilon} \frac{\|\nabla f(x-z-y)\|}{(1+|z|)^{d/r}} + \varepsilon^{-d/r} M_r f(x) \\ &\leqslant \varepsilon 2^{d/r-1} \sup_{z \in \mathbb{R}^d} \frac{\|\nabla f(x-z)\|}{(1+|z|)^{d/r}} + \varepsilon^{-d/r} M_r f(x), \end{aligned}$$

where we used that $(1+|z|) \geqslant \frac{1}{2}(1+|y+z|)$ for $|y| \leqslant \varepsilon \leqslant 1$ and performed a change of variables. Combining this estimate with the first inequality in the statement of the lemma, and taking $\varepsilon \in (0,1]$ small enough, the result follows.

Step 4 – Next let $f \in \mathscr{S}'(\mathbb{R}^d; X)$ and $t > 0$. Let $f_\delta = \psi(\delta \cdot) f$, where $\psi \in \mathscr{S}(\mathbb{R}^d)$ satisfies $\psi(0) = 1$, $\operatorname{supp} \widehat{\psi} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$ and $\delta \in (0, \min\{1, t\})$. Recalling that $f \in C^\infty(\mathbb{R}^d; X)$, clearly we have $f_\delta \in \mathscr{S}(\mathbb{R}^d; X)$, $\widehat{f_\delta}$ has support in B_{2t} and therefore, by the previous steps, the second inequality in the statement of the lemma holds if in the two expressions on the left-hand side f is replaced by f_δ and for the right-hand side we note that $M_r f_\delta(x) \leqslant \|\psi\|_\infty M_r f(x)$. It remains to let $\delta \to 0$ on the left-hand side and note that $f_\delta(x-z) \to f(x-z)$ and similarly for its derivatives. □

Using the pointwise estimate of Lemma 14.6.1, we will now deduce a maximal inequality in $L^p(\mathbb{R}^d; \ell^q)$.

Proposition 14.6.2 (Boundedness of Peetre's maximal function). *Let $p \in [1, \infty)$, $q \in [1, \infty]$, and let $r \in (0, \min\{p, q\})$. Let $f = (f_k)_{k \geqslant 0}$ in $L^p(\mathbb{R}^d; \ell^q(X))$ be such that* $\operatorname{supp}(\widehat{f_k}) \subseteq S_k$ *for all $k \geqslant 0$, where $S_k \subseteq \mathbb{R}^d$ is a* *compact set with diameter* $\delta_k > 0$. *There exists a constant* $C \geqslant 0$, *depending only on* d, p, q, r, *such that*

$$\Big\| \Big(\sup_{z \in \mathbb{R}^d} \frac{\|f_k(\cdot - z)\|}{(1+\delta_k|z|)^{d/r}} \Big)_{k \geqslant 0} \Big\|_{L^p(\mathbb{R}^d;\ell^q)} \leqslant C\|f\|_{L^p(\mathbb{R}^d;\ell^q(X))}.$$

Proof. We use the short-hand notation $f = (f_k)_{k\geqslant 0}$ and $f^*_{d/r} = (f^*_{k,d/r})_{k\geqslant 0}$, where

$$f^*_{k,d/r}(x) = \sup_{z\in\mathbb{R}^d} \frac{\|f_k(x-z)\|}{(1+\delta_k|z|)^{d/r}}, \quad x \in \mathbb{R}^d. \tag{14.68}$$

Multiplying $f_k(x)$ with $e^{2\pi i h_k \cdot x}$ for suitable $h_k \in \mathbb{R}^d$, we may assume that each $\widehat{f}_k$ has support in $B_k = \{\xi \in \mathbb{R}^d : |\xi| \leqslant \delta_k\}$ for $k \geqslant 0$.

Let $g_k(x) := f_k(\delta_k^{-1}x)$. Then $\widehat{g}_k$ has support in a ball of radius 1 centred around the origin. Thus by Lemma 14.6.1 there is a constant c, depending only on d and r, such that for all $k \geqslant 0$ and $x \in \mathbb{R}^d$ we have

$$\sup_{z\in\mathbb{R}^d} \frac{\|g_k(x-z)\|}{(1+|z|)^{d/r}} \leqslant cM_r g_k(x).$$

Rewriting this in terms of f_k gives

$$f^*_{k,d/r}(x) = \sup_{z\in\mathbb{R}^d} \frac{\|f_k(x-z)\|}{(1+\delta_k|z|)^{d/r}} \leqslant cM_r f_k(x).$$

Taking $L^p(\mathbb{R}^d;\ell^q)$ norms and applying the Fefferman–Stein maximal Theorem 3.2.28 in the space $L^{p/r}(\mathbb{R}^d;\ell^{q/r})$, we find that

$$\begin{aligned} \|f^*_{d/r}\|_{L^p(\mathbb{R}^d;\ell^q)} &\leqslant c\|(M_r f_k)_{k\geqslant 0}\|_{L^p(\mathbb{R}^d;\ell^q)} = c\big\|(M(\|f_k\|^r))_{k\geqslant 0}\big\|^{1/r}_{L^{p/r}(\mathbb{R}^d;\ell^{q/r})} \\ &\lesssim_{p,q,r} c\big\|(\|f_k\|^r)_{k\geqslant 0}\big\|^{1/r}_{L^{p/r}(\mathbb{R}^d;\ell^{q/r})} = c\|f\|_{L^p(\mathbb{R}^d;\ell^q(X))}. \end{aligned}$$

□

As a first application we derive a Fourier multiplier theorem for certain functions in $L^p(\mathbb{R}^d;\ell^q)$ for $p \in [1,\infty)$ and $q \in [1,\infty]$ which is essential for later considerations about Triebel–Lizorkin spaces. The main difficulty arises if $p = 1$ or $q = 1$ since the maximal function is not bounded in these cases. The case $q = 1$ turns out to be of particular importance in Section 14.7.a.

The statement of the following theorem, which is needed in the proof of the Mihlin multiplier theorem for Triebel–Lizorkin spaces (theorem 14.6.11) is admittedly somewhat technical. We recall from Subsection 2.4.a that $\check{L}^1(\mathbb{R}^d;X)$ denotes the subspace in $L^\infty(\mathbb{R}^d;X)$ of all functions whose inverse Fourier transform belongs to $L^1(\mathbb{R}^d;X)$.

Theorem 14.6.3. *Let X and Y be Banach spaces and let $p \in [1, \infty)$, $q \in [1, \infty]$, and $r \in (0, \min\{p, q\})$. Let $S_k \subseteq \mathbb{R}^d$, $k \geqslant 0$, be compact sets with diameter $\delta_k > 0$. Then for all sequences $m = (m_k)_{k \geqslant 0}$ in $\check{L}^1(\mathbb{R}^d; \mathscr{L}(X, Y))$ and all $f = (f_k)_{k \geqslant 0} \in L^p(\mathbb{R}^d; \ell^q(X))$ with $\operatorname{supp} \widehat{f_k} \subseteq S_k$ for each $k \geqslant 0$ we have $\mathscr{F}^{-1} m \mathscr{F} f \in L^p(\mathbb{R}^d; \ell^q(Y))$ and*

$$\begin{aligned}
&\|(\mathscr{F}^{-1} m \mathscr{F} f\|_{L^p(\mathbb{R}^d; \ell^q(Y))} \\
&\quad \leqslant C \sup_{k \geqslant 0} \|(1 + \delta_k |\cdot|)^{d/r}) \mathscr{F}^{-1} m_k(\cdot)\|_{L^1(\mathbb{R}^d; \mathscr{L}(X,Y))} \|f\|_{L^p(\mathbb{R}^d; \ell^q(X))} \\
&\quad = C \sup_{k \geqslant 0} \|(1 + |\cdot|)^{d/r}) \mathscr{F}^{-1} [m_k(\delta_k \cdot)]\|_{L^1(\mathbb{R}^d; \mathscr{L}(X,Y))} \|f\|_{L^p(\mathbb{R}^d; \ell^q(X))}
\end{aligned}$$

where the constant $C \geqslant 0$ depends only on d, p, q, r, provided the supremum on the right-hand side is finite.

Proof. The kernels $K_k := \mathscr{F}^{-1} m_k$ are in $L^1(\mathbb{R}^d; \mathscr{L}(X, Y))$ by assumption. Therefore, the functions $\mathscr{F}^{-1}(m_k \widehat{f_k}) = K_k * f_k$ are well defined in $L^p(\mathbb{R}^d; Y)$ by Young's inequality. Let

$$c_m := \sup_{k \geqslant 0} \|(1 + \delta_k |\cdot|)^{d/r} K_k(\cdot)\|_{L^1(\mathbb{R}^d; \mathscr{L}(X,Y))}.$$

Then, using the notation introduced in (14.68),

$$\begin{aligned}
\|K_k * f_k(x)\| &\leqslant \int_{\mathbb{R}^d} \|K_k(x - y)\| (1 + \delta_k |x - y|)^{d/r} \frac{\|f_k(y)\|}{(1 + \delta_k |x - y|)^{d/r}} \, \mathrm{d}y \\
&\leqslant f^*_{n,d/r}(x) \int_{\mathbb{R}^d} \|K_k(x - y)\| (1 + \delta_k |x - y|)^{d/r} \, \mathrm{d}y \leqslant c_m f^*_{n,d/r}(x).
\end{aligned}$$

The required result follows from this by taking $L^p(\mathbb{R}^d; \ell^q)$-norms and applying Proposition 14.6.2.

The final identity of the theorem simply follows by a substitution together with the dilation property $\delta_k^{-1}(\mathscr{F}^{-1} m_k)(\delta_k^{-1} \cdot) = \mathscr{F}^{-1}[m_k(\delta_k \cdot)]$ of the Fourier transform. □

Remark 14.6.4. Lemma 14.6.1 can be used to extend the Bernstein–Nikolskii inequality presented in Lemma 14.4.20 to the full range $0 < p_0 \leqslant p_1 \leqslant \infty$. To this end let ψ be as in the proof of the lemma and note that it suffices to consider the case that $\widehat{f}$ has support in the unit ball.

First consider $0 < p_0 < p_1 \leqslant \infty$ and $\alpha = 0$. If $p_0 \in (0, 1)$ and $p_1 = \infty$, then

$$\begin{aligned}
|f(x)| &\leqslant \int_{\mathbb{R}^d} |\psi(x - y)| \|f(y)\| \, \mathrm{d}y| \\
&\leqslant \|\psi\|_\infty \int_{\mathbb{R}^d} \|f(y)\|^{1-p_0} \|f(y)\|^{p_0} \, \mathrm{d}y| \leqslant \|\psi\|_\infty \|f\|_\infty^{1-p_0} \|f\|_{p_0}^{p_0}
\end{aligned}$$

and consequently $\|f\|_\infty \lesssim_{p_0,\psi} \|f\|_{p_0}$. Since we already knew the result for $p_0 \geqslant 1$, this inequality holds for $p_0 \in (0,\infty)$. In the remaining case $p_0 < p_1 < \infty$, we similarly find that

$$\|f\|_{p_1} \leqslant \|f\|_\infty^{1-p_0/p_1} \|f\|_{p_0}^{p_0/p_1} \lesssim_{p_0,p_1,\psi} \|f\|_{p_0}.$$

The case $p_0 = p_1$ and $\alpha \neq 0$ follows by taking L^{p_1}-norms in the pointwise estimate $\|\partial^\alpha f(x)\| \leqslant C M_r(f)(x)$ with $r \in (0,p_1)$ (see Lemma 14.6.1) and using the $L^{p_1/r}$-boundedness of the Hardy–Littlewood maximal function, to conclude that $\|\partial^\alpha f\|_{p_1} \leqslant C\|f\|_{p_1}$.

If $p_0 < p_1$ and $\alpha \neq 0$ combining the previous two cases gives

$$\|\partial^\alpha f\|_{p_1} \leqslant C\|f\|_{p_1} \leqslant C'\|f\|_{p_0}.$$

14.6.b Definitions and basic properties

We now introduce our main characters. Recall that we have fixed a inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$ in Subsection 14.2.c (see Convention 14.2.8).

Definition 14.6.5 (Triebel–Lizorkin spaces). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. The* Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^d;X)$ *is the space of all $f \in \mathscr{S}'(\mathbb{R}^d;X)$ for which the quantity*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} := \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}$$

is finite.

We comment on the case $p = \infty$ and $q < \infty$ in the Notes, as this exceptional case behaves differently. Below we will check that the above definition is independent on the choice of the Littlewood–Paley sequence up to an equivalent norm and that the resulting spaces are Banach spaces.

It is immediate from Young's inequality that $\psi * f \in F^s_{p,q}(\mathbb{R}^d;X)$ whenever $\psi \in L^1(\mathbb{R}^d)$ and $f \in F^s_{p,q}(\mathbb{R}^d;X)$, and more generally the analogue of Proposition 14.2.3 in valid.

By Fubini's theorem, for all $p \in [1,\infty)$ we have

$$F^s_{p,p}(\mathbb{R}^d;X) = B^s_{p,p}(\mathbb{R}^d;X).$$

We have continuous embeddings

$$F^s_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q_1}(\mathbb{R}^d;X), \quad 1 \leqslant q_0 \leqslant q_1 \leqslant \infty, \tag{14.69}$$

and, by Hölder's inequality for the ℓ^q-norm,

$$F^{s_0}_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p,q_1}(\mathbb{R}^d;X), \quad q_0,q_1 \in [1,\infty],\ s_0 > s_1. \tag{14.70}$$

Next we prove that, up to equivalence of norm, the Triebel–Lizorkin spaces are independent of the choice of the inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$. The corresponding result for Besov spaces, Proposition 14.4.2, was rather easy to prove. The case of Triebel–Lizorkin spaces is not so easy and is based on Proposition 14.6.2. For $p>1$ and $q>1$ the use of this theorem can be avoided by using instead the estimate $\|\varphi_k * f\| \leqslant cMf$ together with the Fefferman–Stein Theorem 3.2.28.

Proposition 14.6.6. *Let $p\in[1,\infty)$, $q\in[1,\infty]$, and $s\in\mathbb{R}$. Up to an equivalent norm, the space $F^s_{p,q}(\mathbb{R}^d;X)$ is independent of the choice of the inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$.*

Proof. Fix inhomogeneous Littlewood–Paley sequences $(\varphi_k)_{k\geqslant 0}$ and $(\psi_k)_{k\geqslant 0}$. For all $j,k\geqslant 0$ with $|j-k|\geqslant 2$ we have $\psi_k*\varphi_j = \mathscr{F}^{-1}(\widehat{\psi}_k\widehat{\varphi}_j)=0$. Therefore, writing $f=\sum_{j\geqslant 0} f_j$ with $f_j=\varphi_j * f$,

$$\big\|(2^{ks}\psi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))} \leqslant \sum_{\ell=-1}^{1} \big\|(2^{ks}\psi_k * f_{\ell+k})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}.$$

Fix an arbitrary $r\in(0,\min\{p,q\})$, say $r=r_{p,q}=\frac{1}{2}\min\{p,q\}$. Applying Theorem 14.6.3 with $\delta_k = 3\cdot 2^k$ and $m_k=\widehat{\psi}_k$ to $(2^{ks}f_{\ell+k})_{k\geqslant 0}$ we obtain

$$\begin{aligned}\big\|(2^{ks}\psi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))} &\leqslant C_{\psi,d,p,q,s}\big\|(2^{ks}f_{\ell+k})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}\\ &\leqslant C'_{\psi,d,p,q,s}\big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}.\end{aligned}$$

Since $(\psi_k)_{k\geqslant 0}$ and $(\varphi_k)_{k\geqslant 0}$ were arbitrary, this completes the proof. □

The same argument and (14.5) lead to the following useful estimate.

Lemma 14.6.7. *Let $f\in F^s_{p,q}(\mathbb{R}^d;X)$, let $(\psi_k)_{k\geqslant 0}$ be a Littlewood–Paley sequence, and set*

$$S_n f := \sum_{k=0}^{n} \psi_k * f, \quad n\geqslant 0.$$

Then $S_n f\in F^s_{p,q}(\mathbb{R}^d;X)$ and there exists a constant $C=C(p,q,d,\psi)$ such that

$$\|S_n f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)}, \quad n\geqslant 0.$$

We have the following analogue of Proposition 14.4.18 for Triebel–Lizorkin spaces:

Proposition 14.6.8 (Sandwiching with Besov spaces). *For all $p\in[1,\infty)$, $q\in[1,\infty]$, and $s\in\mathbb{R}$, we have the natural continuous embeddings*

$$\mathscr{S}(\mathbb{R}^d;X)\hookrightarrow F^s_{p,q}(\mathbb{R}^d;X)\hookrightarrow \mathscr{S}'(\mathbb{R}^d;X),$$

the first of which is dense if $p,q\in[1,\infty)$, and

$$B^s_{p,p\wedge q}(\mathbb{R}^d;X)\hookrightarrow F^s_{p,q}(\mathbb{R}^d;X)\hookrightarrow B^s_{p,p\vee q}(\mathbb{R}^d;X).$$

By combining the first of these inclusions with Lemma 14.2.1 we see that if $p, q \in [1, \infty)$, then $C_{\rm c}^\infty(\mathbb{R}^d) \otimes X$ is dense in $F^s_{p,q}(\mathbb{R}^d; X)$.

Proof. First let $p > q$. For $f \in B^s_{p,q}(\mathbb{R}^d; X)$ it follows from the triangle inequality in $L^{p/q}(\mathbb{R}^d)$ that

$$\begin{aligned}\|f\|^q_{F^s_{p,q}(\mathbb{R}^d;X)} &= \Big\| \sum_{k\geqslant 0} 2^{ksq} \|\varphi_k * f\|^q \Big\|_{L^{p/q}(\mathbb{R}^d)} \\ &\leqslant \sum_{k\geqslant 0} 2^{ksq} \|\varphi_k * f\|^q_{L^p(\mathbb{R}^d;X)} = \|f\|^q_{B^s_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

This gives the first embedding in the second displayed line of the proposition. The second embedding follows from (14.69), which gives $F^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow F^s_{p,p}(\mathbb{R}^d; X) = B^s_{p,p}(\mathbb{R}^d; X)$ continuously. The case $p \leqslant q$ is handled similarly.

The continuous embeddings in the first line now follow from the corresponding result for Besov spaces contained in Proposition 14.4.3.

Let us finally show that $\mathscr{S}(\mathbb{R}^d; X)$ is dense in $F^s_{p,q}(\mathbb{R}^d; X)$. The proof is similar to Step 3 of the proof of Proposition 14.4.3. Let $f \in F^s_{p,q}(\mathbb{R}^d; X)$ and set $\zeta_n := \sum_{k=0}^n \varphi_k$. By (14.6) we have $\|\zeta_n\|_1 \leqslant \|\varphi_0\|_1$.

We will first show that $\zeta_n * f \to f$ in $F^s_{p,q}(\mathbb{R}^d; X)$. Let $\varepsilon > 0$ and choose $K \geqslant 0$ such that

$$\Big\| \Big(\sum_{k>K} 2^{ksq} \|\varphi_k * f\|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} < \varepsilon.$$

By Young's inequality,

$$\Big\| \zeta_n * (2^{ks} \varphi_k * f)_{k>K} \Big\|_{L^p(\mathbb{R}^d;\ell^q(X))} < \varepsilon \|\varphi_0\|_1.$$

It follows that

$$\begin{aligned}&\|f - \zeta_n * f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \\ &\quad\leqslant \varepsilon(1 + \|\varphi_0\|_1) + \Big\| \Big(\sum_{k=0}^K 2^{ksq} \|\varphi_k * f - \zeta_n * \varphi_k * f\|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} \\ &\quad\leqslant \varepsilon(1 + \|\varphi_0\|_1) + \sum_{k=0}^K 2^{ks} \|\varphi_k * f - \zeta_n * \varphi_k * f\|_{L^p(\mathbb{R}^d;X)}\end{aligned}$$

The last term tends to zero as $n \to \infty$ by Proposition 1.2.32.

It remains to approximate each of the functions $\zeta_n * f$ by elements in $\mathscr{S}(\mathbb{R}^d; X)$. This can be done as in Proposition 14.4.3. □

This result enables us to give a quick proof of the completeness of Triebel–Lizorkin spaces:

Proposition 14.6.9. *For $p \in [1, \infty)$, $q \in [1, \infty]$, and $s \in \mathbb{R}$, the space $F^s_{p,q}(\mathbb{R}^d; X)$ is a Banach space.*

Proof. As in the Besov case one proves that for all $p \in [1, \infty)$, $q \in [1, \infty]$, and $s \in \mathbb{R}$, the space $F^s_{p,q}(\mathbb{R}^d; X)$ has the Fatou property. Since Triebel–Lizorkin spaces embed into $\mathscr{S}'(\mathbb{R}^d; X)$ by Proposition 14.6.8, the completeness of $F^s_{p,q}(\mathbb{R}^d; X)$ follows from Lemma 14.4.7. □

14.6.c Fourier multipliers

The main result of this subsection is a version of the Mihlin multiplier theorem for Triebel–Lizorkin spaces. Before we state it we first prove an important lifting property as we saw in Proposition 14.4.15 for Besov spaces.

Proposition 14.6.10 (Lifting). *Let $p \in [1, \infty)$, $q \in [1, \infty]$, and $s \in \mathbb{R}$. Then for all $\sigma \in \mathbb{R}$,*

$$J_\sigma : F^s_{p,q}(\mathbb{R}^d; X) \simeq F^{s-\sigma}_{p,q}(\mathbb{R}^d; X) \quad \textit{isomorphically.} \tag{14.71}$$

Proof. As in Proposition 14.4.15 it suffices to show that J_σ maps $F^s_{p,q}(\mathbb{R}^d; X)$ into $F^{s-\sigma}_{p,q}(\mathbb{R}^d; X)$ and is bounded for each $\sigma \in \mathbb{R}$ We must show that $(2^{n(s-\sigma)}\varphi_n * J_\sigma f)_{n\geqslant 0}$ belongs to $L^p(\mathbb{R}^d; \ell^q(X))$. This will be done by applying the multiplier Theorem 14.6.3 to a multiplier $m = (m_n)_{n\geqslant 0}$ naturally associated with J_σ.

Write

$$2^{-n\sigma}\varphi_n * J_\sigma f = \sum_{\ell=-1}^{1} \mathscr{F}^{-1} m_n \widehat{\varphi}_{n+\ell}\widehat{f},$$

where

$$m_n(\xi) = 2^{-n\sigma}(1 + 4\pi^2|\xi|^2)^{\sigma/2}\widehat{\varphi}_n(\xi).$$

We have $m_n \in C^\infty(\mathbb{R}^d)$ and, putting $\delta_n = 3 \cdot 2^n$,

$$\operatorname{supp} \widehat{\varphi}_n(\delta_n \cdot) \subseteq \big\{\xi \in \mathbb{R}^d : \frac{1}{6} \leqslant |\xi| \leqslant \frac{1}{2}\big\}, \quad (n \geqslant 1)$$
$$\operatorname{supp} \widehat{\varphi}_0(\delta_0 \cdot) \subseteq \big\{\xi \in \mathbb{R}^d : |\xi| \leqslant \frac{1}{2}\big\}.$$

Lemma 14.2.12, applied with $\lambda = d + 1 + \lceil d/r \rceil$ with an arbitrary $r = r_{p,q} \in (0, \min\{p, q\})$, gives the estimate

$$\begin{aligned}\|(1 + |\cdot|)^{d/r}\mathscr{F}^{-1}[m_n(\delta_n \cdot)]\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))} \\ \leqslant C_d \|m_n(\delta_n \cdot)\|_{C^{d+1+\lceil d/r\rceil}(\mathbb{R}^d;\mathscr{L}(X,Y))} \leqslant C_{m,d,r} = C_{m,d,p,q},\end{aligned}$$

where the last inequality is elementary to verify.

Since for $\ell \in \{-1, 0, 1\}$ we have $\operatorname{supp}(\widehat{\varphi}_{n+\ell}\widehat{f}) \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leqslant \delta_n\}$ we are now in a position to apply Theorem 14.6.3 and obtain

$$
\begin{aligned}
\big\|(2^{n(s-\sigma)}\varphi_n * J_\sigma f)_{n\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}
&\leqslant \sum_{\ell=-1}^{1} \|(\mathscr{F}^{-1} m_n 2^{ns}\widehat{\varphi}_{n+\ell}\widehat{f})_{n\geqslant 0}\|_{L^p(\mathbb{R}^d;\ell^q(X))}\\
&\leqslant C_{m,d,p,q}\sum_{\ell=-1}^{1} \|(2^{ns}\varphi_{n+\ell} * f)_{n\geqslant 0}\|_{L^p(\mathbb{R}^d;\ell^q(X))}\\
&\leqslant C'_{m,d,p,q}\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)}.
\end{aligned}
$$

□

We continue with the Mihlin multiplier theorem for Triebel–Lizorkin spaces. Note that the Besov space case was considered in Theorems 14.4.16 and 14.5.6.

Theorem 14.6.11 (Mihlin multiplier theorem for Triebel–Lizorkin spaces). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and X and Y be Banach spaces, and set $N := d + 1 + \lceil \max\{\frac{d}{p}, \frac{d}{q}\}\rceil$. If $m \in C^N(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfies*

$$
K_m := \sup_{|\alpha|\leqslant N} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|})\|\partial^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} < \infty,
$$

then there is a bounded operator $T : F^s_{p,q}(\mathbb{R}^d;X) \to F^s_{p,q}(\mathbb{R}^d;Y)$ with $\|T\| \leqslant C_{d,p,q,s,X,Y} K_m$ such that $Tf = \mathscr{F}^{-1}(m\widehat{f})$ for all $f \in \mathscr{S}(\mathbb{R}^d)\otimes X$.

Note that in the case $q < \infty$, one has that T is the unique bounded extension of $T_m : \mathscr{S}(\mathbb{R}^d)\otimes X \to \mathscr{S}'(\mathbb{R}^d;Y)$.

Proof. We define T in the same was as in (14.59) of the proof of Theorem 14.5.6:

$$
Tf = \sum_{\ell=-1}^{1}\sum_{k\geqslant 0} T_{m_{k+\ell}} f_k,
$$

where $f \in F^s_{p,q}(\mathbb{R}^d;X)$, $f_k = \varphi_k * f$ and $m_k = \widehat{\varphi}_k m$. Since $F^s_{p,q}(\mathbb{R}^d;X) \subseteq B^s_{p,\infty}(\mathbb{R}^d;X)$ it follows from the proof of Theorem 14.5.6 that the above series converges in $\mathscr{S}'(\mathbb{R}^d;Y)$, and that $Tg = \mathscr{F}^{-1}(m\widehat{g})$ for all $g \in \mathscr{S}(\mathbb{R}^d)\otimes X$.

To prove the required boundedness, note that

$$
\|T_m f\|_{F^s_{p,q}(\mathbb{R}^d;Y)} \leqslant \sum_{\ell=-1}^{1} \big\|2^{ks}\mathscr{F}^{-1}(m\widehat{\varphi}_{k+\ell}\widehat{\varphi}_k\widehat{f})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(Y))}.
$$

Fix $\ell \in \{-1,0,1\}$. Then $\operatorname{supp}\widehat{f}_{k+\ell} \subseteq \{|\xi| \leqslant \delta_k\}$, where $\delta_k = 3\cdot 2^k$.

To estimate further it is sufficient to apply Theorem 14.6.3, for which we choose $r = r_{d,p,q} \in (0, \min\{p,q\})$ such that $N = d + 1 + \lceil d/r\rceil$. To check the assumptions of the theorem we have to show that

$$\sup_{k\geqslant 0} \|(1+|\cdot|)^{d/r}\mathscr{F}^{-1}(\widehat{\varphi}_k(\delta_k\cdot)m(\delta_k\cdot))\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))} \leqslant CK_m,$$

where $C \geqslant 0$ is a constant depending only on d and r. Since $\widehat{\varphi}_k(\delta_k\cdot)m(\delta_k\cdot)$ has support in $\{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$, the estimate follows from Lemma 14.2.12. □

The following result is proved in the same way as Proposition 14.4.17.

Proposition 14.6.12. *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. For all $k \in \mathbb{N}$ the expression*

$$|||f|||_{F^s_{p,q}(\mathbb{R}^d;X)} := \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{F^{s-k}_{p,q}(\mathbb{R}^d;X)}$$

defines an equivalent norm on $F^s_{p,q}(\mathbb{R}^d;X)$.

14.6.d Embedding theorems

We have already noted the continuous inclusions

$$\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X)$$

and

$$B^s_{p,p\wedge q}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p\vee q}(\mathbb{R}^d;X)$$

for $s \in \mathbb{R}$, $p \in [1,\infty)$ and $q \in [1,\infty]$. Moreover, for any $q \in [1,\infty]$, it is immediate from the definitions that

$$B^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d;X). \tag{14.72}$$

The next result compares Triebel–Lizorkin spaces with the Bessel potential and Sobolev spaces. It can be improved if X is UMD and has type and cotype properties (see Proposition 14.7.6 below).

Proposition 14.6.13 (Sandwiching with Triebel–Lizorkin spaces). *For $p \in (1,\infty)$, $s \in \mathbb{R}$, and $m \in \mathbb{N}$, we have the following continuous embeddings:*

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X), \tag{14.73}$$

$$F^m_{p,1}(\mathbb{R}^d;X) \hookrightarrow W^{m,p}(\mathbb{R}^d;X) \hookrightarrow F^m_{p,\infty}(\mathbb{R}^d;X). \tag{14.74}$$

In view of the embeddings $B^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,1}(\mathbb{R}^d;X)$ and $F^s_{p,\infty}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d;X)$, (14.73) and (14.74) improve the corresponding embeddings in Proposition 14.4.18.

Proof. For (14.73) and (14.74), by Propositions 5.6.3, 14.6.10 and 14.6.12 it suffices to consider the special case $s = m = 0$, for which $H^{0,p}(\mathbb{R}^d;X) = W^{0,p}(\mathbb{R}^d;X) = L^p(\mathbb{R}^d;X)$. It thus remains to show the continuous embeddings

$$F^0_{p,1}(\mathbb{R}^d;X) \hookrightarrow L^p(\mathbb{R}^d;X) \hookrightarrow F^0_{p,\infty}(\mathbb{R}^d;X). \tag{14.75}$$

The first embedding in (14.75) is true for any $p \in [1,\infty)$: writing $f = \sum_{k\geqslant 0} \varphi_k * f$ it follows that

$$\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant \sum_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} = \|f\|_{F^0_{p,1}(\mathbb{R}^d;X)}.$$

For the second embedding in (14.75) observe that since $\varphi \in \mathscr{S}(\mathbb{R}^d)$, it has a radially decreasing majorant which is integrable. Therefore, by Theorem 2.3.8 there is a constant $C_d \geqslant 0$ such that for all $k \geqslant 0$ and almost all $x \in \mathbb{R}^d$, $\|\varphi_k * f(x)\| \leqslant C_d Mf(x)$. Therefore, by the L^p-boundedness of the Hardy–Littlewood maximal function (Theorem 2.3.2),

$$\|f\|_{F^0_{p,\infty}(\mathbb{R}^d;X)} = \Big\| \sup_{k\geqslant 0} \|\varphi_k * f\| \Big\|_{L^p(\mathbb{R}^d)} \leqslant C_d \|Mf\|_{L^p(\mathbb{R}^d)} \lesssim_p C_d \|f\|_{L^p(\mathbb{R}^d;X)}.$$

This completes the proof. □

We continue with a version of the Sobolev embedding theorem. A surprising feature is that in case of the Triebel–Lizorkin spaces there is an improvement in the microscopic parameter q.

Theorem 14.6.14 (Sobolev embedding for Triebel–Lizorkin spaces). *For given $p_0, p_1 \in [1,\infty)$, $q_0, q_1 \in [1,\infty]$, and $s_0, s_1 \in \mathbb{R}$, we have a continuous embedding*

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)$$

if and only if one of the following two conditions holds:

(i) $p_0 = p_1$ *and* $[s_0 > s_1$ *or* $(s_0 = s_1$ *and* $q_0 \leqslant q_1)]$;
(ii) $p_0 < p_1$ *and* $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$.

The main ingredient is a version of the Gagliardo–Nirenberg inequality with a microscopic improvement.

Proposition 14.6.15 (Gagliardo–Nirenberg inequality for Triebel–Lizorkin spaces). *Let $p, p_0, p_1 \in [1,\infty)$, $q, q_0, q_1 \in [1,\infty]$, let $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$, let $\theta \in (0,1)$, and assume that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s = (1-\theta)s_0 + \theta s_1$. For all $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \cap F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)$ we have $f \in F^s_{p,q}(\mathbb{R}^d;X)$ and*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C \|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)} \|f\|^{\theta}_{F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)},$$

where the constant $C \geqslant 0$ depends only on θ, s_0, s_1.

Proof. Proposition 14.3.5 (applied with $q_0 = q_1 = \infty$) implies that

$$\big\|(2^{ks}a_k)_{k\geqslant 0}\big\|_{\ell^q} \leqslant C_{s_0,s_1,s}\big\|(2^{ks_0}a_k)_{k\geqslant 0}\big\|_{\ell^\infty}^{1-\theta}\big\|(2^{ks_1}a_k)_{k\geqslant 0}\big\|_{\ell^\infty}^{\theta} \tag{14.76}$$

for all sequences of scalars $(a_k)_{k\geqslant 0}$ for which the expression on the right-hand side is finite.

To prove the desired estimate, by (14.69) it suffices to consider the case $q_0 = q_1 = \infty$. Taking $a_k(x) = \|\varphi_k * f(x)\|$ with $x \in \mathbb{R}^d$ in (14.76), raising to the power p and integrating over $\mathbb{R}^d$, by Hölder's inequality (with exponents $\frac{p_0}{(1-\theta)p}$ and $\frac{p_1}{\theta p}$) we obtain

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C_{s_0,s_1,s}\|f\|^{1-\theta}_{F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)}\|f\|^{\theta}_{F^{s_1}_{p_1,\infty}(\mathbb{R}^d;X)}$$

as required. □

In a similar way one can prove the following variant for the end-point $p_1 = \infty$.

Proposition 14.6.16 (Gagliardo–Nirenberg inequality for Triebel–Lizorkin spaces – II). *Let $p, p_0, \in [1,\infty)$, $q, q_0 \in [1,\infty]$, let $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$, let $\theta \in (0,1)$, and assume that $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $s = (1-\theta)s_0 + \theta s_1$. For all $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \cap B^{s_1}_{\infty,\infty}(\mathbb{R}^d;X)$ we have $f \in F^s_{p,q}(\mathbb{R}^d;X)$ and*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{B^{s_1}_{\infty,\infty}(\mathbb{R}^d;X)},$$

where the constant $C \geqslant 0$ depends only on θ, s_0, s_1.

Proof of sufficiency in Theorem 14.6.14. For the sufficiency of (i) first assume that $p_0 = p_1$, $q_0 \leqslant q_1$, and $s_0 \geqslant s_1$. Under these assumptions the result follows from the fact that

$$\big\|(2^{ks_1}a_k)_{k\geqslant 0}\big\|_{\ell^{q_1}} \leqslant \big\|(2^{ks_0}a_k)_{k\geqslant 0}\big\|_{\ell^{q_0}}.$$

If $p_0 = p_1$, $q_0 > q_1$, and $s_0 > s_1$, the result follows from (14.23) and (14.72):

$$\begin{aligned} F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) = F^{s_0}_{p_1,q_0}(\mathbb{R}^d;X) &\hookrightarrow B^{s_0}_{p_1,\infty}(\mathbb{R}^d;X) \\ &\hookrightarrow B^{s_1}_{p_1,\infty}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X). \end{aligned}$$

This completes the proof of (i).

Let us now assume that (ii) holds. By (14.70) it suffices to consider the case $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$. By (14.69) we may furthermore assume that $q_1 = 1$.

First take $f \in \mathscr{S}(\mathbb{R}^d;X)$. Let $\theta_0 \in [0,1)$ be such that $\frac{1}{p_1} - \frac{1-\theta_0}{p_0} = 0$. Choose $\theta \in (\theta_0, 1)$ arbitrary and let r be defined by $\frac{1}{p_1} = \frac{1-\theta}{p_0} + \frac{\theta}{r}$. Note that $p_0 < p_1 < \infty$ implies $r \in (p_1,\infty)$. Let further $t \in \mathbb{R}$ be defined by $t - \frac{d}{r} = s_0 - \frac{d}{p_0}$. Observe that $t < s_0$ and $s_1 = \theta t + (1-\theta)s_0$ (write out the expression for θt and use the formula for θ/r). Therefore, by Proposition 14.6.15,

$$\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)} \leqslant C_{s_0,s_1,\theta}\|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{F^{t}_{r,r}(\mathbb{R}^d;X)}. \tag{14.77}$$

By the case (ii) in Theorem 14.4.19 (using that $r > p_1$),

$$\begin{aligned}\|f\|_{F^{t}_{r,r}(\mathbb{R}^d;X)} &= \|f\|_{B^{t}_{r,r}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{s_1}_{p_1,p_1}(\mathbb{R}^d;X)}\\ &= C\|f\|_{F^{s_1}_{p_1,p_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)},\end{aligned}$$

where in the last step we used (14.69). Substituting the latter estimate into (14.77), we obtain

$$\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)} \leqslant C^{1/(1-\theta)}_{s_0,s_1,\theta}C^{\theta/(1-\theta)}\|f\|_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}. \tag{14.78}$$

Now if $q_0 < \infty$, then the result follows from the density of $\mathscr{S}(\mathbb{R}^d;X)$ in $F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)$.

If $q_0 = \infty$ and $f \in F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)$, we let $S_n f = \sum_{k=0}^{n} \varphi_k * f$. Then by Young's inequality and the fact that $\varphi_j * S_n f = 0$ for $j \geqslant n+1$, we have $S_n f \in B^{s_0}_{p_0,1}(\mathbb{R}^d;X)$. Thus Theorem 14.4.19 implies $S_n f \in B^{s_1}_{p_1,1}(\mathbb{R}^d;X)$. Moreover, by Proposition 14.6.8 and (14.69) we also have $S_n f \in F^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)$ and $S_n f \in F^{s_1}_{p_1,1}(\mathbb{R}^d;X)$. Therefore, by (14.78),

$$\|S_n f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)} \leqslant C^{1/(1-\theta)}_{s_0,s_1}C^{\theta/(1-\theta)}\|S_n f\|_{F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)} \leqslant \tilde{C}\|f\|_{F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)},$$

where the last estimate follows from Lemma 14.6.7. Since $S_n f \to f$ in $\mathscr{S}'(\mathbb{R}^d;X)$ by Lemma 14.2.10, the assertion now follows from the fact that $F^{s}_{p,q}(\mathbb{R}^d;X)$ has the Fatou property. □

Proof of necessity in Theorem 14.6.14. By Proposition 14.6.8,

$$B^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p_1,\infty}(\mathbb{R}^d;X).$$

Therefore, Theorem 14.4.19 implies that $p_0 \leqslant p_1$. If $p_0 = p_1$, then (i) follows from (i). If $p_0 < p_1$, then (ii) follows from (iii) and (ii). □

Proposition 14.4.36 has the following analogue for Triebel–Lizorkin spaces:

Proposition 14.6.17 (Density of compactly supported functions). *Let*

$$\ddot{\mathbb{R}}^d := \mathbb{R}\setminus\{0\}\times\mathbb{R}^{d-1}.$$

Let $p,q \in [1,\infty)$ and $s \in \mathbb{R}$. Then $C^\infty_{\rm c}(\ddot{\mathbb{R}}^d)\otimes X$ is dense in $F^{s}_{p,q}(\mathbb{R}^d;X)$ and $H^{s,p}(\mathbb{R}^d;X)$ in each of the following situations:

(1) *$s < 1/p$;*
(2) *$p \in (1,\infty)$ and $s = 1/p$.*

Proof. First consider the Triebel–Lizorkin case. As in the proof of Proposition 14.4.36 (using Propositions 14.6.8) we can reduce to the smooth and scalar-valued setting. Thus it suffices to show that an arbitrary $f \in C_c^\infty(\mathbb{R}^d)$ there exist functions $f_n \in C_c^\infty(\ddot{\mathbb{R}}^d)$ such that $f_n \to f$ in $F^{1/p}_{p,q}(\mathbb{R}^d)$. By the embedding (14.70) and Theorem 14.6.14, it suffices to prove this for the case (2). However, for this case the result follows from Proposition 14.4.36 and the estimate

$$\|f - f_n\|_{F^{1/p}_{p,q}(\mathbb{R}^d)} \leqslant C\|f - f_n\|_{F^{1/r}_{r,r}(\mathbb{R}^d)} = C\|f - f_n\|_{B^{1/r}_{r,r}(\mathbb{R}^d)}, \quad r \in (1,p),$$

which follows from Theorem 14.6.14.

The same proof for Bessel potential spaces holds, where we note that for the reduction to the scalar situation one can use Proposition 5.6.4, and the embedding $F^{1/r}_{r,r}(\mathbb{R}^d) \hookrightarrow H^{1/p,p}(\mathbb{R}^d)$ follows from Proposition 14.6.13 and Theorem 14.6.14. □

The proof of Theorem 14.5.1 shows that the existence of a continuous embedding

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,r}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X)$$

implies that X has type r, and that the existence of a continuous embedding $\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,r}(\mathbb{R}^d; X)$ implies that X has cotype r. Therefore in the Besov scale the embeddings of Theorem 14.5.1 cannot be improved by using the microscopic parameter r. For the Triebel–Lizorkin spaces the situation is different, as witnessed the following result.

Corollary 14.6.18 (γ-Sobolev embedding – II). *Let $1 \leqslant p_0 \leqslant 2 \leqslant q_0 < \infty$.*

(1) *If X has type p_0, then for all $p \in [1, p_0)$ and all $r \in [1, \infty]$ we have a continuous embedding*

$$F^{(\frac{1}{p}-\frac{1}{2})d}_{p,r}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X).$$

(2) *If X has cotype q_0, then for all $q \in (q_0, \infty)$ and all $r \in [1, \infty]$ we have a continuous embedding*

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow F^{(\frac{1}{q}-\frac{1}{2})d}_{q,r}(\mathbb{R}^d; X).$$

Proof. We give the proof of (1), the proof of (2) being similar. Let $1 \leqslant p < p_0$. Let $s_0 = (\frac{1}{p_0} - \frac{1}{2})d$ and $s = (\frac{1}{p} - \frac{1}{2})d$. By Theorem 14.6.14 we have a continuous embedding

$$F^s_{p,r}(\mathbb{R}^d; X) \hookrightarrow F^{s_0}_{p_0,p_0}(\mathbb{R}^d; X) = B^{s_0}_{p_0,p_0}(\mathbb{R}^d; X).$$

Now the result follows from Theorem 14.5.1. □

14.6.e Difference norms

In Section 14.4.d we have discussed a difference norm characterisation for Besov spaces of positive smoothness. We will now prove a similar result for the Triebel–Lizorkin spaces. Recall the notation

$$\Delta_h f(x) = f(x+h) - f(x)$$

and $\Delta_h^m = (\Delta_h)^m$.

Definition 14.6.19 (Difference norm for Triebel–Lizorkin spaces). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, $s > 0$, $m \in \mathbb{N} \setminus \{0\}$ and $\tau \in [1,\infty)$. For $f \in L^p(\mathbb{R}^d;X)$ we define the* difference norm *by setting*

$$[f]^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)} := \Big\| \Big(\int_0^\infty t^{-sq} \Big(\fint_{\{|h|\leqslant t\}} \|\Delta_h^m f\|_X^\tau \,\mathrm{d}h \Big)^{q/\tau} \frac{\mathrm{d}t}{t} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)},$$

with obvious modifications if $q = \infty$, and

$$|\!|\!| f |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)} := \|f\|_{L^p(\mathbb{R}^d;X)} + [f]^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}.$$

It will be shown shortly that each of the norms $|\!|\!| \cdot |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}$ with $m > s$ and $s > \frac{d}{\min\{p,q\}} - \frac{d}{\tau}$ defines an equivalent norm on $F^s_{p,q}(\mathbb{R}^d;X)$.

The expression for the seminorm simplifies for $\tau = q \in [1,\infty)$. Indeed, by Fubini's theorem we have

$$[f]^{(m,q)}_{F^s_{p,q}(\mathbb{R}^d;X)} = \frac{1}{(sq+d)^{1/q}|B_1|} \Big\| \Big(\int_{\mathbb{R}^d} |h|^{-(s+d)q} \|\Delta_h^m f(x)\|^q \,\mathrm{d}h \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)}.$$

Theorem 14.6.20 (Difference norms for Triebel–Lizorkin spaces). *Let X be a Banach space and let $p, \tau \in [1,\infty)$, $q \in [1,\infty]$, $s > 0$, let $m > s$ be an integer, and suppose that*

$$s > \frac{d}{\min\{p,q\}} - \frac{d}{\tau}. \tag{14.79}$$

Then for all $f \in \mathscr{S}(\mathbb{R}^d;X)$ the following norm equivalence holds:

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,m,p,q,s,\tau} |\!|\!| f |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}, \tag{14.80}$$

whenever one of these expressions is finite.

Note that the condition (14.79) holds trivially holds if $\tau \leqslant \min\{p,q\}$, and in particular if $\tau = 1$. The condition (14.79) is only used in the proof of "$\gtrsim$" of (14.80).

For the proof we will use a discretised version of $|\!|\!| f |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}$. Put

$$J^{m,\tau}(f,k)(x) := \Big(\fint_{|h|\leqslant 1} \|\Delta^m_{2^{-k}h} f(x)\|^\tau \,\mathrm{d}h\Big)^{1/\tau}.$$

As in (14.38) we have

$$[f]^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,s} \big\|(2^{ks}J^{m,\tau}(f,k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}.$$

Therefore, to obtain (14.80) it suffices to prove the norm equivalence

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \eqsim \|f\|_{L^p(\mathbb{R}^d;X)} + \big\|(2^{ks}J^{m,\tau}(f,k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}, \qquad (14.81)$$

where the implicit constant may depend on d, p, q, m, s, τ. The proof of the estimate $\lesssim$ in (14.81) is similar to Step 2 of the proof of Theorem 14.4.24 except that instead of Proposition 14.4.2 one has to use Proposition 14.6.6 and towards the end of the proof one has to take $L^p(\mathbb{R}^d;\ell^q)$-norms instead of $\ell^q(L^p(\mathbb{R}^d))$-norms.

In the remainder of this subsection we will concentrate on proving the inequality $\gtrsim$ in (14.81). We begin with a lemma involving the maximal function

$$M_r := (M(\|f\|^r)(x))^{1/r}$$

introduced in (14.65).

Lemma 14.6.21. *Let $f \in \mathscr{S}'(\mathbb{R}^d;X)$ satisfy* $\mathrm{supp}(\widehat{f}) \subseteq \{|\xi| \leqslant t\}$. *Then $f \in C^\infty(\mathbb{R}^d;X)$ and for all $r \in (0,\infty)$, $m \in \mathbb{N}$, and all $x, h \in \mathbb{R}^d$ we have*

$$\|\Delta^m_h f(x)\| \lesssim_{d,m,r} (t|h|)^{d/r} M_r(f)(x) \qquad \text{if } |h| > t^{-1}; \qquad (14.82)$$

$$\|\Delta^m_h f(x)\| \lesssim_{d,m,r} (t|h|)^{m} M_r(f)(x) \qquad \text{if } |h| \leqslant t^{-1}. \qquad (14.83)$$

Proof. That f belongs to $C^\infty(\mathbb{R}^d;X)$ follows from Lemma 14.2.9.

Recall that by Lemma 14.6.1

$$\|\partial^\alpha f(x+h)\| \lesssim_{|\alpha|,d,r} t^{|\alpha|}(1+t|h|)^{d/r} M_r f(x). \qquad (14.84)$$

The estimate (14.82) follows from (14.84) and Lemma 14.4.22, for if $|h| > t^{-1}$, then

$$\begin{aligned}\|\Delta^m_h f(x)\| &\leqslant \sum_{j=0}^m \binom{m}{j} \|f(x+hj)\| \\ &\lesssim_{d,r} 2^m (1+t|h|m)^{d/r} M_r f(x) \lesssim_{d,m,r} (t|h|)^{d/r} M_r f(x).\end{aligned}$$

To prove (14.83) fix $|h| \leqslant t^{-1}$. Set $\phi(s) := f(x+sh)$ for $s \in \mathbb{R}$. Then $\Delta^m_h f(x) = \Delta^m_1 \phi(0)$. Since for any $g \in C^1(\mathbb{R};X)$ we have $\|\Delta_1 g(s)\| \leqslant \sup_{\theta\in[s,s+1]} \|g'(\theta)\|$, an induction argument gives

$$\|\Delta^m_1 \phi(s)\| \leqslant \sup_{\theta\in[s,s+m]} \|\phi^{(m)}(\theta)\|, \quad s \in \mathbb{R}.$$

In particular,

$$\|\Delta_h^m f(x)\| \leqslant \sup_{\theta\in[0,m]} \|\phi^{(m)}(\theta)\| \leqslant |h|^m \sup_{\theta\in[0,m]} \Big(\sum_{|\alpha|=m} \|\partial^\alpha f(x+\theta h)\|^2\Big)^{1/2}.$$

By (14.84) and the fact that $t|h| \leqslant 1$, for $\theta \in [0,m]$ we have

$$\|\partial^\alpha f(x+\theta h)\| \lesssim_{d,m,r} t^m (1+tm|h|)^{d/r} M_r f(x) \lesssim_{d,r,m} t^m M_r f(x).$$

Substituting this into the previous estimate gives the required estimate. □

Proof of Theorem 14.6.20. It remains to prove the inequality $\gtrsim$ in (14.81).

To begin with, from (i) we have inequality

$$\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant \Big\| \sum_{j=0}^{\infty} \|f_j\|_X \Big\|_{L^p(\mathbb{R}^d)} = \|f\|_{F^0_{p,1}(\mathbb{R}^d;X)} \lesssim_{d,p,q,s} \|f\|_{F^s_{p,q}(\mathbb{R}^d;X)},$$

where $f_j = \varphi_j * f$ as always.

To deal with the seminorm, note that from the assumption (14.79) it follows that we can find $r \in (0,\infty)$ and $\lambda \in (0,1]$ such that

$$p, q > \max\{r, \lambda\tau\} \quad \text{and} \quad s > (1-\lambda)d/r. \tag{14.85}$$

Since $f = \sum_{n\in\mathbb{Z}} f_{n+k}$ in $L^p(\mathbb{R}^d;X)$ for any $k \in \mathbb{Z}$ (recall the convention that we set $\varphi_j = 0$ for $j \leqslant -1$, so that we may put $f_j = 0$ for $j \leqslant 1$), we have

$$\big\|(2^{ks} J^{m,\tau}(f,k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))} \leqslant \sum_{n\in\mathbb{Z}} \big\|(2^{ks} J^{m,\tau}(f_{n+k},k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}.$$

For $n \leqslant 0$, by (14.83) with $t = 2^{k+n+1}$, we have

$$\begin{aligned} J^{m,\tau}(f_{n+k},k)(x) &= \Big(\fint_{|h|\leqslant 1} \|\Delta^m_{2^{-k}h} f_{n+k}(x)\|^\tau \,\mathrm{d}h\Big)^{1/\tau} \\ &\lesssim_{d,m,r} \Big(\fint_{|h|\leqslant 1} (|2^n h|^m M_r(f_{n+k})(x))^\tau \,\mathrm{d}h\Big)^{1/\tau} \\ &\leqslant 2^{nm} M_r(f_{n+k})(x), \end{aligned}$$

and therefore

$$\begin{aligned} &\big\|(2^{ks} J^{m,\tau}(f_{n+k},k)(x))_{k\geqslant 0}\big\|_{\ell^q(\mathbb{Z})} \\ &\qquad \lesssim_{d,m,r} (2^{ks} 2^{nm} M_r f_{n+k}(x))_{k\geqslant 0}\|_{\ell^q(\mathbb{Z})} \\ &\qquad = 2^{n(m-s)} \big\|(2^{s(k+n)} 2^{nm} M_r f_{n+k}(x))_{k\geqslant 0}\big\|_{\ell^q(\mathbb{Z})}. \end{aligned}$$

Since $s < m$ and M_r is bounded on $L^p(\mathbb{R}^d;\ell^q)$ by the Fefferman–Stein maximal Theorem 3.2.28, we obtain

$$\begin{aligned}
&\sum_{n\leqslant 0}\big\|(2^{ks}J^{m,\tau}(f_{n+k},k\geqslant 0))_k\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}\\
&\qquad\lesssim_{d,m,r}\sum_{n\leqslant 0}2^{n(m-s)}\big\|(2^{(k+n)s}2^{nm}M_rf_{n+k})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}\\
&\qquad\lesssim_{d,m,s}\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)}.
\end{aligned}$$

Next take $n\geqslant 1$. Fixing $\lambda\in(0,1]$ for the moment, we have

$$\begin{aligned}
&J^{m,\tau}(f_{n+k},k)(x)\\
&\quad\leqslant\sup_{|h|\leqslant 1}\|\varDelta^m_{2^{-k}h}f_{n+k}(x)\|^{1-\lambda}\Big(\fint_{\{|h|\leqslant 1\}}\|\varDelta^m_{2^{-k}h}f_{n+k}(x)\|^{\tau\lambda}\,\mathrm{d}h\Big)^{1/\tau}\\
&\quad=: T_1(x)\times T_2(x).
\end{aligned}$$

From (14.82) we obtain the pointwise bound

$$T_1(x)\leqslant 2^{dn(1-\lambda)/r}M_r(f_{n+k})(x)^{1-\lambda}.$$

To estimate T_2, we use Lemma 14.4.22 and the inequality $(\sum_{j=1}^m|a_j|)^{\lambda\tau}\lesssim_{\lambda,m,\tau}\sum_{j=1}^m|a_j|^{\lambda\tau}$ to obtain

$$\|\varDelta^m_{2^{-k}h}f_{n+k}(x)\|^{\tau\lambda}\lesssim_{\lambda,m,\tau}\|f_{n+k}(x)\|^{\tau\lambda}+\sum_{j=1}^m\binom{m}{j}\|f_{n+k}(x+2^{-k}hj)\|^{\tau\lambda}.$$

Estimating both terms by the maximal function, we obtain the pointwise bound that $T_2(x)$ is less than a constant depending on λ,m,τ times

$$\begin{aligned}
&\Big(M_{\tau\lambda}(f_{n+k})(x)^{\tau\lambda}+\sum_{j=1}^m\binom{m}{j}\fint_{\{|h|\leqslant 1\}}\|f_{n+k}(x+2^{-k}hj)\|^{\tau\lambda}\,\mathrm{d}h\Big)^{1/\tau}\\
&\quad=\Big(M_{\tau\lambda}(f_{n+k})(x)^{\tau\lambda}+\sum_{j=1}^m\binom{m}{j}\fint_{|y|\leqslant j2^{-k}}\|f_{n+k}(x+y)\|^{\tau\lambda}\,\mathrm{d}y\Big)^{1/\tau}\\
&\quad\leqslant(2^m+1)^{1/\tau}M_{\tau\lambda}(f_{n+k})(x)^{\lambda}.
\end{aligned}$$

Combining the estimates for T_1 and T_2, we conclude that

$$J^{m,\tau}(f_{n+k},2^{-k})\lesssim_{d,\lambda m,r,\tau}2^{dn(1-\lambda)/r}M_r(f_{n+k})^{1-\lambda}M_{\tau\lambda}(f_{n+k})^{\lambda}.$$

Since $s>\frac{(1-\lambda)d}{r}$ (see (14.85)), by Hölder's inequality (applied twice) we obtain

$$\begin{aligned}
&\sum_{n\geqslant 1}\big\|(2^{ks}J^{m,\tau}(f_{n+k},k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}\\
&\quad\lesssim\sum_{n\geqslant 0}2^{-n(s-\frac{(1-\lambda)d}{r})}\big\|\big(2^{(n+k)s}M_r(f_{n+k})^{1-\lambda}M_{\tau\lambda}(f_{n+k})^{\lambda}\big)_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q)}
\end{aligned}$$

$$\lesssim_{d,\lambda,r,s} \big\|\big(2^{js}M_r(f_j)\big)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q)}^{1-\lambda}\big\|\big(2^{js}M_{\tau\lambda}(f_j)\big)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q)}^{\lambda}$$
$$\lesssim_{\lambda,p,q,r,\tau} \big\|\big(2^{js}f_j\big)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q)} = \|f\|_{F^s_{p,q}(\mathbb{R}^d;X)},$$

where in the last estimate we used the boundedness of M_r and $M_{\tau\lambda}$ on and $L^p(\mathbb{R}^d;\ell^q)$ thanks to (14.85). □

14.6.f Interpolation

In order to prove interpolation results for the scale of Triebel–Lizorkin spaces we need the following variation of Lemma 14.4.29.

Lemma 14.6.22. *Let $s \in \mathbb{R}$, $p \in (1,\infty)$ and $q \in (1,\infty]$. For $k \geqslant 0$ set $\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$. Define the operators*

$$R : L^p(\mathbb{R}^d;\ell^q_{w_s}(X)) \to F^s_{p,q}(\mathbb{R}^d;X)$$
$$S : F^s_{p,q}(\mathbb{R}^d;X) \to L^p(\mathbb{R}^d;\ell^q_{w_s}(X))$$

by

$$R(f_k)_{k\geqslant 0} = \sum_{k\geqslant 0}\psi_k * f_k, \quad Sf = (\varphi_k * f)_{k\geqslant 0}.$$

Then S is an isometry, R is bounded, and $RS = I$.

Proof. All assertions follow in the same way as in Lemma 14.4.29, except for the boundedness of R. To see that $\sum_{k\geqslant 0}\psi_k * f_k$ converges in $\mathscr{S}'(\mathbb{R}^d;X)$ note that $L^p(\mathbb{R}^d;\ell^q_{w_s}(X)) \hookrightarrow L^p(\mathbb{R}^d;\ell^p_{w_t}(X)) = \ell^p_{w_t}(L^p(\mathbb{R}^d;X))$ for any $t < s$ by Hölder's inequality, so the convergence follows from Lemma 14.4.29. To see that R is bounded, note that since $\widehat{\psi_k} \equiv 1$ on $\operatorname{supp}(\widehat{\varphi}_k)$ we have

$$\begin{aligned}\|R(f_k)_{k\geqslant 0}\|_{F^s_{p,q}(\mathbb{R}^d;X)} &\leqslant \sum_{|\ell|\leqslant 2}\big\|(\|\varphi_j * \psi_{j+\ell} * f_{j+\ell}\|_X)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})}\\ &\lesssim \sup_{|\ell|\leqslant 2}\big\|(M(\|f_{j+\ell}\|_X))_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})}\\ &\lesssim_{d,p,q} \sup_{|\ell|\leqslant 2}\big\|(\|f_{j+\ell}\|_X)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})}\\ &\leqslant 4^{|s|}\big\|(\|f_j\|_X)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})},\end{aligned}$$

where we used Proposition 2.3.9 and the boundedness of the Hardy–Littlewood maximal function M on $L^p(\mathbb{R}^d;\ell^q_{w_s})$, which is an immediate consequence of the Fefferman–Stein theorem (Theorem 3.2.28); here we use the assumptions $p \in (1,\infty)$ and $q \in (1,\infty]$. □

Using the operators R and S from Lemma 14.6.22 in the same way as in Theorem 14.4.30, the following theorem identifies the complex interpolation spaces of Triebel–Lizorkin.

Theorem 14.6.23 (Complex interpolation of Triebel–Lizorkin spaces). *Let* (X_0, X_1) *be an interpolation couple of Banach spaces and let* $p_0, p_1 \in (1,\infty)$, $q_0, q_1 \in (1,\infty]$ *with* $\min\{q_0, q_1\} < \infty$, $s_0, s_1 \in \mathbb{R}$ *and let* $\theta \in (0,1)$. *Define* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, *and* $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$[F^{s_0}_{p_0,q_0}(\mathbb{R}^d; X_0), F^{s_1}_{p_1,q_1}(\mathbb{R}^d; X_1)]_\theta = F^s_{p,q}(\mathbb{R}^d; X_\theta),$$

isomorphically, where $X_\theta = [X_0, X_1]_\theta$.

The following result on the real interpolation of Triebel–Lizorkin spaces can be derived from the corresponding result for Besov spaces in the same way as Theorem 14.4.31, but now using the sandwich result of Proposition 14.6.13.

Proposition 14.6.24 (Real interpolation of Triebel–Lizorkin spaces). *Let* $p \in [1,\infty)$, $q_0, q_1, q \in [1,\infty]$, *and* $s_0 \neq s_1 \in \mathbb{R}$. *For* $\theta \in (0,1)$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, *and* $s = (1-\theta)s_0 + \theta s_1$ *we have*

$$(F^{s_0}_{p,q_0}(\mathbb{R}^d; X), F^{s_1}_{p,q_1}(\mathbb{R}^d; X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d; X).$$

Our next aim is an interpolation result which will be used improve the Sobolev embedding result of Theorems 14.4.19 and 14.6.14.

Proposition 14.6.25. *Let* $p_0, p_1 \in (1,\infty)$, $q \in (1,\infty]$, *and* $s \in \mathbb{R}$. *For* $\theta \in (0,1)$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *we have*

$$\begin{aligned}
(F^s_{p_0,q}(\mathbb{R}^d; X), F^s_{p_1,q}(\mathbb{R}^d; X))_{\theta,p} &= F^s_{p,q}(\mathbb{R}^d; X),\\
(F^s_{p_0,1}(\mathbb{R}^d; X), F^s_{p_1,1}(\mathbb{R}^d; X))_{\theta,p} &\hookrightarrow F^s_{p,1}(\mathbb{R}^d; X).
\end{aligned}$$

Proof. The first interpolation identity can be proved as in Theorem 14.4.31, using Lemma 14.6.22 and the isomorphic identification

$$(L^{p_0}(\mathbb{R}^d; \ell^q_{w_s}(X)), L^{p_1}(\mathbb{R}^d; \ell^q_{w_s}(X)))_{\theta,p} = L^p(\mathbb{R}^d; \ell^q_{w_s}(X))$$

which follows from Theorem 2.2.10 and Proposition 14.3.5. The case $q = 1$ can be deduced from the proof of Theorem 14.4.31 as well. Indeed, since the operator S of Lemma 14.6.22 is an isometry also for $q = 1$, we find

$$\begin{aligned}
\|f\|_{F^s_{p,1}(\mathbb{R}^d;X)} &= \|Sf\|_{L^p(\mathbb{R}^d;\ell^1_{w_s}(X))}\\
&\eqsim_{p,p_0,p_1,\theta} \|Sf\|_{(L^{p_0}(\mathbb{R}^d;\ell^1_{w_s}(X)),L^{p_1}(\mathbb{R}^d;\ell^1_{w_s}(X)))_{\theta,p}}\\
&\lesssim_{p,p_0,p_1,\theta} \|f\|_{(F^s_{p_0,1}(\mathbb{R}^d;X),F^s_{p_1,1}(\mathbb{R}^d;X))_{\theta,p}}.
\end{aligned}$$

□

As an application we can prove some further embedding results.

Theorem 14.6.26 (Jawerth–Franke). *Let* $p_0, p_1, q \in [1, \infty]$ *and* $s_0, s_1 \in \mathbb{R}$ *satisfy* $1 \leqslant p_0 < p_1 \leqslant \infty$ *and* $s_0 > s_1$. *If* $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$, *then we have continuous embeddings*

$$B^{s_0}_{p_0,p_1}(\mathbb{R}^d; X) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^d; X) \quad \text{if } p_1 < \infty; \tag{14.86}$$

$$F^{s_0}_{p_0,q}(\mathbb{R}^d; X) \hookrightarrow B^{s_1}_{p_1,p_0}(\mathbb{R}^d; X). \tag{14.87}$$

Since the embedding $F^{s_0}_{p_0,p_0}(\mathbb{R}^d; X) \hookrightarrow B^{s_0}_{p_0,p_1}(\mathbb{R}^d; X)$ holds trivially, (14.86) improves the embedding in Theorem 14.6.14. In a similar way one sees that (14.87) is an improvement of Theorem 14.6.14. Consequently, it follows from Theorem 14.6.14 that, under the assumption $p_0 < p_1$, the condition $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$ is also necessary for both (14.86) and (14.87).

Proof. By the trivial embeddings (14.23) and (14.70), it suffices to consider $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$.

To prove (14.86), assume that $p_1 < \infty$. In view of (14.70) it suffices to consider $q = 1$. Fix $p_0 < r_0 < p_1 < r_1$ and $\theta \in (0, 1)$ such that $\frac{1}{p_1} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Let $t_0, t_1 \in \mathbb{R}$ be such that

$$t_0 - \frac{d}{p_0} = s_1 - \frac{d}{r_0} \quad \text{and} \quad t_1 - \frac{d}{p_0} = s_1 - \frac{d}{r_1}.$$

Then $(1 - \theta)t_0 + \theta t_1 = s_0$ and therefore, using Proposition 14.6.24, Theorem 14.6.14, and Proposition 14.6.25,

$$\begin{aligned} B^{s_0}_{p_0,p_1}(\mathbb{R}^d; X) &= (F^{t_0}_{p_0,1}(\mathbb{R}^d; X), F^{t_1}_{p_0,1}(\mathbb{R}^d; X))_{\theta,p_1} \\ &\hookrightarrow (F^{s_1}_{r_0,1}(\mathbb{R}^d; X), F^{s_1}_{r_1,1}(\mathbb{R}^d; X))_{\theta,p_1} \hookrightarrow F^{s_1}_{p_1,1}(\mathbb{R}^d; X), \end{aligned}$$

which implies the embedding (14.86).

To prove (14.87) it suffices to consider $q = \infty$. Moreover, by Theorems 14.4.19 and 14.6.14 we may assume that $1 < p_0 < p_1 < \infty$. Fix $1 < r_0 < p_0 < r_1 < p_1$ and $\theta \in (0, 1)$ such that $\frac{1}{p_0} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Let $t_0, t_1 \in \mathbb{R}$ be such that

$$t_0 - \frac{d}{p_1} = s_0 - \frac{d}{r_0} \quad \text{and} \quad t_1 - \frac{d}{p_1} = s_0 - \frac{d}{r_1}.$$

Then $(1 - \theta)t_0 + \theta t_1 = s_1$. By Proposition 14.6.25, Theorem 14.6.14 and Proposition 14.6.24,

$$\begin{aligned} F^{s_0}_{p_0,\infty}(\mathbb{R}^d; X) &= (F^{s_0}_{r_0,\infty}(\mathbb{R}^d; X), F^{s_0}_{r_1,\infty}(\mathbb{R}^d; X))_{\theta,p_0} \\ &\hookrightarrow (F^{t_0}_{p_1,\infty}(\mathbb{R}^d; X), F^{t_1}_{p_1,\infty}(\mathbb{R}^d; X))_{\theta,p_0} = B^{s_1}_{p_1,p_0}(\mathbb{R}^d; X). \end{aligned}$$

□

As an interesting consequence of Theorem 14.6.26 we have the following improvement of Corollary 14.4.27 (2), extending it to the case $p_0 = 1$. The result is false for integrability exponents $p_0 > 1$. Indeed, if $s - \frac{d}{p_0} \geqslant 0$ and it would hold that $F^s_{p_0,q}(\mathbb{R}^d) \hookrightarrow C^{s-\frac{d}{p_0}}_{\rm ub}(\mathbb{R}^d)$ for $q = \infty$, then it would also hold for all $q \in [1,\infty)$. However, by Proposition 14.6.17 this would imply that every function in $F^s_{p_0,q}(\mathbb{R}^d)$ is zero at $x_1 = 0$, which is of course not true.

Corollary 14.6.27. *If $s \geqslant d$ is an integer, then $F^s_{1,\infty}(\mathbb{R}^d;X) \hookrightarrow C^{s-d}_{\rm ub}(\mathbb{R}^d;X)$ continuously.*

The result also holds in the case where $s > d$ is not integer. However, in this case Corollary 14.4.27 (2) gives a better result.

Proof. By Theorem 14.6.26 and Proposition 14.4.18,

$$F^s_{1,\infty}(\mathbb{R}^d;X) \hookrightarrow B^{s-d}_{\infty,1}(\mathbb{R}^d;X) \hookrightarrow C^{s-d}_{\rm ub}(\mathbb{R}^d;X).$$

□

14.6.g Duality

The next theorem identifies the duals of vector-valued Triebel–Lizorkin spaces.

Theorem 14.6.28. *Let $p,q \in (1,\infty)$ and $s \in \mathbb{R}$. Then*

$$F^s_{p,q}(\mathbb{R}^d;X)^* \simeq F^{-s}_{p',q'}(\mathbb{R}^d;X^*)$$

isomorphically.

The proof is similar to that of Theorem 14.4.34. The restriction $p,q > 1$ comes in through Lemma 14.6.22.

14.6.h Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$ in $B^s_{p,q}$ and $F^s_{p,q}$

In this section we apply the difference norm characterisation of Theorem 14.6.20, as well as the interpolation and duality results proved in this section, to study pointwise multiplication in Triebel–Lizorkin spaces with the non-smooth function $\mathbf{1}_{\mathbb{R}_+}$. The corresponding result for Besov spaces will be derived afterwards by real interpolation.

As a preparation we first deduce several fractional Hardy inequalities.

Proposition 14.6.29 (Hardy–Young inequality). *Let $p \in [1,\infty]$ and $\alpha \in \mathbb{R} \setminus \{0\}$, and let $f : \mathbb{R}_+ \to X$ be strongly measurable and integrable on every finite interval $(0,t)$. Each of the conditions*

(1) $\alpha > 0$ *and* $\lim_{t\to 0} \frac{1}{t}\int_0^t f(\tau)\,\mathrm{d}\tau = 0$
(2) $\alpha < 0$ *and* $\lim_{t\to \infty} \frac{1}{t}\int_0^t f(\tau)\,\mathrm{d}\tau = 0$

implies

$$\begin{aligned}\|t \mapsto t^{-\alpha} f(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t};X)} \\ \leqslant (1+|\alpha|^{-1})\Big\|t \mapsto t^{-\alpha}\Big(f(t) - \fint_0^t f(\tau)\,\mathrm{d}\tau\Big)\Big\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t};X)}\end{aligned}$$

provided the right-hand side is finite.

Proof. (1): Let $F(t) := f(t) - \fint_0^t f(\tau)\,\mathrm{d}\tau$. Integrating by parts on $[t,\sigma]$ we obtain

$$I := \int_t^\sigma \frac{1}{s^2}\int_0^s f(r)\,\mathrm{d}r\,\mathrm{d}s = -\frac{1}{\sigma}\int_0^\sigma f(r)\,\mathrm{d}r + \frac{1}{t}\int_0^t f(r)\,\mathrm{d}r + \int_t^\sigma f(s)\,\frac{\mathrm{d}s}{s}.$$

Therefore,

$$\begin{aligned}\int_t^\sigma F(s)\,\frac{\mathrm{d}s}{s} = \int_t^\sigma f(s)\,\frac{\mathrm{d}s}{s} - I &= \fint_0^\sigma f(r)\,\mathrm{d}r - \fint_0^t f(r)\,\mathrm{d}r \\ &= f(\sigma) - F(\sigma) - \fint_0^t f(r)\,\mathrm{d}r.\end{aligned} \tag{14.88}$$

Letting $t \downarrow 0$ in (14.88) and taking norms, we obtain the estimate

$$\|f(\sigma)\| \leqslant \|F(\sigma)\| + \int_0^\sigma \|F(s)\|\,\frac{\mathrm{d}s}{s}, \quad t > 0.$$

Applying Hardy's inequality (see Lemma L.3.2(1)) with $\widetilde{\alpha} := \alpha - 1 > -1$ to the function $s \mapsto \|F(s)\|$ we obtain

$$\|\sigma \mapsto \sigma^{-\alpha} f(\sigma)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}\sigma}{\sigma};X)} \leqslant (1+\alpha^{-1})\|\sigma \mapsto \sigma^{-\alpha} F(\sigma)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}\sigma}{\sigma};X)}$$

which gives the required estimate.

(2): We argue in the same way, but this time we rewrite the right-hand side of (14.88) as

$$\int_t^\sigma F(s)\,\frac{\mathrm{d}s}{s} = \fint_0^\sigma f(r)\,\mathrm{d}r - f(t) + F(t).$$

Letting $\sigma \to \infty$ and taking norms, we obtain the estimate

$$\|f(t)\| \leqslant \|F(t)\| + \int_t^\infty \|F(s)\|\,\frac{\mathrm{d}s}{s}, \quad t > 0.$$

Now the proof is finished as before, this time applying Lemma L.3.2(2) with $\widetilde{\alpha} := \alpha - 1 < -1$. □

As an immediate consequence we obtain the following result.

Proposition 14.6.30 (Fractional Hardy inequality). *Let $p \in [1, \infty)$ and $\beta \in \mathbb{R}$, and let $f : \mathbb{R}_+ \to X$ is strongly measurable and integrable on every finite sub-interval $(0, t)$. Each of the conditions*

(1) $\beta \in (1/p, \infty)$ *and* $\lim_{t \downarrow 0} \fint_0^t \|f(\tau)\| \, \mathrm{d}\tau = 0$
(2) $\beta \in (-\infty, 1/p)$ *and* $\lim_{t \to \infty} \fint_0^t \|f(\tau)\| \, \mathrm{d}\tau = 0$

implies

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}_+, t^{-\beta p}\,\mathrm{d}t; X)} &\leqslant C \Big\| x \mapsto x^{-\beta} \Big(\fint_{(0,x)} \|f(x) - f(x-h)\| \, \mathrm{d}h \Big) \Big\|_{L^p(\mathbb{R}_+)} \\ &\leqslant C \Big\| x \mapsto \sup_{t>0} \; t^{-\beta} \fint_{(0,x\wedge t)} \|f(x) - f(x-h)\| \, \mathrm{d}h \Big\|_{L^p(\mathbb{R}_+)}\end{aligned}$$

with $C := 1 + \frac{1}{|\beta - \frac{1}{p}|}$, provided the right-hand side is finite.

Proof. By Proposition 14.6.29 with $\alpha = \beta - \frac{1}{p}$,

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}_+, t^{-\beta p}\,\mathrm{d}t; X)} &\leqslant C \Big\| x \mapsto x^{-\beta} \Big\| f(x) - \fint_{(0,x)} f(\tau) \, \mathrm{d}\tau \Big\| \Big\|_{L^p(\mathbb{R}_+; X)} \\ &\leqslant C \Big\| x^{-\beta} \Big(\fint_{(0,x)} \|f(x) - f(x-h)\| \, \mathrm{d}h \|_{L^p(\mathbb{R}_+)} \\ &\leqslant C \Big\| \sup_{t>0} \; t^{-\beta} \Big(\fint_{(0,x\wedge t)} \|f(x) - f(x-h)\| \, \mathrm{d}h \Big\|_{L^p(\mathbb{R}_+)}.\end{aligned}$$

This gives the required estimate in both cases. □

For $p \in [1, \infty)$, $q \in [1, \infty]$, and $s \in (1/p, 1)$ we define the following closed subspaces of $H^{s,p}(\mathbb{R}; X)$ and $F^s_{p,q}(\mathbb{R}; X)$, respectively:

$$\begin{aligned}{}_0H^{s,p}(\mathbb{R}; X) &:= \{f \in H^{s,p}(\mathbb{R}; X) : f(0) = 0\}, \\ {}_0F^s_{p,q}(\mathbb{R}; X) &:= \{f \in F^s_{p,q}(\mathbb{R}; X) : f(0) = 0\}.\end{aligned}$$

Here we use the bounded continuous version for f (which exists by Corollary 14.4.27 combined with Propositions 14.6.8 and 14.6.13) respectively. The continuity of the embeddings in Corollary 14.4.27 gives the closedness of these subspaces.

We can now prove the following fractional Hardy inequality in terms of the spaces $F^s_{p,q}$ and $H^{s,p}$ and their analogues ${}_0F^s_{p,q}$ and ${}_0H^{s,p}$.

Corollary 14.6.31. *Let $p \in [1, \infty)$ and $q \in [1, \infty]$.*

(1) *If $s \in (1/p, 1)$, then each of the spaces ${}_0F^s_{p,q}(\mathbb{R}; X)$ and ${}_0H^{s,p}(\mathbb{R}; X)$ continuously embeds into $L^p(\mathbb{R}, |t|^{-sp}\,\mathrm{d}t; X)$.*
(2) *If $s \in (0, 1/p)$, then each of the spaces $F^s_{p,q}(\mathbb{R}; X)$ and $H^{s,p}(\mathbb{R}; X)$ (if $p \neq 1$) continuously embeds into $L^p(\mathbb{R}, |t|^{-sp}\,\mathrm{d}t; X)$.*

Since $W^{s,p}(\mathbb{R};X) = F^s_{p,p}(\mathbb{R};X)$ for $s \in (0,1)$, the corollary also covers fractional Sobolev spaces.

Proof. By the embeddings (14.69) and (14.73) it suffices to prove the result for ${}_0F^s_{p,\infty}(\mathbb{R};X)$ and $F^s_{p,\infty}(\mathbb{R};X)$.

By Proposition 14.6.30, using that for bounded continuous functions $f : \mathbb{R} \to X$ we have $⨍_0^t f(\tau)\,\mathrm{d}\tau \to f(0) = 0$ as $t \downarrow 0$ in case (1) and $⨍_0^t f(\tau)\,\mathrm{d}\tau \to 0$ as $t \to \infty$ in case (2), we have

$$\begin{aligned}\|\mathbf{1}_{\mathbb{R}_+} f\|_{L^p(\mathbb{R},|t|^{-sp}\,\mathrm{d}t;X)} &\leqslant C\Big\| x \mapsto x^{-s} ⨍_{(0,x)} \|f(x) - f(x-h)\|\,\mathrm{d}h \Big\|_{L^p(\mathbb{R}_+)} \\ &\leqslant 2C\Big\| x \mapsto \sup_{t>0} t^{-s} ⨍_{(-t,t)} \|\Delta_h f(x)\|\,\mathrm{d}h \Big\|_{L^p(\mathbb{R})} \\ &= 2C[f]^{(1)}_{F^s_{p,\infty}(\mathbb{R};X)} \lesssim_{p,s} \|f\|_{F^s_{p,\infty}(\mathbb{R};X)}\end{aligned}$$

where in the last step we used Theorem 14.6.20 with $m = 1$. A similar estimate holds for f on the negative real axis. □

As a consequence we obtain the following result on pointwise multiplication.

Theorem 14.6.32 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in (0,1)$. Each of the two conditions*

(1) *$s \in (0,1/p)$ and $f \in F^s_{p,q}(\mathbb{R};X)$*
(2) *$s \in (1/p,1)$ and $f \in {}_0F^s_{p,q}(\mathbb{R};X)$*

implies that $\mathbf{1}_{\mathbb{R}_+} f \in F^s_{p,q}(\mathbb{R};X)$ and

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{F^s_{p,q}(\mathbb{R};X)} \leqslant C\|f\|_{F^s_{p,q}(\mathbb{R};X)}.$$

Without the condition $f(0) = 0$, the result is false for $s > 1/p$. Indeed, this is clear from the fact that, by combining Corollary 14.4.27 and Proposition 14.6.13, we have a continuous embedding $F^s_{p,q}(\mathbb{R};X) \hookrightarrow C_{\mathrm{ub}}(\mathbb{R};X)$. A counterexample to the case $s = 1/p$ will be discussed in Example 14.6.33. It shows that Propositions 14.6.29, 14.6.30, and Corollary 14.6.31 do not hold for $\alpha = 0$ and $s = 1/p$.

Proof. Clearly, $\|\mathbf{1}_{\mathbb{R}_+} f\|_{L^p(\mathbb{R}^d;X)} \leqslant \|f\|_{L^p(\mathbb{R}^d;X)}$. Therefore, using the difference norm of Theorem 14.6.20 it remains to estimate $[\mathbf{1}_{\mathbb{R}_+} f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)}$ in terms of $\|f\|_{F^s_{p,q}(\mathbb{R};X)}$ and $[f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)}$. We give the proof for $q \in [1,\infty)$; the case $q = \infty$ requires the usual obvious modifications.

By the triangle inequality,

$$\begin{aligned}&[\mathbf{1}_{\mathbb{R}_+} f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)} \\ &\leqslant \Big(\int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}_+} t^{-sq} \Big(\frac{1}{t} \int_{(-t,t)\cap(-x,\infty)} \|f(x+h) - f(x)\|\,\mathrm{d}h \Big)^q \frac{\mathrm{d}t}{t} \Big)^{p/q} \mathrm{d}x \Big)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&+ \Big(\int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}_+} t^{-sq}\Big(\frac{1}{t}\int_{(-t,t)\cap(-\infty,-x)} \|f(x)\|\,\mathrm{d}h\Big)^q \,\frac{\mathrm{d}t}{t}\Big)^{p/q}\,\mathrm{d}x\Big)^{1/p}\\
&+ \Big(\int_{(-\infty,0)} \Big(\int_{\mathbb{R}_+} t^{-sq}\Big(\frac{1}{t}\int_{(-t,t)\cap(-x,\infty)} \|f(x+h)\|\,\mathrm{d}h\Big)^q \,\frac{\mathrm{d}t}{t}\Big)^{p/q}\,\mathrm{d}x\Big)^{1/p}\\
=\!:&\ (I) + (II) + (III).
\end{aligned}
$$

We estimate these three terms separately. Clearly, $(I) \leqslant [f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)}$ and, with $C = 1 + \frac{p}{|sp-1|}$,

$$
\begin{aligned}
(II) &\leqslant \Big(\int_{\mathbb{R}_+} \Big(\int_x^\infty t^{-sq}\,\frac{\mathrm{d}t}{t}\Big)^{p/q} \|f(x)\|^p\,\mathrm{d}x\Big)^{1/p}\\
&\leqslant (sq)^{-1/q}\Big(\int_0^\infty x^{-sp}\|f(x)\|^p\,\mathrm{d}x\Big)^{1/p}\\
&\lesssim_{s,p,q} \|f\|_{F^s_{p,q}(\mathbb{R};X)},
\end{aligned}
$$

using Corollary 14.6.31 in the last step.

To estimate (III) fix $x \in (-\infty, 0)$. By Minkowski's inequality (Theorem 1.2.22),

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}_+} t^{-sq}\Big(\frac{1}{t}\int_{(-t,t)\cap(-x,\infty)} \|f(x+h)\|\,\mathrm{d}h\Big)^q\,\frac{\mathrm{d}t}{t}\Big)^{p/q}\\
&\qquad\leqslant \Big(\int_{\mathbb{R}_+}\Big(\int_{\mathbb{R}_+} t^{-sq-q}\mathbf{1}_{(h,\infty)}(t)\,\frac{\mathrm{d}t}{t}\Big)^{1/q}\mathbf{1}_{(-x,\infty)}(h)\|f(x+h)\|\,\mathrm{d}h\\
&\qquad= K_{q,s}\int_{\mathbb{R}_+} h^{-s-1}\mathbf{1}_{(-x,\infty)}(h)\|f(h+x)\|\,\mathrm{d}h\\
&\qquad= K_{q,s}\int_{\mathbb{R}_+} (y-x)^{-s-1}\|f(y)\|\,\mathrm{d}y,
\end{aligned}
$$

where $K_{q,s} = (sq+q)^{1/q}$. Setting $z = -x$ and $\phi_p(z) = z^{1/p}(1+z)^{-s-1}$, (III) can be estimated using Young's inequality for convolutions for the multiplicative group $\mathbb{R}_+$ with Haar measure $\frac{\mathrm{d}z}{z}$:

$$
\begin{aligned}
(III) &\leqslant K_{q,s}\Big(\int_{\mathbb{R}_+}\Big(\int_{\mathbb{R}_+} (y+z)^{-s-1}\|f(y)\|\,\mathrm{d}y\Big)^p\,\mathrm{d}z\Big)^{1/p}\\
&= K_{q,s}\Big(\int_{\mathbb{R}_+}\Big(\int_{\mathbb{R}_+} \phi_p(z/y)y^{-s+\frac{1}{p}}\|f(y)\|\,\frac{\mathrm{d}y}{y}\Big)^p\frac{\mathrm{d}z}{z}\Big)^{1/p}\\
&\leqslant K_{q,s}\|\phi_p\|_{L^1(\mathbb{R}_+,\frac{\mathrm{d}z}{z})}\Big(\int_{\mathbb{R}_+} y^{-sp}\|f(y)\|^p\,\mathrm{d}y\Big)^{1/p}\\
&\lesssim_{p,q,s} \|f\|_{F^s_{p,q}(\mathbb{R};X)},
\end{aligned}
$$

using Corollary 14.6.31 as in the estimate for (II). □

Example 14.6.33. Theorem 14.6.32 is false for $s = 1/p$ even in the scalar-valued case. Indeed, $f \in C_c^\infty(\mathbb{R})$ is any function satisfying $f \equiv 1$ on $[-1,1]$, then for all $p \in [1,\infty)$ we have $f \in F^{1/p}_{p,q}(\mathbb{R})$. Let us prove that $\mathbf{1}_{\mathbb{R}_+} f \notin F^{1/p}_{p,q}(\mathbb{R})$. To this end it suffices to take $q = \infty$. In case $p \in (1,\infty)$ we can use Theorem 14.6.20 to find

$$\begin{aligned}
\|\mathbf{1}_{\mathbb{R}_+} f\|_{F^{1/p}_{p,\infty}(\mathbb{R}^d;X)} &\eqsim_p \|\mathbf{1}_{\mathbb{R}_+} f\|^{(1)}_{F^{1/p}_{p,\infty}(\mathbb{R}^d;X)} \\
&\geqslant \Big\| x \mapsto \sup_{t>0} t^{-\frac{1}{p}-1} \int_{-t}^{-x} |f(x)|\,\mathrm{d}h \Big\|_{L^p(0,1)} \\
&= \Big\| x \mapsto \sup_{t>x} t^{-\frac{1}{p}-1} (t-x) \Big\|_{L^p(0,1)} \\
&\gtrsim_p \|x \mapsto x^{-\frac{1}{p}}\|_{L^p(0,1)} = \infty.
\end{aligned}$$

For $p = 1$ we note that $F^1_{1,q}(\mathbb{R}) \hookrightarrow F^{1/r}_{r,\infty}(\mathbb{R})$ for all $r \in (p,\infty)$ by Theorem 14.6.14, and therefore $\mathbf{1}_{\mathbb{R}_+} f \notin F^1_{1,q}(\mathbb{R})$.

One could still hope that the boundedness of $f \mapsto \mathbf{1}_{\mathbb{R}_+} f$ for $s = 1/p$ holds on the closure in $F^{1/p}_{p,q}(\mathbb{R})$ of the smooth functions satisfying $f(0) = 0$. This turns out to be false as well. Indeed, in the case $q < \infty$ the latter space coincides with $F^{1/p}_{p,q}(\mathbb{R})$ by Proposition 14.6.17. If $q = \infty$, the boundedness is also fails, as follows from the previous example and the embedding $F^{1/p}_{p,\infty}(\mathbb{R}) \hookrightarrow F^{1/r}_{r,r}(\mathbb{R})$ for all $r \in (p,\infty)$ contained in Theorem 14.6.14.

By duality and interpolation, we now extend Theorem 14.6.32 to smoothness exponents $s \leqslant 0$, which excludes the end-point cases.

Corollary 14.6.34 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in (1,\infty)$, $q \in (1,\infty)$, and $s \in (-1/p', 0]$. For all $f \in F^s_{p,q}(\mathbb{R};X)$ we have $\mathbf{1}_{\mathbb{R}_+} f \in F^s_{p,q}(\mathbb{R};X)$ and*

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{F^s_{p,q}(\mathbb{R};X)} \leqslant C \|f\|_{F^s_{p,q}(\mathbb{R};X)}.$$

Proof. By density it suffices to consider $f \in C^\infty(\mathbb{R} \setminus \{0\}) \otimes X$. We use duality result. By Theorems 14.6.28 and 14.6.32 for any $g \in \mathscr{S}(\mathbb{R}^d;X^*)$ we have

$$\begin{aligned}
|\langle \mathbf{1}_{\mathbb{R}_+} f, g\rangle| = |\langle f, \mathbf{1}_{\mathbb{R}_+} g\rangle| &\leqslant C \|f\|_{F^s_{p,q}(\mathbb{R};X)} \|\mathbf{1}_{\mathbb{R}_+} g\|_{F^{-s}_{p',q'}(\mathbb{R};X^*)} \\
&\leqslant C' \|f\|_{F^s_{p,q}(\mathbb{R};X)} \|g\|_{F^{-s}_{p',q'}(\mathbb{R};X^*)}.
\end{aligned}$$

Since $\mathscr{S}(\mathbb{R}^d;X^*)$ is dense in $F^{-s}_{p',q'}(\mathbb{R};X^*)$, the result follows by another application of Theorem 14.6.28.

The case $s = 0$ follows by complex interpolation between the cases s and $-s$ for $s > 0$ small enough, using Theorems C.2.6 and 14.6.23. □

Applying the real interpolation method instead, we obtain the following for the Besov scale.

Corollary 14.6.35 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in (1,\infty)$, $q \in [1,\infty]$, and $s \in (-1/p, 1/p)$. For all $f \in B^s_{p,q}(\mathbb{R};X)$ we have $\mathbf{1}_{\mathbb{R}_+} f \in B^s_{p,q}(\mathbb{R};X)$ and*

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{B^s_{p,q}(\mathbb{R};X)} \leqslant C\|f\|_{B^s_{p,q}(\mathbb{R};X)}, \quad f \in B^s_{p,q}(\mathbb{R};X).$$

Proof. First let $s > 0$. Since $(F^{s-\varepsilon}_{p,2}, F^{s+\varepsilon}_{p,2})_{1/2,q} = B^s_{p,q}$ by Theorem 14.4.31, the result follows from Theorems 14.6.32 and C.3.3. Here we can allow $p = 1$ as well.

The result for $s < 0$ and $q \in (1,\infty)$ follows from Theorem 14.4.34 in the same way as in Corollary 14.6.34. The cases $q = 1$ and $q = \infty$ can be obtained by another real interpolation argument as we did in Example 14.4.35.

The case $s = 0$ follows by real interpolation between the cases s and $-s$ for $s > 0$ small. □

14.7 Bessel potential spaces

In this section we prove Sobolev embeddings and norm estimates for Bessel potential spaces. Some results will depend on the geometry of X. Real interpolation for $H^{s,p}(\mathbb{R}^d;X)$ has already been considered in Theorem 14.4.31. Duality for $H^{s,p}(\mathbb{R}^d;X)$ has already been considered in Proposition 5.6.7.

14.7.a General embedding theorems

We begin with the following Sobolev embedding theorem.

Theorem 14.7.1 (Sobolev embedding for Bessel potential spaces and Sobolev spaces). *Let $p_0, p_1 \in (1,\infty)$ and $s_0, s_1 \in \mathbb{R}$. We have a continuous embedding*

$$H^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^d;X)$$

if and only if one of the following two conditions holds:

$$p_0 = p_1 \quad \textit{and} \quad s_0 \geqslant s_1; \tag{14.89}$$

$$p_0 < p_1 \qquad \textit{and} \qquad s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}. \tag{14.90}$$

If $s_0, s_1 \in \mathbb{N}$, then the same necessary and sufficient conditions give the existence of a continuous embedding

$$W^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow W^{s_1,p_1}(\mathbb{R}^d;X).$$

Proof. We first prove the result for Bessel potential spaces.

'If': By Proposition 14.6.13, for $p \in (1,\infty)$ and $s \in \mathbb{R}$ we have continuous embeddings

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X). \tag{14.91}$$

From Theorem 14.6.14 we see that if either (14.89) or (14.90) holds, then $F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,1}(\mathbb{R}^d;X)$. Therefore the required embedding follows from (14.91) with $s = s_0, s_1$ and $p = p_0, p_1$.

'Only if': If the stated embedding holds, then by (14.91) with $s = s_0, s_1$ and $p = p_0, p_1$, we also have a continuous embedding $F^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,\infty}(\mathbb{R}^d;X)$. Therefore, either (14.89) or (14.90) must hold by Theorem 14.6.14.

The corresponding result for Sobolev spaces with integer smoothness can be proved in the same way, noting that the analogue of (14.91) holds for these spaces. □

Remark 14.7.2. The embedding of Theorem 14.7.1 for Bessel potential spaces can be restated as the boundedness of $J_{-(s_0-s_1)} = (1-\Delta)^{-(s_0-s_1)}$ from $L^{p_0}(\mathbb{R}^d;X)$ into $L^{p_1}(\mathbb{R}^d;X)$. Since $J_{-(s_0-s_1)}$ is a positive operator by Proposition 5.6.6, we infer from Theorem 2.1.3 that the boundedness in the scalar case is actually equivalent to boundedness in the vector-valued situation.

By the same argument as in Theorem 14.7.1, the following result can be deduced from Proposition 14.6.15.

Proposition 14.7.3 (Gagliardo–Nirenberg inequality for Bessel potential spaces). *Let $p_0, p_1 \in (1,\infty)$, $-\infty < s_0 < s_1 < \infty$, and $\theta \in (0,1)$, and let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1.$$

There exists a constant $C = C_{\theta,p_0,p_1,s_0,s_1} \geqslant 0$ such that for all $f \in H^{s_0,p_0}(\mathbb{R}^d;X) \cap H^{s_1,p_1}(\mathbb{R}^d;X)$ we have $f \in H^{s,p}(\mathbb{R}^d;X)$ and

$$\|f\|_{H^{s,p}(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{H^{s_0,p_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{H^{s_1,p_1}(\mathbb{R}^d;X)}.$$

If, in Proposition 14.7.3, $s_0, s_1 \geqslant 0$ are integers and $p \in (1,\infty)$, the same argument gives that $f \in W^{s_0,p_0}(\mathbb{R}^d;X) \cap W^{s_1,p_1}(\mathbb{R}^d;X)$ implies $f \in W^{s,p}(\mathbb{R}^d;X)$ and

$$\|f\|_{W^{s,p}(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{W^{s_0,p_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^d;X)}. \tag{14.92}$$

The latter estimate extends to $p_0 \in (1,\infty]$ and $p_1 \in (1,\infty]$. Indeed, if only one of the exponents is infinite, then (14.92) is a consequence of Proposition 14.6.16 and the sandwich results of Propositions 14.4.18 (see (14.29)) and

14.6.13. If $p = p_0 = p_1 \in [1, \infty]$, (14.92) can be deduced from these sandwich results and real interpolation and (L.2):

$$\begin{aligned}(W^{s_0,p}(\mathbb{R}^d; X), W^{s_1,p}(\mathbb{R}^d; X))_{\theta,1} &\hookrightarrow (B^{s_0}_{p,\infty}(\mathbb{R}^d; X), B^{s_1}_{p,\infty}(\mathbb{R}^d; X))_{\theta,1} \\ &= B^s_{p,1}(\mathbb{R}^d; X) \qquad \text{(by (14.48))} \\ &\hookrightarrow W^{s,p}(\mathbb{R}^d; X).\end{aligned}$$

Note that this even gives (14.92) for $p = p_0 = p_1 = 1$.

The estimate (14.92) self-improves to the following Gagliardo–Nirenberg type inequality for $W^{s,p}(\mathbb{R}^d; X)$:

Theorem 14.7.4 (Schmeisser–Sickel). *Let $p_0, p_1, p \in (1, \infty]$, $m \in \mathbb{N}$, and $|\alpha| \leqslant m$ satisfy*

$$\theta = \frac{|\alpha|}{m} \quad \textit{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

There exists a constant $C \geqslant 0$ such that for all $f \in L^{p_0}(\mathbb{R}^d; X) \cap W^{m,p_1}(\mathbb{R}^d; X)$ we have

$$\|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{L^{p_0}(\mathbb{R}^d;X)} \Big(\sum_{|\beta|=m} \|\partial^\beta f\|_{L^{p_1}(\mathbb{R}^d;X)} \Big)^\theta.$$

Moreover, the same holds if $p = p_0 = p_1 = 1$.

Proof. For $\theta = \frac{|\alpha|}{m} \in \{0, 1\}$ there is nothing to prove, so we may assume that $\theta \in (0, 1)$. Taking $s = |\alpha|$, $s_0 = 0$, and $s_1 = m$ in (14.92), it follows that

$$\|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{L^{p_0}(\mathbb{R}^d;X)} \Big(\sum_{|\beta|\leqslant m} \|\partial^\beta f\|_{L^{p_1}(\mathbb{R}^d;X)} \Big)^\theta.$$

Applying this to the function $f(\lambda\cdot)$ for $\lambda > 0$, we obtain

$$\begin{aligned}&\lambda^{|\alpha|-\frac{d}{p}} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)} \\ &\qquad \leqslant C(\lambda^{-\frac{d}{p_0}} \|f\|_{L^{p_0}(\mathbb{R}^d;X)})^{1-\theta} \Big(\sum_{|\beta|\leqslant m} \lambda^{|\beta|-\frac{d}{p_1}} \|\partial^\beta f\|_{L^{p_1}(\mathbb{R}^d;X)} \Big)^\theta.\end{aligned}$$

Now divide both sides by $\lambda^{|\alpha|-\frac{d}{p}}$ and pass to the limit $\lambda \to \infty$. □

14.7.b Embedding theorems under geometric conditions

Littlewood–Paley inequality for Bessel potential spaces

The aim of this paragraph is to prove the following Littlewood–Paley inequality with smooth cut-offs for $H^{s,p}(\mathbb{R}^d; X)$.

Theorem 14.7.5 (Littlewood–Paley theorem for Bessel potential spaces). *Let X be a UMD space, $p \in (1,\infty)$, and $s \in \mathbb{R}$. A tempered distribution $f \in \mathscr{S}'(\mathbb{R}^d;X)$ belongs to $H^{s,p}(\mathbb{R}^d;X)$ if and only if*

$$\|\!|f|\!\|_{H^{s,p}(\mathbb{R}^d;X)} := \sup_{n\geqslant 0}\Big\|\sum_{k=0}^{n}\varepsilon_k 2^{ks}\varphi_k * f\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} < \infty.$$

In this situation the sum $\sum_{k\geqslant 0}\varepsilon_k 2^{ks}\varphi_k * f$ converges, both in $L^p(\Omega\times\mathbb{R}^d;X)$ and almost surely in $L^p(\mathbb{R}^d;X)$, and we have an equivalence of norms

$$\|\!|f|\!\|_{H^{s,p}(\mathbb{R};X)} \eqsim_{d,p,s,X} \|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

For $s=0$ the above estimate yields an equivalent norm on $L^p(\mathbb{R}^d;X)$ which is slightly different from the Littlewood–Paley estimate with smooth cut-offs of Theorem 5.5.22, where the summation was taken over $\mathbb{Z}$ and the functions ψ_k were of the form $2^k\psi(2^k\cdot)$ for a Littlewood–Paley function ψ in the sense of Definition 5.5.20.

Proof. 'Only if': Fix $f\in H^{s,p}(\mathbb{R}^d;X)$. Fix a sequence of signs $\epsilon=(\epsilon_k)_{k\geqslant 0}$ in $\{z\in\mathbb{K}: |z|=1\}$. For integers $n\geqslant 0$, define the function $m_n\in C^\infty(\mathbb{R}^d)$ by

$$m_n(\xi) := \sum_{k=0}^{n}\epsilon_k 2^{ks}(1+|\xi|^2)^{-s/2}\widehat{\varphi}_k(\xi).$$

From the location of the supports of the functions $\widehat{\varphi}_k$ one sees three things: first, that for each $\xi\in\mathbb{R}^d$ at most three terms in this sum are non-zero (the sum therefore converges for trivial reasons); second, that $\|\partial^\beta\widehat{\varphi}_k\|_\infty\leqslant C_\beta 2^{-k|\beta|}$; and third, that

$$c_d = \sup_{\epsilon}\sup_{n\geqslant 0}\sup_{\alpha\in\{0,1\}^d}\sup_{\xi\neq 0}|\xi|^{|\alpha|}|\partial^\alpha m_n(\xi)|$$

is finite, the outer supremum being taken over all sequences of signs $\epsilon=(\epsilon_k)_{k\geqslant 0}$.

By the Mihlin multiplier theorem (Theorem 5.5.10), the Fourier multiplier operators T_{m_n} associated with m_n are bounded on $L^p(\mathbb{R}^d;X)$, with estimates uniform in n and signs ϵ, say $\sup_\epsilon\sup_{n\geqslant 0}\|T_{m_n}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}\leqslant C_{X,p,d}$. Since

$$\sum_{k=0}^{n}\epsilon_k 2^{ks}\varphi_k * f = T_{m_n}J_s f, \tag{14.93}$$

we obtain

$$\Big\|\sum_{k=0}^{n}\epsilon_k 2^{ks}\varphi_k * f\Big\|_{L^p(\mathbb{R}^d;X)} \leqslant C_{X,p,d}\|J_s f\|_{L^p(\mathbb{R}^d;X)} = C_{X,p,d}\|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

Taking $\epsilon_k = \varepsilon_k(\omega)$ and passing to the $L^p(\Omega)$-norms, we obtain the estimate

$$\Big\| \sum_{k=0}^{n} \varepsilon_k 2^{ks} \varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \leqslant C_{X,p,d} \|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

'If': Assume now that $f \in \mathscr{S}'(\mathbb{R}^d;X)$ satisfies $|\!|\!| f |\!|\!|_{H^{s,p}(\mathbb{R}^d;X)} < \infty$. We claim that $\sum_{k\geqslant 0} \varepsilon_k 2^{ks}\varphi_k * f$ converges in $L^p(\Omega; L^p(\mathbb{R}^d;X))$ and almost surely in $L^p(\mathbb{R}^d;X)$. Indeed, $L^p(\mathbb{R}^d;X)$ is a UMD space by Proposition 4.2.15, so by Proposition 4.2.19 it does not contain an isomorphic copy of c_0. The convergence of the sum, in $L^p(\Omega\times\mathbb{R}^d;X)$ and almost surely in $L^p(\mathbb{R}^d;X)$, now follows from Corollary 6.4.12. Moreover, by Fatou's lemma and the Kahane contraction principle,

$$|\!|\!| f |\!|\!|_{H^{s,p}(\mathbb{R}^d;X)} = \Big\| \sum_{k\geqslant 0} \varepsilon_k 2^{ks} \varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}.$$

For $k \in \{0,1\}$ choose $\psi_k \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \widehat{\psi}_k \leqslant 1$, $\operatorname{supp}\widehat{\psi}_0 \subseteq \{0 \leqslant |\xi| \leqslant 2\}$ and $\operatorname{supp}\widehat{\psi}_1 \subseteq \{\frac14 \leqslant |\xi| \leqslant 4\}$, and $\widehat{\psi}_k \equiv 1$ on $\operatorname{supp}\widehat{\varphi}_k$. For $k \geqslant 2$ we define $\widehat{\psi}_k := \widehat{\psi}_1(2^{-(k-1)}\cdot)$. For $\omega \in \Omega$ put

$$m_\omega := \sum_{j\geqslant 0} \overline{\varepsilon_j(\omega)} 2^{-js}(1+|\cdot|^2)^{s/2}\widehat{\psi}_j, \qquad g_\omega := \sum_{k\geqslant 0} \varepsilon_k(\omega) 2^{ks}\varphi_k * f.$$

As before,

$$C_m = \sup_{\omega\in\Omega} \sup_{\alpha\in\{0,1\}^d} \sup_{\xi\neq 0} |\xi|^{|\alpha|} |\partial^\alpha m_\omega(\xi)| < \infty.$$

Therefore, by the Mihlin multiplier Theorem 5.5.10,

$$\|T_{m_\omega} g_\omega\|_{L^p(\mathbb{R}^d;X)} \leqslant C \|g_\omega\|_{L^p(\mathbb{R}^d;X)}$$

for almost every $\omega \in \Omega$. Considering finite sums first, one checks that $\omega \mapsto T_{m_\omega} g_\omega$ is strongly measurable. Since $\omega \mapsto g_\omega$ belongs to $L^p(\Omega; L^p(\mathbb{R}^d;X))$, it follows that so does $\omega \mapsto T_{m_\omega} g_\omega$. By the condition $\widehat{\psi}_k \equiv 1$ on $\operatorname{supp}\widehat{\varphi}_k$, as in (14.93) we have

$$\int_\Omega T_{m_\omega} g_\omega \,\mathrm{d}\mathbb{P}(\omega) = J_s f.$$

By Jensen's inequality and Fubini's theorem, $f \in H^{s,p}(\mathbb{R}^d;X)$ and

$$\begin{aligned}
\|f\|^p_{H^{s,p}(\mathbb{R}^d;X)} &= \|J_s f\|^p_{L^p(\mathbb{R}^d;X)} \\
&= \Big\| \int_\Omega T_{m_\omega} g_\omega \,\mathrm{d}\mathbb{P}(\omega) \Big\|^p_{L^p(\mathbb{R}^d;X)} \\
&\leqslant \int_\Omega \|T_{m_\omega} g_\omega\|^p_{L^p(\mathbb{R}^d;X)} \,\mathrm{d}\mathbb{P}(\omega)
\end{aligned}$$

$$\leqslant C \int_{\Omega} \|g_\omega\|^p_{L^p(\mathbb{R}^d;X)} \,\mathrm{d}\mathbb{P}(\omega) = C\|f\|^p_{H^{s,p}(\mathbb{R}^d;X)}.$$

□

We continue with an embedding result under additional geometric assumptions on X. The cases $p_0 = 1$ and $q_0 = \infty$ were proved for general Banach spaces in Propositions 14.4.18 and 14.6.13.

Proposition 14.7.6 (Sandwich theorem under type and cotype). *Let X be a UMD Banach space with type $p_0 \in [1,2]$ and cotype $q_0 \in [2,\infty]$. For all $p \in (1,\infty)$ and $s \in \mathbb{R}$ we have continuous embeddings*

$$F^s_{p,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q_0}(\mathbb{R}^d;X).$$

Proof. We only prove $F^s_{p,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X)$; the other embedding is proved similarly.

Let $f \in F^s_{p,p_0}(\mathbb{R}^d;X)$. By Theorem 14.7.5, the Kahane–Khintchine inequality (Theorem 6.2.4) and the type p_0 property of X, we have

$$\begin{aligned}
\|f\|_{H^{s,p}(\mathbb{R}^d;X)} &\leqslant C \sup_{n\geqslant 1} \Big\| \sum_{k=0}^n \varepsilon_k 2^{ks}\varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\
&\eqsim_p C \sup_{n\geqslant 1} \Big(\int_{\mathbb{R}^d} \Big\| \sum_{k=0}^n \varepsilon_k 2^{ks}\varphi_k * f \Big\|^p_{L^{p_0}(\Omega;X)} \,\mathrm{d}x \Big)^{1/p} \\
&\leqslant C \sup_{n\geqslant 1} \Big(\int_{\mathbb{R}^d} \Big(\sum_{k=0}^n \|2^{ks}\varphi_k * f\|^{p_0} \Big)^{p/p_0} \mathrm{d}x \Big)^{1/p} \\
&= C\|f\|_{F^s_{p,p_0}(\mathbb{R}^d;X)}.
\end{aligned}$$

□

In combination with Proposition 14.6.13 and Corollary 14.6.18 we obtain:

Corollary 14.7.7 (γ-Sobolev embedding – III). *Let $p_0 \in [1,2]$ and $q_0 \in [2,\infty]$.*

(1) *If X has type p_0, then for all $p \in [1,p_0)$ we have a continuous embedding*

$$H^{(\frac{1}{p}-\frac{1}{2})d,p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X).$$

(2) *If X has cotype q_0, then for all $q \in (q_0,\infty)$ we have a continuous embedding*

$$\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow H^{(\frac{1}{q}-\frac{1}{2})d,q}(\mathbb{R}^d;X)$$

By Theorem 9.2.10, for $p_0 = 2$ assertion (1) also holds for $p = 2$, and for $q_0 = 2$ assertion (2) also holds for $q = 2$.

Necessity of the type and cotype assumptions

Proposition 14.7.8. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $m \in \mathbb{N}$. Then the following assertions hold with $\mathcal{A} \in \{B, F\}$:*

(1) *If $\mathcal{A}^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X)$ continuously, then X has type q.*
(2) *If $H^{s,p}(\mathbb{R}^d;X) \hookrightarrow \mathcal{A}^s_{p,q}(\mathbb{R}^d;X)$ continuously, then X has cotype q.*
(3) *If $\mathcal{A}^k_{p,q}(\mathbb{R}^d;X) \hookrightarrow W^{m,p}(\mathbb{R}^d;X)$ continuously, then X has type q.*
(4) *If $W^{m,p}(\mathbb{R}^d;X) \hookrightarrow \mathcal{A}^m_{p,q}(\mathbb{R}^d;X)$ continuously, then X has cotype q.*

Proof. (1): By the lifting properties of Propositions 14.4.15, 14.6.10, and 5.6.3, it suffices to consider $s = 0$. Fix a finitely non-zero sequence $(x_n)_{n\geqslant 1}$ in X. Let $\psi \in \mathscr{S}(\mathbb{R}^d)$ be a non-zero function satisfying $\operatorname{supp}(\widehat{\psi}) \subseteq [-\frac{1}{4}, -\frac{1}{8}]^d$ and put

$$f(t,\omega) := \psi(t) \sum_{n\geqslant 1} \varepsilon_n(\omega) e^{2\pi i 2^n t_1} x_n,$$

where as always $(\varepsilon_n)_{n\geqslant 1}$ is a Rademacher sequence. Since $(\varepsilon_n e^{2\pi i 2^n t_1})_{n\geqslant 1}$ is a Rademacher sequence for each $t \in \mathbb{R}^d$, we have

$$\begin{aligned}\mathbb{E}\|f\|^p_{L^p(\mathbb{R}^d;X)} &= \int_{\mathbb{R}^d} |\psi(t)|^p \mathbb{E}\Big\| \sum_{n\geqslant 1} \varepsilon_n e^{2\pi i 2^n t_1} x_n\Big\|^p \,\mathrm{d}t \\ &= \|\psi\|^p_{L^p(\mathbb{R}^d)} \mathbb{E}\Big\| \sum_{n\geqslant 1} \varepsilon_n x_n\Big\|^p.\end{aligned} \tag{14.94}$$

On the other hand, the Fourier support properties of $\widehat{\psi}(\cdot - 2^n t_1 e_1)$ and $\widehat{\varphi}_n$ (see (14.8) and (14.9)) imply that $\|f(\cdot,\omega) * \varphi_n\|_X = |\psi(t)|\|x_n\|$ and $\|f(\cdot,\omega) * \varphi_0\|_X = 0$. Therefore,

$$\|f(\cdot,\omega)\|_{\mathcal{A}^s_{p,q}(\mathbb{R}^d;X)} = \|\psi\|_{L^p(\mathbb{R}^d)} \|(x_n)_{n\geqslant 1}\|_{\ell^q(X)}. \tag{14.95}$$

Applying the assumption (1) pointwise in Ω, we obtain

$$\begin{aligned}\|\psi\|^p_{L^p(\mathbb{R})} \mathbb{E}\Big\| \sum_{n\geqslant 1} \varepsilon_n x_n\Big\|^p &= \mathbb{E}\|f\|^p_{L^p(\mathbb{R}^d;X)} \\ &\leqslant C^p \mathbb{E}\|f\|^p_{\mathcal{A}^s_{p,q}(\mathbb{R}^d;X)} = C^p \|\psi\|^p_{L^p(\mathbb{R}^d)} \|(x_n)_{n\geqslant 1}\|^p_{\ell^q(X)}.\end{aligned}$$

By the Kahane–Khintchine inequalities, this shows that X has type q.

(2): This follows from the previous proof upon replacing "$\leqslant$" by "$\geqslant$".

(3): The idea of the proof is the same as in (1), but this case is slightly more technical. Let $(x_n)_{n\geqslant 1}$ and ψ be as before and put

$$f(t,\omega) := \psi(t) \sum_{n\geqslant 1} 2^{-mn} \varepsilon_n(\omega) e^{2\pi i 2^n t_1} x_n =: \psi(t) f_m(t,\omega).$$

By Leibniz's rule we obtain

$$\partial^\alpha f(t,\omega) = \sum_{|\beta|+j=|\alpha|} c_{\beta,\gamma}\partial^\beta \psi(t) f_{m-j}(t,\omega),$$

For $j \in \{0,\dots,m-1\}$,

$$\begin{aligned}\|(\partial^\beta\psi) f_{m-j}(\cdot,\omega)\|_{L^p(\mathbb{R}^d;X)} &\leqslant \|\partial^\beta\psi\|_\infty \|f_{m-j}(\cdot,\omega)\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \|\partial^\beta\psi\|_\infty \sum_{n\geqslant 1} 2^{-(m-j)n}\|x_n\| \leqslant \|\partial^\beta\psi\|_\infty \sup_{n\geqslant 1}\|x_n\|.\end{aligned}$$

For $j = m$, as in (14.94) we have

$$\mathbb{E}\|f_0\|^p_{L^p(\mathbb{R}^d;X)} = \|\psi\|^p_{L^p(\mathbb{R})}\mathbb{E}\Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|^p.$$

By the reverse triangle inequality, this shows that there exists a constant $C = C(d,m,p,\psi)$ such that

$$\Big|\|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))} - \|\psi\|_{L^p(\mathbb{R})}\Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|_{L^p(\Omega;X)}\Big| \leqslant C\sup_{n\geqslant 1}\|x_n\|. \quad (14.96)$$

Stated differently, up a relatively small term the norm $\|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))}$ is equivalent to the norm $\|\sum_{n\geqslant 1}\varepsilon_n x_n\|$ of the random sum. As in (14.95) we see that

$$\|f(\cdot,\omega)\|_{\mathcal{A}^m_{p,q}(\mathbb{R}^d;X)} = \|\psi\|_{L^p(\mathbb{R}^d)}\|(x_n)_{n\geqslant 1}\|_{\ell^q(X)}.$$

Now from (14.96) and the assumptions, we obtain

$$\begin{aligned}\|\psi\|_{L^p(\mathbb{R})}\Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|_{L^p(\Omega;X)} &\leqslant \|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))} + C\sup_{n\geqslant 1}\|x_n\| \\ &\lesssim \|f\|_{L^p(\Omega;\mathcal{A}^m_{p,q}(\mathbb{R}^d;X))} + \sup_{n\geqslant 1}\|x_n\| \\ &\lesssim \|(x_n)_{n\geqslant 1}\|_{\ell^q(X)}.\end{aligned}$$

(4): This can be proved in the same way as (3). By (14.96) and the Kahane contraction principle, which implies bound $\sup_{n\geqslant 1}\|x_n\|^p \leqslant \mathbb{E}\|\sum_{n\geqslant 1}\varepsilon_n x_n\|^p$, from the assumption (4) we obtain

$$\begin{aligned}\|\psi\|_{L^p(\mathbb{R}^d)}\|(x_n)_{n\geqslant 1}\|_{\ell^q(X)} &= \|f\|_{L^p(\Omega;\mathcal{A}^m_{p,q}(\mathbb{R}^d;X))} \\ &\lesssim \|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))} \lesssim \Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|_{L^p(\Omega;X)}.\end{aligned}$$

□

A Hilbert space characterisation

The equality $F^s_{p,2}(\mathbb{R}^d;X) = H^{s,p}(\mathbb{R}^d;X)$ with equivalent norms characterises Hilbert spaces:

Theorem 14.7.9 (Han–Meyer). *Let $p \in (1,\infty)$, $s \in \mathbb{R}$, and $m \in \mathbb{N}$. The following assertions are equivalent:*

(1) *$F^m_{p,2}(\mathbb{R}^d;X) = W^{m,p}(\mathbb{R}^d;X)$ with equivalent norms;*
(2) *$F^s_{p,2}(\mathbb{R}^d;X) = H^{s,p}(\mathbb{R}^d;X)$ with equivalent norms;*
(3) *X is isomorphic to a Hilbert space.*

Proof. (1)⇒(3) and (2)⇒(3): By Proposition 14.7.8, X has type 2 and cotype 2. Therefore X is isomorphic to a Hilbert space by Theorem 7.3.1.

(3)⇒(2): This is immediate from Proposition 14.7.6 and the fact that Hilbert spaces are UMD (by Theorem 4.2.14) and have type 2 and cotype 2 (by the result of Example 7.1.2).

(3)⇒(1): This is a special case of the previous implication since Theorem 5.6.11 implies $W^{m,p}(\mathbb{R}^d;X) = H^{m,p}(\mathbb{R}^d;X)$ with equivalent norms. □

14.7.c Interpolation

Real interpolation of vector-valued Bessel potential spaces has already been considered in Theorem 14.4.31. Complex interpolation was considered in Theorem 5.6.9, but only in the case $p_0 = p_1$ and $X_0 = X_1$. In order to treat a more general case we need a variant of the complex interpolation results for $\ell^p_{w_s}(X)$ of Proposition 14.3.3.

Let $(\varepsilon_k)_{k\geqslant 0}$ be a Rademacher sequence on a probability space Ω. Let $p \in (1,\infty)$ and $s \in \mathbb{R}$, and let $\varepsilon^{s,p}(X)$ denote the space of all sequences $(x_k)_{k\geqslant 0}$ in X for which

$$\big\|(x_k)_{k\geqslant 0}\big\|_{\varepsilon^{s,p}(X)} := \sup_{n\geqslant 1}\Big\|\sum_{k=0}^{n}\varepsilon_k 2^{ks}x_k\Big\|_{L^p(\Omega;X)} < \infty.$$

The spaces $\varepsilon^p(X) := \varepsilon^{0,p}(X)$ have been introduced in Section 6.3. Clearly the mapping $(x_k)_{k\geqslant 0} \mapsto (2^{ks}x_k)_{k\geqslant 0}$ defines an isometric isomorphism from $\varepsilon^{s,p}(X)$ onto $\varepsilon^p(X)$. For fixed $s \in \mathbb{R}$ the spaces $\varepsilon^{s,p}(X)$, $1 < p < \infty$, coincide, with pairwise equivalent norms; this follows from the Kahane–Khintchine inequalities as in Proposition 6.3.1. If X does not contain a copy isomorphic to c_0, then Corollary 6.4.12 implies that for any $(x_k)_{k\geqslant 0}$ in $\varepsilon^{s,p}(X)$ the sum $\sum_{k\geqslant 0}\varepsilon_k 2^{ks}x_k$ converges in $L^p(\Omega;X)$ and almost surely in X, and in this case

$$\big\|(x_k)_{k\geqslant 0}\big\|_{\varepsilon^{s,p}(X)} = \Big\|\sum_{k\geqslant 0}\varepsilon_k 2^{ks}x_k\Big\|_{L^p(\Omega;X)}.$$

In particular, the partial sum projections $P_n : (x_k)_{k\geqslant 0} \mapsto (x_k)_{k=0}^n$ are uniformly bounded and strongly convergent to the identity as operators on $\varepsilon^{s,p}(X)$.

The next result extends Theorem 7.4.16, which corresponds to the special case $s = 0$.

Lemma 14.7.10. *For $j \in \{0,1\}$ let X_j be a K-convex space and let $p_j \in (1,\infty)$. For $\theta \in (0,1)$ set $X_\theta := [X_0, X_1]_\theta$. Then*

$$[\varepsilon^{s_0,p_0}(X_0), \varepsilon^{s_1,p_1}(X_1)]_\theta = \varepsilon^{s,p}(X_\theta),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s = (1-\theta)s_0 + \theta s_1$.

Proof. By Proposition 7.4.15, X_θ is K-convex. By Proposition 7.4.5 and Lemma 7.4.11, X_θ, does not contain an isomorphic copy of c_0, and hence the partial sum projections P_n on $\varepsilon^{s,p}(X_\theta)$ are strongly convergent to the identity.

To prove the required identity one can repeat the argument in Theorem 14.3.1 to reduce the result to the unweighted setting considered in Theorem 7.4.16. □

As a final preparation for the complex interpolation of Bessel potential spaces, we prove a version of Lemma 14.4.29 for Bessel potential spaces.

Lemma 14.7.11. *Let X be a UMD space and let $p \in [1,\infty]$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. For $k \geqslant 0$ set $\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$. The operators*

$$R : \varepsilon^{s,p}(L^p(\mathbb{R}^d;X)) \to H^{s,p}(\mathbb{R}^d;X)$$
$$S : H^{s,p}(\mathbb{R}^d;X) \to \varepsilon^{s,p}(L^p(\mathbb{R}^d;X))$$

defined by

$$R(f_k)_{k\geqslant 0} = \sum_{k\geqslant 0} \psi_k * f_k, \quad Sf = (\varphi_k * f)_{k\geqslant 0},$$

are bounded and satisfy $RS = I$.

Proof. The identity $RS = I$ is proved as in Lemma 14.4.29. The boundedness of S follows from Theorem 14.7.5. It remains to prove that R is bounded. Let $E := L^p(\Omega; L^p(\mathbb{R}^d;X))$. By Theorem 14.7.5 and a density argument it suffices to show that, for all finitely non-zero sequences $(f_\ell)_{\ell\geqslant 0}$ in $L^p(\mathbb{R}^d;X)$,

$$\Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * \sum_{j\geqslant 0} \psi_j * f_j \Big\|_E \leqslant C \Big\| \sum_{k\geqslant 0} \varepsilon_k 2^{ks} f_k \Big\|_E, \quad n \geqslant 0.$$

From Theorem 14.7.5 (with $s = 0$) and Proposition 8.4.6(i) we see that the sequence $\{\varphi_k * : k \geqslant 0\}$ is R-bounded in $\mathscr{L}(L^p(\mathbb{R}^d;X))$, with R-bound at most by $C_{p,X}$. Hence also the sequence $\{\psi_k * : k \geqslant 0\}$ is R-bounded in this space, with R-bound at most $3C_{p,X}$. Therefore, by the Fourier support properties (14.8) and (14.9) of φ_k,

$$\Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * \sum_{j\geqslant 0} \psi_j * f_j \Big\|_E \leqslant \sum_{|\ell|\leqslant 2} \Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * \psi_{k+\ell} * f_{k+\ell} \Big\|_E$$

$$\leqslant 3C_{p,X}^2 \sum_{|\ell\leqslant 2} \Big\| \sum_{k=0}^{n} \varepsilon_k 2^{ks} f_{k+\ell} \Big\|_E$$

$$\leqslant 3C_{p,X}^2 4^{|s|} \Big\| \sum_{k\geqslant 0} \varepsilon_k 2^{ks} f_k \Big\|_E,$$

where in the last step we used Kahane's contraction principle. □

Theorem 14.7.12 (Complex interpolation of Bessel potential spaces). *Let (X_0, X_1) be an interpolation couple of UMD Banach spaces and let $p_0, p_1 \in (1, \infty)$, $s_0, s_1 \in \mathbb{R}$, and $\theta \in (0, 1)$. Then*

$$[H^{s_0,p_0}(\mathbb{R}^d; X_0), H^{s_1,p_1}(\mathbb{R}^d; X_1)]_\theta = H^{s,p}(\mathbb{R}^d; X_\theta) \quad \textit{with equivalent norms,}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s = (1-\theta)s_0 + \theta s_1$, and $X_\theta = [X_0, X_1]_\theta$.

Proof. Let R and S be the operator of Lemma 14.7.11. Let

$$E_j := \varepsilon^{s_j,p_j}(L^{p_j}(\mathbb{R}^d; X_j)), \quad F_j := H^{s_j,p_j}(\mathbb{R}^d; X_j), \quad j \in \{0, 1\},$$

and set $E_\theta := (E_0, E_1)_{\theta,q}$ and $F_\theta := (F_0, F_1)_{\theta,q}$. Then, by Theorem 2.2.6 and Lemma 14.7.10, $E_\theta = \varepsilon^{s,p}(L^p(\mathbb{R}^d; X_\theta))$ isomorphically. Now the proof can be completed in the same way as in Theorem 14.4.30, replacing $\ell^q_{w_s}$ by $\varepsilon^{s,p}$ and $B^s_{p,q}$ by $H^{s,p}$ everywhere. □

Theorem 14.7.12 contains several results of Volume I as special cases. To begin with, it contains Theorem 5.6.9, which asserts that if X is a UMD space, $p \in (1, \infty)$, and $s_0 < s_1$, then

$$[H^{s_0,p}(\mathbb{R}^d; X), H^{s_1,p}(\mathbb{R}^d; X)]_\theta = H^{s_\theta,p}(\mathbb{R}^d; X)$$

and, if in addition $s \geqslant 0$,

$$[L^p(\mathbb{R}^d; X), H^{s,p}(\mathbb{R}^d; X)]_\theta = H^{\theta s,p}(\mathbb{R}^d; X)$$

up to equivalent norms. It also contains Theorem 5.6.1, which asserts that if X is a UMD space, $p \in (1, \infty)$, and $k \geqslant 1$ is an integer, then

$$[L^p(\mathbb{R}^d; X), W^{k,p}(\mathbb{R}^d; X)]_\theta = H^{\theta k,p}(\mathbb{R}^d; X)$$

up to an equivalent norm. This result is obtained by taking $X_0 = X_1 = X$, $p_0 = p_1 = p$, $s_0 = 0$, and $s_1 = k$ in Theorem 14.7.12 and noting that $H^{k,p}(\mathbb{R}^d; X) = W^{k,p}(\mathbb{R}^d; X)$ up to equivalent norm by Theorem 5.6.11.

Upon combining Theorem 14.7.12 with Theorem 5.6.11 we obtain another extension of Theorem 5.6.1:

Corollary 14.7.13 (Complex interpolation for Sobolev spaces). *Let (X_0, X_1) an interpolation couple of UMD Banach spaces and let $p_0, p_1 \in (1, \infty)$, $k_0, k_1 \in \mathbb{N}$, and $\theta \in (0, 1)$. Then*

$$[W^{k_0,p_0}(\mathbb{R}^d; X_0), W^{s_1,p_1}(\mathbb{R}^d; X_1)]_\theta = H^{k_\theta,p}(\mathbb{R}^d; X_\theta) \quad \textit{with equivalent norms,}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $k_\theta = (1-\theta)k_0 + \theta k_1$, and $X_\theta = [X_0, X_1]_\theta$.

As in Examples 14.4.33 and 14.4.35, we can use this corollary to prove boundedness of pointwise multiplication by smooth functions:

Example 14.7.14 (Pointwise multiplication by smooth functions – I). Let X and Y be UMD spaces, let $p \in [1, \infty]$ and $s \in \mathbb{R}$, and let $k \in [s, \infty) \cap \mathbb{N}$ be an integer. If $\zeta \in C_{\rm b}^k(\mathbb{R}^d; \mathscr{L}(X, Y))$, then pointwise multiplication

$$f \mapsto \zeta f$$

defines a bounded mapping from $H^{s,p}(\mathbb{R}^d; X)$ into $H^{s,p}(\mathbb{R}^d; Y)$ of norm $\lesssim_{k,s} \|\zeta\|_{C_{\rm b}^k(\mathbb{R}^d;\mathscr{L}(X,Y))}$.

Indeed, the pointwise multiplier $f \mapsto \zeta f$ is bounded as a mapping from $W^{j,p}(\mathbb{R}^d; X)$ into $W^{j,p}(\mathbb{R}^d; Y)$ for each $j \in \{0, \dots, k\}$. Therefore, for $s \in \mathbb{N}$ the result is immediate from Theorem 5.6.11. If $-s \in \mathbb{N}$, then the result follows by the duality result of Proposition 5.6.7 and Theorem 5.6.11. If $s \in (0, \infty)$, then the result follows by interpolation between the cases $j = 0$ and $j = k$ by the complex method $[\cdot, \cdot]_{\frac{s}{k}}$ and applying Theorem C.2.6 and Corollary 14.7.13. Finally, the case $s \in (-\infty, 0)$ follows by duality again.

14.7.d Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$ in $H^{s,p}$

To conclude this section we present a result on pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$ for vector-valued Bessel potential spaces. The cases of vector-valued Besov spaces and Triebel–Lizorkin space have been considered in Section 14.6.h; in both cases, values in general Banach spaces X could be allowed. In the Bessel potential case, the proof below requires the UMD property of the range space X. It seems to be an open problem whether this conditions is actually necessary.

Theorem 14.7.15 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in (1, \infty)$ and $s \in (-1/p', 1/p)$, and let X be a UMD space. For all $f \in H^{s,p}(\mathbb{R}; X)$ we have $\mathbf{1}_{\mathbb{R}_+} f \in H^{s,p}(\mathbb{R}; X)$ and*

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{H^{s,p}(\mathbb{R};X)} \leqslant C \|f\|_{H^{s,p}(\mathbb{R};X)}, \quad f \in H^{s,p}(\mathbb{R}; X).$$

The UMD property of X will only be used through the following proposition.

Proposition 14.7.16. *Let $p \in (1, \infty)$ and $s > 0$, and let X be a UMD space.*

(1) *The operator $(-\Delta)^s : \mathscr{S}(\mathbb{R}^d; X) \to \mathscr{S}'(\mathbb{R}^d; X)$ given by*

$$(-\Delta)^s f = |2\pi \cdot|^s \widehat{f}$$

uniquely extends to $(-\Delta)^s \in \mathscr{L}(H^{s,p}(\mathbb{R}^d; X), L^p(\mathbb{R}^d; X))$.

(2) *For all $f \in H^{s,p}(\mathbb{R}^d; X)$ the following norm equivalence holds*

$$\|f\|_{H^{s,p}(\mathbb{R}^d;X)} \eqsim_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)} + \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^d;X)}.$$

Proof. (1): Let $m_1(\xi) = \frac{|2\pi\xi|^s}{(1+|2\pi\xi|^2)^{s/2}}$. Using Mihlin's multiplier Theorem 5.5.10 one can check that $m_1 \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$. Therefore,

$$\|(-\Delta)^s f\|_p = \|T_{m_1} J_s f\|_p \leqslant \|m_1\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \|J_s f\|_p \leqslant C_{p,X} \|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

(2): Note that since $s > 0$, Proposition 5.6.6 gives that $H^{s,p}(\mathbb{R}^d; X) \hookrightarrow L^p(\mathbb{R}^d; X)$ contractively. This combined with (1) gives the estimate "$\gtrsim$".

The estimate $\lesssim$ follows similarly. Let $m_2(\xi) = \frac{(1+|2\pi\xi|^2)^{s/2}}{1+|2\pi\xi|^s}$. Then $m_2 \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ as before. Therefore,

$$\begin{aligned} \|f\|_{H^{s,p}(\mathbb{R}^d;X)} &= \|T_{m_2}(I + (-\Delta)^{s/2})f\|_p \\ &\leqslant \|m_2\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} (\|f\|_p + \|(-\Delta)^{2/s} f\|_p. \end{aligned}$$

□

We need two more preparatory results. The first one is a concrete formula for $(-\Delta)^{s/2} f$ as an integral operator.

Lemma 14.7.17. *Let $s \in (0, 1)$. For $f \in \mathscr{S}(\mathbb{R}; X)$ we have*

$$(-\Delta)^{s/2} f = c_s \int_{\mathbb{R}} \frac{f(\cdot + h) - f(\cdot)}{|h|^{1+s}} dh, \quad x \in \mathbb{R},$$

where the integral on the right-hand side converges absolutely pointwise $\mathbb{R}$, and as a Bochner integral in $L^p(\mathbb{R}; X)$ for any $p \in [1, \infty)$. Here $c_s \in \mathbb{R} \setminus \{0\}$ is a constant only depending on s.

Proof. The convergence of the integral for $|h| > 1$ is immediate. The convergence for $|h| < 1$ follows by writing $f(x + h) - f(x) = \int_0^1 f'(x + th)h \, dt$.

To prove the stated identity we take Fourier transforms on the right-hand side and use Fubini's theorem to obtain

$$\mathscr{F} \int_{\mathbb{R}} \frac{f(\cdot + h) - f(\cdot)}{|h|^{1+s}} \, dh \, dx = \int_{\mathbb{R}} \frac{e^{2\pi i h\xi} - 1}{|h|^{1+s}} \widehat{f}(\xi) \, dh = k_s |\xi|^s \widehat{f}(\xi),$$

where from the fact that the odd part of the integral cancels we see that $k_s = 2\int_{\mathbb{R}_+} \frac{\cos(2\pi t)-1}{t^{1+s}} \, dt$ is in $(-\infty, 0)$. This proves the result with constant $c_s = k_s^{-1}(2\pi)^s$. □

We also need the following inequality.

Lemma 14.7.18 (Hilbert absolute inequality). *Let $p \in (1, \infty)$. For $f \in L^p(\mathbb{R}_+)$ one has*

$$\Big\| x \mapsto \int_{\mathbb{R}_+} \frac{|f(y)|}{x + y} \, dy \Big\|_{L^p(\mathbb{R}_+)} \leqslant C_p \|f\|_{L^p(\mathbb{R}_+)}.$$

Proof. Letting $\zeta_p(y) = \frac{x^{1/p}}{x+1}$, after rewriting the integral, we can use Young's inequality for the multiplicative group $\mathbb{R}_+$ with Haar measure $\frac{\mathrm{d}x}{x}$ to obtain

$$\Big\| x \mapsto \int_{\mathbb{R}_+} \frac{|f(y)|}{x+y}\,\mathrm{d}y \Big\|_{L^p(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}_+} \zeta_p(x/y) y^{1/p} f(y)\,\frac{\mathrm{d}y}{y}\Big)^p \frac{\mathrm{d}x}{x}\Big)^{1/p}$$
$$\leqslant \|\zeta_p\|_{L^1(\mathbb{R}_+,\frac{\mathrm{d}x}{x})}\|f\|_{L^p(\mathbb{R}_+)}.$$

□

Proof of Theorem 14.7.15. By Proposition 14.6.17 it suffices to prove the desired estimate for f in the dense class $C_c^\infty(\mathbb{R}\setminus\{0\})\otimes X$. In that case one actually has $g := \mathbf{1}_{\mathbb{R}_+} f$ is in the same class and thus is smooth as well.

We claim that

$$\|(-\Delta)^{s/2} g\|_p \leqslant \|(-\Delta)^{s/2} f\|_p + C_{p,s}\|f\|_{H^{s,p}(\mathbb{R};X)}. \tag{14.97}$$

As soon as we proved the claim, then the result follows. Indeed, applying Proposition 14.7.16 twice we obtain

$$\begin{aligned}\|g\|_{H^{s,p}(\mathbb{R};X)} &\eqsim_{p,X} \|g\|_p + \|(-\Delta)^{s/2} g\|_p \\ &\overset{(14.97)}{\leqslant} \|f\|_p + \|(-\Delta)^{s/2} f\|_p + C_{p,s}\|f\|_{H^{s,p}(\mathbb{R};X)}. \\ &\eqsim_{p,X} \|f\|_{H^{s,p}(\mathbb{R};X)}.\end{aligned}$$

To rewrite $(-\Delta)^{s/2} g$ in a suitable way, let

$$S := \{(x,h)\in\mathbb{R}^2 : (x>0 \text{ and } h<-x) \text{ or } (x<0 \text{ and } h>-x)\}.$$

Then applying Lemma 14.7.17 twice, by elementary considerations we see that for all $x\in\mathbb{R}$,

$$\begin{aligned}(-\Delta)^{s/2} g(x) &= c_s \int_{\mathbb{R}} \frac{g(x+h)-g(x)}{|h|^{1+s}} dh \\ &= c_s \int_{\mathbb{R}} \frac{f(x+h)-f(x)}{|h|^{1+s}} dh - c_s \operatorname{sgn}(x) \int_{\mathbb{R}} \mathbf{1}_S(x,h)\frac{f(x+h)}{|h|^{1+s}}\,\mathrm{d}h \\ &= (-\Delta)^{s/2} f(x) - c_s \operatorname{sgn}(x) \int_{\mathbb{R}} \mathbf{1}_S(x,h)\frac{f(x+h)}{|h|^{1+s}}\,\mathrm{d}h.\end{aligned}$$

Taking L^p-norms, we see that (14.97) holds if we can show that

$$\Big\| x\mapsto \int_{\mathbb{R}} \mathbf{1}_S(x,h)\frac{\|f(x+h)\|}{|h|^{1+s}}\,\mathrm{d}h\Big\|_{L^p(\mathbb{R})} \lesssim_{p,s} \|f\|_{H^{s,p}(\mathbb{R};X)}. \tag{14.98}$$

To prove (14.98) we only consider the part $L^p(\mathbb{R}_+)$ as the other one is similar. By elementary considerations

$$\int_0^\infty \Big(\int_{\mathbb{R}} \mathbf{1}_S(x,h)\frac{\|f(x+h)\|}{|h|^{1+s}}\,\mathrm{d}h\Big)^p \mathrm{d}x = \int_0^\infty \Big(\int_{-\infty}^{-x} \frac{\|f(x+h)\|}{|h|^{1+s}}\,\mathrm{d}h\Big)^p \mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^\infty \Big(\int_0^\infty \frac{\|f(-y)\|}{(y+h)^{1+s}} \,\mathrm{d}h \Big)^p \,\mathrm{d}y \\
&\leqslant \int_0^\infty \Big(\int_0^\infty \frac{y^{-s}\|f(-y)\|}{y+h} \,\mathrm{d}h \Big)^p \,\mathrm{d}y \\
&\overset{(i)}{\leqslant} C_p^p \|y \mapsto |y|^{-s} f(y)\|_{L^p(\mathbb{R};X)}^p \\
&\overset{(ii)}{\leqslant} C_p^p C_{p,s}^p \|f\|_{H^{s,p}(\mathbb{R};X)}^p,
\end{aligned}
$$

where in (i) we applied Lemma 14.7.18 to the function $y \mapsto y^{-s}\|f(-y)\|$, and (ii) follows from Corollary 14.6.31(2). This completes the proof of the remaining estimate (14.98). □

14.8 Notes

Early influential monographs on function spaces are those of Adams [1975] (see also Adams and Fournier [2003]), Bergh and Löfström [1976], Peetre [1976], and Triebel [1978]. After these works appeared, a new maximal function argument was discovered by Peetre [1975] which made it possible to study Besov and Triebel–Lizorkin spaces in the full range $p, q \in (0, \infty]$. This theory is presented in detail in the monograph of Triebel [1983] and the more recent works of Triebel [1992, 2006, 2020, 2013, 2014]; further expositions are due to Bahouri, Chemin, and Danchin [2011], Denk and Kaip [2013], Grafakos [2009], Maz'ya [2011], Runst and Sickel [1996], and Sawano [2018].

Standard references for function spaces in the vector-valued setting include the works of Amann [1995, 1997, 2019], Triebel [1997], König [1986], Schmeisser [1987], Schmeisser and Sickel [2001], and Schmeisser and Sickel [2005]. A unified treatment of Besov and Triebel–Lizorkin spaces and related classes of function spaces is given by Lindemulder [2021], where the axiomatic setting of Hedberg and Netrusov [2007] is extended to the vector-valued context. In particular, this covers the weighted and anisotropic settings, and it allows for Banach function space other than the spaces $\ell^q(L^p)$ or $L^p(\ell^q)$ employed in the construction of the Besov and Triebel–Lizorkin spaces.

The theory of function spaces is a vast topic, and by necessity our treatment does not cover a number of important topics such as approximation theory, wavelets, atomic decompositions, weighted spaces, paraproducts, anisotropic spaces, and typical aspects for bounded domains and manifolds such as traces, extension operators, boundary values, and interpolation with boundary conditions (although some of these topics will be briefly visited in these notes). Of the omitted themes, we specifically mention the ϕ-transform of Frazier and Jawerth [1990], which allows the identification of Besov and Triebel–Lizorkin spaces with subspaces of appropriate discrete sequence spaces. In this identification, the question of boundedness of various operators on the original function spaces is transformed into the question of boundedness of infinite matrices on the corresponding sequence spaces, which in turn can be deduced from natural almost diagonality estimates of these matrices, in certain analogy with our proof of the $T(1)$ theorem on $L^p(\mathbb{R}^d; X)$ spaces through estimates of the matrix coefficients of T with respect to the Haar basis. This approach lies behind many of the proofs of $T(1)$ theorems in Besov and Triebel–Lizorkin spaces that we discussed in the Notes of Chapter 12.

The 'classical' Besov and Triebel–Lizorkin spaces considered in this chapter are modelled on the gradient ∇ in the setting of $\mathbb{R}^d$. It is possible to introduce Besov and Triebel–Lizorkin spaces based on different types of sectorial operators and to study them in the setting of manifolds; we refer to Batty and Chen [2020], Haase [2006], Kriegler and Weis [2016], Kunstmann and Ullmann [2014], Taylor [2011a], Taylor [2011b], Taylor [2011c], Taylor [1974], and Voigtlaender [2022].

Section 14.2

Lemma 14.2.1 is taken from Amann [1995]. The other results of this section are standard in the scalar-valued case, and their extensions to the vector-valued setting are straightforward.

Section 14.3

The complex and real interpolation results for vector-valued and weighted L^q-spaces of Theorems 14.3.1 and 14.3.4 extend Theorems 2.2.6 and 2.2.10, where the unweighted case was treated. The scalar-valued case goes back to Stein and Weiss [1958], and the extension to the vector-valued weighted setting is well-known, at least for complex interpolation. The case of real interpolation is included in the work of Kreĭn, Petunīn, and Semënov [1982], and a different approach based on Stein interpolation for the real method is due to Lindemulder and Lorist [2022]. The interpolation results for $q_0 = q_1 = \infty$ are false in general. Indeed, already Triebel [1978, 1.18.1] gave an example where $[\ell^\infty_{w_{s_0}}(X_0), \ell^\infty_{w_{s_1}}(X_1)]_\theta \neq \ell^\infty_{w_s}([X_0, X_1]_\theta)$ with $w_s(n) = 2^{ns}$. Propositions 14.3.3 and 14.3.5 are presented by Triebel [1978], who attributes the real case to Peetre [1967]. More generally, Triebel [1978, Section 1.18] identifies the complex and real interpolation spaces of $\ell^{p_0}((X_j)_{j\geqslant 1})$ and $\ell^{p_1}((Y_j)_{j\geqslant 1})$ for $p_0, p_1 < \infty$ and for sequences of interpolation couples $(X_j, Y_j)_{j\geqslant 1}$; here $\ell^p((Z_j)_{j\geqslant 1})$ is the space of all sequences $(z_j)_{j\geqslant 1}$ with $z_j \in Z_j$ such that $(\|z_j\|_{Z_j})_{j\geqslant 1}$ belongs to ℓ^p, $Z \in \{X, Y\}$. Proposition 14.3.3 then follows by taking $X_j = 2^{js}X$ and $X_j = 2^{js}Y$. It seems that Proposition 14.3.5 can only be stated for a single space X unless further assumptions on q_0 and q_1 are made.

Section 14.4

Our introduction of vector-valued Besov spaces is self-contained up to a modest number of prerequisites from earlier chapters. Part of the section follows the presentation by Schmeisser and Sickel [2001]. For the history of Besov spaces, we refer the reader to Bergh and Löfström [1976] and Triebel [1978, 1983]. Besov spaces appear naturally as real interpolation spaces between L^p and $W^{k,p}$ (see Theorem 14.4.31). As such, they have important applications in the theory of evolution equations (see Chapter 18). Moreover, by choosing the microscopic parameter q suitably, one can often include end-point cases into the considerations.

In contrast to the theory of the spaces $W^{k,p}(\mathbb{R}^d;X)$ and $H^{s,p}(\mathbb{R}^d;X)$, where assumptions on the space X such as the Radon–Nikodým property or the UMD property are often needed, many key results on vector-valued Besov spaces hold for general Banach spaces X.

Lemma 14.4.5 on the sequential completeness of $\mathscr{S}'(\mathbb{R}^d;X)$ is a standard result. It is possible to endow the space $C_c^\infty(U;X)$ with a complete locally convex topology in such a way that sequential convergence in this topology coincides with the *ad hoc* notion of sequential convergence used here. A detailed construction is presented by Rudin [1991].

Fourier multipliers

Fourier multipliers for vector-valued Besov spaces have been discussed by Amann [1997], Weis [1997], Girardi and Weis [2003a], Hytönen [2004], and Hytönen and Weis [2006a]. In Theorem 14.4.16, we only considered smooth m, and this restriction was removed in Theorem 14.5.6. The latter result and related ones can be found in the work of Girardi and Weis [2003a], who showed that the operator T is a continuous extension (with respect to a weaker topology) of T_m also if $\max\{p,q\} = \infty$. Fourier multipliers for vector-valued Besov spaces have been applied by Weis [1997] to obtain sharp exponential stability results of C_0-semigroups in spaces with Fourier type p.

Embedding

The sandwich result of Proposition 14.4.18 is very useful in avoiding additional conditions on the Banach space X. The Sobolev embedding result of Theorem 14.4.19 is standard. Especially the sufficiency is simple to prove via Lemma 14.4.20. For the proof of this lemma and its extension to all $0 < p_0 < p_1 < \infty$ in Remark 14.6.4, we follow Schmeisser and Sickel [2001].

Difference norms

The difference norm characterisation of Besov spaces can be found in many places. It was already used before the Fourier analytic description of Besov spaces was given. We refer the reader to Bergh and Löfström [1976], Triebel [1983], and references therein for historical details. The difference norms have the advantage that in certain cases one can check by hand whether a given function belongs to some given Besov space. By choosing the parameter τ in Theorem 14.4.24 appropriately, the Besov spaces can be identified with other classical spaces, as we have done in Corollaries 14.4.25 and 14.4.26 for $W^{s,p}$ and C^s_{ub}.

In Step 1 of the proof of Theorem 14.4.24 we follow the presentation of Bergh and Löfström [1976], where the case $\tau = \infty$ was given. Step 2 of the proof is based on the presentation of Schmeisser and Sickel [2001].

Interpolation

Interpolation of Besov spaces is discussed by Bergh and Löfström [1976], König [1986], and Triebel [1978, 1983]; further references to the literature can be found in these works. The method to reduce the proofs to interpolation of $\ell^q(L^p)$-spaces fits into a more general retraction–co-retraction scheme explained by [Triebel, 1978, Theorem 1.2.4].

The complex interpolation result of Theorem 14.4.30 is folklore, although we are not aware of a reference containing the general form with an interpolation couple (X_0, X_1) presented here. In the special case $X = X_0 = X_1$, the theorem can be proved in the same way as in the scalar-valued case, and some end-point results are valid as well. For instance, we have

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d; X), B^{s_1}_{p_1,q_1}(\mathbb{R}^d; X)]_\theta = B^s_{p,q}(\mathbb{R}^d; X), \quad p_j, q_j \in [1, \infty],\ s_j \in \mathbb{R},$$

with equivalent norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad s = (1-\theta)s_0 + \theta s_1.$$

The real interpolation result of Theorem 14.4.31 is well known, and the proof is a simple generalisation of the standard proof for the scalar-valued case. Several other real interpolation results can be proved with the same methods. For instance, if $\min\{p_0, p_1\} < \infty$, $\min\{q_0, q_1\} < \infty$, and $s_0, s_1 \in \mathbb{R}$, then

$$(B^{s_0}_{p_0,q_0}(\mathbb{R}^d; X_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^d; X_1))_{\theta,p} = B^s_{p,p}(\mathbb{R}^d; (X_0, X_1)_{\theta,p}),$$

with equivalent norms, where again $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $s = (1-\theta)s_0 + \theta s_1$. This follows Theorem 14.3.4 in a similar way as in Theorem 14.4.30.

Duality

In Theorem 14.4.34, we identified the dual of $B^s_{p,q}(\mathbb{R}^d; X)$ with respect to the duality for $\mathscr{S}(\mathbb{R}^d; X)$ and $\mathscr{S}'(\mathbb{R}^d; X)$. Unlike in the L^p-setting treated in Section 1.3, no conditions on X are needed. A result of this type in a more general abstract setting (including weights and anisotropic function spaces) is presented by Lindemulder [2021]. The proof that we have given follows Agresti, Lindemulder, and Veraar [2023].

Section 14.5

The characterisations in Theorem 14.5.1 of type and cotype in terms of embedding properties of Besov spaces into spaces of γ-radonifying operators are due to Kalton, Van Neerven, Veraar, and Weis [2008]. This paper also contains the γ-Bernstein–Nikolskii inequality of Lemma 14.5.2, as well as optimal embedding results for the smooth spaces $\gamma(H^{-s,2}(\mathbb{R}^d); X)$. The consequences for Bessel potential spaces discussed in Corollary 14.7.7 are taken from Veraar [2013]. This work also contained the following result:

Theorem 14.8.1. *Let X be a Banach lattice, and $1 \leqslant p \leqslant 2 \leqslant q < \infty$. If X is p-convex and q-concave, then*

$$H^{(\frac{1}{p}-\frac{1}{2})d,p}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X),$$
$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow H^{(\frac{1}{q}-\frac{1}{2})d,q}(\mathbb{R}^d; X).$$

It is an open problem to characterise the Banach spaces for which these embeddings hold (see Problem Q.14).

Mapping properties of the Fourier transform

The mapping properties of the vector-valued Fourier transform $\mathscr{F}$ for Banach spaces X with Fourier type p contained in Proposition 14.5.3 appear in the papers by García-Cuerva, Kazaryan, Kolyada, and Torrea [1998], König [1991], and Girardi and Weis [2003a]. Real interpolation of the end-point cases $q = p$ and $q = \infty$ in Proposition 14.5.3 gives an alternative proof of some of the results in the papers just mentioned:

Theorem 14.8.2. *Suppose that X has Fourier type $p \in (1, 2]$. Let $q \in (p, \infty)$, $r \in [1, \infty]$, and $s = \frac{d}{p} - \frac{d}{q}$. Then $\mathscr{F}$ is bounded from $B^s_{p,r}(\mathbb{R}^d; X)$ into the Lorentz space $L^{q',r}(\mathbb{R}^d; X)$.*

Proposition 14.5.3 contains a parallel result under the assumption that X has type p and cotype 2. Recall from Proposition 13.1.35 that, under these assumptions, X has Fourier type r for any $r \in [1, p)$.

The mapping properties of the Fourier transform on vector-valued L^p-spaces with power weights have been recently studied by Dominguez and Veraar [2021], who show that a version of the classical Pitt inequalities holds if and only if X has non-trivial Fourier type. In particular, the following result was proved:

Theorem 14.8.3. *Let X be of Fourier type $p_0 \in (1, 2]$. Let $1 < p \leqslant q < \infty$ and $\beta, \gamma \geqslant 0$. If*

$$\max\left\{0, d\left(\frac{1}{\min\{p, p_0\}} + \frac{1}{q} - 1\right)\right\} < \gamma < \frac{d}{q} \quad \textit{and} \quad \beta - \gamma = d\Big(1 - \frac{1}{p} - \frac{1}{q}\Big),$$

then $\mathscr{F}$ extends boundedly from $L^p(\mathbb{R}^d, |\cdot|^{\beta p}; X)$ into $L^q(\mathbb{R}^d, |\cdot|^{-\gamma q}; X)$.

In the limiting case $\gamma = \max\{0, d(\frac{1}{\min\{p,p_0\}} + \frac{1}{q} - 1)\}$, the above boundedness of $\mathscr{F}$ still holds true under further restrictions on p and q. Surprisingly, if X has non-trivial Fourier type (equivalently, by Theorem 13.1.33, non-trivial type), one can allow $p = q = 2$ by choosing the weights suitably. A similar result holds in the periodic setting, but the problem is open for more general orthogonal systems that have been considered by Stein [1956].

R-boundedness

R-boundedness of smooth operator-valued functions is studied by Girardi and Weis [2003c] under Fourier type conditions, and by Hytönen and Veraar [2009] under (co)type conditions; the latter paper contains Theorems 14.5.8 and 14.5.9.

Section 14.6

In this section, we followed part of the presentation of Schmeisser and Sickel [2001]. For a detailed description of the history of Triebel–Lizorkin spaces, we refer the reader to Bergh and Löfström [1976], and Triebel [1978, 1983]. Below, we only discuss those aspects of Triebel–Lizorkin spaces that are specific for this class of spaces.

Triebel–Lizorkin spaces $F^s_{p,q}$ were originally introduced as a natural variant of Besov spaces, with the roles of L^p and ℓ^q interchanged in the definition. The special case $q = 2$ leads to the equality $F^s_{p,2} = H^{s,p}$ with equivalent norms for $p \in (1, \infty)$, and in the early days of the theory the cases $q \neq 2$ were mostly studied for reasons of mathematical curiosity. The definition of Triebel–Lizorkin spaces given here does not cover the spaces $F^s_{\infty,q}$. The latter are known to be connected to BMO spaces, and require a modification of the definition for which we refer to Triebel [1983]. These spaces are naturally contained, as $F^s_{\infty,q} = F^{s,1/p}_{p,q}$ for any $p \in (0, \infty)$, in the general framework of *Triebel–Lizorkin-type spaces* $F^{s,\tau}_{p,q}$ with a fourth parameter $\tau \in [0, \infty)$, which has been introduced by Yang and Yuan [2008] and studied in several subsequent works.

Genesis of (vector-valued) Triebel–Lizorkin spaces

Vector-valued Triebel–Lizorkin spaces are needed for the treatment of parabolic boundary value problems in the spaces $L^p(0, T; L^q(\mathbb{R}^d_+))$. Such applications first appeared in the works of Weidemaier [2002] for $q \leqslant p$ and scalar second order equations with inhomogeneous Dirichlet boundary conditions, and of Denk, Hieber, and Prüss [2007] for $p, q \in (1, \infty)$ and more general systems and boundary conditions. Kunstmann [2015] introduced a new interpolation method $(\cdot, \cdot)_{\theta,\ell^q}$ and shows that $F^s_{p,q} = (L^p, W^{k,p})_{s/k,\ell^q}$ with equivalent norms. This interpolation method fits into the axiomatic setting of discrete interpolation recently developed by Lindemulder and Lorist [2021].

As in the Besov space case, results for vector-valued Triebel–Lizorkin spaces typically hold without restrictions on the target Banach space X. Thanks to the sandwich result

$$B^s_{p,1} \hookrightarrow F^s_{p,1} \hookrightarrow H^{s,p} \hookrightarrow F^s_{p,\infty} \hookrightarrow B^s_{p,\infty},$$

one can sometimes deduce results about vector-valued Bessel potential spaces as well. Within the Triebel–Lizorkin scale, one can get closer to $H^{s,p}$ than in the Besov scale, which often makes Triebel–Lizorkin spaces more useful. For instance, the sandwich result can be combined with the Sobolev Embedding Theorem 14.6.14, which allows arbitrary microscopic improvement for Triebel–Lizorkin spaces. Further flexibility in sandwiching and embedding theorems can be built in by introducing weights such as $|x|^\gamma$ or $|x_1|^\gamma$ as was done by Meyries and Veraar [2012, 2014a].

The boundedness of the Peetre maximal function proved in Proposition 14.6.2 appears in the book of Triebel [1997]. This proposition extends results of Triebel [1983, Theorem 1.6.3] and Triebel [1997, Formula 15.3(iv)] to the vector-valued setting.

Theorems 14.6.3 and 14.6.11 are presented by Triebel [1997] for scalar-valued multipliers m. An operator-valued extension is due to Bu and Kim [2005].

Gagliardo–Nirenberg inequalities and Sobolev embedding

The Gagliardo–Nirenberg inequalities of Proposition 14.6.15 and 14.6.16 are taken from Brezis and Mironescu [2001]. Our presentation follows Schmeisser and Sickel [2001, 2005]. Proposition 14.6.13 and Theorem 14.6.14 can also be found in these works. Gagliardo–Nirenberg inequalities in the Besov scale can be found in the paper of Brezis and Mironescu [2018]; they do not allow for a microscopic improvement.

Difference norms

Difference norm characterisations of Triebel–Lizorkin spaces appear in the works of Kaljabin [1977, 1979], and Triebel [1983]. Our presentation of Theorem 14.6.20 follows Schmeisser and Sickel [2001], who consider the case $\tau = 1$.

Interpolation and duality

The interpolation and duality results for Triebel–Lizorkin spaces are similar to their Besov space counterparts. In our presentation, the end-point $q = 1$ is excluded, since the Fefferman–Stein inequality for the maximal operator is not valid in $L^p(\mathbb{R}^d; \ell^1)$. This problem can be circumvented by a reduction to interpolation identities for vector-valued Hardy spaces instead of $L^p(\mathbb{R}^d; \ell^q(X))$ (see Triebel [1983]). The embedding (14.87) of Theorem 14.6.26 is due to Jawerth [1977], and the one of (14.86) to Franke [1986].

Fractional Hardy inequalities

The fractional Hardy inequalities of Proposition 14.6.30 and Corollary 14.6.31 are variations of those by Krugljak, Maligranda, and Persson [2000], who proved the results with a fractional Sobolev norm $W^{s,p}$ on the right-hand side. The advantage of our formulation is that both the $H^{s,p}$ and the $W^{s,p}$ cases are consequences of the stronger estimate using the space $F^s_{p,\infty}$. Higher-dimensional versions of fractional Hardy inequalities can be deduced from the work of Meyries and Veraar [2012], where Sobolev embedding with power weights are discussed.

Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$

Pointwise multiplier results such as the one of Theorem 14.6.32 and Corollaries 14.6.34 and 14.6.35 were proved via paraproducts estimates in more generality by Runst and Sickel [1996]. Some of the results from this monograph were extended to the weighted vector-valued setting by Meyries and Veraar [2015]. In particular, some of the end-points can be included, and higher dimensional versions of the results hold. The results of the present section merely serve as an illustration of how the theory can be applied. Since the work of Grisvard [1967] and Seeley [1972], it is known that results on pointwise multipliers stand at the basis of interpolation with boundary conditions. The one-dimensional case is useful for evolution equations, since ${}_0F^1_{p,q}(\mathbb{R}_+;X)$ and ${}_0B^1_{p,q}(\mathbb{R}_+;X)$ can be used as the domain of the time-derivative. As in the work of Lindemulder, Meyries, and Veraar [2018], one can identity the real and complex interpolation spaces between ${}_0F^1_{p,q}(\mathbb{R};X)$ and $F^0_{p,q}(\mathbb{R};X)$ for $p,q \in (1,\infty)$ using the theory of this section, and similarly for Besov spaces for $p \in (1,\infty)$ and $q \in [1,\infty]$.

Section 14.7

The Embedding Theorems 14.7.1, 14.7.3, and 14.7.4 are taken from Schmeisser and Sickel [2001, 2005]. The end-point cases, where $\min\{p_0,p_1\} = 1 < \max\{p_0,p_1\}$, are not completely understood; we refer the reader to Brezis and Mironescu [2018] for a further discussion.

The Littlewood–Paley theorem 14.7.5 is taken from Meyries and Veraar [2015], who also consider a weighted setting.

The improved embeddings for Besov, Triebel–Lizorkin, and Bessel potential spaces under UMD and (co)type assumptions stated in Proposition 14.7.6 are due to Veraar [2013]. The converse result presented in Proposition 14.7.8 seems to be new. In the case $p = q$, Hytönen and Merikoski [2019] have shown the following more precise result.

Theorem 14.8.4. *For $k \in \mathbb{N}$ and $p \in [2,\infty)$, there is a continuous embedding*

$$B^k_{q,q}(\mathbb{R}^d;X) \hookrightarrow W^{k,q}(\mathbb{R}^d;X)$$

if and only if X has martingale cotype q.

In case the embedding constant depend on d in a polynomial way, such results have applications to quantitative affine approximation in infinite dimensions, as discussed by Hytönen, Li, and Naor [2016] and Hytönen and Naor [2019]. The proof of Theorem 14.8.4 is based on ideas from these works and results of Xu [1998] and Martínez, Torrea, and Xu [2006] connecting Littlewood–Paley–Stein inequalities and martingale (co)type. Some of these results have been extended by Xu [2020]. For open problems related to Theorem 14.8.4, we refer the reader to Problem Q.13.

Theorem 14.7.9 is due to Han and Meyer [1996], who obtained it as a consequence of a more general Littlewood–Paley theorem for $L^p(\mathbb{R}^d; X)$. Our approach is more direct.

The interpolation result of Theorem 14.7.12 was discovered independently by Amann [2019], Hummel [2019], and Lindemulder and Veraar [2020]. In the first reference, the anisotropic setting was also covered, and weighted spaces are included in the latter two references.

Pointwise multipliers

Theorem 14.7.15 is due to Meyries and Veraar [2015], where it appears as a special case of a general pointwise multiplier theorem for weighted vector-valued Bessel potential spaces. It is unknown whether the UMD condition is necessary (see Problem Q.12). The proof presented here is simplified from that of Lindemulder, Meyries, and Veraar [2018]. Another proof, based on a difference norm characterisation, is due to Lindemulder [2017]. The scalar case of Theorem 14.7.15 is due to Shamir [1962] and Strichartz [1967]. Their proof extends to the vector-valued setting only when the range space is isomorphic to a Hilbert spaces (see Walker [2003]).

Interpolation with boundary conditions

Applications to complex interpolation with boundary conditions are given by Lindemulder, Meyries, and Veraar [2018]. Among other things, the domains of the fractional powers of the first order derivative with Dirichlet boundary conditions are identified as $\mathsf{D}(\partial_t^s) = {}_0H^{s,p}(\mathbb{R}_+; X)$ for $s \in (0,1)$. This extends a special case of a result of Seeley [1972] to the vector-valued setting. Certain difficulties in obtaining such identities were overlooked in applications to evolution equations for several years. The boundedness of pointwise multiplication by indicator functions was proved recently in the anisotropic setting by Lindemulder [2022]. This solves an open problem of Amann [2019], who used the boundedness to obtain vector-valued and anisotropic extensions of some of the results of Seeley [1972] on interpolation with boundary conditions.

Function spaces on domains and extension operators

Function spaces on domains $\mathcal{O} \subseteq \mathbb{R}^d$ are usually defined by restriction, declaring that $f \in \mathcal{A}^s_{p,q}(\mathcal{O})$ if there exists $g \in \mathcal{A}^s_{p,q}(\mathbb{R}^d)$ such that $f = g|_{\mathcal{O}}$ in the distributional sense; the norm on $\mathcal{A}^s_{p,q}(\mathcal{O})$ is then taken to be the corresponding quotient norm. From this definition, it is often complicated to decide whether a given function belongs to $\mathcal{A}^s_{p,q}(\mathcal{O})$ and to estimate its norm. Extension operators help to get a better grip on this problem. Given a domain $\mathcal{O} \subseteq \mathbb{R}^d$, an extension operator for $\mathcal{O}$ is a bounded linear operator $E_{\mathcal{O}} : \mathcal{A}^s_{p,q}(\mathcal{O}) \to \mathcal{A}^s_{p,q}(\mathbb{R}^d)$ such that

$$(E_{\mathcal{O}} f)|_{\mathcal{O}} = f, \quad f \in \mathcal{A}^s_{p,q}(\mathcal{O}).$$

For Lipschitz domains $\mathcal{O}$, Rychkov [1999] constructed a 'universal' extension operator $E_{\mathcal{O}}$ which enjoys this property for all $s \in \mathbb{R}$, $p, q \in (0, \infty]$, and $\mathcal{A} \in \{B, F\}$. His proof extends to the vector-valued and weighted setting. A crucial ingredient is the work of Bui, Paluszyński, and Taibleson [1996, 1997], where the restriction that the Littlewood–Paley function φ should have compact Fourier support is relaxed to a moment condition on φ and a Tauberian condition on $\widehat{\varphi}$.

Once an extension operator is available, one often tries to obtain an intrinsic characterisation of the functions in $\mathcal{A}^s_{p,q}(\mathcal{O})$, e.g., in terms of differences and moduli of smoothness. As a consequence of the result of Rychkov [1999], a difference characterisation for $B^s_{p,q}(\mathcal{O})$ was obtained in Dispa [2003] for Lipschitz domains $\mathcal{O}$. A difference norm characterisations for $F^s_{p,q}(\mathcal{O})$ was obtained by Prats [2019] for ε-uniform domains (in particular, for Lipschitz domains).

Other ways to construct extension operators can be found in the books of Triebel [1983, 1992]. A classical method is to find an extension operator for $W^{k,p}(\mathcal{O})$, and use real and complex interpolation and duality to obtain an extension operators for $B^s_{p,q}(\mathcal{O})$ and $H^{s,p}(\mathcal{O})$ with $|s| < k$ and $q \in [1, \infty]$. This approach also works for Triebel–Lizorkin spaces if one uses the ℓ^q-interpolation method from Kunstmann [2015] and Lindemulder and Lorist [2021]. These techniques can also be used for vector-valued function spaces.

Another way to define function spaces on domains is by using wavelets; see Triebel [2006].

Weighted function spaces

Bui [1982] defined and studied the spaces $B^s_{p,q}(\mathbb{R}^d, w)$ and $F^s_{p,q}(\mathbb{R}^d, w)$ for all weights w in the class $A_\infty = \bigcup_{p>1} A_p$, where A_p denotes the class of Muckenhoupt weights as defined in Appendix J. Crucial to this approach is the Peetre maximal function and the weighted version of Theorem 3.2.28. The vector-valued setting was introduced and studied by Meyries and Veraar [2012, 2015, 2014b], Lindemulder, Meyries, and Veraar [2018], and, from a more abstract point of view, Lindemulder [2021].

Matrix-weighted Besov spaces have been introduced and investigated by Roudenko [2003, 2004] for $p \in [1, \infty)$, and by Frazier and Roudenko [2004, 2008] for $p \in (0, \infty)$. The special case $F^0_{p,2}(W)$ of matrix-weighted Triebel–Lizorkin spaces, and its identification with $L^p(W)$, was already considered by Nazarov and Treil [1996] and Volberg [1997], and more recently by Isralowitz [2021], but a systematic introduction and study of the full scale of these spaces is only recently due to Frazier and Roudenko [2021]. Matrix-weighted versions $F^{s,\tau}_{p,q}(W)$ of the generalised Triebel–Lizorkin-type spaces of Yang and Yuan [2008] have been subsequently studied by Bu, Hytönen, Yang, and Yuan [2023].

Two-weight Sobolev embedding

Haroske and Skrzypczak [2008] characterised the validity of the continuous embedding

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)$$

in terms of the weights $w_0, w_1 \in A_\infty$, the exponents $p_0, p_1, q_0, q_1 \in (0, \infty]$, and the smoothness parameters $s_0 \geqslant s_1$. The compactness of this embedding was characterised as well. A characterisation for Triebel–Lizorkin spaces was obtained by Meyries and Veraar [2014b] under the additional assumption $p_0 \leqslant p_1$; as in Theorem 14.6.14, a microscopic improvement occurs. In the vector-valued setting, the case of power weights is fully understood; see Meyries and Veraar [2012].

L^p–L^q-multipliers

In the scalar-valued case, L^p–L^q Fourier multiplier theorems for $p < q$ first appeared in the pioneering work of Hörmander [1960]. The scalar-valued case has the advantage that one can often factor through an L^2-space and use Plancherel's identity. In the Banach space-valued case, this is no longer possible unless additional conditions on the spaces are imposed. The singularities in L^p–L^q-multiplier theorems for $p < q$ usually behave in a different way from the case $p = q$. Often they are absolutely integrable in some appropriate sense, and then trivially extend to the vector-valued setting by Proposition 2.1.3. A typical example where this happens is the classical Hardy–Littlewood–Sobolev inequality on the L^p–L^q-boundedness of $f \mapsto |\cdot|^{-s} * f$.

For operator-valued L^p–L^q-Fourier multipliers, different phenomena arise. For details and applications to stability of C_0-semigroups we refer the reader to Rozendaal and Veraar [2018a, 2017, 2018c,b] and the survey by Rozendaal [2023]. The homogeneous version of Corollary 14.7.7 implies the following multiplier result of Rozendaal and Veraar [2018a].

Theorem 14.8.5. *Let X be a Banach space with type $p_0 \in (1, 2]$ and let Y be a Banach space with cotype $q_0 \in [2, \infty)$. Let $p \in (1, p_0)$ and $q \in (q_0, \infty)$, where we allow $p = 2$ if $p_0 = 2$ and $q = 2$ if $q_0 = 2$. Let $r \in [1, \infty]$ satisfy $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. If $m : \mathbb{R}^d \setminus \{0\} \to \mathscr{L}(X, Y)$ is a strongly measurable function in the strong operator topology, and such that*

$$\{|\xi|^{d/r} m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\}$$

is γ-bounded, then T_m uniquely extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^q(\mathbb{R}^d; Y)$).

The proof of this theorem is based on factorisation through $\gamma(L^2(\mathbb{R}^d), X)$ and uses the γ-boundedness of the stated operator family. To obtain a homogeneous condition on m, one needs the homogeneous version of the γ-Sobolev embedding. It is not known whether Theorem 14.8.5 holds for $p = p_0$ and $q = q_0$. An exception is the case where X and Y are p-convex and q-concave Banach lattices, respectively; the result then follows from the homogeneous version of Theorem 14.8.1. Theorem 14.8.5 was used by Rozendaal [2019] to obtain boundedness of the H^∞-calculus on fractional domain spaces for strip type operators. Rozendaal and Veraar [2018a] also prove the following multiplier theorem under Fourier type assumptions.

Theorem 14.8.6. *Let X be a Banach space with Fourier type $p_0 \in (1, 2]$ and let Y be a Banach space with Fourier type $q_0' \in (1, 2]$. Let $p \in (1, p_0)$ and $q \in (q_0, \infty)$, and let $r \in [1, \infty)$ satisfy $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. If $m : \mathbb{R}^d \setminus \{0\} \to \mathscr{L}(X, Y)$ is a strongly measurable functions and $m \in L^{r,\infty}(\mathbb{R}^d; \mathscr{L}(X, Y))$, then T_m uniquely extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^q(\mathbb{R}^d; Y))$.*

The condition $m \in L^{r,\infty}(\mathbb{R}^d; \mathscr{L}(X, Y))$ allows for singularities of the form $|\cdot|^{-d/r}$. The proof in the case $C_{m,r} := \big\| \|m\|_{\mathscr{L}(X,Y)} \big\|_{L^r(\mathbb{R}^d)} < \infty$ with $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{r_0}$ is completely straightforward. Indeed, by Hölder's inequality,

$$\begin{aligned} \|T_m f\|_{q_0} \leqslant \varphi_{q_0',Y}(\mathbb{R}^d) \|m\widehat{f}\|_{q_0'} &\leqslant \varphi_{q_0',Y}(\mathbb{R}^d) C_{m,r} \|\widehat{f}\|_{p_0'} \\ &\leqslant \varphi_{q_0',Y}(\mathbb{R}^d) \varphi_{p_0,Y}(\mathbb{R}^d) C_{m,r} \|f\|_{p_0}. \end{aligned}$$

Theorem 14.8.6 can be deduced from this estimate by an interpolation argument.

The above Fourier multiplier theorems are stated for one specific value of p and q. However, if the kernel (see Hörmander [1960]) or the multiplier (see Rozendaal and Veraar [2017]) satisfies certain Hörmander conditions, boundedness from L^u into L^v can be shown for all $u, v \in (1, \infty)$ satisfying $\frac{1}{u} - \frac{1}{v} = \frac{1}{p} - \frac{1}{q} =: \frac{1}{r}$. For example, a sufficient condition is

$$\sup_{\xi \neq 0} |\xi|^{|\alpha| + d/r} \|\partial^\alpha m(\xi)\| < \infty, \quad |\alpha| \leqslant \lfloor \tfrac{d}{r'} \rfloor + 1.$$

Under Fourier type assumptions on X and Y, the number of derivatives can be further reduced.

Proposition 14.5.7 can be viewed as a mixed Besov–L^q-Fourier multiplier theorem in the same spirit as Theorems 14.8.5 and 14.8.6.