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Multiple and weak Markov properties in Hilbert spaces with applications to fractional stochastic evolution equations

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ABSTRACT

We define a number of higher-order Markov properties for stochastic processes $(X(t))_{t\in\mathbb{T}}$, indexed by an interval $\mathbb{T} \subseteq \mathbb{R}$ and taking values in a real and separable Hilbert space U. We furthermore investigate the relations between them. In particular, for solutions to the stochastic evolution equation $\mathcal{L}X = \dot{W}$, where $\mathcal{L}$ is a linear operator acting on functions mapping from $\mathbb{T}$ to U and $(\dot{W}(t))_{t\in\mathbb{T}}$ is the formal derivative of a U-valued cylindrical Wiener process, we prove necessary and sufficient conditions for the weakest Markov property via locality of the precision operator $\mathcal{L}^*\mathcal{L}$.

As an application, we consider the space–time fractional parabolic operator $\mathcal{L} = (\partial_t + A)^\gamma$ of order $\gamma \in (1/2, \infty)$, where $-A$ is a linear operator generating a C_0-semigroup on U. We prove that the resulting solution process satisfies an Nth order Markov property if $\gamma = N \in \mathbb{N}$ and show that a necessary condition for the weakest Markov property is generally not satisfied if $\gamma \notin \mathbb{N}$. The relevance of this class of processes is twofold: Firstly, it can be seen as a spatiotemporal generalization of Whittle–Matérn Gaussian random fields if $U = L^2(\mathcal{D})$ for a spatial domain $\mathcal{D} \subseteq \mathbb{R}^d$. Secondly, we show that a U-valued analog to the fractional Brownian motion with Hurst parameter $H \in (0,1)$ can be obtained as the limiting case of $\mathcal{L} = (\partial_t + \varepsilon\,\mathrm{Id}_U)^{H+\frac{1}{2}}$ for $\varepsilon \downarrow 0$.

1. Introduction

1.1. Background and motivation

Gaussian Markov random fields play an important role for various applications, such as the analysis of time series or longitudinal data, image processing and spatial statistics, see e.g. [39, Section 1.3]. The latter focuses on the statistical modeling of spatial or spatiotemporal dependence in data collected from phenomena encountered in disciplines such as climatology [1], epidemiology [24] and neuroimaging [31]. The popularity of Gaussian Markov random fields among the larger class of Gaussian random fields is a consequence of their additional conditional independence properties, which entail a sparse precision structure and facilitate efficient computational methods for statistical inference. In particular, hierarchical models based on Gaussian Markov random fields allow for efficient Bayesian inference using Markov chain Monte Carlo methods, see for instance [39, Section 4.1].

Since a Gaussian process is fully characterized by its second-order structure, i.e., the mean and covariance function, a natural way to specify its distribution is to choose a suitable second-order structure. Alternatively, the dynamics of Gaussian random fields

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defined on a Euclidean domain $\mathcal{D} \subseteq \mathbb{R}^d$ can be specified by means of stochastic partial differential equations (SPDEs), such as the white noise $(\mathcal{W}(x))_{x\in\mathcal{D}}$ driven equation

$$LX(x) = \mathcal{W}(x), \quad x \in \mathcal{D}. \tag{1.1}$$

Here, L is a linear operator acting on real-valued functions defined on $\mathcal{D}$. A spatial Gaussian random field $(X(x))_{x\in\mathcal{D}}$ is said to have the Markov property if the subcollections $(X(x))_{x\in\mathcal{D}_1}$ and $(X(x))_{x\in\mathcal{D}_2}$ corresponding to pairs of disjoint subdomains $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{D}$ are independent conditional on $(X(x))_{x\in\mathcal{D}'}$ for some non-trivial 'splitting' set $\mathcal{D}' \subseteq \mathcal{D}$ separating the two. The precise specification of these sets, which respectively carry the intuitive interpretations of *past*, *future* and *present*, leads to various definitions of the Markov property. According to the theory of Rozanov [38], a real-valued Gaussian random field satisfying (1.1) has such a Markov property if and only if its precision operator L^*L is local, where L^* denotes the $L^2(\mathcal{D})$-adjoint of L.

An important example in spatial statistics is the choice of a fractional-order differential operator $L := \tau(\kappa^2 - \Delta)^\beta$ in (1.1), where Δ is the Laplacian, $\mathcal{W}$ is Gaussian white noise and $\tau, \kappa, \beta \in (0, \infty)$. Whittle [43] observed that the covariance function $\varrho(x, y) := \mathbb{E}[X(x)X(y)]$ of the stationary solution $(X(x))_{x\in\mathcal{D}}$ to (1.1) with $\mathcal{D} = \mathbb{R}^d$ then belongs to the widely used *Matérn class* [30]:

$$\varrho(x, y) = C_{\kappa,\tau,\nu,d}(\kappa\|x - y\|_{\mathbb{R}^d})^\nu K_\nu(\kappa\|x - y\|_{\mathbb{R}^d}) \quad \text{for all } x, y \in \mathbb{R}^d, \tag{1.2}$$

where $\nu := 2\beta - d/2$, $C_{\kappa,\tau,\nu,d} := \tau^{-2}(4\pi)^{-d/2}2^{1-\nu}[\Gamma(\nu + d/2)]^{-1}\kappa^{-2\nu}$ and K_ν denotes the modified Bessel function of the second kind. This observation motivated the *SPDE approach* for spatial statistical modeling proposed by Lindgren, Rue and Lindström [27]. Here, one considers (1.1) on a bounded Euclidean domain $\mathcal{D} \subsetneq \mathbb{R}^d$, augmented with boundary conditions, and approximates the resulting *Whittle–Matérn fields* by means of efficient numerical methods available for (S)PDEs. Owing to its ease of generalization and its computational efficiency as compared to covariance-based techniques, this approach has gained widespread popularity, see e.g. [5–7,9,17,26,40]. Since in this case the precision operator is given by $L^*L = \tau^2(\kappa^2 - \Delta)^{2\beta}$, we find that Whittle–Matérn fields are Gaussian *Markov* random fields in the sense of Rozanov [38] precisely when $2\beta \in \mathbb{N}$.

Recently, extensions of the SPDE approach incorporating time dependence have been discussed. A class of space–time equations which has been proposed in this context is

$$(\partial_t + L)^\gamma X(t, x) = \dot{\mathcal{W}}^Q(t, x), \quad (t, x) \in \mathbb{T} \times \mathcal{D}, \qquad \gamma \in (1/2, \infty), \tag{1.3}$$

where $\mathbb{T} \subseteq \mathbb{R}$ represents a time interval and $\dot{\mathcal{W}}^Q$ is spatiotemporal Gaussian noise, which is spatially colored by an operator Q, see [23,25]. In particular, it has been shown in [23] that Eq. (1.3) extends the Matérn model in terms of spatial marginal covariance, and that the interplay of its parameters governs smoothness in space and time as well as the degree of separability.

Spatiotemporal random fields can be viewed as U-valued stochastic processes by letting a Hilbert space U encode the spatial variable, so that (1.3) corresponds to a stochastic fractional evolution equation of the form

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in \mathbb{T}. \tag{1.4}$$

The (temporal) Markov property of solutions to (1.3) is then equivalent to that of the U-valued solution process $(X(t))_{t\in\mathbb{T}}$, where the Markov behavior is considered with respect to the index set $\mathbb{T}$. Moreover, viewing (1.4) as a special case of

$$\mathcal{L}X(t) = \dot{W}^Q(t), \quad t \in \mathbb{T}, \tag{1.5}$$

where $\mathcal{L}$ is now a linear operator acting on functions from $\mathbb{T}$ to U, the theory of Rozanov [38] suggests that locality of the precision operator $\mathcal{L}^*\mathcal{L}$, also acting on functions $f : \mathbb{T} \to U$, can be used to characterize temporal Markov behavior of the solution X.

1.2. Contributions

In this work we define *simple*, *multiple* (*N-ple* for $N \in \mathbb{N}$) and *weak* Markov properties for stochastic processes which take values in a Hilbert space U. These definitions generalize those appearing for instance in [18,37,38] for real-valued processes to infinite dimensions, see Definitions 3.1, 3.2 and 3.4, respectively. Besides gathering them in once place, we establish their interrelations, see Proposition 3.5 and Remark 3.6. The main results are Theorems 3.7 and 3.9, which give necessary and sufficient conditions, in terms of the precision operator $\mathcal{L}^*\mathcal{L}$, for the weakest notion of Markovianity for a U-valued Gaussian process defined via (1.5). These results are proven by a non-trivial extension of the theory by Rozanov [38, Chapters 2 and 3] from the real-valued to the U-valued setting.

In order to consider the more concrete class of processes defined via (1.4), we construct a stochastic integral for deterministic operator-valued integrands defined on the whole of $\mathbb{R}$ with respect to a two-sided (cylindrical) Q-Wiener process $(W^Q(t))_{t\in\mathbb{R}}$, see Section 2.2. We employ this stochastic integral to define the mild solution process $Z_\gamma = (Z_\gamma(t))_{t\in\mathbb{R}}$ to (1.4) on $\mathbb{T} = \mathbb{R}$, see Definition 4.5. Our rigorous definition of the fractional space–time operator $(\partial_t + A)^\gamma$ for $\gamma \in \mathbb{R}$, see Definition 4.3, extends the Weyl fractional calculus in the sense that one recovers the Weyl fractional derivatives and integrals defined in [21, Section 2.3] upon specializing to $U = \mathbb{R}$ and $A = 0$.

We show that the mild solution Z_γ to (1.4) satisfies the N-ple Markov property if $\gamma = N \in \mathbb{N}$, see Theorem 4.10. Conversely, we use Theorem 3.7 to show that, in general, Z_γ is not weakly Markov for $\gamma \notin \mathbb{N}$. This complements [15, Theorem 2.7], which states that any time-homogeneous U-valued Gaussian simple Markov process is the solution to a first-order stochastic evolution equation.

Finally, we discuss another interesting aspect of the SPDE (1.4): A fractional Q-Wiener process $(W^Q_H(t))_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0, 1)$, as defined for instance in [13], can be obtained as a limiting case of (1.4) with $\gamma = H + 1/2$ and $A = \varepsilon\,\mathrm{Id}_U$ as $\varepsilon \downarrow 0$, see

Table 1
Notation used throughout this article.

Elementary sets and operations		**Function spaces**	
$\mathbb{N}$	Positive integers	J	Non-empty (sub)interval of $\mathbb{R}$
$\mathbb{N}_0$	Non-negative integers	$C(J;E)$	Continuous functions from J to E
Id_D	Identity map on a set D	$C_c^\infty(J;E)$	Compactly supported infinitely differentiable functions from J to E
$\mathbf{1}_{D_0}$	Indicator function of a subset $D_0 \subseteq D$	$C_c^\infty(J)$	Abbreviation for $C_c^\infty(J;\mathbb{R})$
$s \wedge t$	Minimum of $s,t \in \mathbb{R}$	$C_b(E)$	Bounded and continuous functions from E to $\mathbb{R}$
$s \vee t$	Maximum of $s,t \in \mathbb{R}$	$(S,\mathscr{A},\mu)$	Measure space
Bounded linear operators		$B_b(S)$	Bounded and measurable functions from S to $\mathbb{R}$
U, $\widetilde{U}$	Real and separable Hilbert spaces	$L^p(S,\mathscr{A},\mu;E)$	Bochner space of p-integrable functions from S to E
$\langle\,\cdot\,,\cdot\,\rangle_U$	Inner product of U	$L^p(S;E)$	Abbreviation for $L^p(S,\mathscr{A},\mu;E)$
E, F	Real and separable Banach spaces	$H^1(J;U)$	Functions in $L^2(J;U)$ with weak derivatives in $L^2(J;U)$
$\|\cdot\|_E$	Norm of E	$H_0^1(0,\infty;U)$	Functions in $H^1(0,\infty;U)$ which vanish at zero
$\mathscr{L}(E;F)$	Bounded linear operators from E to F		
$\mathscr{L}(E)$	Abbreviation for $\mathscr{L}(E;E)$	**Unbounded linear operators**	
T^*	Adjoint of $T \in \mathscr{L}(E;F)$	$\mathsf{D}(A)$	Domain of unbounded linear operator $A\colon \mathsf{D}(A) \subseteq E \to E$ on E
$\mathscr{L}^+(U)$	Self-adjoint and positive definite operators on U	A_S	Bochner space counterpart on $L^2(S;E)$ of $A\colon \mathsf{D}(A) \subseteq E \to E$, see (4.7)
$\operatorname{tr} T$	Trace of $T \in \mathscr{L}^+(U)$		
$\mathscr{L}_1^+(U)$	$T \in \mathscr{L}^+(U)$ with $\operatorname{tr} T < \infty$		
$\mathscr{L}_2(U;\widetilde{U})$	Hilbert–Schmidt operators from U to $\widetilde{U}$		

Proposition 5.3. The proof is based on a Mandelbrot–Van Ness [29] type integral representation of W_H^Q, again using the two-sided stochastic integral from Section 2.2, see Proposition 5.2. The case $H = \frac{1}{2}$ corresponds to a (non-fractional) Q-Wiener process and is thus Markov. Conversely, although the results of Theorems 3.7 and 3.9 do not apply directly, the above observation provides evidence that W_H^Q does not satisfy a weak Markov property for $H \in (0,1) \setminus \{\frac{1}{2}\}$.

1.3. Outline

In Section 2 we begin by establishing the necessary notation, see Section 2.1, followed by the construction of the stochastic integral with respect to a two-sided (cylindrical) Q-Wiener process in Section 2.2. Section 3 is devoted to defining, relating and (for solutions to (1.5)) characterizing various notions of Markov behavior for U-valued stochastic processes. The goal of Section 4 is to define and analyze the mild solution to (1.4) on $\mathbb{T} = \mathbb{R}$. To this end, we first describe the setting and define $(\partial_t + A)^{-\gamma}$ with $\gamma \in (0,\infty)$ in Sections 4.1 and 4.2, respectively. We subsequently define the mild solution process in Section 4.3, and investigate for which values of $\gamma \in (1/2,\infty)$ it exhibits Markov behavior in Section 4.4. In Section 5 we recall the definition from [13] of a Q-fractional Wiener process and prove a Mandelbrot–Van Ness type integral representation, allowing us to exhibit it as a limiting case of (1.4).

This article is supplemented by two appendices: Appendix A contains auxiliary results relating to specific results from the main text whose statements and proofs were postponed for readability; subjects include conditional independence, filtrations indexed by $\mathbb{R}$ and the mean-square differentiability of stochastic convolutions. Appendix B is a short overview of results regarding fractional powers of linear operators and the interpretation of the fractional parabolic operator $(\partial_t + A)^\gamma$.

2. Preliminaries

2.1. Notation

Table 1 lists some notation which is used throughout this article. Positive definite operators $T \in \mathscr{L}^+(U)$ satisfy $\langle Tx, x\rangle_U \geq \theta \|x\|_U^2$ for all $x \in U$ and some $\theta \in (0,\infty)$. The trace of $T \in \mathscr{L}^+(U)$ is defined by $\operatorname{tr} T := \sum_{j\in\mathbb{N}} \langle Te_j, e_j\rangle_U$, where $(e_j)_{j\in\mathbb{N}}$ is any orthonormal basis of U. The space of Hilbert–Schmidt operators $\mathscr{L}_2(U;\widetilde{U})$ is equipped with the inner product $\langle T,S\rangle_{\mathscr{L}_2(U;\widetilde{U})} := \sum_{j\in\mathbb{N}} \langle Te_j, Se_j\rangle_{\widetilde{U}}$. We call $A\colon \mathsf{D}(A) \subseteq E \to E$ closed if its graph is closed with respect to the graph norm, and densely defined if $\mathsf{D}(A)$ is dense in E.

Throughout this work, we assume that a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ is given, meaning that $\mathcal{F}$ contains the collection $\mathcal{N}_\mathbb{P}$ of $\mathbb{P}$-null sets. We abbreviate the phrase "$\mathbb{P}$-almost surely" by "$\mathbb{P}$-a.s". In what follows, we call a function $Z\colon \Omega \to E$ an E-valued random variable if it is strongly $\mathbb{P}$-measurable. We write $Z \sim N(m,Q)$ if Z is a U-valued Gaussian random variable with mean $m \in U$ and covariance $Q \in \mathscr{L}_1^+(U)$; its existence is guaranteed by [4, Theorem 2.3.1]. Two stochastic processes $(X(t))_{t\in\mathbb{T}}$ and $(\widetilde{X}(t))_{t\in\mathbb{T}}$ are said to be modifications of each other if $\mathbb{P}(X(t) = \widetilde{X}(t)) = 1$, for all $t \in \mathbb{T}$, where $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = [0,\infty)$ or $\mathbb{T} = [0,T]$ for some $T \in (0,\infty)$.

Let $\mathcal{G}_1, \mathcal{H}, \mathcal{G}_2 \subseteq \mathcal{F}$ be sub-σ-algebras of $\mathcal{F}$. The join of two σ-algebras is denoted by $\mathcal{G}_1 \vee \mathcal{G}_2 := \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$. We write $\mathcal{G}_1 \perp\!\!\!\perp \mathcal{G}_2$ to indicate that $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent. The expression $\mathbb{E}[Z \mid \mathcal{H}]$ denotes the conditional expectation of a random variable Z given $\mathcal{H}$, and the conditional probability of $A \in \mathcal{F}$ given $\mathcal{H}$ is defined by $\mathbb{P}(A \mid \mathcal{H}) := \mathbb{E}[1_A \mid \mathcal{H}]$, $\mathbb{P}$-a.s. The notation $\mathcal{G}_1 \perp\!\!\!\perp_\mathcal{H} \mathcal{G}_2$ indicates that $\mathcal{G}_1$ and $\mathcal{G}_2$ are conditionally independent given $\mathcal{H}$, i.e., for all $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ we have $\mathbb{P}(G_1 \cap G_2 \mid \mathcal{H}) = \mathbb{P}(G_1 \mid \mathcal{H})\mathbb{P}(G_2 \mid \mathcal{H})$, $\mathbb{P}$-a.s. When conditioning on the σ-algebra $\sigma(Y) = \{\{Y \in B\} : B \in \mathcal{B}(E)\}$ generated by a random variable Y, we write Y instead of $\sigma(Y)$; e.g., $\mathbb{E}[Z \mid Y]$ or $\mathcal{G}_1 \perp\!\!\!\perp_Y \mathcal{G}_2$.

2.2. Stochastic integration with respect to a two-sided Wiener process

Let $(W_1^Q(t))_{t\geq 0}$, $(W_2^Q(t))_{t\geq 0}$ be independent U-valued standard Q-Wiener processes for a given $Q \in \mathscr{L}_1^+(U)$, see for instance [28, Section 2.1], and define

$$W^Q(t) := \begin{cases} W_1^Q(t), & t \in [0,\infty); \\ W_2^Q(-t), & t \in (-\infty,0). \end{cases}$$

Then the *two-sided Q-Wiener process* $W^Q := (W^Q(t))_{t\in\mathbb{R}}$ satisfies the following:

(WP1) $W^Q(t)$ has mean zero and $W^Q(t) - W^Q(s) \sim N(0,(t-s)Q)$ for $t \geq s$;
(WP2) W^Q has continuous sample paths;
(WP3) $W^Q(t_4) - W^Q(t_3) \perp\!\!\!\perp W^Q(t_2) - W^Q(t_1)$ for $t_1 < t_2 \leq t_3 < t_4$.

One can define a stochastic integral with respect to such a process using a construction analogous to the one-sided case, as presented for instance in [28, Section 2.3]. Restricting ourselves to deterministic integrands $\Phi\colon \mathbb{R} \to \mathscr{L}(U;\widetilde{U})$, this procedure yields a square-integrable stochastic integral $\int_{\mathbb{R}} \Phi(t)\,\mathrm{d}W^Q(t)$ belonging to $L^2(\Omega;\widetilde{U})$ which exists if and only if $\Phi(\,\cdot\,)Q^{\frac{1}{2}} \in L^2(\mathbb{R};\mathscr{L}_2(U;\widetilde{U}))$; see Table 1 for the definitions of these (Bochner) spaces. In this case, it satisfies the following Itô isometry:

$$\left\|\int_{\mathbb{R}} \Phi(t)\,\mathrm{d}W^Q(t)\right\|^2_{L^2(\Omega;\widetilde{U})} = \int_{\mathbb{R}} \|\Phi(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U;\widetilde{U})}\,\mathrm{d}t. \tag{2.1}$$

As in the one-sided case, we can extend the definition of the stochastic integral to allow for $Q \in \mathscr{L}^+(U)\setminus\mathscr{L}_1^+(U)$, cf. [28, Section 2.5].

Now we turn to the matter of $\mathbb{R}$-indexed filtrations on $(\Omega,\mathcal{F},\mathbb{P})$ associated to $(W^Q(t))_{t\in\mathbb{R}}$. In the one-sided case, the integral process $\left(\int_0^t \Phi(r)\,\mathrm{d}W_1^Q(r)\right)_{t\geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_t^{W_1^Q} := \sigma\left(W_1^Q(s) : 0 \leq s \leq t\right) \vee \sigma(\mathcal{N}_{\mathbb{P}})$ whenever $\Phi(\,\cdot\,)Q^{\frac{1}{2}} \in L^2(0,t;\mathscr{L}_2(U;\widetilde{U}))$ for all $t \in [0,\infty)$, which is immediate from the definition of the stochastic integral.

In the two-sided case, we instead use the *(completed) filtration* $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$ *generated by the increments of* W^Q, defined by

$$\mathcal{F}_t^{\delta W^Q} := \sigma(W^Q(u) - W^Q(s) : s < u \leq t) \vee \sigma(\mathcal{N}_{\mathbb{P}}), \quad t \in \mathbb{R}. \tag{2.2}$$

Note that we have $\mathcal{F}_t^{\delta W^Q} \subsetneq \mathcal{F}_t^{W^Q}$ for all $t \in \mathbb{R}$ and $\mathcal{F}_t^{W^Q} = \mathcal{F}_t^{\delta W^Q}$ for $t \in [0,\infty)$, where $\mathcal{F}_t^{W^Q}$ is generated by $(W^Q(s))_{s\in(-\infty,t]}$ for each $t \in \mathbb{R}$. We point out that $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$ is normal, cf. [2, Example 3.6]. By (WP3), the two-sided Wiener process $(W^Q(t))_{t\in\mathbb{R}}$ now satisfies that $W^Q(t) - W^Q(s') \perp\!\!\!\perp \mathcal{F}_s^{\delta W^Q}$ for all $s \leq s' < t$, so that, analogously to the one-sided case, $\left(\int_{-\infty}^t \Phi(r)\,\mathrm{d}W^Q(r)\right)_{t\in\mathbb{R}}$ is a martingale with respect to $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$ for every $\Phi(\,\cdot\,)Q^{\frac{1}{2}} \in L^2(\mathbb{R};\mathscr{L}_2(U,\widetilde{U}))$. Unlike $(W_1^Q(t))_{t\geq 0}$, however, the process $(W^Q(t))_{t\in\mathbb{R}}$ itself will not be a martingale with respect to any filtration, see Proposition A.5 in Appendix A. We refer the reader to [2,3] for more details on the subject of real-valued martingale type processes indexed by $\mathbb{R}$ and stochastic integration with respect to them.

3. Markov properties for Hilbert space valued stochastic processes

Let $X = (X(t))_{t\in\mathbb{T}}$ be a U-valued stochastic process indexed by $\mathbb{T}$, see Section 2.1. Intuitively, X is said to be a Markov process if, at any instant, its past and future states are independent conditional on the present. Varying the amount of information from the present gives rise to different Markov properties, which we will list in decreasing order of strength.

3.1. Simple Markov property

The following definition is often just referred to as the Markov property, see also [10, p. 77] or [12, Equation (6.2), p. 81].

Definition 3.1. An $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process $(X(t))_{t\in\mathbb{T}}$ is said to have the *simple Markov property* if for all $s \leq t$ and $B \in \mathcal{B}(U)$, we have that $\mathbb{P}(X(t) \in B \mid \mathcal{F}_s) = \mathbb{P}(X(t) \in B \mid X(s))$ holds $\mathbb{P}$-a.s.

The simple Markov property can also be characterized by means of *transition operators*: The process $(X(t))_{t\in\mathbb{T}}$ is simple Markov if and only if there exists a family $(T_{s,t})_{s\leq t\in\mathbb{T}}$ of linear operators on $B_b(U)$ satisfying

$$\mathbb{E}[\varphi(X(t)) \mid \mathcal{F}_s] = T_{s,t}\varphi(X(s)), \quad \mathbb{P}\text{-a.s.} \tag{3.1}$$

In this case, the *transition operators* $(T_{s,t})_{s\leq t\in\mathbb{T}}$ have the following properties:

(TO1) $T_{s,t}\varphi(x) \geq 0$ for all $x \in U$ if $\varphi \in B_b(U)$ is non-negative,
(TO2) $T_{s,t}\mathbf{1}_U = \mathbf{1}_U$,
(TO3) $T_{s,u}\varphi(X(s)) = T_{s,t}T_{t,u}\varphi(X(s))$, $\mathbb{P}$-a.s., for $\varphi \in B_b(U)$ and $s \leq t \leq u$.

Lastly, we can also characterize the simple Markov property in terms of conditional independence: By Theorem A.1 in Appendix A, the simple Markov property is equivalent to the fact that $\mathcal{F}_s \perp\!\!\!\perp_{X(s)} \sigma(X(t))$ holds for all $s \leq t$. In fact, according to [20, Lemma 11.1], this is in turn equivalent to the statement that $\mathcal{F}_s \perp\!\!\!\perp_{X(s)} \sigma(X(t) : t \geq s)$ for all $s \in \mathbb{T}$.

3.2. Multiple Markov property

The following weaker notion of Markov behavior dates back to Doob, who introduced it in the context of stationary real-valued Gaussian processes [11, pp. 271–272]. We generalize it to square-integrable U-valued processes with some mean-square differentiability, i.e., $(X(t))_{t\in\mathbb{T}} \subseteq L^2(\Omega; U)$ such that the function $t \mapsto X(t)$ is classically differentiable from $\mathbb{T}$ to $L^2(\Omega; U)$.

Definition 3.2. Suppose that $X = (X(t))_{t\in\mathbb{T}} \subseteq L^2(\Omega; U)$ is an $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process and let $N \in \mathbb{N}$. Then X has the *N-ple Markov property* if it has $N-1$ mean square derivatives and, for $s \le t$ in $\mathbb{T}$ and $B \in \mathcal{B}(U)$,

$$\mathbb{P}(X(t) \in B \mid \mathcal{F}_s) = \mathbb{P}(X(t) \in B \mid X(s), X'(s), \dots, X^{(N-1)}(s)), \quad \mathbb{P}\text{-a.s.}$$

Setting $\mathbf{X}(t) := (X^{(k)}(t))_{k=0}^{N-1}$, one defines a process taking values in the direct product Hilbert space $(U^N, \langle \cdot, \cdot \rangle_{U^N})$, where the inner product, given by $\langle \mathbf{x}, \mathbf{y} \rangle_{U^N} := \sum_{j=1}^N \langle x_j, y_j \rangle_U$ for $\mathbf{x} = (x_j)_{j=1}^N$ and $\mathbf{y} = (y_j)_{j=1}^N \in U^N$, induces the product topology on the set U^N. In particular, the Borel σ-algebra of U^N satisfies $\mathcal{B}(U^N) = \bigotimes^N \mathcal{B}(U)$ by [20, Lemma 1.2]. Theorem A.1 in Appendix A again yields an equivalent formulation of the N-ple Markov property in terms of conditional independence:

$$\forall s \in \mathbb{T}: \quad \mathcal{F}_s \perp\!\!\!\perp_{\mathbf{X}(s)} \sigma(X(t) : t \ge s). \tag{3.2}$$

Note that $\sigma(\mathbf{X}(s)) \vee \mathcal{F}_s = \mathcal{F}_s$ since the mean-square derivatives of X can be replaced by left derivatives, see the proof of Proposition 3.5 below. Arguing as in [20, Lemma 11.1], one can show that this is in turn equivalent to the simple Markov property for $\mathbf{X}$. Thus, we can apply the characterization given by (3.1) to derive the following corollary.

Corollary 3.3. *An $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted and square-integrable U-valued stochastic process $X = (X(t))_{t\in\mathbb{T}}$ with $N-1$ mean-square derivatives is N-ple Markov if and only if there exists a family $(T_{s,t})_{s\le t\in\mathbb{T}}$ of linear operators on $B_b(U^N)$ such that $\mathbb{E}[\varphi(\mathbf{X}(t)) \mid \mathcal{F}_s] = T_{s,t}\varphi(\mathbf{X}(s))$ holds $\mathbb{P}$-a.s. for all $s \le t \le u$ in $\mathbb{T}$ and $\varphi \in B_b(U^N)$. In this case, $(T_{s,t})_{s\le t\in\mathbb{T}}$ satisfies (TO1)–(TO3).*

3.3. Weak Markov properties; relations between concepts

We now define two Markov properties for which the "present" at time $s \in \mathbb{T}$ is represented by information from neighborhoods around s. As we will prove in Proposition 3.5 below, these two notions are equivalent. They appear in, e.g., [38, p. 62] and [18, Equation (5.87), p. 115].

Definition 3.4. An $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process $(X(t))_{t\in\mathbb{T}}$ has

(i) the *weak Markov property* if, for every $s \in \mathbb{T}$, there exists $\delta > 0$ such that for all $\varepsilon \in (0, \delta)$ it holds that $\mathcal{F}_s \perp\!\!\!\perp_{\mathscr{A}_\varepsilon(s)} \sigma(X(t) : t \ge s)$, where $\mathscr{A}_\varepsilon(s) := \sigma(X(u) : u \in (s-\varepsilon, s+\varepsilon) \cap \mathbb{T})$;

(ii) the *σ-Markov property* if for all $s \in \mathbb{T}$ we have $\mathcal{F}_s \perp\!\!\!\perp_{\partial\mathscr{A}(s)} \sigma(X(t) : t \ge s)$, where $\partial\mathscr{A}(s) := \bigcap_{\varepsilon>0} \mathscr{A}_\varepsilon(s)$.

Proposition 3.5. *Suppose that $X = (X(t))_{t\in\mathbb{T}}$ is an $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process. We have the following relations between Markov properties:*

$$\textit{simple Markov} \implies \sigma\textit{-Markov} \iff \textit{weak Markov}.$$

If $N, M \in \mathbb{N}$ are such that $N \ge M$ and X has $N-1$ mean-square derivatives, then we moreover have

$$M\textit{-ple Markov} \implies N\textit{-ple Markov} \implies \textit{weak Markov}.$$

Proof. If X has the weak Markov property, then by definition we have the following identity for fixed $s \in \mathbb{T}$, $B_- \in \mathcal{F}_s$ and $B_+ \in \sigma(X(t) : t \ge s)$:

$$\mathbb{P}(B_- \mid \mathscr{A}_{1/n}(s))\mathbb{P}(B_+ \mid \mathscr{A}_{1/n}(s)) = \mathbb{P}(B_- \cap B_+ \mid \mathscr{A}_{1/n}(s)), \quad \mathbb{P}\text{-a.s.}, \tag{3.3}$$

whenever $n \in \mathbb{N}$ is large enough. Note that $(\mathscr{G}_n)_{n\in\mathbb{N}} := (\mathscr{A}_{1/n}(s))_{n\in\mathbb{N}}$ is a non-increasing sequence of sub-σ-algebras of $\mathcal{F}$, i.e., a backward filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, $(\mathbb{P}(B \mid \mathscr{G}_n))_{n\in\mathbb{N}}$ is a backward martingale with respect to $(\mathscr{G}_n)_{n\in\mathbb{N}}$ for any $B \in \mathcal{F}$. Combined with the fact that $\bigcap_{n\in\mathbb{N}} \mathscr{G}_n = \partial\mathscr{A}(s)$, the backward martingale convergence theorem [16, Section 12.7, Theorem 4] implies that we may take the $\mathbb{P}$-a.s. limit as $n \to \infty$ in (3.3) to find that X is σ-Markov.

Now let $N, M \in \mathbb{N}$ with $N \ge M$ be such that X has the M-ple Markov property and $N-1$ mean-square derivatives. When considering $\sigma(X'(s))$ at $s \in \mathbb{T}$, we can restrict ourselves to mean-square left derivatives, i.e., we consider the sequence

$$(\Delta_n(s))_{n\in\mathbb{N}} := (n[X(s) - X(s - n^{-1})])_{n\in\mathbb{N}} \tag{3.4}$$

converging to $X'(s)$ in the $L^2(\Omega; U)$-norm as $n \to \infty$. Consequently, there exists a subsequence $(\Delta_{n_k}(s))_{k\in\mathbb{N}}$ such that $\Delta_{n_k}(s) \to X'(s)$, $\mathbb{P}$-a.s., as $k \to \infty$. Since $\Delta_{n_k}(s)$ is $\mathcal{F}_s$-measurable for each $k \in \mathbb{N}$, we conclude that $X'(s)$ is $\mathcal{F}_s$-measurable and thus $\sigma(X'(s)) \subseteq \mathcal{F}_s$. By induction, this extends to

$$\sigma(X(s), X'(s), \dots, X^{(M-1)}(s)) \subseteq \sigma(X(s), X'(s), \dots, X^{(N-1)}(s)) \subseteq \mathcal{F}_s,$$

so that Lemma A.2(b) yields the N-ple Markov property as formulated in (3.2).

It remains to show that the N-ple Markov property for $N \in \mathbb{N}$ and the σ-Markov property imply weak Markovianity of X. Fix $s \in \mathbb{T}$, $\varepsilon > 0$, and set

$$\begin{aligned}\mathcal{H}_1' &:= \sigma(X(u) : u \in (s-\varepsilon, s] \cap \mathbb{T}) \subseteq \mathcal{F}_s,\\ \mathcal{H}_2' &:= \sigma(X(u) : u \in [s, s+\varepsilon) \cap \mathbb{T}) \subseteq \sigma(X(t) : t \in [s,\infty) \cap \mathbb{T}),\\ \mathcal{H}' &:= \mathcal{H}_1' \vee \mathcal{H}_2' = \mathscr{A}_\varepsilon(s).\end{aligned}$$

Since $\sigma(X(s)) \subseteq \partial\mathscr{A}(s) \subseteq \mathscr{A}_\varepsilon(s)$, by Lemma A.2(c) the simple (i.e., 1-ple) Markov or σ-Markov property of X would imply

$$\mathcal{F}_s \perp\!\!\!\perp_{\mathscr{A}_\varepsilon(s)} \sigma(X(t) : t \in [s,\infty) \cap \mathbb{T}), \tag{3.5}$$

and thus the weak Markov property since $\varepsilon > 0$ was arbitrary. It remains to show that (3.5) also holds if X is N-ple Markov. Picking $K \in \mathbb{N}$ so large that $n_k > \varepsilon^{-1}$ for all $k \geq K$, we find that $(\Delta_{n_k}(s))_{k\geq K}$ (see (3.4)) is a sequence of $\mathscr{A}_\varepsilon(s)$-measurable random variables converging $\mathbb{P}$-a.s. to $X'(s)$. As before, repeating this argument inductively yields $\mathcal{H} := \sigma(X(s), X'(s), \ldots, X^{(N-1)}(s)) \subseteq \mathscr{A}_\varepsilon(s)$. This justifies the use of Lemma A.2(c) to establish (3.5) for the remaining case, and the desired conclusion follows. □

Remark 3.6. An analog to Definition 3.4 for *generalized* U-valued stochastic processes $(X(\phi))_{\phi\in C_c^\infty(\mathbb{T})}$ is obtained by replacing $\sigma(X(u) : u \in J)$ with the σ-algebra generated by X on an open set $J \subseteq \mathbb{T}$, which is given by $\sigma(X(\phi) : \phi \in C_c^\infty(\mathbb{T}), \operatorname{supp}\phi \subseteq J)$. Since pointwise evaluation is not meaningful for such processes, there is no analog to the simple Markov property. Furthermore, although the proof of the fact that weak Markov implies σ-Markov carries over, its converse now fails: The distributional derivative of white noise is a generalized process which is σ-Markov but not weak Markov, see [38, p. 62].

3.4. Characterization of weakly Markov Gaussian processes

A U-valued stochastic process $X = (X(t))_{t\in\mathbb{T}}$ is said to be Gaussian if the U^n-valued random variable $(X(t_1), X(t_2), \ldots, X(t_n))$ is Gaussian, for any $n \in \mathbb{N}$ and $\{t_i\}_{i=1}^n \subseteq \mathbb{T}$. For such processes, we shall characterize the weak Markov property of Definition 3.4 by extending the theory of Rozanov [38] from real-valued to U-valued processes.

We consider the case of a mean-square continuous Gaussian process X which is the solution of a stochastic evolution equation of the form $\mathcal{L}X = \dot{W}$ for some linear operator $\mathcal{L}\colon \mathsf{D}(\mathcal{L}) \subseteq L^2(\mathbb{T};U) \to L^2(\mathbb{T};U)$; here, $\dot{W}$ denotes spatiotemporal Gaussian white noise, cf. (1.1) and (1.4). More precisely, we assume that $\mathcal{L}$ has a bounded inverse $\mathcal{L}^{-1}$ which *colors* X, cf. [9, Definition 3], meaning

$$\langle X, \phi\rangle_{L^2(\mathbb{T};U)} \overset{d}{=} \mathscr{W}([\mathcal{L}^{-1}]^*\phi) \quad \forall \phi \in C_c^\infty(\mathbb{T};U), \tag{3.6}$$

where $\overset{d}{=}$ indicates equality in distribution. Here, $(\mathscr{W}(f))_{f\in L^2(\mathbb{T};U)} \subseteq L^2(\Omega)$ is an $L^2(\mathbb{T};U)$-isonormal Gaussian process, i.e., a family of mean-zero and real-valued Gaussian random variables satisfying

$$\langle \mathscr{W}(f), \mathscr{W}(g)\rangle_{L^2(\Omega)} = \langle f, g\rangle_{L^2(\mathbb{T};U)} \quad \forall f, g \in L^2(\mathbb{T};U).$$

The following theorem then states that the locality of the *precision operator* $\mathcal{L}^*\mathcal{L}$ is necessary for X to be weakly Markov.

Theorem 3.7. *Let $\mathcal{L}\colon \mathsf{D}(\mathcal{L}) \subseteq L^2(\mathbb{T};U) \to L^2(\mathbb{T};U)$ be a boundedly invertible linear operator, and suppose that $X = (X(t))_{t\in\mathbb{T}}$ is a mean-square continuous Gaussian U-valued process colored by $\mathcal{L}^{-1}$. Let F be a dense subset of U for which $C_c^\infty(\mathbb{T};F) \subseteq \mathsf{D}(\mathcal{L})$. Furthermore, suppose that $C_c^\infty(\mathbb{T};F)$ and its image under $\mathcal{L}$ are dense subsets of $L^2(\mathbb{T};U)$.*

If X has the weak Markov property from Definition 3.4 with respect to its natural filtration $(\mathcal{F}_t^X)_{t\in\mathbb{T}}$, then

$$\forall J \in \mathscr{I}: \quad \langle \mathcal{L}\phi, \mathcal{L}\psi\rangle_{L^2(\mathbb{T};U)} = 0 \quad \forall \phi \in C_c^\infty(J;F), \psi \in C_c^\infty(\mathbb{T}\setminus\overline{J};F), \tag{3.7}$$

where $\mathscr{I}$ denotes the set of all open intervals $J \subseteq \mathbb{T}$.

Proof. For all $J \in \mathscr{I}$ we define a closed subspace $\mathfrak{H}(J)$ of $L^2(\Omega)$ by

$$\mathfrak{H}(J) := \overline{\{\langle X, \phi\rangle_{L^2(\mathbb{T};U)} : \phi \in C_c^\infty(J;F)\}}^{L^2(\Omega)}. \tag{3.8}$$

Then the family $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ is a *Gaussian random field* in the sense of [38, Chapter 2, Section 3.1], and we can connect it to the present setting by showing that $\sigma(X(t) : t \in J) = \sigma(\mathfrak{H}(J))$. Indeed, we have $\sigma(\mathfrak{H}(J)) \subseteq \sigma(X(t) : t \in J)$ since $\langle X, \phi\rangle_{L^2(\mathbb{T};U)}$ is measurable with respect to the latter σ-algebra for all $\phi \in C_c^\infty(J;F)$ with $\operatorname{supp}\phi \subseteq J$.

In order to establish the converse inclusion, it suffices to verify the claim that $X(t)$ is $\sigma(\mathfrak{H}(J))$-measurable for each $t \in J$. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of U and write

$$X(t) = \textstyle\sum_{j=1}^\infty \langle X(t), e_j\rangle_U\, e_j \quad \text{in } L^2(\Omega;U). \tag{3.9}$$

Now we will show that $\langle X(t), e_j\rangle_U$ is $\sigma(\mathfrak{H}(J))$-measurable for every $j \in \mathbb{N}$. In fact, by the density of $F \subseteq U$ it suffices to consider $\langle X(t), x\rangle_U$ for $x \in F$. Let $(\phi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(J)$ be a sequence of bump functions concentrating around t, i.e., we have

$\lim_{n\to\infty} \int_{\mathbb{T}} f(s)\phi_n(s)\,\mathrm{d}s = f(t)$ in E for any $f \in C(\mathbb{T}; E)$, where E is an arbitrary Banach space. It follows from the mean-square continuity of X that $f := \langle X(\cdot), x\rangle_U \in C(\mathbb{T}; L^2(\Omega))$, thus with $E := L^2(\Omega)$ we obtain

$$\langle X(t), x\rangle_U = \lim_{n\to\infty} \int_{\mathbb{T}} \langle X(s), x\rangle_U \phi_n(s)\,\mathrm{d}s = \lim_{n\to\infty} \int_{\mathbb{T}} \langle X(s), \phi_n(s)x\rangle_U\,\mathrm{d}s$$

in $L^2(\Omega)$. Passing to a $\mathbb{P}$-a.s.-convergent subsequence in the rightmost expression, we find that $\langle X(t), x\rangle_U$ is a limit of $\sigma(\mathfrak{H}(J))$-measurable random variables. Thus, each summand in (3.9) is $\sigma(\mathfrak{H}(J))$-measurable, and passing to a $\mathbb{P}$-a.s.-convergent subsequence of $(\sum_{j=1}^N \langle X(t), e_j\rangle_U\, e_j)_{N\in\mathbb{N}}$ proves the claim.

The theory of [38, Chapter 2, Section 3.1], now implies that X has the weak Markov property from Definition 3.4 if and only if $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ is Markov in the sense of [38, p. 97]. For a general $B \subseteq \mathbb{T}$, we define

$$\mathfrak{H}_+(B) := \bigcap_{\varepsilon>0} \mathfrak{H}(B^\varepsilon), \tag{3.10}$$

where $B^\varepsilon := \{t \in \mathbb{T} : \operatorname{dist}(t, B) < \varepsilon\}$ denotes an open ε-neighborhood of B. Using the definition (3.10) for $B \in \{\partial J, J, \mathbb{T}\setminus J\}$, the Markov property for $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ implies that [38, Equations (3.14), p. 97] are satisfied for every $J \in \mathscr{I}$:

$$\mathfrak{H}_+(\partial J) = \mathfrak{H}_+(J) \cap \mathfrak{H}_+(\mathbb{T}\setminus J) \quad \text{and} \quad \mathfrak{H}_+(J)^\perp \perp \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp, \tag{3.11}$$

where we take $L^2(\Omega)$-orthogonal complements in $\mathfrak{H}(\mathbb{T})$.

Next we define $X^*\colon C_c^\infty(\mathbb{T}; F) \to L^2(\Omega)$ by $X^*(\phi) := \mathscr{W}(\mathcal{L}\phi)$, $\mathbb{P}$-a.s., for all $\phi \in C_c^\infty(\mathbb{T}; F)$, to which we associate the spaces

$$\mathfrak{H}^*(J) := \overline{\{X^*(\phi) : \phi \in C_c^\infty(J; F)\}}^{L^2(\Omega)}, \quad J \in \mathscr{I}. \tag{3.12}$$

Then X^* is dual to $(\langle X, \psi\rangle_{L^2(\mathbb{T};U)})_{\psi\in C_c^\infty(\mathbb{T};F)}$ in the sense that

$$\mathbb{E}[\langle X, \phi\rangle_{L^2(\mathbb{T};U)}\, X^*(\psi)] = \mathbb{E}[\mathscr{W}([\mathcal{L}^{-1}]^*\phi)\mathscr{W}(\mathcal{L}\psi)] = \langle \phi, \psi\rangle_{L^2(\mathbb{T};U)} \tag{3.13}$$

for $\phi, \psi \in C_c^\infty(\mathbb{T}; F)$. Next we will prove

$$\mathfrak{H}(\mathbb{T}) = \mathfrak{H}^*(\mathbb{T}) \tag{3.14}$$

by showing that both of these sets equal $\mathscr{Z} := \overline{\{\mathscr{W}(f) : f \in L^2(\mathbb{T}; U)\}}^{L^2(\Omega)}$. First, we note that $\mathfrak{H}(\mathbb{T})$ and $\mathfrak{H}^*(\mathbb{T})$ are clearly contained in $\mathscr{Z}$. Now let $Z \in \mathscr{Z}$ and $\varepsilon > 0$ be arbitrary, and let $f \in L^2(\mathbb{T}; U)$ satisfy $\|\mathscr{W}(f) - Z\|_{L^2(\Omega)} < \frac{1}{3}\varepsilon$. Since the image of $C_c^\infty(\mathbb{T}; F)$ under $\mathcal{L}$ is assumed to be dense in $L^2(\mathbb{T}; U)$, we may furthermore choose $\phi \in C_c^\infty(\mathbb{T}; F)$ such that $\|\mathcal{L}\phi - f\|_{L^2(\Omega;U)} < \frac{2}{3}\varepsilon$. Then

$$\begin{aligned}\|Z - X^*(\phi)\|_{L^2(\Omega)} &\le \|Z - \mathscr{W}(f)\|_{L^2(\Omega)} + \|\mathscr{W}(\mathcal{L}\phi - f)\|_{L^2(\Omega)}\\ &= \|Z - \mathscr{W}(f)\|_{L^2(\Omega)} + \|\mathcal{L}\phi - f\|_{L^2(\mathbb{T};U)} < \varepsilon,\end{aligned}$$

which shows $Z \in \mathfrak{H}^*(\mathbb{T})$ since $\varepsilon > 0$ was arbitrary. On the other hand, since $\mathcal{L}$ is densely defined and has a bounded inverse, it is in particular closed, hence $\mathcal{L}^*$ exists and is also densely defined by [42, Proposition 10.22]. It follows that the range of $(\mathcal{L}^{-1})^* = (\mathcal{L}^*)^{-1}$, which equals $\mathsf{D}(\mathcal{L}^*)$, is dense in $L^2(\mathbb{T}; U)$, so that there exists a $g \in L^2(\mathbb{T}; U)$ satisfying $\|f - (\mathcal{L}^{-1})^* g\|_{L^2(\mathbb{T};U)} < \frac{1}{3}\varepsilon$. Finally, we choose $\psi \in C_c^\infty(\mathbb{T}; F)$ such that $\|\psi - g\|_{L^2(\mathbb{T};U)} < \|(\mathcal{L}^{-1})^*\|^{-1}_{\mathscr{L}(L^2(\mathbb{T};U))}\frac{1}{3}\varepsilon$ so that

$$\begin{aligned}\|\, Z - \langle X, \psi\rangle_{L^2(\mathbb{T};U)}\,\|_{L^2(\Omega)} &< \tfrac{1}{3}\varepsilon + \|\mathscr{W}(f - [\mathcal{L}^{-1}]^* g)\|_{L^2(\Omega)} + \|\mathscr{W}([\mathcal{L}^{-1}]^*(g - \psi))\|_{L^2(\Omega)}\\ &= \tfrac{1}{3}\varepsilon + \|f - [\mathcal{L}^{-1}]^* g\|_{L^2(\mathbb{T};U)} + \|[\mathcal{L}^{-1}]^*(g - \psi)\|_{L^2(\mathbb{T};U)} < \varepsilon,\end{aligned}$$

hence also $Z \in \mathfrak{H}(\mathbb{T})$. We conclude that (3.14) holds.

The necessity of (3.7) for the weak Markov property of X will follow from

$$\mathfrak{H}^*(J) \subseteq \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp \quad \forall J \in \mathscr{I}. \tag{3.15}$$

Indeed, if X is weakly Markov, then (3.15) in combination with (3.11) would imply that the random variables defined by

$$\xi := X^*(\phi) \in \mathfrak{H}^*(J) \subseteq \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp \quad \text{and} \quad \eta := X^*(\psi) \in \mathfrak{H}^*(\mathbb{T}\setminus J) \subseteq \mathfrak{H}_+(J)^\perp,$$

are orthogonal, where $\phi \in C_c^\infty(J; F)$ and $\psi \in C_c^\infty(\mathbb{T}\setminus J; F)$. Therefore, we have $0 = \langle \xi, \eta\rangle_{L^2(\Omega)} = \mathbb{E}[X^*(\phi)X^*(\psi)] = \langle \mathcal{L}\phi, \mathcal{L}\psi\rangle_{L^2(\mathbb{T};U)}$, which shows (3.7). Note that by definition (3.12) and density, the orthogonality extends to all $\xi \in \mathfrak{H}^*(J)$ and $\eta \in \mathfrak{H}^*(\mathbb{T}\setminus J)$.

In order to verify (3.15), we again take $\xi = X^*(\phi)$ with ϕ as above. By the compact support of the latter, there exists $\varepsilon > 0$ such that $\langle \phi, \psi\rangle_{L^2(\mathbb{T};U)} = 0$ for all $\psi \in C_c^\infty((\mathbb{T}\setminus J)^\varepsilon; F)$. Hence, for $\eta = \langle X, \psi\rangle_{L^2(\mathbb{T};U)} \in \mathfrak{H}((\mathbb{T}\setminus J)^\varepsilon)$ we have $\xi \perp \eta$ by (3.13), from which we can deduce $\xi \perp \mathfrak{H}_+(\mathbb{T}\setminus J)$, and thus (3.15). □

In order to state and prove sufficient conditions for weak Markovianity of X in terms of the locality of its precision operator, we first need to collect some definitions which are based on objects encountered in the proof of Theorem 3.7. Namely, we will define spaces $(\mathcal{H}(J))_{J\in\mathscr{I}}$ such that $\mathcal{H}(\mathbb{T})$ is unitarily isomorphic to $\mathfrak{H}(\mathbb{T})$ and there exists a dense injection $\iota\colon C_c^\infty(\mathbb{T}; F) \to \mathcal{H}(\mathbb{T})$.

Associating, to each $\eta \in \mathfrak{H}(\mathbb{T})$, a mapping $I^{-1}\eta\colon C_c^\infty(\mathbb{T}; F) \to \mathbb{R}$ given by

$$I^{-1}\eta(\phi) := \mathbb{E}[\langle X, \phi\rangle_{L^2(\mathbb{T};U)}\eta], \quad \phi \in C_c^\infty(\mathbb{T}; F), \tag{3.16}$$

sets up a linear map $I^{-1}\colon \mathfrak{H}(\mathbb{T}) \to \mathcal{H}(\mathbb{T})$, where $\mathcal{H}(\mathbb{T})$ is defined as the range of I^{-1}. It is also injective since $I^{-1}\eta(\phi) = 0$ for all $\phi \in C_c^\infty(\mathbb{T}; F)$ means $\eta \perp C_c^\infty(\mathbb{T}; F)$ in $\mathfrak{H}(\mathbb{T})$, and thus $\eta = 0$ by (3.8).

We equip $\mathcal{H}(\mathbb{T})$ with the inner product $\langle v_1, v_2\rangle_{\mathcal{H}(\mathbb{T})} := \langle I v_1, I v_2\rangle_{L^2(\Omega)}$ for $v_1, v_2 \in \mathfrak{H}(\mathbb{T})$, rendering $I\colon \mathcal{H}(\mathbb{T}) \to \mathfrak{H}(\mathbb{T})$ a unitary isomorphism. For $J \in \mathscr{I}$ we can then define

$$\mathcal{H}(J) := \bigvee_{\varepsilon>0} \{v \in \mathcal{H}(\mathbb{T}) : v(\phi) = 0 \text{ for all } \phi \in C_c^\infty((\mathbb{T}\setminus J)^\varepsilon; F)\}, \tag{3.17}$$

where $\bigvee$ denotes the closed linear span.

A dense injection $\imath\colon C_c^\infty(\mathbb{T}; F) \to \mathcal{H}(\mathbb{T})$ is obtained by defining $\imath v\colon C_c^\infty(\mathbb{T}; F) \to \mathbb{R}$ in the following way, for any $v \in C_c^\infty(\mathbb{T}; F)$:

$$\imath v(\phi) := \langle v, \phi\rangle_{L^2(\mathbb{T};U)}, \quad \phi \in C_c^\infty(\mathbb{T}; F).$$

Indeed, we find $\imath v \in \mathcal{H}(\mathbb{T})$ since the duality relations (3.13) and (3.14) between X and X^* imply that $X^*(v) \in \mathfrak{H}^*(\mathbb{T}) = \mathfrak{H}(\mathbb{T})$ satisfies

$$\imath v(\phi) = \mathbb{E}[\langle X, \phi\rangle_{L^2(\mathbb{T};U)} X^*(v)] = [I^{-1}X^*(v)](\phi) \quad \forall \phi \in C_c^\infty(\mathbb{T}; F).$$

Moreover, the injectivity follows in the same way as for I^{-1}. To show density of the range, fix an arbitrary $v \in \mathcal{H}(\mathbb{T})$. Then $Iv \in \mathfrak{H}(\mathbb{T}) = \mathfrak{H}^*(\mathbb{T})$ and thus there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{T}; F)$ such that $X^*(\psi_n) \to Iv$ in $L^2(\Omega)$. Consequently, we have $\imath\psi_n = I^{-1}X^*(\psi_n) \to v$ in $\mathcal{H}(\mathbb{T})$.

Remark 3.8. For centered, real-valued Gaussian random fields $(Z(x))_{x\in\mathcal{X}}$ which are indexed by a compact metric space $(\mathcal{X}, d_{\mathcal{X}})$ and moreover mean-square continuous, a unitary isomorphism can be established between the $L^2(\Omega; \mathbb{R})$-closure of all linear combinations of point evaluations and the dual of the Cameron–Martin space for its associated Gaussian measure on the space $L^2(\mathcal{X}; \mathbb{R})$, see [22, Lemma 4.1(iii)]. We point out its analogy to the unitary isomorphism I between $\mathcal{H}(\mathbb{T})$ and $\mathfrak{H}(\mathbb{T})$ defined above.

Theorem 3.9. *Let the linear operator $\mathcal{L}\colon \mathsf{D}(\mathcal{L}) \subseteq L^2(\mathbb{T}; U) \to L^2(\mathbb{T}; U)$, the U-valued process $X = (X(t))_{t\in\mathbb{T}}$ and the subset $F \subseteq U$ be as in Theorem 3.7. Recall that $\mathcal{H}(\mathbb{T})$ is the range of the linear mapping I^{-1} defined by (3.16) and $\mathcal{H}(J)$ is given by (3.17) for all $J \in \mathscr{I}$. If*

$$\mathcal{H}(J) = \overline{\imath C_c^\infty(J; F)}^{\mathcal{H}(\mathbb{T})} \quad \forall J \in \mathscr{I}, \tag{3.18}$$

then (3.7) implies that X has the weak Markov property from Definition 3.4.

Proof. We will show that, under the additional assumption (3.18),

$$\mathfrak{H}^*(J) = \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp \quad \forall J \in \mathscr{I}. \tag{3.19}$$

Note that inclusion (3.15) also holds without this assumption, see the proof of Theorem 3.7. Identities (3.14) and (3.19) express that the collections $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ and $(\mathfrak{H}^*(J))_{J\in\mathscr{I}}$ are *dual* in the sense of [38, Chapter 2, Section 3.5]. In this situation, the theorem on [38, p. 100] yields that X is weakly Markov if and only if $(\mathfrak{H}^*(J))_{J\in\mathscr{I}}$ is *orthogonal*, meaning that $\mathfrak{H}^*(J) \perp \mathfrak{H}^*(\mathbb{T}\setminus \overline{J})$ for all $J \in \mathscr{I}$, which is equivalent to (3.7) by the respective definitions of X^* and $(\mathfrak{H}^*(J))_{J\in\mathscr{I}}$.

To verify (3.19), note that for each $\varepsilon > 0$,

$$\begin{aligned}\mathfrak{H}((\mathbb{T}\setminus J)^\varepsilon)^\perp &= \{\eta \in \mathfrak{H}(\mathbb{T}) : \langle \eta, \xi\rangle_{L^2(\Omega)} = 0 \text{ for all } \xi \in \mathfrak{H}((\mathbb{T}\setminus J)^\varepsilon)\}\\ &= \{\eta \in \mathfrak{H}(\mathbb{T}) : U^{-1}\eta(\phi) = 0 \text{ for all } \phi \in C_c^\infty((\mathbb{T}\setminus J)^\varepsilon; F)\}\\ &\cong \{v \in \mathcal{H}(\mathbb{T}) : v(\phi) = 0 \text{ for all } \phi \in C_c^\infty((\mathbb{T}\setminus J)^\varepsilon; F)\},\end{aligned}$$

and it follows that $\mathcal{H}(J) \cong \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp$. On the other hand, definition (3.12) implies $\mathcal{H}(J) \cong \mathfrak{H}^*(J)$, so together we indeed find (3.19). □

Remark 3.10. In order for locality of the precision operator $\mathcal{L}^*\mathcal{L}$ to imply the weak Markovianity of X, one needs to verify the additional condition (3.18). In the real-valued case, two examples of sufficient conditions on $\mathcal{L}$ for (3.18) to hold are [38, Lemmas 1 and 2, pp. 108–111], which are expressed in terms of boundedness of multiplication and translation operators, respectively, with respect to the norms $\|(\mathcal{L}^{-1})^* \cdot\|_{L^2(\mathbb{T})}$ and/or $\|\mathcal{L}\cdot\|_{L^2(\mathbb{T})}$. In [38, Chapter 3, Section 3.2], these results are applied to differential operators with sufficiently regular coefficients.

Although it is expected that analogous results can be derived in the U-valued setting, this subject is out of scope for the processes considered in the remainder of this article, since we establish Markovianity using direct methods instead of Theorem 3.9, see Section 4.4. However, we do use Theorem 3.7 to show when the process lacks Markov behavior in Section 4.4.3.

4. Fractional stochastic abstract Cauchy problem on $\mathbb{R}$

The aim of this section is to define a Hilbert space valued stochastic process $(Z_\gamma(t))_{t\in\mathbb{R}}$ which can be interpreted as a *mild solution* to the equation

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in \mathbb{R}, \qquad \gamma \in (1/2, \infty). \tag{4.1}$$

In Section 4.1 we specify the setting in which equation (4.1) will be considered. The fractional parabolic integral operator $(\partial_t + A)^{-\gamma}$ is defined in Section 4.2, whereas the noise term $\dot{W}^Q$ in (4.1) is the formal time derivative of the two-sided Q-Wiener process defined in Section 2.2. In Section 4.3 we combine these two notions to give a rigorous definition of the process $(Z_\gamma(t))_{t\in\mathbb{R}}$, and we indicate its relation to the fractional Q-Wiener process defined in Section 5. Lastly, in Section 4.4 we prove that $(Z_\gamma(t))_{t\in\mathbb{R}}$ is N-ple Markov if $\gamma = N \in \mathbb{N}$, but in general does not satisfy the weak Markov property when $\gamma \notin \mathbb{N}$.

4.1. Setting

The standing assumption throughout this section on the Hilbert space U and the linear operator A is as follows.

Assumption 4.1. The linear operator $-A\colon \mathsf{D}(A) \subseteq U \to U$ on the separable real Hilbert space U generates an *exponentially stable* C_0-semigroup $(S(t))_{t\ge 0}$ of bounded linear operators on U, i.e.,

$$\exists M_0 \in [1,\infty), w \in (0,\infty) : \forall t \in [0,\infty) : \quad \|S(t)\|_{\mathscr{L}(U)} \le M_0 e^{-wt}. \tag{4.2}$$

In addition, we may assume one or both of the following conditions on the fractional power γ and the linear operator Q:

Assumption 4.2.

(i) There exist $\gamma_0 \in (1/2, \infty)$ and $Q \in \mathscr{L}^+(U)$ such that

$$\int_0^\infty \|t^{\gamma_0 - 1} S(t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)} \,\mathrm{d}t < \infty.$$

(ii) The C_0-semigroup $(S(t))_{t\ge 0}$ is analytic.

For a more extensive overview of the theory of C_0-semigroups, the reader is referred to [14,34]. Note that the results in these works, while stated for complex Hilbert spaces, can be applied to the real setting by employing *complexifications* of (linear operators on) U, see e.g. Subsection B.2.1 of [23, Appendix B].

We remark that $(1/2, \infty)$ is the maximal range on which Assumption 4.2(i) can hold. Moreover, if Assumption 4.2(i) holds for some $\gamma_0 \in (1/2, \infty)$ then the same is true for all $\gamma' \in [\gamma_0, \infty)$; see Appendix A.2.

Under Assumption 4.2(ii), we have $\frac{\mathrm{d}}{\mathrm{d}t} A^j S(t) = -A^{j+1} S(t)$ as the classical derivative from $(0,\infty)$ to $\mathscr{L}(U)$ for all $j \in \mathbb{N}$; moreover,

$$\exists M_j \in [1,\infty) \colon \forall t \in (0,\infty) : \quad \|A^j S(t)\|_{\mathscr{L}(U)} \le M_j t^{-j} e^{-wt} \tag{4.3}$$

by [34, Chapter 2, Theorem 6.13(c)].

4.2. Fractional parabolic calculus and the deterministic problem

In this section we first consider the following deterministic counterpart to Eq. (4.1):

$$(\partial_t + A)^\gamma u(t) = f(t), \quad t \in \mathbb{R}, \qquad \gamma \in (0,\infty), \tag{4.4}$$

where $f \in L^2(\mathbb{R}; U)$. In order to define its mild solution, we introduce the operation of fractional parabolic integration, generalizing the scalar-valued setting with $A = 0$ which is treated in [21, Chapter 2].

Definition 4.3. Let Assumption 4.1 hold. Given $f\colon \mathbb{R} \to U$, we define its *Weyl type fractional parabolic integral* $\mathfrak{I}^\gamma f\colon \mathbb{R} \to U$ of order $\gamma \in (0,\infty)$ by

$$\mathfrak{I}^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_{-\infty}^t (t-s)^{\gamma-1} S(t-s) f(s) \,\mathrm{d}s \tag{4.5}$$

if this Bochner integral exists for almost all $t \in \mathbb{R}$.

Viewing $\mathfrak{I}^\gamma$ as a linear operator, it turns out that for all $p \in [1,\infty]$, we have $\mathfrak{I}^\gamma \in \mathscr{L}(L^p(\mathbb{R}; U))$ with $\|\mathfrak{I}^\gamma\|_{\mathscr{L}(L^p(\mathbb{R};U))} \le \frac{M_0}{w^\gamma}$, which follows by combining estimate (4.2) with Minkowski's integral inequality [41, Section A.1]. Setting $\mathfrak{I}^0 := \mathrm{Id}_U$, the family $(\mathfrak{I}^\gamma)_{\gamma\ge 0}$ is a semigroup of bounded operators on $L^2(\mathbb{R}; U)$. Indeed, the semigroup law $\mathfrak{I}^{\gamma_1+\gamma_2} = \mathfrak{I}^{\gamma_1}\mathfrak{I}^{\gamma_2}$, for all $\gamma_1, \gamma_2 \in [0,\infty)$, follows from an argument involving Fubini's theorem and [33, Equation (5.12.1)]. The adjoint operator $\mathfrak{I}^{\gamma*}$ of $\mathfrak{I}^\gamma$ satisfies the following formula:

$$\mathfrak{I}^{\gamma*} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^\infty (s-t)^{\gamma-1} [S(s-t)]^* f(s) \,\mathrm{d}s \quad \text{for all } t \in \mathbb{R}. \tag{4.6}$$

Given $T\colon \mathsf{D}(T) \subseteq U \to U$ and a measure space $(S, \mathscr{A}, \mu)$, we define the Bochner space counterpart $\mathcal{T}_S\colon \mathsf{D}(\mathcal{T}_S) \subseteq L^2(S; U) \to L^2(S; U)$ of T by

$$\mathsf{D}(\mathcal{T}_S) = L^2(S; \mathsf{D}(T)) \quad \text{and} \quad [\mathcal{T}_S f](s) := T f(s), \text{ a.a. } s \in S,\ f \in \mathsf{D}(\mathcal{T}_S). \tag{4.7}$$

Using the above notation with $T := A$ and $S := \mathbb{R}$, we have

$$(\partial_t + \mathcal{A}_\mathbb{R}) f = \partial_t f + \mathcal{A}_\mathbb{R} f, \qquad f \in \mathsf{D}(\partial_t + \mathcal{A}_\mathbb{R}) = H^1(\mathbb{R}; U) \cap L^2(\mathbb{R}; \mathsf{D}(A)),$$

where ∂_t denotes the Bochner–Sobolev weak derivative, whose domain is given by $\mathsf{D}(\partial_t) = H^1(\mathbb{R}; U) \subset L^2(\mathbb{R}; U)$; see Table 1. Since $\mathfrak{I}^\gamma$ can be interpreted as a negative fractional power of $\partial_t + \mathcal{A}_\mathbb{R}$, see Appendix B, it is natural to call $\mathfrak{I}^\gamma f$ a *mild solution* to (4.4).

4.3. Mild solution process

Combining the spatiotemporal fractional integration theory from Section 4.2 with the stochastic integral defined in Section 2.2, we can give a rigorous definition for the mild solution to (4.1). We first introduce the notion of predictability for a stochastic process indexed by $\mathbb{R}$.

Definition 4.4. An $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted U-valued process $(X(t))_{t\in\mathbb{R}}$ is *predictable* with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}}$ if $(t,\omega) \mapsto X(t,\omega)$ is strongly measurable with respect to the *predictable σ-algebra* $\mathcal{P}_{\mathbb{R}\times\Omega} := \sigma((s,t] \times F_s : s < t,\ F_s \in \mathcal{F}_s) \vee \sigma(\mathcal{N}_{\mathbb{P}})$.

Definition 4.5. Let Assumption 4.1 be satisfied and let $\gamma \in (1/2, \infty)$ be such that Assumption 4.2(i) holds with $\gamma_0 = \gamma$. An $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$-predictable modification of the process $Z_\gamma = (Z_\gamma(t))_{t\in\mathbb{R}}$ defined by

$$Z_\gamma(t) := \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{t} (t-s)^{\gamma-1} S(t-s)\, \mathrm{d}W^Q(s), \quad t \in \mathbb{R}, \tag{4.8}$$

is said to be a *mild solution* to (4.1).

Note that the stochastic integral on the right-hand side of (4.8) is convergent, i.e., it is a well-defined element in $L^2(\Omega; H)$, for each $t \in \mathbb{R}$. This is a direct consequence of Assumption 4.2(i) and the Itô isometry (2.1). Moreover, by Definition 4.5, the mild solution process is unique up to modification.

Proposition 4.6. *Let Assumption 4.1 be satisfied. Suppose that $t_0 \in [-\infty, \infty)$, $\gamma \in (1/2, \infty)$ are given. Define $\mathbb{T} := [t_0, \infty)$ if $t_0 \in \mathbb{R}$ or $\mathbb{T} := \mathbb{R}$ if $t_0 = -\infty$ and let Assumption 4.2(i) hold for $\gamma_0 = \gamma$. The process $(Z_\gamma(t \mid t_0))_{t\in\mathbb{T}}$ defined by*

$$Z_\gamma(t \mid t_0) := \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} (t-s)^{\gamma-1} S(t-s)\, \mathrm{d}W^Q(s), \quad t \in \mathbb{T}, \tag{4.9}$$

where $Z_\gamma(\,\cdot \mid -\infty) := Z_\gamma$, is mean-square continuous on $\mathbb{T}$.

If in addition Assumption 4.2(ii) is satisfied and there exists $N \in \mathbb{N}$ such that Assumption 4.2(i) holds for $\gamma_0 = \gamma - N$, then $(Z_\gamma(t \mid t_0))_{t\in\mathbb{T}}$ has N mean square derivatives and, for all $t \in [t_0, \infty)$ and $n \in \{0, \dots, N-1\}$, we have

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right) \tfrac{\mathrm{d}^n}{\mathrm{d}t^n} Z_\gamma(t \mid t_0) = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n} Z_{\gamma-1}(t \mid t_0). \tag{4.10}$$

Remark 4.7. The first part of Proposition 4.6 asserts that Z_γ is mean-square continuous, and thus continuous in probability. Combined with the fact that $(Z_\gamma(t))_{t\in\mathbb{R}}$ is $\left(\mathcal{F}_t^{\delta W^Q}\right)_{t\in\mathbb{R}}$-adapted by definition, we can apply [10, Proposition 3.7(ii)] (the proof of which can be generalized to unbounded index sets) to find an $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$-predictable modification of $(Z_\gamma(t))_{t\in\mathbb{R}}$. This modification is a mild solution in the sense of Definition 4.5.

Proof of Proposition 4.6. The mean-square continuity follows by Lemma A.6 in Appendix A, hence we turn to the mean-square differentiability. Define

$$Z_{\beta,j}(t) := \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} A^j S(t-s)\, \mathrm{d}W^Q(s), \quad t \in [t_0, \infty),$$

for $j \in \mathbb{N}_0$ and $\beta \in (1/2, \infty)$ such the right-hand side exists.

We claim that, under Assumption 4.2(i)–(ii) with $\gamma_0 = \gamma - N$, the function $t \mapsto t^{\beta-1} A^j S(t) Q^{\frac{1}{2}}$ belongs to $H_0^1(0, \infty; \mathscr{L}_2(U))$ if $\beta - j - \gamma + N \in [1, \infty)$. To this end, we first note that the product rule for the (classical) derivative yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t} t^{\beta-1} A^j S(t) Q^{\frac{1}{2}} = (\beta-1) t^{\beta-2} A^j S(t) Q^{\frac{1}{2}} - t^{\beta-1} A^{j+1} S(t) Q^{\frac{1}{2}}$$

with values in $\mathscr{L}(U)$ for all $t \in (0, \infty)$. Combining (4.3) with an argument involving a change of variables and the semigroup property (cf. the proof of Lemma A.4 in Appendix A), one can show that the $L^2(0, \infty; \mathscr{L}_2(U))$-norms of the two functions on the right-hand side can be estimated by that of the function $t \mapsto t^{\beta-j-2} S(t) Q^{\frac{1}{2}}$, which is finite since $\beta - j - 1 \geq \gamma_0$. Again by (4.3), we have

$$\|t^{\beta-1} A^j S(t) Q^{\frac{1}{2}}\|_{\mathscr{L}(U)} \leq M_j t^{\beta-j-1} \|Q^{\frac{1}{2}}\|_{\mathscr{L}(U)} \to 0 \quad \text{as } t \downarrow 0$$

since $\beta - j - 1 \geq \gamma_0 > 0$. Noting that $\mathscr{L}_2(U) \hookrightarrow \mathscr{L}(U)$ and using [23, Lemma A.9] then proves the claim.

Thus, we may apply Lemma A.6 from Appendix A, write the result as two separate integrals, and pull the closed operator A out of the stochastic integral defining $Z_{\beta,j+1}$ (cf. [10, Proposition 4.30]) to find

$$Z'_{\beta,j}(t) = Z_{\beta-1,j}(t) - Z_{\beta,j+1}(t). \tag{4.11}$$

$$= Z_{\beta-1,j}(t) - A Z_{\beta,j}(t). \tag{4.12}$$

Rearranging equation (4.12) for $\beta = \gamma$ and $j = 0$ implies (4.10) for $n = 0$. Applying (4.11) iteratively, we find that $Z_{\beta,j}$ has the nth mean-square derivative

$$Z^{(n)}_{\beta,j}(t) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} Z_{\beta-n+m,j+m}(t), \tag{4.13}$$

provided that $\beta - j - \gamma + N \in [n, \infty)$. Now we again let $\beta = \gamma$ and $j = 0$ and apply (4.12) with $\beta' = \gamma - n + m$ and $j' = m$ to each term on the right-hand side to derive (4.10) for the remaining values of n. $\square$

The next result concerns the covariance structure of the process Z_γ. Define the covariance operators $(Q_{Z_\gamma}(s,t))_{s,t\in\mathbb{R}} \subseteq \mathscr{L}_1^+(U)$ of Z_γ via the relation

$$\langle Q_{Z_\gamma}(s,t)x, y\rangle_U = \mathbb{E}\left[\left\langle Z_\gamma(s) - \mathbb{E}\big[Z_\gamma(s)\big], x\right\rangle_U \left\langle Z_\gamma(t) - \mathbb{E}[Z_\gamma(t)], y\right\rangle_U\right] \quad \text{for all } s,t\in\mathbb{R} \text{ and } x,y\in U; \tag{4.14}$$

note that $\mathbb{E}\big[Z_\gamma(\,\cdot\,)\big] \equiv 0$ in this case. The following proposition states that if $A := \kappa\,\mathrm{Id}_U$, then $Q_{Z_\gamma}(s,t)$ is separable, i.e., it can be decomposed into a (scalar) covariance function depending only on the 'time' variables $s,t\in\mathbb{R}$, and a 'spatial' covariance operator acting on U. Moreover, the temporal factor takes the form of a Matérn covariance function (1.2). In the language of spatiotemporal statistics, Z_γ has a marginal temporal covariance structure of Matérn type. This motivates the statistical relevance of Z_γ.

Proposition 4.8. *Let $\gamma \in (1/2,\infty)$, $A := \kappa\,\mathrm{Id}_U$ with $\kappa\in(0,\infty)$ and suppose that Assumption 4.2(i) is satisfied for $\gamma_0 = \gamma$. Then the covariance of Z_γ is separable and its temporal part is of Matérn type, i.e.,*

$$\forall s,t\in\mathbb{R},\ s\neq t: \quad Q_{Z_\gamma}(s,t) = \frac{2^{\frac12-\gamma}\kappa^{1-2\gamma}}{\sqrt{\pi}\,\Gamma(\gamma)}(\kappa|t-s|)^{\gamma-\frac12}K_{\gamma-\frac12}(\kappa|t-s|)\,Q.$$

Proof. For $A = \kappa\,\mathrm{Id}_U$, Assumption 4.1 is trivially satisfied and the definition of Z_γ takes on the following form for all $t\in\mathbb{R}$:

$$Z_\gamma(t) = \frac{1}{\Gamma(\gamma)}\int_{-\infty}^t (t-r)^{\gamma-1}e^{-\kappa(t-r)}\,\mathrm{d}W^Q(r) = \int_{\mathbb{R}} k_{\gamma,\kappa}(t-r)\,\mathrm{d}W^Q(r),$$

with real-valued convolution kernel $k_{\gamma,\kappa}(t) := \frac{1}{\Gamma(\gamma)}t_+^{\gamma-1}e^{-\kappa t}$, where $t_+^{\gamma-1} := t^{\gamma-1}$ if $t\in[0,\infty)$ and $t_+^{\gamma-1} := 0$ otherwise. Define $\widetilde{k}(s,r;x)\in\mathscr{L}(U;\mathbb{R})$ for $s,r\in\mathbb{R}$ and $x\in U$ by $\widetilde{k}(s,r;x)h := k_{\gamma,\kappa}(s-r)\langle x,h\rangle_U$ for all $h\in U$. Then combining the Itô isometry (2.1) and the polarization identity yields

$$\begin{aligned}\mathbb{E}[\langle Z_\gamma(s),x\rangle_U\,\langle Z_\gamma(t),y\rangle_U] &= \mathbb{E}\left[\int_{\mathbb{R}}\widetilde{k}(s,r;x)\,\mathrm{d}W^Q(r)\int_{\mathbb{R}}\widetilde{k}(t,r;y)\,\mathrm{d}W^Q(r)\right]\\ &= \int_{\mathbb{R}}\langle\widetilde{k}(s,r;x)Q,\widetilde{k}(t,r;y)\rangle_{\mathscr{L}_2(U;\mathbb{R})}\,\mathrm{d}r = \left\langle\int_{\mathbb{R}}k_{\gamma,\kappa}(s-r)k_{\gamma,\kappa}(t-r)\,\mathrm{d}r\,Qx,y\right\rangle_U.\end{aligned}$$

Since $x,y\in U$ were arbitrary, we find for all $h\in\mathbb{R}\setminus\{0\}$ the covariance operators

$$Q_{Z_\gamma}(t+h,t) = Q_{Z_\gamma}(t,t+h) = \int_{\mathbb{R}}k_{\gamma,\kappa}(t-r)k_{\gamma,\kappa}(t+h-r)\,\mathrm{d}r\,Q.$$

Using the change of variables $u(r) := h + 2(t-r)$ in the integral, we obtain

$$\begin{aligned}\int_{\mathbb{R}}k_{\gamma,\kappa}(t-r)k_{\gamma,\kappa}(t+h-r)\,\mathrm{d}r &= \frac12\int_{\mathbb{R}}k_{\gamma,\kappa}(\tfrac{u-h}{2})k_{\gamma,\kappa}(\tfrac{u+h}{2})\,\mathrm{d}u\\ &= \frac{2^{1-2\gamma}}{[\Gamma(\gamma)]^2}\int_{|h|}^\infty(u^2-h^2)^{\gamma-1}e^{-\kappa u}\,\mathrm{d}u = \frac{2^{\frac12-\gamma}\kappa^{1-2\gamma}}{\sqrt{\pi}\,\Gamma(\gamma)}(\kappa|h|)^{\gamma-\frac12}K_{\gamma-\frac12}(\kappa|h|),\end{aligned}$$

where the last identity follows by [32, Part I, Equation (3.13)]. $\square$

4.4. Markov behavior

In this section we consider the Markov behavior of the process Z_γ defined in Section 4.3. Namely, we will show that Z_N is N-ple Markov for $N\in\mathbb{N}$ (Theorem 4.10), whereas in general Z_γ is not weak Markov if $\gamma\notin\mathbb{N}$ (Example 4.16).

4.4.1. Integer case; main results

We first introduce the necessary notation and intermediate results leading up to the main theorem asserting the N-ple Markov property of Z_N. The proofs are postponed to Section 4.4.2.

If $\gamma\in(1/2,\infty)$ is such that Assumptions 4.1 and 4.2(i) hold with $\gamma_0=\gamma$, then we define for $t_0\in\mathbb{R}$ the truncated integral process $(\widetilde{Z}_\gamma(t\mid t_0))_{t\in\mathbb{R}}$ by

$$\widetilde{Z}_\gamma(t\mid t_0) := \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t\wedge t_0}(t-s)^{\gamma-1}S(t-s)\,\mathrm{d}W^Q(s), \tag{4.15}$$

so that $\widetilde{Z}_\gamma(\,\cdot\mid t_0) = Z_\gamma$ on $(-\infty,t_0]$ and $Z_\gamma = \widetilde{Z}_\gamma(\,\cdot\mid t_0) + Z_\gamma(\,\cdot\mid t_0)$ on (t_0,∞), where we recall the process $(Z_\gamma(t\mid t_0))_{t\in[t_0,\infty)}$ from (4.9). From these two identities, it follows that $t\mapsto\widetilde{Z}_\gamma(t\mid t_0)$ has the same mean-square differentiability at $t\in\mathbb{R}\setminus\{t_0\}$ as $Z_\gamma(t)$ (and $Z_\gamma(t\mid t_0)$ if $t\in(t_0,\infty)$). In the case $\gamma=N\in\mathbb{N}$, both have $N-1$ mean-square derivatives by Proposition 4.6 if Assumption 4.2(i) is satisfied for $\gamma_0 = \gamma-(N-1) = 1$. The same holds at the critical point $t=t_0$ since the first $N-1$ mean-square (right) derivatives of $Z_\gamma(\,\cdot\mid t_0)$ vanish there, see (4.13) in the proof of Proposition 4.6.

Under Assumption 4.2(ii), it holds that $A^j S(t) \in \mathscr{L}(U)$ for all $j \in \mathbb{N}_0$ and $t \in (0, \infty)$, see (4.3). Therefore, we can define the function $\overline{\Gamma}(n, (\,\cdot\,)A) : [0, \infty) \to \mathscr{L}(U)$ by

$$\overline{\Gamma}(n, tA) := \begin{cases} \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j S(t), & t \in (0, \infty); \\ \mathrm{Id}_U, & t = 0, \end{cases} \tag{4.16}$$

for $n \in \mathbb{N}$. Note the analogy with integer-order scalar-valued normalized upper incomplete gamma functions [33, Equations (8.4.10) and (8.4.11)]. We will use these functions to derive an expression for $(\widetilde{Z}_N(t \mid t_0))_{t\in[t_0,\infty)}$ in terms of Z_N and its mean-square derivatives at t_0. Recall that the U^N-valued process $\mathbf{Z}_N = (Z_N^{(n)})_{n=0}^{N-1}$ consists of Z_N and its first $N-1$ mean-square derivatives.

Proposition 4.9. *Let Assumptions 4.1 and 4.2(ii) be satisfied and suppose that Assumption 4.2(i) holds for $\gamma_0 = 1$. Then for all $N \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $t \in [t_0, \infty)$,*

$$\widetilde{Z}_N(t \mid t_0) = \zeta_N(t \mid t_0)\mathbf{Z}_N(t_0), \quad \mathbb{P}\text{-a.s.}, \tag{4.17}$$

where we define, for any $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^N)$ with $\mathcal{F}_{t_0} := \mathcal{F}_{t_0}^{\delta W^Q}$,

$$\zeta_N(t \mid t_0)\xi := \sum_{k=0}^{N-1} \frac{(t-t_0)^k}{k!} \overline{\Gamma}(N-k, (t-t_0)A)\,\xi_k, \tag{4.18}$$

using the incomplete gamma functions defined in (4.16).

In particular, adding $Z_N(t \mid t_0)$ on both sides of Eq. (4.17) yields

$$\forall t \in [t_0, \infty): \quad Z_N(t) = Z_N(t \mid t_0, \mathbf{Z}_N(t_0)), \quad \mathbb{P}\text{-a.s.}, \tag{4.19}$$

where the process $(Z_N(t \mid t_0, \xi))_{t\in[t_0,\infty)}$ is defined by

$$Z_N(t \mid t_0, \xi) := \zeta_N(t \mid t_0)\xi + Z_N(t \mid t_0), \quad t \in [t_0, \infty). \tag{4.20}$$

By (4.19), it suffices to show that $(Z_N(t \mid t_0, \xi))_{t\in[t_0,\infty)}$ has the N-ple Markov property in the sense of Definition 3.2 for any $t_0 \in \mathbb{R}$ and $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^N)$. In fact, we will show that it is N-ple Markov using Corollary 3.3; this is the subject of the following result, which is the main theorem of this section.

Theorem 4.10. *Let $N \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $\xi = (\xi_k)_{k=0}^{N-1} \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^{N+1})$ be given. Let Assumptions 4.1 and 4.2(ii) hold and suppose that Assumption 4.2(i) is satisfied for $\gamma_0 = 1$. Then the process $(Z_N(t \mid t_0, \xi))_{t\in[t_0,\infty)}$ from (4.20) has the N-ple Markov property in the sense of Definition 3.2 with respect to the transition operators $(T_{s,t})_{t_0\le s\le t}$ on $B_b(U^N)$ defined by*

$$T_{s,t}\varphi(\mathbf{x}) := \mathbb{E}[\varphi(\mathbf{Z}_N(t \mid s, \mathbf{x}))], \quad \varphi \in B_b(U^N),\ \mathbf{x} \in U^N,$$

and the increment filtration $(\mathcal{F}_t)_{t\in[t_0,\infty)} := (\mathcal{F}_t^{\delta W^Q})_{t\in[t_0,\infty)}$ from (2.2). The process $(Z_N(t))_{t\in\mathbb{R}}$ from (4.8) has $N-1$ mean-square derivatives and is N-ple Markov with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}} := (\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$, see Definition 3.2.

The statements and proofs of Proposition 4.9 and Theorem 4.10 use the following result regarding the mean-square differentiability of $(\zeta_N(t \mid t_0)\xi)_{t\in[t_0,\infty)}$, which is similar to Proposition 4.6.

Proposition 4.11. *Let $N \in \{2, 3, \dots\}$, $t_0 \in \mathbb{R}$ and $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^N)$ be given, where $\xi_k \in L^2(\Omega; \mathsf{D}(A))$ for $k \in \{0, \dots, N-2\}$. Suppose that Assumptions 4.1 and 4.2(ii) hold. Then the process $(\zeta_N(t \mid t_0)\xi)_{t\in[t_0,\infty)}$ defined by (4.18) is infinitely mean-square differentiable at any $t \in (t_0, \infty)$ and, for $n \in \{0, \dots, N-2\}$,*

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}\zeta_N(t \mid t_0)\xi = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n}\zeta_{N-1}(t \mid t_0)(\xi_{k+1} + A\xi_k)_{k=0}^{N-2}, \quad \mathbb{P}\text{-a.s.} \tag{4.21}$$

Moreover, we have $\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}\zeta_1(t \mid t_0)\xi = 0$, $\mathbb{P}$-a.s., for $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U)$.

Combining Propositions 4.6, 4.9 and 4.11 yields the following corollary.

Corollary 4.12. *Let $N \in \{2, 3, \dots\}$, $t_0 \in \mathbb{R}$ and $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^N)$ be given, where $\xi_k \in L^2(\Omega; \mathsf{D}(A))$ for $k \in \{0, \dots, N-2\}$. Let Assumptions 4.1 and 4.2(ii) hold and suppose that Assumption 4.2(i) is satisfied for $\gamma_0 = 1$. Then the process $(Z_N(t \mid t_0, \xi))_{t\in[t_0,\infty)}$ from (4.20) is $N-1$ times mean-square differentiable at any $t \in (t_0, \infty)$ and satisfies, for $n \in \{0, \dots, N-2\}$,*

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_N(t \mid t_0, \xi) = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_{N-1}\big(t \mid t_0, (\xi_{k+1} + A\xi_k)_{k=0}^{N-2}\big), \quad \mathbb{P}\text{-a.s.}$$

In particular, it holds for all $t \in \mathbb{R}$ and $n \in \{0, \dots, N-2\}$ that

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_N(t) = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_{N-1}(t), \quad \mathbb{P}\text{-a.s.}$$

Remark 4.13. Corollary 4.12 can be interpreted as saying that the process $(Z_N(t \mid t_0, \xi))_{t\in[t_0,\infty)}$ for $N \in \{2,3,\dots\}$ solves the $L^2(\Omega,\mathcal{F},\mathbb{P};U)$-valued initial value problem

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}t} + A\right)X(t) = Z_{N-1}(t \mid t_0, (\xi_{k+1} + A\xi_k)_{k=0}^{N-2}) & \forall t \in (t_0,\infty), \\ \qquad X(t_0) = \xi_0, \end{cases}$$

whenever $\xi_k \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathsf{D}(A))$ for $k \in \{0,\dots,N-2\}$. This observation is the key to the proofs of Propositions 4.9 and 4.14 below. It is also of interest for computational methods, as it implies that the computation of $Z_N(t \mid t_0,\xi)$ amounts to solving a first-order problem $N-1$ times. In fact, inductively applying this result and the fact that $(\frac{\mathrm{d}}{\mathrm{d}t} + A)\zeta_1(t \mid t_0)\eta = 0$ for $\eta \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U)$, we see that for $N\in\mathbb{N}$ we may interpret $(Z_N(t \mid t_0,\xi))_{t\in[t_0,\infty)}$ as the mild solution to the Nth order initial value problem

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}t} + A\right)^N X(t) = \dot{W}^Q(t) & \forall t \in (t_0,\infty), \\ \qquad \frac{\mathrm{d}^k}{\mathrm{d}t^k}X(t_0) = \xi_k & \forall k \in \{0,\dots,N-1\}. \end{cases}$$

Another key step in the proof of in Theorem 4.10 is given by the following result, which essentially amounts to the fact that $(T_{s,t})_{s\le t}$ satisfies (TO3).

Proposition 4.14. *Let $N \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $\xi \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U^N)$ be given. Let Assumptions 4.1 and 4.2(ii) hold and suppose that Assumption 4.2(i) is satisfied for $\gamma_0 = 1$. The stochastic process $(Z_N(t \mid t_0,\xi))_{t\in[t_0,\infty)}$ from (4.20) has the N-ple Chapman–Kolmogorov property, i.e., for all $t_0 \le s \le t$ we have*

$$\mathbf{Z}_N(t \mid t_0,\xi) = \mathbf{Z}_N(t \mid s, \mathbf{Z}_N(s \mid t_0,\xi)), \quad \mathbb{P}\text{-a.s.} \tag{4.22}$$

4.4.2. Integer case; proofs

As indicated in Section 4.4.1 above, the statements and proofs of Proposition 4.9 and Theorem 4.10 rely on Proposition 4.11, which we prove first.

Proof of Proposition 4.11. We first make some general remarks regarding the operators $\overline{\Gamma}(n,tA)$ from (4.16). Under Assumption 4.2(ii), estimate (4.3) implies that the set $\{t^jA^jS(t) : t \in (0,\infty)\} \subseteq \mathscr{L}(U)$ is uniformly bounded. It follows that $t \mapsto \overline{\Gamma}(n,tA)$ is a strongly continuous function from $[0,\infty)$ to $\mathscr{L}(U)$ for any $n\in\mathbb{N}$, which at $t\in(0,\infty)$ admits a classical derivative satisfying

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\overline{\Gamma}(n,tA) = \begin{cases} 0, & n = 1; \\ A\overline{\Gamma}(n-1,tA), & n \in \{2,3,\dots\}. \end{cases}$$

To prove the proposition, we may assume $t_0 = 0$, so fix $t \in (0,\infty)$. For arbitrary $M \in \mathbb{N}$, $j \in \mathbb{N}_0$ and $\boldsymbol{\eta} \in L^2(\Omega;U^M)$, we define $\zeta_{M,j}(t)\boldsymbol{\eta} := A^j\zeta_M(t \mid 0)\boldsymbol{\eta}$. Combining the product rule in the form $(\frac{\mathrm{d}}{\mathrm{d}t} + A)(uv) = u'v + u[(\frac{\mathrm{d}}{\mathrm{d}t} + A)v]$ with the above recurrence relation yields for $M\in\mathbb{N}$

$$\begin{aligned} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\zeta_{M,j}(t)\boldsymbol{\eta} &= \sum_{k=1}^{M-1} \frac{t^{k-1}}{(k-1)!}A^j\overline{\Gamma}(M-k,tA)\eta_k + \sum_{k=0}^{M-2}\frac{t^k}{k!}A^{j+1}\overline{\Gamma}(M-1-k,tA)\eta_k \\ &= \zeta_{(M-1),j}(t)(\eta_{k+1})_{k=0}^{M-1} + \zeta_{(M-1),(j+1)}(t)(\eta_k)_{k=0}^{M-1}. \end{aligned} \tag{4.23}$$

This shows in particular that (4.21) holds for integers $N\ge 2$ and $n=0$, by applying (4.23) with $M=N$, $\boldsymbol{\eta} = \xi$ and $j=0$. Moreover,

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\zeta_{1,j}(t)\,\eta = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)A^jS(t)\,\eta = 0.$$

Iteratively applying (4.23) and the latter identity then yields that $\zeta_{M,j}\boldsymbol{\eta}$ is $M-1$ times (mean-square) differentiable with an nth derivative of the form

$$\zeta_{M,j}^{(n)}(t)\boldsymbol{\eta} = \sum_{\ell=0}^{n}\sum_{m=0}^{\ell} C_{\ell,m}\zeta_{(M-\ell),(j+n-m)}(t)B_{\ell,m}\boldsymbol{\eta}, \tag{4.24}$$

where $C_{\ell,m}\in\mathbb{R}$, $B_{\ell,m}\in\mathscr{L}(U^M;U^{M-\ell})$ and $\zeta_{(M-\ell),(j+n-m)} := 0$ if $M-\ell<1$. In particular, $\zeta_N(\,\cdot \mid t_0)\xi$ is $N-1$ times (mean-square) differentiable as claimed. In order to deduce that (4.21) also holds for $n\in\{1,\dots,N-2\}$, we need to justify taking the nth derivative on both sides and commuting it with A. Since A is closed, it suffices to verify that $\zeta_N'(\,\cdot \mid t_0)\xi$, $A^j\zeta_N(\,\cdot \mid t_0)\xi$, $\zeta_{N-1}(\,\cdot \mid t_0)(\xi_{j+1})_{j=0}^{N-1}$ and $A^j\zeta_{N-1}(\,\cdot \mid t_0)(\xi_j)_{j=0}^{N-1}$ admit nth derivatives for $j\in\{0,1\}$. Indeed, these assertions follow from (4.24). $\square$

We can now prove Propositions 4.9 and 4.14. For the proof of the latter we may use Corollary 4.12, since it only combines Propositions 4.6, 4.9 and 4.11.

Proof of Proposition 4.9. We use induction on $N\in\mathbb{N}$. For $N=1$, $t\in(t_0,\infty)$,

$$\widetilde{Z}_1(t \mid t_0) = S(t-t_0)\int_{-\infty}^{t_0} S(t_0-s)\,\mathrm{d}W^Q(s) = \zeta_1(t \mid t_0)Z_1(t_0), \quad \mathbb{P}\text{-a.s.}$$

Now suppose that the statement is true for a given $N \in \mathbb{N}$. By Proposition 4.6 and the discussion below (4.15), Z_{N+1} and $\widetilde{Z}_{N+1}(\,\cdot \mid t_0)$ have N mean-square derivatives which satisfy $\left(\frac{\mathrm{d}}{\mathrm{d}t}+A\right)\widetilde{Z}_{N+1}(t \mid t_0) = \widetilde{Z}_N(t \mid t_0)$ and $\left(\frac{\mathrm{d}}{\mathrm{d}t}+A\right)Z^{(k)}_{N+1}(t) = Z^{(k)}_N(t)$ for all $k \in \{0,\dots,N-1\}$ and $t \in (t_0,\infty)$. Combined with Proposition 4.11 and the induction hypothesis, we find

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A\right)\zeta_{N+1}(t \mid t_0)\mathbf{Z}_{N+1}(t_0) = \zeta_N(t \mid t_0)\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A\right)Z^{(k)}_{N+1}(t_0)\right]_{k=0}^{N-1} = \zeta_N(t \mid t_0)\mathbf{Z}_N(t_0) = \widetilde{Z}_N(t \mid t_0) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A\right)\widetilde{Z}_{N+1}(t \mid t_0).$$

Since $\widetilde{Z}_{N+1}(t_0 \mid t_0) = Z_{N+1}(t_0) = \zeta_{N+1}(t_0 \mid t_0)\mathbf{Z}_{N+1}(t_0)$, we find that (4.17) with $N+1$ holds on $[t_0,\infty)$ by the uniqueness of solutions to $L^2(\Omega,\mathcal{F},\mathbb{P};U)$-valued abstract Cauchy problems, see [34, Chapter 4, Theorem 1.3]. □

Proof of Proposition 4.14. Let $t_0 \le s \le t$. We use induction on $N \in \mathbb{N}$. For the base case $N = 1$ we have

$$\begin{aligned} Z_1(t \mid s, Z_1(s \mid t_0,\xi)) &= S(t-s)Z_1(s \mid t_0,\xi) + \int_s^t S(t-r)\,\mathrm{d}W^Q(r) \\ &= S(t-s)S(s-t_0)\xi + S(t-s)\int_{t_0}^s S(s-r)\,\mathrm{d}W^Q(r) + \int_s^t S(t-r)\,\mathrm{d}W^Q(r) \\ &= S(t-t_0)\xi + \int_{t_0}^s S(t-r)\,\mathrm{d}W^Q(r) + \int_s^t S(t-r)\,\mathrm{d}W^Q(r) = Z_1(t \mid t_0,\xi), \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

for $\xi \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathsf{D}(A))$. Now suppose that the result holds for $N \in \mathbb{N}$ and let $\xi \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathsf{D}(A)^N)$. Then, for any $t \in (s,\infty)$,

$$\begin{aligned} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A\right)Z_{N+1}(t \mid s,\mathbf{Z}_{N+1}(s \mid t_0,\xi)) &= Z_N\left(t \mid s, \left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A\right)Z^{(k)}_{N+1}(s \mid t_0,\xi)\right]_{k=0}^{N-1}\right) = Z_N\left(t \mid s, \mathbf{Z}_N(s \mid t_0, [\xi_{k+1}+A\xi_k]_{k=0}^{N-1})\right) \\ &= Z_N(t \mid t_0, [\xi_{k+1}+A\xi_k]_{k=0}^{N-1}) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+A\right)Z_{N+1}(t \mid t_0,\xi), \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

where we applied Corollary 4.12 in every identity except the third, which uses the induction hypothesis. Moreover, the relation $Z_{N+1}(s \mid s,\mathbf{Z}_{N+1}(s \mid t_0,\xi)) = Z_{N+1}(s \mid t_0,\xi)$ is evident from the definitions. Together, these facts imply that the difference process $Y := Z_{N+1}(\,\cdot \mid s,\mathbf{Z}_{N+1}(s \mid t_0,\xi)) - Z_{N+1}(\,\cdot \mid t_0,\xi)$ solves

$$\begin{cases} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}+\mathcal{A}_\Omega\right)Y(t) = 0 & \forall t \in (s,\infty); \\ \qquad\qquad Y(s) = 0, \end{cases}$$

where $\mathcal{A}_\Omega\colon L^2(\Omega;\mathsf{D}(A)) \subseteq L^2(\Omega;U) \to L^2(\Omega;U)$ is as in (4.7). Since $-\mathcal{A}_\Omega$ is the generator of a C_0-semigroup on $L^2(\Omega;U)$, see Lemma B.2 in Appendix B, the uniqueness result [34, Chapter 4, Theorem 1.3] shows that $Y \equiv 0$ on $[s,\infty)$, meaning that $Z_{N+1}(t \mid t_0,\xi) = Z_{N+1}(t \mid s,\mathbf{Z}_{N+1}(s \mid t_0,\xi))$ holds $\mathbb{P}$-a.s. for all $t \in [s,\infty)$. Taking the nth mean-square derivative for $n \in \{0,\dots,N\}$ on both sides, which is justified by Corollary 4.12, we find (4.22). In order to establish this identity for general $\xi \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U^N)$, we use the density of $\mathsf{D}(A)$ in U [34, Chapter 1, Corollary 2.5], which implies the density of $L^2(\Omega;\mathsf{D}(A)^N)$ in $L^2(\Omega;U^N)$, hence it suffices to argue that $\boldsymbol{\eta} \mapsto \mathbf{Z}_{N+1}(t \mid t_0,\boldsymbol{\eta})$ is continuous on $L^2(\Omega;U^N)$ for any fixed $t \in [t_0,\infty)$. Continuity of $\boldsymbol{\eta} \mapsto \zeta_{N+1}(t \mid t_0)\boldsymbol{\eta}$ follows from the fact that $\eta \mapsto \overline{\Gamma}(N+1-k,(t-t_0))\eta$ is bounded on $L^2(\Omega;U)$ for any $k \in \{0,\dots,N\}$. The same holds for the derivatives of $\zeta_{N+1}(\,\cdot \mid t_0)\boldsymbol{\eta}$ since they are of the same form by Proposition 4.11. Together, the conclusion follows. □

With these intermediate results in place, we are ready to prove the main theorem asserting the N-ple Markovianity of Z_N. Its proof is a generalization of [10, Theorem 9.14] and [35, Theorem 9.30], which concern the case $N = 1$.

Proof of Theorem 4.10. Step 1: Well-definedness of $(T_{s,t})_{t_0 \le s \le t}$. We have to show that $T_{s,t}\varphi = \mathbb{E}[\varphi(\mathbf{Z}_N(t \mid s,\,\cdot\,))]$ is measurable for $\varphi \in B_b(U^N)$. Let $\mathscr{H}$ be the linear space of bounded $\varphi\colon U^N \to \mathbb{R}$ such that $\mathbb{E}[\varphi(\mathbf{Z}_N(t \mid s,\,\cdot\,))]$ is measurable. Arguing as in the proof of Proposition 4.14, we find that $[\mathbf{Z}_N(t \mid s,\,\cdot\,)](\omega)$ is continuous on U^N for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then, for $\varphi \in \mathscr{C} := C_b(U^N)$, the dominated convergence theorem implies that $\mathbb{E}[\varphi(\mathbf{Z}_N(t \mid s,\,\cdot\,))]$ is also continuous, hence Borel measurable, and thus $\mathscr{C} \subseteq \mathscr{H}$. Moreover, $\mathscr{H}$ contains all constant functions and given $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}$ such that $0 \le \varphi_n \uparrow \varphi$ pointwise for some bounded limit function φ, we find $\varphi \in \mathscr{H}$ by monotone convergence. Since $\mathscr{C}$ is closed under pointwise multiplication, we find $B_b(U^N,\sigma(\mathscr{C})) = B_b(U^N) \subseteq \mathscr{H}$ by the monotone class theorem [37, Chapter 0, Theorem 2.2].

Step 2: N-ple Markovianity. For $t_0 \le s \le t$ and $\varphi \in B_b\left(U^N\right)$, we show

$$\mathbb{E}[\varphi(\mathbf{Z}_N(t \mid t_0,\boldsymbol{\xi})) \mid \mathcal{F}_s] = T_{s,t}\varphi(\mathbf{Z}_N(s \mid t_0,\boldsymbol{\xi})), \quad \mathbb{P}\text{-a.s.},$$

for all $\boldsymbol{\xi} \in L^2(\Omega,\mathcal{F}_s,\mathbb{P};U^N)$. By Proposition 4.14, it suffices to verify that $\mathbb{E}[\varphi(\mathbf{Z}_N(t \mid s,\boldsymbol{\xi})) \mid \mathcal{F}_s] = T_{s,t}\varphi(\boldsymbol{\xi})$ holds $\mathbb{P}$-a.s. By a monotone class argument similar to that of Step 1, it suffices to consider $\varphi \in C_b(U^N)$. As in [35, Theorem 9.30], one can first verify this identity directly for simple $\boldsymbol{\xi} = \sum_{j=1}^n \mathbf{x}_j \mathbf{1}_{A_j}$, with $n \in \mathbb{N}$, $\{\mathbf{x}_1,\dots,\mathbf{x}_n\} \subseteq U^N$ and disjoint events $\{A_1,\dots,A_n\} \subseteq \mathcal{F}_s$ covering Ω, and subsequently extend it to general $\boldsymbol{\xi} \in L^2(\Omega,\mathcal{F}_s,\mathbb{P};U^N)$ by an approximation argument, using the continuity of the functions $\mathbf{Z}_N(t \mid s,\,\cdot\,)$ and φ. Finally, the statement regarding $(Z_N(t))_{t\in\mathbb{R}}$ follows from Proposition 4.9. □

4.4.3. Non-Markovianity in the fractional case

We conclude this section by showing how Theorem 3.7 can be used to deduce that Z_γ is not weakly Markov (see Definition 3.4) if $\gamma \notin \mathbb{N}$. To this end, we determine the coloring operator of Z_γ.

Proposition 4.15. *Let Assumption 4.1 hold and suppose that $\gamma \in (1/2,\infty)$ is such that Assumption 4.2(i) holds for $\gamma_0 = \gamma$. Then the coloring operator of Z_γ, see (3.6), is given by $\mathcal{L}_\gamma^{-1} = \mathfrak{I}^\gamma Q_{\mathbb{R}}^{\frac{1}{2}} \in \mathscr{L}(L^2(\mathbb{R};U))$, where $\mathfrak{I}^\gamma$ is as in (4.5).*

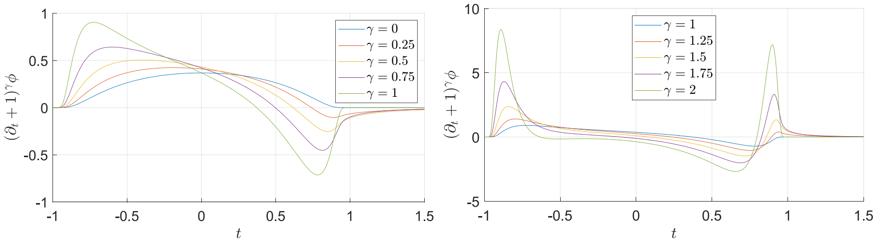

Fig. 1. Graphs of the fractional parabolic derivative $(\partial_t + 1)^\gamma \phi$ with ϕ from (4.26) for certain values of $\gamma \in [0, 2]$. Note the different scales on the y-axes.

Table 2
Numerically approximated values of $F_\gamma(\phi, \psi)$, see (4.25), with ϕ from (4.26) and $\psi := \phi(\cdot - 2 - \delta)$ for certain values of γ and δ.

δ \ γ	0.25	0.50	0.75	1	1.25	1.50	1.75	2	2.25	2.50	2.75	3
10^{-1}	0.004	0.007	0.007	0	0.016	0.042	0.059	0	0.298	1.078	2.111	0
10^{-2}	0.005	0.009	0.009	0	0.024	0.065	0.098	0	0.622	2.568	5.829	0
10^{-3}	0.005	0.009	0.009	0	0.025	0.068	0.104	0	0.678	2.850	6.601	0

Proof. Using the stochastic Fubini theorem as in the proof of [23, Proposition 3.11], we find that $\langle Z_\gamma, f\rangle_{L^2(\mathbb{R};U)} = \int_{\mathbb{R}} \widetilde{\Phi}_f(s)\,\mathrm{d}W^Q(s)$ holds $\mathbb{P}$-a.s. for every $f \in C_c^\infty(\mathbb{R}; U)$, where $\widetilde{\Phi}_f : \mathbb{R} \to \mathscr{L}(U; \mathbb{R})$ is given by

$$\widetilde{\Phi}_f(s)u := \frac{1}{\Gamma(\gamma)} \int_s^\infty \langle (t-s)^{\gamma-1} S(t-s)u, f(t)\rangle_U \,\mathrm{d}t = \langle u, \mathfrak{I}^{\gamma *} f(s)\rangle_U, \quad s \in \mathbb{R},\ u \in U,$$

and we recall equation (4.6) for the adjoint of $\mathfrak{I}^\gamma$. Hence, we have $\|\langle Z_\gamma, f\rangle_{L^2(\mathbb{R};U)}\|^2_{L^2(\Omega)} = \int_{\mathbb{R}} \|\widetilde{\Phi}_f(t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U;\mathbb{R})}\,\mathrm{d}t = \|\mathcal{Q}_{\mathbb{R}}^{\frac{1}{2}} \mathfrak{I}^{\gamma *} f\|^2_{L^2(\mathbb{R};U)}$ by (2.1), and thus (3.6) with $(\mathcal{L}_\gamma^{-1})^* = \mathcal{Q}_{\mathbb{R}}^{\frac{1}{2}} \mathfrak{I}^{\gamma *}$ by the polarization identity. $\square$

Example 4.16. Let Assumptions 4.1 and 4.2(ii) be satisfied and suppose that $\gamma \in (1/2, \infty)$ is such that Assumption 4.2(i) holds for $\gamma_0 = \gamma$. The latter implies that $\partial_t + \mathcal{A}_{\mathbb{R}} = \mathcal{B}$, and we always have $\mathcal{B}^{-\gamma} = \mathfrak{I}^\gamma$, see Section 4.2. Thus,

$$\mathcal{L}_\gamma^* \mathcal{L}_\gamma = \left(\mathcal{Q}_{\mathbb{R}}^{-\frac{1}{2}} \mathcal{B}^\gamma\right)^* \mathcal{Q}_{\mathbb{R}}^{-\frac{1}{2}} \mathcal{B}^\gamma = \mathcal{B}^{\gamma *} \mathcal{Q}_{\mathbb{R}}^{-1} \mathcal{B}^\gamma = (\partial_t + \mathcal{A}_{\mathbb{R}})^{\gamma *} \mathcal{Q}_{\mathbb{R}}^{-1} (\partial_t + \mathcal{A}_{\mathbb{R}})^\gamma.$$

Moreover, this assumption implies that $\mathsf{D}(A^n)$ is dense in U for all $n \in \mathbb{N}$ by [34, Chapter 2, Theorem 6.8(c)]; choosing n large enough, we also find that $C_c^\infty(\mathbb{T}; \mathsf{D}(A^n))$ is dense in $\mathsf{D}(\mathcal{B}^\gamma)$, so we can take $F = \mathsf{D}(A^n)$ in Theorem 3.7.

Although Q^{-1} may be a nonlocal spatial operator, $\mathcal{Q}_{\mathbb{R}}^{-1}$ is always local in time. Thus for $\gamma \in \mathbb{N}$, the precision operator is local as a composition of three local operators, in accordance with the Markovianity shown in Section 4.4.

For $\gamma \notin \mathbb{N}$, we will show that the precision operator is not local in general. Suppose that A has an eigenvector $v \in U$ with corresponding eigenvalue $\lambda \in \mathbb{R}$. Such eigenpairs exists for example if $A = (\kappa^2 - \Delta)^\beta$ with $\kappa, \beta \in (0, \infty)$ and Δ the Dirichlet Laplacian on a bounded Euclidean domain $\mathcal{D} \subsetneq \mathbb{R}^d$. If we moreover assume that $v \in \mathsf{D}(Q^{-\frac{1}{2}})$, then we find $\mathcal{Q}_{\mathbb{R}}^{-\frac{1}{2}} \mathcal{B}^\gamma(\phi \otimes v) = [(\partial_t + \lambda)^\gamma \phi] \otimes Q^{-\frac{1}{2}} v$ for $\phi \in C_c^\infty(\mathbb{R})$ since the spectral mapping theorem implies $S(t)v = e^{-\lambda t} v$ for all $t \in [0, \infty)$. It thus suffices to consider the case $A = \lambda \in \mathbb{R}$, i.e., we wish to find disjointly supported $\phi, \psi \in C_c^\infty(\mathbb{R})$ such that

$$F_\gamma(\phi, \psi) := |\langle (\partial_t + \lambda)^\gamma \phi, (\partial_t + \lambda)^\gamma \psi\rangle_{L^2(\mathbb{R})}| \neq 0. \tag{4.25}$$

We will discuss this by means of a numerical experiment for the case $\lambda = 1$, using the following smooth function $\phi \in C_c^\infty(\mathbb{R})$ supported on $[-1, 1]$:

$$\phi(t) := \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & x \in (-1, 1), \\ 0, & x \in \mathbb{R} \setminus (-1, 1), \end{cases} \tag{4.26}$$

and taking $\psi := \phi(\cdot - 2 - \delta)$ for some $\delta \in (0, \infty)$. In Fig. 1, we see that the parabolic derivatives of ϕ consists of (positive or negative) peaks. For $\gamma \notin \mathbb{N}$, the support of the last of these peaks appears to include the whole of $[1, \infty)$, with its absolute value taking rapidly decaying yet nonzero values there. Therefore, the idea is to take δ small enough, making the right-hand side tail of ϕ overlap with the first peak of ψ to obtain a non-zero $L^2(\mathbb{R})$-inner product. Table 2 shows the approximate outcomes of this process for various values of γ and δ, using symbolic differentiation and numerical integration.

Note the contrast with the merely spatial Matérn case, where the self-adjointness of the shifted Laplacian $\kappa^2 - \Delta$ causes $L_\beta^* L_\beta = \tau^2(\kappa^2 - \Delta)^{2\beta}$, thus we find a weak Markov property also for half-integer values of $\beta \in (0, \infty)$.

5. Fractional Q-Wiener process

In this final section, we further motivate our interest in solutions to (4.1) by relating them to *fractional Q-Wiener processes*, which are U-valued generalizations of the widely studied (real-valued) fractional Brownian motion. In Section 5.1 we show that, analogously to the real-valued case (see [29, Definition 2.1]), a fractional Q-Wiener process can also be expressed as a Mandelbrot–Van Ness type stochastic integral over $\mathbb{R}$. Using this representation, we show that fractional Q-Wiener processes are limiting cases of mild solutions to (4.1) as introduced in Definition 4.5. Finally, in Section 5.2 we comment on the Markov behavior of fractional Q-Wiener processes, and we propose possible directions in which to extend or complement the present results of Section 3 to establish necessary and sufficient conditions for (weak or N-ple) Markovianity.

The following definition was introduced in [13, Definition 2.1].

Definition 5.1. Let $Q \in \mathscr{L}_1^+(U)$. A U-valued Gaussian process $(W_H^Q(t))_{t\in\mathbb{R}}$ is called a *fractional Q-Wiener process with Hurst parameter* $H \in (0,1)$ if

(f-WP1) $\mathbb{E}[W_H^Q(t)] = 0$ for all $t \in \mathbb{R}$;
(f-WP2) $Q_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})Q$ for all $s,t \in \mathbb{R}$;
(f-WP3) W_H^Q has continuous sample paths.

Here, $(Q_H(s,t))_{s,t\in\mathbb{R}} \subseteq \mathscr{L}_1^+(U)$ are the covariance operators of W_H^Q, cf. (4.14).

Note that for $H = \frac{1}{2}$, the above definition reduces to a characterization of a standard (non-fractional) Q-Wiener process when restricted to $[0,\infty)$.

5.1. Integral representation and relation to Z_γ

Let $Q \in \mathscr{L}_1^+(U)$ and let $(W^Q(t))_{t\in\mathbb{R}}$ be a two-sided Q-Wiener process, see Section 2.2. For $H \in (0,1)$, define $(\widehat{W}_H^Q(t))_{t\in\mathbb{R}}$ by

$$\widehat{W}_H^Q(t) := \int_{\mathbb{R}} K_H(t,r)\,\mathrm{d}W^Q(r), \quad t \in \mathbb{R}, \tag{5.1}$$

where the Mandelbrot–Van Ness [29] type kernel $K_H : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$K_H(t,r) := \frac{1}{C_H}\left[(t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right], \quad (t,r) \in \mathbb{R}^2. \tag{5.2}$$

The constant $C_H := \int_{\mathbb{R}} \left|(1-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right|^2 \mathrm{d}r = \frac{3-2H}{4H}\mathrm{B}(2-2H, H+\frac{1}{2})$ (where B is the beta function) [36, Theorem B.1], ensures that $\widehat{Q}_H(1,1) = Q$, where $(\widehat{Q}_H(t,s))_{t,s\in\mathbb{R}}$ denote the covariance operators of $\widehat{W}_H^Q$. Then $\widehat{W}_H^Q$ has a modification which is a fractional Q-Wiener process:

Proposition 5.2. *For every $t \in \mathbb{R}$, (5.1) yields a well-defined random variable $\widehat{W}_H^Q(t) \in L^2(\Omega, \mathcal{F}_t^{\delta W^Q}, \mathbb{P}; U)$, and there exists a modification of $(\widehat{W}_H^Q(t))_{t\in\mathbb{R}}$ which is a fractional Q-Wiener process in the sense of Definition 5.1.*

Proof. To see that $\widehat{W}_H^Q(t) \in L^2(\Omega;U)$, note that $\mathbb{E}[\|\widehat{W}_H^Q(t)\|_U^2] = \int_{\mathbb{R}} \|K_H(t,r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(U)}^2 \mathrm{d}r = \operatorname{tr} Q \int_{\mathbb{R}} |K_H(t,r)|^2 \mathrm{d}r < \infty$ by the Itô isometry (2.1). Since K_H is a deterministic kernel integrated with respect to a mean-zero Gaussian process W^Q, it readily follows that $\widehat{W}_H^Q$ is also mean-zero Gaussian. For the covariance operators $(\widehat{Q}_H(t,s))_{t,s\in\mathbb{R}}$ of $\widehat{W}_H^Q$, we can argue as in the proof of Proposition 4.8 to find $\widehat{Q}_H(t,s) = \int_{\mathbb{R}} K_H(s,r)K_H(t,r)\,\mathrm{d}r\,Q = \mathbb{E}[B_H(t)B_H(s)]\,Q$ for all $t,s \in \mathbb{R}$, where $B_H = (B_H(t))_{t\in\mathbb{R}}$ denotes (real-valued) fractional Brownian motion. Hence (f-WP2) holds by the properties of B_H. Lastly, the existence of a (Hölder) continuous modification of (5.1) can be established analogously to the real-valued case, by using that $\widehat{W}_H^Q$ is self-similar with stationary increments and applying the Kolmogorov–Chentsov theorem [8, Corollary 3.10]. □

Now we consider the relation between the fractional Q-Wiener process and the process Z_γ considered in Section 4. For $\varepsilon \in (0,\infty)$, let Z_γ^ε denote the mild solution to (4.1) with $A = \varepsilon\,\mathrm{Id}_U$ and define the process $\left(\overline{Z}_\gamma^\varepsilon(t)\right)_{t\in\mathbb{R}}$ by

$$\overline{Z}_\gamma^\varepsilon(t) := C_{\gamma-1/2}^{-1}\Gamma(\gamma)(Z_\gamma^\varepsilon(t) - Z_\gamma^\varepsilon(0)), \quad t \in \mathbb{R}. \tag{5.3}$$

Note that $\widehat{W}_H^Q(t)$ can formally be written as the following "convergent difference of divergent integrals" (see [29, Footnote 3]): $\frac{1}{C_H}[\int_{-\infty}^t (t-s)^{H-\frac{1}{2}}\,\mathrm{d}W^Q(s) - \int_{-\infty}^0 (-s)^{H-\frac{1}{2}}\,\mathrm{d}W^Q(s)]$. This expression would correspond to $\varepsilon = 0$ in (5.3), which is ill-defined as Assumption 4.2(i) cannot be satisfied. However, the next result shows that a fractional Q-Wiener process can be seen as a limiting case of $\overline{Z}_\gamma^\varepsilon$ as $\varepsilon \downarrow 0$.

Proposition 5.3. *Let $Q \in \mathscr{L}_1^+(U)$ and $\gamma \in (1/2, 3/2)$. The family of stochastic processes $(\overline{Z}_\gamma^\varepsilon)_{\varepsilon\in(0,\infty)}$ defined by (5.3) converges uniformly on compact subsets of $\mathbb{R}$ in mean-square sense to the fractional Q-Wiener process $\widehat{W}_H^Q$ in (5.1) with Hurst parameter $H = \gamma - \frac{1}{2}$ as $\varepsilon \downarrow 0$:*

$$\forall T \in (0,\infty): \quad \lim_{\varepsilon\downarrow 0} \sup_{t\in[-T,T]} \left\| \widehat{W}_{\gamma-1/2}^Q(t) - \overline{Z}_\gamma^\varepsilon(t) \right\|_{L^2(\Omega;U)} = 0.$$

Proof. For $t \in [0,\infty)$, we can write

$$\widehat{W}^{Q}_{\gamma-1/2}(t) - \overline{Z}^{\varepsilon}_{\gamma}(t) = \frac{1}{C_{\gamma-1/2}} \int_0^t (t-s)^{\gamma-1}\left(1-e^{-\varepsilon(t-s)}\right) \mathrm{d}W^Q(s) + \frac{1}{C_{\gamma-1/2}} \int_{-\infty}^0 \left[(t-s)^{\gamma-1}\left(1-e^{-\varepsilon(t-s)}\right) - (-s)^{\gamma-1}\left(1-e^{\varepsilon s}\right)\right] \mathrm{d}W^Q(s).$$

Applying the Itô isometry to each of these integrals and using the respective changes of variables $s' := -s$ and $s' := t-s$ yields

$$\left\|\widehat{W}^{Q}_{\gamma-1/2}(t) - \overline{Z}^{\varepsilon}_{\gamma}(t)\right\|^2_{L^2(\Omega;U)} = \frac{I_1(t)+I_2(t)}{C_{\gamma-1/2}} (\operatorname{tr} Q)^2, \qquad \begin{aligned} I_1(t) &:= \int_0^{|t|} s^{2\gamma-2}\left(1-e^{-\varepsilon s}\right)^2 \mathrm{d}s, \\ I_2(t) &:= \int_0^\infty \left|(|t|+s)^{\gamma-1}\left(1-e^{-\varepsilon(|t|+s)}\right) - s^{\gamma-1}\left(1-e^{-\varepsilon s}\right)\right|^2 \mathrm{d}s. \end{aligned} \tag{5.4}$$

For $t \in (-\infty,0)$ we find (5.4) by instead splitting into integrals over $(-\infty,t)$ and $(t,0)$ and changing variables $s' := t-s$ and $s' := -s$, respectively. For $t \in [-T,T]$, the elementary inequality $1-e^{-x} \le x$ for $x \in [0,\infty)$ yields $I_1(t) \le \varepsilon^2 \int_0^{|t|} s^{2\gamma}\,\mathrm{d}s = \frac{\varepsilon^2 |t|^{2\gamma+1}}{2\gamma+1} \le \frac{\varepsilon^2 T^{2\gamma+1}}{2\gamma+1}$. For I_2, applying the fundamental theorem of calculus to the function $u \mapsto (u+s)^{\gamma-1}(1-e^{-\varepsilon(u+s)})$, followed by Minkowski's integral inequality [41, Section A.1], yields

$$\begin{aligned} I_2(t) &= \int_0^\infty \left| \int_0^{|t|} [(\gamma-1)(u+s)^{\gamma-2}(1-e^{-\varepsilon(u+s)}) + \varepsilon(u+s)^{\gamma-1} e^{-\varepsilon(u+s)}]\,\mathrm{d}u\right|^2 \mathrm{d}s \\ &\le \left| \int_0^T \left[\int_0^\infty \left|(\gamma-1)(u+s)^{\gamma-2}(1-e^{-\varepsilon(u+s)}) + \varepsilon(u+s)^{\gamma-1} e^{-\varepsilon(u+s)}\right|^2 \mathrm{d}s\right]^{\frac{1}{2}} \mathrm{d}u\right|^2 \\ &= \varepsilon^{3-2\gamma} \left| \int_0^T \left[\int_{\varepsilon u}^\infty \left|(\gamma-1)v^{\gamma-2}(1-e^{-v}) + v^{\gamma-1}e^{-v}\right|^2 \mathrm{d}v\right]^{\frac{1}{2}} \mathrm{d}u\right|^2 \\ &\le \varepsilon^{3-2\gamma} T^2 \int_0^\infty \left|(\gamma-1)v^{\gamma-2}(1-e^{-v}) + v^{\gamma-1}e^{-v}\right|^2 \mathrm{d}v, \end{aligned}$$

where we performed the change of variables $v(s) := \varepsilon(u+s)$ on the third line.

The improper integral on the last line converges: As $v \downarrow 0$, the squares of both terms are of order $\mathcal{O}(v^{2\gamma-2})$, where we again use $1-e^{-v} \le v$ for the first term, and we have $2\gamma-2 \in (-1,1)$; the square of the first term is of order $\mathcal{O}(v^{2\gamma-4})$ as $v \to \infty$, with $2\gamma-4 \in (-3,-1)$, whereas the second term decays exponentially. The convergence thus follows by letting $\varepsilon \downarrow 0$. □

5.2. Remarks on Markov behavior

Now we consider the Markov behavior of fractional Q-Wiener processes with Hurst parameter $H \in (0,1)$. Since the case $H = \frac{1}{2}$ corresponds to a standard Q-Wiener process, we find that $W^Q_{1/2}$ is simple Markov, whereas we can expect that W^Q_H is not weakly Markov for $H \ne \frac{1}{2}$.

In the real-valued case, the first published proof of non-Markovianity appears to be [19], which shows that B_H is not simple Markov for $H \ne \frac{1}{2}$ using a characterization in terms of its covariance function. This result can be improved by applying the theory of [18, Chapter V] for Gaussian N-ple Markov processes to the Mandelbrot–Van Ness representation of B_H. Namely, by [18, Theorem 5.1], any real-valued process of the form $(\int_{-\infty}^t K(t,s)\,\mathrm{d}B(s))_{t\in\mathbb{R}}$ is N-ple Markov for $N \in \mathbb{N}$ only if there exist functions $(f_j)_{j=1}^N, (g_j)_{j=1}^N$ such that $K(s,t) = \sum_{j=1}^N f_j(s) g_j(t)$ for $s,t \in \mathbb{R}$. The real-valued kernel K_H in (5.2) satisfies this condition only if $H = \frac{1}{2}$.

The question arises if one can generalize this characterization of real-valued N-ple Markovianity to the case of Hilbert space valued Gaussian processes, such as W^Q_H and Z_γ (see Definitions 5.1 and 4.5, respectively). However, it is not evident from the proof in the real-valued case what the analogous condition would be in the Hilbertian setting. For instance, the kernel $K(t,s) = S(t-s)$ of the simple Markov process Z_1 factorizes as $K(t,s) = S(t)S(-s)$ provided that $(S(t))_{t\ge 0}$ extends to a C_0-*group* $(S(t))_{t\in\mathbb{R}}$. Otherwise it is not guaranteed that such a factorization exists, and it is not clear either whether this condition remains necessary for simple Markovianity in the Hilbert space case.

In order to establish that W^Q_H (or B_H) does not have the weak Markov property for $H \ne \frac{1}{2}$, one could also attempt to associate a nonlocal precision operator to the process and apply the necessary condition from Theorem 3.7. Formally, its coloring operator $\mathcal{L}_H^{-1}$ acts on $f\colon \mathbb{R} \to U$ as $\mathcal{L}_H^{-1} f(t) = \frac{1}{C_H} \int_{\mathbb{R}} K_H(t,s) f(s)\,\mathrm{d}s$ for all $t \in \mathbb{R}$. For certain ranges of H, see for instance [36, Equation (31)], an explicit formula of its inverse $\mathcal{L}_H$ can also be determined. The operator $\mathcal{L}_H^{-1}$ is bounded on some weighted Hölder space by [36, Theorem 6], but there is no reason to expect that it is bounded on a Hilbert space such as $L^2(\mathbb{R};U)$. Therefore, Theorem 3.7 is not directly applicable, as it would need to be extended to the Banach space setting, which is beyond the scope of this work.

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Declaration of competing interest

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Appendix A. Auxiliary results

This Appendix collects auxiliary results which are needed in the main text but have been postponed for the sake of readability.

A.1. Conditional independence

Let $\mathcal{G}_1, \mathcal{H}, \mathcal{G}_2 \subseteq \mathcal{F}$ be σ-algebras on $(\Omega, \mathcal{F}, \mathbb{P})$. We recall a characterization of conditional independence, see [20, Theorem 8.9], from which we derive a lemma which is useful for establishing relations between various (equivalent formulations of) Markov properties defined in Section 3.

Theorem A.1 (*Doob Conditional Independence Property*). *We have $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}} \mathcal{G}_2$ if and only if $\mathbb{P}(G_2 \mid \mathcal{G}_1 \vee \mathcal{H}) = \mathbb{P}(G_2 \mid \mathcal{H})$ holds $\mathbb{P}$-a.s., for all $G_2 \in \mathcal{G}_2$.*

Lemma A.2. *If $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}} \mathcal{G}_2$, then*

(a) $\mathcal{G}_1 \vee \mathcal{H} \perp\!\!\!\perp_{\mathcal{H}} \mathcal{G}_2$;
(b) $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}'} \mathcal{G}_2$ *for any σ-algebra $\mathcal{H}' \supseteq \mathcal{H}$ such that $\mathcal{H}' \subseteq \mathcal{G}_1$;*
(c) $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}'} \mathcal{G}_2$ *for any σ-algebra $\mathcal{H}' \supseteq \mathcal{H}$ of the form $\mathcal{H}' = \mathcal{H}'_1 \vee \mathcal{H}'_2$, where $\mathcal{H}'_1, \mathcal{H}'_2$ are σ-algebras satisfying $\mathcal{H}'_1 \subseteq \mathcal{G}_1 \vee \mathcal{H}$ and $\mathcal{H}'_2 \subseteq \mathcal{G}_2 \vee \mathcal{H}$.*

Proof. Part (a) is [20, Corollary 8.11(i)]; combining it with $\mathcal{G}_1 = \mathcal{G}_1 \vee \mathcal{H}'$ and $\mathcal{H}' = \mathcal{H} \vee \mathcal{H}'$ yields (b). To prove (c), first note that $\mathcal{G}_1 \vee \mathcal{H} \perp\!\!\!\perp_{\mathcal{H} \vee \mathcal{H}'_1} \mathcal{G}_2 \vee \mathcal{H}$ by parts (a) and (b). Applying (a) again, we find $\mathcal{G}_1 \vee \mathcal{H} \vee \mathcal{H}'_1 \perp\!\!\!\perp_{\mathcal{H} \vee \mathcal{H}'_1} \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1$. Since $\mathcal{H} \vee \mathcal{H}'_1 \subseteq \mathcal{H} \vee \mathcal{H}' = \mathcal{H}' = \mathcal{H}'_2 \vee \mathcal{H}'_1 \subseteq \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1$, part (b) yields $\mathcal{G}_1 \vee \mathcal{H} \vee \mathcal{H}'_1 \perp\!\!\!\perp_{\mathcal{H}'} \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1$, which proves (c) since $\mathcal{G}_1 \subseteq \mathcal{G}_1 \vee \mathcal{H} \vee \mathcal{H}'_1$ and $\mathcal{G}_2 \subseteq \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1$. $\square$

A.2. Results related to Assumption 4.2(i)

Lemma A.3. *Let Assumption 4.1 be satisfied, i.e., suppose the linear operator $-A\colon \mathsf{D}(A) \subseteq U \to U$ on the separable real Hilbert space U generates an exponentially stable C_0-semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on U. If $Q \in \mathscr{L}^+(U)$ and $\gamma_0 \in (-\infty, 1/2]$, then $\int_0^\infty \|t^{\gamma_0 - 1} S(t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}t = \infty$, that is, Assumption 4.2(i) cannot hold for $\gamma_0 \in (-\infty, 1/2]$.*

Proof. Fix some $x \in U$ with $\|x\|_U = 1$. Then $\|S(t)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(U)} \geq \|S(t)Q^{\frac{1}{2}}x\|_U$ for all $t \in [0,\infty)$. Since $t \mapsto S(t)Q^{\frac{1}{2}}x$ is continuous at zero and $S(0)Q^{\frac{1}{2}}x = Q^{\frac{1}{2}}x$, we can choose $\delta \in (0,\infty)$ so small that $\|S(t)Q^{\frac{1}{2}}x\|_U \geq \frac{1}{2}\|Q^{\frac{1}{2}}x\|_U$ for all $t \in [0,\delta]$. If $\gamma_0 \in (-\infty, 1/2]$, we then obtain

$$\int_0^\infty \|t^{\gamma_0-1} S(t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}t \geq \frac{1}{2}\|Q^{\frac{1}{2}}x\|_U^2 \int_0^\delta t^{2(\gamma_0-1)}\,\mathrm{d}t = \infty. \quad \square$$

Lemma A.4. *Let Assumption 4.1 be satisfied. If Assumption 4.2(i) holds for some $\gamma_0 \in (1/2, \infty)$, then it also holds for all $\gamma' \in [\gamma_0, \infty)$.*

Proof. The change of variables $\tau := t/2$, the semigroup property and (4.2) yield

$$\int_0^\infty \|t^{\gamma'-1} S(t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}t \leq 2^{2\gamma'-1} M_0^2 \int_0^\infty e^{-2w\tau}\|\tau^{\gamma'-1} S(\tau) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau.$$

For the latter integral, we split up the domain of integration and estimate each of the resulting integrands to find

$$\begin{aligned}\int_0^\infty e^{-2w\tau}\|\tau^{\gamma'-1} S(\tau) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau &= \sum_{k=1}^\infty \int_{k-1}^k e^{-2w\tau}\|\tau^{\gamma'-1} S(\tau) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau \\ &\leq \sum_{k=1}^\infty e^{-2w(k-1)} k^{2(\gamma'-\gamma_0)} \int_0^\infty \|\tau^{\gamma_0-1} S(\tau) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau < \infty,\end{aligned}$$

where the series converges since $|e^{-2wk}k^{2(\gamma'-\gamma_0)}|^{\frac{1}{k}} \to e^{-2w} < 1$ as $k \to \infty$. $\square$

A.3. Filtrations indexed by the real line

Proposition A.5. *A process $(W^Q(t))_{t\in\mathbb{R}}$ satisfying (WP1) cannot be a martingale with respect to any filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$.*

Proof. Suppose that $(W^Q(t))_{t\in\mathbb{R}}$ is a martingale with respect to some filtration $(\mathcal{F})_{t\in\mathbb{R}}$. Then the same holds for the real-valued process $W_h^Q(t) := \langle W^Q(t), h\rangle_U$, where we choose $h \in U$ such that $\langle Qh, h\rangle_U^2 > 0$ to ensure that $(W_h^Q(t))_{t\in\mathbb{R}}$ has nontrivial increments. In particular, $(W_h^Q(-n))_{n\in\mathbb{N}}$ is a backward martingale with respect to $(\mathcal{F}_{-n})_{n\in\mathbb{N}}$, implying that it converges $\mathbb{P}$-a.s. and in $L^1(\Omega)$ as $n\to\infty$ by the backward martingale convergence theorem [16, Section 12.7, Theorem 4]. But this contradicts (WP1), since $(W_h^Q(-n))_{n\in\mathbb{N}}$ cannot be a Cauchy sequence in $L^1(\Omega)$ as it has (non-trivial) stationary increments. □

A.4. Mean-square differentiability of stochastic convolutions

The following lemma concerning mean-square continuity and differentiation under the integral sign is a straightforward generalization of [23, Propositions 3.18 and 3.21] to stochastic convolutions with respect to a two-sided Wiener process. Its proof is therefore omitted.

Lemma A.6. *Let $t_0 \in [-\infty,\infty)$ be such that $\Psi(\,\cdot\,)Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(U))$ and set $\mathbb{T} := [t_0,\infty)$ if $t_0\in\mathbb{R}$ or $\mathbb{T} := \mathbb{R}$ if $t_0 = -\infty$. Then the stochastic convolution $(\int_{t_0}^t \Psi(t-s)\,\mathrm{d}W^Q(s))_{t\in\mathbb{T}}$ is mean-square continuous.*

If $\Psi(\,\cdot\,)Q^{\frac{1}{2}} \in H_0^1(0,\infty;\mathscr{L}_2(U))$, then the process $(\int_{t_0}^t \Psi(t-s)\,\mathrm{d}W^Q(s))_{t\in\mathbb{T}}$ is mean-square differentiable on $\mathbb{T}$ and, for all $t\in\mathbb{T}$, the identity $\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^t \Psi(t-s)\,\mathrm{d}W^Q(s) = \int_{t_0}^t \partial_t\Psi(t-s)\,\mathrm{d}W^Q(s)$ holds $\mathbb{P}$-a.s.

Appendix B. Fractional powers of the parabolic operator

Let $A\colon \mathsf{D}(A)\subseteq U \to U$ be a linear operator on a real Hilbert space U.

Definition B.1. Let Assumption 4.1 be satisfied. We define the negative fractional power operator $A^{-\alpha}$ of order $\alpha\in(0,\infty)$ as the $\mathscr{L}(U)$-valued Bochner integral $A^{-\alpha} := \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}S(t)\,\mathrm{d}t$. Then $A^{-\alpha}$ is injective and we define $A^\alpha x := (A^{-\alpha})^{-1}x$ for $x\in\mathsf{D}(A^\alpha)$. For $\alpha = 0$ we set $A^0 := \mathrm{Id}_U$.

See [34, Section 2.6] for more details on this definition of fractional powers. The proof of the following lemma is analogous to that of [23, Proposition A.3] and is therefore omitted.

Lemma B.2. *Let $(S,\mathscr{A},\mu)$ be a measure space such that $L^2(S;\mathbb{R})$ is nontrivial and consider the linear operator $\mathcal{A}_S$ on $L^2(S;U)$ defined by (4.7). If $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on U, then $-\mathcal{A}_S$ generates the C_0-semigroup $(\mathcal{T}_S(t))_{t\geq0}$ on $L^2(S;U)$.*

The next results are respectively analogous to [23, Propositions A.5 and 3.2].

Proposition B.3. *For $t\in\mathbb{R}$, define the shift operator $\mathcal{T}(t)\in\mathscr{L}(L^2(\mathbb{R};U))$ by $\mathcal{T}(t)f := f(\,\cdot\,-t)$ for $f\in L^2(\mathbb{R};U)$. The family $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is a C_0-group whose infinitesimal generator is given by $-\partial_t$, where ∂_t is the Bochner–Sobolev weak derivative on $\mathsf{D}(\partial_t) = H^1(\mathbb{R};U)$.*

Proposition B.4. *Suppose that Assumption 4.1 holds. The closure $\mathcal{B}$ of the sum operator $\partial_t + \mathcal{A}_\mathbb{R}$ exists and $-\mathcal{B}$ generates the C_0-semigroup $(\mathcal{S}_\mathbb{R}(t)\mathcal{T}(t))_{t\geq0}$ on $L^2(\mathbb{R};U)$ satisfying $\|\mathcal{S}_\mathbb{R}(t)\mathcal{T}(t)\|_{L^2(\mathbb{R};U)} = \|\mathcal{T}(t)\mathcal{S}_\mathbb{R}(t)\|_{L^2(\mathbb{R};U)} = \|S(t)\|_{\mathscr{L}(U)}$ for all $t\in\mathbb{R}$, where $(\mathcal{S}_\mathbb{R}(t))_{t\geq0}$ and $(\mathcal{T}(t))_{t\geq0}$ are as in (4.7) and Proposition B.3, respectively.*

It follows that $(\mathcal{S}_\mathbb{R}(t)\mathcal{T}(t))_{t\geq0}$ inherits the exponential stability of $(S(t))_{t\geq0}$, so that fractional powers of $\mathcal{B}$ can be defined using Definition B.1. Therefore, under Assumption 4.1, Definition B.1 implies

$$\mathcal{B}^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^\infty r^{\gamma-1}\mathcal{S}_\mathbb{R}(r)\mathcal{T}(r)f(t)\,\mathrm{d}r = \frac{1}{\Gamma(\gamma)}\int_0^\infty r^{\gamma-1}S(r)f(t-r)\,\mathrm{d}r$$

for $\gamma\in(0,\infty)$, $f\in L^2(\mathbb{R};U)$ and almost all $t\in\mathbb{R}$. We conclude that $\mathcal{B}^{-\gamma} = \mathfrak{I}^\gamma$ for all $\gamma\in[0,\infty)$, where the latter is defined by (4.5).

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