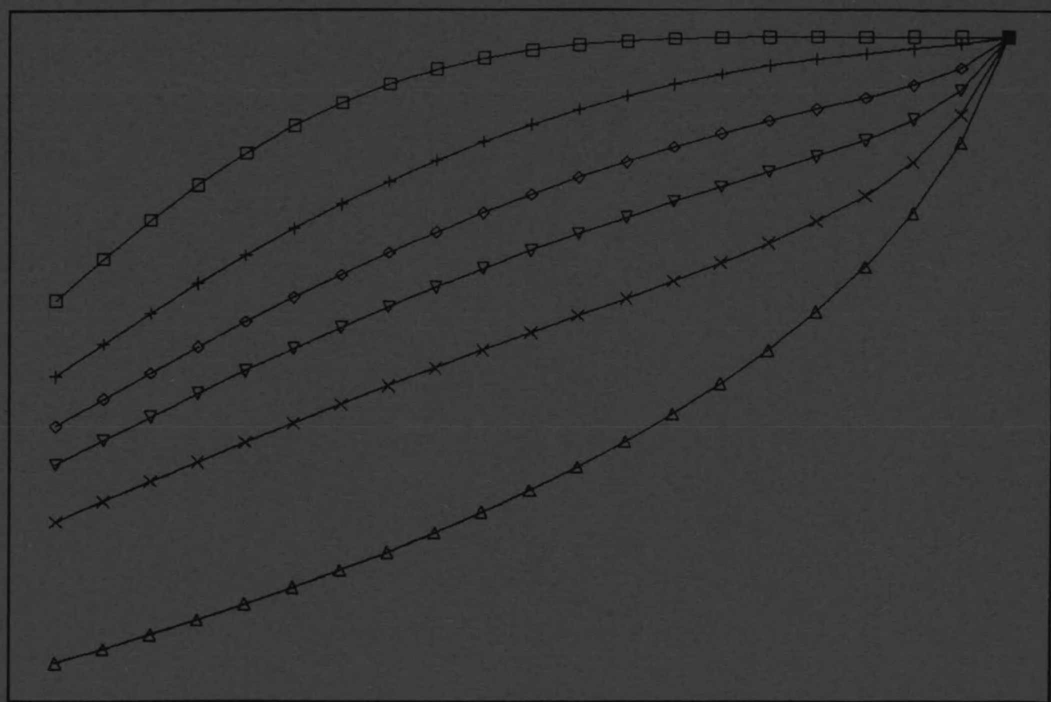
INTERNATIONAL INSTITUTE FOR HYDRAULIC AND ENVIRONMENTAL ENGINEERING
DELFT NETHERLANDS



Self-weight Consolidation on Impervious Bases

Shaoling HU

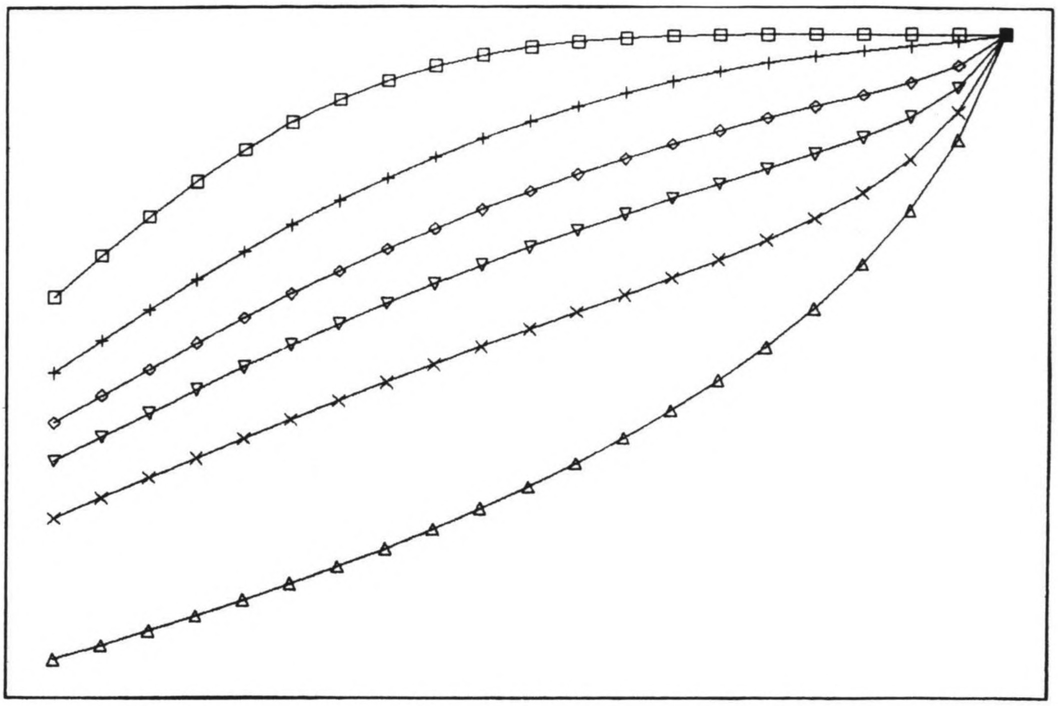
M.Sc. Thesis H.H. 42

1990

IHE
DELFT

delft hydraulics

INTERNATIONAL INSTITUTE FOR HYDRAULIC AND ENVIRONMENTAL ENGINEERING
DELFT NETHERLANDS



Self-weight Consolidation on Impervious Bases

Shaoling HU

M.Sc. Thesis H.H. 42

1990

IHE
DELFT

delft hydraulics

Abstract

This paper presents the study on the self-weight consolidation, which is referred to the consolidation problem of cohesive deposits in reservoirs and based on Gibson´s theory of non-linear finite-strain consolidation.

The analytical solution of the linearized equation is carried out. The solution shows that the consolidation is dominated by the dimensionless thickness of soil Zd. When Zd is large, consolidation progresses faster.

A mathematical model based on the full equation is set up, which is verified by data and can predict the self-weight consolidation with the thickness increasing with time.

The final profile of void ratio is also obtained theoretically. Subsequently, the final thickness of deposits and the final gradient of void ratio are obtained.

The comparisons between the analytical solution of linearized equation and the numerical solution of full equation show that the linearization is valid for the small thickness.

In addition existing literature on consolidation are reviewed and the Gibson's theory which is based in this study is presented in detail.

CONTENTS

CONTENTS (continued)

1. Introduction

The self-weight consolidation of cohesive sediments is one of the major problems in reservoir sedimentation.

As long as the cohesive sediments deposit on the bed of a reservoir , consolidation starts to undergo. Therefore the continuous deposition raises the bed level, whereas the consolidation reduces this raising at the same time.

On the other hand, the sluicing operations are adopted in some reservoirs (e.g. the reservoirs on Yellow River), so as to reduce the sedimentation (Cao,1983; Cao & Du,1986; Zhang & Chien,1985; Zhang & Du,1984; Bruk et al,1983; Bruk,1985; Cavor & Slavic, 1983; Wang & Wang,1983). The prediction of erosion due to sluicing has to take the consolidation into account, because consolidation strongly influences the erodibility of the cohesive deposits (Bouchard et al, 1989; Ariathurai et al, 1976,1977,1978; Kelly & Gularte, 1981; Kuijper et al, 1989; Mehta, 1989; Mehta et al, 1979, 1989, 1989; Otsubo & Muraoka, 1988; Partheniades, 1965, 1972).

Therefore, to predict the morphological processes in reservoirs correctly, self-weight consolidation of cohesive sediments has to be well-investigated.

This work, which focusses on the self-weight consolidation on the impervious bottom (in other words, it is assumed that the bed of reservoir is impervious) comprises the following aspects:

- The previous works on consolidation are reviewed and Gibson's theory on which this work is based is presented in detail;

- The analytical solution of the linearized Gibson's equation is carried out in order to gain the insight into the physics of the self-weight consolidation;

- A mathematical model is set up and verified with data, which can predict not only the consolidation processes after sedimentation but also simultaneous sedimentation and consolidation;

- The theoretical final profile of void ratio is obtained and is used to compare the results of mathematical model;

- Conclusions are drawn and recommendations for future study are given.

2. **Reviews of the Previous Works on Self-weight Consolidation**

2.1 General

Some major theories which represent the self-weight consolidation problems are reviewed in section 2-2. Since it has been widely applied elsewhere and also in this work, Gibson's theory is presented in detail. The predictive approaches of self-weight consolidation are discussed in section 2-3 .

2.2 **Some theories describing consolidation**

i) Terzaghi's theory

The first theoretical model of one-dimensional consolidation was developed by Terzaghi (1923). This theory is based on the following assumptions:

(a) The soil is completely saturated with water;
(b) The soil particles and the pore water are incompressible;
(c) The fluid flow equations follow Darcy's law;
(d) The soil structure is homogeneous. The permeability k is then constant;
(e) The strains are small and the compressibility Mv is constant.

From above assumptions, if only the excess porewater pressure contributes to the progress of consolidation and the applied load is time-independent, Terzaghi's theory is represented by the following diffusion equation,

$$\frac{\delta Pe}{\delta t} = Cf \frac{\delta^2 Pe}{\delta S^2} \qquad (2\text{-}1)$$

in which, $Cf = k/(Mv.g.\rho)$,

Pe = excess porewater pressure;
S = Eulerian co-ordinate, at the bottom, S=0, at surface, S=So(t);
ρ = density of water;
g = the acceleration of gravity;
Mv = compressibility of the soil structure;
k = Darcy's coefficient, or permeability.

In practice, the assumptions are only approximately satisfied. For the hydraulically deposited cohesive-sediments, which usually have high initial void ratio and the large range of void ratio during consolidation, eq(2-1) could not be plausible. In other words, Terzaghi's theory is valid only for the infinitesimal-strain consolidation problems.

ii) Modified Terzaghi's theory

Schiffman and Gibson (1964) followed Terzaghi's idea only to some extent but assumed that the permeability k and compressibility Mv vary with S (or non-homogeneous clay layer), and they derived the following equation.

$$\frac{1}{Cf(S)} \cdot \frac{\delta Pe}{\delta t} = \frac{1}{k} \cdot \frac{dk}{dS} \cdot \frac{\delta Pe}{\delta S} + \frac{\delta^2 Pe}{\delta S^2} \qquad (2-2)$$

Basically k and Mv are time-dependent rather than time-independent.

Yong and Elmonayeri (1984) also analysed the consolidation after sedimentation processes and formulated a convection-diffusion relationship. However, the parameters in the relation have to be well-defined and experimentally determined before it is applied in practice. As mentioned by the authors, "the relationship can model sedimentation of the pure clays-suspensions tested to void ratios of about 3", which is much lower than those of the hydraulically deposited cohesive-sediments.

iii) Gibson's theory

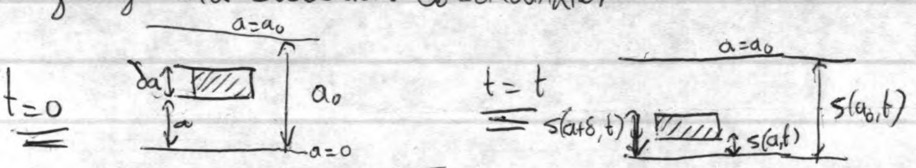
It is a comprehensive theory, either from its theoretical background or from its verifications and applications (Lee & Sills, 1981; Znidarcic, 1986; Bromwell, 1984; Scully et al,1984; Krizek et al,1984; Lin, et al,1984).

Gibson et al (1967) developed a non-linear consolidation theory. It was applied to a thin layer clay at the very beginning (1967) and extended to thick layer with finite strain (1981) owing to the achievements on the property studies and the validation investigations for the theory. Now it has been widely used in sedimentation/consolidation as well as underload consolidation problems (Gibson et al, 1984).

For the convenience of description in the next chapters, Gibson's theory is presented in detail here.

Shaoling Hu (1990) Self-weight Consolidation on Impervious Bases
M.Sc. Thesis H.H. 42, Intern. Institute for Hydr. and Environ. Eng., Delft Netherlands

- Self-weight consolidation of coh. sediments in reservoirs (important for flushing)
- Assume bed of reservoir impervious ; this work based on Gibson's theory
- Existing theories:
  1 Terzaghi's theory (1923) ; diffusion eq. $\frac{\partial P_e}{\partial t} = C_f \frac{\partial^2 P_e}{\partial s^2}$ $\left(C_f = \frac{k}{M_v \gamma_g \rho}\right)$ (k → permeability, Darcy; M_v → compressibility of the soil structure (const))
  P_e = excess porewater pressure (porew. + grains incompressible)
  S = eulerian co-ordinate (bottom S=0; surface = $S_0(t)$)
  2 Modified Terzaghi theory (Schiffman, Gibson, '64) : M_v an k vary with S (non homogeneous layer) → time dependent
  $$\frac{1}{C_f(s)} \frac{\partial P_e}{\partial t} = \frac{1}{k} \frac{dk}{ds} \frac{\partial P_e}{\partial s} + \frac{\partial^2 P_e}{\partial s^2}$$
  3 Gibson's theory (1967, 1981)
  Non linear consolidation theory Here described:
- Lagrange and Eulerian co-ordinates



a, t = independent var.
S = dependent var.

- Derivations
  Vert. equilibr.:

a) Lagrange and Eulerian co-ordinates

"In the derivation which follows we shall adopt consistently the second standpoint and consider an element of the soil structure of unit cross-section area normal to the direction of pore fluid low which at time t=0 lies between planes embedded datum plane (Fig.2-1(a)). At some subsequent time t these same planes will be located at (unknown) distances S(a,t) and S(a+δa,t) from this datum plane. We have here chosen a and t as independent variables, while S is a dependent variable. Each plane of particles is labelled through its subsequent motion by its initial distance a from the datum plane; for example the upper boundary of the layer is always AT a=a0 (Fig. 2-1b). By using these Lagrange co-ordinates we have secured the following advantage: the boundary can always be identified ($a=a_0$), and the boundary conditions on it introduced into the analysis, although we are ignorant of its exact location: $S(a_0,t)$."

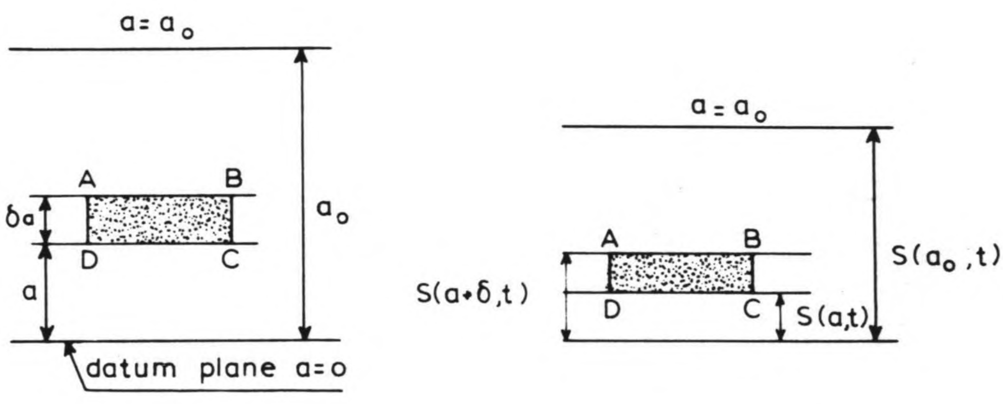


Fig.2-1a. Initial configuration at time t=0

Fig.2-1b. Current conf. at time t

Accordingly, a is a Lagrange coordinate, while S is an Eulerian coordinate.

b) Derivations

The vertical equilibrium of the soil grains and fluid currently occupying the element ABCD (Fig. 2-1(b)), it follows that

$$\frac{\delta\sigma}{\delta a} + [n.\ \rho + (1-n).\ \rho_s].g.\ \frac{\delta S}{\delta a} = 0 \tag{2-3}$$

where,

σ = the total vertical stress, n = porosity,

ρ, ρ_s = densities of water and soil particles, respectively.

The fact that the chosen co-ordinate element always embraces the same mass of solids, leads to the following continuity

$$\rho_s \cdot [1-n(a,0)] = \rho_s \cdot [1-n(a,t)] \cdot \frac{\delta S}{\delta a} \tag{2-.4}$$

To determine the equation of continuity for the fluid phase we denote the velocity of the solid phase by Vs (=δS/δt) and that of the pore fluid by Vw. The rate of mass of fluid flowing into the element ABCD is then

$$n.(Vw-Vs).\rho.g$$

The rate of mass of fluid outflow is the above quantity augmented by

$$\frac{\delta}{\delta a}[n.(Vw-Vs).\rho.g].\delta a$$

but this must equal the rate of change of mass of fluid in the element, so that

$$\frac{\delta}{\delta a}[n.(Vw-Vs).\ \rho.g] + \frac{\delta}{\delta t}[n.\ \rho.g.\frac{\delta S}{\delta a}] = 0 \tag{2-5}$$

The pore fluid movement follows Darcy's law, which is expressed by

$$n.(Vw-Vs) = -\frac{k}{\rho.g} \cdot \frac{\delta Pe}{\delta S} \tag{2-6}$$

where, again, Pe is the excess porewater pressure and P denotes the total porewater pressure. And since

$$Pe = P - \rho.(So - S).g$$

the gradient of excess porewater pressure is that

$$\frac{\delta Pe}{\delta S} = \frac{\delta P}{\delta S} + \rho.g \tag{2-7}$$

with

$$\frac{\delta P}{\delta S} = \frac{\delta P}{\delta a} \cdot \frac{\delta a}{\delta S} \tag{2-8}$$

From (2-7), (2-8), equation (2-6) can be rewritten as follows

$$n.(Vw - Vs).\frac{\delta S}{\delta a} = - \frac{k}{\rho.g} . [\frac{\delta P}{\delta a} + \rho.g. \frac{\delta S}{\delta a}] \qquad (2\text{-}9)$$

Therefore, the constitutive relationships (2-3), (2-5) and (2-9) have been established.

c) Governing equation

The governing equation is transformed into material co-ordinate Z which is introduced by

$$Z(a) = \int_{o}^{a} [1-n\ (a',0)]da' \qquad (2\text{-}10)$$

this implies that a point of the soil structure is now identified as the volume of solids Z in a prism of unit (bulk) horizontal area lying between the datum plane and the point. Clearly, Z is time-independent and Zo (at the surface) is the total solid volume per unit horizontal area.

In addition, $n = Vr/(1+Vr)$ (2-11)

where, Vr is the void ratio.

By introducing (2-11), the previously established equations can be rewritten in Z:

$$\frac{\delta \sigma}{\delta Z} + \frac{Vr.\ \rho + \rho_s}{1 + Vr}.g.\frac{\delta S}{\delta Z} = 0 \qquad (2\text{-}3)bis$$

$$\frac{\delta S}{\delta Z} - (1+Vr).\frac{\rho_s(a,0)}{\rho_s(a,t)} = 0 \qquad (2\text{-}4)bis$$

$$\frac{\delta}{\delta Z} [\frac{Vr.\ \rho.g}{1+Vr}.\ (Vw-Vs)] + \frac{\delta}{\delta t} [\frac{Vr.\ \rho.g}{1 + Vr}.\ \frac{\delta S}{\delta Z}] = 0 \qquad (2\text{-}5)bis$$

$$[\frac{Vr.(Vw-Vs)}{k.(1+Vr)} + 1].\ \frac{\delta S}{\delta Z} + \frac{1}{\rho.g} . \frac{\delta P}{\delta Z} = 0 \qquad (2\text{-}9)bis$$

From (2-4)bis the relationship between Eulerian coordinate S and material coordinate Z is that

$$S = \int_{o}^{Z} (1+Vr)dZ,$$

provided the soil particles are incompressible.

If the soil structure is homogeneous and has no creep effects and the consolidation is monotonic, then k may be expected to depend upon the void ratio,

$$k=k(Vr) \qquad (2\text{-}12)$$

while the vertical effective stress

$$\sigma' = \sigma - P \qquad (2\text{-}13)$$

controls the void ratio,

$$\sigma' = \sigma'(Vr). \qquad (2\text{-}14)$$

Then the governing equation for the void ratio is obtained by combining eqs(2-3)bis--(2-5)bis and (2-9)bis,

$$\left(\frac{\rho_s}{\rho} - 1\right) \cdot \frac{d}{dVr}\left[\frac{k(Vr)}{1+Vr}\right] \cdot \frac{\delta Vr}{\delta Z} + \frac{\delta}{\delta Z}\left[\frac{k(Vr)}{\rho.g.(1+Vr)} \cdot \frac{d\sigma'}{dVr} \cdot \frac{\delta Vr}{\delta Z}\right] + \frac{\delta Vr}{\delta t} = 0$$

(2-15)

In (2-15), two constitutive relationships for $\sigma'(Vr)$ and $k(Vr)$ eq(2-12) and (2-14) are required. Many experiments have been done by different authors and they are summarized by Krizek & Somogyi (1984).

2.3 **Predictive Approaches**

In general, there are analytical, computational and centrifugal-experimental approaches to predict the self-weight consolidation. Here only some previous predictions based on Gibson's theory (or, equation (2-15)) are discussed.

i) Analytical solution

Analytical solution gives the insight of the physical processes, even though substantial simplifications and linearisations are always made.

For underload consolidation Gibson (1967) omitted the first term of eq (2-15) and defined

$$C_v = - \frac{k(Vr)}{\rho.g.(1+Vr)} \cdot \frac{d\sigma'}{dVr} \qquad (2\text{-}16)$$

The analytical solutions for the cases of constant C_v and linear relationship between C_v and Vr, were then obtained. The results show that the linearity of the equation for the thin homogeneous layer is maintained, or his theory is converged to Terzagyi's law in the case of infinitesimal strain. Lee & Sills (1984) followed Gibson's simplification to (2-15) and employed the following initial condition

$$Vr(Z,0) = Vrini \qquad (2\text{-}17)$$

and boundary conditions,

- on the surface,

$$Vr(Zo,t) = Vrini \qquad (2\text{-}18)$$

- and on the impervious bottom,

$$\left.\frac{\delta Vr}{\delta Z}\right|_{Z=0} = \beta \qquad (2\text{-}19)$$

where, constant β is the final gradient of void-ratio profile. Analytical solutions for the dredged-fill consolidation and sedimantation/-consolidation (i.e. the thickness of sediments is increasing in time, whereas consolidation is simultaneously progressing) were obtained. However, some remarks have to be made.

- In the course of self-weight consolidation, the magnitude of the first term is decreasing, while the second term is increasing. The first term cannot be omitted particularly for cohesive sediments with very high void ratios.
- Impervious bottom boundary is obtained from assuming constant k/(1+Vr) and (2-16). The authors did not quantify the final gradient on the bottom β, which can actually be obtained analytically as mentioned in chapter 6.
- The validation analysis of the formulations was left out by the authors.

ii) Mathematical modelling

Before Gibson's theory, mathematical modellings were based on Terzaghi's theory (Abbott,1960; De Leeuw & Abbott,1966; Abbott & Shrivastava, 1967), but only the mathematical modelling based on (2-15) are mentioned here.

Gibson (1981) linearised (2-15) as follows

$$\frac{\delta Vr}{\delta T} + Zd \cdot \frac{\delta Vr}{\delta \eta} - \frac{\delta^2 Vr}{\delta \eta^2} = 0 \qquad (2\text{-}20)$$

where, T, η and Zd (as mentioned in Chapter 3) are dimensionless. His computations of (2-20) show that when Zd is larger, consolidation progresses much faster. The computation had to investigate the physical roles of the first two terms in (2-15).

For practical purposes, mathematical models based on (2-15) were developed by Monte & Krizek (1976) and Somogyi (1984). The applicability of mathematical models for very high void ratio was also investigated by Scully (1984) and Gibson (1984).

Either implicit or explicit finite difference schemes were employed in the models. Although the limited imformation did not give the insight into existing models, the following points arise in literature (Koppula & Morgenstern, 1982, 1984; Krizek & Somogyi,1984; Bromwell, 1984).

- Initial condition for the sedimentation/consolidation problem. The determination of initial void ratio can be made by sedimentation experiment. Alternatively, it can be taken seven times the void ratio at the liquid limit (Carrier et al,1983). However, Carrier states that the initial void ratio usually has a minor effect on the predicted rate of consolidation and only has an effect on the profile during consolidation.

- The linearizations of non-linear terms. The non-linearity of (2-15) should not cause much trouble in mathematical modelling, because there are no shocks during consolidation.
- The boundary condition on the surface (Z=Zo),

$$Pe(Zo,t) = 0$$

leads to (2-18), which has been applied both in mathematical models and analytical solutions. However, for the problem of consolidation after sedimentation this boundary condition can cause numerical instability, particularly when the spatial step is large and the initial void ratio is very high.
- The boundary condition on the undrained bottom,

$$\left.\frac{\delta Pe}{\delta Z}\right|_{Z=0} = 0$$

which leads to

$$\left.\frac{\delta Vr}{\delta Z}\right|_{Z=0} = \left.f(Vr)\right|_{Z=0} \qquad (2\text{-}21)$$

f(Vr) is of high non-linearity since two consitutive relatioships σ'--Vr and Vr--k are of high non-linearity.

2.4 **Experimental results**

Normally the experimental results are in terms of dry density profiles. Owen (1970,1975) measured density profiles during consolidation after sedimentation and revealed that dry density on the surface layer has similar change processes to those of mean dry density. Hayter (1983,1984) re-arranged the data of the authors and suggested a power-law formula to estimate dry-density profiles during consolidation.

Krone (1962) found the order of aggregate is reduced to the next lower order due to overburden thickness, and empirically determined the properties of cohesive-sediment aggregates in quantity. He used his results in the modelling for estuarine-morphological computations.

It should be mentioned that the experimental approaches are valid only for the consolidation processes AFTER sedimentation. In other words, deposition in reservoirs could be intensive and sedimentation and consolidation take place simultaneously. In this case theoretically based mathematical modelling is required.

3. Analytical Solution for the Linearized Case

3.1 General

Although the based equation for analysis is linearized, the solution can give the insight into the physical phenomena of self-weight consolidation. In this chapter, the Laplace's transform is employed to solve the mathematical problem, but only the main idea of the analysis is presented, while a special series expansion is applied to obtain the inverse Laplace's transform (or, the Vr distribution), that is presented in Appendix I.

3.2 Basic equation

Gibson's equation is adopted, but it is supposed that

$$\frac{\rho_s - \rho}{\rho} \cdot \frac{d}{dVr}\left[\frac{k}{1+Vr}\right] = Co = \text{Constant} \qquad (3\text{-}1)$$

and

$$-\frac{k}{\rho . g . (1+Vr)} \cdot \frac{d\sigma'}{dVr} = Cv = \text{Constant} \qquad (3\text{-}2)$$

such that

$$\frac{\delta Vr}{\delta t} + Co . \frac{\delta Vr}{\delta Z} - Cv . \frac{\delta^2 Vr}{\delta Z^2} = 0 \qquad (3\text{-}3)$$

3.3 Initial and boundary conditions

i) Initial condition

Normally, initial condition is written as

$$Vr\ (Z\ ,t)\Big|_{t=0} = Vr\ (Z\ ,0),$$

provided, Vr(Z ,0) is given. The following initial condition is here applied for reason of simplification.

$$Vr\ (Z,0) = Vrini = \text{constant} \qquad (3\text{-}4)$$

Eq(3-4) physically implies that a total amount of considered sediments are immediately deposited just at time t=0.

ii) Boundary condition

On the surface boundary condition (2-18) is used.
And on the undrained bottom,

$$\left.\frac{\delta Pe}{\delta Z}\right|_{Z=0} = 0 \tag{3-5}$$

Since the total vertical stress

$$\sigma = [n.\rho + (1-n)\ \rho_s].g.(So-S)$$

and the effective stress

$$\sigma' = \sigma - Pe - \rho.g.(So-S) = (1-n).(\rho_s - \rho).g.(So-S) - Pe =$$

$$= (\rho_s - \rho).g.(Zo-Z) - Pe \tag{3-6}$$

Therefore, from (3-5) and (3-6), we have

$$\left.\frac{\delta\sigma'}{\delta Z}\right|_{Z=0} = \left.\frac{\delta\sigma}{\delta Z}\right|_{Z=0} = -\ (\rho_s - \rho).g \tag{3-7}$$

that yields on the bottom (or,Z=0)

$$\left.\frac{\delta Vr}{\delta Z}\right|_{Z=0} = \frac{dVr}{d\sigma'}\cdot\left.\frac{\delta\sigma'}{\delta Z}\right|_{Z=0} = -\ (\rho_s - \rho).g.\frac{dVr}{d\sigma'} \tag{3-8}$$

Moreover, from (3-1)

$$\frac{\rho_s - \rho}{\rho}\cdot\frac{k}{1+Vr} = Co.Vr + C \tag{3-9}$$

where, C is integral constant.

Substituting (3-9) into (3-2) yields

$$-\frac{1}{\rho.g}\cdot(Co.Vr + C) = Cv\cdot\frac{\rho_s - \rho}{\rho}\cdot\frac{dVr}{d\sigma'} \tag{3-10}$$

Substituting (3-10) into (3-8) yields

$$\left.\frac{\delta Vr}{\delta Z}\right|_{Z=0} = \left.\frac{Co.Vr + C}{Cv}\right|_{Z=0} \tag{3-11}$$

Since

$$\lim_{t \to \infty} \left. \frac{\delta Vr}{\delta Z} \right|_{Z=0} = \beta,$$

where, β is, again, the final gradient of Vr on the bottom (β is derived in Chapter 6), thence

$$C = -Co.Vr\infty + \beta.Cv$$

in which, $Vr\infty$ denotes the final void ratio on the impervious bottom. So that

$$\left. \frac{\delta Vr}{\delta Z} \right|_{Z=0} = \beta + \left. \frac{Co.\ (Vr-Vr\infty)}{Cv} \right|_{Z=0} \qquad (3\text{-}12)$$

In dimensionless form (3-12) is written as

$$\left. \frac{\delta Vr}{\delta \eta} \right|_{\eta=0} = \beta.Zo + \left. \frac{Co.Zo.(Vr-Vr\infty)}{Cv} \right|_{\eta=0} \qquad (3\text{-}13)$$

in which, $\eta = Z/Zo$.

3.4 **Analytical solution**

i) Mathematical problem

The statement of the linearized self-weight consolidation processes is the following

$$\frac{\delta Vr}{\delta t} + Co\ .\ \frac{\delta Vr}{\delta Z} - Cv\ .\ \frac{\delta^2 Vr}{\delta Z^2} = 0 \qquad (3\text{-}3)$$

$$Vr(Z,0) = Vrini \qquad (3\text{-}4)$$

$$Vr(Zo,t) = Vrini \qquad (2\text{-}18)$$

$$\left. \frac{\delta Vr}{\delta Z} \right|_{Z=0} = \beta + \left. \frac{Co.(Vr-Vr\infty)}{Cv} \right|_{Z=0} \qquad (3\text{-}12)$$

ii) The solution of the corresponding Laplace's transform of the above mathematical problem

a) The Laplace's transform

Let
$$V = \int_{o}^{\infty} Vr.\exp(-p.t).dt,$$

in which, p is the Laplace's constant; and V is Laplace's transform of Vr.

We then have

$$\int_{o}^{\infty} \frac{\delta Vr}{\delta t} . \exp(-p.t).dt = - Vrini + p.V .$$

Therefore, we have the Laplace's transform of the mathematical problem as follows

$$Cv . \frac{\delta^2 V}{\delta Z^2} - Co . \frac{\delta V}{\delta Z} - p.V + Vrini = 0 \qquad (3\text{-}14a)$$

$$V(Zo,t) = Vrini/p \qquad (3\text{-}14b)$$

$$\left.\frac{\delta V}{\delta Z}\right|_{Z=0} = \frac{\beta}{p} - \frac{Co}{Cv} . \frac{Vr\infty}{p} + \frac{Co}{Cv} . V\Big|_{Z=0} \qquad (3\text{-}15c)$$

b) The solution of eqs (3-15)

The general solution of eqs(3-15) is that

$$V = Vrini/p + C_1.\exp(r_1.Z) + C_2.\exp(r_2.Z) \qquad (3\text{-}16)$$

with

$$r_{1,2} = \frac{Co \pm \sqrt{Co^2 + 4.Cv.p}}{2.Cv} \qquad (3\text{-}17)$$

and the boundary conditions (3-14b) and (3-14c) determine the coefficients C_1 and C_2 , i.e.

$$C_1.\exp(r_1.Zo) + C_2.\exp(r_2.Zo) = 0 \qquad (3\text{-}18a)$$

$$C_1.r_1 + C_2.r_2 = \frac{\beta}{p} + \frac{Co.(Vrini-Vr\infty)}{Cv.p} + (C_1+C_2) . \frac{Co}{Cv} \qquad (3\text{-}18b)$$

Therefore, the solution of eqs(3-15) reads

$$V = \frac{\exp(r_1.Z + r_2.Zo) - \exp(r_2.Z + r_1.Zo)}{(r_1 - \frac{Co}{Cv}).\exp(r_2.Zo) - (r_2 - \frac{Co}{Cv}).\exp(r_1.Zo)} *$$

$$* \frac{Vrini - Vr\infty + \beta.Cv/Co}{p} \cdot \frac{Co}{Cv} + \frac{Vrini}{p} \qquad (3\text{-}19)$$

c) Analytical solutions

The void ratio distributions of linearized self-weight consolidation, Vr, is the inverse Laplace's transform of eq(3-19), which is obtained through applying a special series expansion (for the details, see Appendix I)

$$Vr = Vrini - 2.(Vrini - Vr\infty + \beta.Zo/Zd).\exp(\eta.Zd/2) *$$

$$* \left[\sum_{j=1}^{\infty} \frac{2.\sin[b_j Zd.(1-\eta)/2].[b_j \exp(-Zd^2.(1+b_j^2).T/4 + 1/b_j]}{\{Zd.[\cos(b_j Zd/2) - b_j \sin(b_j Zd/2)]/2.0 + \cos(b_j Zd/2)\}.(1+b_j^2)} + \right.$$

$$\left. + \frac{Zd.(1-\eta)}{2+Zd} \right] \qquad (3\text{-}20)$$

in which, $Zd = Co.Zo/Cv$, $T = t.Cv/Zo^2$ and $\eta = Z/Zo$ are dimensionless; b_j (j=1, 2, ...) are dimensionless and the roots of the following algebraic equation (see, Fig.3-1)

$$tg(Zd.b/2.0) + b = 0 \qquad (3\text{-}21)$$

However, it should be noted that b=0 is excluded from (3-20), which is also mentioned in Appendix I.

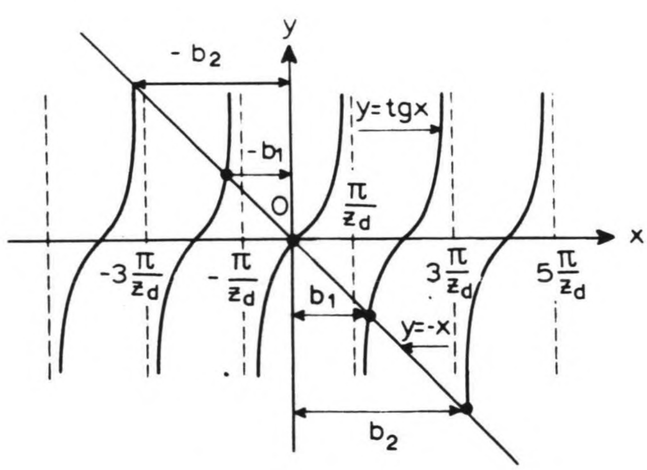


Fig.3-1 Sketch for b_j, roots of eq(3-21)

3.5 **Evidence for the analytical result**

From eq(3-20), it can be seen that the consolidation processes are dominated by the value of Zd, which is shown in the following profiles (Fig.3-2) with different values of Zd.

If the void ratio is approximated by first order (or j=1), then the degree of consolidation Dc(t) can be written as

$$Dc(t) = \frac{So(0) - So(T)}{So(0) - So(\infty)} = 1.0- \exp(-Zd^2.(1+b_1^2).T) \qquad (3\text{-}22)$$

in which, So is the thickness of deposits.

Thus it also can be seen from eq(3-22) (or, Fig.3-3) that when Zd is large, the consolidation is sooner completed.

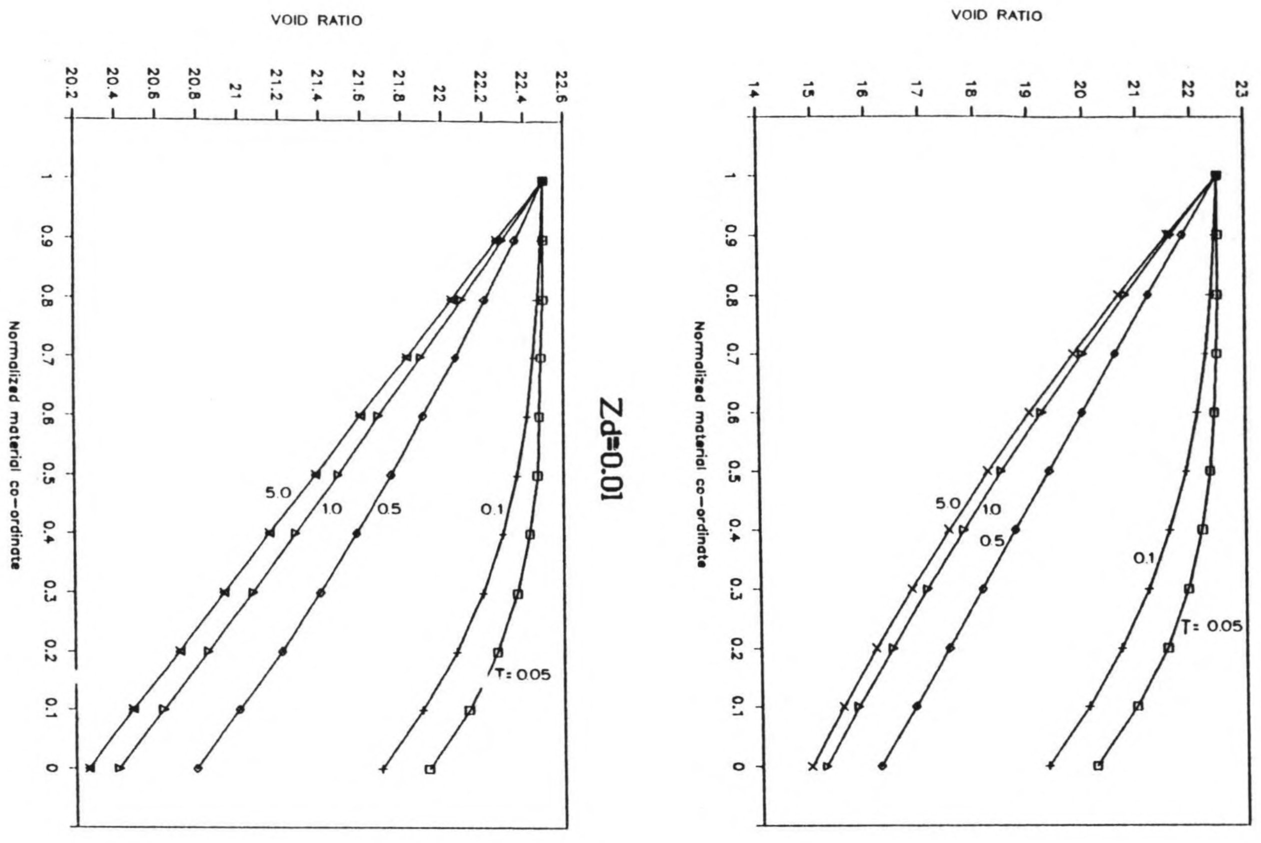


Fig.3-2a The Vr-profiles vs T with fixed Zd

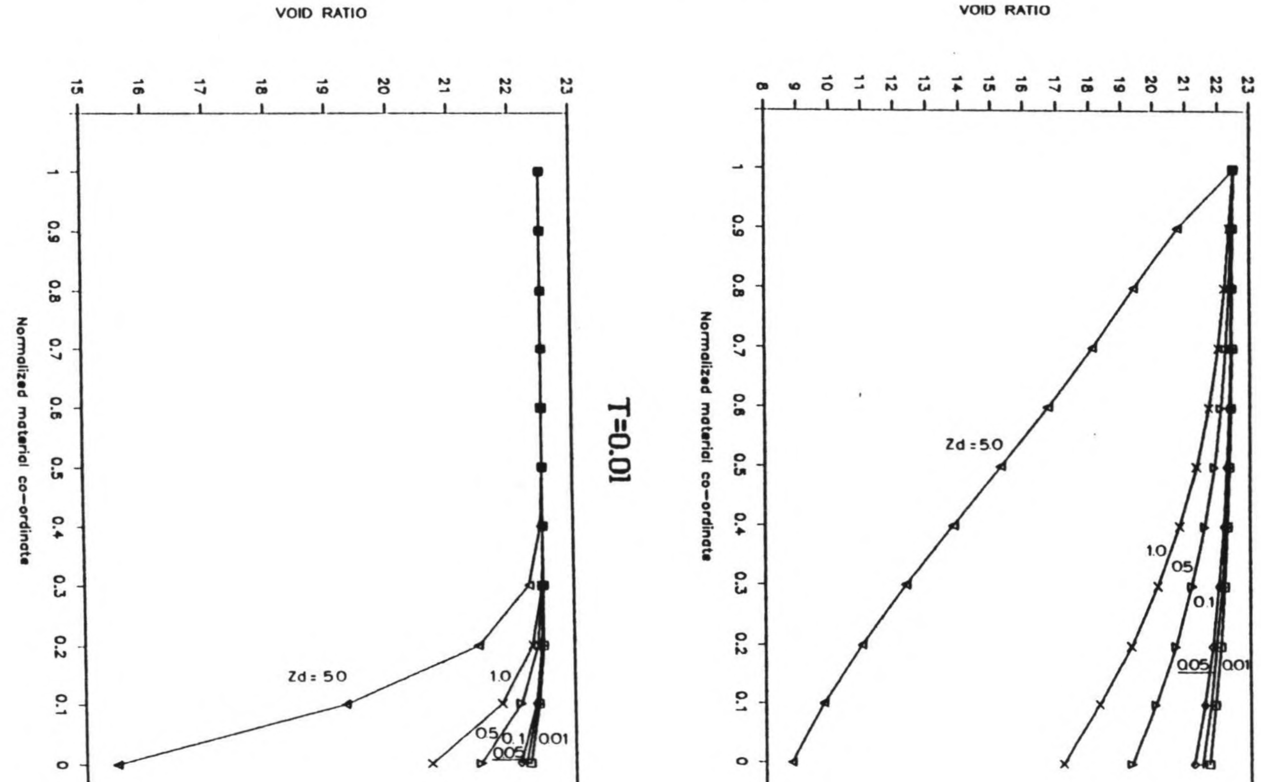


Fig.3-2b The Vr-profile vs Zd with fixed T

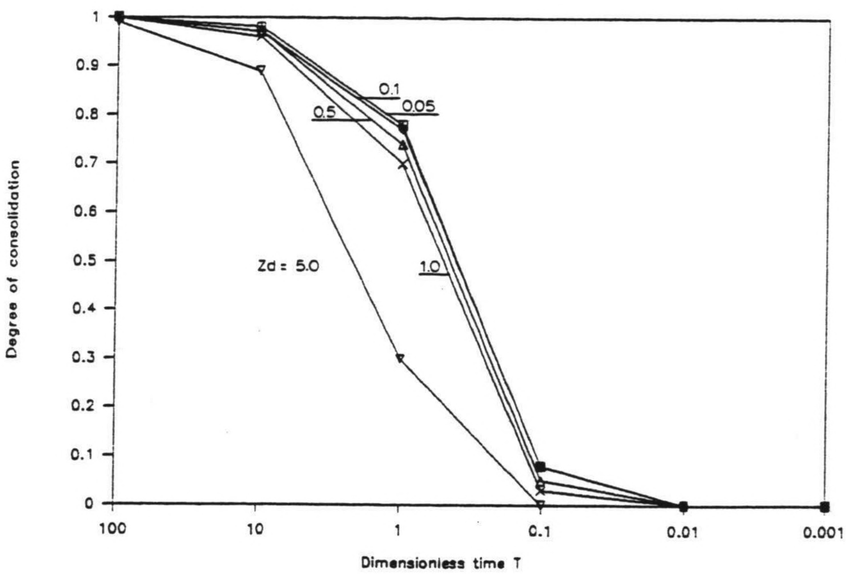


Fig. 3-3 The consolidation degree vs Zd

4. Mathematical Modelling for the Self-weight Consolidation, Case 1, Linearized Model

4.1 General

For the convenience of description of the general model, some fundamental aspects of numerical discretization are described in this chapter, while the initial and boundary conditions are kept the same as mentioned in chapter 3.

4.2 Basic equation

For the reason of simplification, the linearized equation is rewritten in dimensionless form such that

$$\frac{\delta Vr}{\delta T} + Zd \cdot \frac{\delta Vr}{\delta \eta} - \frac{\delta^2 Vr}{\delta \eta^2} = 0 \qquad (4\text{-}1)$$

in which, T, Zd and η are as defined previously in Chapter 3.

4.3 Numerical scheme

The following numerical scheme is applied (Fig.4-1) to decritize eq(4-1),

At grid point j,

$$\frac{\delta Vr}{\delta T} = \frac{Vr_j^{n+1} - Vr_j^n}{\Delta T} \qquad (4\text{-}2)$$

$$\frac{\delta Vr}{\delta \eta} = \frac{\Theta.(Vr_{j+1}^{n+1} - Vr_{j-1}^{n+1})}{2.\Delta\eta} + \frac{(1-\Theta).(Vr_{j+1}^{n} - Vr_{j-1}^{n})}{2.\Delta\eta} \qquad (4\text{-}3)$$

$$\frac{\delta^2 Vr}{\delta \eta^2} = \frac{\Theta.(Vr_{j+1}^{n+1} - 2.Vr_j^{n+1} + Vr_{j-1}^{n+1})}{\Delta\eta^2} +$$

$$+ \frac{(1-\Theta).(Vr_{j+1}^{n} - 2.Vr_j^{n} + Vr_{j-1}^{n})}{\Delta\eta^2} \qquad (4\text{-}4)$$

where,

Θ = weight in time;

$\Delta\eta$ = spatial step;

ΔT = time step;

n,(n+1) = denote the values at time level n and (n+1), respectively.

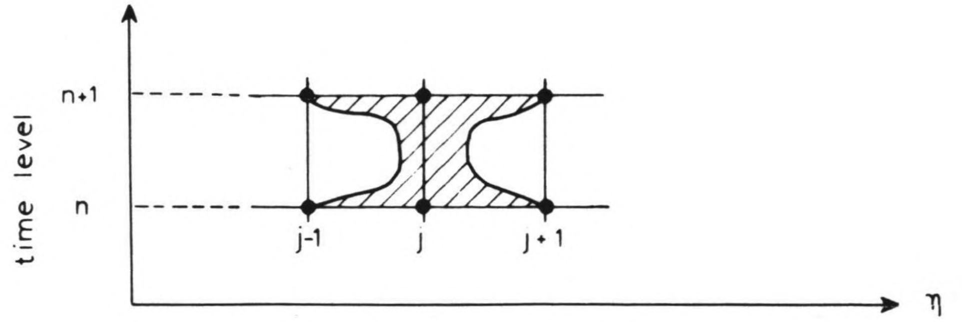


Fig.4-1 Numerical operator

Substituting (4-2), (4-4) into (4-1) and re-arranging yield

$$A(j).Vr_{j+1}^{n+1} + B(j).Vr_{j}^{n+1} + C(j).Vr_{j-1}^{n+1} = D(j) \qquad (4\text{-}5)$$

in which,

$$A(j) = \frac{\Theta.Zd.\Delta T}{2.\Delta\eta} - \frac{\Theta.\Delta T}{\Delta\eta^2},$$

$$B(j) = 1 + \frac{2.\Theta.\Delta T}{\Delta\eta^2},$$

$$C(j) = -\left(\frac{\Theta.Zd}{2.\Delta\eta} + \frac{\Theta}{\Delta\eta^2}\right).\Delta T,$$

$$D(j) = Vr_j^n - \frac{(1-\Theta).Zd.\Delta T}{2.\Delta\eta} . (Vr_{j+1}^n - Vr_{j-1}^n) + $$
$$+ \frac{(1-\Theta).\Delta T}{\Delta\eta^2} . (Vr_{j+1}^n - 2.Vr_j^n + Vr_{j-1}^n)$$

4.4 **Numerical accuracy**

The following Taylor's expansions at (j,n) are employed in the accuracy analysis,

$$Vr_j^{n+1} = Vr_j^n + \frac{\delta Vr}{\delta \eta}_j . \Delta\eta + \frac{1}{2!} . \frac{\delta^2 Vr}{\delta \eta^2}_j . \Delta\eta^2 + \ldots , \ldots ;$$

$$Vr_{j\pm1}^{n+1} = Vr_j^n \pm \frac{\delta Vr}{\delta \eta}_j . \Delta\eta + \frac{\delta Vr}{\delta T}_j . \Delta T + \frac{1}{2!} . \frac{\delta^2 Vr}{\delta \eta^2}_j . \Delta\eta^2 +$$

$$+ \frac{1}{2!} . \frac{\delta^2 Vr}{\delta T^2}_j . \Delta T^2 \pm \frac{\delta^2 Vr}{\delta T \delta \eta}_j . \Delta\eta . \Delta T + \ldots , \ldots ;$$

$$Vr_{j\pm1}^{n} = Vr_j^n \pm \frac{\delta Vr}{\delta \eta}_j . \Delta\eta + \frac{1}{2!} . \frac{\delta^2 Vr}{\delta \eta^2}_j . \Delta\eta^2 + \ldots , \ldots$$

Substituting above expansions into (4-5), re-arranging and removing the subscript j yield

$$\frac{\delta Vr}{\delta T} + Zd. \frac{\delta Vr}{\delta \eta} - \frac{\delta^2 Vr}{\delta \eta^2} =$$

$$= - \Delta T. \frac{\delta}{\delta T} (\frac{1}{2} . \frac{\delta Vr}{\delta T} + \Theta.Zd . \frac{\delta Vr}{\delta \eta} - \Theta . \frac{\delta^2 Vr}{\delta \eta^2}) -$$

$$- \frac{1}{6} . \frac{\delta^3 Vr}{\delta T^3} . \Delta T^2 - \frac{1}{2} .\Theta.\Delta T^2.Zd. \frac{\delta^3 Vr}{\delta \eta \delta T^2} - \frac{1}{6} .Zd.\Delta\eta^2 . \frac{\delta^3 Vr}{\delta \eta^3} +$$

$$+ \frac{1}{6} . (1-\Theta).\Delta T.\Delta\eta^2 . \frac{\delta^4 Vr}{\delta \eta^4} + h.o.t, \qquad (4-6)$$

where, h.o.t stands for higher order terms.

Clearly, when $\Theta \geq 1/2$, the numerical scheme (4-5) is unconditionally stable and the truncation error

$$T.E. = - \Delta T . \frac{\delta}{\delta T} (\frac{1}{2} . \frac{\delta Vr}{\delta T} + \Theta.Zd. \frac{\delta Vr}{\delta \eta} - \Theta. \frac{\delta^2 Vr}{\delta \eta^2}) -$$

$$- \frac{1}{6} . \frac{\delta^3 Vr}{\delta T^3} . \Delta T^2 - \frac{1}{2} . \Theta . \Delta T^2.Zd . \frac{\delta^3 Vr}{\delta \eta \delta T^2} - \frac{1}{6} . Zd . \Delta\eta^2 . \frac{\delta^3 Vr}{\delta \eta^3} +$$

$$+ \frac{1}{6} . (1-\Theta) . \Delta T . \Delta\eta^2 . \frac{\delta^4 Vr}{\delta \eta^4} + h.o.t, \qquad (4-7)$$

that is followed by $T.E. = O(\Delta T^2, \Delta\eta^2)$, for $\Theta = 1/2$.

4.5 **Numerical stability**

The solutions are decomposed into Fourier series as follows,

$$Vr_j^n = \Sigma A_k^n \,.\, \exp[i\frac{k.j.\Delta\eta}{(N-1).\Delta\eta}\pi] \tag{4-8}$$

$$Vr_j^{n+1} = \Sigma A_k^{n+1} \,.\, \exp[i\frac{k.j.\Delta\eta}{(N-1).\Delta\eta}\pi] \tag{4-9}$$

$$Vr_{j\pm1}^n = \Sigma A_k^n \,.\, \exp[i\frac{k.(j\pm1).\Delta\eta}{(N-1).\Delta\eta}\pi] \tag{4-10}$$

$$Vr_{j\pm1}^{n+1} = \Sigma A_k^{n+1} \,.\, \exp[i\frac{k.(j\pm1).\Delta\eta}{(N-1).\Delta\eta}\pi] \tag{4-11}$$

in which, j = grid point (j=1,2,...... N), and A_k^n is the amplitude of the kth component at time level (n+1).

Substituting eqs(4-8)--(4-11) and introducing $C = \frac{Zd.\Delta T}{\Delta\eta}$ and $E = \frac{\Delta T}{\Delta\eta^2}$ into eq(4-5), we finaliy have for the kth component of Fourier series

$$A = \frac{A_k^{n+1}}{A_k^n} = \frac{1-(1-\theta).2E.[1-\cos(k\pi/N-1)] - i(1-\theta).C.\sin(k\pi/N-1)}{1+\theta.2.E.[1-\cos(k\pi/N-1)] + i\theta.C.\sin(k\pi/N-1)} \tag{4-12}$$

From $|A|^2 \leq 1$, we therefore have the following stability condition (for the details, see appendix II),

$$\theta \geq \frac{1}{2} - \min \{\frac{2E}{4E^2 + C^2 \pm |4E^2 - C^2|}\} \tag{4-13}$$

4.6 Computational algorithm

The "double sweep algorithm" is used to solve the tri-diagonal matrix (expressed by eq(4-5)).

Introducing

$$Vr_j^n = L(j).Vr_{j+1}^{n+1} + M(j) \tag{4-14}$$

into (4-5) yields the following re-occurrence coefficients

$$L(j) = -\frac{A(j)}{B(j) + C(j).L(j-1)} \tag{4-15}$$

$$M(j) = \frac{D(j) - C(j).M(j-1)}{B(j) + C(j).L(j-1)} \tag{4-16}$$

where, L(j) and M(j) are determined by boundary condition on the bottom (where, the grid point is j=1).

Provided A(j),B(j),C(j) and D(j) (j=1,2,...,N) are determined beforehand, in the first sweep L(j) and M(j) are calculated progressively from j=1 to j=N, while in the second sweep Vr_j^{n+1} is calculated backward from j=N-1 to j=1 (Note, Vr_N^{n+1} is determined by the boundary condition on the deposits' surface). This procedure can be represented by Fig.(4-2)

4.7 Comparison between numerical and anlytical results

Fig.(4-3) shows the good agreement between the numerical results and analytical results.

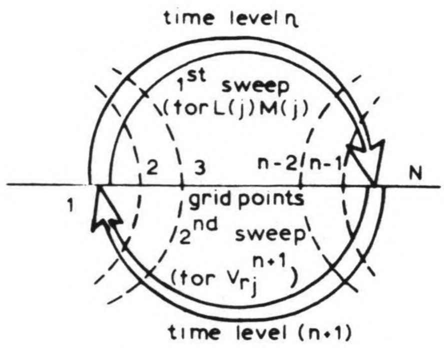


Fig. 4-2. The "double sweep algorithm" procedure

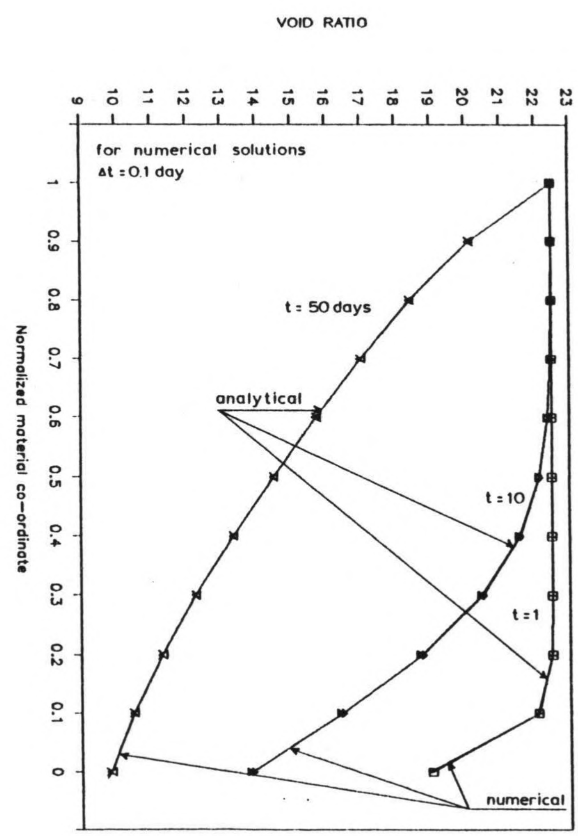


Fig.4-3 The numerical and analytical results of linearized equation (Zo=0.15 m)

5. **Mathematical Modelling for the Self-weight Consolidation, Case 2, General model**

5.1 **General**

Except for being based on the non-linear equation (2-15), this general model includes the simultaneous deposition and consolidation. The main considerations of this model are the followings:

- The numerical scheme described in Chapter 4 is employed.
- Taking the time-dependence of the thickness Zo(t) into account, the vertical material co-ordinate Z in eq (2-15) is normalized by introducing $\eta = Z/Zo(t)$.
- The constitutive relationships Vr--σ' and k--Vr are assumed to be given and to follow the power law.

5.2 **Basic equation and constitutive relationships**

i) Basic equation

The non-linear equation (2-15) is based in this model,

$$\frac{\delta Vr}{\delta t} + \frac{\rho_s - \rho}{\rho} \cdot \frac{d}{dVr}\left[\frac{k}{1+Vr}\right] \cdot \frac{\delta Vr}{\delta Z} + \frac{\delta}{\delta Z}\left[\frac{k}{\rho . g . (1+Vr)} \cdot \frac{d\sigma'}{dVr} \cdot \frac{\delta Vr}{\delta Z}\right] = 0 \qquad (2\text{-}15)$$

Considering Zo is changing with time due to sedimentation, the co-ordinate Z in (2-15) is normalized by introducing $\eta = Z/Zo$. Equation (2-15) then reads

$$\frac{\delta Vr}{\delta t} + \frac{\rho_s - \rho}{\rho . Zo} \cdot \frac{d}{dVr}\left[\frac{k}{1+Vr}\right]\frac{\delta Vr}{\delta \eta} + \frac{1}{\rho . g . Zo^2} \cdot \frac{\delta}{\delta \eta}\left[\frac{k}{1+Vr} \cdot \frac{d\sigma'}{dVr} \cdot \frac{\delta Vr}{\delta \eta}\right] = 0 \qquad (2\text{-}15')$$

As mentioned previously,

$$Zo = Zo(t) \qquad (5\text{-}1)$$

The grid points is now presented by the following Fig.(5-1).

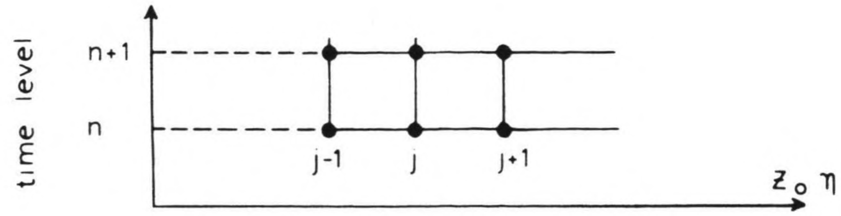


Fig.5-1a Grid points (Zo=constant)

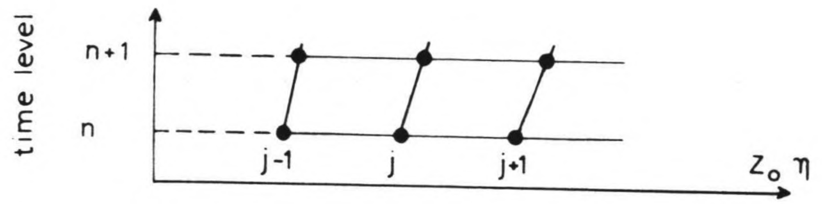


Fig.5-1b Grid point (Zo=Zo(t))

ii) The constitutive relationships

As mentioned in Chapter 2, the expressions of eq(2-12) and (2-14) have to be established to make eq(2-15') close. There are many empirical relationships suggested by different authors (summarized by Krizek and Somogyi, 1984). However, from some experimental results (Krizek and Somogyi, 1984; Znidarcic et al 1984) the more general expressions seem to follow the power law.

a) Compressibility (Vr--σ')

$$Vr = Ap*(\sigma'/\sigma'_c)^{-Bp} \qquad (5\text{-}2)$$

in which, Ap and Bp are empirical constants and positive, σ'_c is the reference effective stress.

b) Permeability (k--Vr)

$$k/kc = Ak.(Vr)^{Bk} \qquad (5\text{-}3)$$

in which Ak and Bk are empirical constants and positive, kc is the reference permeability.

Constants Ak, Bk, Ap and Bp depend upon the properties of the deposites. Table (5-1) (roughly obtained from some literature) shows the values of the constants with different soils.

CLAY NAME	Ak	Bk	AP	BP
FLORIDA CLAY	$1.4*10^{-11}$	4.11	90.37	0.29
KINGS BAY	$2.0*10^{-11}$	5.40	26.07	0.19
SODIUM MONTMORILLONITE	$1.0*10^{-14}$	3.0	9567.0	1.00
CALCIUM MONTMORILLONITE	$1.0*10^{-12}$	6.0	31.92	0.3
MAUMEE RIVER,TOLEDO	$5.0*10^{-12}$	5.70	5.16	0.14

Notes: kc = 1.0 m/s; σ'_c = 1.0 Pa.

Table 5-1. The values of Ak, Bk, Ap and Bp

5.3 **Linearizations of the nonlinear terms**

Provided that eq. (5-2) and (5-3) are given before hand, eq(2-15') can be re-written as follows

$$\frac{\delta Vr}{\delta t} + \frac{F(Vr)}{Zo(t)} \cdot \frac{\delta Vr}{\delta \eta} - \frac{1}{Zo^2(t)} \cdot \frac{\delta}{\delta \eta}\left[G(Vr) \cdot \frac{\delta Vr}{\delta \eta}\right] = 0 \qquad (5\text{-}4)$$

where,

$$F(Vr) = (\frac{\rho_s}{\rho} - 1) \frac{d}{dVr}\left[\frac{k}{1+Vr}\right],$$

$$G(Vr) = - \frac{k}{\rho.g.(1+Vr)} \cdot \frac{d\sigma'}{dVr}$$

Then the nonlinear terms (the second and third terms of (5-4)) are linearized as follows.

At grid point j (Fig.5-1)

$$\frac{F(Vr)}{Zo^2(t)} \cdot \frac{\delta Vr}{\delta \eta} = \frac{\frac{1}{2}\left[F(Vr_j^{n+1}) + F(Vr_j^n)\right]}{\frac{1}{2}\left[Zo(\Delta t.n) + Zo[\Delta t(n+1)]\right]} *$$

$$* \left[\frac{\theta.(Vr_{j+1}^{n+1} - Vr_{j-1}^{n+1})}{2.\Delta\eta} + \frac{(1-\theta).(Vr_{j+1}^{n} - Vr_{j-1}^{n})}{2.\Delta\eta}\right]$$

Or, F(Vr) is centred at (j, n+1/2), Zo(t) is centred at time level (n+1/2).

$$\frac{1}{Zo^2(t)} \cdot \frac{\delta}{\delta\eta}\left[G(Vr) \cdot \frac{\delta Vr}{\delta\eta}\right] = \frac{1}{\{Zo(\Delta t.n)/2 + Zo[\Delta t.(n+1)]/2\}^2} *$$

$$*\left\{\left[G2.\frac{\theta.(Vr_{j+1}^{n+1} - Vr_{j}^{n+1})}{\Delta\eta^2} - G1.\frac{\theta.(Vr_{j}^{n+1} - Vr_{j-1}^{n+1})}{\Delta\eta^2}\right] - \right.$$

$$\left. - \left[G2.\frac{(1-\theta).(Vr_{j+1}^{n} - Vr_{j}^{n})}{\Delta\eta^2} - G1.\frac{(1-\theta).(Vr_{j}^{n} - Vr_{j-1}^{n})}{\Delta\eta^2}\right]\right\}$$

where,

$$G2 = \frac{1}{4} . \left[G(Vr_{j+1}^{n+1}) + G(Vr_{j}^{n+1}) + G(Vr_{j+1}^{n}) + G(Vr_{j}^{n})\right]$$

and $$G1 = \frac{1}{4} . \left[G(Vr_{j}^{n+1}) + G(Vr_{j-1}^{n+1}) + G(Vr_{j}^{n}) + G(Vr_{j-1}^{n})\right]$$

The values of Vr at time level (n+1) in G1 and G2 are to be determined by iteration. See Section 5-5.

5.4 **Initial and boundary conditions**

i) <u>Initial condition</u>

For void ratio at time t=0, the initial condition eq(2-17) is applied. It has to be mentioned that for the case of self-weight consolidation with Zo(t) increasing with time, the initial thickness Zo(0) can be very small but not zero.

ii) Boundary condition

a) At the surface ($\eta = 1.0$), the following boundary conditions are applied for both cases of constant Zo and Zo(t) increasing with time during consolidation.

$$Vr\ (\eta,\ t)\Big|_{\eta=1} = Vrini \tag{2-18}$$

b) At the bottom, the boundary condition has to be derived in the following:

From Section 3-3,

$$\left.\frac{\delta\sigma'}{\delta Z}\right|_{Z=0} = -\ (\rho_s - \rho).g \tag{3-7}$$

and from (5-2),

$$\frac{dVr}{d\sigma'} = -\ Ap.Bp.(\sigma'/\sigma'_c)^{-Bp-1}/\sigma'_c =$$

$$= -\ Ap.Bp.(Ap/Vr)^{-(1+Bp)/Bp}/\sigma'_c \tag{5-6}$$

so that we have the boundary condition at the impervious bottom

$$\left.\frac{\delta Vr}{\delta Z}\right|_{Z=0} = (\rho_s - \rho).g.Ap.Bp.(Vr/Ap)^{(1+Bp)/Bp}/\sigma'_c \tag{5-7}$$

or,

$$\left.\frac{\delta Vr}{\delta \eta}\right|_{\eta=0} = Zo(t).(\rho_s - \rho).g.Ap.Bp.(Vr/Ap)^{(1+Bp)/Bp}/\sigma'_c \tag{5-8}$$

Therefore, suppose at bottom (η=0) grid point j=1, eq(5-8) is then discretized as follows

$$\frac{Vr_2^{n+1} - Vr_1^{n+1}}{\Delta\eta} = Zo^{n+1}\ .\ \frac{\rho_s - \rho}{\sigma'_c}.g.\frac{Bp}{Ap^{1/Bp}} *$$

$$* \left[-\ \frac{\left(Vr'_j\right)^{(1+Bp)/Bp}}{Bp} + \frac{(1+Bp).\left(Vr'_1\right)^{1/Bp} * Vr_1^{n+1}}{Bp} \right] \tag{5-9}$$

in which, $Zo^{n+1} = Zo[\Delta t.(n+1)]$, and Vr'_1 is determined by iteration.

5.5 Iteration processes

The values of Vr at time level (n+1) in G1 and G2 and Vr' in eq. (5-9) are determined by the following iteration processes.

The values are initiated by those of the last step (at time level n) for the current computations of Vr at time level (n+1). Then they are replaced by the newly computed values at time level (n+1), whereas the computations of Vr at time level (n+1) are repeated.

5.6 Computational results

The data of Florida Clay (Gibson et al, 1984) are used to verify this modelling. Fig. (5-2) shows the good agreement between the computational results and the data. In addition, Fig.(6-1) also shows the good agreement between the computed final Vr-profile and the analyzed final Vr-profile.

Fig. (5-3) and (5-4) show the different consolidation processes of constant Zo and Zo increasing with time.

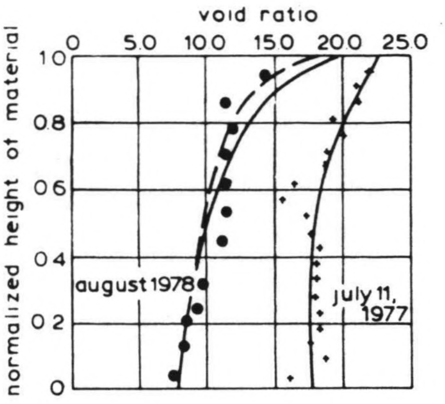
Fig. (5-5), (5-8) show the consolidation processes in terms of void ratio profiles and dry-density profiles for the cases of different Zo.

5.7 Remarks

i) From Fig.(5-2), the boundary condition at the surface does not look so reasonable, but this problem is avoided in the case of consolidation with simultaneous deposition (or, the thickness Zo is increasing with time) which is common in reservoir. To improve on this shortcoming the finer grid is recommanded.

ii) The linearizations of eq (2-15') do not cause any numerical instabilities in the computations. The non-linearity of boundary condition at the impervious bottom causes a large error in the very first steps (Fig.5-9). Therefore the time-step cannot be too large if the void ratio at very small t is important.

iii) This modelling is to be modified for the case of alternative erosion and deposition during consolidation. The idea is, in this case, as the same as the multi-layer problem (Abbott, 1960).



Legend:

________	Gibson's results;
- - - - -	Author's results;
. . . .	Data;
+ + + +	Data.

Fig. 5-2 Void ratio profile - Tank test, Florida Clay (After Gibson et al, 1984)

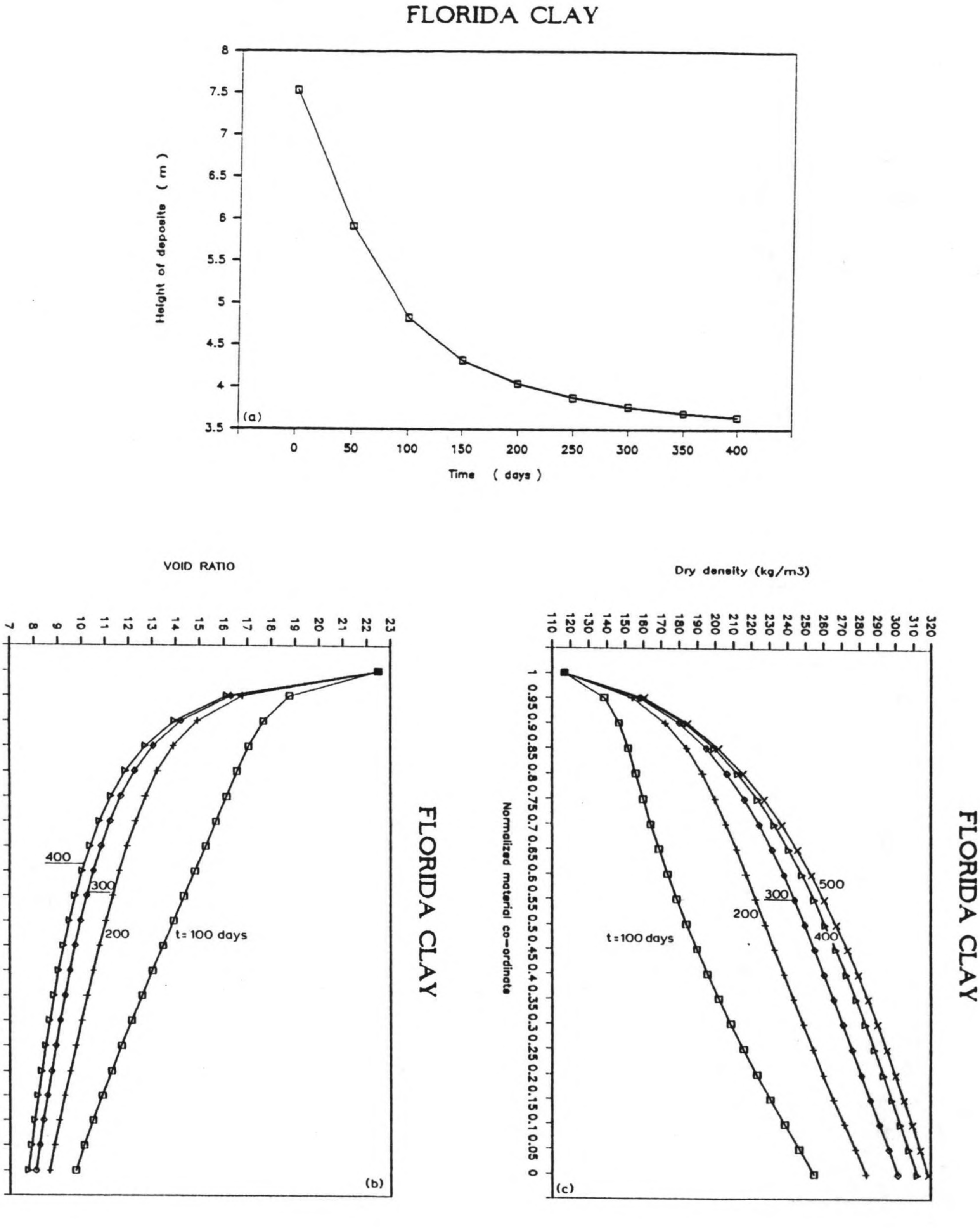


Fig. 5-3 Consolidation processes (Zo=0.32m)

(a) Deposit's thickness processes

(b) Void ratio profiles

(c) Dry-density profiles

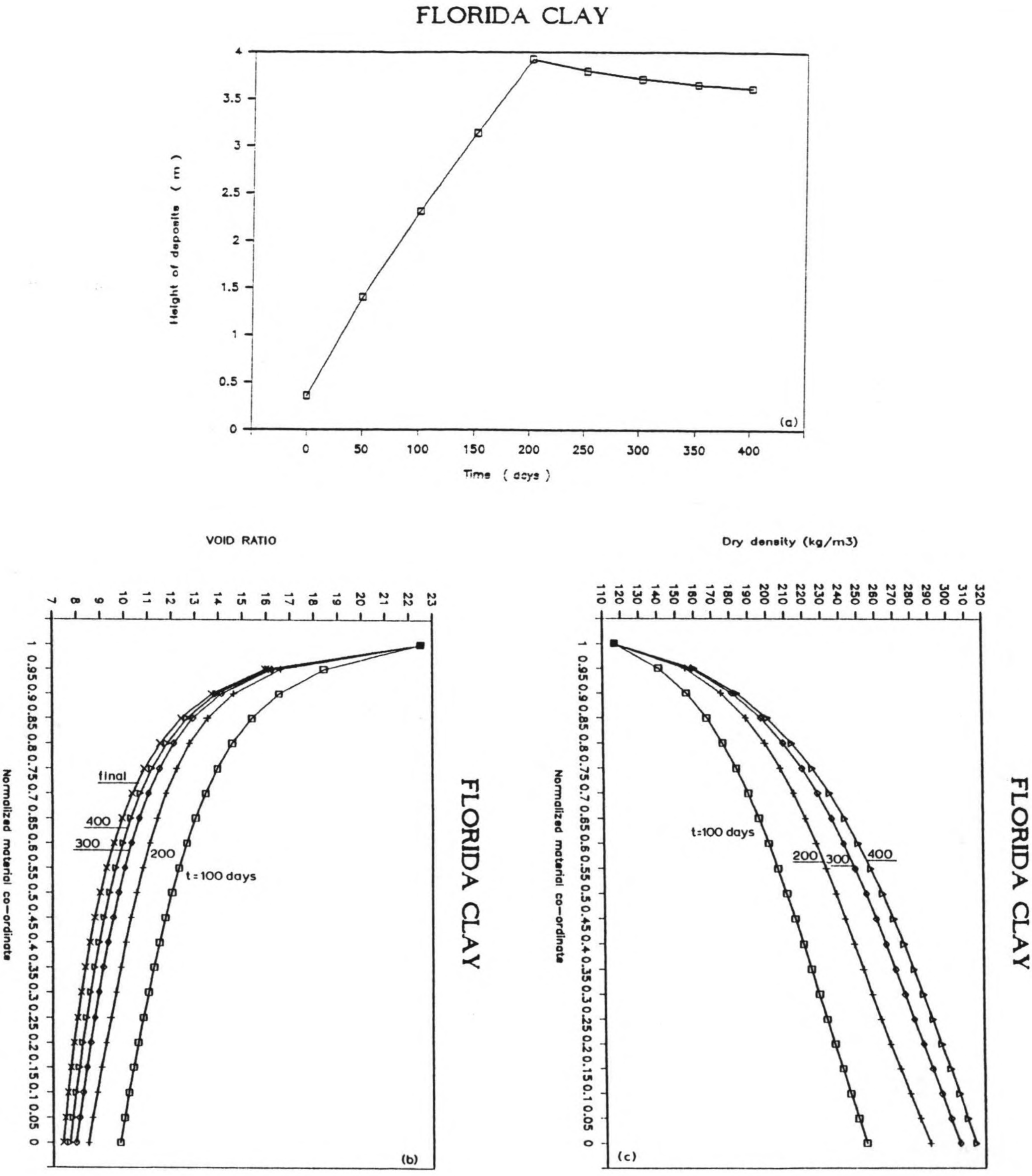


Fig. 5-4 Consolidation processes (Zo=0.015+ Md.t)

Md =0.001525 m/day, for t≤200 days

Md =0 , for t>200 days

(a) Deposit's thickness processes

(b) Void ratio profiles

(c) Dry-density profiles

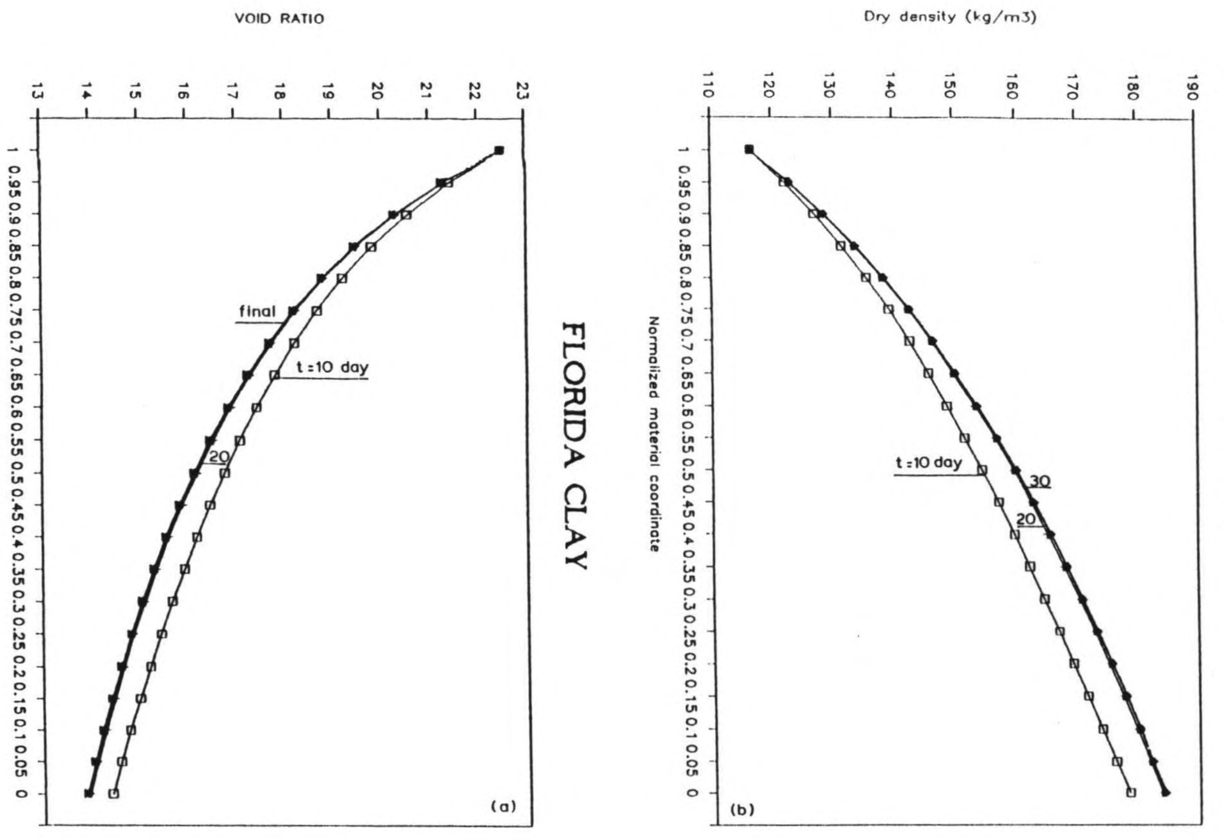


Fig. 5-5 Consolidation processes (Zo=0.03 m)
(a) Void ratio profiles
(b) Dry-density profiles

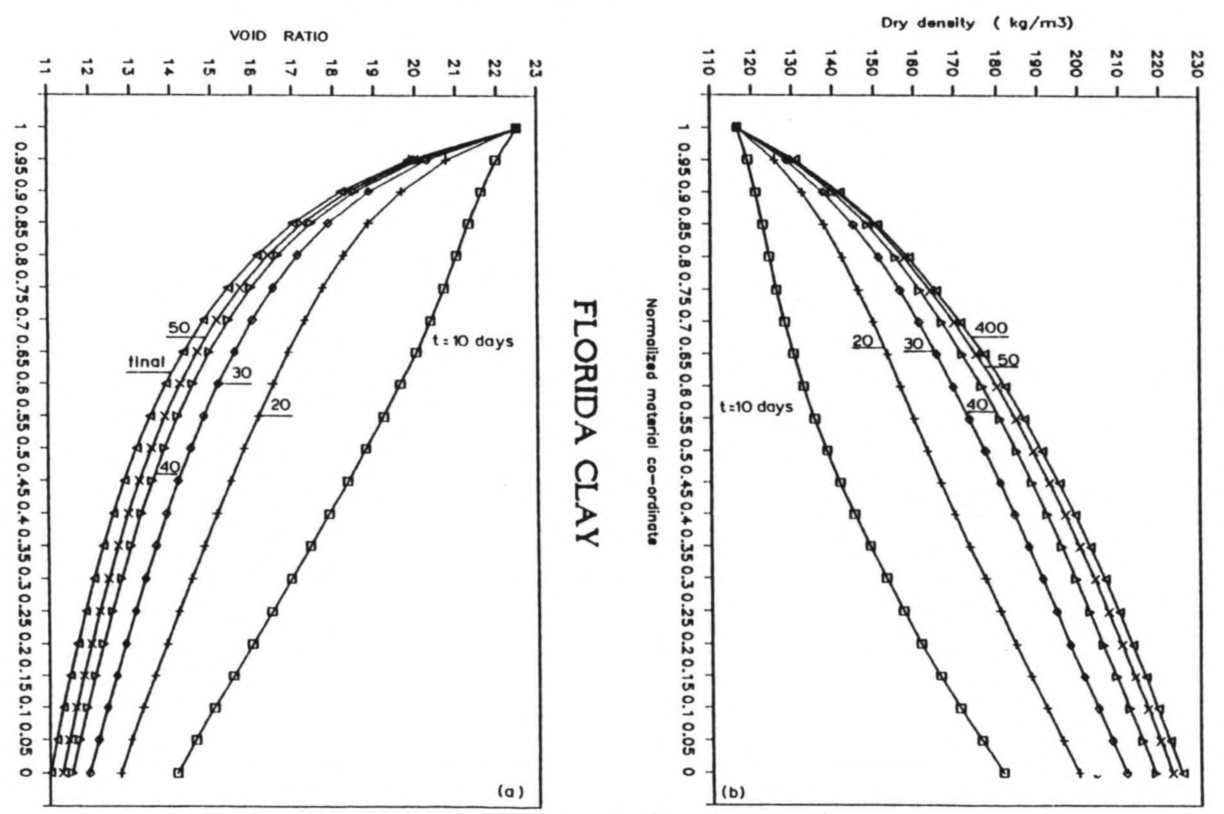


Fig. 5-6 Consolidation processes (Zo=0.076 m)
(a) Void ratio profiles
(b) Dry-density profiles

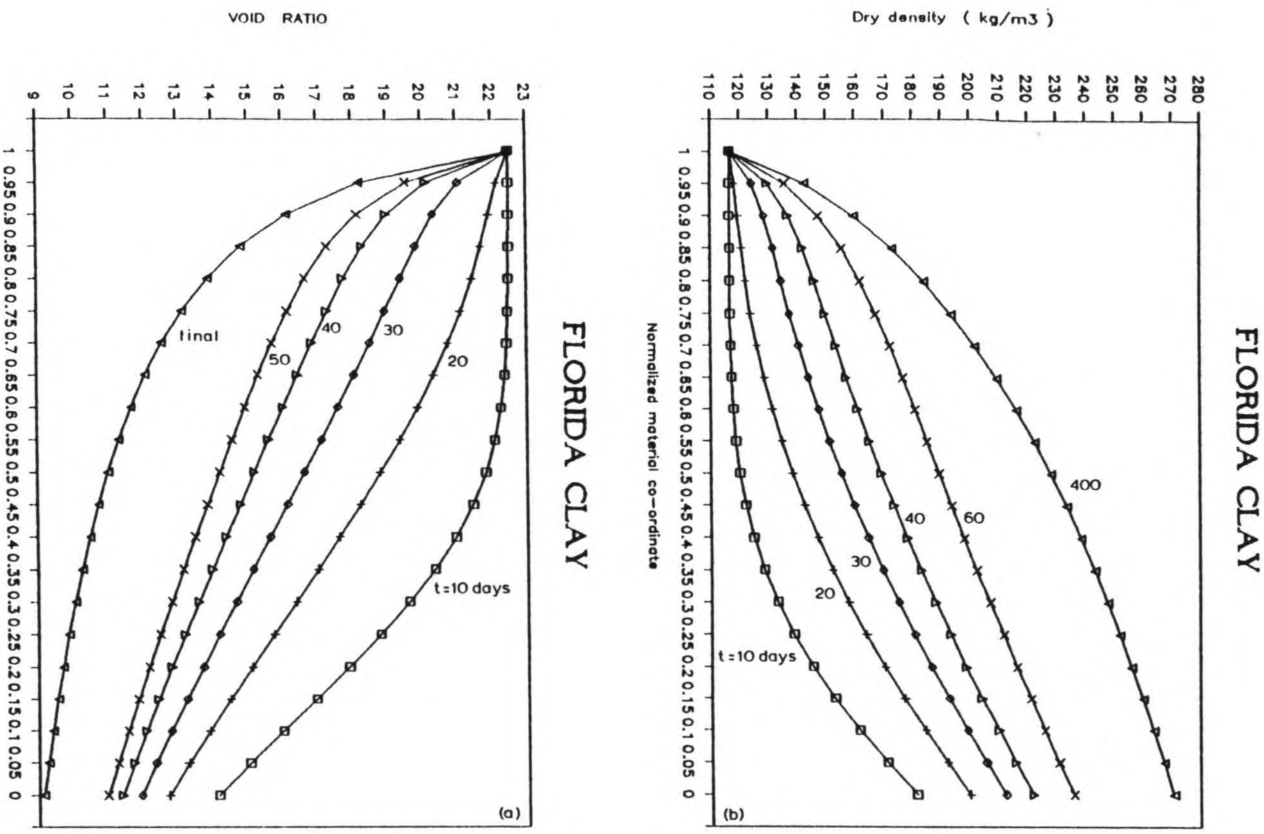


Fig. 5-7 Consolidation processes (Zo=0.15 m)
(a) Void ratio profiles
(b) Dry-density profiles

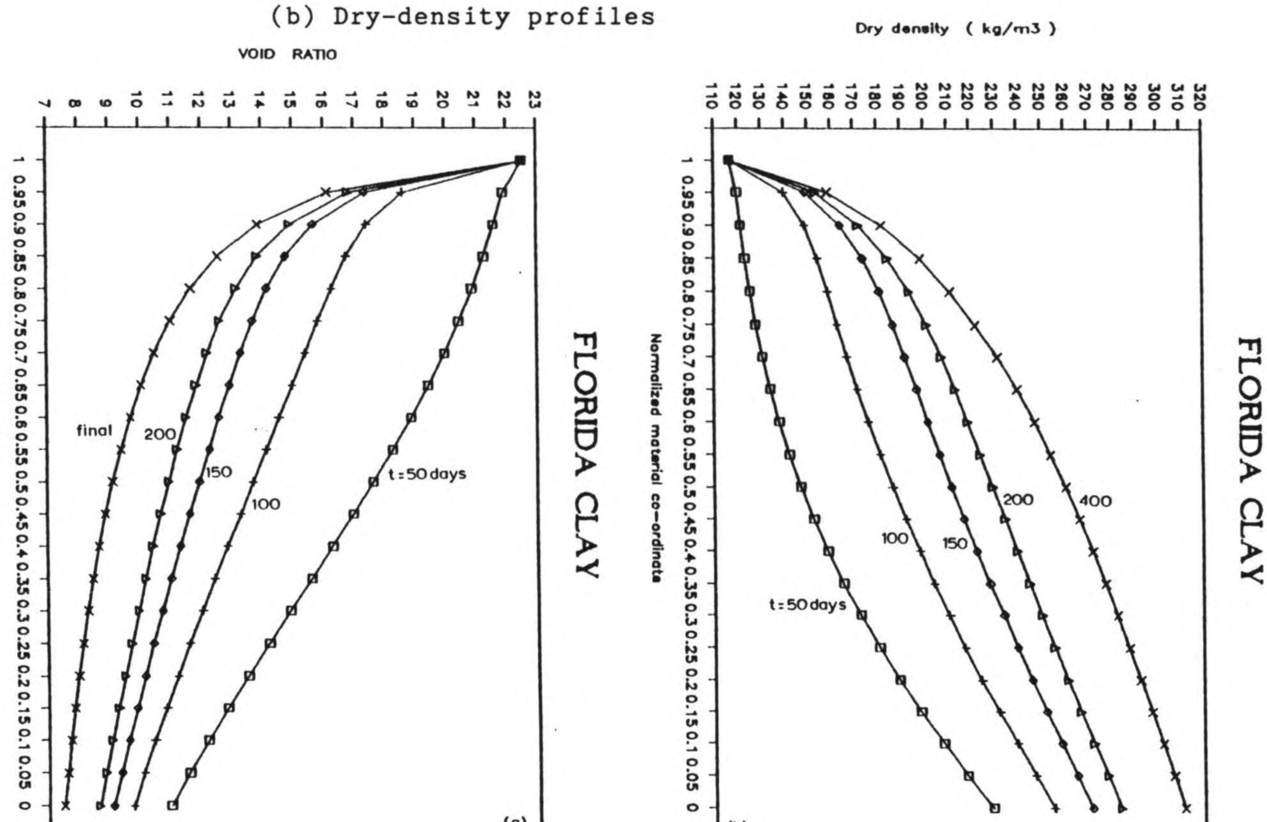


Fig. 5-8 Consolidation processes (Zo=0.30 m)
(a) Void ratio profiles
(b) Dry-density profiles

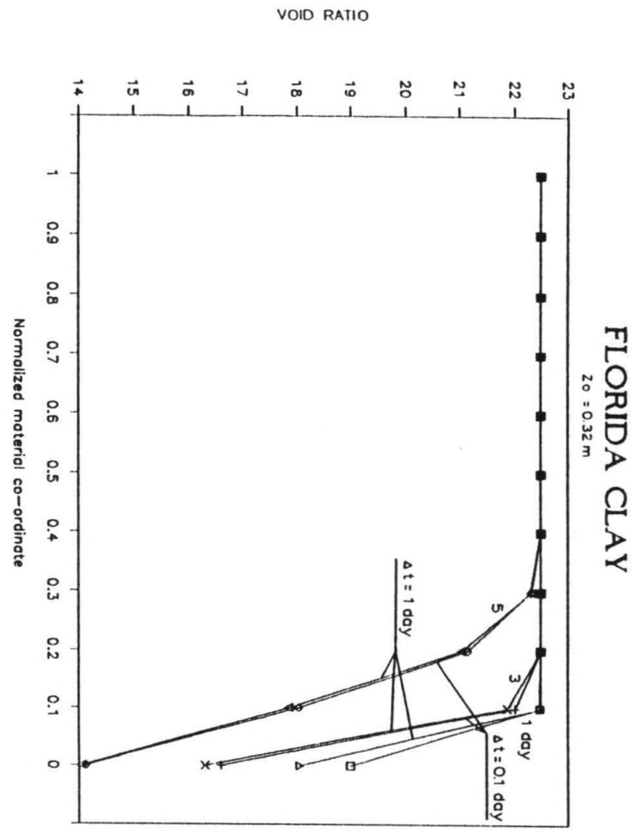


Fig. 5-9 Comparison between the results computed by using different time steps (Zo=0.32 m)

6. Discussions on Results

6.1 General

In Section 6-2, the analytical final profile of void ratio for the non-linear case is carried out. Subsequently, the final thickness of deposits and β (the final gradient of void ratio on the impervious bottom, introduced in Chapter 2,3 and 4) are formulated. In this section the comparison between analytical and non-linearly computational final profile of void ratio, is made as well. In Section 6-3, the results of analytical solution of the linearized equation and the non-linear mathematical modelling are compared.

6.2 Final profile of void ratio

i) The analytical final profile

After sedimentation, self-weight consolidation will become complete and the void ratio reaches to the final profile. Since the physical meanings of eq. (2-15) are not so precise, we start with the following original equations to obtain this final profile.

$$\frac{\delta\sigma}{\delta Z} + \frac{Vr.(\rho_s + \rho).g}{1 + Vr} \cdot \frac{\delta S}{\delta Z} = 0 \qquad \text{(2-3)bis}$$

$$\frac{\delta S}{\delta Z} - (1 + Vr) = 0 \qquad \text{(2-4)bis}$$

and

$$\left[\frac{Vr.(Vw - Vs)}{k.\ (1 + Vr)} + 1\right] \cdot \frac{\delta S}{\delta Z} + \frac{1}{\rho.g} \cdot \frac{\delta p}{\delta Z} = 0 \qquad \text{(2-9)bis}$$

For the FINAL case, there is no solid particles movement in the layer, such that

$$Vw - Vs = 0 \qquad \text{(6-1)}$$

Therefore, from equation (2-4)bis equation (2-9)bis becomes

$$\frac{\delta p}{\delta Z} = - \rho.g.(1 + Vr) \qquad \text{(6-2)}$$

Substracting eq. (2-3)bis by eq. (6-2) yields

$$\frac{\delta(\sigma-p)}{\delta Z} = - Vr.\rho.g - \rho_s.g + \rho.g.\ (1 + Vr)$$

or,

$$\frac{\delta\sigma'}{\delta Z} = - (\rho_s - \rho).g \qquad (6\text{-}3)$$

Equation (6-3) implies that when consolidation is completed, the excess porewater pressure Pe has a uniform distribution over the deposits' thickness. That is

$$\frac{\delta Pe}{\delta Z} = 0$$

From eq(6-3), we have

$$\sigma' = - (\rho_s - \rho).g.Z + C \qquad (6\text{-}4)$$

where, C-- integral constant.

At surface,

$$Vr(Zo) = Vrini$$

which from eq(5-2) leads to

$$\sigma'(Zo) = (Ap/Vrini)^{1/Bp}.\sigma'_c\ ,$$

Therefore,

$$C = (\rho_s - \rho).g.Zo + (Ap/Vrini)^{1/Bp}.\sigma'_c \qquad (6\text{-}5)$$

Substituting eq. (6-5) into eq. (6-4), from eq. (5-2) we have the following expression of void ratio,

$$Vr = \frac{Ap}{\left[\frac{(\rho_s - \rho).g.(Zo-Z)}{\sigma'_c} + (Ap/Vrini)^{1/Bp}\right]^{Bp}} \qquad (6\text{-}6)$$

The final thickness of the deposits is then given by

$$H(\infty) = \int_{o}^{Zo} (1+Vr).dZ =$$

$$= Zo + \frac{\sigma'_c}{(\rho_s - \rho).g} . \frac{Bp}{1-Bp} \left[\left[\frac{\rho_s - \rho}{\sigma'_c} .g.Zo + (\frac{Ap}{Vrini})^{1/Bp} \right]^{1-Bp} - \right.$$

$$\left. - (\frac{Ap}{Vrini})^{(1-Bp)/Bp} \right] \qquad (6-7)$$

ii) <u>The final gradient of Vr-profile</u>

$$\frac{dVr}{dZ} = \frac{Ap.Bp.(\rho_s - \rho).g/ \sigma'_c}{[(\rho_s - \rho).g.(Zo-Z)/\sigma'_c + (Ap/Vrini)^{1/Bp}]^{1+Bp}}$$

So that β, the gradient of Vr-profile on the impervious bottom is then written as

$$\beta = \left. \frac{dVr}{dZ} \right|_{Z=0} = \frac{Ap.Bp.(\rho_s - \rho).g/ \sigma'_c}{[(\rho_s - \rho).g.Zo/\sigma'_c + (Ap/Vrini)^{1/Bp}]^{1+Bp}}$$

that shows β is inversus to Zo.

iii) <u>Comparison between the analyzed and computed final Vr-profiles</u>

The data of Florida clay is used to make this comparison. Fig.(6-1) shows the good agreement between the computational results (of nonlinear model) and the analytical results (calculated from eq(6-6)).

6.3 **The validation of the analytical solution**

In order to make comparisons between the analytical solution of linearized eq. (3-3) and non-linear computational solution, the following parameters are determined beforehand by using average void ratio $\bar{V}r$; they are listed in Table (6-1).

Zo (m)	$\bar{V}r$	Vr∞	β (1/m)	Co (m/s)	Cv (m^2/s)	Zd
0.031	18.3	13.87	107.06	$3.394*10^{-8}$	$5.317*10^{-10}$	1.98
0.076	16.7	11.01	38.34	$2.788*10^{-8}$	$5.983*10^{-10}$	3.54
0.153	15.0	9.12	16.56	$2.213*10^{-8}$	$6.865*10^{-10}$	4.93

Table 6-1. The parameters for the Analytical solution

Fig.(6-2a) shows good agreement between the two results throughout the consolidation process for Zo=0.031 meter (Fig.(6-2b) shows that at t=10 days, consolidation is nearly completed), while Fig.(6-3) and Fig.(6-4) show that the analytical solution of the linearized equation (2-20) can represent the consolidation process only for a small t for the case of Zo=0.076 meter.

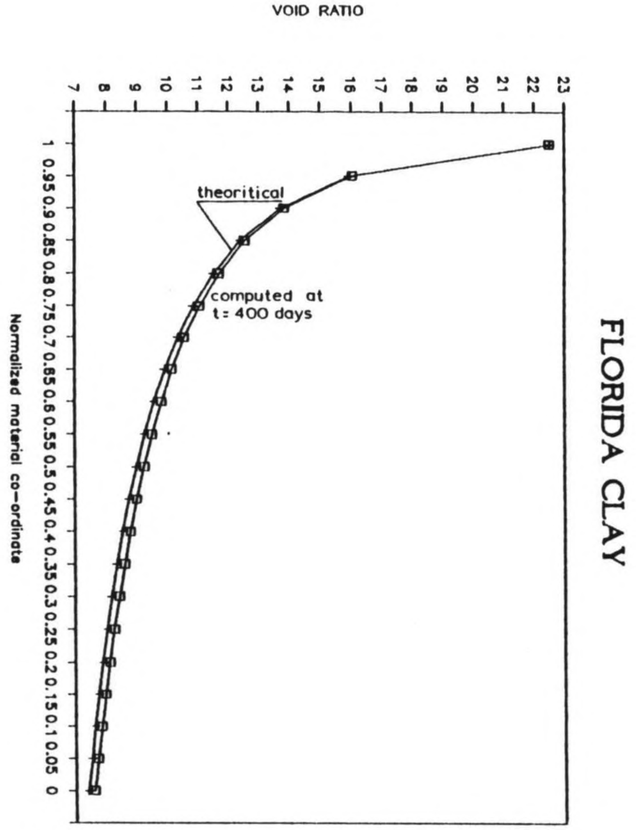


Fig. 6-1 The final Vr-profile (Zo= 0.32 m)

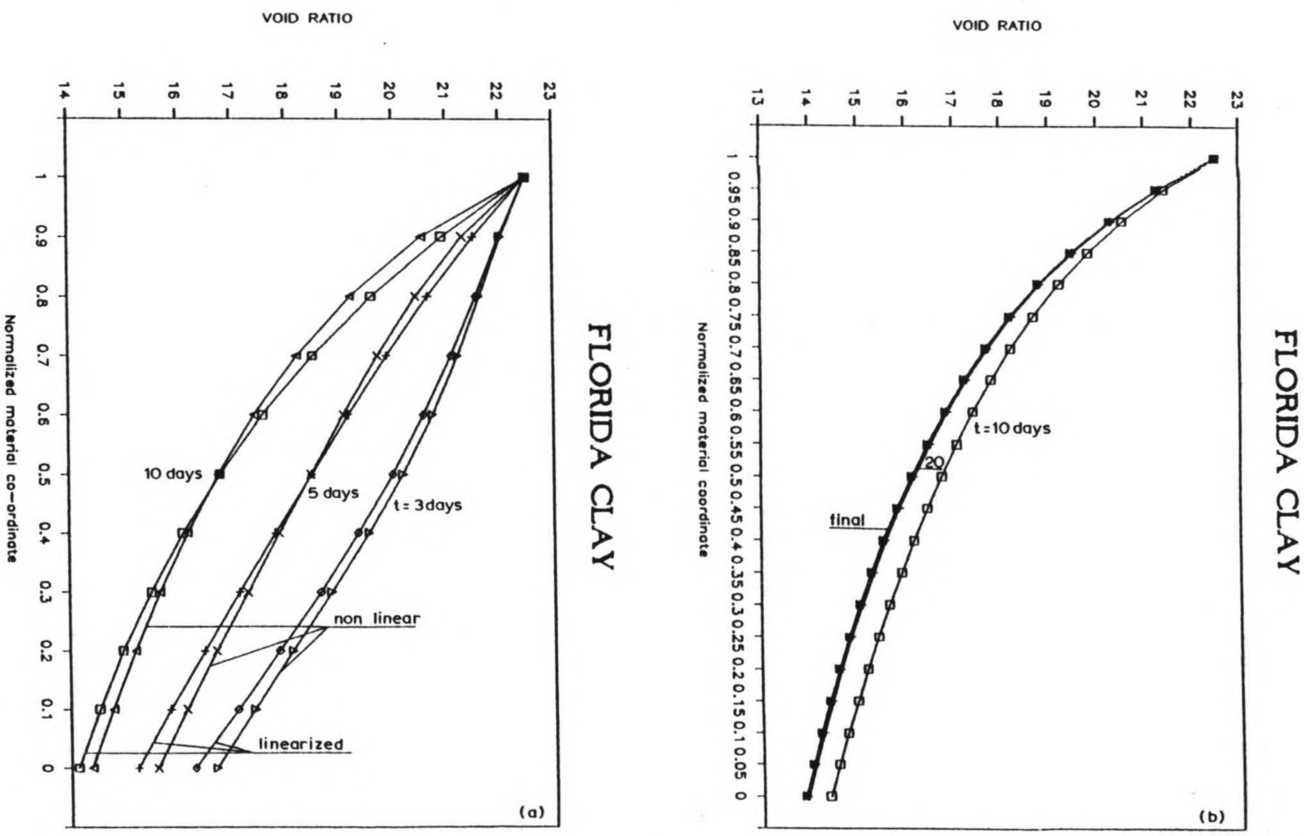


Fig. 6-2 Comparison between the analytical solution and numerical result (Zo=0.03 m)

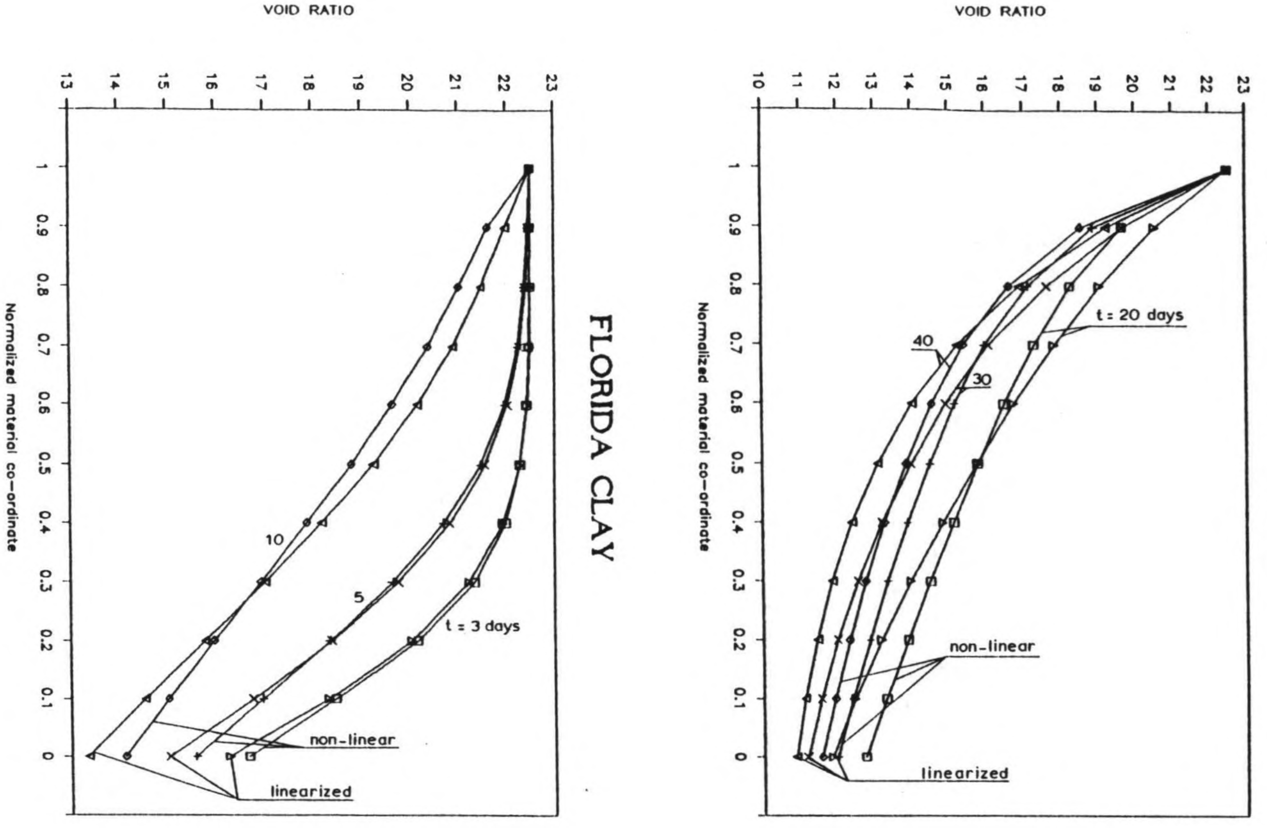


Fig. 6-3 Comparison between the analytical solution and numerical result (Zo=0.076 m)

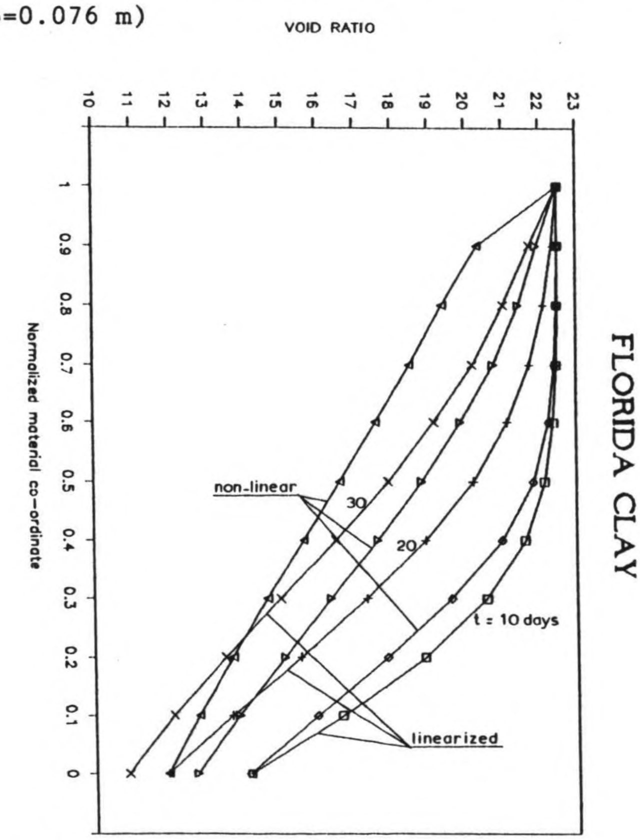


Fig. 6-4 Comparison between the analytical solution and numerical results (Zo=0.15 m)

7. **Conclusions and Recommendations**

i) Conclusions

a) The analytical solution of linearized eq(2-20) for self-weight consolidation is carried out. The solution shows that the self-weight consolidation is mainly dominated by the dimensionless thickness Zd. The larger Zd is, the faster consolidation progresses.

b) A non-linear mathematical modelling for self-weight consolidation is built up, which can simulate the self-weight consolidation of clay with high initial void ratio both for the case of constant Zo and for the case of Zo increasing with time. The latter case is common in reservoirs.

c) The analytical final Vr-profile is carried out. Subsequently, the final thickness of a certain amount of deposits and the final gradient of Vr on the impervious bottom, β, are formulated. The final thickness formula could be applicable in evaluating the lifetime of the mining-waste fills.

d) Comparison between non-linearly computational and analytical final Vr-profiles shows that the mathematical modelling is verified for the given data of Florida clay.

e) For Florida clay, the analytical solution can represent the self-weight consolidation process only for small Zd. When Zd is large, it is valid only for small t. This implies that the mathematical modelling is the effective tool to predict the self-weight consolidation. Besides, the analytical solutions are only for constant Zo, while mathematical modelling can definitly simulate the self-weight consolidation with Zo(t) which is increasing with time.

ii) Recommendations

a) The constitutive relationships Vr--σ' and k--Vr are to be determined beforehand. But these empirical relationships are often not so accurately determined. The low accuracies directly influence the prediction.

b) Alternative erosion and deposition in reservoirs are also possible due to flushing and impounding operations. In this case, the self-weight consolidation becomes the "multi-layer self-weight consolidation" problem. Therefore, the mathematical modelling as presented in Chapter 5 has to be modified.

8. **Acknowledgement**

I wish to express my deep gratitude to Prof. dr. M. de Vries for his recommending me studing in IHE and concerning to this study.

I also wish to express my gratitude to:
DELFT HYDRAULICS for their financial support;
Mr. C. Flokstra for his great help in solving the analytical solution and in setting up the mathematical modelling;
Mr. J. Lindenberg for his suggestions in this study;
Other staff working at the River, Canal and Navigation Division of DELFT HYDRAULICS for their kindness.

9. **References**

ABBOTT, M. B. (1960)
"One-dimensional consolidation of multi-layered soils"
The Institution of Civil Engineers, Dec.

ABBOTT, M. B. & SHRIVASTAVA, S. P. (1967a)
"A note on one-dimensional consolidation of spatially nonhomogeneous soils"
Laboratorium Voor Grondmechanica, Apr.

ABBOTT, M. B. & SHRIVASTAVA, S. P. (1967b)
"A note on one-dimensional consolidation of a clay layer that increases in thickness with time"
Laboratorium Voor Grondmechanica, Apr.

ARIATHURAI, R. & ARULANUNDAN, K. (1978)
"Erosion ratess of cohesive soils"
J. Hydraulics Division, ASCE. Hy2.

ARIATHURAI, R. & KRONE, R.B. (1976)
"Finite element method for cohesive sediment transport"
J. Hydraulics Division, ASCE. Hy3.

ARIATHURAI, R. , MACARTHUR, R. & KRONE, R. B. (1977)
"Mathematical model of estuarial sediment transport"
Dept Civil Eng. Un of Davis, CA, USA Oct.

BOUCHARD J-P et al (1989)
"Numerical simulation of mud erosion in reservoirs by floods application to reservoirs of the durance river"
Proc. 23rd IAHR, Ottawa, Canada.

BROMWELL, L. G. (1984)
"Consolidation of mining wastes"
Sedimentation/Consolidation Models, Prediction and Validation
Proc. symp. ASCE, Oct.

BRUK, S. (Rapporteur) (1985)
"Methods of computing sedimentation in lakes and research"
UNESCO, IHP-PROJECT IIA.2.6, Paris, Feb.

<u>**References**</u> (continued)

BRUK, S. CAVOR, R. & SIMONOVIC, S. (1983)
"Analysis of storage recovery by sediment flushing"
Proc. The Second Inter. Symp. on River Sedimentation, Nanjing, China, Oct.

CAO, S. Y. (1983)
"A study of the retrogressive erosion on deposited fine sediment"
Collected Research papers,Institute of Water Conservancy and Hydropower Research, Beiling, China.

CAO, S. & DU, G. (1986)
"Experiments on erosion and deposition of cohesive sediments"
J. Sediment Research (in Chinese) No.4.

CAVOR, R. & SLAVIC, M. (1983)
"Mathematical model of reservoir flushing"
Inter. Congress on Numerical Methods in Laminar and Turbulent Flow, University of Washington,Seattle, USA.

GIBSON, R. E., ENGLAND, G. L. & HUSSEY, M. J. L. (1967)
"The theory of one-dimensional consolidation of saturated clays.
I. finite non-linear consolidation of thin homogeneous layers"
Geotechnique, Vol.17.

GIBSON, R. E., SCHIFFMAN, R. L. & CARGILL, K. W. (1981)
"The theory of one-dimensional consolidation of saturated clays.
II. finite nonlinear consolidation of thick homogeneous layers"
Canadian Geotechanical Journal, Vol.18.

HAYTER, E. J. (1983)
"Prediction of cohesive sediment movement in estuarial waters"
A Dissertation Presented to the graduate Council of the Un. of Florida in Partial Fulfillment of the Reqirements for the Degree of Ph.D. Un. of Florida, USA.

References (continued)

HAYTER, E. J. (1984)
Chap 5: "estuarial sediment bed model"
Estuarine Cohesive Sediment Dynamics---Lecture Notes on Coastal and Estuarine Studies, edited by Mehta,A.J.
Springer-Verlag.

HAYTER, E. J. & MEHTA, A. J. (1982)
"Modeling of estuarial fine sediment transport for tracking pollution movement "
University of Florida, Gaineville,Florida,USA, Dec.

HAYTER, E. J. (1986)
"Mathematical modelling of cohesive sediment transport"
Proc. 3rd Inter. Symp. on River Sedimentation,
The University of Mississippi.

KELLY, W. E. & GULARTE, R. C. (1981)
"Erosion resistance of cohesive soils"
J. Hydraulic Division ASCE, Oct.

KOPPULA, S. D. & MORGENSTERN, N. R. (1982)
"On the consolidation of sedimenting clays"
Canadian Geotechanical Journal, Vol.19.

KRIZEK, R. J. & SOMOGYI, F. (1984)
"Perspectives on modelling consolidation models"
Sedimentation/consolidation Models, Prediction and Validation
Proc.Symp.ASCE,Oct.

KUIJPER, C. CORNELISSE, J.M. & WINTERWERP, J.C. (1989)
"Research on erosive properties of cohesive sediments"
J. Geophysical Research, Vol.94, No.c10, Oct.

LEE, K. & SILLS, G. C. (1981)
"The consolidation of a soil stratum, including self-weight effects and large strains"
Inter. Journal for Numerical and Analysis Methods in
Geomechanics, Vol.5.

References (continued)

LEEUW, E. H. DE & ABBOTT, M. B. (1966)
"On the non-linear consolidation of multi-layered soils"
Laboratorium Voor Grondamechanica, Jan.

LIN, T. W. & LOHNES, R. A. (1984)
"Sedimentation and self weight consolidation of dredge spoil"
Sedimentation/consolidation Models, Prediction and Validation
Proc. Symp. ASCE, Oct.

MEHTA, A. J. & PARTHENIADES, E. (1979)
"Koalinite resuspension properties"
J. Hydraulics Division ASCE, Hy4.

MEHTA, A. J., HAYTER, E. J. et al (1989)
"Cohesive sediment transport I: process description"
J. Hydraulic Eng. ASCE, No.8.

MEHTA, A. J., MCANALLY,W. H. Jr et al (1989)
"Cohesive sediment transport II: application"
J. Hydraulic Eng. ASCE, No. 8.

MEHTA, A. J. (1989)
"On estuarine cohesive sediment suspension behavior"
J. Geophysical Research, Vol.94, No.C10, Oct.

NICHOLSON, J. & O'CONNER,B.A. (1986)
"Cohesive sediment transport model"
J. Hydraulic Eng. ASCE, No.7.

OTSUBO, K. & MURAOKA, K. (1988)
"Critical shear stress of cohesive bottom sediments"
J. Hydraulic Eng. ASCE, No.10.

OWEN, M. W. (1970)
"Properties of a consolidated mud"
Report of Hydraulics Research Station, England. No.INT83, Dec.

References (continued)

PARCHURE, T.M. & MEHTA, A.J. (1985)
"Erosion of soft cohesive sediment deposits"
J. Hydraulic Eng., ASCE, No.10.

PARTHENIADES, E. (1965)
"Erosion and deposition of cohesive soils"
J. Hydraulics Division,ASCE, Hy1.

PARTHENIADES, E. (1972)
Chap.20, "Results of recent investigations on erosion and deposition of cohesive sediments "
Sedimentation, Symp. to Honor Professor H.A.Einstein,Fort Collins, Colorado, USA.

SCHIFFMAN, R. L. & GIBSON, R. E. (1964)
"Consolidation of nonhomogeneous clay layers"
J. Soil Mechanics and Foundations Division, ASCE, Sep.

SCHIFFMAN, R. L. & PANE, V. & GIBSON, R. E. (1984)
"The theory of one-dimensional consolidation of saturated clays"
Sedimentation/consolidation Models, Prediction and Validation
Proc. symp. ASCE, Oct.

SCOTT, R. F. (1963)
"Principles of soil mechanics"
Addison-wesley Publishing Company,Inc.

SCULLY, R. W., SCHIFFMAN, R. L. et al (1984)
"Validation of consolidation properties of phosphatic clay at very high void ratios"
Sedimentation/consolidation Models, Prediction and Validation
Proc. Symp. ASCE, Oct.

THORN, M. F. C. & PARSON, J. G.(1980)
"Erosion of cohesive sediments in estuaries"
Proc. 3rd Inter. Symp. of Dredging Technology. Bordeaux, Mar.

References (continued)

WANG, M. S. & WANG, S.C. et al (1983)
"The way of operation and sedimentation of the Gongzui Reservoir"
J. Sediment Research (in Chinese), No.1.

YONG, R. N. & ELMONAYERI, D. S. (1984)
"Convection-diffusion analysis of sedimentation in initially dilute solids-suspensions"
Sedimentation/Consolidation Models, Prediction and Validation
Proc. Symp. ASCE, Oct.

ZHANG, R. & NING,CHIEN (1985)
"Reservoir Sedimentation"
Lecturenotes of the Training Course on Reservoir Sedimentation,
IRTCES, Beijing, China, Nov.

ZHANG, Z.Q. & DU, G.H (1984)
"The rational operation of drawdown flushing in the Shuicaozi Reservior"
J. Sediment Research (in Chinese), No.4.

ZNIDARCIC, D., SCHIFFMAN,R. L. et al (1986)
"The theory of one-dimensional consolidation of saturated clays:
Part V, constant rate of deformation testing and analysis"

Appendix I. <u>The Inverse Laplace's Transform of Eq(3-19)</u>

I-1. General

The theories of complex variables are applied in these derivations of the inverse Laplace's transform. Equation (3-19) has only singularities of simple poles. Therefore Mittag-Leffler's expansion theorem is applied. The following represents the prepared knowledge on complex variables and the details of the derivations

I-2. Prepared knowledge on complex variables

i) <u>Eulerian formulas</u>

$$\exp(ia) = \cos(a) + i\sin(a), \qquad \text{(I-1)}$$

$$\exp(-ia) = \cos(a) - i\sin(a), \qquad \text{(I-2)}$$

where, i--imaginary unit. From (I-1) and (I-2) we have

$$\cos(a) = (\exp(ia) + \exp(-ia))/2, \qquad \text{(I-3)}$$

$$\sin(a) = (\exp(ia) - \exp(-ia))/2i, \qquad \text{(I-4)}$$

and

$$\sinh(ia) = (\exp(ia) - \exp(-ia))/2 = i\sin(a) \qquad \text{(I-5)}$$

$$\cosh(ia) = (\exp(ia) + \exp(-ia))/2 = \cos(a) \qquad \text{(I-6)}$$

ii) <u>Determination of the residue of $f(a)$ at simple pole b</u>

Suppose the singularity of $f(a)$ in a finite complex plane a is simple pole of b, then the residue of $f(a)$ at b is written as

$$\mathrm{Re}(b) = \lim_{a \to b} (a-b).f(a) \qquad \text{(I-7)}$$

iii) Mittag-Leffler's expansion theorem

a) Suppose that the only singularities of $f(a')$ in the finite complex a' plane are the simple poles $b_1, b_2, b_3, \ldots, \ldots$ arranged in the order of increasing absolute value.

b) Let the residues of $f(a')$ at $b_1, b_2, \ldots\ldots$ be $Re(b_1), Re(b_2), \ldots, \ldots$

c) Let C_N be circle of radius R_N which do not pass any poles and upon which $|f(a')| < M$, where M is independent of N and $R \to \infty$ as $N \to \infty$.

Then Mittag-Leffler's expansion theorem states that

$$f(a') = f(0) + \sum_{j=1}^{\infty} Re(b_j).\{ 1/(a'-b_j) + 1/b_j \} \qquad (I-8)$$

I-2. The inverse Laplace's transform of eq(3-19)

Eq(3-19) reads

$$V = \frac{\exp(r_1.Z + r_2.Zo)-\exp(r_2.Z + r_1.Zo)}{(r_1 - \frac{Co}{Cv}).\exp(r_2.Zo) - (r_2 - \frac{Co}{Cv}).\exp(r_1.Zo)} *$$

$$* \frac{Co}{Cv} . \frac{Vrini - Vr\infty + \beta.Co/Cv}{p} + \frac{Vrini}{p} \qquad (3-19)$$

Let

$$factor(1) = \frac{\exp(r_1.Z + r_2.Zo)-\exp(r_2.Z + r_1.Zo)}{(r_1 - \frac{Co}{Cv}).\exp(r_2.Zo) - (r_2 - \frac{Co}{Cv}).\exp(r_1.Zo)} \qquad (2-9)$$

Let $a = \sqrt{4CvP + Co^2}$, $a' = a/Co$, $\alpha = \frac{Co.Zo}{2.Cv}$ and $\alpha' = \frac{Co(Zo-Z)}{2.Cv}$

then

$$r_1 = \frac{Co + a}{2.Cv} \text{ and } r_2 = \frac{Co-a}{2.Cv} \quad .$$

Therefore,

$$factor(1) = \frac{2.Cv}{Co}.\ \exp(\frac{Co.Z}{2.Cv}).\ \frac{\exp(-a'\alpha')-\exp(a'\alpha')}{(a'-1).\exp(-a'\alpha)+ (a'+1).\exp(a'\alpha)} \tag{2-10}$$

Let

$$f(a') = \frac{\exp(-a'\alpha') - \exp(a'\alpha')}{(a'-1).\exp(-a'\alpha) +(a'+1).\exp(a'\alpha)}$$

Then

$$factor(1) = \frac{2.Cv}{Co} .\ \exp(\frac{Co.Z}{2.Cv}).f(a') \tag{I-11}$$

If $f(a')$ is transformed to complex plane, then $f(a')$ has simple poles ib at the complex plane, i.e.

$$[(a'-1).\exp(-a'\alpha) + (a'+1).\exp(a'\alpha)]\Big|_{a'=ib} =0$$

or,

$$ib.\exp(-ib\alpha) + ib.\exp(ib\alpha) - \exp(-ib\alpha) + \exp(ib\alpha) =0 \quad .$$

From eqs(I-3) and (I-4), above equation becomes

$$ib\cos(b\alpha) + i\sin(b\alpha) = 0$$

so that

$$tg(b\alpha) + b = 0 \tag{I-12}$$

Eq(I-12) determines b_j for the simple poles $\pm ib_j$ $(j=1,2,\cdot...,...)$ (see Fig. 3-1).

From (I-7), the residue of $f(a')$ at ib_j is as follows

$$Re(ib_j) = \frac{\exp(-ib_j\alpha') - \exp(ib_j\alpha')}{\alpha.[(-ib_j+1)\exp(-ib_j\alpha)+(1+ib_j)\exp(ib_j\alpha)] + \exp(-ib_j\alpha)+\exp(ib_j\alpha)}$$

By applying eqs(I-3)--(I-6), we have

$$Re(ib_j) = \frac{-2i\sin(b_j\alpha')}{\alpha.[-2b_j\sin(b_j\alpha) + 2\cos(b_j\alpha)] + 2\cos(b_j\alpha)} \qquad (I-13)$$

Similarly,

$$Re(-ib_j) = \frac{2i\sin(b_j\alpha')}{\alpha.[-2b_j\sin(b_j\alpha) + 2\cos(b_j\alpha)] + 2\cos(b_j\alpha)}$$

or,

$$Re(-ib_j) = - Re(ib_j) \qquad (I-14)$$

and

$$b_j > 0$$

Moreover,

$$f(0) = \lim_{a'\to o} \frac{\exp(-a'\alpha')-\exp(a'\alpha')}{(a'-1)\exp(-a'\alpha) +(1+a')\exp(a'\alpha)} =$$

$$= \lim_{a'\to o} \frac{-\alpha'\exp(-a'\alpha') - \alpha'\exp(a'\alpha')}{\exp(-a'\alpha) -\alpha(a'-1)\exp(-a'\alpha)+\alpha(1+a')\exp(a'\alpha) +\exp(a'\alpha)} =$$

$$= \frac{-2\alpha'}{2+2\alpha} = - \frac{\alpha'}{1+\alpha} \qquad (I-15)$$

So that $f(a')$ can now be written in Mittag-Leffler's expansion,

$$f(a') = f(0) + \sum_{j=1}^{\infty} Re(ib_j).[1/(a'-ib_j) + 1/b_j] +$$

$$+ \sum_{j=1}^{\infty} Re(-ib_j).[1/(a'+ib_j) -1/ib_j]$$

Substituting eqs(I—13)—(I—15) into the above equation yields

$$f(a') = f(0) + \sum_{j=1}^{\infty}\left[\frac{\mathrm{Re}(ib_j)}{a'-ib_j} - \frac{\mathrm{Re}(ib_j)}{a'+ib_j} + \frac{2\mathrm{Re}(ib_j)}{ib_j}\right] =$$

$$= -\frac{\alpha'}{1+\alpha} + \sum_{j=1}^{\infty}\frac{2\sin(b_j\alpha')}{\alpha.[-2b_j\sin(b_j\alpha) + 2\cos(b_j\alpha)] + 2\cos(b_j\alpha)} *$$

$$* \left[\frac{2b_j}{(a')^2+b_j^2} - \frac{2}{b_j}\right] \qquad \text{(I—16)}$$

So that,

$$\text{factor}(1) = \frac{2Cv}{Co}.\exp\left(\frac{Co.Z}{2Cv}\right).f(a') =$$

$$= \frac{2Cv}{Co}.\exp\left(\frac{Co.Z}{2Cv}\right).\ \Big\{-\frac{\alpha'}{1+\alpha} +$$

$$+ \sum_{j=1}^{\infty}\frac{2\sin(b_j\alpha')}{\alpha.[-b_j\sin(b_j\alpha) + \cos(b_j\alpha)] + \cos(b_j\alpha)} * \left[\frac{b_j}{(a')^2+b_j^2} - \frac{1}{b_j}\right]\Big\}$$

$$\text{(I—17)}$$

Therefore, eq(3—19) is rewritten as

$$V = \frac{Vrini}{p} + \frac{Co}{Cv}\ .\ (Vrini - Vr\infty + Cv.\beta/Co)\ .\ \frac{1}{p}\ .\ \text{factor}(1) =$$

$$= \frac{Vrini}{p} + \frac{Co}{Cv}\ .\ (Vrini - Vr\infty + Cv.\beta/Co)\ .\ \frac{2Cv}{Co}\ .\ \exp\left(\frac{Co.Z}{2Cv}\right) *$$

$$* \Big\{-\frac{\alpha'}{p(1+\alpha)} + \sum_{j=1}^{\infty}\frac{2\sin(b_j\alpha')}{\alpha.[-b_j\sin(b_j\alpha) + \cos(b_j\alpha)] + \cos(b_j\alpha)} *$$

$$* \left[\frac{b_j}{p((a')^2+b_j^2)} - \frac{1}{p.b_j}\right]\Big\} \qquad \text{(I—18)}$$

Substituting $a' = \dfrac{\sqrt{4Cvp + Co^2}}{Co}$ into the above equation yields

$$V = \frac{Vrini}{p} + \frac{Co}{Cv}.(Vrini - Vr\infty + Cv.\beta/Co).\frac{1}{p}.factor(1)$$

or,

$$V = \frac{Vrini}{p} + \frac{Co}{Cv}.(Vrini - Vr\infty + Cv.\beta/Co).\frac{2Cv}{Co}.\exp(\frac{Co.Z}{2Cv}) *$$

$$* \{- \frac{\alpha'}{p(1+\alpha)} + \sum_{j=1}^{\infty} \frac{2\sin(b_j\alpha')}{\alpha.[-b_j\sin(b_j\alpha) + \cos(b_j\alpha)] + \cos(b_j\alpha)} *$$

$$* [\frac{- \frac{b_j}{1+b_j^2}}{p + \frac{Co^2(1+b_j^2)}{4Cv}} - \frac{1}{b_j(1+b_j^2)p}]\} \qquad \text{(I-19)}$$

From eq(I–19), we finally obtain the inverse Laplace's transform of eq(3–19) as follows

$$Vr = Vrini - 2.(Vrini - Vr\infty + \beta.Zo/Zd).\exp(\eta.Zd/2) *$$

$$* \{\sum_{j=1}^{\infty} \frac{2\sin[b_j(1-\eta)Zd/2].[b_j\exp(-Zd^2(1+b_j^2)*T/4) + 1/b_j]}{Zd.[-b_j\sin(b_jZd/2)+\cos(b_jZd/2)]/2 + \cos(b_jZd/2)}.\frac{1}{1+b_j^2} +$$

$$+ \frac{Zd.(1-\eta)}{2+Zd}\} \qquad \text{(3–20)}$$

where, $Zd = Co.Zo/Cv$, $T = Cv.t/Zo^2$ and $\eta = Zo/Z$.

Appendix II. Stability Analysis

The stability condition for the numerical scheme presented in Chapter 4 is derived in the following.

For the eq(4–1)

$$\frac{\delta Vr}{\delta T} + Zd \, . \, \frac{\delta Vr}{\delta \eta} - \frac{\delta^2 Vr}{\delta \eta^2} = 0 \qquad (4\text{–}1)$$

the finite difference equation is written as

$$\frac{Vr_j^{n+1} - Vr_j^n}{\Delta T} + \theta . Zd . \frac{Vr_j^{n+1} - Vr_j^{n+1}}{2.\Delta\eta} + (1-\theta).Zd.\frac{Vr_j^n - Vr_j^n}{2.\Delta\eta} -$$

$$- \theta . \frac{Vr_{j+1}^{n+1} - 2.Vr_j^{n+1} + Vr_{j-1}^{n+1}}{\Delta\eta^2} - (1-\theta).\frac{Vr_{j+1}^n - 2.Vr_j^n + Vr_{j-1}^n}{\Delta\eta^2} = 0 \qquad (II\text{–}1)$$

The solution is decomposed into Fourier series as follows

$$Vr_j^n = \Sigma \, A_k^n \, .\exp[i \frac{k.j.\Delta\eta}{(N-1).\Delta\eta} . \pi] \qquad (II\text{–}2)$$

$$Vr_j^{n+1} = \Sigma \, A_k^{n+1} .\exp[i \frac{k.j.\Delta\eta}{(N-1).\Delta\eta} . \pi] \qquad (II\text{–}3)$$

$$Vr_{j\pm1}^n = \Sigma \, A_k^n \, .\exp[i \frac{k.(j\pm1).\Delta\eta}{(N-1).\Delta\eta} . \pi] \qquad (II\text{–}4)$$

$$Vr_{j\pm1}^{n+1} = \Sigma \, A_k^{n+1} .\exp[i \frac{k.(j\pm1).\Delta\eta}{(N-1).\Delta\eta} . \pi] \qquad (II\text{–}5)$$

in which, j is the grid point (j=1, 2, ... , N) and A_k^n is the amplitude of the kth component of Fourier series at time level n .

Introducing $C = \frac{Zd.\Delta T}{\Delta\eta}$, $E = \frac{\Delta T}{\Delta\eta^2}$, and substituting eqs(II-2)— (II-5) into eq. (II-1), we have for the kth component

$$2.\ [A_k^{n+1}.\exp(i.\frac{k.j.\pi}{N-1}) - A_k^n.\exp(i.\frac{k.j.\pi}{N-1})] +$$

$$+\ \theta.C.\ [A_k^{n+1}.\exp(i.\frac{k(j+1).\pi}{N-1}) - A_k^{n+1}.\exp(i.\frac{k(j-1)\pi}{N-1})]\ +$$

$$+(1-\theta).C.\ [A_k^n.\exp(i.\frac{k(j+1).\pi}{N-1}) - A_k^n\ .\exp(i.\frac{k(j-1)\pi}{N-1})]\ -$$

$$-\ 2.\theta.E.\ [A_k^{n+1}.\exp(i.\frac{k(j+1)\pi}{N-1}) - 2.A_k^{n+1}.\exp(i\frac{kj\pi}{N-1}) + A_k^{n+1}.\exp(i\frac{k(j-1)\pi}{N-1})]-$$

$$-\ 2(1-\theta).E.\ [A_k^n.\exp(i.\frac{k(j+1)\pi}{N-1}) - 2.A_k^n.\exp(i\frac{kj\pi}{N-1}) + A_k^n.\exp(i\frac{k(j-1)\pi}{N-1})] = 0.$$

leading to:

$$(A_k^{n+1} - A_k^n) + i.\theta.C.A_k^{n+1}.\sin(\frac{k\pi}{N-1}) + i.(1-\theta).C.A_k^n.\sin\ (\frac{k\pi}{N-1}) +$$

$$+\ 2.\theta.E.A_k^{n+1} - 2.\theta.E.A_k^{n+1}.\cos(\frac{k\pi}{N-1}) + 2.(1-\theta).E.A_k^n\ -$$

$$-\ 2.(1-\theta).E.A_k^n.\cos(\frac{k\pi}{N-1}) = 0$$

Let

$$A = \frac{A_k^{n+1}}{A_k^n}$$

from the above equation we then have

$$A.\ \{1 + i.\theta.C.\sin(\frac{k\pi}{N-1}) + 2.\theta.E.\ [1 - \cos(\frac{k\pi}{N-1})]\} =$$

$$= 1 - i.(1-\theta).\sin(\frac{k\pi}{N-1}) - 2.(1-\theta).E.[1 - \cos(\frac{k\pi}{N-1})$$

$$A = \frac{A_k^{n+1}}{A_k^n} = \frac{1-2.(1-\theta).E.[1-\cos(\frac{k\pi}{N-1})] - i.(1-\theta).\sin(\frac{k\pi}{N-1})}{1+2.\theta.E.[1-\cos(\frac{k\pi}{N-1})] + i.\theta.C.\sin(\frac{k\pi}{N-1})} \qquad (4\text{-}12)$$

$$|A|^2 = \frac{\{\ 1-2.(1-\theta).E.[1-\cos(\frac{k\pi}{N-1})]\ \}^2 + \{\ (1-\theta).\sin(\frac{k\pi}{N-1})\ \}^2}{\{\ 1+2.\theta.E.[1-\cos(\frac{k\pi}{N-1})]\ \}^2 + \{\theta.C.\sin(\frac{k\pi}{N-1})\ \}^2}$$

Therefore, $|A|^2 \leq 1$ leads to

$$\theta \geq \frac{1}{2} - \frac{2.E.\{1-\cos(k\pi/N-1)]\}}{4.E^2.[1-\cos(k\pi/N-1)]^2 + C^2\ \{1-\cos^2(k\pi/N-1)\}}$$

or,

$$\theta \geq \frac{1}{2} - \frac{2.E}{4.E^2 + C^2 - (4.E^2 - C^2).\cos(k\pi/N-1)}$$

So that the stability condition for the numerical scheme is that

$$\theta \geq \frac{1}{2} - \min\{\frac{2.E}{4.E^2 + C^2 \pm |4.E^2 - C^2|}\} \qquad (4\text{-}13)$$

in which, again, $C = \frac{Zd.\Delta T}{\Delta\eta}$ and $E = \frac{\Delta T}{\Delta\eta^2}$

Appendix III. List Of Notation Symbols

A = amplification, $= \dfrac{A_k^{n+1}}{A_k^n}$

Ak, Ap = empirical constants for relationship Vr—k .

A_k^n = amplitude of the kth component of Fourier series at time level n .

A(j) = coefficients for "double sweep" algorithm.

a = Lagrange co-ordinate; or, complex variable.

a' = complex variable.

Bk,Bp = empirical constants for relationship σ'—Vr .

B(j) = coefficients for "double sweep" algorithm.

b = simple pole; or, for simple poles $\pm ib_j$.

C = integral constant; or, C=Zd.ΔT/Δη .

Co = constant, $Co = \dfrac{\rho_s - \rho}{\rho} \cdot \dfrac{d}{dVr} \left[\dfrac{k}{1+Vr}\right]$

C_1 , C_2 = integral constants for the analytical solution.

Cf = Terzaghi's constants, Cf=k/(Mv.g.ρ).

C_N = circle.

C(j) = coefficients for "double sweep" algorithm.

D(j) = Coefficients for "double sweep" algorithm.

E = constant, $= \Delta T/\Delta\eta^2$.

$$F(Vr) = \left(\frac{\rho_s}{\rho} - 1\right) \cdot \frac{d}{dVr} \left[\frac{k}{1+Vr}\right]$$

f(a), *f*(a') and f(Vr) = functions of a, a' and Vr respectively.

$$G(Vr) = - \frac{k}{\rho.g.(1+Vr)} \cdot \frac{d\sigma'}{dVr}$$

G1	=	the value of G(Vr) centred at (j −1/2, n +1/2).
G2	=	the value of G(Vr) centred at (j +1/2, n +1/2).
g	=	gravity acceleration.
H(∞)	=	final thickness of deposits.
i	=	imaginary unit.
j	=	grid point.
k	=	wave conponent number of Fourier series; or, permeability.
kc	=	reference permeability.
L(j)	=	coefficients for "double sweep" algorithm.
Md	=	deposition rate.
M(j)	=	coefficients for "double sweep" algorithm.
Mv	=	compressibility of soil skeleton.
N	=	the maximum grid point; or, series number of b_j.
p	=	Laplace's constant.
P	=	porewater pressure.
Pe	=	excess porewater pressure.
R_N	=	radius of circle C_N.
$r_{1,2}$	=	$\frac{Co \pm \sqrt{Co^2 + 4Cvp}}{2Cv}$
S	=	Eulerian co-ordinate.
So	=	deposit's thickness.
t	=	time.
T	=	dimensionless time.
ΔT	=	time-step (dimensionless).
Δt	=	time-step.
V	=	Laplace's transform of Vr .
Vr	=	Void ratio.
Vr'	=	Value of void ratio determined by iteration.
Vrini	=	initial void ration.
Vr∞	=	final void ratio on the impervious bottom.
Vr_j^n	=	void ratio of grid point j at time level n.
Vs	=	velocity of solid.
Vw	=	velocity of porewater.
Z	=	material co-ordinate.
Zo	=	material height of deposits.
Zd	=	dimensionless thickness of deposits, =Co.Zo/Cv .

α = $\dfrac{Co.Zo}{2.Cv}$

α' = $\dfrac{Co.(Zo-Z)}{2.Cv}$

β = final gradient of void ratio on the impervious bottom.

σ = total vertical stress.

σ' = effective vertical stress.

σ'_c = reference effective stress.

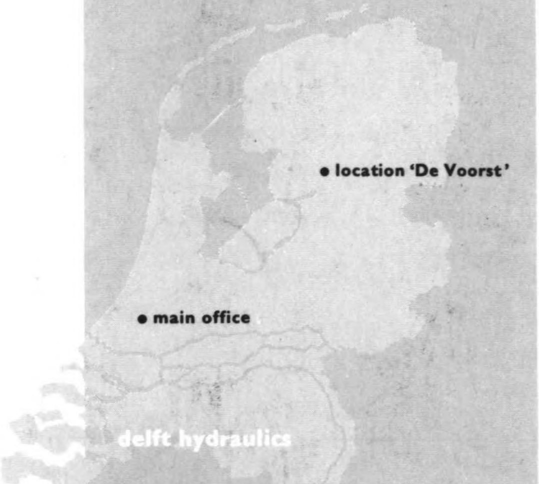
ρ = water density.

ρ_s = solid density.

η = normalized material co-ordinate, =Z/Zo .

$\Delta\eta$ = spatial step (normalized).

Θ = weight in time (in numerical scheme)



main office
Rotterdamseweg 185
p.o. box 177
2600 MH Delft
The Netherlands
telephone (31) 15 - 56 93 53
telefax (31) 15 - 61 96 74
telex 38176 hydel-nl

location ' De Voorst'
Voorsterweg 28, Marknesse
p.o. box 152
8300 AD Emmeloord
The Netherlands
telephone (31) 5274 - 29 22
telefax (31) 5274 - 35 73
telex 42290 hylvo-nl