Theorem 2.3: Suppose that D satisfies H_1 and H_2. Then there exists a common Lyapunov function for D.

Remark 2.3: Theorem 2.3 generalizes previous results about the existence of common Lyapunov functions for switched systems.

It generalizes to the nonlinear case the results obtained in [11] for a linear switched system associated with a finite family of pairwise commuting Hurwitz matrices.

It also extends the results obtained [13] for a switching system composed of a finite family of exponentially stable subsystems whose flows commute, and shows that in order to assure global stability, the global Lipschitz condition of the fields is superfluous.

III. Conclusions

In this paper, we have presented a sufficient condition for the global uniform asymptotic stability of an equilibrium of a switched system. We have shown that when the switched system is composed of a finite family of subsystems, the global asymptotic stability of each subsystem and the pairwise commutativity of their flows are sufficient for the global asymptotic stability of the switched system. We have also shown, by combining this result with the converse Lyapunov theorem obtained in [9], that these conditions are also sufficient for the existence of a common Lyapunov function. The results here presented generalize those obtained in [11] for linear systems and those local ones obtained in [13] for exponentially stable systems.

Acknowledgment

The author would like to express his thanks to R. A. García for his help in preparing the final version of the manuscript.

References

[1] A. A. Agrachev and D. Liberzon, "Lie-algebraic conditions for exponential stability of switched systems," in *Proc. 38th Conf. Decision and Control*, Phoenix, AZ, 1999, pp. 2679–2684.

[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994, vol. 15, Studies in Applied Mathematics.

[3] W. P. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical system which undergo switching," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 751–760, 1999.

[4] W. Hahn, *Stability of Motion*. Berlin, Germany: Springer-Verlag, 1967.

[5] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Contr. Syst. Mag.*, vol. 19, pp. 59–70, 1999.

[6] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched systems: A Lie-algebraic condition," *Syst. Contr. Lett.*, vol. 37, pp. 117–122, 1999.

[7] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Contr. Optim.*, vol. 34, pp. 124–160, 1996.

[8] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer-Verlag, 1990.

[9] J. L. Mancilla-Aguilar and R. A. Garcìa, "A converse Lyapunov theorem for nonlinear switched systems," *Syst. Contr. Lett.*, vol. 44, pp. 67–71, 2000.

[10] Y. Mori, T. Mori, and Y. Kuroe, "A solution to the common Lyapunov function problem for continuous-time systems," in *Proc. 36th Conf. Decision and Control*, San Diego, CA, 1997, pp. 3530–3531.

[11] K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2469–2471, 1994.

[12] T. Ooba and Y. Funahashi, "On a common quadratic Lyapunov function for widely distant systems," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1697–1699, 1997.

[13] H. Shim, D. J. Noh, and J. H. Seo, "Common Lyapunov function for exponentially stable nonlinear systems," in *4th SIAM Conf. Control and its Applications*, Jacksonville, FL, 1998.

[14] R. N. Shorten and K. S. Narendra, "A sufficient condition for the existence of a common Lyapunov function for two second-order linear systems," in *Proc. 36th Conf. Decision and Control*, San Diego, CA, 1997, pp. 3521–3522.

[15] R. N. Shorten and K. S. Narendra, "On the existence of a common Lyapunov for linear stable switching systems," in *Proc. 10th Yale Workshop Adaptive and Learning Systems*, 1998, pp. 130–140.

[16] E. D. Sontag, "Comments on integral variants of ISS," *Syst. Contr. Lett.*, vol. 34, pp. 93–100, 1998.

Minimality and Local State Decompositions of a Nonlinear State Space Realization Using Energy Functions

Jacquelien M. A. Scherpen and W. Steven Gray

***Abstract*—In this paper a set of sufficient conditions is developed in terms of controllability and observability functions under which a given state-space realization of a formal power series is minimal. Specifically, it is shown that positivity of these functions, in addition to a stability requirement and a few technical conditions, implies minimality. Using the nonlinear analogue of the Kalman decomposition, connections are then established between minimality, singular value functions, balanced realizations, and various notions of reachability and observability for nonlinear systems.**

***Index Terms*—Controllability and observability functions, formal power series, minimal realizations, nonlinear systems.**

I. Introduction

The problem of determining when the dimension of a state-space realization for a given input–output map is minimal is a fundamental problem in systems. It connects to many other topics in realization theory like controllability and observability properties, similarity invariants, balanced realizations, and model reduction. The theory is quite complete in the case of linear systems. For example, it is well known that minimality is equivalent to joint controllability and observability, and for stable systems, this is further equivalent to the positive definiteness of the controllability and observability Gramians. These Gramian matrices naturally appear in balanced realization theory and optimal control problems. In the nonlinear setting, minimality theory is not nearly as well developed. For example, there are several existing theories for minimality depending on the exact nature in which the input–output mapping is described, i.e., in terms of a set of input–output differential equations (see [20] and the references therein), a Volterra series [6], [11], [12] or a formal power series/Chen-Fliess functional expansion [6]. At present, the exact connections between these different approaches are not completely understood. Furthermore, motivated by the linear case, we might expect that minimality should have

Manuscript received December 19, 1997; revised April 1, 1999 and March 12, 2000. Recommended by Associate Editor, L. S. Wang. This work was supported in part by the North Atlantic Treaty Organization through the NATO Collaborative Research Grant CRG-971135.

J. M. A. Scherpen is with the Delft University of Technology, Fac. of Information Technology and Systems, Department of Electrical Engineering, 2600 GA Delft, The Netherlands (e-mail: J.M.A.Scherpen@its.tudelft.nl).

W. S. Gray is with the Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, VA 23529-0246 USA (e-mail: gray@ece.odu.edu).

Publisher Item Identifier S 0018-9286(00)10017-0.

connections to the nonlinear extensions of the Gramians, which have been developed for nonlinear balancing [3], [4], [14]–[16]. But these connections are also largely unknown at present.

The primary purpose of this paper is to develop a set of sufficient conditions in terms of controllability and observability functions under which a given state-space realization of a formal power series is minimal. Specifically, it will be shown that positivity of these so-called *energy functions*, plus a few technical conditions, implies minimality. Of course there exists well-known necessary and sufficient conditions for minimality in terms of Kalman-type rank conditions on the accessibility and observability distributions [6], [11]. So the novelty of the approach taken here is in establishing a connection between these differential geometric type minimality conditions and properties of energy functions, which are connected with Hamilton–Jacobi type optimal control theory. Then, using the nonlinear analogue of the Kalman decomposition, we establish connections between minimality, singular value functions and the various notions of reachability and observability for nonlinear systems which preliminarily appeared in [16].

The paper is organized as follows. In Section II, the background material pertaining to all the relevant subjects is briefly reviewed. In Section III, we then develop relationships between positivity of the energy functions and the accessibility/observability rank conditions that are related to minimality. Then in Section IV, we introduce as an application of the new minimality results, the decomposition material. Section V concludes with two examples, where one includes some related computational issues.

Notation: The mathematical notation used throughout is fairly standard. Vector norms are represented by $\|x\| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. $L_2(a,b)$ represents the set of Lebesgue measurable functions, possibly vector-valued, with finite L_2 norm $\|x\|_{L_2} = \sqrt{\int_a^b \|x(t)\|^2\,dt}$. If $L\colon \mathbb{R}^n \mapsto \mathbb{R}$ is a differentiable function, then its partial derivative $\partial L/\partial x$ will be the row vector of partial derivatives $\partial L/\partial x_i$ where $i = 1, \ldots, n$. Furthermore, $x(t_2) = \varphi(t_2, t_1, x_1, u)$ denotes the solution at time t_2 of the system $\dot{x} = f(x) + g(x)u$ with initial condition $x(t_1) = x_1$ and input $u\colon [t_1, t_2] \to \mathbb{R}^m$. A condition about 0 means that this conditions holds for a neighborhood of 0. Finally, $x(-\infty)$ is an abbreviation for $\lim_{t\to-\infty} x(t)$.

II. Background

A. Controllability and Observability Functions for Stable Nonlinear Systems

Controllability and observability functions play an important role in balancing and model reduction for stable nonlinear systems [14]. In this section we give a brief review of the results that are important for the minimality theory presented in Section III.

Consider a smooth, i.e., C^∞, nonlinear system of the form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u\\ y &= h(x)\end{aligned}\tag{1}$$

where $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$, and $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state space manifold denoted by M. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium to be at 0, i.e., $f(0) = 0$, and we also take $h(0) = 0$.

Definition 2.1 [14]: The *controllability* and *observability functions* of (1) are defined as

$$L_c(x_0) = \min_{\substack{u\in L_2(-\infty,0)\\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2\,dt \tag{2}$$

$$L_o(x_0) = \frac{1}{2}\int_0^\infty \|y(t)\|^2\,dt, \qquad x(0) = x_0, \quad u(t) \equiv 0,\ 0 \le t < \infty \tag{3}$$

respectively. □

The value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0, and the value of the observability function at x_0 is the amount of output energy generated by x_0. Obviously, $L_c(x)$ and $L_o(x)$ are nonnegative. It is assumed throughout that L_c and L_o are *finite* and *smooth* functions of x.

Theorem 2.2 [14]: If $f(x)$ is asymptotically stable on a neighborhood W of 0, then for all $x \in W$, $L_o(x)$ is the unique smooth solution of the following Lyapunov-type equation:

$$\frac{\partial L_o}{\partial x}(x) f(x) + \frac{1}{2} h^T(x) h(x) = 0, \qquad L_o(0) = 0. \tag{4}$$

Furthermore for all $x \in W$, $L_c(x)$ is the unique smooth solution of the following Hamilton–Jacobi equation:

$$\frac{\partial L_c}{\partial x}(x) f(x) + \frac{1}{2}\frac{\partial L_c}{\partial x}(x) g(x) g^T(x) \frac{\partial^T L_c}{\partial x}(x) = 0, \quad L_c(0) = 0 \tag{5}$$

with $-(f(x) + g(x)g^T(x)(\partial^T L_c/\partial x)(x))$ asymptotically stable on W. □

Remark 2.3: If we assume that $f(x)$ is asymptotically stable and that (4) has a smooth solution, it then follows that L_o, as in (3), exists, i.e., is finite, [14]. See [1] for more results about the existence and the continuity of L_o (in [1] L_o also plays an important role in the context of stability and invariance). Furthermore, if we assume that (5) has a smooth solution L_c that is antistabilizing (i.e., $-(f(x) + g(x)g(x)^T(\partial^T L_c/\partial x)(x))$ is asymptotically stable), it follows that L_c, as in (2), exists [14]. □

Theorem 2.4 [14]: Assume $f(x)$ is asymptotically stable on a neighborhood W of 0 and (5) has a smooth solution $\overline{L}_c$ on W. Then $\overline{L}_c(x) > 0$ for $x \in W$, $x \neq 0$, if and only if $-(f(x) + g(x)g^T(x)(\partial^T \overline{L}_c/\partial x)(x))$ is asymptotically stable on W. □

For the analysis in this paper the definitions of local *reachability*, *(strong) accessibility*, and *observability* are needed. We refer to standard references like [5], [6], [11], and [13]. These definitions are usually given in the context where only piecewise constant inputs are admissible. However, the effects of approximations of more general inputs by piecewise constant inputs have been considered in earlier work [18], and statements about these properties holding for larger classes of inputs can be found in [17], [21], and [19]. For clarity we mention a special case of observability, though also well known, it is less standard, namely, *zero-state observability*. System (1) is *zero-state observable* if any trajectory where $u(t) \equiv 0$, $y(t) \equiv 0$ implies $x(t) \equiv 0$. We say that (1) is *locally zero-state observable*, if there exists a neighborhood W of 0 where the system is zero-state observable. The following theorem is closely related to results that appear in [5] and [13]. It reveals an important relationship between zero-state observability and positive definiteness of the observability function.

Theorem 2.5 [14]: Assume $f(x)$ is asymptotically stable on a neighborhood W of 0. If (1) is zero-state observable on W, then $L_o(x) > 0, \forall\, x \in W, x \neq 0$. □

It is well known, e.g., [11], that for the accessibility distribution, C, the strong accessibility distribution, C_0, and the observation space, $\mathcal{O}$, with its corresponding codistribution, $d\mathcal{O}$, there exist rank conditions implying local (strong) accessibility and local observability. For local zero-state observability a similar rank condition exists with the zero-state observation space $\mathcal{O}_0$ defined by the linear space of functions on M containing $h_1, \ldots, h_p$ and all repeated Lie derivatives $L_f^k h_j$, $j \in 1, \ldots, p$, $k = 1, 2, \ldots$. As a consequence, local zero-state observability implies local observability at 0. Furthermore, it follows that local strong accessibility at x_0 implies local accessibility at x_0.

Local zero-state observability is certainly more restrictive than local observability. The previous results in a more general observability setting require the input to play a role. Given the system (f, g, h), the corresponding homogeneous system is denoted by (f, g_h, h), where $g_h(x) = g(x) - g(0)$. Thus, under our general assumptions $(f(0), g_h(0), h(0)) = (0,0,0)$. It is easily shown that (f,g,h) and its homogeneous counterpart always have the same observability spaces, and thus have basically the same observability properties. Consider the following definition.

Definition 2.6 [3], [4]: The *natural observability function* for (1) is defined as

$$L_o^N(x_0) = \max_{\substack{u\in B_\alpha \\ x(0)=x_0,\, x(\infty)=0}} \frac{1}{2}\int_0^\infty \|\tilde{y}(t)\|^2\,dt \tag{6}$$

where $B_\alpha := \{u \in L_2[0,\infty) : \|u\|_{L_2} \le \alpha\}$ with $\alpha \ge 0$ a fixed real number, and $\tilde{y}$ is the output response of the corresponding homogeneous system. □

Clearly $L_o^N(x_0)$ is the maximum output energy one could expect from initializing the homogeneous system at $x(0) = x_0$ and applying any input with energy bounded by α. When $\alpha = 0$, we have the observability function given in Definition 2.1. A defining equation for L_o^N analogous to (4) exists. We refer to [3] and [4] for the details. A smooth solution to this equation implies the existence of L_o^N and as is also the case for L_o, the converse can be stated. The following theorem gives the relation between observability and positivity of L_o^N.

Theorem 2.7 [3], [4]: Suppose 0 is an asymptotically stable equilibrium of the system (f,g,h) on a neighborhood W of 0 and $h(0) = 0$. If the system (f,g,h) is observable with respect to B_α then $L_o^N(x) > 0$ when $x \in W$, $x \ne 0$. □

B. Balanced Realizations

Balanced realizations play an important role in a variety of realization and control problems. The classic linear case was first introduced by Moore in [10]. The extension to the nonlinear case appears in [14] and [15]. Consider a nonlinear system of the form (1) with smooth and well-defined controllability and observability function L_c and L_o, respectively, as in Definition 2.1. Additionally, assume the following.

1) $f(x)$ is asymptotically stable on some neighborhood Y of 0.
2) The system is zero-state observable on Y.
3) $(\partial^2 L_c/\partial x^2)(0) > 0$ and $(\partial^2 L_o/\partial x^2)(0) > 0$.

From Morse's lemma, e.g., [8], one can bring the system into input normal form. Furthermore, by applying the Fundamental Theorem of Integral Calculus and smoothness results from [7], the following input-normal/output-diagonal results are obtained.

Theorem 2.8 [14]: Consider (1) with certain technical conditions (see [7] and [8]). Then there exists a neighborhood U of 0 and a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, such that in the new coordinates $z \in W := \psi^{-1}(U)$ the functions L_c, and L_o are of the form

$$\check{L}_c(z) := L_c(\psi(z)) = \tfrac{1}{2} z^T z$$

$$\check{L}_o(z) := L_o(\psi(z)) = \tfrac{1}{2} z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z$$

where $\tau_1(z) \ge \ldots \ge \tau_n(z)$ *are the smooth* singular value functions. □

The form of the controllability and observability function is not yet entirely balanced. For that we need a simple additional coordinate transformation. We refer to [14] for the details on this matter.

C. Minimal Realizations Via Formal Power Series

In this section we briefly review a theory of minimal state space realizations for input–output systems that can be represented by a formal power series (Chen–Fliess functional expansion). A detailed treatment may be found in [6]. Ultimately this leads to the well-known rank conditions, which are necessary and sufficient conditions for a realization to be minimal.

Let S be a given input–output map represented by a convergent generating series

$$S: u \to y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta[u](t,t_0)$$

$$E_{i_k\ldots i_0}[u](t,t_0) = \int_{t_0}^t u_{i_k}(\tau) E_{i_{k-1}\ldots i_0}[u](\tau,t_0)\,d\tau \tag{7}$$

where I^* is the set of multi-indices for the index set $I = \{0,1,\ldots,m\}$, $c(\eta) \in \mathbb{R}^p$, for $t \in [t_0,T]$ with $E_\emptyset(t,t_0)[u] := 1$ and $u_0(t) := 1$. The mapping S can then also be represented by a formal power series in noncommuting monomials $\mathcal{Z} = \{z_0, z_1, \ldots, z_m\}$ via $c = \sum_{\eta\in I^*} c(\eta) z_\eta$, where $z_\eta = z_{i_k}\ldots z_{i_0}$ when $\eta = (i_k \ldots i_0)$. Now define $\mathbb{R}\langle\mathcal{Z}\rangle$ as the set of polynomials in $\mathcal{Z}$ over $\mathbb{R}$, and $\mathbb{R}^p\langle\langle\mathcal{Z}\rangle\rangle$ as the set of formal power series in $\mathcal{Z}$ over $\mathbb{R}^p$. The (block) Hankel mapping associated with c is defined as the $\mathbb{R}$-vector space morphism $\mathcal{H}: \mathbb{R}\langle\mathcal{Z}\rangle \to \mathbb{R}^p\langle\langle\mathcal{Z}\rangle\rangle$, uniquely specified by the generalized shifting property $[\mathcal{H}(z_\zeta)](\eta) = c(\eta\ \zeta)$, where $\eta, \zeta \in I^*$. In this context we have the following definition.

Definition 2.9: The *Lie rank* of a formal power series c is defined as $\rho_L(c) := \dim(\mathcal{H}(\mathcal{L}(\mathcal{Z})))$, where $\mathcal{L}(\mathcal{Z})$ denotes the smallest Lie algebra containing $\mathcal{Z}$. □

An analytic state space realization (f,g,h) defined locally about x_0 is said to *realize* a formal power series c if $c(i_k \ldots i_0) = L_{X_{i_0}} L_{X_{i_1}} \ldots L_{X_{i_k}} h(x_0)$ for every $(i_k\ldots i_0) \in I^*$, where $X_i \in \{f, g_1, \ldots, g_m\}$. It is well known that if a certain growth condition on the coefficients $\{c(\eta)\}_{\eta\in I^*}$ is satisfied, then there exists a realization of c if and only if the Lie rank of c is finite. A realization (f,g,h) about x_0 of a formal power series c is *minimal* if its dimension is less than or equal to the dimension of any other realization of c. The following results characterize minimality.

Theorem 2.10 [6]: An analytic realization (f,g,h) about x_0 of a formal power series c is minimal if and only if its dimension is equal to the Lie rank $\rho_L(c)$. □

Theorem 2.11 [6]: An analytic realization (f,g,h) about x_0 of a formal power series c is minimal if and only if dim $C(x_0) = n$ and dim $d\mathcal{O}(x_0) = n$. □

III. Minimality and Energy Functions

A. The Controllability Function and the Accessibility Rank Condition

In this section we develop connections between the controllability function and the accessibility rank condition in order to apply Theorem 2.11. It is assumed throughout that the system (1) is asymptotically stable on a neighborhood Y of 0.

The following relation is easily deduced (following the lines of the proof of [13, Th. 13])

$$L_c(x_0) = L_r(x_0) := \inf_{\substack{u\in L_2(-\bar{t},0) \\ \bar{t}\ge 0 \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\bar{t}}^0 \|u(t)\|^2\,dt \tag{8}$$

and thus reachability from x_0 implies well-definedness of L_r for all $x \in M$, and likewise for L_c. However, reachability is not implied from a well-defined and positive definite L_c. For our application it is sufficient (as observed from Theorem 2.4) to consider only the antistabilizability of the solution of the Hamilton–Jacobi equation (5), which is a condition that can be seen as reachability from 0 in infinite time (so-called asymptotic reachability from 0). This notion is formally defined below.

Definition 3.1: System (1) is said to be *asymptotically reachable from* x_0 on a neighborhood W of x_0 if $\forall x \in W$ there exists a $u \in L_2(0,\infty)$ such that $\varphi(\tau,0,x_0,u) \in W$ for $\tau \ge 0$, and $\lim_{t\to\infty}\varphi(t,0,x_0,u) = x$.

System (1) is said to be *locally asymptotically reachable from* x_0 if there exists a neighborhood W of x_0 such that the system is asymptotically reachable from x_0 on every neighborhood $V \subset W$ of x_0. □

Clearly, this notion of asymptotic reachability corresponds to the notion of antistabilizability, which is related to the positivity and finiteness of L_c in Remark 2.3 and Theorem 2.4. By $R^V(x_0, T)$ we denote the reachable set from x_0 at time $T > 0$, following the trajectories which remain in the neighborhood V of x_0 for $t \leq T$, and define $R_T^V(x_0) := \cup_{\tau \leq T} R^V(x_0, \tau)$. In the following theorem, we obtain the relation between local asymptotic reachability from x_0 and local accessibility from x_0.

Theorem 3.2: Assume that the accessibility distribution C has constant dimension about x_0. Then local asymptotic reachability from x_0 implies that the system is locally accessible from x_0.

Proof: Suppose that the system is not locally accessible from x_0, then we know from standard results in the literature (e.g., [11]) that dim $C(x_0) = k < n$. Hence from [11, Proposition 3.12] there must exist a neighborhood V of x_0 and local coordinates $x_1, \ldots, x_n$ such that the submanifold $S_{x_0} = \{q \in V | x_i(q) = x_i(x_0),\ i = k+1, \ldots, n\}$ contains $R_T^{\overline{V}}(x_0)$ for any neighborhood $\overline{V} \subset V$ of x_0 and for all $T > 0$. This implies that all $q \in V$ such that $q \notin S_{x_0}$ are not asymptotically reachable from x_0 on V, and thus the local asymptotic reachability from x_0 is contradicted. ■

Our main aim now is to relate the positive definiteness and finiteness of the controllability function to the accessibility rank condition. Note that having $L_c(x)$ finite on W implicitly implies that $\varphi(\tau, -\infty, 0, u) \in W$ for all $\tau \leq 0$. This, combined with Remark 2.3 and Theorem 2.4 gives rise to the following corollary.

Corollary 3.3: Assume that the accessibility distribution C has constant dimension about 0, and assume that f is locally asymptotically stable. If there exists a neighborhood W of 0 such that the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in V$, $x \neq 0$, for all $V \subset W$, then dim $C(0) = n$. □

Remark 3.4: The above corollary is restricted by local requirements on L_c, since we need local asymptotic reachability from 0 in order to use Theorem 3.2. Only asymptotic reachability on a neighborhood W of 0 does not suffice. An example of a smooth system that is asymptotically reachable on a neighborhood W of 0 and that is not locally accessible is easy to construct. However, if we assume that (1) is *analytic*, then we can relax the local requirements on L_c to requirements on a neighborhood W of 0. This is due to the fact that asymptotic reachability from x_0 implies local accessibility from x_0 for analytic systems, e.g., [17] and [21]. Analyticity is actually not a strong restriction in our setting, since it is also a standing assumption for the realization theory in Section II-C. □

The analysis in Remark 3.4 results in the following corollary.

Corollary 3.5: Let (1) be analytic. Assume that the accessibility distribution C has constant dimension about 0, and assume that f is asymptotically stable on a neighborhood W of 0. If the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in W$, $x \neq 0$, then dim $C(0) = n$. □

So far, the focus has been on the concept of local accessibility. However, for the state space analysis presented in Section IV, we use the nonlinear counterpart of the Kalman decomposition, and thus we need to use the concept of local strong accessibility. The local strong accessibility version of Theorem 3.2 is given below.

Theorem 3.6: Assume that the strong accessibility distribution C_0 has constant dimension about x_0. Then local asymptotic reachability from x_0 implies that the system is locally strongly accessible from x_0.

Proof: Suppose that the system is not locally strongly accessible from x_0, then we know from standard results in the literature (e.g., [11]) that dim $C_0(x_0) = k < n$. Hence from [11, Proposition 3.22] there are two possibilities:

i) If $f(x_0) \in C_0(x_0)$, then the proof here follows similar to the Proof of Theorem 3.2.

ii) If $f(x_0) \notin C_0(x_0)$, then by continuity $f(q) \notin C_0(q)$ for all $q \in \tilde{U}$, $\tilde{U} \subset U$ is a neighborhood of x_0, and dim $C(q) =$ dim $C_0(q) + 1$ for all $q \in \tilde{U}$. In this case, one can select the coordinates $\tilde{x}_{k+1}, \ldots, \tilde{x}_n$ in such a way that $S_{x_0}^T = \{q \in \tilde{U} | \tilde{x}_{k+1}(q) = T,\ \tilde{x}_{k+2}(q) = \ldots = \tilde{x}_n(q) = 0\}$ contains $R^{\tilde{U}}(x_0, T)$ for any $T > 0$. Again, we have two cases: (a) If dim $C_0(x_0) < n-1$, then this implies that all $q \in \tilde{U}$ such that $q \notin S_{x_0}^T$ are not asymptotically reachable from x_0 on V, and thus the local asymptotic reachability from x_0 is contradicted. (b) If dim $C_0(x_0) = n-1$, then all $q \in \tilde{U}$ such that $\tilde{x}_n = -K$, $K > 0$, are not asymptotically reachable from x_0 on $\tilde{U}$. This concludes the proof. ■

This theorem gives rise to corollaries similar to Corollaries 3.3 and 3.5, except with accessibility replaced by strong accessibility.

B. The Observability Function and the Observability Rank Condition

For the observability counterpart of the previous section we consider the observability functions as defined in (3) and (6). It is assumed throughout that (1) is asymptotically stable on a neighborhood Y of 0. We start with the observability function in (6) for which observability with respect to the input class B_α plays an important role. The corresponding results for the observability function (3) then follow as a special case when $\alpha = 0$.

Lemma 3.7: Let L_o^N be the natural observability function (6) for some fixed $\alpha > 0$. Assume that $L_o^N(x)$ is smooth and finite for system (1) on a neighborhood W of 0. Then $L_o^N(x) > 0$ for $x \in W$, $x \neq 0$, implies that (1) is locally observable at 0 with respect to B_α.

Proof: Assume that (1) is not locally observable at 0 with respect to B_α. Then the corresponding homogeneous system is also not locally observable at 0 with respect to B_α. Hence there exists an initial state $x_a \neq 0$ such that $h(\tilde{\varphi}(t, 0, 0, u)) = h(\tilde{\varphi}(t, 0, x_a, u))$, $t \geq 0$, $\forall u \in B_\alpha$, where $\tilde{\varphi}(\cdot)$ denotes the solution to homogeneous system. By definition of the natural observability function, we have that $L_o^N(0) = 0$ and by the positivity of L_o^N it follows that $L_o^N(x_a) > 0$. However, from (6) it follows immediately that the maximum over $u \in B_\alpha$ for both states 0 and x_a results in the same optimal input u. This implies that $L_o^N(0) = L_o^N(x_a)$, and yields the desired contradiction to prove the lemma. ■

Motivated by the minimality conditions of Theorem 2.11, we next obtain the following corollary, which follows straightforwardly from the previous lemma and some standard results from [6] and [11].

Corollary 3.8: Assume that the observability codistribution $d\mathcal{O}$ has constant dimension about 0. If the natural observability function (6) is smooth, finite and satisfies $L_o^N(x) > 0$ for $x \in W$, $x \neq 0$, then dim $d\mathcal{O}(0) = n$. □

Now, if we let $\alpha = 0$, then we obviously return to the observability function of (3), and the observability with respect to the input class B_α becomes zero-state observability. The following special case of Corollary 3.8 is useful in Section IV.

Corollary 3.9: Assume that the zero-observability codistribution $d\mathcal{O}_0$ has constant dimension about 0. If the observability function (3) is smooth, finite and satisfies $L_o(x) > 0$, $x \in W$, $x \neq 0$, then dim $d\mathcal{O}_0(0) = n$.

Remark 3.10: It is interesting to compare the results of this section to those of the previous section. They do not completely follow along similar or "dual" lines. Specifically, the results related to the observability functions as given by (3) and (6) are given in terms of the zero-state observability and observability rank condition, respectively. Starting with the rank conditions the converse of Corollaries 3.8 and 3.9 also hold by applying Theorem 2.5. However, for the controllability function, we are considering asymptotic reachability which implies local accessibility, which in turn can be related to the accessibility rank condition. The reverse direction is far less obvious in this case, however, because accessibility from 0 is not sufficient for asymptotic reachability from 0. If asymptotic reachability can somehow be assumed for a given system, then the converse of Corollaries 3.3 and 3.5 would follow for the controllability function.

C. Sufficient Conditions for Minimality

Briefly summarized below is a main result of the paper.

Theorem 3.11: Assume that the observability codistribution $d\mathcal{O}$ (or the zero-observability codistribution $d\mathcal{O}_0$, respectively) and the accessibility distribution C of a system (f, g, h) each have constant dimension about 0. Furthermore, assume that the analytic system (f, g, h) is a realization of the formal power series c, and that $f(x)$ is asymptotically stable. Then, if $0 < L_c(x) < \infty$ and $0 < L_o^N(x) < \infty$ (or $0 < L_o(x) < \infty$, respectively) for $x \in W$, $x \neq 0$, then (f, g, h) is a minimal realization of c. □

These conditions are not necessary due to the fact that, contrary to the linear case, accessibility and controllability are not equivalent in general. Only under additional assumptions can a converse result be obtained.

IV. Local State Decompositions

For linear systems it is well known that the Hankel singular values are independent of the chosen state space realization and only depend on the input–output behavior of the system. In fact, they are the singular values of the Hankel operator of the system (e.g., Glover [2]). If we consider a nonminimal linear state space system with controllability Gramian W and observability Gramian M, the nonzero eigenvalues of MW correspond exactly to the squared Hankel singular values, and the number of zero eigenvalues of MW equals the difference between the state-space dimension of the given system and the state space dimension of any minimal representation. In this section we extend these observations to the nonlinear setting. We are interested in the *Hankel* structure of the system and the related nonlinear balancing concept presented in Section II-B. Since the system Hankel operator corresponds to the mapping from past inputs to future outputs (where the input is zero for positive time) we consider the controllability function as defined in (2) and the observability function as defined in (3).

Consider the nonlinear system (1) and assume that it is locally asymptotically stable. In this section we do *not* assume local zero-state observability, and hence the observability function is not necessarily positive definite. Furthermore, we do *not* assume that $-(f(x)+g(x)g(x)^T(\partial^T L_c/\partial x)(x))$ is locally asymptotically stable (or in other words: we do *not* assume asymptotic reachability from 0), and thus the controllability function need not be finite for all x.

One can use Frobenius' theorem to construct the zero-state observable "part" of the system. In order to be able to do the same for the asymptotically reachable "part" of the system, one must consider the part of the state space system that is asymptotically reachable from 0, i.e., where

$$-\left(f(x) + g(x)g^T(x)\,\frac{\partial^T L_c}{\partial x}(x)\right) \tag{9}$$

is asymptotically stable. In the linear case this part equals the controllable part of the system. In the nonlinear case, the converses of Theorems 3.2 and 3.6 are not always true. So, in order to be able to construct a decomposition analogous to the known nonlinear generalization of the Kalman decomposition (e.g., [11, Th. 3.51]), we must consider the strongly accessible part of the system.

Theorem 4.1: Assume that the distributions C_0, ker $d\mathcal{O}_0$ and $C_0 +$ ker $d\mathcal{O}_0$ all have constant dimension and that $C_0 +$ ker $d\mathcal{O}_0$ is involutive. Then one can find local coordinates $x = (x^1, x^2, x^3, x^4)$ such that $C_0 = \text{span}\{\partial/\partial x^1, \partial/\partial x^2\}$ and ker $d\mathcal{O}_0 = \text{span}\{\partial/\partial x^2, \partial/\partial x^4\}$. The system takes the form

$$\dot{x}^1 = f^1(x^1, x^3) + \sum_{j=1}^{m} g_j^1(x^1, x^2, x^3, x^4)\, u_j \tag{10}$$

$$\dot{x}^2 = f^2(x^1, x^2, x^3, x^4) + \sum_{j=1}^{m} g_j^2(x^1, x^2, x^3, x^4)\, u_j \tag{11}$$

$$\dot{x}^3 = f^3(x^3) \tag{12}$$

$$\dot{x}^4 = f^4(x^3, x^4) \tag{13}$$

$$y = h(x^1, x^3). \tag{14}$$

Proof: The proof is similar to that given in [11, Th. 3.51], which uses Frobenius' theorem. The primary difference is that here we deal with the zero-observable part instead of the observable part. Therefore, for this proof it is enough to observe that the codistribution $d\mathcal{O}_0$ is invariant for the dynamics $\dot{x} = f(x)$ since $L_f d\mathcal{O}_0 \subset d\mathcal{O}_0$. Hence ker $d\mathcal{O}_0 = \text{span}\{\partial/\partial x^2\}$ is an invariant distribution for $\dot{x} = f(x)$. Since ker $d\mathcal{O}_0 \subset$ ker dh, the theorem is proven. ■

Remark 4.2: Another way to view the difference between the decomposition above and that given by [11, Th. 3.51] is in the form of the input vector field in (10). For zero-state observability, the input vector field does not matter, while for the more general concept of observability it *may* matter. That means that x_1 and x_3 are zero-state observable, and thus observable, and that x_2 and x_4 are not zero-state observable, but they still may be observable! However, since we are interested only in the Hankel structure, and specifically in the singular value functions of the nonlinear system, the above decomposition is the most suitable. □

Let n_i be the dimension of x^i, $i = 1, 2, 3, 4$, and let Y be a neighborhood of 0 where the decomposition above is valid. Then clearly (10), (12), and (14) form the zero-state observable part of the system, while (10) and (11) is the strongly accessible part of the system. To assure that for (10), (12), and (14) the observability function exists, we assume that in these local coordinates equation (4) in Theorem 2.2 has a smooth solution for $(x^1, 0, x^3, 0) \in Y$. Furthermore, note that $(f^3(x^3)^T, f^4(x^3, x^4)^T)^T$ is asymptotically stable, and by the form of (12) and (13) it is impossible for $-(f(x)+g(x)g(x)^T(\partial^T L_c/\partial x)(x))$ to be asymptotically stable on Y. To assure that for (10) and (11) the controllability function exists, we assume that in these local coordinates equation (5) has an antistabilizing solution as in Theorem 2.2 for $(x^1, x^2, 0, 0) \in Y$. In fact, the assumption on the existence of the controllability function for the strongly accessible part of the system implies that *the part of the system that is asymptotically reachable from 0 corresponds exactly to the strongly accessible part of the system*.

Theorem 4.3: If the above assumptions on existence of solutions and antistabilizing solutions to (4) and (5), respectively, on parts of the state space are fulfilled, then:

1) $L_o(x^1, x^2, x^3, x^4) > 0$ whenever $(x^1, x^4, x^3, x^4) \in Y$, and $(x^1, x^3) \neq (0, 0)$;
2) $L_o(0, x^2, 0, x^4) = 0$ for all $(0, x^2, 0, x^4) \in Y$;
3) $L_c(x^1, x^2, x^3, x^4)$ is infinite whenever $(x^1, x^2, x^3, x^4) \in Y$, $(x^3, x^4) \neq (0, 0)$;
4) $0 < L_c(x^1, x^2, 0, 0) < \infty$ for all $(x^1, x^2, 0, 0) \in Y$, $(x^1, x^2) \neq (0, 0)$.

Proof—Proof of 1) and 2): It is clear that $h(0, x^2(\tau), 0, x^4(\tau)) = 0$ for all $\tau \geq 0$. By the form of (10) and (12) we obtain that

$$L_o(0, x^2, 0, x^4) = \frac{1}{2}\int_0^\infty h(0, x^2(\tau), 0, x^4(\tau))^T \cdot h(0, x^2(\tau), 0, x^4(\tau))\, d\tau = 0$$

for $u \equiv 0$, and for all $(0, x^2, 0, x^4) \in Y$. Again, by the form of (10), (12), and (14) we then have $\check{L}_o(x^1, x^3) = L_o(x^1, x^2, x^3, x^4)$ for $u \equiv 0$, where $\check{L}_o$ is the observability function of (10), (12), and (14). By assumption $\check{L}_o = L_o$ exists and is smooth. By Theorem 2.5 $L_o(x^1, x^2, x^3, x^4) = \check{L}_o(x^1, x^3) > 0$ for $(x^1, x^3) \neq (0, 0)$.

Proof of 3) and 4): The controllability function L_c must satisfy (2). Since the system formed by (12) and (13) is asymptotically stable, it follows immediately that $L_c(x^1, x^2, x^3, x^4) = \infty$ for all $(x^1, x^2, x^3, x^4) \in Y$, with $(x^3, x^4) \neq (0, 0)$. By Theorem 2.2 $L_c(x^1, x^2, 0, 0) < \infty$ for all $(x^1, x^2, 0, 0) \in Y$. Furthermore, by Theorem 2.4 it follows that $L_c(x^1, x^2, 0, 0) > 0$ for all $(x^1, x^2, 0, 0) \in Y$, $(x^1, x^2) \neq (0, 0)$. ■

Remark 4.4: L_c is infinite on the subsystem that is *not* strongly accessible. Hence, that subsystem is also *not* asymptotically reachable from 0. This in essence yields another proof of Theorem 3.6. □

Remark 4.5: Now assume that the full system is locally accessible (remember that this is, together with local observability, a condition that implies minimality), but *not* locally strongly accessible. We know from Theorem 4.3 that the states which are not locally strongly accessible force the controllability function L_c to become infinite. Thus, one can conclude, contrary to the linear case, that minimality for a nonlinear system as discussed in the previous sections does *not* ensure that the controllability function is finite. □

Remark 4.6: The observability counterpart to Remark 4.5 is similar, but in fact easier to describe. Assume that the full system is locally observable, but *not* locally zero-state observable. We know from Theorem 4.3 that the part of the system which is not locally zero-state observable corresponds to the observability function L_o being zero. Thus one can conclude, again contrary to the linear case, that minimality for a nonlinear system does *not* ensure the observability function to be positive. However, we have introduced the natural observability function L_o^N in Section II-A. For this function to be positive definite, we only need observability with respect to B_α, and not the more restrictive zero-state observability. If observability with respect to B_α is equivalent to observability (which is not very restrictive, since we only require the input to have finite energy), we can repeat the analysis of this section for L_o^N with zero-state observability replaced by observability. The new analysis results in the generalized Kalman decomposition as found in [11], and straightforwardly we obtain similar results as for L_o, with the additional property that for L_o^N the results do coincide with the usual results for the observability function in the linear case. □

If additionally one assumes that $(\partial^2 L_o)/(\partial x^1)^2(0) > 0$ and $(\partial^2 L_c)/(\partial x^1)^2(0) > 0$, then it becomes clear from Theorem 4.3 that $L_o(x^1, 0, 0, 0)$ and $L_c(x^1, 0, 0, 0)$ may be transformed into the form of Theorem 2.8. In fact, there exists a local x^1 coordinate transformation $x^1 = \psi(z)$, $\psi(0) = 0$, $(\psi^{-1}(x^1), 0, 0, 0) \in Y$, such that $L_c(\psi(z), 0, 0, 0)$ and $L_o(\psi(z), 0, 0, 0)$ are in the form of Theorem 2.8. Thus this part of the system may be balanced on a neighborhood of 0 with singular value functions $\tau_1(z) \geq \ldots \geq \tau_{n_1}(z)$. Furthermore, if we also consider x^2, then there exist local coordinates $(z^1, z^2) = \phi^{-1}(x^1, x^2)$ such that $L_c(\phi(z^1, z^2), 0, 0) = (1/2){z^1}^T z^1 + (1/2){z^2}^T z^2$. Now write $L_o(\phi(z^1, z^2), 0, 0) = (1/2)({z^1}^T\ {z^2}^T) M(z^1, z^2)({z^1}^T\ {z^2}^T)^T$. If the assumptions of Theorem 2.8 are fulfilled, one may diagonalize $M(z^1, z^2)$. The functions on the diagonal: $\overline{\tau}_1(z^1, z^2) \geq \ldots \geq \overline{\tau}_{n_1+n_2}(z^1, z^2)$ are such that $\overline{\tau}_i(z^1, 0) = \tau_i(z)$, $i = 1, \ldots, n_1$, and $\overline{\tau}_j(0, z^2) = 0$, $j = n_1 + 1, \ldots, n_1 + n_2$. This is analogous to the linear case, where the unobservable part corresponds to zero Hankel singular values. Note that it is not possible to transform the whole system into the form of Theorem 2.8, since $L_c(0, 0, x^3, x^4)$ is infinite, but this is still in agreement with the linear theory, since here we are dealing with the "inverse of the controllability Gramian." Hence that part of the system that is not strongly accessible yields an "inverse of the controllability Gramian" that is infinite, and thus a "controllability Gramian" that is singular.

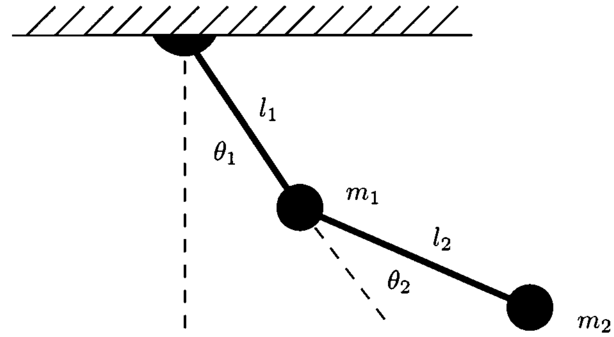

Fig. 1. The double pendulum.

V. EXAMPLES

The first example is an academic one meant to simply illustrate the basic theory presented in this paper. The second example is physical in nature and reveals some computational issues related to solving Hamilton–Jacobi equations.

Example 5.1: Consider the following system (1), where:

$$f(x) = \begin{pmatrix} -x_1 + x_2^3 \\ -x_2 - x_2^3 \\ -x_3 + x_1 x_2^2 + x_3 x_2^2 - x_2^3 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} 0 & -\sqrt{2 - 2(x_1 + x_3)^2 + 2x_2^2} \\ 0 & \sqrt{2 - 2(x_1 + x_3)^2 + 2x_2^2} \\ \sqrt{2} & \sqrt{2 - 2(x_1 + x_3)^2 + 2x_2^2} \end{pmatrix}$$

$$h(x) = \begin{pmatrix} 2x_1 + 2x_3 \\ \sqrt{2}x_2 \end{pmatrix}.$$

This system is asymptotically stable and analytic on a neighborhood of 0. The rank of the accessibility distribution C at 0 is 2 (it is easily seen that the Lie bracket directions are already given by $g_1(0)$, and $g_2(0)$). The accessibility distribution equals in this case the strong-accessibility distribution. The rank of the observability codistribution $d\mathcal{O}$ at 0 is also 2 (the two directions of the zero-state observability codistribution in 0 are given by $dh_1(0)$, and $dh_2(0)$). The observability codistribution equals in this case the zero-state observability codistribution. By Corollaries 3.5 and 3.9 we know now that there exist $x \in \mathbb{R}^3$ such that $L_o(x) = 0$ with $x \neq 0$ and $L_c(x)$ infinite, with x finite. Now, to bring the system in the form of Theorem 4.1 apply the transformation

$$x = Tz = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} z$$

then
$$\begin{cases} \dot{z}_1 = -z_1 + z_1 z_2^2 + u_1\sqrt{2}, \\ \dot{z}_2 = -z_2 - z_2^3 + u_2\sqrt{2 - 2z_1^2 + 2z_2^2}, \\ \dot{z}_3 = -z_3, \end{cases} \quad \begin{cases} y_1 = 2z_1 \\ y_2 = \sqrt{2}z_2 \end{cases}.$$

Obviously, z_3 is the nonaccessible and nonobservable part of the system. By Theorem 4.3, we have that $L_o(0, 0, z_3) = 0$, that $L_o(z_1, z_2, z_3) > 0$ for $(z_1, z_2) \neq (0, 0)$, that $0 < L_c(z_1, z_2, 0) < \infty$, and that $L_c(z_1, z_2, z_3)$ is infinite for $z_3 \neq 0$. This also directly follows from trying to solve the corresponding Hamilton–Jacobi equations (4) and (5). For the (z_1, z_2) subsystem (i.e., the minimal subsystem) note that L_c and L_o are already in the form of Theorem 2.8, i.e., $L_c(z) = (1/2)z^T z$ and $L_o(z) = (1/2)z^T\ \mathrm{diag}(2, 1 + z_1^2)z$. The singular value functions are therefore $\tau_1(z) = 2$ and $\tau_2(z) = 1 + z_1^2$. □

Example 5.2: Consider a frictionless double pendulum (or two-link robot manipulator) with control torque u applied at the first joint; see Fig. 1. The dynamics of such a double pendulum may be obtained via

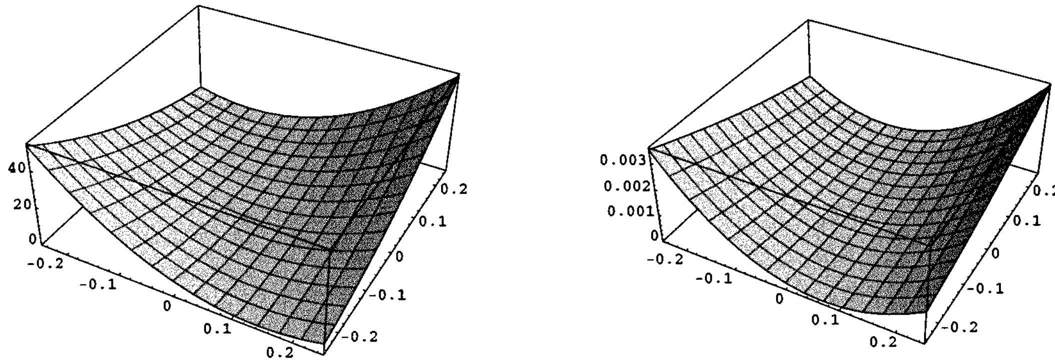

Fig. 2. The controllability (left) and observability (right) functions for Example 5.2.

the Hamiltonian formalism. We derive the equations for the simple Hamiltonian form in order to be able to consider the associated gradient system, which is of smaller order, and therefore computationally easier to handle, but still captures the physical properties of the system. Furthermore, the frictionless system is only Lyapunov stable, but not asymptotically stable, while the associated gradient system is asymptotically stable, and thus fulfills the requirements of this paper. Let $\theta = (\theta_1, \theta_2)$ and $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2)$. The kinetic coenergy is given by the sum of the kinetic coenergies of the masses m_1 and m_2, respectively. This yields the equations shown at the bottom of the page. Note that $M(\theta)$ is a positive definite matrix for every θ. Similarly the potential energy V is the sum of the potential energies of the two masses, i.e., $V(\theta) = -m_1 g l_1 \cos\theta_1 - m_2 g l_1 \cos\theta_1 - m_2 g l_2 \cos(\theta_1 + \theta_2)$. Define $q := \theta$, and $p := M(\theta)\dot{\theta}$, thus $\dot{q} = M(q)^{-1} p$. Furthermore, denote by Q the manifold with local coordinates q_1, q_2. The Hamiltonian H can be written as $H(q,p) = \frac{1}{2} p^T M(q)^{-1} p + V(q)$, where the kinetic energy in the (q,p) coordinates is given by the Riemannian metric $M(q)$ on Q and $V(q)$ is the potential energy. We obtain that the output map C is given by $C(q) = q_1$. In the (q,p) coordinates the equations of the double pendulum in simple Hamiltonian form are given by

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (1 \quad 0) q. \end{aligned} \tag{15}$$

Let $P(q) = M(q)^{-1}$. The associated gradient system is given by

$$\dot{x} = -P(x)\frac{\partial^T V}{\partial x}(x) + P(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = (1 \quad 0)x. \tag{16}$$

Here, we only consider the case where $l_1 = l_2 = 1$, and $m_1 = m_2 = 1$. Mathematica software was employed to approximately solve the Hamilton–Jacobi equations for the observability and controllability function, L_o and L_c, for the gradient system (16). Specifically, (4) and (5), are solved up to order 4 using an iterative procedure from Lukes [9]. If we write

$$N = \frac{\partial^2 L_o}{\partial x^2}(0), \quad R = P(0)$$
$$Q = \frac{\partial^2 V}{\partial x^2}(0), \quad H = \frac{\partial C}{\partial x}(0)$$

then

$$\begin{aligned} L_o(x) &= \frac{1}{2} x^T N x + L_o^h(x) \\ -P(x)\frac{\partial^T V}{\partial x}(x) &= -RQx + f^h(x) \\ \frac{1}{2} C(x)^T C(x) &= \frac{1}{2} x^T H^T H x + \theta^h(x) \end{aligned} \tag{17}$$

where $L_o^h(x)$, $f^h(x)$ and $\theta^h(x)$ contain higher-order terms (beginning with degrees 3, 2, and 3, respectively). The Lyapunov type equation (4) splits into two parts: the first part is the Lyapunov equation of the observability Gramian of the linearized gradient system, while the second part is a higher-order equation. The mth order terms $L_o^{(m)}(x)$ of $L_o(x)$ can now be computed inductively for $m \geq 3$. Denote the mth order terms in $(\partial L_o / \partial x) f^h(x) + \theta^h(x)$ by $K_m(x)$. Then, since $-RQ$ has all eigenvalues in the left half-plane, it follows that

$$\begin{aligned} \frac{\partial L_o^{(m)}}{\partial x}(x) RQx = K_m(x) \Rightarrow L_o^{(m)}(x) \\ = \int_0^\infty K_m\left(e^{-RQt}x\right) dt. \end{aligned} \tag{18}$$

It is easily seen that $K_m(x)$ only depends on $L_o^{(m-1)}$, $L_o^{(m-2)}, \ldots, L_o^{(2)}$, and therefore (18) determines $L_o^{(m)}$ inductively starting from $L_o^{(2)} = (1/2)x^T N x$. This procedure can also be followed for the controllability function L_c. It yields for our gradient system the following result:

$$\begin{aligned} L_o(x_1, x_2) = {} & 0.034375 x_1^2 + 0.00212286 x_1^4 + 0.01875 x_1 x_2 \\ & - 0.0046596 x_1 x_2^3 - 0.00168806 x_1^2 x_2^2 \\ & + 0.00015811 x_1^3 x_2 + 0.003125 x_2^2 \\ & - 0.000909133 x_2^4 \\ L_c(x_1, x_2) = {} & 360 x_1^2 - 107.411 x_1^4 + 400 x_1 x_2 \\ & - 24.1667 x_1 x_2^3 - 191.25 x_1^2 x_2^2 \\ & - 230.595 x_1^3 x_2 + 120 x_2^2 + 21.875 x_2^4. \end{aligned}$$

Examining these functions near the origin (see Fig. 2) it is evident that they are strictly positive, and hence, the system is minimal. This corresponds to our physical intuition. Observe that the observability function is quite close to zero at some values. This gives us a kind of measure for "weak" zero-observability. Likewise, for the controllability function we can make a similar observation for "weak" asymptotic reachability. □

$$T(\dot{\theta}) := \tfrac{1}{2}\dot{\theta}^T M(\theta), \qquad M(\theta) = \begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 \end{pmatrix}$$

REFERENCES

[1] C. I. Byrnes and C. F. Martin, "An integral-invariance principle for nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 983–994, 1995.
[2] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞}-error bounds," *Int. J. Contr.*, vol. 39, pp. 1115–1193, 1984.
[3] W. S. Gray and J. P. Mesko, "Controllability and observability functions for model reduction of nonlinear systems," in *Proc. 1996 Conf. Inf. Sci. Syst.*, Princeton, NJ, 1996, pp. 1244–1249.
[4] ——, "Observability functions for linear and nonlinear systems," *Syst. Contr. Lett.*, vol. 38, pp. 99–113, 1999.
[5] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 708–711, 1976.
[6] A. Isidori, *Nonlinear Control Systems*, 3rd ed. London: Springer-Verlag, 1995.
[7] T. Kato, *A Short Introduction to Perturbation Theory for Linear Operators*. New York: Springer-Verlag, 1982.
[8] J. Milnor, *Morse Theory*: Princeton Univ. Press, 1963.
[9] D. L. Lukes, "Optimal regulation of nonlinear dynamical systems," *SIAM J. Contr.*, vol. 7, pp. 75–100, 1969.
[10] B. C. Moore, "Principal component analysis in linear systems: Controllability, observability and model reduction," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 17–32, 1981.
[11] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer-Verlag, 1990.
[12] W. J. Rugh, *Nonlinear System Theory: The Volterra–Wiener Approach*: John Hopkins Univ. Press, 1981.
[13] A. J. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 770–784, 1992.
[14] J. M. A. Scherpen, "Balancing for nonlinear systems," *Syst. Contr. Lett.*, vol. 21, pp. 143–153, 1993.
[15] ——, "Balancing for nonlinear systems," in *Proc. 1993 Eur. Contr. Conf.*, vol. 4, Groningen, The Netherlands, 1993, pp. 1838–1843.
[16] ——, "On the similarity invariance of balancing for nonlinear systems," in *Preprints IFAC Nonl. Contr. Syst. Des. (NOLCOS)*, vol. 2, Tahoe City, CA, 1995, pp. 783–788.
[17] E. D. Sontag, "Integrability of certain distributions associated to actions on manifolds and an introduction to Lie-algebraic control," Rutgers Center SYCON Rep. 88-04.
[18] H. J. Sussman, "Existence and uniqueness of minimal realizations of nonlinear systems," *Math. Syst. Theory*, vol. 10, pp. 263–284, 1977.
[19] ——, "A general theorem on local controllability," *SIAM J. Contr. Opt.*, vol. 25, pp. 158–194, 1987.
[20] Y. F. Zheng, P. Liu, A. S. I. Zinober, and C. H. Moog, "What is the dimension of the minimal realization of a nonlinear system?," in *Proc. 34th IEEE CDC*, New Orleans, LA, 1995, pp. 4239–4244.
[21] E. D. Sontag, *Nonlinear Controllability and Optimal Control*, H. J. Sussmann, Ed., 1990, pp. 1–41.

Matrix Approach to Deadlock-Free Dispatching in Multi-Class Finite Buffer Flowlines

A. Gürel, S. Bogdan, and F. L. Lewis

***Abstract*—For finite-buffer manufacturing systems, the major stability issue is "deadlock," rather than "bounded-buffer-length stability." The paper introduces the concept of "system deadlock," defined rigorously in Petri net terms, and system operation with uninterrupted part-flow is characterized in terms of the absence of this condition. For a large class of finite-buffer multi-class re-entrant flowline systems, an analysis of "circular waits" yields necessary and sufficient conditions for the occurrence of "system deadlock." This allows the formulation of a maximally permissive one-step-look-ahead deadlock-avoidance control policy for dispatching jobs, while maximizing the percent utilization of resources. The result is a generalized kanban dispatching strategy, which is more general than the standard multi-class last buffer first serve (LBFS) dispatching strategies for finite buffer flowlines that typically under-utilize the resources. The problem of computational complexity associated with Petri net (PN) applications is overcome by using certain sub-matrices of the PN incidence matrix. Computationally efficient matrix techniques are given for implementing the deadlock-free dispatching policy.**

***Index Terms*—Control policy, deadlock, dispatching, flexible manufacturing system, kanban, matrix methods, Petri net, stability.**

I. INTRODUCTION

In *flexible manufacturing systems* (FMS) [2], resource sharing is ubiquitous. A given resource may be common to the production processes of several part-types (*parallel sharing*), and/or may be used multiple times during the production process of a given part-type (*sequential sharing or reentrance*). A key role in job routing/dispatching is played by the FMS *controller*, which allocates resources to perform jobs for customers or on parts. Failure by the controller to suitably assign resources during job dispatching can lead to serious performance problems. There are numerous formal job-dispatching rules, such as first-in-first-out (FIFO), first-buffer-first-serve (FBFS), last-buffer-first-serve (LBFS), earliest due date (EDD), least slack (LS), and so on [11], [13].

One fundamental question that needs to be addressed in connection with any FMS dispatching policy is whether or not it is *stable*. Studies of stability for FMS often focus on stability in the sense of *bounded buffer lengths* [9], [11]. However, in practice, the buffer lengths are *finite*, and such stability results are inapplicable, since it is not obvious how to keep the buffer lengths below some *fixed finite value*. For finite-buffer multi-class reentrant flowline (MRF) systems [9], which constitute a large class of FMSs, the issue is stability, not in the sense of bounded buffer lengths, but in the sense of absence of *deadlock*. A flowline for a given part-class is said to be deadlocked if it holds a part that cannot complete its processing sequence. Many popular dispatching rules can result in deadlock if care is not taken (for instance, see [14]). In [11], the FBFS and LBFS policies have been shown to

Manuscript received June 17, 1998; revised March 2, 1999. Recommended by Associate Editor, F. Lin.

A. Gürel is with the Department of Electrical and Electronic Engineering, The Eastern Mediterranean University, Famagusta, via Mersin 10 Turkey (e-mail: gurel@mu.edu.tr).

S. Bogdan is with the Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, HR-10000 Zagreb, Croatia (e-mail: stjepan.bogdan@fer.hr).

F. L. Lewis is with the Automation and Robotics Research Institute, The University of Texas at Arlington, Fort Worth, TX 76118-7115 USA (e-mail: flewis@controls.uta.edu).

Publisher Item Identifier S 0018-9286(00)10030-3.