Elephant random walk

A comparison with the normal random walk with a focus on the gambler's ruin problem

by

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to obtain the degree of Bachelor of Science

at the Delft University of Technology,

to be defended publicly on Wednesday July 2, 2025 at 11:00 AM.

Student number:	5876117	
Project duration:	April 22, 2025 – July 2, 2025	
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An electronic version of this thesis is available at http://repository.tudelft.nl/.

TU Delft

Abstract

In this bachelor thesis, we will look a variant of the random walk, namely the elephant random walk. The elephant random walk, named after the elephant due to its excellent memory, is a random walk where future steps are based on previous steps taken in the walk based on the memory parameter p. To that end, we will be looking at the differences between the normal random walk and elephant random walk: we will look at the differences in behavior and properties of both types of random walks. In particular, we will focus on the gambler's ruin problem, where we discuss the chances of player A winning, $\nu(a)$, and the expected number of steps until either player wins, $e(a)$. To do so, we will make use of martingales and known theorems about martingales to prove the theorems about the elephant random walk. For the gambler's ruin problem for the elephant random walk, we use simulations to look at the behavior of the gambler's ruin for the elephant random walk.

We found that in general, the elephant random walk behaves differently from the normal random walk: while the expected value of the normal random walk increases or decreases linearly or stays at zero, the elephant random walk either diverges to infinity less than linearly or converges to zero based on the memory parameter p. In addition to this, one finds that the elephant random walk has three different diffusion regimes based on p: compared to the normal random walk, which only has a diffusive regime, the elephant random walk has a diffusive, marginally superdiffusive and superdiffusive regime. Looking at the properties, we see that some of the results of the normal random walk also hold for the elephant random walk, in particular the law of large numbers. However, it turns out that the central limit theorem only holds for certain values of p, and that the central limit theorem stops holding for larger p due to the dependence of steps on each other being too high. In addition, we found that the memory parameter p and initial parameter q for the gambler's ruin for the elephant random walk affects how $\nu(a)$ and $e(a)$ behaves, with the influence of q increasing as p increases on both $\nu(a)$ and $e(a)$. The total capital N has little influence on the behavior of the gambler's ruin for the elephant random walk, with N only affecting the maximal expected number of steps.

Layman abstract

Normally, a random walk consists of a sum of steps that randomly go the left or to the right. In this bachelor thesis, we look at a variation of this random walk called the elephant random walk, where the steps of the elephant random walk are based on the previous steps that have already been taken. In particular, we are interested in the differences in behavior between the random walk and elephant random walk, and whether or not important properties of the random walk also translate to the elephant random walk. In addition to this, we look at an application called the gambler's ruin problem for both types of random walks, which concerns a gambling game between two players.

It turns out that, while the elephant random walk does behave quite differently from the random walk, the elephant random walk still has a lot of important properties that the normal random walk has. Furthermore, the behavior of the gambler's ruin for the elephant random walk can be split into two cases based on the parameters of the gambler's ruin.

Contents

1 Introduction

The random walk is one of the most well known processes: in its simplest form, the random walk is a stochastic process which consists of multiple random steps with a value of either -1 or 1. The steps are influenced on the parameter p: higher values of p makes it more likely for steps to have the value 1, whereas lower values of p causes more steps to have the value -1. Many aspects of the random walk are well known, which include topics such as hitting times and hitting probabilities, recurrence and transience, and long term behavior of the random walk, among various other topics. The random walk has various applications in various disciplines, which includes particle movements, stock market behavior and animal behavior (Hughes, 1995; Pokojovy et al., 2024; Codling et al., 2008). One of the more interesting applications of the random walk has to do with the gambler's ruin problem. In this problem, two players with a certain starting capital play a game using a coin. After doing a coin flip, the players exchange money based on who won the coin flip. This gets repeated until one of the players is "ruined" and has no money left.

However, by introducing memory to the random walk, the random walk becomes a lot more complex when compared to its memoryless counterpart. This variant, called the elephant random walk, was first introduced by Schütz and Trimper in 2004. In this version of the random walk, which was named the elephant random walk due to the fact that elephants have a really good memory, the elephant bases its steps based on the previous steps taken: the first step is based on the initial parameter q, whereas later steps are randomly based on a previous step of the elephant random walk with memory parameter p, where higher values of p causes the step to have a higher chance to go the same direction as the previous step that is chosen. It turns out that the elephant random walk acts different when it comes to the behavior of each step and the long term behavior, and that the elephant random walk has different properties. But what changes exactly?

The goal of this bachelor thesis is to explore the differences between the normal random walk and the elephant random walk. In particular, we will focus on the expected behavior, the asymptotic behavior of both random walks, which includes the law of large numbers and the central limit theorem, and the different diffusion processes in the two types of random walks. In addition to this, we will look at the gambler's ruin problem for the elephant random walk, which includes studying the behavior of both the chance of winning the game and the expected numbers of steps until either player wins.

In Chapter 2, we will look at known literature about the elephant random walk and their results. In Chapter 3, we will discuss the aspects of the normal random walk, which consists of the expectation of the random walk, diffusion processes, the Markov property, the law of large numbers, the central limit theorem and the gambler's ruin problem. In Chapter 4, we will discuss how the elephant random walk works, and we will discuss the same aspects that were discussed in Chapter 3.

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Literature study

The elephant random walk is a relatively new subject: the term ”elephant random walk”, which is used to indicate a discrete-time random walk where each step is dependent on the complete memory of the random walk, was coined by a paper in 2004 from Schütz and Trimper due to ”...the traditional saying that elephants can always remember” (Schütz & Trimper, 2004). In the paper, Schütz and Trimper (2004) calculated the expected value and expected second moment of the displacement for the elephant random walk based on the memory parameter p, which determines how later steps steps get determined based on previous steps. They found that there are two regimes for the expected value of the elephant random walk[1]: for $p < 1/2$, the expected value of the elephant random walk converges to zero, whereas the expected value of the elephant random walk diverges to infinity for $p > 1/2$. The direction where it diverges to is dependent on the initial parameter q, which determines the first step of the elephant random walk. Meanwhile, it was found by Schütz and Trimper (2004) that the type of diffusion process of the elephant random walk can be split into three different regimes. Interestingly, the critical value is not at $p = 1/2$, but rather at $p = 3/4$: for $p < 3/4$, the elephant random walk is a diffusive process, which means the variance of the elephant random walk grows linearly as the number of steps increases. On the other hand, the elephant random walk becomes superdiffusive for $p > 3/4$, which means the variance of the elephant random walk increases more than linearly, but less than quadratically. At the critical value $p = 3/4$, the elephant random walk behaves like a marginally superdiffusive process, which is characterized by a linearlogarithmically growing variance. Furthermore, the paper of Schütz and Trimper (2004) explored the probability distribution of the elephant random walk. By transforming the elephant random walk into a Fokker-Planck equation, Schütz and Trimper (2004) conjectured that the elephant random walk converges towards a normal distribution when looking at the odd and even moments of the Fokker-Planck equation.

However, a paper from Coletti et al. (2016) found that the probability distribution only converges to the normal distribution for certain values of the memory parameter p. The paper proved that, even though the steps are not independent and identically distributed, the law of large numbers still applies to the elephant random walk. Unlike the results from Schütz and Trimper (2004), Coletti et al. (2016) found that the central limit theorem only applies to the diffusive and marginally superdiffusive regime, i.e. $p \leq 3/4$. For the superdiffusive regime, it is instead found that the probability distribution does converge, but to a non-normal distribution. Coletti et al. (2016) suggest that this result is due to ”...cross-correlations between the increments [that] are in some sense “too strong”” (Coletti et al., 2016). A later paper from Coletti et al. (2017) showed that that, aside from the central limit theorem, the law of iterated logarithms also holds for the elephant random walk in the diffusive and marginally superdiffusive regime.

[1]At the transition point $p = 1/2$, the elephant random walk reverts back to a normal random walk.

Aside from these results, results regarding the return time to the origin have also been proven: A paper from Bertoin (2021), which looked at the diffusive regime of the elephant random walk, showed that the number of times the elephant random walk returns to the origin grows like the square root of the number of steps as the number of steps increases. Interestingly, for the marginally superdiffusive regime of the elephant random walk, a paper of Fang (2024) showed that most of the times the elephant random walk returns to the origin happens just before the last time the elephant random walk returns to the origin, and that the number of returns to the origin divided by the logarithm of the number of steps approaches the arcsine distribution as the number of steps approaches infinity.

Multiple extensions of the elephant random walk have also been studied: one of the more obvious extensions would be to extend the elephant random walk to multiple dimensions. A paper from Bercu and Laulin (2017) showed that similar results hold for the elephant random walk for the diffusive, marginally superdiffusive and superdiffusive regime, but that the critical point $p_d = (2d + 1)/4d$ shifts towards $p = 1/2$ as the number of dimensions increases.

Another studied extension is making the elephant random walk not depend on the whole history of the walk, but rather a part of it. A paper by Gut and Stadtmüller (2021) studied several variations of this, which includes basing the current step on the first steps, latest steps, or a combination of both. As is shown in the paper, the known results for the elephant random walk also hold for the variations that Gut and Stadtmüller (2021) discussed, which include the law of large numbers, law of iterated logarithm and central limit theorem.

Closely related to this is the elephant random walk with delays, where each step can take the value $-1, 0$ or 1, which was studied by Gut and Stadtmüller (2019). Just like the original elephant random walk, the paper shows that the elephant random walk with delays has three different diffusion regimes, where the critical value is dependent on the chance of going the same direction as a previous step and the chance of going the opposite direction as a previous step. Additionally, similar results hold for this variation as the general elephant random walk.

It is important to note that the elephant random walk is connected to a number of other subjects. The most important connection of the elephant random walk is to Pólya-type urns: Pólya-type urns in its most common form consists of an urn with different types of balls in it. At each time step, a ball is taken from the urn. Afterwards, two of the same type of ball is put back into the urn, after which the process repeats. Baur and Bertoin (2020) showed that the elephant random walk can be modeled using Pólya-type urns by representing the steps with value of -1 and 1. At each time step, one ball gets removed from the urn and then two balls get returned: the ball that got removed and another ball based on the removed ball. By modeling the elephant random walk in such a way, many of the results presented in this section can be proved by using theorems from Pólya-type urns.

3

Random walk

In this chapter, the general version of the elephant random walk, the random walk, will be discussed, and characteristics of the random walk such as the expected value, law of large numbers and central limit theorem will be discussed. Lastly, we will briefly look at the gambler's ruin. Note that the focus for this chapter is to look at some of the results for the random walk that we will also discuss for the elephant random walk, so that we can compare the behavior and results between the random walk and elephant random walk. Thus, other results for the random walk will not be discussed. However, for more information about the random walk one may look at Lawler and Limic (2010) and Hughes (1995).

3.1. The model

In this section, we will discuss the general model for the random walk. In particular, we will discuss the one-dimensional random walk on $\mathbb{Z}$, although this can be extended to multiple dimensions or to a continuous random walk, among other variations. The random walk is defined as follows: The walk starts at a point $X_0 \in \mathbb{Z}$ at time $n = 0$. At each discrete time step, the walker decides to move a step to the left or to the right, which is given the value of -1 and 1 respectively. The value of each step, η_n, is determined as follows: given a value $p \in [0, 1]$, η_n is a random variable with

$$\mathbb{P}[\eta_n = 1] = p, \tag{3.1}$$
$$\mathbb{P}[\eta_n = -1] = 1 - p. \tag{3.2}$$

In Figure 3.1, the process for each step η_{n+1} with $n \geq 0$ is illustrated.

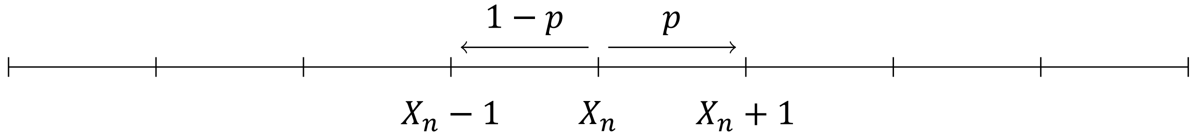

Figure 3.1: Illustration of the random walk for each step η_{n+1} with $n \geq 0$, with p and $1 - p$ denoting the probabilities to take a step with value 1 and -1 respectively.

It is important to note that the value of X_n can be represented in two ways, both recursively and non-recursively:

$$X_n = X_{n-1} + \eta_n \tag{3.3}$$

$$= X_0 + \sum_{k=1}^{n} \eta_k. \tag{3.4}$$

For simplicity, we will assume that $X_0 = 0$, as results for general $X_0 \in \mathbb{Z}$ can be obtained by shifting the starting point of the random walk with $X_0 = 0$.

3.2. Results of the random walk

In this section, we will discuss some results of the random walk. In particular, we will discuss the Markov property, expected values and asymptotic behavior of the random walk.

3.2.1. Markov chain

One of the most important properties of the random walk is that the random walk is an example of a homogeneous Markov chain, which means that the random walk satisfies the Markov property, and that the conditional probability for each step is independent of the time. The Markov property is defined as follows:

Definition 3.2.1 (Markov chain)**.** A sequence $(X_n)_{n\geq 0}$ is a Markov chain if it satisfies the Markov property:

$$\mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0] = \mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n] \tag{3.5}$$

for all $n \geq 0$ and $x_0, \dots, x_n \in S$ points in the state space. Additionally, the Markov chain is homogeneous if for all $i, j \in S$, the probability $\mathbb{P}[X_{n+1} = i \mid X_n = j]$ is independent of n.

We get the following result for the random walk:

Proposition 3.2.2. Let $(X_n)_{n\geq 0}$ be a random walk with $p \in [0, 1]$. Then, the random walk is a homogeneous Markov chain.

Proof. First, it is important to note that each step η_n is independent of X_k with $k \in \{0, \dots, n-1\}$. We denote the step we take at time n with σ_n. Thus, we get:

$$\begin{aligned}\mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_0 = x_0] &= \mathbb{P}[X_{n+1} = x_n + \sigma_{n+1} \mid X_n = x_n, \dots, X_0 = x_0] \\ &= \mathbb{P}[X_{n+1} = x_n + \sigma_{n+1} \mid X_n = x_n] \\ &= \mathbb{P}[\eta_{n+1} = \sigma_{n+1} \mid X_n = x_n] = \mathbb{P}[\eta_{n+1} = \sigma_{n+1}],\end{aligned} \tag{3.6}$$

where the homogeneity property follows from the fact that $P[\eta_{n+1} = \sigma_{n+1}]$ is independent from n, which completes the proof. □

While the Markov property has plenty of interesting consequences, we will not go into much more detail about Markov chains. However, more information can for example be found in Grimmett and Welsh (2014).

3.2.2. Expected value

Next, we look at the expected value of the random walk. The reason we look at the expected value of the random walk is because it gives us a way to analyze the behavior of the average of the random walk, which gives us insight into whether or not the random walk will drift away from the origin and to which direction it will drift to. Furthermore, it will be useful later on when discussing the variance of the random walk.

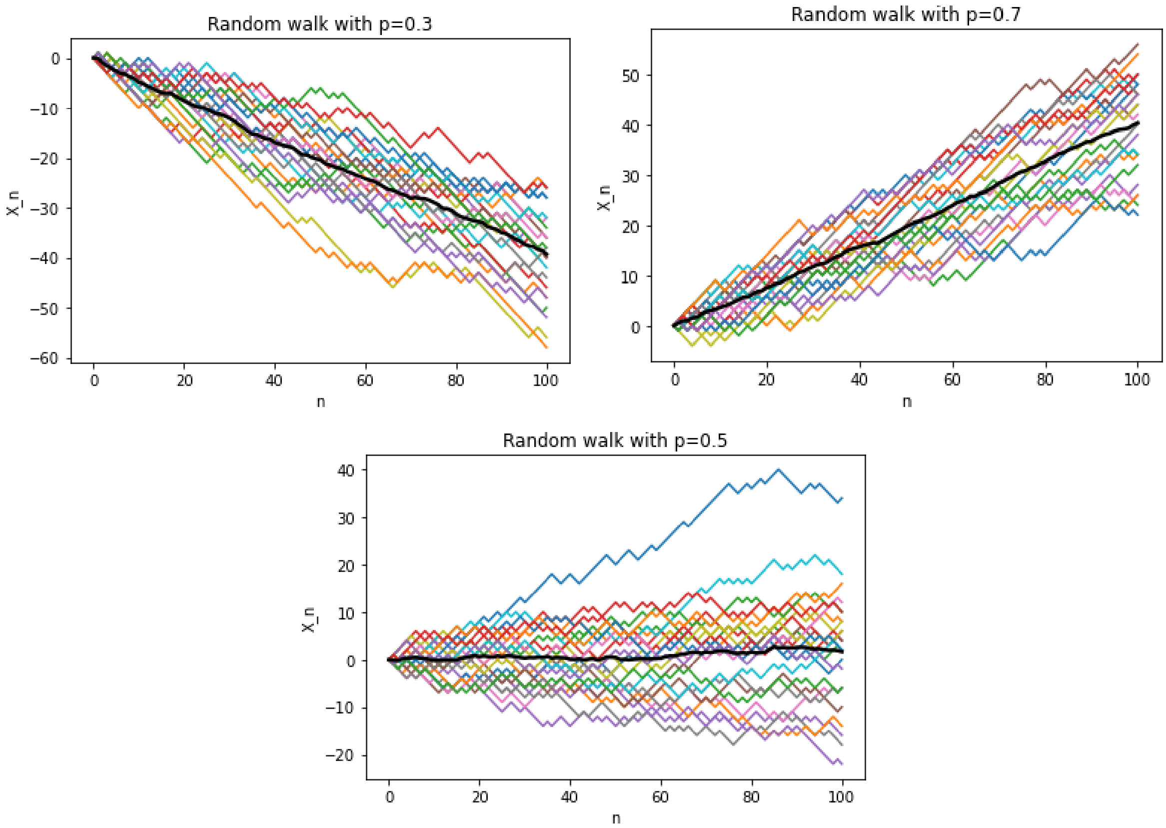

Figure 3.2: Visualization of 25 iterations of the random walk for $p = 0.3$, $p = 0.5$ and $p = 0.7$. The mean value at each step is shown with a black line.

In Figure 3.2, 25 iterations for the random walk are shown for $p = 0.3$, $p = 0.5$ and $p = 0.7$, along with the sample mean value of each step illustrated with a black line. As can be seen from Figure 3.2, the random walk X_n tends to go to negative values for $p = 0.3$ and to positive values for $p = 0.7$. For $p = 0.5$, the random walk tends to stay around 0. Intuitively, this can be explained by the fact that for $p = 0.3$, each step has a higher probability to be negative than positive, while for $p = 0.7$ each step has a higher probability to be positive than negative. The behavior for $p = 0.5$ can be explained by the fact that the positive and negative step have equal chance of happening. Proposition 3.2.3 formally states this observation:

Proposition 3.2.3. Let $(X_n)_{n \geq 0}$ be a random walk with $p \in [0, 1]$. Then, the expected value of X_n is equal to

$$\mathbb{E}[X_n] = n(2p - 1). \tag{3.7}$$

Proof. To find the expected value of X_n, it is important to note that $X_n = \sum_{k=1}^{n} \eta_k$. Thus, we must first find the expected value of each step, as the expected value of X_n follows by taking the expectation on both sides. To that end, note that for each step η_k, it holds that

$$\begin{aligned}\mathbb{E}[\eta_k] &= \mathbb{P}[\eta_k = 1] - \mathbb{P}[\eta_k = -1] \\ &= p - (1-p) \\ &= 2p - 1.\end{aligned} \tag{3.8}$$

Thus, we can find the expected value of X_n by taking the expected value on both sides of the equation:

$$\mathbb{E}[X_n] = \mathbb{E}\left[\sum_{k=1}^{n} \eta_k\right] = \sum_{k=1}^{n} \mathbb{E}[\eta_k] = n(2p-1), \tag{3.9}$$

as desired. □

3.2.3. Second moment of displacement

Besides the expected displacement $\mathbb{E}[X_n]$, we are also interested in the second moment of displacement X_n^2. The reason for this, is that the expected value of the random walk does not give information about how much the random walk is expected to deviate from its expected value. In addition to this, the second moment of displacement allows us to calculate the variance of the random walk, which makes us able to make statements about whether or not it is a diffusive process. In Figure 3.3, 25 iterations of the second moment of displacement of the random walk are shown for $p = 0.3$, $p = 0.5$ and $p = 0.7$, similarly to figure 3.1, along with the sample mean value of all the iterations at each step with a black line.

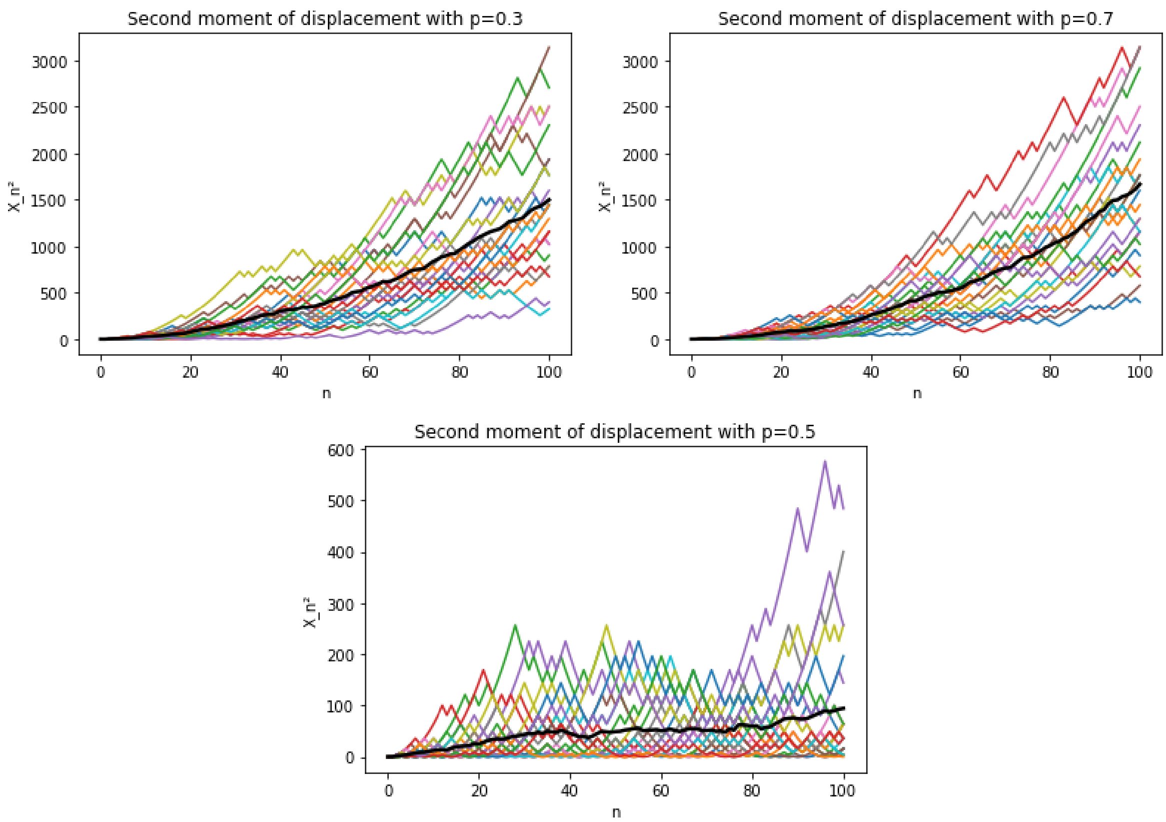

Figure 3.3: Visualisation of the second moment of displacement of 25 iterations of the random walk for $p = 0.3$, $p = 0.5$ and $p = 0.7$. The mean second moment of displacement at each step is shown with a black line.

When looking at the second moment of displacement of $p = 0.3$ and $p = 0.7$, it is important to note that the mean second moment of displacement for both figures are similar, and that the mean second moment of displacement seems to increase quadratically for each step taken. Intuitively, this can be explained by the symmetrical nature of the random walk, where a random walk $(X_n)_{n\geq 0}$ with $p = 0.3$ is the same in distribution as a random walk $-(Y_n)_{n\geq 0}$ with $p = 0.7$, causing the expected second moment of displacement $\mathbb{E}[X_n^2] = \sum_{k\in\mathbb{Z}} k^2 \mathbb{P}[X_n = k]$ to be the same. Meanwhile, for $p = 0.5$ the mean second moment of displacement seems to increase linearly for each step taken. Theorem 3.2.4 confirms this observation.

Theorem 3.2.4. Let $(X_n)_{n\geq 0}$ be a random walk with $p \in [0, 1]$. Then, the expected second moment of displacement of X_n is equal to

$$\mathbb{E}[X_n^2] = n + (2p - 1)^2(n^2 - n). \tag{3.10}$$

Proof. First off, we note that using the recursive representation of X_n gives us

$$\begin{aligned} X_n^2 &= (X_{n-1} + \eta_n)^2 \\ &= X_{n-1}^2 + 2X_{n-1}\eta_n + \eta_n^2. \end{aligned} \tag{3.11}$$

Thus, taking the expectation on both sides we get that $\mathbb{E}[X_n^2] = \mathbb{E}[X_{n-1}]^2 + 2\mathbb{E}[X_{n-1}\eta_n] + \mathbb{E}[\eta_n^2]$. Note that $\mathbb{E}[\eta_n^2] = 1$ for all $n \geq 1$. Furthermore, due to each step being independent, we get

$$\begin{aligned} \mathbb{E}[X_{n-1}\eta_n] &= \mathbb{E}[X_{n-1}]\mathbb{E}[\eta_n] \\ &= (n-1)(2p-1)(2p-1) \\ &= (n-1)(2p-1)^2. \end{aligned} \tag{3.12}$$

Thus, we get the following:

$$\begin{aligned} \mathbb{E}[X_n^2] &= \mathbb{E}[X_{n-1}^2] + 2(n-1)(2p-1)^2 + 1 \\ &= \mathbb{E}[X_1^2] + \sum_{k=1}^{n-1} \left[2k(2p-1)^2 + 1\right] \\ &= 1 + (n-1) + 2(2p-1)^2 \sum_{k=1}^{n-1} k \\ &= n + (2p-1)^2(n^2 - n), \end{aligned} \tag{3.13}$$

as desired. □

Remark 3.2.5. As a result of Theorem 3.2.3 and Theorem 3.2.4, we find the variance of X_n:

$$\begin{aligned} \text{Var}(X_n) &= \mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2 \\ &= n + (2p-1)^2(n^2 - n) - (n(2p-1))^2 \\ &= n(1 - (2p-1)^2). \end{aligned} \tag{3.14}$$

As can be seen from Remark 3.2.5, the variance of the random walk increases linearly as n increases[1], where the choice of p influences how quickly the variance grows. Values of p closer to $p = \frac{1}{2}$ have a faster growing variance than values of p closer to zero or one. As a result, the random walk is known as a diffusive process, which are characterized by a linearly increasing variance. Diffusion processes in which the variance is not linearly increasing are known as anomalous diffusion. An example of such a diffusion process will be discussed in Chapter 4.

[1]Unless $p \in \{0, 1\}$, in which case the variance is zero.

3.3. Law of large numbers and central limit theorem

Aside from the expected value of the random walk, it is also important to study the long term behavior of the random walk. To that end, we look at how the random walk behaves for larger step numbers, and how the distribution of multiple random walks of the same length look like. Luckily, each step of the random walk is independent and identically distributed. As a result, we can use the strong law of large numbers and central limit theorem and apply them to the random walk, both of which can be found in Appendix A. Applying these theorems to the random walk, we obtain Corollary 3.3.1 and Corollary 3.3.2:

Corollary 3.3.1 (Law of large numbers for random walks)**.** Let $(X_n)_{n\geq 0}$ be a random walk with $p \in [0,1]$. Then:

$$\frac{X_n - \mathbb{E}[X_n]}{n} = \frac{X_n - n(2p-1)}{n} \to 0 \text{ a.s.} \tag{3.15}$$

Corollary 3.3.2 (Central limit theorem for random walks)**.** . Let $(X_n)_{n\geq 0}$ be a random walk with $p \in (0,1)$. Then:

$$\frac{X_n - \mathbb{E}[X_n]}{\sigma\sqrt{n}} = \frac{X_n - n(2p-1)}{\sqrt{n(1-(2p-1)^2)}} \xrightarrow{d} \mathcal{N}(0,1), \tag{3.16}$$

where $\sigma = \sqrt{\text{var}(\eta)}$ is the standard deviation.

Remark 3.3.3. It is of note that in Theorem 3.3.2, we leave out the case $p \in \{0,1\}$. The reason for this, is that the central limit theorem cannot be applied, as the variance is equal to zero. However, in these two cases we know that $X_n = n$ for $p = 1$ and $X_n = -n$ for $p = 0$, making the use of Theorem 3.3.2 unnecessary.

These two corollaries give us important information about the behavior of the random walk: Corollary 3.3.1 states that the random walk will tend towards the expected value of the random walk as the number of steps go to infinity, that is $\frac{X_n}{n} \to 2p-1$ almost surely, and that the limit is non-random as a result. Corollary 3.3.2 states that distribution of the shifted random walk will converge in distribution to the normal distribution, which gives us a way of predicting where a random walk will end up at after a given number of steps.

Proof of Corollary 3.3.1. Since each step is independent and identically distributed and $\mathbb{E}[\eta] < \infty$, we can use the strong law of large numbers (Theorem A.2.3) to get

$$\frac{X_n - n\mathbb{E}[\eta]}{n} \to 0 \text{ a.s.} \tag{3.17}$$

Furthermore, note that $n\mathbb{E}[\eta] = n(2p-1) = \mathbb{E}[X_n]$, thus we get that

$$\frac{X_n - \mathbb{E}[X_n]}{n} = \frac{X_n - n(2p-1)}{n} \to 0 \text{ a.s.} \tag{3.18}$$

as desired. □

Proof of Corollary 3.3.2. Since each step is independent and identically distributed with finite mean $\mathbb{E}[\eta] < \infty$ and non-zero variance Var$(\eta) = 1-(2p-1)^2$, we can use the central limit theorem (Theorem A.2.4) to get:

$$\frac{X_n - n\mathbb{E}[\eta]}{\sigma\sqrt{n}} = \frac{X_n - n(2p-1)}{\sqrt{n(1-(2p-1)^2)}} \xrightarrow{d} \mathcal{N}(0,1), \tag{3.19}$$

as desired. □

3.4. Gambler's ruin

In the last section of this chapter, we will discuss the gambler's ruin problem, which is based on the material on the gambler's ruin presented in Grimmett and Welsh (2014). The gambler's ruin problem is as follows: two players, player A and B, play a game. Player A starts with a euros, and player B starts with $N-a$ euros. They play the following game: player A and player B repeatedly flip a not necessarily fair coin that comes up heads with probability p and tails with probability $1-p$. If the coin comes up heads, player B gives player A one euro. If the coin comes up tails, player A gives player B one euro instead. This continues until either player A or player B has zero euros and is "ruined".

We can transform this game into a random walk by only looking at the money of player A. Doing so, we can see that it is a random walk on the numbers $\{0,\ldots,N\}$, where 0 and N are called "absorbing barriers": the random walk stays at 0 or N if it lands there. Interestingly, the chance that this game continues forever is zero. If we say that heads correspond to a step value of 1 and tails correspond to a step value of -1, then the chance of A winning with starting capital of a euros corresponds to the chance $v(a)$ of the random walk being absorbed at N from starting point a. To that end, we can describe the chance of player A winning with Theorem 3.4.1:

Theorem 3.4.1 (Gambler's Ruin)**.** Let $(X_n)_{n\geq 0}$ be a random walk with $p \in [0,1]$ and starting point $a \in \{0,\ldots,N\}$, where 0 and N are absorbing barriers. Then, the probability $v(a)$ of the random walk being absorbed at N is equal to

$$v(a) = \begin{cases} \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^N - 1} & \text{if } p \neq \frac{1}{2}, \\ \frac{a}{N} & \text{if } p = \frac{1}{2}. \end{cases} \tag{3.20}$$

Proof of Theorem 3.4.1. To begin, we condition $v(a)$ on the result of the first step taken, namely the step being equal to 1 or -1. This results in the following:

$$\begin{aligned} \mathbb{P}[\text{A wins}] &= \mathbb{P}[\text{A wins} \mid \eta_1 = 1]\mathbb{P}[\eta_1 = 1] + \mathbb{P}[\text{A wins} \mid \eta_1 = -1]\mathbb{P}[\eta_1 = -1], \\ \Rightarrow v(a) &= v(a+1)p + v(a-1)(1-p) \\ v(a+1)p - v(a) &+ v(a-1)(1-p) = 0, \end{aligned} \tag{3.21}$$

where we use the fact that the chance of A winning given that the first step is known is the same as looking at the chance of A winning from the position after the first step. As a result, we find a difference equation, which can be solved with the following auxiliary equation with the following boundary conditions[2]:

$$p\theta^2 - \theta + (1-p) = 0, \quad v(0) = 0, \quad v(N) = 1, \tag{3.22}$$

where we derive the boundary condition by noting that the chance for player A to win with no money is equal to zero, and that the chance for player A to win with the total capital is equal to one. Solving for the roots, we find:

$$\begin{aligned} \theta_{1,2} &= \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm |2p-1|}{2p}, \\ \Rightarrow \theta_{1,2} &= \begin{cases} \{1, \frac{1-p}{p}\} & \text{if } p \neq \frac{1}{2}, \\ \{1, 1\} & \text{if } p = \frac{1}{2}. \end{cases} \end{aligned} \tag{3.23}$$

Thus, we find that:

$$v(a) = \begin{cases} c_1 + c_2\left(\frac{1-p}{p}\right)^a & \text{if } p \neq \frac{1}{2}, \\ c_3 + c_4 a & \text{if } p = \frac{1}{2}, \end{cases} \tag{3.24}$$

[2]For more info about difference equations one may look at Grimmett and Welsh (2014).

where c_1, c_2, c_3, c_4 are constants. We determine these constants using the boundary conditions. For $p \neq \frac{1}{2}$, we get

$$\begin{aligned} v(0) &= c_1 + c_2 = 0, \\ v(N) &= c_1 + c_2 \left(\frac{1-p}{p}\right)^N = 1, \\ \Rightarrow c_2 &\left(\left(\frac{1-p}{p}\right)^N - 1\right) = 1 \\ c_2 &= \frac{1}{\left(\frac{1-p}{p}\right)^N - 1}, \\ \Rightarrow c_1 &= \frac{-1}{\left(\frac{1-p}{p}\right)^N - 1}. \end{aligned} \tag{3.25}$$

Thus, we find

$$v(a) = \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^N - 1}. \tag{3.26}$$

For $p = \frac{1}{2}$, we will do similar calculations:

$$\begin{aligned} v(0) &= c_3 = 0, \\ v(N) &= c_3 + c_4 N = 1, \\ \Rightarrow c_4 &= \frac{1}{N}. \end{aligned} \tag{3.27}$$

Thus, we find

$$v(a) = \frac{a}{N}. \tag{3.28}$$

Combining the two results, we get

$$v(a) = \begin{cases} \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^N - 1} & \text{if } p \neq \frac{1}{2}, \\ \frac{a}{N} & \text{if } p = \frac{1}{2}, \end{cases} \tag{3.29}$$

as desired. □

In Figure 3.4, $v(a)$ is given for various probabilities p and starting points a, as well as for different total capitals N. When looking at Figure 3.4, we can roughly see what we expect: for $p = 1/2$, we find that the chance of a winning increases linearly. For $p < 1/2$, we see that the chance of player A winning increases exponentially as a increases. Besides the parameter p, the total capital N also plays a role in $v(a)$: if the total capital is low, then the probability p plays a moderate role when it comes to $v(a)$. However, if the starting capital is high, then changes of the probability p result in more ”extreme changes” in the chances of winning for player A.

We can also find the expected number of steps the random walk takes until it reaches either zero or N. This is given in Theorem 3.4.2:

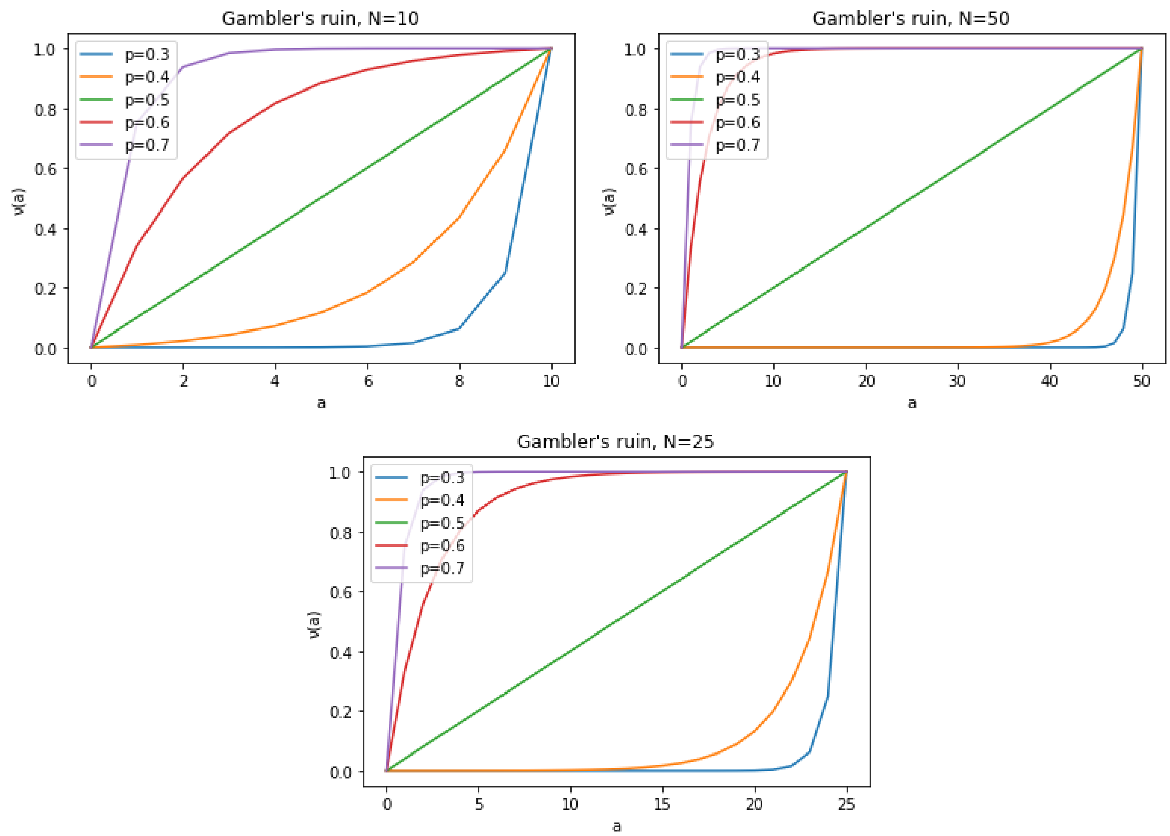

Figure 3.4: Chance of player A to win for different total capitals N

Theorem 3.4.2. Let $(X_n)_{n\geq 0}$ be a random walk with $p \in [0,1]$ and starting point $a \in \{0,\dots,N\}$, with 0 and N absorbing barriers. Then, the expected number of steps $e(a)$ until the walk gets absorbed at 0 or N is given by

$$e(a) = \begin{cases} \frac{1}{2p-1}\left(N\frac{((1-p)/p)^a-1}{((1-p)/p)^N-1} - a\right) & \text{if } p \neq 1/2, \\ a(N-a) & \text{if } p = 1/2. \end{cases} \tag{3.30}$$

Proof. We use a similar approach as the proof of Theorem 3.4.1. To that end, let F be the number of steps taken until the random walk gets absorbed at 0 or N. By conditioning the expectation of F on the result of the first step, we get:

$$\begin{aligned} \mathbb{E}[F] &= \mathbb{E}[F \mid \eta_1 = 1]\mathbb{P}[\eta_1 = 1] + \mathbb{E}[F \mid \eta_1 = -1]\mathbb{P}[\eta_1 = -1] \\ \Rightarrow e(a) &= p(1 + e(a+1)) + (1-p)(1 + e(a-1)). \end{aligned} \tag{3.31}$$

Here, we use the fact that the expected number of steps of the random walk is equal to the expected number of steps given we started after the first step is taken, plus the step that is already taken. As a result, we get the following equation with the boundary conditions:

$$pe(a+1) - e(a) + (1-p)e(a-1) = -1, \quad e(0) = e(N) = 0, \tag{3.32}$$

where we note that the expected number of steps is zero if either player A or player B starts with the total capital. We solve this by solving for the homogeneous and particular solution. For the homogeneous solution $e_h(a)$, we find the following auxiliary equation:

$$\begin{aligned} &p\theta^2 - \theta + (1-p) = 0, \\ &\Rightarrow e_h(a) = \begin{cases} c_1 + c_2\left(\frac{1-p}{p}\right)^a & \text{if } p \neq \frac{1}{2}, \\ c_1 + c_2 a & \text{if } p = \frac{1}{2}. \end{cases} \end{aligned} \tag{3.33}$$

Here, we use the fact that we have already solved this equation in the proof of Theorem 3.4.1. Next, we look at the particular solution. We split this into two cases based on the value of p:

- If $p \neq 1/2$, we try substituting $e_p(a) = ka$ with $k \in \mathbb{R}$ into Equation (3.32). Thus, we get:

$$\begin{aligned} pk(a+1) - ka + (1-p)k(a-1) &= -1 \\ pka + pk - ka + ka - pka - k + pk &= -1 \\ (2p-1)k &= -1 \\ k &= \frac{1}{1-2p}. \end{aligned} \tag{3.34}$$

- If $p = 1/2$, we try substituting $e_p(a) = ka^2$ with $k \in \mathbb{R}$ into Equation (3.32). Thus, we get:

$$\begin{aligned} \frac{1}{2}k(a+1)^2 - ka^2 + \frac{1}{2}k(a-1)^2 &= -1 \\ \frac{1}{2}k\left(a^2+2a+1\right) - ka^2 + \frac{1}{2}k\left(a^2-2a+1\right) &= -1 \\ \left(\frac{1}{2}k - k + k - \frac{1}{2}k\right)a^2 + (k-k)\,a + 2k &= -1 \\ k &= -1. \end{aligned} \tag{3.35}$$

As a result, we get the following equation:

$$e(a) = \begin{cases} c_1 + c_2\left(\frac{1-p}{p}\right)^a + \frac{a}{1-2p} & \text{if } p \neq 1/2, \\ c_3 + c_4 a - a^2 & \text{if } p = 1/2. \end{cases} \tag{3.36}$$

We can solve for the constants using the boundary conditions:

- For $p \neq 1/2$ we get the following:

$$\begin{aligned} e(0) &= c_1 + c_2 = 0 \\ e(N) &= c_1 + c_2\left(\frac{1-p}{p}\right)^N + \frac{N}{1-2p} = 0, \\ \Rightarrow c_2&\left(\left(\frac{1-p}{p}\right)^N - 1\right) = -\frac{N}{1-2p} \\ c_2 &= \frac{N}{2p-1}\frac{1}{\left(\frac{1-p}{p}\right)^N - 1}, \\ \Rightarrow c_1 &= -\frac{N}{2p-1}\frac{1}{\left(\frac{1-p}{p}\right)^N - 1} \end{aligned} \tag{3.37}$$

- For $p = 1/2$ we get the following:

$$\begin{aligned} e(0) &= c_3 = 0 \\ e(N) &= c_4 N - N^2 = 0, \\ \Rightarrow c_4 &= \frac{1}{N} \end{aligned} \tag{3.38}$$

Finally, we get the value of $e(a)$:

$$e(a) = \begin{cases} \frac{1}{2p-1}\left(N\frac{((1-p)/p)^a-1}{((1-p)/p)^N-1} - a\right) & \text{if } p \neq 1/2, \\ a(N-a) & \text{if } p = 1/2, \end{cases} \tag{3.39}$$

as desired. □

In Figure 3.5, the expected number of steps is given for different total capitals $N = 10, N = 25$ and $N = 50$. In general, we see that the maximum expected number of steps for a given value of p shifts to the left as p increases and to the right as p decreases. Intuitively, this can be explained by the fact that the gambler's ruin tends to be absorbed at zero for lower values of p, which is why the maximum expected number of steps for lower values of p shifts to the right, as the walks that go to the left will take more steps. Furthermore, we can see that the the maximum expected number of steps for a given p increases as it approaches $p = 1/2$, and that the proportional difference between the maximum expected value for $p = 1/2$ and other values of p increases as the number of steps increases. This can be explained by the fact that the gambler's ruin with value of p close to zero or one tend to go to zero or N faster than gambler's ruins with values of p close to $1/2$, resulting a in lower maximum expected number of steps. Lastly, one can see that the total capital does not influence the general shape of the graphs, but the effect of N on the maximum expected steps is greater near $p = 0.5$.

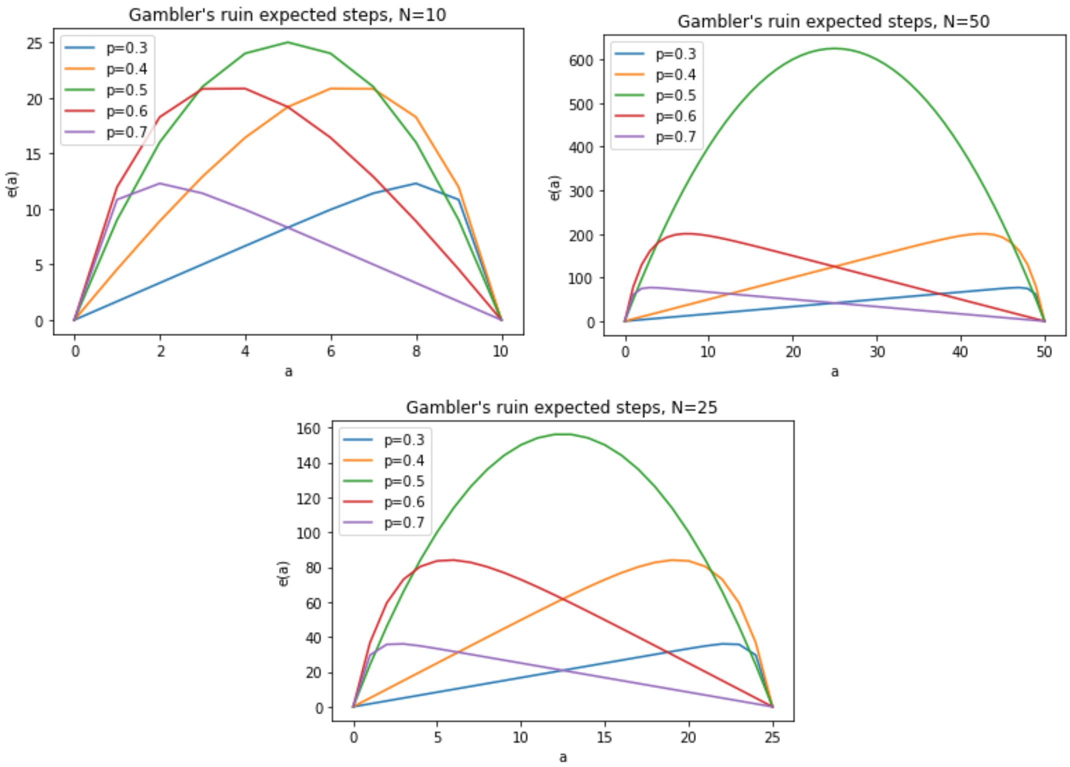

Figure 3.5: Expected number of steps until either player A or player B wins for different starting capitals N. Note that the scale of the y-axis increases as N increases.

4

Elephant random walk

In this chapter, we will discuss the elephant random walk, and we will discuss several powerful results of the elephant random walk, such as the law of large numbers and the central limit theorem. Lastly, we will discuss the gambler's ruin problem for the elephant random walk. Note that the content of this chapter is largely based on published results by Schütz and Trimper (2004) and Colletti et al. (2016).

4.1. The model

In this section, we will discuss the general model of the elephant random walk. In particular, we will discuss the one-dimensional elephant random walk on $\mathbb{Z}$. In summary, the elephant random walk is a random walk with complete memory: similarly to the normal random walk, the elephant random walk consists of a sum of steps from a certain starting point X_0. However, the way each step is taken differs from the normal random walk: while the first step is the same, later steps are dependent on the previous steps taken. This is unlike the normal random walk, where each step was independent and identically distributed. The elephant random walk is defined as follows: the walk starts at a point $X_0 \in \mathbb{Z}$ at time $n = 0$. At each discrete time step, the elephant decides to move a step to the left or to the right, which is given the value of -1 and 1 respectively. This gives rise to two ways to express X_n, similar to the random walk:

$$X_n = X_{n-1} + \eta_n = X_0 + \sum_{k=1}^{n} \eta_k, \tag{4.1}$$

where X_n denotes the location of the elephant at time n and $\eta_n = \pm 1$ is a random variable which denotes the step taken at time n. For $p, q \in [0,1]$, which we will call the memory parameter and initial parameter respectively, the value of η_{n+1} is determined as follows:

- η_1 is a random variable with

$$\begin{aligned} \mathbb{P}[\eta_1 = 1] &= q, \\ \mathbb{P}[\eta_1 = -1] &= 1 - q. \end{aligned} \tag{4.2}$$

- For η_{n+1} with $n \neq 0$ we first choose a previous step $\eta' \in \{\eta_1, \eta_2, \ldots, \eta_n\}$ with uniform probability. Then, η_{n+1} is a random variable with

$$\begin{aligned} \mathbb{P}[\eta_{n+1} = \eta'] &= p, \\ \mathbb{P}[\eta_{n+1} = -\eta'] &= 1 - p. \end{aligned} \tag{4.3}$$

In Figure 4.1 and 4.2, the process of the elephant random walk is shown for $n = 1$ and for $n > 1$.

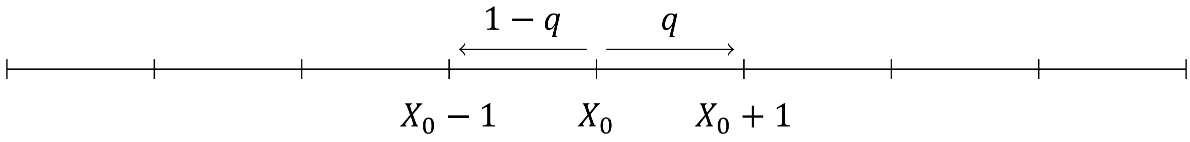

Figure 4.1: Illustration of the elephant random walk for $n = 1$.

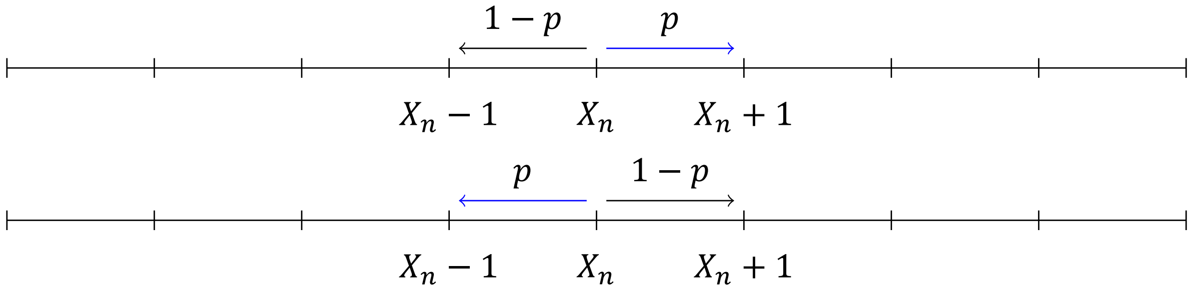

Figure 4.2: Illustration of the elephant random walk for $n > 1$. The blue arrow represents the direction of the chosen previous step.

Similarly to the random walk, we will from now on assume that $X_0 = 0$ for simplicity, as results for $X_0 \in \mathbb{Z}$ can be obtained by shifting the results of the elephant random walk with $X_0 = 0$.

4.2. Step distribution and expectation

In this section, we will discuss the probability function of each step in the elephant random walk, as well as the mean increment and mean displacement. Lastly, we will discuss the second moment of displacement for the elephant random walk.

4.2.1. Step distribution

Before we can discuss the expected displacement of the elephant random walk, we must first discuss the probability distribution of each step. It is too complex to write the probability distribution of each step, due to the fact that each step is based of a random previous step. However, it is possible to give a conditional based probability density function for each step, where we condition on the previous steps of the walk. This gives rise to Theorem 4.2.1:

Theorem 4.2.1. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0, 1]$, and let $\eta \in \{-1, 1\}$. For the first step, it holds that

$$\mathbb{P}[\eta_1 = \eta] = \frac{1}{2}\left[1 + (2q-1)\eta\right]. \tag{4.4}$$

Furthermore, the conditional probability of a step η_{n+1} given a previous history $\{\eta_1, \dots, \eta_n\}$ is given by

$$\mathbb{P}[\eta_{n+1} = \eta \mid \eta_1, \dots, \eta_n] = \frac{1}{2n}\sum_{k=1}^{n}\left[1 + (2p-1)\eta_k\eta\right]. \tag{4.5}$$

It may be confusing as to why we write the conditional probability this way. The reason for this, is that we are able to write the probability of a step using the same formula, where the probability of a step with value -1 or 1 only differ by changing the sign within the formula as seen in equation (4.4). As a result, we are able to take the sum over all steps as seen in equation (4.5), as we don't have to split the probability into multiple cases based on the step chosen. This will also be useful in keeping the computations clear later in this chapter.

Proof of Theorem 4.2.1. The first part of Theorem 4.2.1 follows from the fact that we can write the two cases, $\eta_1 = 1$ and $\eta_1 = -1$, as the following:

$$\begin{aligned} \mathbb{P}[\eta_1 = 1] = q &= \frac{1}{2}\left[1 + (2q-1)(1))\right] \\ \mathbb{P}[\eta_1 = -1] = 1 - q &= \frac{1}{2}\left[1 + (2q-1)(-1)\right], \\ \Rightarrow \mathbb{P}[\eta_1 = \eta] &= \frac{1}{2}\left[1 + (2q-1)\eta\right]. \end{aligned} \tag{4.6}$$

For the second part of theorem 4.2.1, we can use the law of total probability (Theorem A.2.1) to condition on the chosen previous step η'. In other words,

$$\mathbb{P}[\eta_{n+1} = \eta] = \sum_{k=1}^{n} \mathbb{P}[\eta' = \eta_k]\mathbb{P}[\eta_{n+1} = \eta \mid \eta' = \eta_k]. \tag{4.7}$$

Since each previous step has uniform chance of being chosen, we get that $\mathbb{P}[\eta' = \eta_k] = \frac{1}{n}$ for all $k \in \{1, \dots, n\}$. To determine $\mathbb{P}[\eta_{n+1} = \eta \mid \eta' = \eta_k]$, we can split the conditional probability into two cases where $\eta, \eta_k \in \{-1, 1\}$:

1. $\eta_k\eta = 1 : \mathbb{P}[\eta_{n+1} = \eta \mid \eta' = \eta_k] = p = \frac{1}{2}\left[1 + (2p-1)(1)\right] = \frac{1}{2}\left[1 + (2p-1)\eta_k\eta\right],$
2. $\eta_k\eta = -1 : \mathbb{P}[\eta_{n+1} = \eta \mid \eta' = \eta_k] = 1 - p = \frac{1}{2}\left[1 + (2p-1)(-1)\right] = \frac{1}{2}\left[1 + (2p-1)\eta_k\eta\right].$

As a result, we obtain that $\mathbb{P}[\eta_{n+1} = \eta \mid \eta' = \eta_k] = \frac{1}{2}\left[1 + (2p-1)\eta_k\eta\right]$. Combining both parts, we get

$$\begin{aligned} \mathbb{P}[\eta_{n+1} = \eta] &= \sum_{k=1}^{n} \mathbb{P}[\eta' = \eta_k]\mathbb{P}[\eta_{n+1} = \eta \mid \eta' = \eta_k] \\ &= \sum_{k=1}^{n} \frac{1}{n}\frac{1}{2}\left[1 + (2p-1)\eta_k\eta\right] \\ &= \frac{1}{2n}\sum_{k=1}^{n}\left[1 + (2p-1)\eta_k\eta\right], \end{aligned} \tag{4.8}$$

as desired. □

4.2.2. Markov chain

In Section 3.2.1 of Chapter 3, we looked at the Markov property for the random walk. It turns out that the Markov property still holds for the elephant random walk, but that the walk is time inhomogeneous. Intuitively, this can be explained by the fact that it does not matter how the steps are taken, but instead it matters how many steps there are of values -1 and 1. The inhomogeneous part stems from the fact that the previous steps play a role in the probability of taking a step with value -1 or 1. We get the following theorem:

Theorem 4.2.2. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0,1], p \neq 1/2$. Then, the elephant random walk is a inhomogeneous Markov chain.

Proof. Just like in Proposition 3.2.2 we define σ_n to be the step at time n. We begin the proof by noting that we can rewrite the conditional property as follows:

$$\begin{aligned}\mathbb{P}[X_{n+1} = x_{n+1} \,|\, X_n = x_n, \dots, X_1 = x_1] &= \mathbb{P}[X_{n+1} = x_n + \sigma_{n+1} \,|\, X_n = x_n, \dots, X_1 = x_1] \\ &= \mathbb{P}[X_{n+1} = x_n + \sigma_{n+1} \,|\, X_n = x_n, \dots, X_1 = x_1] \\ &= \mathbb{P}[\eta_{n+1} = \sigma_{n+1} \,|\, X_n = x_n, \dots, X_1 = x_1].\end{aligned} \tag{4.9}$$

Thus, we need to show that $\mathbb{P}[\eta_{n+1} = \sigma_{n+1} \,|\, X_n = x_n, \dots, X_1 = x_1] = \mathbb{P}[\eta_{n+1} = \sigma_{n+1} \,|\, X_n = x_n]$. To that end, note that if $X_n = x_n$, then $(n + x_n)/2$ steps have to be steps with value 1, and $(n - x_n)/2$ steps have to be steps with value -1. As a result, we immediately find that the equation holds, with

$$\mathbb{P}[\eta_{n+1} = \sigma_{n+1} \,|\, X_n = x_n] = \begin{cases} p\frac{n+x_n}{2n} + (1-p)\frac{n-x_n}{2n} & \eta'_{n+1} = 1, \\ (1-p)\frac{n+x_n}{2n} + p\frac{n-x_n}{2n} & \eta'_{n+1} = -1 \end{cases} \tag{4.10}$$

for $n \geq 1$. To prove the inhomogeneity of the elephant random walk, we look at the following three conditional probabilities:

$$\mathbb{P}[X_2 = 2 \,|\, X_1 = 1] = p\frac{1+1}{2} + (1-p)\frac{1-1}{2} = p, \tag{4.11}$$

$$\mathbb{P}[X_4 = 2 \,|\, X_3 = 1] = p\frac{3+1}{6} + (1-p)\frac{3-1}{6} = \frac{1}{3}(1+p), \tag{4.12}$$

$$\mathbb{P}[X_6 = 2 \,|\, X_5 = 1] = p\frac{5+1}{10} + (1-p)\frac{5-1}{10} = \frac{1}{5}(2+p). \tag{4.13}$$

These three equations cannot all be satisfied simultaneously for a given p, which shows that the elephant random walk is not homogeneous, as desired. □

Remark 4.2.3. The reason we exclude the case $p = 1/2$ is due to the fact that the elephant random walk reverts to the normal random walk if p is equal to $1/2$, of which we know that this case would be a homogeneous markov chain because of Proposition 3.2.2.

4.2.3. Expectation

Using the probabilities calculated in subsection 4.2.1, we can now discuss the expected values of each step, as well as the expected value of the elephant random walk at each time step. For ease of reading, we define $\alpha \coloneqq 2p - 1$ from now on. To begin, we first look at the expected value of each step:

Theorem 4.2.4. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0, 1]$. The conditional expected value of a step η_{n+1} given a previous history $\{\eta_1, \dots, \eta_n\}$ is given by

$$\mathbb{E}[\eta_{n+1} \,|\, \eta_1, \dots \eta_n] = \frac{\alpha}{n} X_n. \tag{4.14}$$

Furthermore, the expected value of a step η_{n+1} is given by

$$\mathbb{E}[\eta_{n+1}] = \frac{\alpha}{n} \mathbb{E}[X_n]. \tag{4.15}$$

Proof. Using the definition of expectation in combination with Theorem 4.2.1, we get

$$\begin{aligned}\mathbb{E}[\eta_{n+1} \mid \eta_1, \dots \eta_n] &= \mathbb{P}[\eta_{n+1} = 1 \mid \eta_1, \dots, \eta_n] - \mathbb{P}[\eta_{n+1} = -1 \mid \eta_1, \dots, \eta_n] \\ &= \frac{1}{2n}\sum_{k=1}^{n}[1 + (2p-1)\eta_k] - \frac{1}{2n}\sum_{k=1}^{n}[1 - (2p-1)\eta_k] \\ &= \frac{1}{2n}\sum_{k=1}^{n}[2(2p-1)\eta_k] \\ &= \frac{2p-1}{n}\sum_{k=1}^{n}\eta_k = \frac{\alpha}{n}X_n.\end{aligned} \tag{4.16}$$

By taking the expectation of both sides and using the law of total expectation (Theorem A.2.2), we obtain

$$\mathbb{E}[\eta_{n+1}] = \mathbb{E}[\mathbb{E}[\eta_{n+1} \mid \eta_1, \dots \eta_n]] = \mathbb{E}\left[\frac{\alpha}{n}X_n\right] = \frac{\alpha}{n}\mathbb{E}[X_n], \tag{4.17}$$

as desired. □

Since we now know the conditional expected value of each step, we can now discuss the expected value of X_n at each time step n. While we are not able to obtain a direct formula for $\mathbb{E}[X_n]$ using just Theorem 4.2.4, Theorem 4.2.4 does allow us to find a recursive formula for the expected value by using the fact that $X_{n+1} = X_n + \eta_{n+1}$:

Theorem 4.2.5. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0, 1]$. For $n \geq 1$, the conditional expected value of X_{n+1} given a previous history $\{\eta_1, \dots, \eta_n\}$ is given by

$$\mathbb{E}[X_{n+1} \mid \eta_1, \dots, \eta_n] = \left(1 + \frac{\alpha}{n}\right)X_n. \tag{4.18}$$

Furthermore, the expected value of X_{n+1} is given by

$$\mathbb{E}[X_{n+1}] = \left(1 + \frac{\alpha}{n}\right)\mathbb{E}[X_n]. \tag{4.19}$$

Proof. We begin the proof by noting that $X_{n+1} = X_n + \eta_{n+1}$. By taking the expectation on both sides and conditioning on the previous history, we get the following by using Theorem 4.2.4:

$$\begin{aligned}\mathbb{E}[X_{n+1} \mid \eta_1, \dots, \eta_n] &= \mathbb{E}[X_n + \eta_{n+1} \mid \eta_1, \dots, \eta_n] \\ &= \mathbb{E}[X_n \mid \eta_1, \dots, \eta_n] + \mathbb{E}[\eta_{n+1} \mid \eta_1, \dots, \eta_n] \\ &= X_n + \frac{\alpha}{n}X_n \\ &= \left(1 + \frac{\alpha}{n}\right)X_n.\end{aligned} \tag{4.20}$$

By taking the expectation of both sides and using the law of total expectation, we obtain

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[\mathbb{E}[X_{n+1} | \eta_1, \dots \eta_n]] = \mathbb{E}\left[\left(1 + \frac{\alpha}{n}\right)X_n\right] = \left(1 + \frac{\alpha}{n}\right)\mathbb{E}[X_n], \tag{4.21}$$

as desired. □

Before we give a non-recursive formula for the expected displacement, we will discuss the value of $\prod_{k=1}^{n-1}(k + \alpha)$. The reason for this is that this term will show up frequently in the remainder of this section, as well as in Subsection 4.2.4. The goal is to simplify the product of terms as a fraction of gamma functions. To that end, we have Definition 4.2.6 and Proposition 4.2.8:

Definition 4.2.6. The **gamma function** is defined as follows:

$$\Gamma(x) = \int_0^1 t^{x-1}e^{-t}\,dt \tag{4.22}$$

Remark 4.2.7. The gamma function has three important properties which we will make us of in the remainder of the chapter:

- For every positive integer n, we have that $\Gamma(n) = (n-1)!$,
- We have that $\Gamma(x+1) = x\Gamma(x)$ for $x \in \mathbb{R} \setminus (\mathbb{Z}^- \cup \{0\})$,
- The gamma function is closely related to the beta function[1], which can be defined as follows using gamma functions:

$$B(x_1, x_2) = \frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1 + x_2)}. \tag{4.23}$$

Proposition 4.2.8. For $\alpha \in \mathbb{R} \setminus (\mathbb{Z}^- \cup \{0\})$, we have that

$$\prod_{k=1}^{n-1}(k+\alpha) = \frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)}. \tag{4.24}$$

Proof. Using the fact that $x\Gamma(x) = \Gamma(x+1)$, we get:

$$\begin{aligned}\prod_{k=1}^{n-1}(k+\alpha) &= (1+\alpha)(2+\alpha)\dots(n-1+\alpha)\\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)}(1+\alpha)(2+\alpha)\dots(n-1+\alpha)\\ &= \frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)},\end{aligned} \tag{4.25}$$

as desired. □

Using the recursive formula from Theorem 4.2.5, we can now describe the expected displacement of the elephant random walk by using Proposition 4.2.8:

Theorem 4.2.9. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0, 1]$. The expected displacement of X_{n+1} is given by

$$\mathbb{E}[X_n] = (2q-1)\frac{\Gamma(n+\alpha)}{\Gamma(2p)\Gamma(n)}. \tag{4.26}$$

Furthermore, for the asymptotic behavior, we obtain

$$\mathbb{E}[X_n] \sim (2q-1)\frac{n^{2p-1}}{\Gamma(2p)}. \tag{4.27}$$

[1]The beta function is a special function which is used to define the probability function of the beta distribution.

Proof. To obtain a non-recursive formula for the expected displacement, we will use Theorem 4.2.5. To that end, note that

$$
\begin{aligned}
\mathbb{E}[X_n] &= \left(1 + \frac{\alpha}{n-1}\right) \mathbb{E}[X_{n-1}] \\
&= \left(\frac{n+\alpha-1}{n-1}\right)\left(1 + \frac{\alpha}{n-2}\right) \mathbb{E}[X_{n-2}] \\
&= \left(\frac{n+\alpha-1}{n-1}\right)\left(\frac{n+\alpha-2}{n-2}\right) \dots \left(1 + \frac{\alpha}{1}\right) \mathbb{E}[X_1] \\
&= \mathbb{E}[X_1] \prod_{k=1}^{n-1} \left(\frac{k+\alpha}{k}\right) \\
&= \frac{\mathbb{E}[X_1]}{(n-1)!} \prod_{k=1}^{n-1} (k+\alpha) \\
&= \frac{\mathbb{E}[X_1]}{\Gamma(n)} \frac{\Gamma(n+\alpha)}{\Gamma(2p)} \\
&= (2q-1) \frac{\Gamma(n+\alpha)}{\Gamma(2p)\Gamma(n)}.
\end{aligned}
\tag{4.28}
$$

To obtain the asymptotic behavior of X_n, we use the asymptotic approximation $\Gamma(n+\alpha) \sim \Gamma(n)\, n^{\alpha}$ for $\alpha \in \mathbb{R}$ (Proposition A.1.2) to get

$$
\begin{aligned}
\mathbb{E}[X_n] &= (2q-1) \frac{\Gamma(n+\alpha)}{\Gamma(2p)\Gamma(n)} \\
&\sim (2q-1) \frac{\Gamma(n)\, n^{\alpha}}{\Gamma(2p)\Gamma(n)} \\
&= (2q-1) \frac{n^{2p-1}}{\Gamma(2p)},
\end{aligned}
\tag{4.29}
$$

as desired. □

Remark 4.2.10. By using both Theorem 4.2.4 and Theorem 4.2.9, we can find the expected value of each step:

$$
\mathbb{E}[\eta_{n+1}] = \frac{\alpha}{n} \mathbb{E}[X_n] \sim \frac{(2p-1)(2q-1)}{\Gamma(2p)} n^{2p-2}. \tag{4.30}
$$

As a result of Theorem 4.2.9, we see that the behavior of the displacement of the random walk can be split into 3 cases based on the value of p:

1. If $p < 1/2$, then we see that $2p - 1 < 0$, and thus

$$
\begin{aligned}
\mathbb{E}[X_n] &\sim (2q-1) \frac{n^{2p-1}}{\Gamma(2p)} \\
&= \frac{2q-1}{n^{1-2p}\Gamma(2p)} \to 0
\end{aligned}
\tag{4.31}
$$

   as $n \to \infty$. As a result, we see that the elephant random walk will stay around its starting point on average. Intuitively, this is what we expect: if the elephant random walk has a lot of step with value 1 proportionally, then the chance of the elephant random walk taking a step with value -1 becomes a lot larger, effectively ”correcting” itself to its starting point.

2. If $p = 1/2$, we end up with a usual random walk with a fair coin after one step, as each step has equal chance of happening regardless of the step chosen after the first one. As such, we find that the expected displacement of the elephant random walk is equal to $2q - 1$.

3. If $p > 1/2$, then we see that $0 < 2p - 1 \leq 1$. We again split into 3 cases depending on the value of q:

   (a) If $q < 1/2$, then
$$\mathbb{E}[X_n] \sim (2q-1)\frac{n^{2p-1}}{\Gamma(2p)} \to -\infty, \tag{4.32}$$

   (b) If $q = 1/2$, then
$$\mathbb{E}[X_n] = (2q-1)\frac{\Gamma(n+\alpha)}{\Gamma(2p)\Gamma(n)} = 0. \tag{4.33}$$

   (c) If $q > 1/2$, then
$$\mathbb{E}[X_n] \sim (2q-1)\frac{n^{2p-1}}{\Gamma(2p)} \to \infty. \tag{4.34}$$

   as $n \to \infty$. As a result, we see that the displacement of elephant random walk increases indefinitely as long as $q \neq 1/2$, and that the direction in which it diverges is dependent on the expected value of the first step taken. It is of note that, unless $p = 1$, the displacement of the elephant random walk grows slower than linear, making it so that the speed at which the displacement grows slows down as the number of steps increases.

4.2.4. Second moment of displacement

Similarly to the random walk, we will now look at the second moment of displacement X_n^2 of the elephant random walk. Unlike the random walk, where we could split the second moment of displacement into two cases, the elephant random walk has three different cases. However, before we can discuss these, we must first find the second moment of displacement of the elephant random walk. To that end, we first find a recursive formula for the second moment of displacement of the elephant random walk:

Theorem 4.2.11. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0,1]$. For $n \geq 1$, the expected second moment of displacement is recursively given by
$$\mathbb{E}[X_{n+1}^2] = 1 + \left(1 + \frac{2\alpha}{n}\right)\mathbb{E}[X_n^2]. \tag{4.35}$$

Proof. To determine the expected second moment of displacement, note that
$$\begin{aligned} X_{n+1} &= X_n + \eta_{n+1}, \\ \Rightarrow X_{n+1}^2 &= (X_n + \eta_{n+1})^2 \\ &= X_n^2 + 2X_n\eta_{n+1} + \eta_{n+1}^2. \end{aligned} \tag{4.36}$$

Taking the expectation on both sides, we obtain $\mathbb{E}[X_{n+1}^2] = \mathbb{E}[X_n^2] + 2\mathbb{E}[X_n\eta_{n+1}] + \mathbb{E}[\eta_{n+1}^2]$ by linearity. Furthermore, note that $\mathbb{E}[\eta_n^2] = 1$ for all n. Thus, it remains to express $\mathbb{E}[X_n\eta_{n+1}]$ in terms of X_n. To that end, we use the law of total expectation to condition on the previous steps:
$$\begin{aligned} \mathbb{E}[X_n\eta_{n+1}] &= \mathbb{E}[\mathbb{E}[X_n\eta_{n+1} \mid \eta_1, \dots, \eta_n]] \\ &= \mathbb{E}[X_n\mathbb{E}[\eta_{n+1} \mid \eta_1, \dots, \eta_n]] \\ &= \mathbb{E}\left[X_n \cdot \frac{\alpha}{n}X_n\right] \\ &= \frac{\alpha}{n}\mathbb{E}[X_n^2]. \end{aligned} \tag{4.37}$$

This results in the recursive formula

$$\begin{aligned} \mathbb{E}[X_{n+1}^2] &= \mathbb{E}[X_n^2] + 2\left(\frac{\alpha}{n}\mathbb{E}[X_n^2]\right) + 1 \\ &= 1 + \left(1 + \frac{2\alpha}{n}\right)\mathbb{E}[X_n^2], \end{aligned} \tag{4.38}$$

as desired. □

Before we look at the non-recursive form for the expected second moment of displacement for the elephant random walk, we will first discuss a possible solution for the recursive formula of Theorem 4.2.11. To that end, note that the formula is of the form

$$M_{n+1} = f_n + g_n M_n, \tag{4.39}$$

with $f_n = 1, g_n = 1 + \frac{2\alpha}{n}$ and $M_n = \mathbb{E}[X_n]$ for $n \geq 1$. Thus, if we can find a general solution for Equation (4.39), then we can find a non-recursive form for the second moment of displacment of the elephant random walk. Luckily, there does exist a general solution, as is shown in Theorem 4.2.12:

Theorem 4.2.12. Let f_n, g_n and M_n be functions, and suppose for $n \geq 1$ they obey the form

$$M_{n+1} = f_n + g_n M_n. \tag{4.40}$$

Then, the general solution is given by

$$M_n = M_1 \prod_{k=1}^{n-1} g_k + \sum_{k=1}^{n-1}\left[f_k \prod_{i=k+1}^{n-1} g_i\right]. \tag{4.41}$$

Proof. We proof this theorem using induction. To that end, note that for $n = 1$, we get

$$M_1 = M_1 \prod_{k=1}^{0} g_k + \sum_{k=1}^{0}\left[f_k \prod_{i=k+1}^{0} g_i\right] = M_1, \tag{4.42}$$

where we use that the empty product is equal to 1 and empty sum is equal to 0. Thus, the induction hypothesis holds. For the induction step, suppose that

$$M_m = M_1 \prod_{k=1}^{m-1} g_k + \sum_{k=1}^{m-1}\left[f_k \prod_{i=k+1}^{m-1} g_i\right] \tag{4.43}$$

holds for a $m \geq 1$. Then, we have that

$$\begin{aligned} M_{m+1} &= f_m + g_m M_m \\ &= f_m + g_m\left[M_1 \prod_{k=1}^{m-1} g_k + \sum_{k=1}^{m-1}\left[f_k \prod_{i=k+1}^{m-1} g_i\right]\right] \\ &= f_m + g_m\left[M_1 \prod_{k=1}^{m-1} g_k\right] + g_m \sum_{k=1}^{m-1}\left[f_k \prod_{i=k+1}^{m-1} g_i\right] \\ &= M_1 \prod_{k=1}^{m} g_k + \left[f_m + \sum_{k=1}^{m-1}\left[f_k \prod_{i=k+1}^{m} g_i\right]\right] \\ &= M_1 \prod_{k=1}^{m} g_k + \sum_{k=1}^{m}\left[f_k \prod_{i=k+1}^{m} g_i\right], \end{aligned} \tag{4.44}$$

where in the last line, we use the fact that putting $k = m$ in the last term would result in an empty product, making the next term f_t. As a result, the induction is complete. □

We can now look at the general form of the expected second moment of displacement of the elephant random walk by using Theorem 4.2.12 on the recursive formula found in Theorem 4.2.11:

Theorem 4.2.13. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0,1]$. The expected second moment of displacement is given by

$$\mathbb{E}[X_n^2] = \begin{cases} \frac{n}{2\alpha-1}\left(\frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)} - 1\right) & \text{if } p \neq \frac{3}{4}, \\ n\sum_{k=1}^{n}\frac{1}{k} & \text{if } p = \frac{3}{4}. \end{cases} \tag{4.45}$$

Proof. Applying Theorem 4.2.12 directly to the recursive second moment of displacement of Theorem 4.2.11, we get

$$\begin{aligned} \mathbb{E}[X_n^2] &= 1\prod_{k=1}^{n-1}\left(1+\frac{2\alpha}{k}\right) + \sum_{k=1}^{n-1}\left[1\prod_{i=k+1}^{n-1}\left(1+\frac{2\alpha}{i}\right)\right] \\ &= \prod_{k=1}^{n-1}\left(\frac{k+2\alpha}{k}\right) + \sum_{k=1}^{n-1}\prod_{i=k+1}^{n-1}\left(\frac{i+2\alpha}{i}\right) \\ &= \frac{\Gamma(n+2\alpha)}{\Gamma(2\alpha+1)\Gamma(n)} + \sum_{k=1}^{n-1}\left(\frac{\Gamma(n+2\alpha)\Gamma(k+1)}{\Gamma(n)\Gamma(2\alpha+k+1)}\right) \\ &= \sum_{k=0}^{n-1}\left(\frac{\Gamma(n+2\alpha)}{\Gamma(n)}\frac{\Gamma(k+1)}{\Gamma(2\alpha+k+1)}\right) \\ &= \frac{\Gamma(n+2\alpha)}{\Gamma(n)}\sum_{k=0}^{n-1}\left(\frac{\Gamma(k+1)}{\Gamma(2\alpha+k+1)}\right). \end{aligned} \tag{4.46}$$

Thus, it remains to find an expression for $\sum_{k=0}^{n-1}\left(\frac{\Gamma(k+1)}{\Gamma(2\alpha+k+1)}\right)$. To that end, note that multiplying the sum with $\Gamma(2\alpha)$ gives us an expression that is similar to that of a beta function:

$$\begin{aligned} \sum_{k=0}^{n-1}\left(\frac{\Gamma(2\alpha)\Gamma(k+1)}{\Gamma(2\alpha+k+1)}\right) &= \sum_{k=0}^{n-1} B(2\alpha, k+1) \\ &= \sum_{k=0}^{n-1}\int_0^1 u^{2\alpha-1}(1-u)^k\,du \\ &= \int_0^1\sum_{k=0}^{n-1} u^{2\alpha-1}(1-u)^k\,du \\ &= \int_0^1 u^{2\alpha-1}\left(\frac{1-(1-u)^n}{u}\right)du \\ &= \int_0^1 u^{2\alpha-2}\,du - \int_0^1 (u^{2\alpha-2}(1-u)^n)\,du \\ &= \frac{1}{2\alpha-1} - B(2\alpha-1, n+1) \\ &= \frac{1}{2\alpha-1} - \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(n+2\alpha)}, \end{aligned} \tag{4.47}$$

where $B(2\alpha, k+1)$ denotes the beta function. It is important to note that we assume that $\alpha \neq 1/2$ in Equation (4.47). The case $\alpha = \frac{1}{2}$ will be discussed separately. Thus, we get:

$$\begin{aligned}
\mathbb{E}[X_n^2] &= \frac{\Gamma(n+2\alpha)}{\Gamma(n)} \frac{1}{\Gamma(2\alpha)} \left(\frac{1}{2\alpha-1} - \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(n+2\alpha)} \right) \\
&= \frac{\Gamma(n+2\alpha)}{\Gamma(n)\Gamma(2\alpha)} \left(\frac{1}{2\alpha-1} - \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(n+2\alpha)} \right) \\
&= \frac{\Gamma(n+2\alpha)}{\Gamma(n)\Gamma(2\alpha)} \frac{1}{2\alpha-1} - \frac{\Gamma(n+2\alpha)}{\Gamma(n)\Gamma(2\alpha)} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(n+2\alpha)} \\
&= \frac{1}{2\alpha-1} \frac{\Gamma(n+2\alpha)}{\left(\frac{\Gamma(n+1)}{n}\right)\Gamma(2\alpha)} - \frac{\Gamma(n+1)\Gamma(2\alpha-1)}{\Gamma(n)\Gamma(2\alpha)} \\
&= \frac{n}{2\alpha-1} \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)} - \frac{n\Gamma(n)\Gamma(2\alpha-1)}{(2\alpha-1)\Gamma(n)\Gamma(2\alpha-1)} \\
&= \frac{n}{2\alpha-1} \left(\frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)} - 1 \right).
\end{aligned} \tag{4.48}$$

For $\alpha = \frac{1}{2}$, we obtain the following:

$$\begin{aligned}
\sum_{k=0}^{n-1} \left(\frac{\Gamma(k+1)}{\Gamma(k+2)} \right) &= \sum_{k=0}^{n-1} \left(\frac{\Gamma(k+1)}{(k+1)\Gamma(k+1)} \right) = \sum_{k=1}^{n} \frac{1}{k}, \\
\Rightarrow \mathbb{E}[X_n^2] &= \frac{\Gamma(n+1)}{\Gamma(n)} \sum_{k=1}^{n} \frac{1}{k} \\
&= \frac{n\Gamma(n)}{\Gamma(n)} \sum_{k=1}^{n} \frac{1}{k} \\
&= n \sum_{k=1}^{n} \frac{1}{k},
\end{aligned} \tag{4.49}$$

as desired. □

From Theorem 4.2.13, we can see that the diffusion behavior can be split into three cases, depending on the value of p, where we recall that $a := 2p - 1$:

1. If $p < 3/4$, we then have that $2\alpha < 1$. As a result, we find

$$\begin{aligned}
\mathbb{E}[X_n^2] &= \frac{n}{2\alpha-1} \left(\frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)} - 1 \right) \\
&\sim \frac{n}{2\alpha-1} \left(\frac{n^{2\alpha}}{n} \frac{\Gamma(n)}{\Gamma(n)\Gamma(2\alpha)} - 1 \right) \\
&\sim -\frac{n}{2\alpha-1} = \frac{n}{3-4p}.
\end{aligned} \tag{4.50}$$

for large n. As a result, we see that the expected second moment of displacement grows asymptotically linear. In comparison, note that squaring the asymptotic expected displacement found in 4.2.9 gives the following:

$$\mathbb{E}[X_n]^2 \sim \left((2q-1)\frac{n^{2p-1}}{\Gamma(2p)}\right)^2 = \frac{(2q-1)^2}{\Gamma(2p)^2}n^{4p-2}. \tag{4.51}$$

Since the mean second order of displacement is of a higher order than the square of the expected displacement, we find that the variance Var$(X_n) = \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2$ grows linearly, which means that the elephant random walk is a diffusive process for $p < 3/4$, similar to the normal random walk.

2. If $p = 3/4$, we then have that

$$\begin{aligned} \mathbb{E}[X_n^2] &= n\sum_{k=1}^{\infty}\frac{1}{k} \sim n\log(n), \\ \mathbb{E}[X_n]^2 &= (2q-1)^2 \end{aligned} \tag{4.52}$$

Surprisingly, we find that for $p = 3/4$, the variance of the elephant random walk is not linear or quadratic, or n^x for $1 < x < 2$, but linearlogarithmic. As a result, we have a marginally superdiffusive process.

3. If $p > 3/4$, then we have that $2\alpha > 1$. As a result, we find

$$\begin{aligned} \mathbb{E}[X_n^2] &= \frac{n}{2\alpha-1}\left(\frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)} - 1\right) \\ &\sim \frac{n}{2\alpha-1}\left(\frac{n^{2\alpha}}{n}\frac{\Gamma(n)}{\Gamma(n)\Gamma(2\alpha)} - 1\right) \\ &\sim \frac{n}{2\alpha-1}\frac{n^{2\alpha}}{n\Gamma(2\alpha)} \\ &= \frac{n^{2\alpha}}{(2\alpha-1)\Gamma(2\alpha)} = \frac{n^{4p-2}}{(4p-3)\Gamma(4p-2)} \end{aligned} \tag{4.53}$$

for large n. Comparing this to the square of the expected displacement we found in (4.51), we note that, while both are of the same order, they are not equal to each other[2]. As a result, we see that the variance of the elephant random walk is n^x for $1 < x < 2$. As a result, the elephant random walk is called a superdiffusive process for $p > 3/4$.

4.3. Asymptotic properties of large scale elephant random walks

In this section, we will discuss the characterization of the elephant random walk when it comes to its asymptotic properties. Even though the elephant random walk does not have independent steps, many of the results of the normal random walk still hold, which we will discuss in this section.

4.3.1. Prerequisites

To be able to discuss the results regarding the long term behavior of the elephant random walk, it is important to introduce some definitions and theorems that will be used in the proofs later in this section. To that end, we look at the following two definitions regarding filtrations and martingales:

Definition 4.3.1 (Filtration)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. Then $\{\mathcal{F}_n\}_{n\geq 0}$ with $\mathcal{F}_n$ a sub-σ-algebra of $\mathcal{F}$ is called a **filtration** if $\mathcal{F}_i \subseteq \mathcal{F}_j$ for all $i \leq j$.

[2]Unless if $(p,q) = (1,\pm 1)$, in which it is trivial due to the fact that $X_n = n$ for $(p,q) = (1,1)$ and $X_n = -n$ for $(p,q) = (1,-1)$.

Remark 4.3.2. Given a measurable space $(\Omega, \mathcal{F})$ and a sequence of real-valued random variables $(X_n)_{n\geq 0}$, the natural filtration $(\mathcal{F}_n)_{n\geq 0}$ is defined by

$$\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n) \quad n \geq 0, \tag{4.54}$$

where $\sigma(X_0, X_1, \dots, X_n)$ is the σ-algebra generated by $(X_0, X_1, \dots, X_n)$.

Definition 4.3.3 (Martingale)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **martingale** with respect to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ is a discrete-time stochastic process $\{X_n\}_{n\geq 0}$ such that, for all $n \geq 0$,

$$\mathbb{E}[|X_n|] < \infty, \tag{4.55}$$

$$\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = X_n. \tag{4.56}$$

Essentially, a martingale is characterized by the fact that the conditional expectation of the next value, given that we know the previous values, is equal to its most recent value. An example of such a martingale is the random walk with $p = 1/2$, as we know that $\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = 0 = \mathbb{E}[X_n]$ for $n \geq 0$. Furthermore, it is important to note that, unless mentioned, we are working with the natural filtration $(\mathcal{F}_n)_{n\geq 0}$ for the elephant random walk. The reason we look at martingales, is because there are many known theorems regarding martingales which will be useful, particularly when looking at the central limit theorem for the elephant random walk. However, to make use of these theorems, we will need to transform the elephant random walk into a martingale. To that end, for $n \geq 1$ let a_n be defined by

$$a_1 = 1, \qquad a_n = \prod_{k=1}^{n-1}\left(1 + \frac{\alpha}{k}\right) = \prod_{k=1}^{n-1}\frac{k+\alpha}{k} = \frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(2p)} \sim \frac{n^{\alpha}}{\Gamma(2p)}. \tag{4.57}$$

Furthermore, let $M_n = \frac{X_n - \mathbb{E}[X_n]}{a_n}$. We claim that M_n is a martingale with respect to the natural filtration $\{\mathcal{F}_n\}_{n\geq 1} = \sigma(\eta_1, \dots, \eta_n)$:

Theorem 4.3.4. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p, q \in [0, 1]$. Then, $M_n = \frac{X_n - \mathbb{E}[X_n]}{a_n}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1} = \sigma(\eta_1, \dots, \eta_n)$.

Proof. To show that M_n is a martingale, we must show that $\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] = M_n$ for all $n \geq 1$. To that end, note that we can write the conditional expectation of M_{n+1} as follows by using Theorem 4.2.4 and the law of total expectation:

$$\begin{aligned}
\mathbb{E}[M_{n+1} \,|\mathcal{F}_n] &= \frac{\mathbb{E}[X_{n+1} - \mathbb{E}[X_{n+1}] \,|\, \mathcal{F}_n]}{a_{n+1}} \\
&= \frac{\mathbb{E}[X_n + \eta_{n+1} - \mathbb{E}[X_n + \eta_{n+1}] \,|\, \mathcal{F}_n]}{a_{n+1}} \\
&= \frac{\mathbb{E}[X_n|\mathcal{F}_n] - \mathbb{E}[\mathbb{E}[X_n]|\mathcal{F}_n]}{a_{n+1}} + \frac{\mathbb{E}[\eta_{n+1}|\mathcal{F}_n] - \mathbb{E}[\mathbb{E}[\eta_{n+1}]|\mathcal{F}_n]}{a_{n+1}} \\
&= \frac{X_n - \mathbb{E}[X_n]}{a_{n+1}} + \frac{\mathbb{E}[\eta_{n+1} \,|\, \mathcal{F}_n] - \mathbb{E}[\eta_{n+1}]}{a_{n+1}} \\
&= \frac{X_n - \mathbb{E}[X_n]}{a_{n+1}} + \frac{\left(\frac{2p-1}{n}\right) X_n - \left(\frac{2p-1}{n}\right)\mathbb{E}[X_n]}{a_{n+1}} \\
&= \frac{X_n - \mathbb{E}[X_n]}{a_{n+1}} + \frac{\left(\frac{2p-1}{n}\right)(X_n - \mathbb{E}[X_n]))}{a_{n+1}} \\
&= (X_n - \mathbb{E}[X_n])\frac{1 + \left(\frac{2p-1}{n}\right)}{a_{n+1}} \\
&= (X_n - \mathbb{E}[X_n])\frac{1}{a_n} \\
&= M_n,
\end{aligned} \tag{4.58}$$

as desired. □

In addition to this martingale, we let $(D_n)_{n\geq 1}$ be the martingale differences of M_n, defined by $D_1 := M_1$ and $D_n := M_n - M_{n-1}$. It will be important to bound D_n later in this section. Thus, notice that for $n \geq 2$ we find the following:

$$\begin{aligned} D_n &= \frac{X_n - \mathbb{E}[X_n]}{a_n} - \frac{X_{n-1} - \mathbb{E}[X_{n-1}]}{a_{n-1}} \\ &= \frac{X_{n-1} + \eta_n - \mathbb{E}[X_{n-1} + \eta_n]}{a_n} - \frac{X_{n-1} - \mathbb{E}[X_{n-1}]}{\left(\frac{a_n}{1+\frac{2p-1}{n-1}}\right)} \\ &= \frac{\eta_n - \mathbb{E}[\eta_n]}{a_n} + \frac{X_{n-1} - \mathbb{E}[X_{n-1}]}{a_n} - \left(1 + \frac{2p-1}{n-1}\right)\frac{X_{n-1} - \mathbb{E}[X_{n-1}]}{a_n} \\ &= \frac{\eta_n - \mathbb{E}[\eta_n]}{a_n} - \frac{X_{n-1} - \mathbb{E}[X_{n-1}]}{n-1}\frac{2p-1}{a_n}. \end{aligned} \tag{4.59}$$

Taking the absolute value on both sides, we can bound the function in the following way by using the triangle inequality and noting that $a_n > 0$ for all $n \geq 1$:

$$\begin{aligned} \Rightarrow |D_n| &= \left|\frac{\eta_n - \mathbb{E}[\eta_n]}{a_n} - \frac{X_{n-1} - \mathbb{E}[X_{n-1}]}{n-1}\frac{2p-1}{a_n}\right| \\ &\leq \left|\frac{\eta_n - \mathbb{E}[\eta_n]}{a_n}\right| + \left|\frac{X_{n-1} - \mathbb{E}[X_{n-1}]}{n-1}\frac{2p-1}{a_n}\right| \\ &\leq \frac{|\eta_n - \mathbb{E}[\eta_n]|}{a_n} + \frac{|X_{n-1} - \mathbb{E}[X_{n-1}]|}{n-1}\frac{|2p-1|}{a_n} \\ &\leq \frac{2}{a_n} + \frac{2(n-1)}{n-1}\frac{1}{a_n} = \frac{4}{a_n}. \end{aligned} \tag{4.60}$$

Thus we find that $|D_n|$ can be bounded by $\frac{4}{a_n}$ for $n \geq 1$. Similarly, we will need to make statements later in this section using the value of $\sum_{k=1}^{n} \frac{1}{a_k^2}$. To that end, we look at the following lemma:

Lemma 4.3.5. $\sum_{k=1}^{n} \frac{1}{a_k^2}$ converges if and only if $p > 3/4$.

Proof. First off, we define $b_n := \frac{1}{a_n^2}$. We split the proof up in 2 cases based on the value of p:

1. If $p \neq 3/4$, then we may use the Raabe's test (Theorem A.5.1) to look at the convergence of the series. But before that, note that

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(2p)}}{\frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(2p)}} \\ &= \frac{\Gamma(n+\alpha+1)\Gamma(n)\Gamma(2p)}{\Gamma(n+\alpha)\Gamma(n+1)\Gamma(2p)} \\ &= \frac{n+\alpha}{n}\frac{\Gamma(n+\alpha)\Gamma(n)\Gamma(2p)}{\Gamma(n+\alpha)\Gamma(n)\Gamma(2p)} \\ &= \frac{n+\alpha}{n}, \end{aligned} \tag{4.61}$$

where we use $\Gamma(z+1) = z\Gamma(z)$ to simplify the fraction. Using Raabe's test, we get:

$$\begin{aligned} \rho_n &= n\left(\frac{b_n}{b_{n+1}} - 1\right) \\ &= n\left(\left(\frac{a_{n+1}}{a_n}\right)^2 - 1\right) \\ &= n\left(\frac{(n+\alpha)^2 - n^2}{n^2}\right) \\ &= \frac{2n\alpha + \alpha^2}{n} \\ \Rightarrow \rho &= \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \frac{2n\alpha+\alpha^2}{n} = 2\alpha. \end{aligned} \tag{4.62}$$

Looking at the value of ρ, we see that the series converges if $\alpha > 1/2$, which corresponds to $p > 3/4$, and diverges if $\alpha < 1/2$, which corresponds to $p < 3/4$.

2. If $p = 3/4$, then we cannot use the Raabe criteria, as the test is inconclusive. Instead, we note the following:

$$\begin{aligned} a_n &= \frac{\Gamma(n+1/2)}{\Gamma(n)\Gamma(2p)} \sim \frac{\sqrt{n}}{\Gamma(2p)}, \\ \Rightarrow \sum_{k=1}^{n} \frac{1}{a_k^2} &\sim \sum_{k=1}^{n} \frac{\Gamma(2p)^2}{n}, \end{aligned} \tag{4.63}$$

the latter of which diverges by comparison with the harmonic series as $n \to \infty$, and as a result we find that $\sum_{k=1}^{n} \frac{1}{a_k^2}$ diverges as well.

Thus, we find that $\sum_{k=1}^{\infty} 1/a_k^2$ converges if and only if $p > 3/4$, as desired. □

Lastly, we will introduce a notation to indicate the asymptotic growth of a function, namely the little-o notation. This will be useful when estimating expectations when proving the central limit theorem for the elephant random walk in Theorem 4.3.14. In addition to this, we will also introduce a proposition regarding sums of little-o's. Thus, we look at Definition 4.3.6 and Proposition 4.3.8:

Definition 4.3.6 (Little-o notation)**.** A sequence $(x_n)_{n\geq 1}$ is $o(y_n)$ if for every $\epsilon > 0$ there exists a $N \geq 1$ such that:

$$|x_n| \leq \epsilon\, y_n \quad \text{for } n \geq N. \tag{4.64}$$

Remark 4.3.7. If the sequence $(x_n)_{n\geq 1}$ is nonzero for all $n \geq N$ for a certain N, then $x_n = o(y_n)$ is equivalent to

$$\lim_{x\to\infty} \frac{x_n}{y_n} = 0. \tag{4.65}$$

Proposition 4.3.8. Let $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ be two sequences such that $x_n = o(y_n)$ and $\sum_{k=1}^{n} y_k \to \infty$ as $n \to \infty$. Then,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} y_k} = 0. \tag{4.66}$$

Proof. As $x_n = o(y_n)$, we get that for all $\epsilon > 0$ there exists a $N_1 \geq 1$ such that $|x_n| < \frac{\epsilon}{2} y_n$ for $n \geq N_1$. Thus, we can write the following for $n \geq N_1$:

$$\begin{aligned}\frac{\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} y_k} &= \frac{\sum_{k=1}^{N_1} x_k}{\sum_{k=1}^{n} y_k} + \frac{\sum_{k=N_1+1}^{n} x_k}{\sum_{k=1}^{n} y_k} \\ &< \frac{\sum_{k=1}^{N_1} x_k}{\sum_{k=1}^{n} y_k} + \frac{\epsilon}{2} \frac{\sum_{k=N_1+1}^{n} y_k}{\sum_{k=1}^{n} y_k} \\ &\leq \frac{\sum_{k=1}^{N_1} x_k}{\sum_{k=1}^{n} y_k} + \frac{\epsilon}{2}.\end{aligned} \tag{4.67}$$

Next, notice that $\sum_{k=1}^{n} y_k \to \infty$ as $n \to \infty$. In other words, for all $\epsilon > 0$ there exists a $N_2 \geq 1$ such that for all $n \geq N_2$

$$\begin{aligned}\sum_{k=1}^{n} y_k &> \frac{2}{\epsilon} \sum_{k=1}^{N_1} x_k, \\ \Rightarrow \frac{1}{\sum_{k=1}^{n} y_k} &< \frac{\epsilon}{2} \frac{1}{\sum_{k=1}^{N_1} x_k}.\end{aligned} \tag{4.68}$$

Thus, we find for $n \geq \max\{N_1, N_2\}$,

$$\frac{\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} y_k} < \frac{\sum_{k=1}^{N_1} x_k}{\sum_{k=1}^{n} y_k} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{4.69}$$

as desired. □

4.3.2. Law of large numbers

As was mentioned in the introduction of this section, we are unable to use the usual law of large numbers to make statements about the behavior of the elephant random walk, due to the fact that each step is dependent on the history of the elephant random walk. As it turns out, we can still show that the law of large numbers applies to the elephant random walk, as is shown in Theorem 4.3.9:

Theorem 4.3.9. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p \in [0, 1), q \in [0, 1]$. Then

$$\frac{X_n - \mathbb{E}[X_n]}{n} \to 0 \text{ a.s.} \tag{4.70}$$

Remark 4.3.10. Note that Theorem 4.3.9 does not hold for the case $p = 1$. However, in that case we know that:

$$\mathbb{P}[X_n = n] = q, \tag{4.71}$$

$$\mathbb{P}[X_n = n] = 1 - q. \tag{4.72}$$

This result basically shows that, even though the steps or not independent and identically distributed, we still expect the average of a large number of iterations of the elephant random walk to converge to the expected value. To prove this theorem, we will make use of the following lemma:

Lemma 4.3.11 (Kronecker's lemma)**.** Suppose $(x_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers such that $\sum_{n=1}^{\infty} x_n$ converges and b_n is a monotone sequence of positive constants such that $b_n \uparrow \infty$. Then,

$$\frac{1}{b_n} \sum_{k=1}^{\infty} x_k b_k \to 0. \tag{4.73}$$

Remark 4.3.12. Alternatively, Lemma 4.3.11 may be restated as follows by letting $y_n = \frac{x_n}{b_n}$: Suppose $(y_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers such that $\sum_{n=1}^{\infty} \frac{y_n}{b_n}$ converges and b_n is a monotone sequence of positive constants such that $b_n \uparrow \infty$. Then,

$$\frac{1}{b_n}\sum_{k=1}^{\infty} y_k \to 0. \tag{4.74}$$

Furthermore, it is important to note that $\frac{X_n - \mathbb{E}[X_n]}{n} = \frac{a_n}{n} M_n = \frac{a_n}{n}\sum_{k=1}^{n} D_k$. Thus, the proof boils down to showing that $\sum_{k=1}^{\infty} D_k \frac{a_k}{k}$ converges and that $\frac{n}{a_n}$ diverges, after which we may use the lemma to show that it goes to 0.

Proof of Theorem 4.3.9. We first prove that $\frac{n}{a_n}$ diverges, which is shown by proving that $\frac{a_n}{n}$ converges to 0. To that end, note that

$$\begin{aligned}
\frac{a_n}{n} &= \frac{1}{n}\prod_{k=1}^{n-1}\left(1+\frac{\alpha}{k}\right) \\
&= \frac{1}{n}\prod_{k=1}^{n-1}\left(\frac{k+2p-1}{k}\right) \\
&= \prod_{k=1}^{n-1}\left(\frac{k+2p-1}{k+1}\right) \\
&= \frac{\Gamma(n+2p-1)}{\Gamma(n+1)} \\
&\sim \frac{\Gamma(n)n^{2p-1}}{\Gamma(n)n} = n^{2p-2}.
\end{aligned} \tag{4.75}$$

Note that in the third line, we use the fact that

$$n\prod_{k=1}^{n-1} k = \prod_{k=1}^{n} k = 1\cdot\prod_{k=2}^{n} k = \prod_{k=1}^{n-1}(k+1). \tag{4.76}$$

Since $\lim_{n\to\infty} n^{2p-2} = 0$ for $p \in [0,1)$, we find that $\lim_{n\to\infty}\frac{a_n}{n} = 0$ as well. In addition, note that $0 \leq \frac{k+2p-1}{k+1} < 1$, thus $\left(\frac{a_n}{n}\right)_{n\geq 1}$ is a monotone sequence.

The other part to prove is that $\sum_{k=1}^{\infty} D_n \frac{a_n}{n}$ converges. To that end, we may use the following Theorem:

Theorem 4.3.13 (Hall and Heyde, 1980)**.** Let $\{S_n = \sum_{k=1}^{n} X_i, \mathcal{F}_n, n \geq 1\}$ be a martingale and let $p \in [1,2]$, then S_n converges almost surely on the set $\{\sum_{i=1}^{\infty}\mathbb{E}(|X_i|^p \mid \mathcal{F}_{i-1}) < \infty\}$.

Thus, we can show that $\sum_{k=1}^{\infty} D_k \frac{a_k}{k}$ converges if we can show that $\sum_{k=1}^{\infty}\mathbb{E}\left[\left(D_k\frac{a_k}{k}\right)^2 \middle| \mathcal{F}_{k-1}\right] < \infty$. Note that we can use the bound for D_n found earlier in Equation (4.60) to get the following:

$$\left|\sum_{k=1}^{\infty}\mathbb{E}\left[\left(D_k\frac{a_k}{k}\right)^2 \middle| \mathcal{F}_{k-1}\right]\right| \leq \sum_{k=1}^{\infty}\left(\frac{a_k}{k}\right)^2\left(\frac{4}{a_k}\right)^2 = \sum_{k=1}^{\infty}\frac{16}{k^2} < \infty. \tag{4.77}$$

Thus, we may use Lemma 4.3.11 to get that

$$\frac{X_n - \mathbb{E}[X_n]}{n} = \frac{a_n}{n} M_n = \frac{a_n}{n} \sum_{k=1}^{n} D_k \to 0 \text{ a.s.,} \tag{4.78}$$

as desired. □

4.3.3. Central limit theorem

Aside from the law of large numbers, we are also interested in whether or not the central limit theorem also holds for the elephant random walk. Similarly to the law of large numbers, the usual theorem used to prove the central limit theorem for the normal random walk does not hold anymore, as the steps are neither independent nor identically distributed.

As it turns out, we can prove the central limit using the martingale differences we defined in Section 4.3.1. However, the central limit theorem only holds for the diffusive and marginally superdiffusive regimes, i.e. $p \leq 3/4$, and does not hold for the superdiffusive regime. The result for the diffusive and marginally superdiffusive regimes is shown in Theorem 4.3.14:

Theorem 4.3.14. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p \in \left[0, \frac{3}{4}\right], q \in [0, 1]$. Then,

$$\frac{X_n - \frac{2q-1}{\Gamma(2p)} n^{2p-1}}{\sqrt{\frac{n}{3-4p}}} \xrightarrow{d} \mathcal{N}(0,1), p < \frac{3}{4}, \tag{4.79}$$

$$\frac{X_n - \frac{2q-1}{\Gamma(\frac{3}{2})} n^{\frac{3}{2}}}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(0,1), p = \frac{3}{4}. \tag{4.80}$$

Before we prove this theorem, we first look at the theorem that will be used in the proof, namely a central limit theorem for martingale difference arrays based on the one presented in Hall and Heyde (1980)[3]:

Theorem 4.3.15 (Hall and Heyde, 1980)**.** Let $\{S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a zero-mean, square integrable martingale array with differences X_{ni}. Suppose that for all $\epsilon > 0$,

$$\forall \epsilon > 0, \sum_{i=1}^{n} \mathbb{E}[X_{ni}^2 \mathbb{I}[|X_{ni} > \epsilon] \,|\, \mathcal{F}_{n,i-1}] \xrightarrow{p} 0, \tag{4.81}$$

$$\sum_{i=1}^{n} \mathbb{E}[X_{ni}^2 \,|\, \mathcal{F}_{n,i-1}] \xrightarrow{p} 1, \tag{4.82}$$

then $S_{nk_n} = \sum_{i=1}^{k_n} X_{ni} \xrightarrow{p} \mathcal{N}(0,1)$.

This theorem gives us a way to prove that the probability distribution converges to a normal distribution if the two conditions are satisfied, which are the Lindenberg condition (4.81) and a condition on the conditional variance of the martingale (4.82).

[3]A general version of this theorem can be found in the appendix (Theorem A.5.3).

To that end, the proof will consist of checking if these two conditions hold for the elephant random walk. Before we do so, we will first look at the function $(s_n)_{n\geq 1}$, which is defined as follows:

$$s_1^2 \coloneqq 4q(1-q), \tag{4.83}$$

$$s_n^2 \coloneqq 4q(1-q) + \sum_{k=2}^{n} \frac{1}{a_k^2}. \tag{4.84}$$

The reason we look at s_n, is so that we can combine the two statements from Theorem 4.3.14 into one formula. To that end, note the following:

$$\begin{aligned} s_n^2 &= 4q(1-q) + \sum_{k=2}^{n} \frac{1}{a_k^2} \\ &\sim 4q(1-q) + \sum_{k=2}^{n} \left(\frac{\Gamma(2p)}{n^{2p-1}}\right)^2 \\ &\sim \Gamma(2p)^2 \int_0^n x^{2-4p}\, dx \\ &= \begin{cases} \Gamma(2p)^2 \frac{n^{3-4p}}{3-4p} & p < 3/4, \\ \Gamma(2p)^2 \log n & p = 3/4. \end{cases} \end{aligned} \tag{4.85}$$

Here, we first use the fact that $a_n \sim n^{2p-1}/\Gamma(2p)$, after which we approximate the finite sum with an integral, which is possible as $\sum_{k=2}^{n} \frac{1}{a_n^2} \to \infty$. Thus, we find

$$a_n s_n \sim \begin{cases} \sqrt{\frac{n}{(3-4p)}} & p < 3/4, \\ \sqrt{n \log(n)} & p = 3/4. \end{cases} \tag{4.86}$$

Thus, we may look at following expression to prove Theorem 4.3.14:

$$\frac{X_n - \mathbb{E}[X_n]}{a_n s_n} = \frac{M_n}{s_n} = \frac{\sum_{k=1}^{n} D_k}{s_n}. \tag{4.87}$$

Proof of Theorem 4.3.14. We will first look at the first condition of Theorem 4.3.15. To that end, define $D_{nj} \coloneqq \frac{D_j}{s_n}$ for $1 \leq j \leq n$. We need to show that for all $\epsilon > 0$,

$$\sum_{j=1}^{n} \mathbb{E}[D_{nj}^2 \mathbb{I}(|D_{nj}| > \epsilon) \,|\, \mathcal{F}_{n-1}] \to 0 \text{ a.s.} \tag{4.88}$$

as $n \to \infty$, where $\mathbb{I}$ denotes the indicator function. Recall that $a_1 = 1, a_n = \prod_{k=1}^{n}(1 + \alpha/k)$. We split Equation (4.88) into two cases based on the value of p:

- If $1/2 \leq p \leq 3/4$, then we have that $a_n \geq 1$ for $n \geq 1$ and $s_n \to \infty$ as $n \to \infty$ by Lemma 4.3.5. Thus, using the bound found in Equation (4.60) we get that:
$$|D_{nj}| \leq \frac{4}{a_j s_n} \leq \frac{4}{s_n}. \tag{4.89}$$
Thus, given an epsilon $\epsilon > 0$ we can find a large enough n such that D_{nj} is smaller than ϵ for all $1 \leq j \leq n$.
- If $p < 1/2$, then we notice that a_n is a decreasing sequence, and thus a_n^{-1} is an increasing sequence. If we combine this with the fact that $a_n s_n \to \infty$ for $n \to \infty$, we get the following estimation:
$$|D_{nj}| \leq \frac{4}{a_j s_n} \leq \frac{4}{a_n s_n}. \tag{4.90}$$
Similar to the previous case, for a given epsilon $\epsilon > 0$ we can find a large enough n such that D_{nj} is smaller than ϵ for all $1 \leq j \leq n$.

In both cases, we find that $\mathbb{I}[D_{nj} > \epsilon] = 0$ for all $0 \leq j \leq n$ given we choose a n large enough. Thus, we find that Equation (4.88) holds.

Next, we check the second condition of Theorem 4.3.14, that is

$$\sum_{k=1}^{n} \mathbb{E}[D_{nk}^2 \mid \mathcal{F}_{k-1}] = \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}[D_k^2 \mid \mathcal{F}_{k-1}] \to 1 \text{ a.s.} \tag{4.91}$$

If we prove that Equation (4.91) holds, then we have proven Theorem 4.3.14. Thus, we will need to look at the value of $\mathbb{E}[D_k^2 \mid \mathcal{F}_{k-1}]$. To that end, we use Equation (4.59) to get the following for $k \geq 2$:

$$\begin{aligned}
D_k^2 &= \left(\frac{\eta_k - \mathbb{E}[\eta_k]}{a_k} - \frac{X_{k-1} - \mathbb{E}[X_{k-1}]}{k-1}\frac{2p-1}{a_k}\right)^2 \\
&= \frac{1}{a_k^2}\left((\eta_k - \mathbb{E}[\eta_k]) - \frac{(2p-1)(X_{k-1} - \mathbb{E}[X_{k-1}])}{k-1}\right)^2 \\
&= \frac{1}{a_k^2}\left((\eta_k - \mathbb{E}[\eta_k])^2 - 2(\eta_k - \mathbb{E}[\eta_k])\frac{(2p-1)(X_{k-1} - \mathbb{E}[X_{k-1}])}{k-1} + \left(\frac{(2p-1)(X_{k-1} - \mathbb{E}[X_{k-1}])}{k-1}\right)^2\right) \\
&= \frac{1}{a_k^2}\left(1 - 2\eta_k \mathbb{E}[\eta_k] + \mathbb{E}[\eta_k]^2\right) + \frac{1}{a_k^2}\left((2p-1)^2\left(\frac{X_{k-1} - \mathbb{E}[X_{k-1}]}{k-1}\right)^2\right) \\
&- \frac{2}{a_k^2}\left((\eta_k - \mathbb{E}[\eta_k])(2p-1)\frac{X_{k-1} - \mathbb{E}[X_{k-1}]}{k-1}\right).
\end{aligned} \tag{4.92}$$

Now, we may use the law of large numbers for the elephant random walk. Recall that the law of large numbers stated that $(X_n - \mathbb{E}[X_n])/n \to 0$ almost surely. As a result, we find that

$$\frac{1}{a_k^2}\left((2p-1)^2\left(\frac{X_{k-1} - \mathbb{E}[X_{k-1}]}{k-1}\right)^2\right) = o\left(\frac{1}{a_k^2}\right) \text{ a.s.} \tag{4.93}$$
$$\frac{2}{a_k^2}\left((\eta_k - \mathbb{E}[\eta_k])(2p-1)\frac{X_{k-1}\mathbb{E}[X_k - 1]}{k-1}\right) = o\left(\frac{1}{a_k^2}\right) \text{ a.s.} \tag{4.94}$$

It remains to find the value of $\frac{1}{a_j^2}\mathbb{E}[1-2\eta_k\mathbb{E}[\eta_k]+\mathbb{E}[\eta_k]^2\,|\,\mathcal{F}_{k-1}]$. To that end, we use Theorem 4.2.4 along with the law of large numbers to get the following:

$$\begin{aligned}\frac{1}{a_k^2}\mathbb{E}[1-2\eta_k\mathbb{E}[\eta_k]+\mathbb{E}[\eta_k]^2\,|\,\mathcal{F}_{k-1}] &= \frac{1}{a_k^2}+\frac{1}{a_k^2}\left(\left(\frac{(2p-1)(2q-1)}{\Gamma(2p)}(k-1)^{2p-2}\right)^2\right)\\ &-\frac{2}{a_k^2}\left(2\frac{(2p-1)X_{k-1}}{k-1}\frac{(2p-1)(2q-1)}{\Gamma(2p)}(k-1)^{2p-2}\right)\\ &=\frac{1}{a_k^2}+o\left(\frac{1}{a_k^2}\right),\end{aligned} \tag{4.95}$$

where the second and third term are both $o\left(\frac{1}{a_k^2}\right)$ due to $(k-1)^{2p-2}$ converging to zero for $p\le 3/4$. Thus, we get for $k\ge 2$, we get

$$\mathbb{E}[D_k^2\,|\,\mathcal{F}_{k-1}]=\frac{1}{a_k^2}+o\left(\frac{1}{a_k^2}\right). \tag{4.96}$$

For $k=1$, we calculate $\mathbb{E}[D_1^2\,|\,\mathcal{F}_0]$ explicitly:

$$\begin{aligned}\mathbb{E}[D_1^2\,|\,\mathcal{F}_0] &= \mathbb{E}\left[\left(\frac{\eta_1-\mathbb{E}[\eta_1]}{a_1}\right)^2\,|\,\mathcal{F}_0\right]\\ &=\frac{1}{a_1^2}\mathbb{E}[\eta_1^2-2\eta_1\mathbb{E}[\eta_1]+\mathbb{E}[\eta_1]^2\,|\,\mathcal{F}_0]\\ &=\mathbb{E}[\eta_1^2\,|\,\mathcal{F}_0]-2(2q-1)\mathbb{E}[\eta_1\,|\,\mathcal{F}_0]]+(2q-1)^2\\ &=1-2(2q-1)^2+(2q-1)^2\\ &=1-(4q^2-4q+1)=4q(1-q).\end{aligned} \tag{4.97}$$

Lastly, it is important to note that we found that $\sum_{k=1}^n\frac{1}{a_k^2}\to\infty$ in Lemma 4.3.5. Thus, we can use Proprosition 4.3.8 to get

$$\frac{1}{s_n^2}\sum_{k=2}^n\mathbb{E}[D_k^2\,|\,\mathcal{F}_{k-1}]=\frac{4q(1-q)+\sum_{k=1}^n\left(\frac{1}{a_k^2}+o\left(\frac{1}{a_n^2}\right)\right)}{4q(1-q)+\sum_{k=2}^n\frac{1}{a_k^2}} \tag{4.98}$$

$$=1+\frac{\sum_{k=2}^n o\left(\frac{1}{a_n^2}\right)}{4q(1-q)+\sum_{k=2}^n\frac{1}{a_k^2}}\to 1\text{ a.s.} \tag{4.99}$$

Thus, we may use Theorem 4.3.15 to conclude that the central limit theorem holds for the elephant random walk for $p\le 3/4$, as desired. □

Unlike the diffusive and marginally superdiffusive regime, we find that a different result holds for the superdiffusive regime. In Figure 4.3, a sampling of 10 000 random walks of length 1000 are shown for $p = 0.9$. As can be seen in the figure, the distribution of the sampling does not follow a normal distribution. Rather, the distribution has two peaks, the size of which is influenced by the value of q.

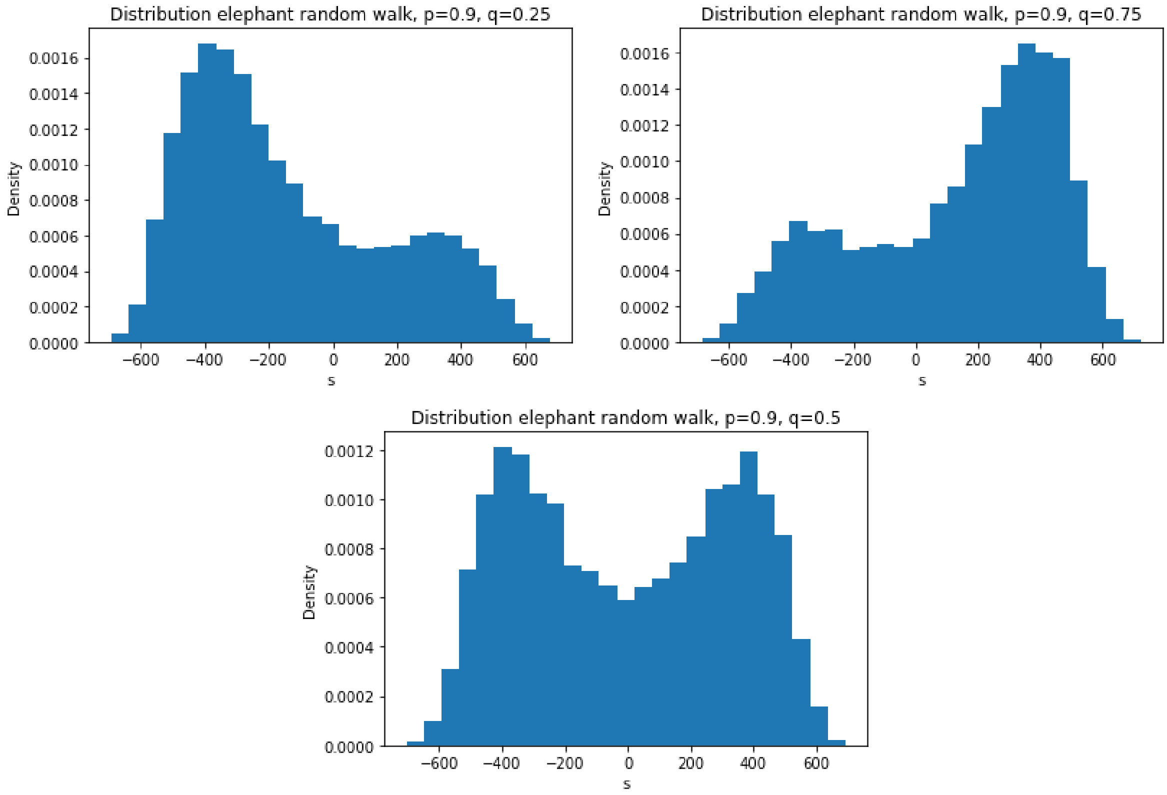

Figure 4.3: Probability distribution of the elephant random walk for $p = 0.9$ and different values of q.

While it turns out in theory that the elephant random walk does indeed not converge to a normal distribution, the elephant random walk does converge to a non-degenerate distribution for the superdiffusive regime:

Theorem 4.3.16. Let $(X_n)_{n\geq 1}$ be an elephant random walk with $p \in \left(\frac{3}{4}, 1\right], q \in [0, 1]$. Then

$$\frac{X_n}{n^{2p-1}\Gamma(2p)^{-1}} - (2q-1) \overset{a.s.}{\to} M, \tag{4.100}$$

where M is a non-degenerate random variable with zero mean, but M is not normally distributed.

The proof of Theorem 4.3.16 will consist of three parts:

- Showing that M_n converges almost surely to a distribution M,
- Showing that M is non-degenerate and zero-mean, which means showing that M has a non-zero variance and $\mathbb{E}[M] = 0$,
- Showing that M is not a normal distribution. To show this, we will use the fact that a normal distribution has skewedness zero and excess kurtosis zero, i.e. $\mathbb{E}\left[\left(\frac{M-\mathbb{E}[M]}{\sigma}\right)^3\right] = 0$ and $\mathbb{E}\left[\left(\frac{M-\mathbb{E}[M]}{\sigma}\right)^4\right] = 0$. For this, we will use the results presented in the article by Paraan and Esguerra (2006).

To show that M_n converges almost surely to M, we will use the following definition and lemma, which give us a way to check for almost sure convergence for martingales:

Definition 4.3.17 ($\mathcal{L}^p$-spaces)**.** For $1 \leq p < \infty$, the random variable X is in $\mathcal{L}^p$ if:

$$||X||_p = \mathbb{E}[|X^p|]^{\frac{1}{p}} < \infty. \tag{4.101}$$

Lemma 4.3.18 (Williams, 1991)**.** Let M be a martingale such that $M_n \in \mathcal{L}^2$ for all $n \geq 1$. Then, M is bounded in $\mathcal{L}^2$ if and only if $\sum_{k=1}^{\infty} \mathbb{E}[(M_k - M_{k-1})^2] < \infty$. If this is the case, then $M_n \to M$ in $\mathcal{L}^2$ and almost surely.

Proof of 4.3.16. To make use of Lemma 4.3.18, we will first show that M_n is indeed in $\mathcal{L}^2$ for all $n \geq 1$. Thus, note that

$$\begin{aligned} ||M_n||_2 &= \mathbb{E}\left[\left|\frac{X_n - \mathbb{E}[X_n]}{a_n}\right|^2\right]^{\frac{1}{2}} \\ &= \mathbb{E}\left[\frac{|X_n - \mathbb{E}[X_n]|^2}{|a_n|^2}\right]^{\frac{1}{2}} \\ &\leq \mathbb{E}\left[\frac{2n^2}{|a_n|^2}\right]^{\frac{1}{2}} < \infty, \end{aligned} \tag{4.102}$$

thus $M_n \in \mathcal{L}^2$. For almost sure convergence, note that Lemma 4.3.18 states that it is sufficient to check if $\sum_{k=1}^{\infty} \mathbb{E}[D_k^2] < \infty$. To that end, note that in lemma 4.3.5 we found that $\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty$ for $p > \frac{3}{4}$. Combining this with the bound $|D_n| < \frac{4}{a_n}$ we found earlier in Equation 4.60, we find:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[D_k^2] < \lim_{n\to\infty} \sum_{k=1}^{n} \frac{16}{a_k^2} < \infty. \tag{4.103}$$

Since the mean sum of squares of martingale differences is bounded, we may use Lemma 4.3.18 to conclude that $M_n \to M$ almost surely. To show M is a non-degenerate zero-mean random variable, note that

$$\mathbb{E}[M_n] = \mathbb{E}\left[\frac{X_n - \mathbb{E}[X_n]}{a_n}\right] = \frac{\mathbb{E}[X_n] - \mathbb{E}[X_n]}{\mathbb{E}[a_n]} = 0. \tag{4.104}$$

Thus, using the $\mathcal{L}^2$-convergence of M_n and Jensen's inequality (Theorem A.1.4), we get that

$$\begin{aligned} |\mathbb{E}[M]| &= |\mathbb{E}[M - M_n]| \\ &\leq \mathbb{E}[|M - M_n|] \\ &\leq \mathbb{E}[|M - M_n|^2]^{\frac{1}{2}} \to 0. \end{aligned} \tag{4.105}$$

For the variance, we see that

$$
\begin{aligned}
\mathrm{Var}(M) &= \lim_{n\to\infty} \mathrm{Var}(M_n) \\
&= \lim_{n\to\infty} \mathrm{Var}\left(\sum_{k=1}^{n} D_k\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n}\left(\mathrm{Var}(D_k) + 2\sum_{1\le j<k\le n} \mathrm{cov}(D_j, D_k)\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n}(\mathbb{E}[D_k^2] - (\mathbb{E}[D_k])^2) + 0\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[D_k^2] < \infty.
\end{aligned}
\tag{4.106}
$$

Here, we use the fact that $\mathbb{E}[D_k] = \mathbb{E}[M_k - M_{k-1}] = 0$. The fact that the covariance for D_j and D_k for $1 \le j < k \le n$ is equal to zero follows by using the law of total expectation to condition on the natural filter $\mathcal{F}_{k-1}$:

$$
\begin{aligned}
\mathrm{cov}(D_j, D_k) &= \mathrm{cov}(M_j - M_{j-1}, M_k - M_{k-1}) \\
&= \mathbb{E}[(M_j - M_{j-1})(M_k - M_{k-1})] \\
&= \mathbb{E}[\mathbb{E}[(M_j - M_{j-1})(M_k - M_{k-1})|\mathcal{F}_{k-1}]] \\
&= \mathbb{E}[(M_j - M_{j-1})(\mathbb{E}[M_k|\mathcal{F}_{k-1}] - M_{k-1})] \\
&= \mathbb{E}[(M_j - M_{j-1})(M_{k-1} - M_{k-1})] = 0,
\end{aligned}
\tag{4.107}
$$

where we note that the last line of Equation (4.107) follows from the definition of the martingale. We have now shown that M is a non-degenerate zero-mean random variable, but it remains to show that M does not have a normal random distribution. To that end, we look at the kurtosis and skewedness of M. In the article from Paraan and Esguerra, 2006, the asymptotic values of the kurtosis and skewedness of M has been explicitly calculated for $\alpha > \frac{1}{2}$ and $\beta := 2q - 1$, which is shown in Figure 4.4. As can be seen from Figure 4.4, there exists combinations of α and β such that either the kurtosis or skewedness of M is 0, but there are no combinations of α and β such that both the kurtosis and skewedness of M is 0. As a result, we find that M cannot have a normal distribution for $p > \frac{3}{4}$, which completes the proof.

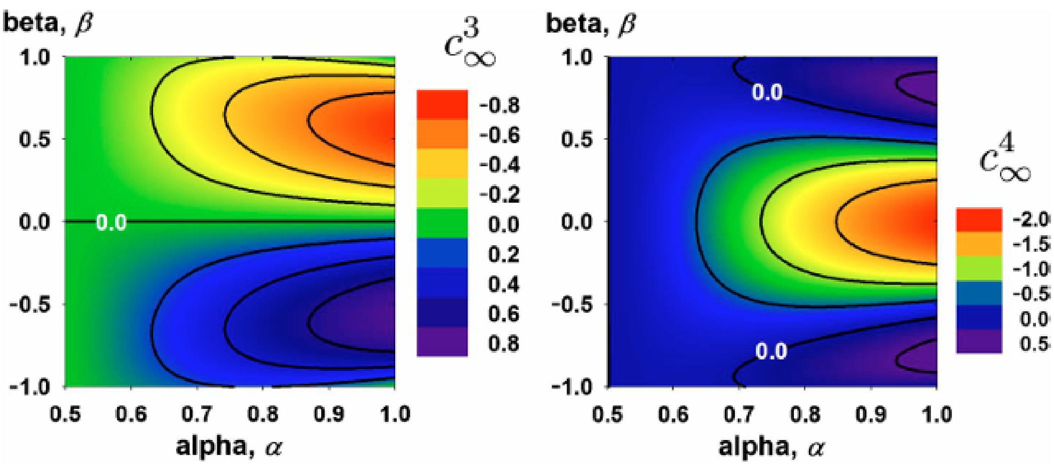

Figure 4.4: Asymptotic values of the skewness (left) and kurtosis (right) of the elephant random walk (Paraan and Esguerra, 2006).

□

4.4. Elephant gambler's ruin

In Chapter 3, we have seen the gambler's ruin problem for the random walk, and have seen and proved results for the gambler's ruin, which were the chance for player A to win and the expected amount of steps until either player won. In this section, we will extend the gambler's ruin to the elephant random walk, and we will study the effect of the three parameters, N, p and q, on $\nu(a)$ and $e(a)$, where we recall that we denote the chance of player A winning given a starting capital a with $\nu(a)$ and the expected number of steps until the random walk gets absorbed at either 0 or N given that player A starts with starting capital a with $e(a)$.

As we have seen in Section 3.4, we are able to determine the chance for player A to win and the expected amount of steps until either player won using difference equations for the random walk. However, this approach cannot be used for the elephant random walk due to the dependence on the amount of steps that have already been taken: the resulting difference equation involves both the amount of steps taken and the place of the elephant random walk at a certain time. As a result, we get a difference equation that we are unable to solve due to the presence of both time and place.

It is worth noting that we are also able to prove the theorems of Section 3.4.1 with martingales. While we will not go into detail about this approach, the main idea behind the proof is to transform the random walk into two martingales, with one of them only depending on the location of the random walk, after which both $\nu(a)$ and $e(a)$ follows. This approach can also not be used for the elephant random walk due to being unable to find a martingale which does not depend on the amount of steps taken.

Thus, we look at simulations of the gambler's ruin for the elephant random walk. To that end, we look at 10 000 simulations for the elephant random walk and look at the sample mean success rate, that is the average win rate for player A, and the sample mean amount of steps needed until either player A or B wins. Furthermore, we look at the influence of the three parameters, namely the total capital N, p and q, on $\nu(a)$ and $e(a)$.

4.4.1. Initial parameter q and memory parameter p

We first look at the influence of p and q on $\nu(a)$, where we fix $N = 50$. It turns out we can split the behavior of $\nu(a)$ into three cases, namely $p < 1/2$, $p = 1/2$ and $p > 1/2$. To that end, we look at the case $p = 1/2$ first. In Figure 4.5, the gambler's ruin is simulated for $p = 1/2$ and varying values of q. As can be seen from the figure, $\nu(a)$ increases linearly as a increases, with the value of q having minimal effect on $\nu(a)$. This is to be expected, as the elephant random walk behaves like a fair random walk if $p = 1/2$, as $\mathbb{P}[\eta_k = 1] = \mathbb{P}[\eta_k = -1] = 1/2$ regardless of the previous step chosen, with the value of q only affecting the first step.

When looking at the expected number of steps for $p = 1/2$, we get Figure 4.6. Due to the fact that the elephant random walk behaves like a normal random walk with $p = 1/2$, the expected number of steps is similar to that of the normal random walk with $p = 1/2$, with the value of q having minimal influence due to only having an affect on the first step.

Next, we look at the case $p < 1/2$. Figure 4.7 illustrates $\nu(a)$ for $p = 0$, $p = 1/8$, $p = 1/4$ and $p = 3/8$. As can be seen in Figure 4.7, the behavior of $\nu(a)$ turns into a sigmoid-like shape as p decreases. Also, the influence of q on $\nu(a)$ is nihil. Intuitively, this result can be explained by the fact that it is harder for the elephant random walk to reach the further point between 0 and N, as the elephant random walk behaves like a reformer and thus stays near its starting location a when looking at $\mathbb{E}[X_n]$ in combination with the law of large numbers. The minimal effect of q follows from the behavior of $\mathbb{E}[X_n]$ not depending on q for $p < 1/2$. In Figure 4.8, we see the expected number of steps for $p < 1/2$. As can be seen in the figure, the general behavior of $e(a)$ resembles a normal distribution as p decreases, with the sample mean number of steps being maximal at $a = 25$, although a decrease in the value of p does make the maximum sample mean number of steps higher. This can be explained by the elephant random walk

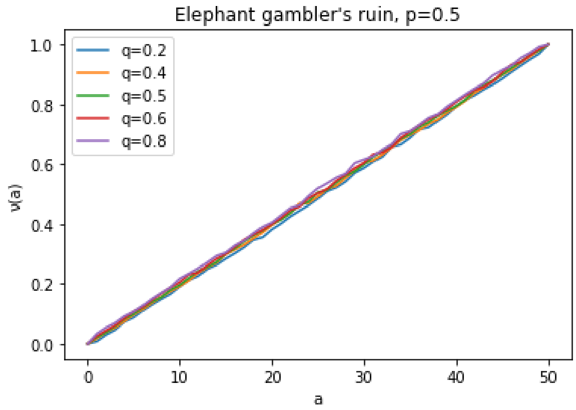

Figure 4.5: Chance of player A to win for $p = 0.5$.

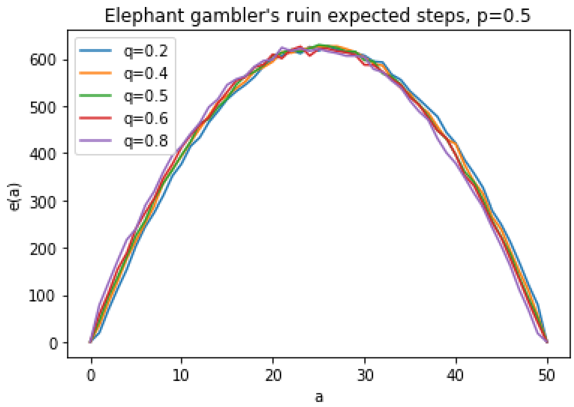

Figure 4.6: Expected number of steps of the elephant gambler's ruin for $p = 0.5$

having a higher chance to stay near its initial position as p decreases, and as a result will take longer to reach either 0 or N.

Lastly, we look at the case $p > 1/2$. In Figure 4.9, we see $v(a)$ for $p = 5/8$, $p = 3/4$, $p = 7/8$ and $p = 1$. In the figure, we can see that the increase of $v(a)$ as a increases becomes slower as p increases, with $p = 1$ resulting in $v(a)$ being a constant value. Furthermore, q has a larger effect as the value of p increases, with an increase in q increasing $e(a)$ for all a. Intuitively, we expect that the elephant random walk has a higher chance to go to the direction of the first step as p increases due to the elephant random walk choosing the same direction of a step from its history more often, and as a result we expect the behavior of $v(a)$ to depend less on the value of p and more on the value of q. In Figure 4.10, we see the expected number of steps for $p > 1/2$. As can be seen from the figure, the behavior for $p = 1$ is linear, with the values $1/2 < p < 1$ being a combination of the cases $p = 1/2$ and $p = 1$. Intuitively, we expect for $p = 1$ that the elephant random walk has q chance to go to N and $1 - q$ chance to go to 0, and we can look at the cases between $p = 1/2$ and $p = 1$ as a combination of the elephant random walk with $p = 1/2$ and $p = 1$, with values of p closer to $1/2$ behaving more like the elephant random walk with $p = 1/2$, and values of p closer to 1 behaving more like the elephant random walk with $p = 1$.

4.4.2. Total capital N

Lastly, we will look at the influence of the total capital N on both $v(a)$ and $e(a)$. In particular, we will look at the cases where $N = 25$, $N = 50$ and $N = 75$, where $N = 50$ will serve as the reference point as we have already discussed the behavior for the case $N = 50$ in the previous subsection. It is important to note that in both figures, namely Figure 4.11 and Figure 4.12, $q = 0.5$ has been chosen.

In Figure 4.11, $v(a)$ is shown for different values of p and N. When we look at the influence of N on $v(a)$, we see that in all cases N does not affect $v(a)$. This suggests that the chance for player A to win depends on the percentage of the total capital that player A has. This result is unlike the gambler's ruin for the random walk, where we saw that an increase in N made the behavior of $v(a)$ more extreme. Unlike for $v(a)$, N does have an effect on $e(a)$, as can be seen in Figure 4.12, which shows $e(a)$ for different values of p and N. As can be seen in Figure 4.12, an increase in the number of steps causes the expected number of steps to increase as the total capital increases. It is of note that p affects this: the expected number of steps for $p = 0$ shows that $e(a)$ grows quadratically as N increases, whereas $p = 1$ shows linear growth in $e(a)$ as N increases. This is similar to the gambler's ruin for the random walk, where we recall that an increase in total capital N had a bigger effect on the expected number of steps if p was closer to $1/2$.

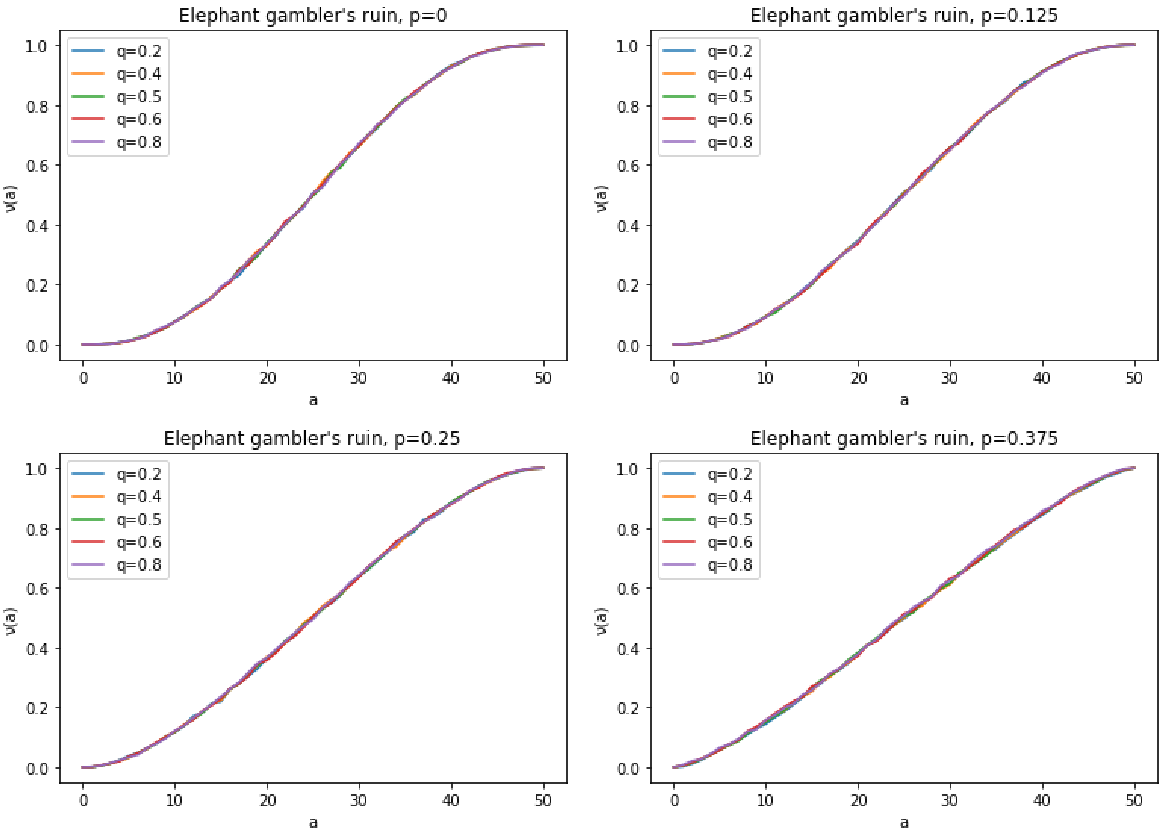

Figure 4.7: Chance of player A to win for various values of $p < 0.5$.

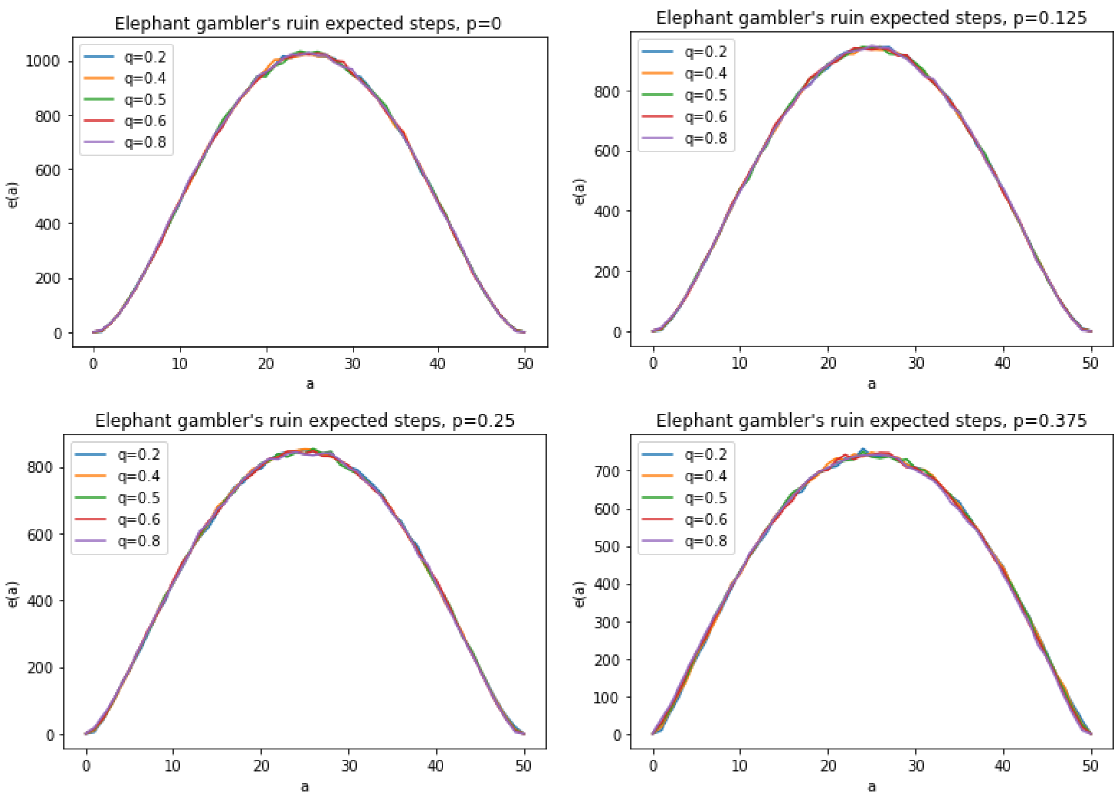

Figure 4.8: Expected number of steps of the elephant gambler’s ruin for various values of $p < 0.5$ and $q = 0.5$. Note that the scale of the y-axis changes as p changes, with lower values of p resulting in a higher maximal expected number of steps.

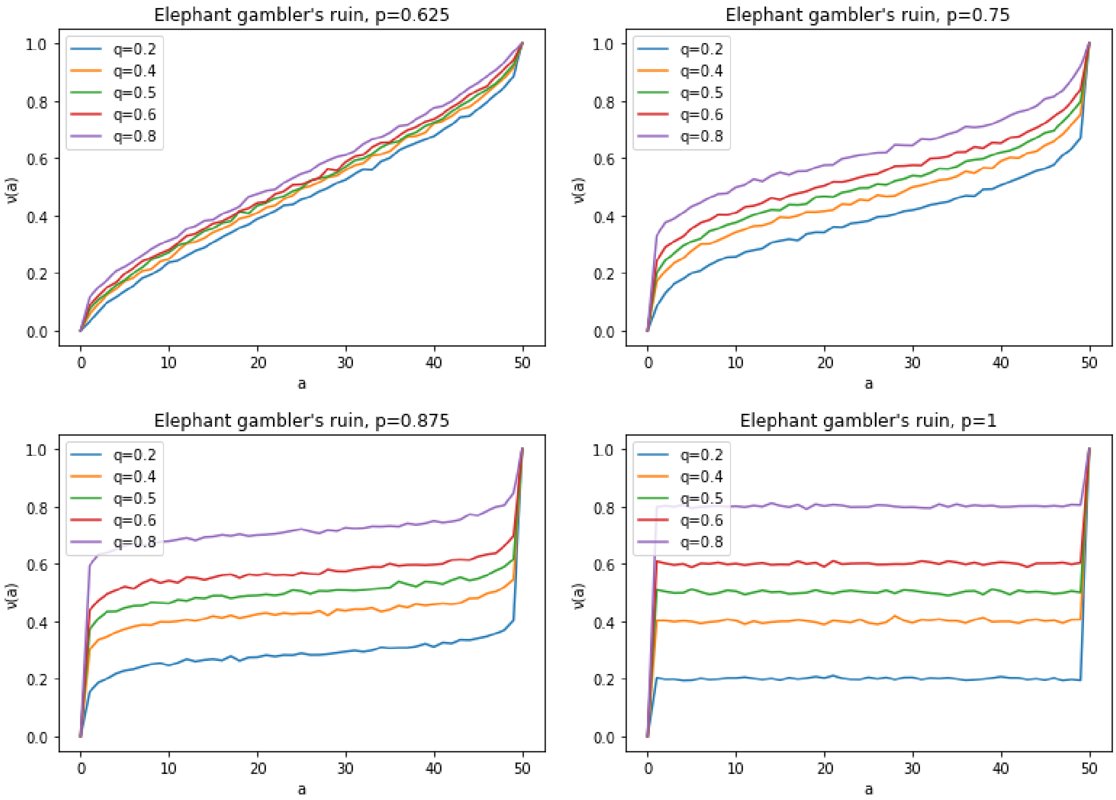

Figure 4.9: Chance of player A to win for various values of $p > 0.5$ and $q = 0.5$.

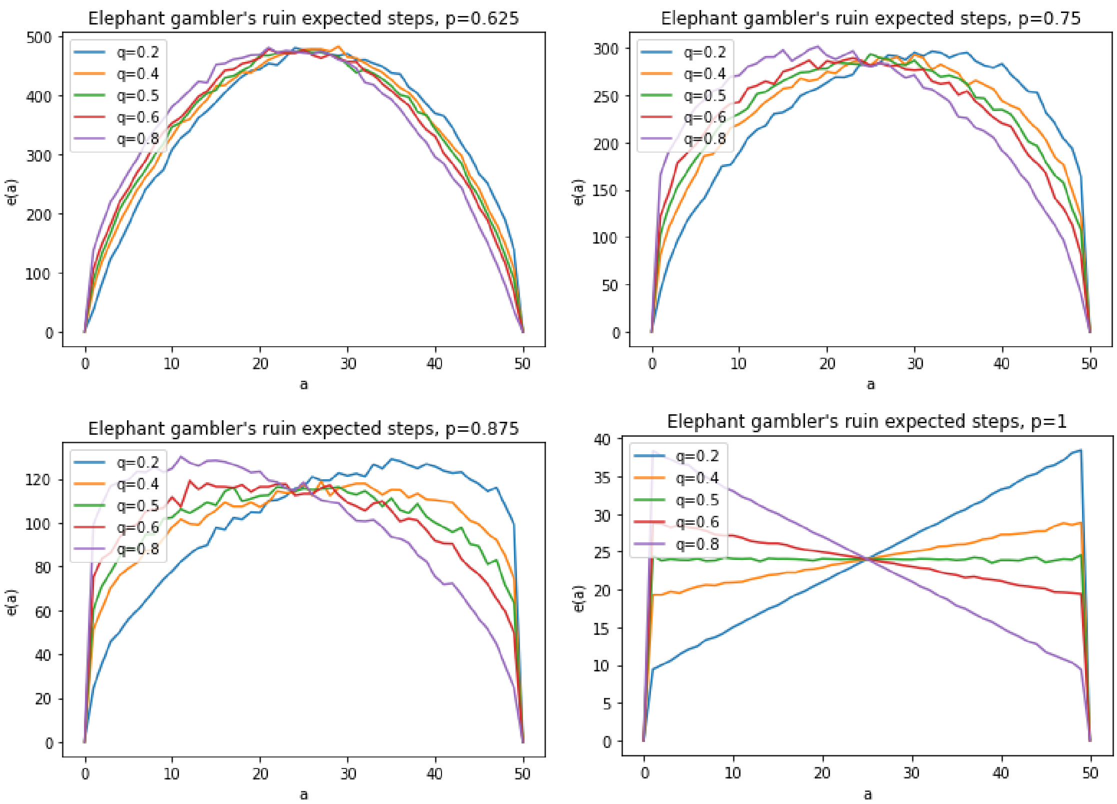

Figure 4.10: Expected number of steps of the elephant gambler's ruin for various values of $p > 0.5$. Note that the scale of the y-axis changes as p changes.

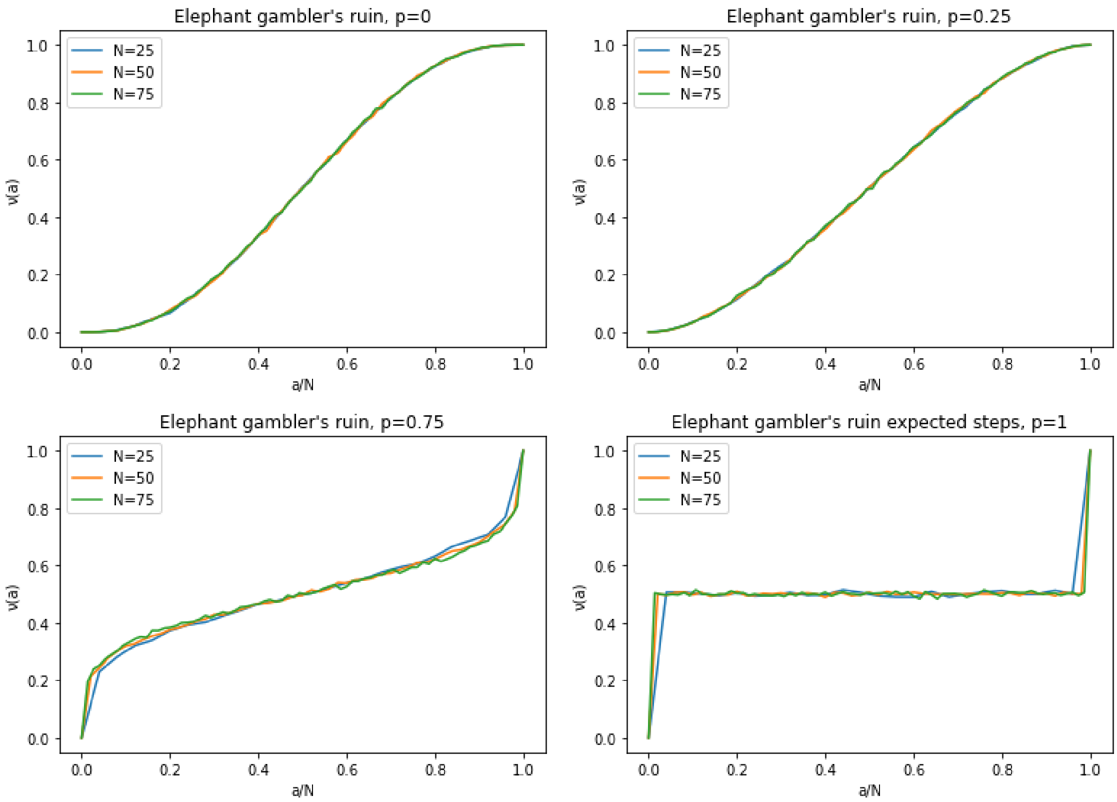

Figure 4.11: Chance of player A to win for various total capital N

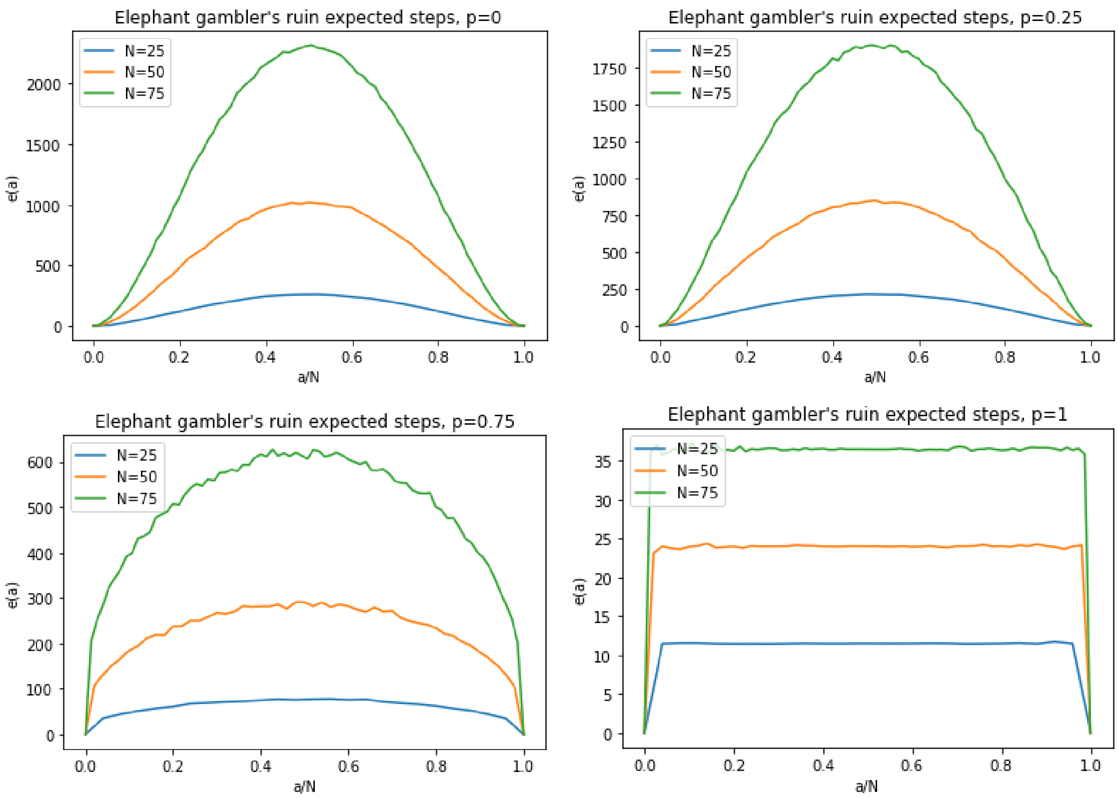

Figure 4.12: Expected number of steps of the elephant gambler's ruin for various total capital N. Note that the scale of the y-axis changes as p changes.

5

Conclusion

In this thesis, we explored the differences between the random walk and the elephant random walk, with a focus on the gambler's ruin.

All in all, we have found that there are several differences when comparing the normal random walk and elephant random walk. The most obvious difference is in how each step is taken: the random walk consists of steps where each step is dependent on the parameter p. For the elephant random walk, the first step depends on the initial parameter q, whereas later steps are dependent on earlier steps taken in the walk, where the value of the step is dependent on the memory parameter p.

When looking at the expected value, we found that the behavior of the expected value of the elephant random walk can be split into two cases: for $p < 1/2$ the expected value converges to zero, and for $p > 1/2$ the expected value diverges to infinity with the direction of divergence being dependent on the initial parameter q.

Unlike the random walk, which is a diffusion process regardless of the chosen value of p, the elephant random walk shows three different diffusion regimes based on the value of p: a diffusive regime, a superdiffusive regime and a marginally superdiffusive regime. The critical point at which the elephant random walk changes regimes is based off the memory parameter p. Interestingly, the critical point is not located at $p = 1/2$, but rather at $p = 3/4$: values of $p < 3/4$ result in a diffusive process, values of $p > 3/4$ result in a superdiffusive process and a marginally superdiffusive process occurs at $p = 3/4$.

When looking at the long term behavior of the elephant random walk, there are several similarities and differences between the normal random walk and elephant random walk. We find that the law of large numbers holds for both random walks. Unlike the law of large numbers, the central limit theorem can only be applied for certain values for the elephant random walk: for the normal random walk the central limit theorem almost always holds, but for the elephant random walk the central limit theorem only holds for the diffusive and marginally superdiffusive regime. For the superdiffusive regime, one finds that the probability distribution does converge to a non-degenerate distribution, but that the distribution is not a normal distribution.

Lastly, we looked at the elephant random walk variant of the gambler's ruin: when looking at the chance of player A to win given starting capital a, $v(a)$, and the expected number of steps until either player A or player B wins given that player A has starting capital a, $e(a)$, we see that $v(a)$ behaves linearly for $p = 1/2$ exponentially for the random walk for $p \neq 1/2$, and that this exponential behavior becomes more pronounced as the total capital N increases. In addition to this, we find that both the location and value of the maximal expected number of steps $e(a)$ are influenced by p and N.

This is in contrast to the gambler's ruin for the elephant random walk. Firstly, $\nu(a)$ behaves differently depending on the value of p: for $p = 1/2$, we obtain the normal random walk, and as a result we find similar results as the normal random walk with $p = 1/2$. For $p < 1/2$, $\nu(a)$ turns into a sigmoid function-like shape as p decreases, with q having minimal influence on $\nu(a)$. For $p > 1/2$, we find the effect of q on $\nu(a)$ increases as p increases, with $\nu(a)$ being solely dependent on q when $p = 1$. Looking at $e(a)$, we once again find similar results for $p = 1/2$ as the normal random walk with $p = 1/2$. For $p < 1/2$, we find that the the behavior of $e(a)$ turns into a shape similar to the normal distribution, with q having minimal affect on the shape. In comparison, for $p = 1$ we find that the $e(a)$ behaves linearly, with q being the only factor in the shape of $e(a)$, and the cases $1/2 < p < 1$ behaving like a combination of the cases $p = 1/2$ and $p = 1$. Lastly, we found that N has no significant effect on $\nu(a)$, and that N only influences the maximal value of $e(a)$ and does not affect the general behavior of $e(a)$.

6
Discussion

As can be seen in Section 3.4, it was not possible to derive a explicit formula for $v(a)$ and $e(a)$ with the two proof methods that were presented, namely the difference equation approach and the martingale approach. As a result, we instead looked at simulations of the gambler's ruin, after which we looked at the sample mean values of $v(a)$ and $e(a)$. However, we only looked at a select few combinations of values of p, q and N due to simulation times.

A future study could look at other combinations of p, q and N to see whether or not interesting behavior emerges for intermediate values for the elephant gambler's ruin. Even more of note would be to look at a possible way to derive an explicit formula for the chance of player A to win $v(a)$ and the expected number of steps $e(a)$, which could possibly involve a numerical analysis or transforming the elephant random walk into a partial differential equation with boundaries.

Another possible extension would be to look at the exact distribution of M from Theorem 4.3.16, which was about the convergence to a non-normal distribution of the elephant random walk for the superdiffusive regime. As we saw in Theorem 4.3.16, the probability distribution does converge, but to a non-normal non-degenerate distribution. As we saw from Figure 4.3, the probability distribution seems to be a distribution with two peaks, of which the shape depends on p and q. As a result, one may be interested in whether or not we can describe this probability distribution in terms of p and q using an already known distribution.

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A

Used theorems

Below, one will find a collection of the theory, asymptotics and other formulas used in Chapters 3 and 4.

A.1. Asymptotics & funtions

Definition A.1.1. We write $a_n \sim b_n$ if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1. \tag{A.1}$$

Proposition A.1.2. For $x \to +\infty$, it holds that

$$\Gamma(x+\alpha) \sim \Gamma(x)\, x^{\alpha}. \tag{A.2}$$

Lemma A.1.3 (Kronecker's lemma)**.** Suppose $(x_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers such that $\sum_{n=1}^{\infty} x_n$ converges and b_n is a monotone sequence of positive constants such that $b_n \uparrow \infty$. Then

$$\frac{1}{b_n}\sum_{k=1}^{\infty} x_n b_n \to 0. \tag{A.3}$$

Theorem A.1.4 (Jensen's inequality)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, and X a random variable on that probability space. If g is a convex function, then

$$g(\mathbb{E}[X]) \leq \mathbb{E}[g(X)]. \tag{A.4}$$

Definition A.1.5 (Indicator functions)**.** Given a constant $\epsilon > 0$, the **indicator function** of x is the function

$$\mathbb{I}[x > \epsilon] = \begin{cases} 0 & x \leq \epsilon, \\ 1 & x > \epsilon. \end{cases} \tag{A.5}$$

Definition A.1.6 (Little-o notation)**.** A sequence $(x_n)_{n\geq 1}$ is $o(y_n)$ if for every $\epsilon > 0$ there exists a $N \geq 1$ such that:

$$|x_n| \leq \epsilon\, y_n \quad \text{for } n \geq N. \tag{A.6}$$

A.2. Probability & expectation

Theorem A.2.1 (Law of total probability)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space with $A \in \mathcal{F}$. If $\{B_n : n \geq 1\} \in \mathcal{F}$ is a partition of Ω with $\mathbb{P}[B_k] > 0$ for all k, then

$$\mathbb{P}[A] = \sum_{k=1}^{n} \mathbb{P}[A \mid B_k]\mathbb{P}[B_k]. \tag{A.7}$$

Theorem A.2.2 (Law of total expectation)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. If X and Y are random variables on the probability space, then

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]. \tag{A.8}$$

Theorem A.2.3 (Strong law of large numbers)**.** Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables with finite mean and finite variance, then

$$\frac{S_n - n\mathbb{E}[X]}{n} \to 0 \text{ a.s.} \tag{A.9}$$

as $n \to \infty$, where $S_n \coloneqq \sum_{k=1}^{n} X_k$.

Theorem A.2.4 (Central limit theorem)**.** Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables with finite mean and non-zero variance. Then,

$$\frac{S_n - n\mathbb{E}[X]}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \tag{A.10}$$

as $n \to \infty$, where $S_n \coloneqq \sum_{k=1}^{n} X_k$.

A.3. $\mathcal{L}^p$ spaces

Definition A.3.1. For $1 \leq p < \infty$, the random variable X is in $\mathcal{L}^p$ if

$$||X||_p = \mathbb{E}[|X^p|]^{\frac{1}{p}} < \infty. \tag{A.11}$$

A.4. Filtrations & martingales

Definition A.4.1 (Filtration)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. Then $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t$ a sub-σ-algebra of $\mathcal{F}$ is called a **filtration** if $\mathcal{F}_i \subseteq \mathcal{F}_j$ for all $i \leq j$.

Remark A.4.2. Given a measurable space $(\Omega, \mathcal{F})$ and a sequence of real-valued random variables $(X_t)_{t\geq 0}$, the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ is defined by

$$\mathcal{F}_t = \sigma(X_0, X_1, \ldots, X_t) \quad t \geq 0, \tag{A.12}$$

where $\sigma(X_0, X_1, \ldots, X_t)$ is the σ-algebra generated by $(X_0, X_1, \ldots, X_t)$.

Definition A.4.3 (Martingale)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **martingale** with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is a discrete-time stochastic process $\{X_t\}_{t\geq 0}$ such that, for all $t \geq 0$,

$$\mathbb{E}[|X_t|] < \infty, \tag{A.13}$$

$$\mathbb{E}[X_{t+1} \mid \mathcal{F}_t] = X_t. \tag{A.14}$$

A.5. Convergence theorems

Theorem A.5.1 (Raabe's test). Let $(a_n)_{n\geq 1}$ be a sequence of real valued numbers. Define the following:

$$\rho_n := n\left(\frac{a_n}{a_{n+1}} - 1\right), \tag{A.15}$$

$$\rho := \lim_{n\to\infty} \rho_n. \tag{A.16}$$

Then, the following holds:

- If $\rho > 1$, then $\sum_{k=1}^{\infty} a_k$ converges,
- If $\rho < 1$, then $\sum_{k=1}^{\infty} a_k$ diverges,
- If $\rho = 1$, then the he test is inconclusive.

Theorem A.5.2 (Hall and Heyde, 1980). Let $\{S_n = \sum_{i=1}^{n} X_i, \mathcal{F}_n, n \geq 1\}$ be a martingale and let $p \in [1, 2]$. Then S_n converges almost surely on the set $\{\sum_{i=1}^{\infty} \mathbb{E}(|X_i|^p \mid \mathcal{F}_{i-1}) < \infty\}$.

Theorem A.5.3 (Hall and Heyde, 1980). Let $\{S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be a zero-mean, square integrable martingale array with differences X_{ni}, and let η^2 be an almost surely finite random variable. Suppose that

$$\forall\epsilon > 0, \sum_{i=1}^{k_n} \mathbb{E}[X_{ni}^2 \mathbb{I}[|X_{ni} > \epsilon] \mid \mathcal{F}_{n,i-1}] \xrightarrow{p} 0, \tag{A.17}$$

$$\sum_{i=1}^{k_n} \mathbb{E}[X_{ni}^2 \mid \mathcal{F}_{n,i-1}] \xrightarrow{p} \eta^2. \tag{A.18}$$

Then $S_{nk_n} = \sum_{i=1}^{k_n} X_{ni} \xrightarrow{p} Z$, where Z has characteristic function $e^{-\frac{1}{2}\eta^2 t^2}$.

Lemma A.5.4 (Williams, 1991). Let M be a martingale such that $M_n \in \mathcal{L}^2$ for all $n \geq 1$. Then, M is bounded in $\mathcal{L}^2$ if and only if $\sum_{k=1}^{\infty} \mathbb{E}[(M_k - M_{k-1})^2] < \infty$. If this is the case, then $M_n \to M$ in $\mathcal{L}^2$ and almost surely.

B

Code used

Below, one finds the code used for the figures in Chapters 3 and 4. In particular, it contains the following:

- Simulations of X_n and X_n^2 for the random walk,
- Simulations of $\nu(a)$ and $e(a)$ for the random walk,
- Probability distribution from multiple samples of the elephant random walk,
- Simulations of X_n and X_n^2 for the elephant random walk,
- Simulations of $\nu(a)$ and $e(a)$ for the elephant random walk.

```
import matplotlib.pyplot as plt
import random as rnd
import math as m

#%%

#Random walk
def random_walk(p,n,s):
    walks = []
    mean = []
    for j in range(n):
        X = [0]
        S = [0]
        chance1 = rnd.random()
        if p > chance1:
            X += [1]
            S += [1]
        else:
            X += [-1]
            S += [-1]
        for i in range(s-1):
            chance2 = rnd.random()
            if p > chance2:
                X += [1]
            else:
```

```
                X += [-1]
            S += [sum(X)]
        plt.plot(S)
        walks += [S]
    for i in range(s+1):
        mean+=[0]
        for j in range(n):
            mean[-1] +=walks[j][i]
        mean[-1] = mean[-1]/n
    plt.plot(mean,color="black", linewidth = 2.5)

#Second moment of displacement random walk
def random_walk2(p,n,s):
    walks = []
    mean = []
    for j in range(n):
        X = [0]
        S = [0]
        chance1 = rnd.random()
        if p > chance1:
            X += [1]
            S += [1]
        else:
            X += [-1]
            S += [-1]
        for i in range(s-1):
            chance2 = rnd.random()
            if p > chance2:
                X += [1]
            else:
                X += [-1]
            S += [sum(X)**2]
        plt.plot(S)
        walks += [S]
    for i in range(s+1):
        mean+=[0]
        for j in range(n):
            mean[-1] +=walks[j][i]
        mean[-1] = mean[-1]/n
    plt.plot(mean,color="black", linewidth = 2.5)

#%%

#Gambler's ruin RW probability

def gambler_ruin(p,n):
    q=1-p
    values = []
    for i in range(n+1):
        if p == 1/2:
            values += [i/n]
        else:
            values += [((q/p)**i-1)/((q/p)**n-1)]
    return values

```

```
#%%

#Gambler's ruin RW expected steps

def gambler_ruin_steps(p,n):
    q= 1-p
    ratio = q/p
    values = []
    for i in range(n+1):
        if p == 1/2:
            values += [i*(n-i)]
        else:
            values += [(1/(p-q))*((n*ratio**i- n)/(ratio**n-1)-i)]
    return values

plt.plot(gambler_ruin_steps(0.1, 100))

#%%

#Distribution of the elephant random walk

def elephant_sample(Number,N,p,q):
    Lst = []
    for i in range(Number):
        X = []
        chance1 = rnd.random()
        if q > chance1:
            X += [1]
        else:
            X += [-1]
        for i in range(N):
            choose = rnd.randint(0,i)
            chance2 = rnd.random()
            if p > chance2:
                X += [X[choose]]
            else:
                X += [-X[choose]]
        Lst += [sum(X)]
    plt.hist(Lst, density=True, bins = 25)
    plt.xlabel("s")
    plt.ylabel("Density")
    plt.title("Distribution elephant random walk, p="+str(p)+",
     ↪ q="+str(q))

#%%
#Elephant random walk
def elephant_random_walk(p,q,n,s):
    walks = []
    mean = []
    for j in range(n):
        chance1 = rnd.random()
        X = [0]
        S = [0]
        if q > chance1:
            X += [1]
```

```
                S += [1]
            else:
                X += [-1]
                S += [-1]
            for i in range(s-1):
                choose = rnd.randint(1, i+1)
                chance2 = rnd.random()
                if p > chance2:
                    X += [X[choose]]
                else:
                    X += [-X[choose]]
                S += [sum(X)]
            plt.plot(S)
            walks += [S]
        for i in range(s+1):
            mean+=[0]
            for j in range(n):
                mean[-1] +=walks[j][i]
            mean[-1] = mean[-1]/n
        plt.plot(mean,color="black", linewidth = 2.5)

#Second moment displacement
def elephant_random_walk2(p,q,n,s):
    for j in range(n):
        chance1 = rnd.random()
        X = [0]
        S = [0]
        if q > chance1:
            X += [1]
            S += [1]
        else:
            X += [-1]
            S += [-1]
        for i in range(s-1):
            choose = rnd.randint(1, i+1)
            chance2 = rnd.random()
            if p > chance2:
                X += [X[choose]]
            else:
                X += [-X[choose]]
            S += [sum(X)**2]
        plt.plot(S)

#%%

#Elephant Gambler ruin

def elephant_ruin(N,a,p,q):
    if a==0:
        return 0,0
    if a==N:
        return N,0
    chance1 = rnd.random()
    S=a
    if q>chance1:
```

```
            X = [1]
            S += 1
        else:
            X = [-1]
            S += -1
        i=0
        while (S!=0 and S!=N):
            choose = rnd.randint(0, i)
            chance2 = rnd.random()
            if p > chance2:
                X += [X[choose]]
                S += X[choose]
            else:
                X += [-X[choose]]
                S += -X[choose]
            i+=1
        return S,i

    #Simulating multiple elephant ruins for a certain starting capital a
    def elephant_ruin_simulation(Number,N,a,p,q):
        counter = 0
        steps = 0
        for i in range(Number):
            Values = elephant_ruin(N, a, p, q)
            if Values[0]==N:
                counter+=1
            steps+=Values[1]
        return counter/Number,steps/Number

    #Plotting the elephant ruin
    def plot_elephant_ruin(Number,N,p,q):
        Y1 = []
        Y2 = []
        for i in range(N+1):
            Y1 += [elephant_ruin_simulation(Number, N, i, p, q)[0]]
            Y2 += [elephant_ruin_simulation(Number, N, i, p, q)[1]]
        return(Y1,Y2)

    def plot_elephant_ruin_N(Number,N,p,q):
        X = []
        Y1 = []
        Y2 = []
        for i in range(N+1):
            elephant_sim = elephant_ruin_simulation(Number, N, i, p, q)
            X += [i/N]
            Y1 += [elephant_sim[0]]
            Y2 += [elephant_sim[1]]
            print(i)
        return (X,Y1,Y2)
```