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A streamline tracking algorithm for semi-Lagrangian advection schemes based on the analytic integration of the velocity field

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Abstract

A new scheme for the construction on an unstructured grid of the streamlines of the three-dimensional shallow water equations is presented. The qualitative advantages of the scheme, notably closed streamlines and realistic treatment of closed boundaries, are derived and the spatial accuracy is demonstrated.

Semi-Lagrangian advection schemes offer the computational cost advantage of being explicit but also unconditionally stable with respect to time step. However, semi-Lagrangian methods based on the numerical integration of the discretised velocity field frequently have difficulty in meeting physically significant criteria such as the closure of streamlines and the inviolability of closed boundaries. Here a streamline tracking scheme based on the analytic integration of the discretised velocity field is presented.

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Keywords: Computational fluid dynamics; Semi-Lagrangian; Advection; Streamlines; Finite volume; Unstructured mesh

1. Introduction

The streamline tracking scheme presented here has been developed as a part of a scheme for the three-dimensional shallow water equations [3]. The mesh used is composed of triangular prisms and

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the variables are discretised in a generalised Arakawa C grid scheme with the cell face normal velocity component at the centre of each face. The continuity equation is discretised using a finite volume scheme so the integral of the divergence of the velocity over a cell is guaranteed to be zero. A feature of this discretisation is that if a single layer of cells is used, then the scheme reduces to a solver for the two-dimensional shallow water equations. The tracking of streamlines in two-dimensional flow is therefore also of interest.

The concept behind a semi-Lagrangian scheme is that the material derivative of the velocity may be discretised by tracing a streamline back through one time step to calculate where the fluid that will arrive at a given point at the end of a time step has come from. The focus of this paper is on a method for constructing the streamlines rather than on the interpolation of the advected velocity. The importance of the quality of streamline tracking has been recognised elsewhere as significant in the results of semi-Lagrangian advection algorithms [4].

2. Streamlines

The streamlines of a velocity field are obtained by solving the ordinary differential equation:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}). \tag{1}$$

It is an obvious requirement that the continuous velocity field used to calculate the streamlines be consistent with the discretised velocity field produced by the flow model. Since the flow model is intended to model physical behaviour it is desirable that the streamlines produced should exhibit physically realistic behaviour. In particular, we shall require that the streamlines produced not cross each other or a closed boundary (the no crossing condition).

3. Integration techniques

The conventional way to integrate the velocity field would be using a numerical ODE solver (see, for example [4]). However, due to the errors inherent in numerical integration techniques, it is difficult or impossible to guarantee the no crossing condition. At a minimum, very small time steps are required when calculating streamlines near closed boundaries. Instead, we generalise the approach in [2] to our unstructured grid. The basis for this approach is the construction of a continuous velocity field which is then integrated analytically to produce an analytic expression for the streamline starting at a given point. Since the integration is analytic, no errors (up to machine precision) are introduced by the integration process.

The analytic integration approach imposes another constraint on the continuous velocity field: it must be analytically integrable at a reasonable computational cost. The obvious candidate is a cell-wise linear field. Assume $\mathbf{x}$ lies in cell i then:

$$\mathbf{u}(\mathbf{x}) = A_i\mathbf{x} + b_i, \tag{2}$$

where A_i is a constant matrix and $\mathbf{b}_i$ is a constant vector. The integration problem reduces in this case to the solution of a three-dimensional system of linear ordinary differential equations. To determine the unknown matrix and vector, we first impose the constraints given by the discretised flow field. For each face j of cell i, if $\mathbf{n}_j$ is the normal to that face and u_j is the normal velocity component then:

$$(A_i\mathbf{x}_j + b_i) \cdot \mathbf{n}_j = u_j. \tag{3}$$

This is a single linear scalar constraint which therefore costs one degree of freedom per face. In both two and three dimensions this leaves Eq. (2) underdetermined. It will therefore be necessary to find further constraints in order to uniquely specify a velocity field.

4. Crossing streamlines

In addition to failing to produce a well posed velocity field problem, the constraints so far imposed fail to guarantee that streamlines will not meet. In other words, the existence and uniqueness of solutions to (1) is not guaranteed at cell boundaries.

A two-dimensional example which illustrates this problem may be constructed by considering the case where the flux through an interior face is zero. If the flux through the other faces is not zero then solutions exist for which the normal velocity component to the zero flux face is not uniformly zero. This raises the possibility of the mirror image case presented in Fig. 1 in which streamlines meet. Not only does this pose severe conservation problems for a semi-Lagrangian scheme, it makes the tracking of streamlines generally impossible since streamlines may be followed into dead end situations. However, by specifying the normal component of velocity at every point on the cell face, this difficulty may be avoided. Clearly this requirement will also ensure that the inviolability of closed boundaries is preserved.

The most obvious manner in which this new constraint may be applied is by requiring that the face normal velocity component be everywhere equal to the average flux through the face (the flux constraint):

$$\frac{\partial u_i}{\partial z} = 0, \quad \frac{\partial v_i}{\partial z} = 0, \tag{4}$$

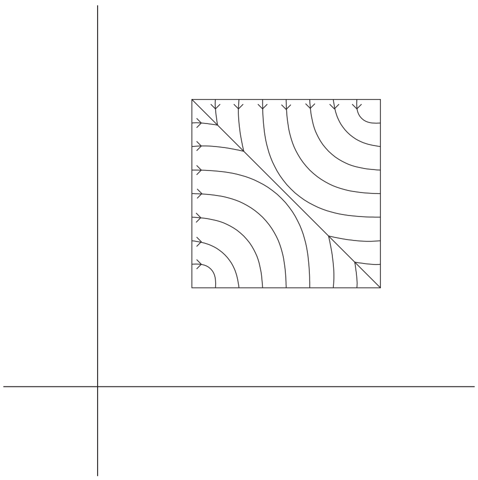

Fig. 1. Sketch of streamlines in two mirror image cells illustrating the possibility of crossing streamlines. The flux through the diagonal cell face is 0.

$$\frac{\partial w_i}{\partial x} = 0, \quad \frac{\partial w_i}{\partial y} = 0, \tag{5}$$

where u_i, v_i and w_i are the components of $\mathbf{u}_i$. In addition, if $\mathbf{t}_j$ is the horizontal tangent vector to side face j of cell i then the imposition of constant normal velocity along the line $\mathbf{x}_y + k\mathbf{t}_j$ amounts to a further constraint on each cell side. In the two-dimensional case this last constraint defines the remaining 3 degrees of freedom while in the three-dimensional case, these 3 and the 4 above in addition to the 5 imposed by (3) provide the 12 required constraints. In addition, constraints (4) and (5) make the matrix A_i block diagonal. This reduces the integration problem to a two-dimensional linear first-order ODE and a one-dimensional linear first-order ODE.

5. The continuity equation

Given that the discretised flow field is mass conservative, it is known that the normal flux integrated over the surface of one three-dimensional cell is zero. In the two-dimensional case, this is true if the flow is steady. It will be shown that the velocity field constructed above satisfies the continuity equation in the weak sense everywhere. We start with a simple proof concerning the situation within each grid cell.

Proposition 1. *If a linear velocity field satisfies the continuity equation integrated over some region of non-zero volume then the velocity field satisfies the continuity equation at every point.*

Proof. Let $\mathbf{u} = A\mathbf{x} + \mathbf{b}$ be a velocity field such that on some region V with $\mathrm{vol}(V) > 0$:

$$\int\int_{\Gamma V} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}A = 0. \tag{6}$$

Observe that:

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad \nabla \cdot \mathbf{u} = \mathrm{trace}(A) \tag{7}$$

which is constant. Then by Gauss' theorem:

$$\int\int_{\Gamma V} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}A = \mathrm{vol}(V)\mathrm{trace}(A).$$

Hence from (6) and (7) we conclude that $\forall \mathbf{x} \in \mathbb{R}^3$, $\nabla \cdot \mathbf{u} = 0$. □

We now wish to prove continuity everywhere. However, the velocity field we have constructed is not smooth (or even continuous) at cell boundaries so its divergence is undefined at those points. The strongest result we can therefore hope to prove is that continuity is satisfied when integrated over any region (possibly about such a point).

Proposition 2. *The integral over any non-trivial region of the divergence of a velocity field defined as specified in Sections* 3 *and* 4 *is* 0. *In two dimensions the velocity field is also assumed to be steady.*

Proof. Let $\mathbf{u}(\mathbf{x})$ be a two-dimensional velocity field satisfying the conditions above and let V be some region with $\mathrm{vol}(V) > 0$.

If V lies wholly within one grid cell, then $\mathbf{u}$ is linear throughout V and the result follows from Proposition 1. If V intersects more than one cell then we decompose V into a number of sub-regions $V_1, \ldots, V_n$ such that each sub-region intersects only one cell. Under Gauss' theorem we must show that

$$\int\int_{\Gamma V} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\Gamma = 0. \tag{8}$$

We know from the previous case that this statement holds if we replace V with V_i for any i in $1, \ldots, n$. Since the V_i are a decomposition of V, this is equivalent to requiring that the integral of the flux through the boundary shared between V_i and V_j is 0. Since these boundaries follow the cell boundaries, this follows immediately from the flux constraint. □

6. Closure of streamlines

A further physically significant property of streamlines in steady two-dimensional flow is that (except in unimportant degenerate cases) streamlines form closed loops in the flow. The usual proof of this result (see, for example [1, pp. 75–77]) assumes that the velocity field is continuous and differentiable, however the results above will allow us to generalise it to this case.

Proposition 3. *The streamlines of a steady two-dimensional velocity field defined as specified in Sections* 3 *and* 4 *are closed.*

Proof. To prove that the streamlines are closed, it is sufficient to demonstrate the existence of a stream function. The streamlines are then the contours of that function and are therefore closed. If $\mathbf{u} := (u, v)$ is the velocity field then the stream function is the scalar potential of the field $\mathbf{F} := (-v, u)$. That is, the stream function is defined (up to a constant) by

$$\Psi(\mathbf{x}) - \Psi_0 = \int_0^{\mathbf{x}} \mathbf{F} \, \mathrm{d}\mathbf{r}. \tag{9}$$

Clearly Ψ is only well defined if this integral is path independent. This in turn is equivalent to requiring that:

$$\oint_C \mathbf{F} \, \mathrm{d}\mathbf{r} = 0 \tag{10}$$

for any loop C. However, by construction of $\mathbf{F}$:

$$\oint_C \mathbf{F} \, \mathrm{d}\mathbf{r} = \oint_C \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s. \tag{11}$$

The right-hand side of this equation is simply the two-dimensional version of (8) so this result follows from Proposition 2. □

7. Increased accuracy

The assumption that the normal velocity component is constant over each cell face is first order in space. In the two-dimensional case, we may improve this by specifying that the normal velocity may vary linearly along each cell side. The appropriate gradient for this variation may be obtained using finite differences on nearby velocity points. This constraint incurs the same cost in degrees of freedom as specifying a constant normal velocity along each face however the resulting velocity field is now exact for linear velocity fields rather than only for constant fields. In other words, the approximation to the field is second order.

For three-dimensional models based on tetrahedral discretisations, this approach can be directly generalised. In this case, the tangent space to each face is two-dimensional so the total flux and two-flux gradients on each of the four triangular faces together fix the 12 degrees of freedom of the general linear three-dimensional velocity field. In this case, solving Eq. (1) primarily consists of solving a three-dimensional eigenproblem: still a relatively straightforward and inexpensive operation.

Whereas in [3], the grid is made up of layers of triangular prisms, each cell has 5 faces so the linear system obtained by directly generalising the two-dimensional case will be overdetermined. It is therefore necessary to reduce the number of constraints applied to the velocity. In a shallow water model, the vertical velocity is typically much less important than the horizontal velocities. With this in mind, we may choose to forgo second-order accuracy in the representation of this velocity. In other words, we continue to impose constraint (5). This results in the expression:

$$w_i = az + b \tag{12}$$

and hence

$$z_i(t) = -b/a + (z(0) + b/a)\,\mathrm{e}^{at}. \tag{13}$$

Now, just as the grid itself consists of layers of prisms, we may construct a two-dimensional field in the plane through the centre of each layer and then interpolate linearly between those layers. The horizontal velocity at a point between layers α and β is then given by

$$\mathbf{u}_{xy}(t) = \frac{z - z_\alpha}{z_\beta - z_\alpha}(A_\alpha \mathbf{x}_{xy} + \mathbf{b}_\alpha) + \frac{z - z_\beta}{z_\alpha - z_\beta}(A_\beta \mathbf{x}_{xy} + \mathbf{b}_\beta). \tag{14}$$

This is not a linear expression in $\mathbf{x}$, however when (13) is substituted into (14) then the resulting ODE for position is linear, albeit with variable coefficients. The form of the equation is

$$\mathbf{u}_{xy}(t) = (A\mathbf{x}_{xy} + \mathbf{b})(c + \mathrm{e}^{\mathrm{d}t}). \tag{15}$$

If $\{e_1, e_2\}$ are the generalised eigenvectors of A and $\{\lambda_1, \lambda_2\}$ are the corresponding eigenvalues, and $Z(t) = \mathrm{e}^{ct} + c\,\mathrm{d}t$ then a fundamental matrix for (15) is

$$\mathrm{X}(t) = [e_1 \mathrm{e}^{Z(t)\lambda_1} | e_2 \mathrm{e}^{Z(t)\lambda_2}]. \tag{16}$$

8. Results

The streamline closure property proven in the preceding sections may be demonstrated by tracking streamlines in a simple rotating flow:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

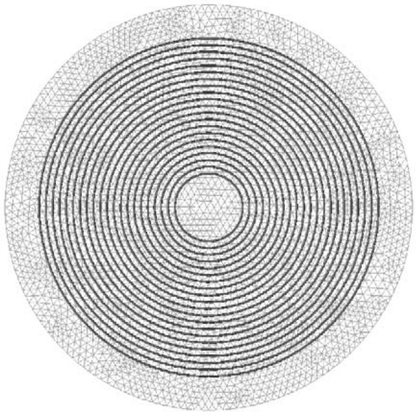

Fig. 2. Tracking of closed streamlines in a rotating flow. The errors in the connection of the loops are on the order of machine precision and are not visible at this scale.

The flow domain was 30 m in diameter and a typical grid length of 1 m was specified. Twenty one equally spaced streamlines at radiuses from 5 to 25 m were tracked for a full revolution (60πs) using the two-dimensional second-order scheme described in Section 7. Since the prescribed velocity field is linear, the second-order scheme was exact up to machine precision with the relative error (connection error divided by streamline length) being between 4×10^{-8} and 6×10^{-8} in each case. Fig. 2 illustrates the streamlines generated.

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