A Stochastic Parametrically-Forced NLS Equation

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Cover image: Kasezo (2018). *Glowing fiber optic strings* [3d illustration].

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Abstract

In this thesis, a variation on the nonlinear Schrödinger (NLS) equation with multiplicative noise is studied. In particular, we consider a stochastic version of the parametrically-forced nonlinear Schrödinger equation (PFNLS), which models the effect of linear loss and the compensation thereof by phase-sensitive amplification in pulse propagation through optical fibers. We establish global existence and uniqueness of mild solutions for initial data in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$.

The proof is an adaptation of a fixed-point argument employed by de Bouard and Debussche [Comm. Math. Phys., 205:161-181, 1999] for the nonlinear Schrödinger equation with multiplicative noise. The fixed-point argument relies on space-time estimates on the semigroup generated by the linear parametrically-forced Schrödinger operator. We prove these so-called Strichartz estimates, originally proven for the Schrödinger operator, using Fourier methods. A key difference between the Schrödinger operator and its parametrically-forced version is that the latter is not self-adjoint. We overcome this complication by establishing fixed-time estimates on the semigroup and its adjoint, based on their Fourier representations.

We also briefly discuss possible future research in the direction of stability of solitary standing wave solutions of the PFNLS equation under the influence of multiplicative noise. Using informal calculations, we demonstrate an approach to track the displacement of a soliton due to small stochastic forcing.

Preface

This thesis forms the conclusion of my time as a master's student in Applied Mathematics at the EEMCS faculty. In fact, this thesis marks the end of a fruitful and much enjoyed period as a student at the Delft University of Technology. Initially, I entered the department as a freshman to embark on the undergraduate program to complement my undergraduate studies in Applied Physics. However, it soon became clear that my interest lied principally with mathematics. Therefore, it is only natural that this thesis treats the rigorous mathematics behind an equation that originated from physics.

I would like to take this opportunity to thank Dr. Manuel Gnann for providing me with the opportunity to complete my thesis under his supervision. I enjoyed our weekly meetings in which he patiently advised me on encountered difficulties and provided insights into problems that previously seemed hopeless. Additionally, I much appreciate his advice on PhD positions abroad, and his support for my application process at Leiden University. I look forward to future collaboration. I would also like to thank Prof. Mark Veraar for his advice on PhD positions in the Netherlands, and his encouragement towards a successful application at Leiden University. Furthermore, I am grateful for his help with figuring out a course list for my exchange to Singapore, and many inspiring lectures throughout my undergraduate and graduate program. I also thank Dr. Frank van der Meulen for taking a seat on my thesis committee.

To Yanna, thank you for proofreading my work and providing a listening ear during study sessions. To Anand, thanks for your help with the cover image, for setting up a stream of the thesis defense and for proofreading my work. Finally, I would like to thank my roommates (both in Delft and Rotterdam), friends, and family for their continued interest in the project, even though my accounts of the progress were sometimes hard to follow.

I hope reading this thesis brings you joy and insight.

R.W.S. Westdorp
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Contents

Introduction

Nonlinear Schrödinger equations

The nonlinear Schrödinger (NLS) equation is a well-studied nonlinear partial differential equation that models the propagation of nonlinear dispersive waves if dissipative processes are negligible [33]. The equation is

$$z_t = i\Delta z + i\lambda |z|^{2\sigma} z \quad \text{for } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}^+. \tag{1}$$

Here, z is a complex-valued function of space and time, Δ is the Laplacian, and $\lambda, \sigma > 0$. The condition $\lambda > 0$ corresponds to the so-called 'focusing' NLS equation, and in case $\lambda < 0$ the equation is called 'defocusing'. The one dimensional focusing NLS equation finds application in the modeling of pulse propagation trough optical fibers, where the nonlinear term arises through the Kerr effect. Optical fibers can be used to set up systems for fiber-optic communications, enabling long-distance communication at high bandwidth [36]. In this application, z models a complex wave envelope of the electric field in a propagating pulse. In contrast to the linear Schrödinger equation from quantum mechanics, the variable t represents the physical distance along the fiber and x corresponds to physical time in the description of an optical wave through a nonlinear medium. For a detailed review of this application of the NLS equation, we refer to [6, 30].

The NLS equation is known to be globally well-posed for initial data in L^2 if $0 < \sigma < \frac{2}{n}$ [35], and in H^1 if $0 < \sigma < \frac{2}{n-2}$ [12]. These well-posedness results are based on dispersive properties of the Schrödinger evolution. In particular, a set of space-time estimates, called Strichartz estimates, provides an essential tool for the analysis of the (nonlinear) Schrödinger equation. For an extensive overview of the mathematical properties of the NLS equation, we refer the reader to [33].

The NLS equation supports solitary standing wave solutions (solitons) of the form

$$z(x) = \sqrt{c/2}\,\mathrm{sech}(\sqrt{c}x), \tag{2}$$

which are key to physical applications. Here, sech denotes the secant hyperbolic function and $c > 0$. In more realistic models of optical fibers dissipative processes can not be neglected, and in that situation, the soliton no longer remains. To compensate for the loss, an optical fiber loop that makes use of amplification can be considered. The parametrically-forced nonlinear Schrödinger (PFNLS) equation describes an optical fiber loop in which the linear loss is compensated by phase-sensitive amplification [17, 24, 25]. The PFNLS equation reads

$$z_t = i\Delta z - i\nu z - \epsilon(\gamma z - \mu \bar{z}) + 4i|z|^2 z \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}^+, \tag{3}$$

where the constants ϵ, γ, μ are all positive and ν is real-valued. The constant γ models the presence of dissipation, and the constant μ models the phase-sensitive gain. In order to prevent phase-sensitive effects arising from the phase-sensitive gain, the model includes a periodic conjugation constant ν. In [23], it was shown that the PFNLS equation supports solitons of the same form as (2). Moreover, this soliton is exponentially stable, meaning that a small perturbation of the soliton will decay exponentially to the same pattern.

A stochastic equation

For some physical systems, a more realistic model is obtained by considering perturbations of the system by noise, giving rise to a stochastic version of the NLS equation and variations thereof. See for instance the model proposed in [1] describing molecular aggregates with thermal fluctuations. In [3], de Bouard and Debussche prove existence of global mild solutions in L^2, in case the noise is sufficiently regular, for an NLS equation with multiplicative noise. In the subsequent paper [4], the same authors also show well-posedness for initial data in H^1. More recent works extend and generalize these existence results, for instance, to treat nonlinear noise terms [20], or to analyze the equation in the setting of compact manifolds [7, 8].

Although many variations and generalizations of the stochastic NLS equation are analyzed in the literature, these works do not cover the non-self-adjoint linear operation $i\Delta z - i\nu z - \epsilon(\gamma z - \mu\overline{z})$ found in the PFNLS equation. In this thesis, we analyze the following PFNLS equation with multiplicative noise:

$$\mathrm{d}z = (i\Delta z - i\nu z - \epsilon(\gamma z - \mu\overline{z}))\,\mathrm{d}t + 4i|z|^2 z\,\mathrm{d}t - i(z \circ \mathrm{d}W) \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}^+. \tag{4}$$

Here, the noise W is real-valued and $\circ$ denotes the Stratonovitch product. This choice of multiplicative noise is natural in the context of NLS equations, as it conserves the L^2-norm in the absence of dissipation and amplification ($\epsilon = 0$); a property that is motivated by physical arguments. Indeed, the Stratonovitch product follows the rules of classical calculus, and upon multiplying (4) with $\epsilon = 0$ by $\overline{z}$ and integrating we find

$$\mathrm{d}|z|^2_{L^2} = i(\langle \Delta z, z\rangle_{L^2} - \nu|z|^2_{L^2} + 4|z|^3_{L^3})\,\mathrm{d}t - i(|z|^2_{L^2} \circ \mathrm{d}W).$$

By taking the real part we informally see that the L^2-norm is conserved.

In this thesis we prove global existence and uniqueness of mild solutions to (4) for initial data in L^2 and H^1, thereby extending the works of de Bouard and Debussche [3, 4] to incorporate parametric forcing. These mild solutions take values in the spaces

$$L^r\left(\Omega; C([0,T]; L^2(\mathbb{R})) \cap L^r(0,T; L^p(\mathbb{R}))\right),$$

and

$$L^r\left(\Omega; C([0,T]; H^1(\mathbb{R})) \cap L^r(0,T; W^{1,p}(\mathbb{R}))\right),$$

for initial data in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ respectively. Here, p is a constant satisfying $p \geq 4$, which depends on the regularity of the noise. The exponent r is chosen such that (r,p) is an admissible pair for the Strichartz inequalities. The results hold under the assumption that the noise is suitably regular, to be specified in what follows.

We also briefly embark upon a discussion on the stability of solitons of the form (2) in the stochastic PFNLS equation. We demonstrate an approach to track the displacement of a soliton due to the stochastic perturbation, and discuss future research directions. With this aim in mind, we restrict ourselves in this work to one spatial dimension, in which an explicit representation of the soliton and the stability result of [23] are available.

The proof of existence and uniqueness directly follows the works of de Bouard and Debussche on the stochastic NLS equation in L^2 and H^1 [3, 4]. The proof is based on a fixed-point argument, employing the Banach fixed point theorem. As the nonlinearity is non-Lipschitz, we first consider an equation in which the nonlinearity is truncated. In this way, we obtain global mild solutions for the truncated equation, which give rise to a local mild solution for the original problem. We then formulate a blow-up criterion, stating that a finite existence time can only occur due to blow-up of the L^2-norm and H^1-norm of solutions in the cases of initial data in L^2 and H^1, respectively. We conclude by proving a bound on the L^2-norm and H^1-norm of solutions on finite time intervals. This bound is derived from a formula describing the evolution of the L^2 and H^1 norm of the mild solutions to (4). The proof of this formula relies on an application of Itô's formula, which we justify using a rather technical regularization procedure.

The original fixed-point argument of de Bouard and Debussche relies on the Strichartz estimates that are available for the Schrödinger evolution. As such, a significant part of this thesis is devoted to a proof of Strichartz estimates for the evolution of the linear parametrically-forced Schrödinger (PFS) equation. This result forms the main contribution of this work. The linear evolution of the Schrödinger equation is given by a multiplication in the Fourier space with the exponential $e^{-it|\xi|^2}$. Fixed-time estimates easily follow from this Fourier solution, which are needed to prove the Strichartz estimates. To recover these fixed-time estimates in the case of the PFS equation, we explicitly compute its Fourier solution, which takes a more complicated form. Based on these fixed-time estimates, we then prove the Strichartz estimates for the PFS evolution, using an additional estimate on the adjoint of the semigroup. We require this additional estimate because the operator of the linear parametrically-forced Schrödinger equation is not self-adjoint, as opposed to the Schrödinger operator.

Notation

Throughout this thesis, the following notation is used.

Function spaces

For $p \in [1,\infty]$, we denote by $L^p(\mathbb{R})$ the Lebesgue space of complex valued functions on the real line, or we use the shorthand notation L^p_x. We write

$$\|\cdot\|_{L^p_x} = \left(\int_{\mathbb{R}} |\cdot|^p \,\mathrm{d}x\right)^{1/p}$$

for its norm and in case $p = 2$ we denote the inner product by

$$\langle f, g\rangle_{L^2_x} = \int_{\mathbb{R}} f(x)\overline{g}(x)\,\mathrm{d}x.$$

We denote the norm of general normed spaces X by $\|\cdot\|_X$ and the norm of general inner product spaces by $\langle\cdot,\cdot\rangle_H$. The weak derivative of a weakly differentiable function $f \in L^p(\mathbb{R})$ is denoted by $\partial_x f$ and we write $\Delta = \partial_x^2$ for the Laplacian. For $k \in \mathbb{N}_0$ and $p \in [1,\infty]$, $W^{k,p}(\mathbb{R})$ (or the shorthand $W^{k,p}_x$) is the Sobolev space of functions $f \in L^p(\mathbb{R})$ that are k times weakly differentiable with weak derivatives up to order k belonging to $L^p(\mathbb{R})$. We equip this space with the norm

$$\|\cdot\|_{W^{k,p}_x} = \sum_{\alpha=0}^{k} \|\partial_x^\alpha \cdot\|_{L^2_x}.$$

We write $H^k_x = W^{k,2}_x$ for $k \in \mathbb{N}_0$ and equip this space with the inner product

$$\langle\cdot,\cdot\rangle_{H^k} = \sum_{\alpha=0}^{k} \langle \partial_x^\alpha \cdot, \partial_x^\alpha \cdot\rangle_{L^2_x}.$$

For $r \in [0,\infty]$, we will also use the Lebesgue-Bochner spaces of the form $L^r(I;X)$, where I is an interval in the real line and X is a Banach space. These are the strongly Lebesgue-measurable functions $f : I \to X$ such that $t \mapsto \|f(t)\|_X$ is in $L^r(I)$. In case $I = [0,T]$ for a fixed $T > 0$, we use the shorthand $L^r_t(X)$. Hence, we also combine shorthand notations as $L^r_t(L^p_x) = L^r(0,T;L^p(\mathbb{R}))$.

Lastly, we write $\mathscr{S}(\mathbb{R})$ for the Schwartz class of functions whose derivatives are rapidly decreasing. A more precise definition will follow.

Operator classes

By $\mathscr{L}(X,Y)$ we denote the class of bounded linear operators from a normed space X into a normed space Y. Let $H, \tilde{H}$ be separable Hilbert spaces and let X be a Banach space. We denote the class of Hilbert-Schmidt operators from H into $\tilde{H}$ as $\mathscr{L}_2(H;\tilde{H})$ and the class of γ-radonifying operators from H into X as $\gamma(H;X)$. Both operator classes will be introduced later.

Thesis outline

This thesis is composed of four themed chapters.

Chapter 1 provides an overview of the theory underlying the topic, which we will make use of in subsequent chapters. First, we briefly review the basic properties C_0-semigroups and their application to partial differential equations in Section 1.1. Then, in Section 1.2, we introduce the concept of Fourier multipliers and present the Mikhlin multiplier theorem. As an important example, we consider the Riesz potential and we state the related Hardy-Littlewood-Sobolev inequality. In Section 1.3 we give a compressed overview of the theory of stochastic integration in the Banach-valued setting. We start by introducing the concept of Brownian motion in Hilbert spaces, which we use to define the stochastic integral. We then discuss various properties of the stochastic integral and state a few useful inequalities.

Chapter 2 is devoted to the linear parametrically-forced Schrödinger equation, since an understanding of the linear equation is essential to the analysis of the stochastic nonlinear equation. We first show in Section 2.1 that the parametrically-forced Schrödinger operator gives rise to a Fourier solution, and we furthermore show that it generates a C_0-group on $L^2(\mathbb{R})$ in Section 2.2. We then use the Fourier solution to derive fixed-time estimates on the semigroup in Section 2.3. Finally, we prove that the semigroup satisfies the Strichartz estimates in Section 2.4.

In Chapter 3, we turn to the analysis of the stochastic equation (4), in which we give a combined presentation of the results in [3] and [4]. We start by describing the setting and assumptions in more detail in Section 3.1. Then, in Section 3.2, we prove a few useful estimates on the stochastic convolution with the semigroup of the parametrically-forced Schrödinger equation which are required later on. We proceed by proving global existence and uniqueness of mild solutions for a truncated equation in Section 3.3 using a fixed-point argument. The solutions to the truncated problem are then used to define a local solution to the original problem in Section 3.4. Here we also formulate a blow-up criterion, stating that a finite existence time can only occur in case the H^s_x-norm of a solution blows up. Finally, we prove in Section 3.5 that blow-up cannot occur, by analysis of the evolution of the H^s_x-norm using Itô's formula.

In the final Chapter, Chapter 4, we discuss the stability of solitons in the (stochastic) PFNLS equation. We start in Section 4.1 by showing that the deterministic equation admits solitons, and present a stability result due to Kapitula and Sandstede [23]. Then, in Section 4.2, we display an approach to tracking the position of a stochastically perturbed soliton. We analyze the leading-order behavior of this position correction in Section 4.3. Lastly, we discuss directions for future research in Section 4.4.

1
Theory

This chapter aims to give a concise overview of the theory required for the analysis of the stochastic PFNLS equation. The proofs in subsequent chapters rely heavily on the properties of the C_0-semigroup generated by the linear operator of Equation (3). We therefore briefly recollect the properties of C_0-semigroups and their generators, and we discuss their relation to partial differential equations in Section 1.1. We will also require many tools from Fourier analysis, and we set out the basic concepts of the Fourier transform and related theory in Section 1.2, restricting ourselves to results needed for subsequent proofs. As Equation (4) is a stochastic partial differential equation, its analysis belongs to the field of stochastic integration theory. In particular, a bit of knowledge on the integration of operator-valued processes is required to understand the formal meaning of (4). In Section 1.3, we, therefore, give a compressed overview of the theory of stochastic integration in the Banach-valued setting.

1.1. Strongly continuous (semi)groups

The concept of strongly continuous semigroups generalizes the exponential solution of a finite-dimensional system of linear ODEs to the infinite-dimensional setting. Consider a system of linear first-order ODEs of the form

$$\begin{cases} u_t = Au, \\ u(0) = u_0 \in \mathbb{R}^n, \end{cases}$$

where $u(t) \in \mathbb{R}^n$ and A is an $n \times n$ matrix. Such a system has a unique solution $u \in C^\infty(\mathbb{R}; \mathbb{R}^n)$, given by $u(t) = e^{tA}u_0$. Here, e^{tA} is the matrix exponential of tA, defined as

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \ldots + \frac{1}{n!}t^nA^n + \ldots,$$

for $t \in \mathbb{R}$. More generally, we can define the operator exponential of a bounded operator A on a Banach space X by the same series. The operator exponential e^{tA} is then well-defined as a bounded linear operator on X, and it enjoys the properties

- $e^{0A} = I$;
- $e^{sA}e^{tA} = e^{(s+t)A}$ for all $s, t \in \mathbb{R}$;
- $t \mapsto e^{tA}$ is continuous;
- $\frac{\mathrm{d}}{\mathrm{d}t}e^{tA} = Ae^{tA}$.

Consider now the evolution equation

$$\begin{cases} u_t = Au, \\ u(0) = u_0 \in X, \end{cases}$$

where $u(t) \in X$ and $A \in \mathscr{L}(X)$. The unique solution $u \in C^\infty(\mathbb{R}; X)$ of this problem is given by the operator exponential, as $u(t) = e^{tA}u_0$. In applications to partial differential equations, the operator A is usually a differential operator acting on a suitable function space X. In most relevant cases, the operator A is, however, unbounded on the Banach space X. Take for instance the heat equation posed on $L^p(\mathbb{R})$, where the Laplacian is unbounded. With the aim of generalizing the notion of solution operators for linear evolution equations to unbounded linear operators, we introduce the following definition of a strongly continuous semigroup.

Definition 1.1.1 (C_0-semigroup)
*A family of bounded linear operators $S = \{S(t)\}_{t\geq 0}$ acting on a Banach space X is called a C_0-**semigroup** if:*

1. $S(0) = I$,
2. $S(s)S(t) = S(s+t)$ *for all* $t, s \geq 0$ *(Semigroup property),*
3. $\lim_{t\downarrow 0} \|S(t)x - x\|_X = 0$ *for all* $x \in X$ *(Strong continuity).*

If furthermore $\|S(t)\|_{\mathscr{L}(X)} \leq 1$ *for all* $t \geq 0$*, then we call the semigroup contractive.*

The infinitesimal generator of S is the linear operator A with domain $D(A)$ defined by

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \tfrac{1}{t}(S(t)x - x) \quad \text{exists} \right\},$$
$$Ax = \lim_{t\downarrow 0} \tfrac{1}{t}(S(t)x - x), \quad x \in D(A).$$

If instead, we consider a family of bounded linear operators parameterized by the real line instead of the half-line, we may analogously define the notion of a C_0-group.

Definition 1.1.2 (C_0-group)
*A family of bounded linear operators $S = \{S(t)\}_{t\in\mathbb{R}}$ acting on a Banach space X is called a C_0-**group** if:*

1. $S(0) = I$,
2. $S(s)S(t) = S(s+t)$ *for all* $t, s \in \mathbb{R}$ *(Group property),*
3. $\lim_{t\to 0} \|S(t)x - x\|_X = 0$ *for all* $x \in X$ *(Strong continuity).*

The infinitesimal generator of a C_0-group S is also defined in a similar way, by

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \tfrac{1}{t}(S(t)x - x) \quad \text{exists} \right\},$$
$$Ax = \lim_{t\to 0} \tfrac{1}{t}(S(t)x - x), \quad x \in D(A),$$

the only difference with the generator of a C_0-semigroup being the convergence $t \to 0$ instead of $t \downarrow 0$.

Remark 1.1.3. The family of operators $\{S(t)\}_{t\in\mathbb{R}}$ is a strongly continuous group if and only if $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup of invertible operators and $S(-t) = S^{-1}(t)$ for all $t \geq 0$.

1.1.1. Characterization of generators and properties

The following theorem characterizes when an unbounded linear operator A generates a C_0-semigroup.

Theorem 1.1.4 (Hille-Yosida)
An unbounded linear operator $A : D(A) \subseteq X \to X$ on a Banach space X generates a C_0-semigroup on X if and only if there exist constants $M \geq 1$ and $a \in \mathbb{R}$ such that

1. *the domain $D(A)$ is dense in X and A is closed;*
2. *every $\lambda \in \mathbb{R}$ such that $\lambda > a$ belongs to the resolvent set of A;*

3. *if $\lambda > a$ and $n \in \mathbb{N}$, then*
$$\|(\lambda I - A)^{-n}\|_X \leq \frac{M}{(\lambda - a)^n}.$$

In that case, $\|S(t)\|_{\mathscr{L}(X)} \leq Me^{at}$ *for all* $t \geq 0$.

A proof of the Hille-Yosida theorem can for instance be found in [29, Theorem 5.3, p. 20]. The condition on powers of the resolvent is often hard to check in practice. The Lumer-Phillips theorem provides a more useful condition for unbounded operators to generate a contraction semigroup, especially in the Hilbert space setting. For a proof, we refer to [29, Theorem 4.3, p. 14].

Theorem 1.1.5 (Lumer-Phillips)
An operator $A : D(A) \subset H \to H$ in a Hilbert space H is the generator of a contraction semigroup on H if and only if:

1. *the domain $D(A)$ is dense in H and A is closed;*
2. $\mathrm{Re}\langle Ax, x\rangle_H \leq 0$ *for all $x \in D(A)$ (dissipativity);*
3. *the range of $\lambda I - A$ is equal to H for some $\lambda > 0$.*

We collect the following elementary properties of C_0-semigroups.

Proposition 1.1.6
Let S be a C_0-semigroup on X with generator A.

1. *If T is another C_0-semigroup generated by A, then $T(t) = S(t)$ for $t \geq 0$.*
2. *The semigroup generates continuous orbits from all starting points in X, that is $t \mapsto S(t)x$ is a continuous X-valued function for all $x \in X$ and $t \geq 0$.*
3. *The semigroup generates continuously differentiable orbits for all starting points in $D(A)$, that is $t \mapsto S(t)x$ is a differentiable X-valued function for all $x \in D(A)$. We furthermore have $S(t)x \in D(A)$, and*
$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)x = AS(t)x = S(t)Ax, \quad t \geq 0.$$
4. *For all $x \in X$ we have $\int_0^t S(s)x\,\mathrm{d}s \in D(A)$, and*
$$A\int_0^t S(s)x\,\mathrm{d}s = S(t)x - x.$$
If $x \in D(A)$, then both sides are equal to $\int_0^t S(s)Ax\,\mathrm{d}s$.

The proofs of these properties can for instance be found in [29, Section 1.2]. We also refer the interested reader to this work for a detailed treatment of the topic.

1.1.2. Application to evolution equations

From Proposition 1.1.6 it follows that $u(t) = S(t)u_0$ solves the evolution equation
$$\begin{cases} u_t = Au, \\ u(0) = u_0 \in X, \end{cases}$$
where $u(t) \in X$, $A : D(A) \subseteq X \to X$ is an unbounded operator on X that generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$, and $u_0 \in D(A)$. For arbitrary $u_0 \in X$, the orbits generated by $\{S(t)\}_{t\geq 0}$ are not necessarily differentiable, and we can therefore not expect $u(t) = S(t)u_0$ to solve the evolution equation in the classical sense. It is, however, true that $u(t) = S(t)u_0$ solves the integrated version
$$u(t) = u_0 + \int_0^t Au(s)\,\mathrm{d}s, \quad t \in [0, T].$$

Let us now consider a nonlinear evolution equation

$$\begin{cases} u_t = Au + f(t, u(t)), \\ u(0) = u_0 \in X, \end{cases} \tag{1.1}$$

where f is a nonlinearity depending on both time and u. In this situation we can also not expect classical solutions for arbitrary initial data $u_0 \in X$. Instead, we introduce the concept of a mild solution.

Definition 1.1.7 (Mild solution)
$u : [0, T] \to X$ is called a mild solution of (1.1) if u is continuous and satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s) f(s, u(s))\,\mathrm{d}s.$$

Under suitable conditions on the nonlinearity f (e.g. Lipschitz continuity), it can be shown that the semi-linear evolution equation admits unique mild solutions. Moreover, every classical solution is a mild solution. For proofs of these claims and a more detailed discussion, we refer to [29, Chapter 6]. This shows why the notion of a mild solution is convenient for semi-linear evolution equations with suitable nonlinearities. Definition 1.1.7 is general enough so that for each initial datum u_0 there exists a unique mild solution, and this class of solutions contains the classical solutions.

1.2. Fourier analysis

Fourier analysis forms a powerful tool for the study of partial differential equations. Below we present the fundamental properties of the Fourier transform, and a few (loosely) related results. Most of the material in this section is standard, and we refer to [18, Chapter 2] for the results where a proof or reference to a proof is omitted.

Definition 1.2.1 (Schwartz space)
The Schwartz space $\mathscr{S}(\mathbb{R})$ is the set of smooth functions $f:\mathbb{R}\to\mathbb{C}$, such that for each $\alpha,\beta\in\mathbb{N}$ we have

$$x^\alpha \partial_x^\beta f \to 0 \quad \text{as} \quad |x|\to\infty.$$

Proposition 1.2.2
The Schwartz space $\mathscr{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ if $1\le p<\infty$.

Definition 1.2.3 (Fourier transform)
For a function $f\in\mathscr{S}(\mathbb{R})$, the Fourier transform of f is the function $\hat{f}:\mathbb{R}\to\mathbb{C}$ defined as

$$\hat{f}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{-i\xi x}\,\mathrm{d}x, \quad \xi\in\mathbb{R}. \tag{1.2}$$

The operator $\mathscr{F}: f\mapsto \hat{f}$ is called the Fourier transform. The inverse Fourier transform of f is the function $\check{f}:\mathbb{R}\to\mathbb{C}$ defined by

$$\check{f}(x) := \int_{\mathbb{R}} f(x)e^{i\xi x}\,\mathrm{d}x, \quad x\in\mathbb{R}. \tag{1.3}$$

Proposition 1.2.4
The Fourier transform $\mathscr{F}$ is a continuous, one-to-one map of $\mathscr{S}(\mathbb{R})$ onto itself. Its inverse $\mathscr{F}^{-1}$ is given by $\mathscr{F}^{-1}: f\mapsto\check{f}$.

We may also define the Fourier transform on $L^1(\mathbb{R})$ via the formula in (1.2), since

$$\left|\frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{-i\xi x}\mathrm{d}x\right| \le \frac{1}{2\pi}\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x = \frac{1}{2\pi}|f|_{L^1_x}.$$

By taking the supremum, we see that the Fourier transform $\mathscr{F}$ is a bounded operator from $L^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$. If $f\in L^2(\mathbb{R})\setminus L^1(\mathbb{R})$, then we can in general not define the Fourier transform of f via the formula in (1.2), since the integral does not converge. Instead, we use that the Schwartz space is dense in $L^2(\mathbb{R})$ (Proposition 1.2.2) and define the Fourier transform of $f\in L^2(\mathbb{R})$ as the L^2-limit of the Fourier transforms of an approximating sequence of Schwartz functions. The next theorem shows that the Fourier transform is an isometry on $L^2(\mathbb{R})$.

Theorem 1.2.5 (Plancherel theorem)
If $f\in L^2(\mathbb{R})$, then $\hat{f}\in L^2(\mathbb{R})$ and

$$|\hat{f}|_{L^2_\xi} = \frac{1}{2\pi}|f|_{L^2_x}.$$

We can combine Parseval's theorem with the bound $|f|_{L^\infty_x}\le\frac{1}{2\pi}|f|_{L^1_x}$ to define the Fourier transform on $L^p(\mathbb{R})$ for all $1\le p\le 2$. This follows from the following interpolation theorem due to Riesz and Thorin [34].

Theorem 1.2.6 (Riesz-Thorin)
Let Ω be a measure space and $1\le p_0,p_1\le\infty, 1\le q_0,q_1\le\infty$. Suppose that

$$T: L^{p_0}(\Omega)+L^{p_1}(\Omega)\to L^{q_0}(\Omega)+L^{q_1}(\Omega)$$

is a linear map such that $T : L^{p_i}(\Omega) \to L^{q_i}(\Omega)$ for $i = 0,1$ and

$$|Tf|_{L^{q_0}(\Omega)} \le M_0 |f|_{L^{p_0}(\Omega)}, \quad |Tf|_{L^{q_1}(\Omega)} \le M_1 |f|_{L^{p_1}(\Omega)},$$

for some constants M_0, M_1. If $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then $T : L^p(\Omega) \to L^q(\Omega)$ maps $L^p(\Omega)$ into $L^q(\Omega)$ and

$$|Tf|_{L^q(\Omega)} \le M_0^{1-\theta} M_1^{\theta} |f|_{L^p(\Omega)}.$$

As an immediate consequence, we obtain the Hausdorff-Young theorem, which shows that the Fourier transform maps $L^p(\mathbb{R})$ into $L^{p'}(\mathbb{R})$ for $1 \le p \le 2$.

Theorem 1.2.7 (Hausdorff-Young)
If $f \in L^p(\mathbb{R})$ with $1 \le p \le 2$, then $\hat{f} \in L^{p'}(\mathbb{R})$ and

$$|\hat{f}|_{L_\xi^{p'}} \le \frac{1}{2\pi} |f|_{L_x^p}.$$

The Fourier transform can be used to define operators on $L^p(\mathbb{R})$ by multiplying its Fourier transform with a function.

Definition 1.2.8 (Fourier multipliers)
For a function $m : \mathbb{R} \to \mathbb{C}$, we define the Fourier multiplier operator T_m as

$$T_m f : f \mapsto \mathscr{F}^{-1}\{m\hat{f}\}.$$

The function m is called the symbol of the Fourier multiplier T_m. If a Fourier multiplier is furthermore bounded on $L^p(\mathbb{R})$, then we call it an L^p-multiplier.

Example 1.2.9. For a differentiable function f, we can compute the derivative using the Fourier inversion formula (1.3) as

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} \,\mathrm{d}\xi = \int_{\mathbb{R}} i\xi \hat{f}(\xi) e^{i\xi x} \,\mathrm{d}\xi = \mathscr{F}^{-1}\{i\xi \hat{f}(\xi)\}(x).$$

This shows that differentiation is a Fourier multiplier, with symbol $m(\xi) = i\xi$.

Remark 1.2.10. If T is an L^p-multiplier, we can write

$$|Tf|_{W_x^{k,p}} = \sum_{\alpha=0}^{k} |\partial_x^\alpha (Tf)|_{L_x^p} = \sum_{\alpha=0}^{k} |T(\partial_x^\alpha f)|_{L_x^p} \le C \sum_{\alpha=0}^{k} |\partial_x^\alpha f|_{L_x^p} = C|f|_{W_x^{k,p}},$$

since Fourier multipliers commute. This shows that an L^p-multiplier is also bounded on $W^{k,p}(\mathbb{R})$.

As a consequence, we have the following characterization of the H^k-norm.

Proposition 1.2.11 (Fourier characterization of Sobolev spaces)
If $k \in \mathbb{N}_0$, and $f \in H^k(\mathbb{R})$, then

$$|f|_{H_x^k} \simeq 2\pi \left(\int_{\mathbb{R}} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \right)^{1/2}.$$

Here, the symbol '≃' denotes norm equivalence.

Proof. We compute

$$4\pi^2\int_{\mathbb{R}}(1+|\xi|^2)^k|\hat{f}(\xi)|^2\,\mathrm{d}\xi = 4\pi^2\int_{\mathbb{R}}\sum_{\alpha=0}^{k}\binom{k}{\alpha}|\xi|^{2\alpha}|\hat{f}(\xi)|^2\,\mathrm{d}\xi = 4\pi^2\sum_{\alpha=0}^{k}\binom{k}{\alpha}\int_{\mathbb{R}}|(i\xi)^\alpha\hat{f}(\xi)|^2\,\mathrm{d}\xi$$
$$= \sum_{\alpha=0}^{k}\binom{k}{\alpha}|\partial_x^\alpha f|^2_{L^2_x},$$

which is equivalent to the square of the $H^k(\mathbb{R})$-norm. □

We can also consider the Fourier multiplier with symbol $\xi \mapsto |\xi|^{-\alpha}$, which corresponds to the Riesz potential operator and belongs to the topic of fractional integration.

Definition 1.2.12 (Riesz potential)
*For $0<\alpha<1$, we define the **Riesz potential** $I_\alpha : \mathbb{R}\to\mathbb{R}$ as the function*

$$I_\alpha(x) = \frac{1}{\gamma_\alpha|x|^{1-\alpha}},$$

where γ_α is the constant

$$\gamma_\alpha = \frac{2^\alpha\sqrt{\pi}\,\Gamma(\alpha/2)}{\Gamma(1/2-\alpha/2)}.$$

*We also define the **Riesz potential** of order α of a function $\phi\in\mathcal{S}(\mathbb{R})$ as*

$$(I_\alpha*\phi)(x) = \frac{1}{\gamma_\alpha}\int_{\mathbb{R}}\frac{\phi(y)}{|x-y|^{1-\alpha}}\,\mathrm{d}y.$$

The next theorem shows that this operator is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, where $\frac{1}{q}=\frac{1}{p}-\alpha$.

Theorem 1.2.13 (Hardy-Littlewood-Sobolev)
Suppose that $0<\alpha<1$, $1<p<1/\alpha$, and q is defined by $\frac{1}{q}=\frac{1}{p}-\alpha$. Then there exists a constant $C(\alpha,p)$ so that

$$|I_\alpha*\phi|_{L^q_x} \le C(\alpha,p)|\phi|_{L^p_x} \quad \textit{for all } \phi\in L^p(\mathbb{R}).$$

A proof can be found in [19, Theorem 6.1.3, p. 3]. The following theorem provides a useful condition to assert that a Fourier multiplier is a bounded operator on $L^p(\mathbb{R})$

Theorem 1.2.14 (Mikhlin multiplier theorem)
If $m:\mathbb{R}\setminus\{0\}\to\mathbb{C}$ satisfies

$$|m(\xi)| \le C_0, \quad \textit{for all } \xi\in\mathbb{R}\setminus\{0\}$$

and

$$|\xi|\left|\frac{\mathrm{d}m}{\mathrm{d}\xi}(\xi)\right| \le C_1, \quad \textit{for all } \xi\in\mathbb{R}\setminus\{0\}$$

and some $C_0, C_1>0$, then the Fourier multiplier T_m with symbol m is an L^p-multiplier for all $1<p<\infty$. Moreover, the operator norm of T_m depends only on C_0, C_1, and p.

For a proof, see [18, Theorem 6.2.7, p. 446]. We present also a few useful properties of the operator $(I-\epsilon\Delta)^{-1}$, which will serve as a regulizing operator. By the Lax-Milgram theorem, there exists for every $f\in H^{-1}(\mathbb{R})$ a unique element $u\in H^1(\mathbb{R})$ such that

$$u-\epsilon\Delta u = f, \quad \text{in } H^{-1}(\mathbb{R}).$$

Hence, we see that $(I-\epsilon\Delta)^{-1}$ is well-defined as a mapping from $H^{-1}(\mathbb{R})$ to $H^1(\mathbb{R})$. We may also define $(I-\epsilon\Delta)^{-1}$ on $L^p(\mathbb{R})$ with $p\ge 1$ as a Fourier multiplier with symbol $(1+\epsilon\xi^2)^{-1}$. We collect the following useful properties of the operator $(I-\epsilon\Delta)^{-1}$.

Lemma 1.2.15 (Regulization)
Let $p \in (1,\infty)$, $k \in \mathbb{N}$, *and* $f \in W^{k,p}$.

1. *For all* $\epsilon > 0$, $(I-\epsilon\Delta)^{-1}$ *is a bounded operator from* $W^{k,p}(\mathbb{R})$ *to* $W^{k+2,p}(\mathbb{R})$, *with*
$$|(I-\epsilon\Delta)^{-1}f|_{W_x^{k+2,p}} \le C_{\epsilon,p}|f|_{W_x^{k,p}}. \tag{1.4}$$
2. $(I-\epsilon\Delta)^{-1}$ *is also bounded from* $W^{k,p}(\mathbb{R})$ *to* $W^{k,p}(\mathbb{R})$, *with*
$$|(I-\epsilon\Delta)^{-1}f|_{W_x^{k,p}} \le C_p|f|_{W_x^{k,p}}, \tag{1.5}$$
where C_p *does not depend on* ϵ.
3. $(I-\epsilon\Delta)^{-1}f$ *converges to* f *in* $W^{k,p}(\mathbb{R})$ *as* $\epsilon \downarrow 0$.

The proof can be found in Appendix A. We conclude the section with a useful embedding result.

Proposition 1.2.16
If $p \ge 2$, *then* $H^1(\mathbb{R})$ *embeds continuously into* $L^p(\mathbb{R})$ *and* $L^{p'}(\mathbb{R})$ *embeds continuously into* $H^{-1}(\mathbb{R})$. *That is,*
$$H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}), \quad \textit{and} \quad L^{p'}(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R}). \tag{1.6}$$

Proof. We first show that $H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$. Let therefore $u \in C_c^\infty(\mathbb{R})$ and $p \ge 2$. Then
$$|u|_{L_x^p}^p = \int_{\mathbb{R}} |u|^p \,\mathrm{d}x = \int_{\mathbb{R}} |u|^{p-2}|u|^2 \,\mathrm{d}x \le |u|_{L_x^\infty}^{p-2}|u|_{L_x^2}^2,$$
so that
$$|u|_{L_x^p} \le |u|_{L_x^\infty}^{\frac{p-2}{p}} |u|_{L_x^2}^{\frac{2}{p}} \le |u|_{L_x^\infty}^{\frac{p-2}{p}} |u|_{H_x^1}^{\frac{2}{p}}.$$
As u is smooth, we can write
$$(u(x))^2 = 2\int_{-\infty}^x u(y)\partial_x u(y)\,\mathrm{d}y,$$
so that
$$|u|_{L_x^\infty}^2 \le 2|u|_{L^2}|\partial_x u|_{L_x^2} \le 2|u|_{H_x^1}^2,$$
and we conclude that
$$|u|_{L_x^p} \le 2^{\frac{p-2}{p}} |u|_{H_x^1}^{\frac{p-2}{p}} |u|_{H_x^1}^{\frac{2}{p}} = 2^{\frac{p-2}{p}} |u|_{H_x^1}. \tag{1.7}$$

The general case follows by density of $C_c^\infty(\mathbb{R})$ in $H^1(\mathbb{R})$. For the dual estimate, let $v \in L^{p'}(\mathbb{R})$. Since $H^{-1}(\mathbb{R})$ is the dual of $H^1(\mathbb{R})$, and $L^{p'}(\mathbb{R})$ is isometrically isomorphic to the dual of $L^p(\mathbb{R})$ [31, Theorem 4.1, p. 13], we can write
$$|v|_{H_x^{-1}} = \sup_{u\in H_x^1} \frac{\langle u,v\rangle_{H_x^1\times H_x^{-1}}}{|u|_{H_x^1}} \le \sup_{u\in L_x^p} 2^{\frac{p-2}{p}} \frac{\langle u,v\rangle_{H_x^1\times H_x^{-1}}}{|u|_{L_x^p}} = 2^{\frac{p-2}{p}} \sup_{u\in L_x^p} \frac{\langle u,v\rangle_{L_x^p\times L_x^{p'}}}{|u|_{L_x^p}} = 2^{\frac{p-2}{p}} |v|_{L_x^{p'}},$$
where we have used (1.7) in the second step. This shows the result. □

1.3. Stochastic integration in Banach spaces

In this section, we give a condensed introduction to the topic of stochastic integration in Banach spaces. A detailed treatment of the subject can be found in [27], as well as in [14] and [26]. Before we can give the definition of the stochastic integral, we first need to introduce the concepts of type 2 Banach spaces, γ-radonifying operators and cylindrical Brownian motion. Throughout this section, let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of independent normal real-valued random variables on a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$, and let E be a Banach space.

1.3.1. Type p Banach spaces and γ-radonifying operators

We start by introducing the following geometric property of a Banach space, related to random sums.

Definition 1.3.1 (Type *p* Banach space)
*Let $p\in[1,2]$. The Banach space E has the **type** p **property** if there exists a constant $C>0$ such that for all finite sequences $x_1,\dots,x_n\in E$*

$$\mathbb{E}\left\|\sum_{n=1}^N \gamma_n x_n\right\|_E^p \le C\sum_{n=1}^N \|x_n\|_E^p.$$

The following result can be interpreted as a norm-equivalence result in the context of random sums.

Theorem 1.3.2 (Kahane-Khintchine inequality)
For all $1\le p,q<\infty$ there exists a constant $C_{p,q}$ such that for all finite sequences $x_1,\dots,x_n\in E$ we have

$$\left(\mathbb{E}\left\|\sum_{n=1}^N \gamma_n x_n\right\|_E^p\right)^{\frac{1}{p}} \le C_{p,q}\left(\mathbb{E}\left\|\sum_{n=1}^N \gamma_n x_n\right\|_E^q\right)^{\frac{1}{q}}.$$

A proof can be found in [22, Theorem 6.2.6, p. 23]. With help of the previous theorem, we can give an example of a type 2 Banach space.

Proposition 1.3.3
For $k\in\mathbb{N}$ and $p\ge 2$, the Banach space $W^{k,p}(\mathbb{R})$ has the type 2 property.

Proof. Using Hölder's inequality and Fubini's theorem, we may write for $f_1,\dots,f_N\in W^{k,p}(\mathbb{R})$

$$\left|\sum_{n=1}^N \gamma_n f_n\right|_{L^2(\Omega;W_x^{1,p})} \le \left|\sum_{n=1}^N \gamma_n f_n\right|_{L^p(\Omega;W_x^{1,p})} = \sum_{\alpha=0}^k\left(\int_{\mathbb{R}}\mathbb{E}\left|\sum_{n=1}^N \gamma_n\partial_x^\alpha f_n\right|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

By using the Kahane-Khintchine inequality, and $L^2(\Omega)$ orthogonality of the sequence $(\gamma_k)_{k\in\mathbb{N}}$ it follows that

$$\left|\sum_{n=1}^N \gamma_n f_n\right|_{L^2(\Omega;W_x^{1,p})} \le C\sum_{\alpha=0}^k\left(\int_{\mathbb{R}}\mathbb{E}\left(\left|\sum_{n=1}^N \gamma_n\partial_x^\alpha f_n\right|^2\right)^{\frac{p}{2}} \mathrm{d}x\right)^{\frac{1}{p}} = C\sum_{\alpha=0}^k\left|\sum_{n=1}^N |\partial_x^\alpha f_n|^2\right|_{L_x^{p/2}}^{\frac{1}{2}},$$

and via the triangle inequality

$$\left|\sum_{n=1}^N \gamma_n f_n\right|_{L^2(\Omega;W_x^{1,p})} \le C\sum_{\alpha=0}^k\left(\sum_{n=1}^N |(\partial_x^\alpha f_n)^2|_{L_x^{p/2}}\right)^{\frac{1}{2}} = C\sum_{\alpha=0}^k\left(\sum_{n=1}^N |\partial_x^\alpha f_n|_{L_x^p}^2\right)^{\frac{1}{2}} \le \tilde{C}\left(\sum_{n=1}^N |f_n|_{W_x^{k,p}}^2\right)^{\frac{1}{2}}. \qquad \square$$

1.3.2. Hilbert-Schmidt and γ-radonifying operators

Recall the following definition of a Hilbert-Schmidt operator.

Definition 1.3.4 (Hilbert-Schmidt operator)
*Let $H, \tilde{H}$ be separable Hilbert spaces. An operator $\Phi \in \mathscr{L}_2(H;\tilde{H})$ is called a **Hilbert-Schmidt operator** if the series $\sum_{k\in\mathbb{N}} \|\Phi e_k\|^2_{\tilde{H}}$ converges for any orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H. We denote by $\mathscr{L}_2(H;\tilde{H})$ the space of such operators, with norm*

$$\|\Phi\|^2_{\mathscr{L}_2(H;\tilde{H})} = \mathrm{Tr}(\Phi^*\Phi) = \sum_{k\in\mathbb{N}} \|\Phi e_k\|^2_{\tilde{H}}. \tag{1.8}$$

This norm is well-defined since the quantity $\sum_{k\in\mathbb{N}} \|\Phi e_k\|^2_{\tilde{H}}$ does not depend on the choice of $(e_k)_{k\in\mathbb{N}}$. With the aim to generalize this notion to Banach-valued operators, we introduce the notion of γ-radonifying operators. The following definition of γ-radonifying operators is tailored to operators defined on a separable Hilbert space, see for example [22, Chapter 9] for a more general definition and treatment of the topic.

Definition 1.3.5 (γ-radonifying operator)
*Let H be a separable Hilbert space. A bounded operator K is called a γ-**radonifying operator** if the series $\sum_{k\in\mathbb{N}} \gamma_k K e_k$ converges in $L^2(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}},E)$ for any orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H and any sequence $(\gamma_k)_{k\in\mathbb{N}}$ of independent normal real-valued random variables on a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$. We denote by $\gamma(H;E)$ the space of such operators, with norm*

$$\|K\|^2_{\gamma(H;E)} = \tilde{\mathbb{E}} \left\| \sum_{k\in\mathbb{N}} \gamma_k K e_k \right\|^2_E. \tag{1.9}$$

This norm does not depend on the choice of $(e_k)_{k\in\mathbb{N}}$ and $(\gamma_k)_{k\in\mathbb{N}}$, and the space $\gamma(H;E)$ is a Banach space. In the case that E is a separable Hilbert space, the space of γ-radonifying operators is equal to the space of Hilbert-Schmidt operators, with equality of norms. We have the following result on the composition of a linear operator and a γ-radonifying operator.

Lemma 1.3.6 (Left ideal property)
Let H be a separable Hilbert space and E, F separable Banach spaces. If $K \in \gamma(H;E)$ and $L \in \mathscr{L}(E;F)$, then $LK \in \gamma(H;F)$ and

$$\|LK\|_{\gamma(H;F)} \le \|L\|_{\mathscr{L}(E;F)} \|K\|_{\gamma(H;E)}. \tag{1.10}$$

Proof. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of independent normal real-valued random variables on a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$, and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of H. Then, for all $m \in \mathbb{N}$

$$\tilde{\mathbb{E}} \left\| \sum_{n=1}^{m} \gamma_k L K e_k \right\|^2_F = \tilde{\mathbb{E}} \left\| L \sum_{n=1}^{m} \gamma_k K e_k \right\|^2_F \le \|L\|^2_{\mathscr{L}(E;F)} \tilde{\mathbb{E}} \left\| \sum_{n=1}^{m} \gamma_k K e_k \right\|^2_E \le \|L\|^2_{\mathscr{L}(E;F)} \|K\|^2_{\gamma(H;E)}.$$

The conclusion follows by letting $m \to \infty$. □

1.3.3. Cylindrical Brownian motion

Recall that a real-valued random variable γ is called Gaussian if it has a probability density of the form

$$f_\gamma(t) = \frac{1}{\sqrt{2\pi q}} e^{-t^2/2q}.$$

Definition 1.3.7 (Gaussian random variable)
*An E-valued random variable X is called **Gaussian** if for all $x^* \in E^*$ the real-valued random variable $\langle X, x^*\rangle$ is Gaussian.*

Recall also that an E-valued stochastic process is a family of E-valued random variables $(X(i))_{i\in I}$ for some index-set I defined on a common probability space. An important class of stochastic processes is characterized by the following definition.

Definition 1.3.8 (Gaussian process)
An E-valued stochastic process $(X(i))_{i\in I}$ is called a ***Gaussian process*** *if for all finite sequences $i_1,\dots,i_N \in I$ the E^N-valued random variable $(X(i_1),\dots,X(i_N))$ is Gaussian.*

It will also be useful to introduce the concept of an isonormal process.

Definition 1.3.9 (H-isonormal process)
Let H be a Hilbert space. An H- ***isonormal process*** *on Ω is a mapping $W : H \to L^2(\Omega)$ such that:*

1. *For all $h \in H$ the random variable Wh is Gaussian;*
2. *For all $h_1, h_2 \in H$ we have $\mathbb{E}(Wh_1 \cdot Wh_2) = \langle h_1, h_2\rangle_H$.*

Example 1.3.10. If W is an $L^2(0,T)$-isonormal process, then $\tilde{W}(t) := W\mathbb{1}_{[0,t]}$ defines a real-valued Brownian motion on $[0,T]$.

With these definitions in place, we can define the Hilbert space analogue of a real-valued Brownian motion.

Definition 1.3.11 (H-cylindrical Brownian motion)
An $L^2(0,T;H)$-isonormal process is called an H- ***cylindrical Brownian motion*** *on $[0,T]$.*

1.3.4. The Itô integral

In this section, we present the stochastic integral for operator-valued processes. We first give its definition for a class of simple processes, called finite-rank adapted step processes. For the remainder of this section, let W_H be an H-cylindrical Brownian motion on some Hilbert space H. With this aim in mind, we introduce for $h \in H$ and $x \in X$ the notation $h \otimes x$ for the element of $\mathscr{L}(H,X)$ defined as

$$(h\otimes x)k := \langle h,k\rangle_H x, \quad \text{for } k \in H.$$

Definition 1.3.12 (Finite-rank adapted step process)
A process $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,X)$ is called a ***finite-rank adapted step process*** *with respect to a filtration $(\mathscr{F}_t)_{t\in[0,T]}$, if it is of the form*

$$\Phi(t,\omega) = \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbb{1}_{(t_{n-1},t_n]}(t) h_m \otimes \xi_{mn}(\omega),$$

where $h_1,\dots,h_M$ is an orthonormal system in H, and for all $(m,n)\in M\times N$, ξ_{nm} is an $\mathscr{F}_{t_n}$-measurable step function from $\Omega \to X$.

For this class of operator-valued processes, we define the stochastic integral as follows.

Definition 1.3.13 (Itô integral)
For Φ as above, we define the ***Itô integral*** *as*

$$\int_0^T \Phi(t)\,\mathrm{d}W_H(t) := \sum_{n=1}^{N}\sum_{m=1}^{M}(W_H(t_n)h_m - W_H(t_{n-1})h_m)\xi_{mn}.$$

The integral extends to processes that can be approximated by a sequence of finite-rank adapted step processes. In case E is a type 2 and so-called UMD Banach space, then the processes $\Phi \in L^p(\Omega; L^2(0,T;\gamma(H;E)))$, where $1<p<\infty$, that are adapted to $(\mathscr{F}_t)_{t\in[0,T]}$ have this property. We call such processes stochastically integrable with respect to W_H. The abbreviation 'UMD' is short for 'unconditional martingale differences'. It was shown in [5, 10] by Burkholder and Bourgain that the class of UMD Banach spaces are exactly the Banach spaces E for which the Hilbert transform extends to a bounded linear operator on $L^p(\mathbb{R};E)$. For the precise definition of the UMD class of Banach spaces and related theory, we refer to [21, Chapter 4]. The following theorem reveals that the Itô integral is an isometry from the space of integrable processes to the space of square-integrable E-valued random variables.

Theorem 1.3.14 (Itô isometry)
Let E be a UMD and type 2 Banach space and let $1 < p < \infty$. Then for stochastically W_H-integrable processes Φ, we have

$$\mathbb{E}\left\|\int_0^T \Phi(t)\,\mathrm{d}W_H(t)\right\|_E^p \lesssim \mathbb{E}|\Phi|^p_{L^2(0,T;\gamma(H;E))}$$

See [27, Theorem 13.2, p. 184] for a proof. Using Doob's maximal inequality, this result can be used to prove the following inequality on the supremum of the stochastic integral.

Theorem 1.3.15 (Burkholder-Davis-Gundy inequality)
Let H be a separable Hilbert space, E a UMD and type 2 Banach space. If $p \in [1,\infty]$ and Φ is a W_H-integrable process, then

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t \Phi(t)\,\mathrm{d}W_H(t)\right\|_E^p \le C\mathbb{E}|\Phi|^p_{L^2(0,T;\gamma(H;E)))}. \tag{1.11}$$

A proof of this inequality can be found in [28, Theorem 4.4]. The following lemma provides an estimate for the stochastic convolution with a contractive semigroup. See [14, Theorem 6.10, p. 166] for a proof.

Lemma 1.3.16
Let $\{S(t)\}_{t\ge0}$ be a contractive C_0-semigroup on a Hilbert space $\tilde{H}$. Then for a W_H-integrable process Φ we have

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t S(t-s)\Phi(s)\,\mathrm{d}W_H(s)\right\|_{\tilde{H}}^p \le C\mathbb{E}|\Phi|^p_{L^2(0,T;\gamma(H;\tilde{H}))}.$$

We now state a property of the stochastic integral which is familiar from classic integration theory.

Theorem 1.3.17 (Stochastic Fubini)
If for each $0 \le s \le S$ we have that Φ_s is a W_H-integrable process and

$$\int_0^S (\mathbb{E}\|\Phi_s\|^2_{L^2(0,T;\gamma(H,E))})^{1/2}\,\mathrm{d}s < \infty,$$

then almost surely

$$\int_0^S\int_0^T \Phi_s(t)\,\mathrm{d}W_H(t)\,\mathrm{d}s = \int_0^T\int_0^S \Phi_s(t)\,\mathrm{d}s\,\mathrm{d}W_H(t).$$

For a proof, we refer to [14, Theorem 4.33, p. 110]. We conclude the section with a helpful formula describing the evolution of a bilinear pairing between two stochastic processes. Although usually stated in a more general fashion (see [14, Theorem 4.32, p. 106]), we state this version because it is useful for characterizing the evolution of the squared norm of a stochastic process.

Theorem 1.3.18 (Itô's formula (for bilinear maps))
Let E_1, E_2 and F be UMD Banach spaces and $(e_k)_{k\ge1}$ an orthonormal basis of H. Let furthermore $b : E_1 \times E_2 \to F$ be a bilinear map and let Θ_1 be stochastically integrable w.r.t. W_H and have paths in $L^2(0,T;\gamma(H,E_1))$ almost surely. Let $\psi_1 : [0,T]\times\Omega \to E_1$ be $(\mathscr{F}_t)_{t\in[0,T]}$-adapted and have paths in $L^1(0,T;E_1)$ almost surely. Let also $\xi_1 : \Omega \to E_1$ be $(\mathscr{F}_t)_{t\in[0,T]}$-adapted, and we assume all of the above for Θ_2, ψ_2 and ξ_2, where we replace E_1 by E_2. If

$$\zeta_i(t) = \xi_i + \int_0^t \psi_i(s)\,\mathrm{d}s + \int_0^t \Theta_i(s)\,\mathrm{d}W_H(s),$$

for $i = 1, 2$, then almost surely

$$
\begin{aligned}
b(\zeta_1(t), \zeta_2(t)) - b(\zeta_1(0), \zeta_2(0)) = \int_0^t b(\zeta_1(s), \psi_2(s)) + b(\psi_1(s), \zeta_2(s))\, \mathrm{d}s + \int_0^t b(\zeta_1(s), \Theta_2(s)) + b(\Theta_1(s), \zeta_2(s))\, \mathrm{d}W_H(s) \\
+ \int_0^t \sum_{k \geq 1} b(\Theta_1(s) e_k, \Theta_2(s) e_k)\, \mathrm{d}s.
\end{aligned}
$$

See [9, Corollary 2.6] for a proof.

2

The linear parametrically-forced Schrödinger equation

This chapter is devoted to the linear parametrically-forced Schrödinger equation. We begin by considering the Fourier transform of the equation, from which we deduce a characterization of the Fourier transform of solutions. We proceed by showing that the linear operator of the equation generates a C_0-group in $L^2(\mathbb{R})$ in Section 2.2. In Section 2.3, we then derive a set of fixed-time estimates on the semigroup using the representation of solutions in Fourier space. In the last section, Section 2.4, we show that the semigroup satisfies the Strichartz estimates, which are key to proving the existence and uniqueness of mild solutions to the nonlinear stochastic equation.

Recall that the initial value problem for the parametrically-forced Schrödinger equation (PFS) is

$$\begin{aligned} z_t = i\Delta z - i\nu z - \epsilon(\gamma z - \mu\overline{z}) \quad & \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}^+, \\ z(x,0) = z_0(x) \quad & \text{for } x \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $z : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a complex-valued function of space and time. The constants ϵ, γ and μ are all positive and ν is real-valued. Throughout this work, we will assume that $|\nu| > \epsilon\mu$, meaning that the conjugation occurs 'fast enough' to prevent phase-sensitive effects due to the phase-sensitive forcing term of strength $\epsilon\mu$.

By considering only the local effects in equation (2.1), meaning if we ignore for a moment the Laplacian, we can informally make sense of (2.1) as a physical model for the compensation of dissipative effects. More precisely, if we consider

$$z_t = -i\nu z - \epsilon(\gamma z - \mu\overline{z}),$$

with $z : \mathbb{R}^+ \to \mathbb{C}$, then we can summarize the effects of the various parameters as follows. The parameter $\gamma > 0$, modelling dissipation, draws z towards zero. The phase-sensitive gain parameter $\mu > 0$ pulls z towards the direction of its conjugate. This amplifies real-valued $z \in \mathbb{C}$ and draws purely imaginary $z \in \mathbb{C}$ to the origin. Their combined effect is dissipative or amplifying, depending only on the phase of $z \in \mathbb{C}$. Averaged over all phases, the net effect of γ and μ is always dissipative, regardless of the value of $\mu > 0$. The conjugation parameter ν acts as a rotation in the complex plane, preventing blow-up of $z \in \mathbb{C}$ that happen to be in a phase-region where the net effect is amplifying, by rotating z to a phase-region where the net effect is dissipative.

2.1. Fourier solution

Recall that the solution to the linear Schrödinger equation is given by the Fourier multiplier $e^{-it|\xi|^2}$. By considering the Fourier transform of (2.1), we will obtain an analogous exponential multiplier that serves as a solution operator to (2.1). It will therefore be useful to rewrite the equation first. Let us write $z(x,t) =: a(x,t) + ib(x,t)$, with $a, b : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ real-valued. Then we can recast (2.1) as the following real-valued system

$$\begin{aligned} a_t &= -\Delta b + \nu b - \epsilon(\gamma-\mu)a && \text{for } x\in\mathbb{R} \text{ and } t\in\mathbb{R}^+, \\ b_t &= \Delta a - \nu a - \epsilon(\gamma+\mu)b && \text{for } x\in\mathbb{R} \text{ and } t\in\mathbb{R}^+, \\ a(x,0) &= a_0(x) && \text{for } x\in\mathbb{R}, \\ b(x,0) &= b_0(x) && \text{for } x\in\mathbb{R}, \end{aligned} \tag{2.2}$$

or in matrix form

$$\begin{aligned} \partial_t \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} -\epsilon(\gamma-\mu) & -\Delta+\nu \\ \Delta-\nu & -\epsilon(\gamma+\mu) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} && \text{for } x\in\mathbb{R} \text{ and } t\in\mathbb{R}^+, \\ \begin{bmatrix} a \\ b \end{bmatrix}(x,0) &= \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}(x) && \text{for } x\in\mathbb{R}. \end{aligned} \tag{2.3}$$

We have the following result for the Fourier solution of (2.2).

Theorem 2.1.1
If $a_0, b_0 \in \mathscr{S}(\mathbb{R})$, then there is a unique solution $a, b \in C^1([0,\infty); \mathscr{S}(\mathbb{R}))$ of (2.2). The spatial Fourier transform of the solution is given by

$$\begin{bmatrix} \hat a \\ \hat b \end{bmatrix}(t,\xi) = e^{-\epsilon\gamma t} \begin{bmatrix} \cos(\phi(\xi)t) + \epsilon\mu t \operatorname{sinc}(\phi(\xi)t) & t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t) \\ -t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t) & \cos(\phi(\xi)t) - \epsilon\mu t \operatorname{sinc}(\phi(\xi)t) \end{bmatrix} \begin{bmatrix} \hat a \\ \hat b \end{bmatrix}(0,\xi) \quad \text{for } \xi\in\mathbb{R} \text{ and } t\in\mathbb{R}^+, \tag{2.4}$$

where $\phi : \mathbb{R} \to \mathbb{R}^+$ is defined as

$$\phi(\xi) = (\xi^2+\nu)\sqrt{1 - \frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2}} \quad \text{for } \xi \in \mathbb{R}. \tag{2.5}$$

Proof. We apply a Fourier transform to (2.3) in the spatial variable

$$\partial_t \begin{bmatrix} \hat a \\ \hat b \end{bmatrix} = \begin{bmatrix} -\epsilon(\gamma-\mu) & \xi^2+\nu \\ -\xi^2-\nu & -\epsilon(\gamma+\mu) \end{bmatrix} \begin{bmatrix} \hat a \\ \hat b \end{bmatrix} =: \hat A(\xi) \begin{bmatrix} \hat a \\ \hat b \end{bmatrix} \quad \text{for } \xi\in\mathbb{R} \text{ and } t\in\mathbb{R}^+,$$

and obtain a matrix ODE which is solved by

$$\begin{bmatrix} \hat a \\ \hat b \end{bmatrix}(t,\xi) = e^{t\hat A(\xi)} \begin{bmatrix} \hat a \\ \hat b \end{bmatrix}(0,\xi) \quad \text{for } \xi\in\mathbb{R} \text{ and } t\in\mathbb{R}^+.$$

We calculate the matrix exponential $e^{t\hat A(\xi)}$ by diagonalization, i.e. we write $\hat A(\xi) = U(\xi)D(\xi)U^{-1}(\xi)$ with $D(\xi) =: \begin{bmatrix} \lambda_1(\xi) & 0 \\ 0 & \lambda_2(\xi) \end{bmatrix}$. Then, the matrix exponential takes the form

$$e^{t\hat A(\xi)} = U(\xi) \begin{bmatrix} e^{t\lambda_1(\xi)} & 0 \\ 0 & e^{t\lambda_2(\xi)} \end{bmatrix} U^{-1}(\xi).$$

The eigenvalues $\lambda_1(\xi), \lambda_2(\xi)$ are the roots of the characteristic equation

$$\begin{aligned} 0 = \det(\hat A(\xi) - \lambda I) &= \begin{vmatrix} -\epsilon(\gamma-\mu)-\lambda & \xi^2+\nu \\ -\xi^2-\nu & -\epsilon(\gamma+\mu)-\lambda \end{vmatrix} \\ &= \lambda^2 + \epsilon(\gamma+\mu)\lambda + \epsilon(\gamma-\mu)\lambda + \epsilon^2(\gamma-\mu)(\gamma+\mu) + (\xi^2+\nu)^2 \\ &= \lambda^2 + 2\epsilon\gamma\lambda + \epsilon^2(\gamma^2-\mu^2) + (\xi^2+\nu)^2. \end{aligned}$$

Its roots are

$$\lambda_{1,2}(\xi) = \frac{-2\epsilon\gamma \pm \sqrt{4\epsilon^2\gamma^2 - 4(\epsilon^2(\gamma^2-\mu^2)+(\xi^2+\nu)^2)}}{2} = -\epsilon\gamma \pm \sqrt{\epsilon^2\gamma^2 - (\epsilon^2(\gamma^2-\mu^2)+(\xi^2+\nu)^2)}$$
$$= -\epsilon\gamma \pm \sqrt{\epsilon^2\mu^2 - (\xi^2+\nu)^2} = -\epsilon\gamma \pm i(\xi^2+\nu)\sqrt{1-\frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2}}$$
$$=: -\epsilon\gamma \pm i\phi(\xi),$$

where the last square root is real valued by the assumption $|\nu| > \epsilon\mu$. The corresponding eigenvectors are

$$v_{1,2}(\xi) = \begin{bmatrix} \xi^2+\nu \\ -\epsilon\gamma \pm i\phi(\xi)+\epsilon(\gamma-\mu) \end{bmatrix} = \begin{bmatrix} \xi^2+\nu \\ -\epsilon\mu \pm i\phi(\xi) \end{bmatrix}.$$

Hence, we obtain the diagonalization

$$\hat{A}(\xi) = U(\xi)D(\xi)U^{-1}(\xi)$$
$$= \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu+i\phi(\xi) & -\epsilon\mu-i\phi(\xi) \end{bmatrix} \begin{bmatrix} -\epsilon\gamma+i\phi(\xi) & 0 \\ 0 & -\epsilon\gamma-i\phi(\xi) \end{bmatrix} \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu+i\phi(\xi) & -\epsilon\mu-i\phi(\xi) \end{bmatrix}^{-1},$$

and upon exponentiating

$$e^{t\hat{A}(\xi)} = U(\xi)\begin{bmatrix} e^{t\lambda_1(\xi)} & 0 \\ 0 & e^{t\lambda_2(\xi)} \end{bmatrix} U^{-1}(\xi)$$
$$= \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu+i\phi(\xi) & -\epsilon\mu-i\phi(\xi) \end{bmatrix} \begin{bmatrix} e^{-\epsilon\gamma t}e^{i\phi(\xi)t} & 0 \\ 0 & e^{-\epsilon\gamma t}e^{-i\phi(\xi)t} \end{bmatrix} \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu+i\phi(\xi) & -\epsilon\mu-i\phi(\xi) \end{bmatrix}^{-1}. \tag{2.6}$$

After simplification, this yields the result

$$e^{t\hat{A}(\xi)} = e^{-\epsilon\gamma t}\begin{bmatrix} \cos(\phi(\xi)t)+\epsilon\mu t\,\mathrm{sinc}(\phi(\xi)t) & t(\xi^2+\nu)\,\mathrm{sinc}(\phi(\xi)t) \\ -t(\xi^2+\nu)\,\mathrm{sinc}(\phi(\xi)t) & \cos(\phi(\xi)t)-\epsilon\mu t\,\mathrm{sinc}(\phi(\xi)t) \end{bmatrix}. \tag{2.7}$$

□

By inspecting the Fourier multiplier of (2.4) at $\xi = 0$, we can again discuss the roles of the parameters γ, μ and ν in (2.1). Using that $\phi(0) = \sqrt{\nu^2-\epsilon^2\mu^2}$, the multiplier $e^{t\hat{A}(\xi)}$ takes the form

$$e^{t\hat{A}(0)} = e^{-\epsilon\gamma t}\begin{bmatrix} \cos\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right)+\epsilon\mu t\,\mathrm{sinc}\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) & t\nu\,\mathrm{sinc}\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) \\ -t\nu\,\mathrm{sinc}\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) & \cos\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right)-\epsilon\mu t\,\mathrm{sinc}\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) \end{bmatrix},$$

at $\xi = 0$. By expanding the terms containing sinc-functions around $\frac{\epsilon\mu}{\nu} = 0$ as

$$t\nu\,\mathrm{sinc}\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) = \left(1-\tfrac{\epsilon^2\mu^2}{\nu^2}\right)^{-1/2}\sin\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) = \left(1+\tfrac{\epsilon^2\mu^2}{2\nu^2}+O(\tfrac{\epsilon^4\mu^4}{\nu^4})\right)\sin\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right),$$

and

$$\epsilon\mu t\,\mathrm{sinc}\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) = \left(\tfrac{\nu^2}{\epsilon^2\mu^2}-1\right)^{-1/2}\sin\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) = \left(\tfrac{\epsilon\mu}{\nu}+O(\tfrac{\epsilon^3\mu^3}{\nu^3})\right)\sin\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right),$$

we may rewrite

$$e^{t\hat{A}(0)} = e^{-\epsilon\gamma t}\begin{bmatrix} \cos\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) & \sin\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) \\ -\sin\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) & \cos\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right) \end{bmatrix} + e^{-\epsilon\gamma t}\sin\left(\sqrt{\nu^2-\epsilon^2\mu^2}\,t\right)\begin{bmatrix} \frac{\epsilon\mu}{\nu}+O(\frac{\epsilon^3\mu^3}{\nu^3}) & \frac{\epsilon^2\mu^2}{2\nu^2}+O(\frac{\epsilon^4\mu^4}{\nu^4}) \\ -\frac{\epsilon^2\mu^2}{2\nu^2}+O(\frac{\epsilon^4\mu^4}{\nu^4}) & -\frac{\epsilon\mu}{\nu}+O(\frac{\epsilon^3\mu^3}{\nu^3}) \end{bmatrix}.$$

As expected, the parameter γ gives rise to a dissipative factor $e^{-\epsilon\gamma t}$. The effect of the amplification parameter μ and the conjugation parameter ν seem to be more intertwined. The first matrix in the previous decomposition acts as a rotation with frequency $\sqrt{\nu^2-\epsilon^2\mu^2}$. The second matrix captures the phase-sensitive effect, which is oscillatory in time and becomes small if $\nu \gg \epsilon\mu$. Since the system in (2.2) is equivalent to (2.1), we immediately obtain the following corollary to Theorem 2.1.1.

Corollary 2.1.2
If $z_0 \in \mathscr{S}(\mathbb{R})$, then there is a unique solution $z \in C^1([0,\infty);\mathscr{S}(\mathbb{R}))$ of (2.1). The spatial Fourier transform of the solution is given by

$$\begin{aligned}\hat{z}(t,\xi) =& e^{-\epsilon\gamma t}\left[(\cos\left(\phi(\xi)t\right) + \epsilon\mu t\operatorname{sinc}\left(\phi(\xi)t\right))\hat{a}(0,\xi) + t(\xi^2+\nu)\operatorname{sinc}\left(\phi(\xi)t\right)\hat{b}(0,\xi)\right] \\ &+ i e^{-\epsilon\gamma t}\left[-t(\xi^2+\nu)\operatorname{sinc}\left(\phi(\xi)t\right)\hat{a}(0,\xi) + (\cos\left(\phi(\xi)t\right) - \epsilon\mu t\operatorname{sinc}\left(\phi(\xi)t\right))\hat{b}(0,\xi)\right],\end{aligned} \tag{2.8}$$

for $\xi \in \mathbb{R}$ and $t \in \mathbb{R}^+$, where $\phi : \mathbb{R} \to \mathbb{R}^+$ is defined as in Theorem 2.1.1.

2.2. The forced Schrödinger semigroup

We show in this section that the linear operator associated with the parametrically-forced Schrödinger equation generates a semigroup on $L^2(\mathbb{R})$. To this end we define the operator A on $L^2(\mathbb{R})$ by

$$\begin{cases} D(A) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \\ Az = i\Delta z - i\nu z - \epsilon(\gamma z - \mu\overline{z}) \text{ for } z \in D(A). \end{cases} \tag{2.9}$$

As before, ϵ, γ, μ are positive constants and ν is real-valued.

Proposition 2.2.1
The operator $A: H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is closed and densely defined.

Proof. We first verify that the unbounded operator A is well defined. It is indeed easy to check that A is bounded from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$

$$|Az|_{L^2_x} \overset{(2.9)}{=} |i\Delta z - i\nu z - \epsilon(\gamma z - \mu\overline{z})|_{L^2_x} \leq |\Delta z|_{L^2_x} + (|\nu| + \epsilon(\gamma + \mu))|z|_{L^2_x} \leq C|z|_{H^2_x}.$$

Furthermore, it follows from the density of the test functions $C_c^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$ that $H^2(\mathbb{R})$ is also dense in $L^2(\mathbb{R})$. Therefore, A is a densely defined unbounded linear operator. To show that A is also closed, we let $z_n \to z$ in $L^2(\mathbb{R})$ with $z_n \in H^2(\mathbb{R})$ for all $n \in \mathbb{N}$, and $Az_n \to y$ in $L^2(\mathbb{R})$. In order to show that $z \in H^2(\mathbb{R})$, we first prove a convenient auxiliary inequality. Via the triangle inequality, we have the point-wise estimate

$$|\Delta z| \leq |Az| + C|z|,$$

with C a sufficiently large constant. By squaring the inequality, and applying Young's inequality with exponent 2 we obtain

$$|\Delta z|^2 \leq |Az|^2 + C^2|z|^2 + 2|Az||z| \leq (1+C)|Az|^2 + (C + C^2)|z|^2 \tag{2.10}$$

We then write

$$\begin{aligned} |z|^2_{H^2_x} &\simeq \int_{\mathbb{R}} (|\xi|^4 + 2|\xi|^2 + 1)|\hat{z}(\xi)|^2 \, \mathrm{d}\xi \leq 2\int_{\mathbb{R}} (|\xi|^4 + 1)|\hat{z}(\xi)|^2 \, \mathrm{d}\xi \\ &= \frac{1}{\pi}|\Delta z|^2_{L^2_x} + \frac{1}{\pi}|z|^2_{L^2_x} \overset{(2.10)}{\leq} \tilde{C}(|Az|^2_{L^2_x} + |z|^2_{L^2_x}), \end{aligned}$$

where we have again used Young's inequality to eliminate the $|\xi|^2$ term. We now obtain

$$|z_n - z_m|_{H^2_x} \leq \tilde{C}(|A(z_n - z_m)|_{L^2_x} + |z_n - z_m|_{L^2_x}),$$

and it follows from $z_n \to z$ in $L^2(\mathbb{R})$ and $Az_n \to y$ in $L^2(\mathbb{R})$ that $(z_n)_{n\geq 1}$ is a Cauchy sequence in $H^2(\mathbb{R})$. Therefore, it has a limit in $H^2(\mathbb{R})$. As limits are unique, we must have $z \in H^2(\mathbb{R})$. Furthermore, it follows from

$$|Az - y|_{L^2_x} \leq |A(z - z_n)|_{L^2_x} + |Az_n - y|_{L^2_x} \leq C|z - z_n|_{H^2_x} + |Az_n - y|_{L^2_x}$$

that $Az = y$ by taking the limit $n \to \infty$. This shows that A is closed and concludes the proof. □

We will use Theorem 1.1.5 to prove that the operator A generates a semigroup on $L^2(\mathbb{R})$.

Theorem 2.2.2
The operator A defined in (2.9) generates a strongly continuous semigroup on $L^2(\mathbb{R})$, and for all $z \in D(A) = H^2(\mathbb{R})$ we have

$$\mathrm{Re}\langle Az, z\rangle_{L^2_x} = -\epsilon\gamma|z|^2_{L^2_x} + \epsilon\mu(|\mathrm{Re}\, z|^2_{L^2_x} - |\mathrm{Im}\, z|^2_{L^2_x}). \tag{2.11}$$

If $\mu \leq \gamma$, then the semigroup is furthermore contractive.

Proof. We first show the inner product identity. Let therefore $z \in H^2(\mathbb{R})$. Then

$$\begin{aligned}\operatorname{Re}\langle Az, z\rangle_{L_x^2} &\overset{(2.9)}{=} \operatorname{Re}\left[i\langle \Delta z, z\rangle_{L_x^2} - i\nu\langle z, z\rangle_{L_x^2} + \int_{\mathbb{R}} (-\epsilon\gamma|z|^2 + \epsilon\mu\overline{z}^2)\,\mathrm{d}x\right] \\ &= \operatorname{Re}\left[-i|\partial_x z|_{L_x^2}^2 - i\nu|z|_{L_x^2}^2 + \int_{\mathbb{R}} (-\epsilon\gamma|z|^2 + \epsilon\mu\overline{z}^2)\,\mathrm{d}x\right].\end{aligned}$$

Let us write $z = a + ib$, with $a, b \in H^2(\mathbb{R})$ real valued. We then find (2.11) as follows

$$\begin{aligned}\operatorname{Re}\langle Az, z\rangle_{L_x^2} &= \operatorname{Re}\int_{\mathbb{R}} -\epsilon\gamma(a^2+b^2) + \epsilon\mu(a^2 - b^2 - i2ab)\,\mathrm{d}x \\ &= -\epsilon\gamma|z|_{L_x^2}^2 + \epsilon\mu(|a|_{L_x^2}^2 - |b|_{L_x^2}^2).\end{aligned}$$

To show that A generates a C_0-semigroup on $L^2(\mathbb{R})$, we consider the shifted operator

$$A - \epsilon(\mu-\gamma)I : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

and we write $\tilde{A} := A - \epsilon(\mu-\gamma)I$. It follows from the previous inequality that

$$\operatorname{Re}\langle \tilde{A}z, z\rangle_{L_x^2} \overset{(2.9)}{\leq} -\epsilon\gamma|z|_{L_x^2}^2 + \epsilon\mu|z|_{L_x^2}^2 - \epsilon(\mu-\gamma)|z|_{L_x^2}^2 = 0,$$

so that $\tilde{A}$ satisfies the dissipativity condition of Theorem 1.1.5. The shifted operator $\tilde{A}$ also satisfies condition 1 of Theorem 1.1.5, as it is the sum of the closed and densely defined operators A and $-\epsilon(\mu-\gamma)I$. To verify the last condition, set $\tilde{\lambda} := \lambda + \epsilon(\mu-\gamma)$ so that $\lambda I - \tilde{A} = \tilde{\lambda}I - A$. We now show that $\tilde{\lambda}I - A$ has range $L^2(\mathbb{R})$ for all $\tilde{\lambda} \in \mathbb{R}$. Therefore, consider the Fourier transform of the equation $(\tilde{\lambda}I - A)z = f$ (or rather an equivalent system), for some $f \in L^2(\mathbb{R})$

$$\begin{bmatrix} -\epsilon(\gamma-\mu) - \tilde{\lambda} & \xi^2 + \nu \\ -\xi^2 - \nu & -\epsilon(\gamma+\mu) - \tilde{\lambda}\end{bmatrix}\begin{bmatrix}\hat{a} \\ \hat{b}\end{bmatrix}(\xi) = \begin{bmatrix}\hat{f}_1 \\ \hat{f}_2\end{bmatrix}(\xi) \quad \text{for all } \xi \in \mathbb{R},$$

where $f = f_1 + if_2$. Recall from the proof of Theorem 2.1.1 that this matrix has determinant zero if and only if $\tilde{\lambda} = -\epsilon\gamma \pm i\phi(\xi)$. Since $\phi(\xi) > 0$ for all $\xi \in \mathbb{R}$, the matrix is invertible for all $\xi \in \mathbb{R}$ when $\tilde{\lambda} \in \mathbb{R}$. So $(\tilde{\lambda}I - A)z = f$ has a solution for all $\xi \in \mathbb{R}$. In the Fourier space, the solution can be computed via the matrix inverse

$$\begin{bmatrix}\hat{a} \\ \hat{b}\end{bmatrix}(\xi) = \frac{1}{(\xi^2+\nu)^2 + \tilde{\lambda}^2 + 2\epsilon\gamma\tilde{\lambda} + \epsilon^2(\gamma^2-\mu^2)}\begin{bmatrix} -\epsilon(\gamma+\mu) - \tilde{\lambda} & -\xi^2 - \nu \\ \xi^2 + \nu & -\epsilon(\gamma-\mu) - \tilde{\lambda}\end{bmatrix}\begin{bmatrix}\hat{f}_1 \\ \hat{f}_2\end{bmatrix}(\xi) \quad \text{for all } \xi \in \mathbb{R}, \tag{2.12}$$

which scales as $\frac{1}{\xi^2}\begin{bmatrix}\hat{f}_1 \\ \hat{f}_2\end{bmatrix}(\xi)$ as $|\xi| \to \infty$. Via the Fourier representation of the Sobolev norm (Proposition 1.2.11)

$$|f|_{H_x^2}^2 \simeq 4\pi^2\int_{\mathbb{R}} (1 + |\xi|^2)^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi,$$

and the fact that $f \in L^2(\mathbb{R})$ we then see that the solution is an element of $H^2(\mathbb{R})$. It follows that $\tilde{\lambda}I - A = \lambda I - \tilde{A}$ has range $L^2(\mathbb{R})$ for any $\lambda, \tilde{\lambda} \in \mathbb{R}$, and in particular for any $\lambda > 0$. We can then conclude by Theorem 1.1.5 that $\tilde{A}$ generates a contraction semigroup on $L^2(\mathbb{R})$. It is now easily verified that $e^{At} = e^{\epsilon(\mu-\gamma)t}e^{\tilde{A}t}$ is also a strongly continuous semigroup on $L^2(\mathbb{R})$, with generator A. If $\mu \leq \gamma$, then

$$|e^{At}|_{\mathscr{L}(L_x^2)} = e^{\epsilon(\mu-\gamma)t}|e^{\tilde{A}t}|_{\mathscr{L}(L_x^2)} \leq 1,$$

which shows that in this case the semigroup $(e^{At})_{t\geq 0}$ is contractive. □

We will denote the semigroup generated by A as $\{S(t)\}_{t\geq 0}$. Since the initial value problem (2.1) is an evolution equation of the linear operator A, the semigroup $\{S(t)\}_{t\geq 0}$ is the solution operator for the PFS equation and is given by the multiplier of Equation (2.7). Using the Fourier solution, we can in fact show that the initial value problem (2.1) can also be solved backwards in time, i.e. the semigroup $\{S(t)\}_{t\geq 0}$ can be extended to a

C_0-group $\{S(t)\}_{t\in\mathbb{R}}$. Via Remark 1.1.3, it suffices to show that $S(t)$ is invertible for all $t \geq 0$, and that its inverse is $S(-t)$. Recall from the proof of Theorem 2.1.1 (Equation (2.6)) that $S(t)$ is a multiplier with symbol

$$e^{t\hat{A}(\xi)} = \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu + i\phi(\xi) & -\epsilon\mu - i\phi(\xi) \end{bmatrix} \begin{bmatrix} e^{-\epsilon\gamma t}e^{i\phi(\xi)t} & 0 \\ 0 & e^{-\epsilon\gamma t}e^{-i\phi(\xi)t} \end{bmatrix} \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu + i\phi(\xi) & -\epsilon\mu - i\phi(\xi) \end{bmatrix}^{-1}.$$

It follows that $S(-t)$ is a multiplier with symbol

$$e^{-t\hat{A}(\xi)} = \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu + i\phi(\xi) & -\epsilon\mu - i\phi(\xi) \end{bmatrix} \begin{bmatrix} e^{\epsilon\gamma t}e^{-i\phi(\xi)t} & 0 \\ 0 & e^{\epsilon\gamma t}e^{i\phi(\xi)t} \end{bmatrix} \begin{bmatrix} \xi^2+\nu & \xi^2+\nu \\ -\epsilon\mu + i\phi(\xi) & -\epsilon\mu - i\phi(\xi) \end{bmatrix}^{-1}.$$

Indeed, we see that the matrix $e^{-t\hat{A}(\xi)}$ is the inverse of $e^{t\hat{A}(\xi)}$. To conclude that the operator $S(-t)$ is also the inverse of the operator $S(t)$, we note that $S(-t)$ is a bounded operator on $L^2(\mathbb{R})$ for all $t \geq 0$, since the transformation $t \mapsto -t$ only rearranges the matrix entries in (2.7). In what follows, however, we often work with the semigroup $\{S(t)\}_{t\geq 0}$ instead of the C_0-group $\{S(t)\}_{t\in\mathbb{R}}$ as the exponential factor $e^{-\epsilon\gamma t}$ can be used to control any polynomial growth in time for $t \geq 0$.

2.3. Dispersive properties

From the Fourier solution operator of the linear Schrödinger equation, i.e. the multiplier with symbol $e^{-it|\xi|^2}$, we can immediately deduce dispersive properties of the corresponding C_0-group $\{T(t)\}_{t\in\mathbb{R}}$. Indeed, we can for instance see, using Theorem 1.2.5 that,

$$|T(t)z_0|_{L^2_x} = 2\pi|e^{-it|\xi|^2}\hat{z}_0(\xi)|_{L^2_\xi} = 2\pi|\hat{z}_0(\xi)|_{L^2_\xi} = |z_0|_{L^2_x},$$

for $z_0 \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$. Hence, the Schrödinger evolution conserves the L^2-norm. Furthermore, by taking the inverse transform of the Gaussian $e^{-it|\xi|^2}$, it follows that the multiplier corresponds to the convolution with the kernel

$$K_t(x) := \frac{1}{\sqrt{4\pi i t}} e^{-ix^2/4t},$$

so that we have the estimate

$$|T(t)z_0|_{L^\infty_x} = \sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}} K_t(x)z_0(x-y)\,\mathrm{d}y\right| \le \frac{1}{\sqrt{4\pi|t|}}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\left|e^{-iy^2/4t}z_0(x-y)\right|\mathrm{d}y \le \frac{1}{\sqrt{4\pi|t|}}|z_0|_{L^1_x},$$

for $z_0 \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$. This L^∞-L^1 estimate reveals the dispersive nature of the Schrödinger group: solutions are uniformly bounded by a function decreasing with time and converging to 0. Meanwhile, the mass (L^2-norm) of solutions is conserved but spreads out over a larger region. In this section, we prove analogues of the previous two estimates for the linear parametrically-forced Schrödinger equation in forward time. By interpolation, we then obtain the full range of L^p-L^q estimates on the semigroup in Subsection 2.3.3.

2.3.1. An L^∞-L^1-estimate

We first prove the L^∞-L^1-estimate, which holds for $t \ge 0$. For that, we will need the following lemma about the functions in the convolution kernels of the C_0-group.

Lemma 2.3.1
Let $\phi : \mathbb{R} \to \mathbb{R}^+$ be defined as in (2.5), i.e. $\phi(\xi) = (\xi^2+\nu)\sqrt{1-\frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2}}$. Then there exist constants $c_1, c_2, c_3, c_4 > 0$ such that:

1. $|\mathscr{F}^{-1}\{\cos(\phi(\xi)t)\}|_{L^\infty_x} \le c_1/\sqrt{|t|} + c_3|t|$,
2. $|\mathscr{F}^{-1}\{t\,\mathrm{sinc}(\phi(\xi)t)\}|_{L^\infty_x} \le c_2 + c_4|t|^2$,
3. $|\mathscr{F}^{-1}\{t(\xi^2+\nu)\,\mathrm{sinc}(\phi(\xi)t)\}|_{L^\infty_x} \le c_1/\sqrt{|t|} + c_3|t| + c_4|t|^2$,

for all $t \in \mathbb{R}$.

Proof. Assume without loss of generality that $t \ge 0$.

1. First, we rewrite the square root in ϕ using the mean value theorem

$$\sqrt{1-x} = 1 - \frac{x}{2\sqrt{1-\alpha}} \quad \text{for } x \in (0,1),$$

with $\alpha \in (0,x)$. This allows us to rewrite ϕ as follows

$$\phi(\xi)t = (\xi^2+\nu)\sqrt{1-\frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2}}\,t = (\xi^2+\nu)t - \frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)\sqrt{1-\alpha_\xi}} \quad \text{for } \xi\in\mathbb{R} \text{ and } t\in\mathbb{R}, \tag{2.13}$$

with $\alpha_\xi \in (0, \frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2})$. Similarly

$$\cos(a-y) = \cos(a) + y\sin(\beta) \quad \text{for } a\in\mathbb{R} \text{ and } y\in\mathbb{R}^+,$$

with $\beta \in (a-y, a)$. We combine these reformulations as

$$\cos(\phi(\xi)t) = \cos((\xi^2+\nu)t) + \frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)}\frac{\sin(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}} \quad \text{for } \xi\in\mathbb{R} \text{ and } t\in\mathbb{R}^+,$$

with $\alpha_\xi \in (0, \frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2})$ and $\alpha_{\xi,t} \in \left((\xi^2+\nu)t - \frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)\sqrt{1-\alpha_\xi}}, (\xi^2+\nu)t\right)$. We can now estimate the L^∞-norm of the inverse Fourier transform by taking the inverse of the leading term and treating the second term as a correction

$$|\mathscr{F}^{-1}\{\cos(\phi(\xi)t)\}|_{L^\infty_x} \leq |\mathscr{F}^{-1}\{\cos((\xi^2+\nu)t)\}|_{L^\infty_x} + \left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)}\frac{\sin(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L^\infty_x},$$

in which the second term satisfies

$$\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)}\frac{\sin(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L^\infty_x} \leq C|t|, \quad \text{(Lemma A.0.1)}.$$

The Fourier inverse of the leading term can be explicitly calculated

$$\mathscr{F}^{-1}\{\cos((\xi^2+\nu)t)\}(x) = \frac{1}{2\sqrt{t}}(\cos(x^2/4t+\nu t)+\sin(x^2/4t+\nu t)), \quad \text{(Lemma B.0.1)}.$$

Thus, we arrive at the desired estimate

$$|\mathscr{F}^{-1}\{\cos(\phi(\xi)t)\}|_{L^\infty_x} \leq \frac{C}{\sqrt{|t|}} + C|t|.$$

2. Again, we rewrite using the mean value theorem

$$\operatorname{sinc}(a-y) = \operatorname{sinc}(a) - y\operatorname{sinc}'(\beta) \quad \text{for } a\in\mathbb{R} \text{ and } y\in\mathbb{R}^+,$$

with $\beta \in (a-y, a)$. We combine this reformulation with (2.13) as

$$t\operatorname{sinc}(\phi(\xi)t) = t\operatorname{sinc}((\xi^2+\nu)t) - \frac{\epsilon^2\mu^2t^2}{2(\xi^2+\nu)}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}} \quad \text{for } \xi\in\mathbb{R} \text{ and } t\in\mathbb{R}^+$$

where again $\alpha_\xi \in \left(0, \frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2}\right)$ and $\alpha_{\xi,t} \in \left((\xi^2+\nu)t - \frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)\sqrt{1-\alpha_\xi}}, (\xi^2+\nu)t\right)$. We then estimate the L^∞-norm of the inverse Fourier transform as

$$|\mathscr{F}^{-1}\{t\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x} \leq t|\mathscr{F}^{-1}\{\operatorname{sinc}((\xi^2+\nu)t)\}|_{L^\infty_x} + \left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2t^2}{2}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right\}\right|_{L^\infty_x},$$

in which the second term is again bounded, this time quadratic in t

$$\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2t^2}{2}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right\}\right|_{L^\infty_x} \leq C|t|^2, \quad \text{(Lemma A.0.1)}.$$

We can then conclude that

$$|\mathscr{F}^{-1}\{t\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x} \leq |t||\operatorname{sinc}((\xi^2+\nu)t)|_{L^1_\xi} + C|t|^2 \leq C + C|t|^2,$$

as desired.

3. Again, we rewrite using the mean value theorem

$$t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t) = \sin((\xi^2+\nu)t) - \frac{\epsilon^2\mu^2t^2}{2}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}} \quad \text{for } \xi\in\mathbb{R} \text{ and } t\in\mathbb{R}^+,$$

and find for the L^∞-norm of the inverse Fourier transform

$$|\mathscr{F}^{-1}\{t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x} \leq |\mathscr{F}^{-1}\{\sin((\xi^2+\nu)t)\}|_{L^\infty_x} + \left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2t^2}{2}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L^\infty_x}.$$

The second term satisfies a bound

$$\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2 t^2}{2}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L^\infty_x} \le C|t| + C|t|^2, \quad \text{(Lemma A.0.1)},$$

and the Fourier inverse of the leading term can be explicitly calculated

$$\mathscr{F}^{-1}\{\sin((\xi^2+\nu)t)\}(x) = \frac{1}{2\sqrt{t}}(\cos(x^2/4t+\nu t) - \sin(x^2/4t+\nu t)), \quad \text{(Lemma B.0.1)}.$$

Thus, we arrive at the estimate

$$|\mathscr{F}^{-1}\{t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty(\mathbb{R})} \le C/\sqrt{|t|} + C|t| + C|t|^2. \qquad \square$$

We are now ready to prove the following $L^\infty - L^1$ estimate.

Proposition 2.3.2
Let $z_0 \in L^1(\mathbb{R})$, then $S(t)z_0$ satisfies

$$|S(t)z_0|_{L^\infty_x} \le \frac{C}{\sqrt{t}}|z_0|_{L^1_x}. \tag{2.14}$$

for $t \ge 0$.

Proof. By density, it suffices to consider $z_0 \in \mathscr{S}(\mathbb{R})$. As usual, we write $z_0 =: a_0 + ib_0$, with $a_0, b_0 : \mathbb{R} \to \mathbb{R}$ real-valued. Then we have via the Fourier representation (2.8)

$$\begin{aligned} e^{\epsilon\gamma t}|S(t)z_0|_{L^\infty_x} \le{}& |\mathscr{F}^{-1}\{\cos(\phi(\xi)t) + \epsilon\mu t\operatorname{sinc}(\phi(\xi)t)\} * a_0|_{L^\infty_x} + |\mathscr{F}^{-1}\{t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\} * b_0|_{L^\infty_x} \\ &+ |\mathscr{F}^{-1}\{-t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\} * a_0|_{L^\infty_x} + |\mathscr{F}^{-1}\{\cos(\phi(\xi)t) - \epsilon\mu t\operatorname{sinc}(\phi(\xi)t)\} * b_0|_{L^\infty_x}, \end{aligned}$$

and upon estimating the convolution kernels by their supremum

$$\begin{aligned} e^{\epsilon\gamma t}|S(t)z_0|_{L^\infty_x} \le{}& |\mathscr{F}^{-1}\{\cos(\phi(\xi)t) + \epsilon\mu t\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}|a_0|_{L^1_x} + |\mathscr{F}^{-1}\{t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}|b_0|_{L^1_x} \\ &+ |\mathscr{F}^{-1}\{-t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}|a_0|_{L^1_x} + |\mathscr{F}^{-1}\{\cos(\phi(\xi)t) - \epsilon\mu t\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}|b_0|_{L^1_x}. \end{aligned}$$

Then, via the triangle inequality

$$\begin{aligned} e^{\epsilon\gamma t}|S(t)z_0|_{L^\infty_x} \le{}& \big(|\mathscr{F}^{-1}\{\cos(\phi(\xi)t)\}|_{L^\infty_x} + \epsilon\mu|\mathscr{F}^{-1}\{t\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}\big)|a_0|_{L^1_x} \\ &+ |\mathscr{F}^{-1}\{t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}|b_0|_{L^1_x} + |\mathscr{F}^{-1}\{t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}|a_0|_{L^1_x} \\ &+ \big(|\mathscr{F}^{-1}\{\cos(\phi(\xi)t)\}|_{L^\infty_x} + \epsilon\mu|\mathscr{F}^{-1}\{t\operatorname{sinc}(\phi(\xi)t)\}|_{L^\infty_x}\big)|b_0|_{L^1_x}, \end{aligned}$$

and with help of Lemma 2.3.1

$$|S(t)z_0|_{L^\infty_x} \le e^{-\epsilon\gamma t}\Big(\frac{C}{\sqrt{t}} + C + Ct + Ct^2\Big)|z_0|_{L^1_x}.$$

We now distinguish two cases. First, we have for $t \le 1$

$$|S(t)z_0|_{L^\infty_x} \le \Big(\frac{C}{\sqrt{t}} + 3C\Big)|z_0|_{L^1_x} \le \frac{4C}{\sqrt{t}}|z_0|_{L^1_x}.$$

If otherwise $t > 1$, then

$$|S(t)z_0|_{L^\infty_x} \le e^{-\epsilon\gamma t}(2C + Ct + Ct^2)|z_0|_{L^1_x} \le e^{-\epsilon\gamma t}4Ct^2|z_0|_{L^1_x} \le \frac{\tilde{C}}{\sqrt{t}}|z_0|_{L^1_x},$$

where $\tilde{C}$ is chosen large enough. So we see that upon taking the maximum of the two constants we obtain (2.14), as desired. $\square$

2.3.2. An L^2-estimate

We now show that the L^2-operator norm of $S(t)$ is uniformly bounded for positive times. This is obvious if $\mu \le \gamma$, in which case the semigroup is contractive and the operator norm is uniformly bounded by 1. The uniform bound in the non-contractive case follows again from a consideration of the Fourier solution.

Proposition 2.3.3
Let $z_0 \in L^2(\mathbb{R})$, then $S(t)z_0$ satisfies

$$|S(t)z_0|_{L^2_x} \le C|z_0|_{L^2_x}, \tag{2.15}$$

for $t \ge 0$.

Proof. By density, it suffices to consider $z_0 \in \mathscr{S}(\mathbb{R})$. As usual, we write $z_0 =: a_0 + i b_0$, with $a_0, b_0 : \mathbb{R} \to \mathbb{R}$ real-valued. Then again by (2.8)

$$\begin{aligned} e^{\epsilon\gamma t}|S(t)z_0|_{L^2_x} \le\ & |\mathscr{F}^{-1}\{\cos(\phi(\xi)t) + \epsilon\mu t \operatorname{sinc}(\phi(\xi)t)\} * a_0|_{L^2_x} + |\mathscr{F}^{-1}\{t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\} * b_0|_{L^2_x} \\ & + |\mathscr{F}^{-1}\{-t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\} * a_0|_{L^2_x} + |\mathscr{F}^{-1}\{\cos(\phi(\xi)t) - \epsilon\mu t \operatorname{sinc}(\phi(\xi)t)\} * b_0|_{L^2_x}, \end{aligned}$$

and via Parseval's theorem

$$\begin{aligned} e^{\epsilon\gamma t}|S(t)z_0|_{L^2_x} \le\ & 2\pi|(\cos(\phi(\xi)t) + \epsilon\mu t \operatorname{sinc}(\phi(\xi)t))\hat{a}_0|_{L^2_\xi} + 2\pi|t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\,\hat{b}_0|_{L^2_\xi} \\ & + 2\pi|-t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)\,\hat{a}_0|_{L^2_\xi} + 2\pi|(\cos(\phi(\xi)t) - \epsilon\mu t \operatorname{sinc}(\phi(\xi)t))\hat{b}_0|_{L^2_\xi}. \end{aligned}$$

By estimating the multipliers with their supremum, we obtain

$$\begin{aligned} e^{\epsilon\gamma t}|S(t)z_0|_{L^2_x} \le\ & 2\pi|\cos(\phi(\xi)t) + \epsilon\mu t \operatorname{sinc}(\phi(\xi)t)|_{L^\infty_\xi}|\hat{a}_0|_{L^2_\xi} + 2\pi|t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)|_{L^\infty_\xi}|\hat{b}_0|_{L^2_\xi} \\ & + 2\pi|-t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)|_{L^\infty_\xi}|\hat{a}_0|_{L^2_\xi} + 2\pi|\cos(\phi(\xi)t) - \epsilon\mu t \operatorname{sinc}(\phi(\xi)t)|_{L^\infty_\xi}|\hat{b}_0|_{L^2_\xi} \\ \le\ & 2\pi C|\hat{z}_0|_{L^2_\xi} = C|z_0|_{L^2_x}, \end{aligned}$$

where we have used that

$$\begin{aligned} |\cos(\phi(\xi)t) \pm \epsilon\mu t \operatorname{sinc}(\phi(\xi)t)|_{L^\infty_\xi}| &= \left|\cos(\phi(\xi)t) \pm \epsilon\mu \frac{\sin(\phi(\xi)t)}{\phi(\xi)} t\right|_{L^\infty_\xi} \\ &\le 1 + \epsilon\mu \left|\frac{1}{\phi(\xi)}\right|_{L^\infty_\xi} = 1 + \epsilon\mu \left|\frac{1}{(\xi^2+\nu)\sqrt{1 - \frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2}}}\right|_{L^\infty_\xi} \\ &= 1 + \frac{\epsilon\mu}{\sqrt{\nu^2 - \epsilon^2\mu^2}} \le C, \end{aligned}$$

and

$$\begin{aligned} |\pm t(\xi^2+\nu)\operatorname{sinc}(\phi(\xi)t)|_{L^\infty_\xi} &= \left|\frac{\xi^2+\nu}{\phi(\xi)} \sin(\phi(\xi)t)\right|_{L^\infty_\xi} \\ &\le \left|\frac{\xi^2+\nu}{\phi(\xi)}\right|_{L^\infty_\xi} = \left|\frac{1}{\sqrt{1 - \frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2}}}\right|_{L^\infty_\xi} = \frac{1}{\sqrt{1 - \frac{\epsilon^2\mu^2}{\nu^2}}}. \end{aligned}$$

□

2.3.3. L^p-$L^{p'}$-estimates

We now combine the $L^\infty - L^1$ estimate of Proposition 2.3.2 and the L^2-estimate of Proposition 2.3.3 into an $L^p - L^{p'}$-estimate for all $p \in [2,\infty]$, via Theorem 1.2.6. Here, we denote by p' the conjugate exponent of $p \in [2,\infty]$, meaning that p' is the unique exponent for which $\frac{1}{p} + \frac{1}{p'} = 1$. We can now formulate the full range of dispersive estimates for the forced Schrödinger semigroup.

Theorem 2.3.4
If $p \in [2,\infty]$ and $t > 0$, then $S(t)$ maps continuously $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$ and there exist a constant C such that

$$|S(t)z_0|_{L^p_x} \leq Ct^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L^{p'}_x}, \tag{2.16}$$

for all $z_0 \in L^{p'}(\mathbb{R})$.

Proof. Using Theorem 1.2.6, it follows by interpolation between $(p,p') = (2,2)$ and $(p,p') = (\infty,1)$, that for $2 \leq p \leq \infty$

$$|S(t)z_0|_{L^p_x} \leq Ct^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L^{p'}_x}.$$

□

As a consequence of Theorem 2.1.1, the C_0-group $\{S(t)\}_{t\in\mathbb{R}}$ commutes with Fourier multipliers such as the weak derivatives ∂_x^α, where $\alpha \in \mathbb{N}$. Indeed, the 2×2 matrix in (2.4) commutes with the symbol of the weak derivatives, which are $(i\xi)^\alpha I_2$ for $\alpha \in \mathbb{N}$. Therefore, the fixed-time estimates of Theorem 2.3.4 also apply to Sobolev spaces, and we obtain the following corollary.

Corollary 2.3.5
If $p \in [2,\infty]$, $k \in \mathbb{N}$ and $t > 0$, then $S(t)$ maps continuously $W^{k,p'}(\mathbb{R})$ to $W^{k,p}(\mathbb{R})$ and there exist a constant C such that

$$|S(t)z_0|_{W^{k,p}_x} \leq Ct^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{W^{k,p'}_x}, \tag{2.17}$$

for all $z_0 \in W^{k,p'}(\mathbb{R})$.

Proof. We write out the Sobolev norm as

$$|S(t)z_0|_{W^{k,p}_x} = \sum_{\alpha=0}^{k} |\partial_x^\alpha S(t)z_0|_{L^p_x},$$

and use that the semigroup commutes with the weak derivative, so that

$$|S(t)z_0|_{W^{k,p}_x} = \sum_{\alpha=0}^{k} |S(t)\partial_x^\alpha z_0|_{L^p_x} \overset{(2.16)}{\leq} \sum_{\alpha=0}^{k} Ct^{-(\frac{1}{2}-\frac{1}{p})}|\partial_x^\alpha z_0|_{L^{p'}_x} = Ct^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{W^{k,p'}_x}.$$

□

2.4. Strichartz estimates

The estimates of the previous section are, although insightful, not particularly useful to work with. Instead, we show that the semigroup $\{S(t)\}_{t\geq 0}$ satisfies a set of space-time smoothing estimates, called Strichartz estimates. This type of estimate was first proved for the linear Schrödinger equation, and can be used to obtain local existence results for solutions of the nonlinear Schrödinger equation [16]. Likewise, we will use the Strichartz estimate for the forced Schrödinger semigroup to obtain existence of mild solutions for the stochastic parametrically-forced NLS equation. In general Strichartz estimates hold for dispersive equations that satisfy the dispersive estimates of the previous section. The estimates apply to Bochner-Lebesgue-norms in which the exponents are admissible pairs. These are defined as follows.

Definition 2.4.1 (Admissible pairs)
A pair (r,p) of exponents is called ***admissible*** *if $2 \leq p \leq \infty$ and*

$$\frac{2}{r} = \frac{1}{2} - \frac{1}{p}. \tag{2.18}$$

The original proof of Strichartz estimates can be found in [32]. We, however, present an adaptation of the approach of Cazenave [11, Theorem 3.2.5, p. 35]. This approach makes use of the unitarity of the Schrödinger semigroup, which does not apply to the forced Schrödinger semigroup. Instead, we will work with the adjoint of the semigroup and its Fourier representation. We now state and prove the Strichartz estimates.

Theorem 2.4.2 (Strichartz estimates)
Let $\{S(t)\}_{t\geq 0}$ be the semigroup associated to the forced Schrödinger equation, and let $T \in (0,\infty)$. Let furthermore (r,p) and (γ,q) be admissible pairs.

1. *(Convolution estimates) If $f \in L^{\gamma'}(0,T;L^{q'}(\mathbb{R}))$, then the function*

$$t \mapsto \Phi_f(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s \quad \text{for } t \in [0,T), \tag{2.19}$$

belongs to $L^r(0,T;L^p(\mathbb{R})) \cap C([0,T];L^2(\mathbb{R}))$. Furthermore, there exists a constant C, depending only on r and γ such that

$$|\Phi_f|_{L^r(0,T;L^p_x)} \leq C|f|_{L^{\gamma'}(0,T;L^{q'}_x)}, \tag{2.20}$$

for every $f \in L^{\gamma'}(0,T;L^{p'}(\mathbb{R}))$.

2. *(Homogeneous estimates) For every $\phi \in L^2(\mathbb{R})$, the function $t \mapsto S(t)\phi$ belongs to $L^r(0,T;L^p(\mathbb{R})) \cap C([0,T];L^2(\mathbb{R}))$. Furthermore, there exists a constant C, depending only on q such that*

$$|S(\cdot)\phi|_{L^r(0,T;L^p_x)} \leq C|\phi|_{L^2_x}. \tag{2.21}$$

Proof. The proof is divided into seven steps, the first six of which prove the convolution estimates and the last of which proves the homogeneous estimates. It will be helpful to define the operators Θ_t, Λ_s and Ψ as

$$\Theta_{t,f}(s) = \int_0^t S^*(t-s)S(t-\sigma)f(\sigma)\,\mathrm{d}\sigma \quad \text{for } s \in [0,T), \tag{2.22}$$

$$\Lambda_{s,f}(t) = \int_s^T S(t-s)S^*(\tau-s)f(\tau)\,\mathrm{d}\tau \quad \text{for } t \in [0,T), \tag{2.23}$$

and

$$\Psi_f(s) = \int_s^T S^*(t-s)f(t)\,\mathrm{d}t \quad \text{for } s \in [0,T). \tag{2.24}$$

Step 1.
We show that for every admissible pair (r,p), the operator Φ is continuous from $L^{r'}(0,T;L^{p'}(\mathbb{R}))$ to $L^r(0,T;L^p(\mathbb{R}))$,

with norm depending only on r. Therefore, by density it suffices to consider $f \in C_c([0,T), L^{p'}(\mathbb{R}))$. Theorem 2.3.4 then shows that

$$|\Phi_f(t)|_{L^p_x} \overset{(2.19)}{=} \left|\int_0^t S(t-s)f(s)\,\mathrm{d}s\right|_{L^p_x} \le \int_0^t |S(t-s)f(s)|_{L^p_x}\,\mathrm{d}s \overset{(2.16)}{\le} C\int_0^t |t-s|^{-(\frac{1}{2}-\frac{1}{p})}|f(s)|_{L^{p'}_x}\,\mathrm{d}s$$
$$\overset{(2.18)}{=} C\int_0^t |t-s|^{-\frac{2}{r}}|f(s)|_{L^{p'}_x}\,\mathrm{d}s \le C\int_0^T |t-s|^{-\frac{2}{r}}|f(s)|_{L^{p'}_x}\,\mathrm{d}s.$$

We recognize this last integral as the Riesz potential of order $1-\frac{2}{r}$ of the function $s \mapsto |f(s)|_{L^{p'}}$, defined for $s \in [0,T)$. From the Hardy-Littlewood-Sobolev inequality (Theorem 1.2.13), it then follows that

$$|\Phi_f|_{L^r(0,T;L^p_x)} \le C|f|_{L^{r'}(0,T;L^{p'}_x)},$$

where C only depends on r.

Step 2.
By the same argument, the operators Θ_t and Λ_s are continuous from $L^{r'}(0,T;L^{p'}(\mathbb{R}))$ to $L^r(0,T;L^p(\mathbb{R}))$, with norm depending only on q. Instead of the L^p-estimate on the semigroup, one now uses the following L^p-estimates on the product of the adjoint and the semigroup.

$$|S^*(t_1)S(t_2)z_0|_{L^p_x} \le C|t_2-t_1|^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L^{p'}_x} \quad \text{for all } z_0 \in L^{p'}(\mathbb{R}) \text{ and } t_1, t_2 > 0, \tag{2.25}$$

and

$$|S(t_1)S^*(t_2)z_0|_{L^p_x} \le C|t_2-t_1|^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L^{p'}_x} \quad \text{for all } z_0 \in L^{p'}_x(\mathbb{R}) \text{ and } t_1, t_2 > 0. \tag{2.26}$$

These estimates imply that

$$|S^*(t-s)S(t-\sigma)z_0|_{L^p_x} \le C|s-\sigma|^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L^{p'}_x} \quad \text{for all } z_0 \in L^{p'}_x(\mathbb{R}),$$

and

$$|S(t-s)S^*(\tau-s)z_0|_{L^p_x} \le C|t-\tau|^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L^{p'}_x} \quad \text{for all } z_0 \in L^{p'}(\mathbb{R}).$$

One then follows the argument of step 1, and we obtain

$$|\Theta_{t,f}|_{L^r(0,T;L^p_x)} \le C|f|_{L^{r'}(0,T;L^{p'}_x)}, \quad \text{and} \tag{2.27}$$
$$|\Lambda_{s,f}|_{L^r(0,T;L^p_x)} \le C|f|_{L^{r'}(0,T;L^{p'}_x)}. \tag{2.28}$$

The proofs of (2.25) and (2.26) are analogous to that of Theorem 2.3.4, and can be found in Appendix A (Lemma A.0.2).

Step 3.
We show that for every admissible pair (r,p), the operator Φ is continuous from $L^{r'}(0,T;L^{p'}(\mathbb{R}))$ to $C([0,T],L^2(\mathbb{R}))$, with norm depending only on r. Therefore, by density it suffices to consider $f \in C_c([0,T),L^{p'}(\mathbb{R}))$. Via the embedding $L^{p'}(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R})$ (Proposition 1.2.16) it follows that $f \in C_c([0,T),L^{p'}(\mathbb{R})) \cap C_c([0,T),H^{-1}(\mathbb{R}))$. By applying the operator $(I-\epsilon\Delta)^{-1}$, we may approximate f with functions $f_\epsilon := (I-\epsilon\Delta)^{-1}f \in C_c([0,T),H^1(\mathbb{R}))$. It follows from

$$|\Phi_{f_\epsilon}(t)|_{L^2_x} \overset{(2.19)}{=} \left|\int_0^t S(t-s)f_\epsilon(s)\,\mathrm{d}s\right|_{L^2_x} \le \int_0^t |S(t-s)f_\epsilon(s)|_{L^2_x}\,\mathrm{d}s \le Ct|f_\epsilon|_{C([0,t];L^2_x)}$$

that $\Phi_{f_\epsilon} \in C([0,T),L^2(\mathbb{R}))$. This allows us to write

$$\begin{aligned}
|\Phi_{f_\epsilon}(t)|^2_{L^2_x} &= \left\langle \int_0^t S(t-s)f_\epsilon(s)\,\mathrm{d}s, \int_0^t S(t-\sigma)f_\epsilon(\sigma)\,\mathrm{d}\sigma \right\rangle_{L^2_x} \\
&= \int_0^t\int_0^t \langle S(t-s)f_\epsilon(s), S(t-\sigma)f_\epsilon(\sigma)\rangle_{L^2_x}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&= \int_0^t\int_0^t \langle f_\epsilon(s), S^*(t-s)S(t-\sigma)f_\epsilon(\sigma)\rangle_{L^2_x}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&= \int_0^t \langle f_\epsilon(s), \int_0^t S^*(t-s)S(t-\sigma)f_\epsilon(\sigma)\,\mathrm{d}\sigma\rangle_{L^2_x}\,\mathrm{d}s \\
&= \int_0^t \langle f_\epsilon(s), \Theta_{t,f_\epsilon}(s)\rangle_{L^2_x}\,\mathrm{d}s \le \int_0^t |f_\epsilon(s)\Theta_{t,f_\epsilon}(s)|_{L^1_x}\,\mathrm{d}s
\end{aligned}$$

By applying Hölder's inequality first in space with exponents p, p', and then in time with exponents r, r', we obtain

$$\begin{aligned} |\Phi_{f_\epsilon}(t)|^2_{L^2_x} &\le \int_0^t |f_\epsilon(s)|_{L^{p'}_x} |\Theta_{t,f_\epsilon}(s)|_{L^p_x} \, \mathrm{d}s \le |f_\epsilon|_{L^{r'}(0,T;L^{p'}_x)} |\Theta_{t,f_\epsilon}|_{L^r(0,T;L^p_x)} \\ &\overset{(2.27)}{\le} |f_\epsilon|^2_{L^{r'}(0,T;L^{p'}_x)} \overset{(1.5)}{\le} C|f|^2_{L^{r'}(0,T;L^{p'}_x)}, \end{aligned} \tag{2.29}$$

where we have used step 2 to obtain the second to last inequality, and the uniform bound on the regulizing operator for the last inequality. We now pass to the limit $\epsilon \downarrow 0$ in (2.29), resulting in

$$|\Phi_f(t)|^2_{L^2_x} \le C|f|^2_{L^{r'}(0,T;L^{p'}_x)}.$$

Step 4.
We show that for every admissible pair (r, p), the operator Ψ is also continuous from $L^{r'}(0,T;L^{p'}(\mathbb{R}))$ to $C([0,T],L^2(\mathbb{R}))$, with norm depending only on r. Therefore, by density it suffices to consider $f \in C_c([0,T),L^{p'}(\mathbb{R}))$, and as in the previous step, we may approximate f with functions $f_\epsilon := (I-\epsilon\Delta)^{-1} f \in C_c([0,T),H^1(\mathbb{R}))$. We then proceed in the same manner

$$\begin{aligned} |\Psi_{f_\epsilon}(s)|^2_{L^2_x} &\overset{(2.24)}{=} \left\langle \int_s^T S^*(t-s) f_\epsilon(t) \, \mathrm{d}t, \int_s^T S^*(\tau - s) f_\epsilon(\tau) \, \mathrm{d}\tau \right\rangle_{L^2_x} \\ &= \int_s^T \int_s^T \langle S^*(t-s) f_\epsilon(t), S^*(\tau-s) f_\epsilon(\tau) \rangle_{L^2_x} \, \mathrm{d}\tau \, \mathrm{d}t \\ &= \int_s^T \int_s^T \langle f_\epsilon(t), S(t-s) S^*(\tau - s) f_\epsilon(\tau) \rangle_{L^2_x} \, \mathrm{d}\tau \, \mathrm{d}t \\ &= \int_s^T \langle f_\epsilon(t), \int_s^T S(t-s) S^*(\tau-s) f_\epsilon(\tau) \, \mathrm{d}\tau \rangle_{L^2_x} \, \mathrm{d}t \\ &= \int_s^T \langle f_\epsilon(t), \Lambda_{s,f_\epsilon}(t) \rangle_{L^2_x} \, \mathrm{d}t \le \int_s^T |f_\epsilon(t) \Lambda_{s,f_\epsilon}(t)|_{L^1_x} \, \mathrm{d}t. \end{aligned}$$

By applying Hölder's inequality first in space with exponents p, p', and then in time with exponents r, r', we obtain

$$|\Psi_{f_\epsilon}(s)|^2_{L^2_x} \le \int_s^T |f_\epsilon(t)|_{L^{p'}_x} |\Lambda_{s,f_\epsilon}(t)|_{L^p_x} \, \mathrm{d}t \le |f_\epsilon|_{L^{r'}(0,T;L^{p'}_x)} |\Lambda_{s,f_\epsilon}|_{L^r(0,T;L^p_x)} \overset{(2.28)}{\le} |f_\epsilon|^2_{L^{r'}(0,T;L^{p'}_x)}, \tag{2.30}$$

where we have used step 2 to obtain the last inequality. Since s is arbitrary, the result follows upon letting $\epsilon \downarrow 0$.

Step 5.
We show that for every admissible pair (r, p), the operator Φ is continuous from $L^1(0,T;L^2(\mathbb{R}))$ to $L^r(0,T;L^p(\mathbb{R}))$, with norm depending only on r. Therefore, let $f \in L^1(0,T;L^2(\mathbb{R}))$ and consider $\phi \in C_c([0,T),C_c^\infty(\mathbb{R}))$. We have

$$\begin{aligned} \int_0^T \langle \Phi_f(t), \phi(t) \rangle_{L^2_x} \, \mathrm{d}t &= \int_0^T \left\langle \int_0^t S(t-s) f(s) \, \mathrm{d}s, \phi(t) \right\rangle_{L^2_x} \mathrm{d}t \\ &= \int_0^T \int_s^T \langle S(t-s) f(s), \phi(t) \rangle_{L^2_x} \, \mathrm{d}t \, \mathrm{d}s \\ &= \int_0^T \left\langle f(s), \int_s^T S^*(t-s) \phi(t) \, \mathrm{d}t \right\rangle_{L^2_x} \mathrm{d}s \\ &= \int_0^T \langle f(s), \Psi_\phi(s) \rangle_{L^2_x} \, \mathrm{d}s. \end{aligned}$$

Now, by the Cauchy-Schwarz inequality

$$\begin{aligned} \left| \int_0^T \langle \Phi_f(t), \phi(t) \rangle_{L^2_x} \, \mathrm{d}t \right| &\le \int_0^T |f(s)|_{L^2_x} |\Psi_\phi(s)|_{L^2_x} \, \mathrm{d}s \le |f|_{L^1(0,T;L^2_x)} |\Psi_\phi|_{L^\infty(0,T;L^2_x)} \\ &\overset{(2.30)}{\le} C|f|_{L^1(0,T;L^2_x)} |\phi|_{L^{r'}(0,T;L^{p'}_x)}, \end{aligned} \tag{2.31}$$

where we have used step 4 to obtain the last inequality. Using that the dual of the Bochner space $L^r(0,T;L^p_x)$ is isometrically isomorphic to $L^{r'}(0,T;L^{p'}_x)$ (see for instance [13]), we obtain the following characterization of the $L^r(0,T;L^p_x)$-norm of functions $g \in L^1(0,T;L^2_x)$.

$$|g|_{L^r(0,T;L^p_x)} = \sup\Big\{ \int_0^T \langle g(t), \phi(t)\rangle_{L^2_x}\, \mathrm{d}t; \quad \phi \in C_c([0,T), C_c^\infty(\mathbb{R})), |\phi|_{L^{r'}(0,T;L^{p'}_x)} = 1\Big\}. \tag{2.32}$$

The result follows from (2.31), and the above relation applied with $g = \Phi_f$.

Step 6.
Assume that (γ, q) is another admissible pair, i.e. $2 \le q \le \infty$ and

$$\frac{2}{\gamma} = \frac{1}{2} - \frac{1}{q}. \tag{2.33}$$

From step 3 we obtain that Φ is continuous from $L^{\gamma'}(0,T;L^{q'}_x)$ to $L^\infty(0,T;L^2_x)$ and from step 1 we obtain that Φ is continuous from $L^{\gamma'}(0,T;L^{q'}_x)$ to $L^\gamma(0,T;L^q_x)$. Therefore there exists a constant C so that

$$|\Phi_f|_{L^\gamma(0,T;L^q_x)} \le C|f|_{L^{\gamma'}(0,T;L^{q'}_x)}, \quad \text{and} \tag{2.34}$$

$$|\Phi_f|_{L^\infty(0,T;L^2_x)} \le C|f|_{L^{\gamma'}(0,T;L^{q'}_x)}. \tag{2.35}$$

We now distinguish two cases. First, assume that $2 \le p \le q$, and pick $\theta \in [0,1]$ so that

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}. \tag{2.36}$$

This implies for r and γ that

$$\frac{1}{r} \overset{(2.18)}{=} \frac{1}{4} - \frac{1}{2p} \overset{(2.36)}{=} \frac{1}{4} - \frac{\theta}{2q} - \frac{1-\theta}{4} = \frac{\theta}{4} - \frac{\theta}{2q} \overset{(2.33)}{=} \frac{\theta}{\gamma} + \frac{1-\theta}{\infty}, \tag{2.37}$$

where we have used the admissibility condition (Definition 2.4.1) in the first and last equality. We now apply Hölder's inequality to the product $\Phi_f(t)^\theta \Phi_f(t)^{1-\theta}$, which gives

$$|\Phi_f(t)|_{L^p_x} \overset{(2.36)}{\le} |\Phi_f(t)|^\theta_{L^q_x} |\Phi_f(t)|^{1-\theta}_{L^2_x}. \tag{2.38}$$

Then, we apply Hölder's inequality in time to find

$$\begin{aligned} |\Phi_f|_{L^r(0,T;L^p_x)} &= \left(\int_0^T |\Phi_f(t)|^r_{L^p_x}\, \mathrm{d}t\right)^{\frac{1}{r}} \overset{(2.38)}{\le} \left(\int_0^T |\Phi_f(t)|^{\theta r}_{L^q_x} |\Phi_f(t)|^{(1-\theta)r}_{L^2_x}\, \mathrm{d}t\right)^{\frac{1}{r}} \\ &\overset{(2.37)}{\le} |\Phi_f|^\theta_{L^\gamma(0,T;L^q_x)} |\Phi_f|^{1-\theta}_{L^\infty(0,T;L^2_x)}. \end{aligned} \tag{2.39}$$

Using the previous continuity results for Φ, we finally obtain

$$|\Phi_f|_{L^r(0,T;L^p_x)} \overset{(2.34),(2.35)}{\le} C|f|_{L^{\gamma'}(0,T;L^{q'}_x)}.$$

We conclude that Φ is a continuous map from $L^{\gamma'}(0,T;L^{q'}_x)$ to $L^r(0,T;L^p_x)$.

Now assume that $q < p$. It follows from step 1 that Φ is continuous from $L^{r'}(0,T;L^{p'}_x)$ to $L^r(0,T;L^p_x)$ and from step 5 that Φ is continuous from $L^1(0,T;L^2_x)$ to $L^r(0,T;L^p_x)$. By an interpolation theorem due to Bergh and Löfström [2, Theorem 5.1.2, p. 107], we obtain that there exists a one-parameter family of exponent pairs (σ, δ) for which it holds that Φ is continuous from $L^\sigma(0,T;L^\delta_x)$ to $L^r(0,T;L^p_x)$. These are the pairs (σ,δ) that satisfy

$$\frac{1}{\sigma} = \frac{\theta}{1} + \frac{1-\theta}{r'} \text{ and } \frac{1}{\delta} = \frac{\theta}{2} + \frac{1-\theta}{p'}$$

for some $\theta \in [0,1]$. We now show that (γ', q') is part of this family of exponent pairs. Pick $\theta \in [0,1]$ so that

$$\frac{1}{q'} = \frac{\theta}{2} + \frac{1-\theta}{p'}. \tag{2.40}$$

This implies for γ' and r' that

$$\frac{1}{\gamma'} = \frac{5}{4} - \frac{1}{2q'} \overset{(2.40)}{=} \frac{5}{4} - \frac{\theta}{4} - \frac{1-\theta}{2p'} = \frac{5}{4} - \frac{\theta}{4} + \frac{1-\theta}{2}\left(\frac{2}{r'} - \frac{5}{2}\right) = \frac{\theta}{1} + \frac{1-\theta}{r'},$$

where we have used that the admissibility condition (Definition 2.4.1) can be formulated as $\frac{2}{\gamma'} = \frac{5}{2} - \frac{1}{q'}$ and $\frac{2}{r'} = \frac{5}{2} - \frac{1}{p'}$ for the admissible pairs (γ, q) and (r, p) in the first and third equality. It follows that Φ is a continuous map from $L^{\gamma'}(0,T;L_x^{q'})$ to $L^r(0,T;L_x^p)$. This completes the proof of the first part of the theorem.

Step 7. We now prove the homogeneous estimate, which is proved in the same way as the convolution estimate. Using the characterization of the $L^q(0,T;L_x^r)$-norm of functions $g \in L^1(0,T;L_x^2)$, we may write

$$|g|_{L^r(0,T;L_x^p)} = \sup\left\{\int_0^T \langle g(t), \psi(t)\rangle_{L_x^2}\, \mathrm{d}t; \quad \psi \in C_c([0,T), C_c^\infty(\mathbb{R})), |\psi|_{L^{r'}(0,T;L_x^{p'})} = 1\right\},$$

as in (2.32) and we note that is suffices to show the bound

$$\left|\int_0^T \langle S(t)\phi, \psi(t)\rangle_{L_x^2}\, \mathrm{d}t\right| \le C|\phi|_{L_x^2},$$

for $\phi \in L_x^2$ and $\psi \in C_c([0,T), C_c^\infty(\mathbb{R}))$ with $|\psi|_{L^{r'}(0,T;L_x^{p'})} = 1$. Therefore we write

$$\left|\int_0^T \langle S(t)\phi, \psi(t)\rangle_{L_x^2}\, \mathrm{d}t\right| = \left|\left\langle \phi, \int_0^T S^*(t)\psi(t)\, \mathrm{d}t\right\rangle_{L_x^2}\right|,$$

and it follows from the Cauchy-Schwarz inequality that

$$\left|\int_0^T \langle S(t)\phi, \psi(t)\rangle_{L_x^2}\, \mathrm{d}t\right| \le |\phi|_{L_x^2}\left|\int_0^T S^*(t)\psi(t)\, \mathrm{d}t\right|_{L_x^2}. \tag{2.41}$$

This last factor can in turn be written as

$$\begin{aligned}
\left|\int_0^T S^*(t)\psi(t)\, \mathrm{d}t\right|_{L_x^2}^2 &= \left\langle \int_0^T S^*(t)\psi(t)\, \mathrm{d}t, \int_0^T S^*(s)\psi(s)\, \mathrm{d}s\right\rangle_{L_x^2} \\
&= \int_0^T\int_0^T \langle S^*(t)\psi(t), S^*(s)\psi(s)\rangle_{L_x^2}\, \mathrm{d}t\, \mathrm{d}s \\
&= \int_0^T\int_0^T \langle \psi(t), S(t)S^*(s)\psi(s)\rangle_{L_x^2}\, \mathrm{d}t\, \mathrm{d}s \\
&= \int_0^T \left\langle \psi(t), \int_0^T S(t)S^*(s)\psi(s)\, \mathrm{d}s\right\rangle_{L_x^2} \mathrm{d}t \\
&\le \int_0^T \left|\psi(t), \int_0^T S(t)S^*(s)\psi(s)\, \mathrm{d}s\right|_{L_x^1} \mathrm{d}t.
\end{aligned}$$

By applying Hölder's inequality, first in space and then in time, we find

$$\begin{aligned}
\left|\int_0^T S^*(t)\psi(t)\, \mathrm{d}t\right|_{L_x^2}^2 &\le \int_0^T |\psi(t)|_{L_x^{p'}} \left|\int_0^T S(t)S^*(s)\psi(s)\, \mathrm{d}s\right|_{L_x^p} \mathrm{d}t \\
&\le |\psi|_{L^{r'}(0,T;L_x^{p'})}\left|\int_0^T S(\cdot)S^*(s)\psi(s)\, \mathrm{d}s\right|_{L^r(0,T;L_x^p)} \le C|\psi|_{L^{r'}(0,T;L_x^{p'})}^2 = C,
\end{aligned}$$

where the last inequality follows as in step 2, and the last equality follows from the assumption $|\psi|_{L^{r'}(0,T;L_x^{p'})} = 1$. Combining this with (2.41), we get

$$\left|\int_0^T \langle S(t)\phi, \psi(t)\rangle_{L_x^2}\, \mathrm{d}t\right| \le C|\phi|_{L_x^2},$$

as desired. □

As in Corollary 2.3.5, we may use the fact that the semigroup commutes with weak derivatives ∂_x^α, with $\alpha \in \mathbb{N}$, to observe that

$$\partial_x^\alpha \Phi_f(t) = \partial_x^\alpha \int_0^t S(t-s)f(s)\,\mathrm{d}s = \int_0^t \partial_x^\alpha S(t-s)f(s)\,\mathrm{d}s = \int_0^t S(t-s)\partial_x^\alpha f(s)\,\mathrm{d}s = \Phi_{\partial_x^\alpha f}(t), \tag{2.42}$$

and obtain Strichartz estimates for Sobolev spaces of the spatial variable. At various points in the proof we use the equivalence of p-norms on $\mathbb{R}^n$, i.e. for all $p \geq 1$ there exist constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{i=1}^n |x_i| \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \leq C_2 \sum_{i=1}^n |x_i|, \tag{2.43}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$. We will also write $\lesssim$ to denote that the term to the left of $\lesssim$ is smaller than or equal to a positive constant times the term to the right of $\lesssim$.

Corollary 2.4.3 (Strichartz estimates for Sobolev norms)
Let $\{S(t)\}_{t\geq 0}$ be the semigroup associated to the forced Schrödinger equation, and let $T \in (0,\infty)$. Let furthermore $k \in \mathbb{N}$ and let (r,p) and (γ,q) be admissible pairs.

1. *(Convolution estimates) If $f \in L^{\gamma'}(0,T;W^{k,q'}(\mathbb{R}))$, then the function*

$$t \mapsto \Phi_f(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s \quad \text{for } t \in [0,T),$$

belongs to $L^r(0,T;W^{k,p}(\mathbb{R})) \cap C([0,T];H^k(\mathbb{R}))$. Furthermore, there exists a constant C, depending only on r and γ such that

$$|\Phi_f|_{L^r(0,T;W_x^{k,p})} \leq C|f|_{L^{\gamma'}(0,T;W_x^{k,q'})}, \tag{2.44}$$

for every $f \in L^{\gamma'}(0,T;W^{k,p'}(\mathbb{R}))$.

2. *(Homogeneous estimates) For every $\phi \in H^k(\mathbb{R})$, the function $t \mapsto S(t)\phi$ belongs to $L^r(0,T;W^{k,p}(\mathbb{R})) \cap C([0,T];H^k(\mathbb{R}))$. Furthermore, there exists a constant C, depending only on q such that*

$$|S(\cdot)\phi|_{L^r(0,T;W_x^{k,p})} \leq C|\phi|_{H_x^k}. \tag{2.45}$$

Proof. We first show (2.44), and write out the $L^r(0,T;W_x^{k,p})$-norm as

$$|\Phi_f|^r_{L^r(0,T;W_x^{k,p})} = \int_0^T |\Phi_f|^r_{W_x^{k,p}}\,\mathrm{d}t = \int_0^T \left(\sum_{\alpha=0}^k |\partial_x^\alpha \Phi_f|_{L_x^p}\right)^r \mathrm{d}t,$$

and use the observation in (2.42), so that

$$|\Phi_f|^r_{L^r(0,T;W_x^{k,p})} \overset{(2.42)}{=} \int_0^T \left(\sum_{\alpha=0}^k |\Phi_{\partial_x^\alpha f}|_{L_x^p}\right)^r \mathrm{d}t \overset{(2.43)}{\lesssim} \sum_{\alpha=0}^k \int_0^T |\Phi_{\partial_x^\alpha f}|^r_{L_x^p}\,\mathrm{d}t = \sum_{\alpha=0}^k |\Phi_{\partial_x^\alpha f}|^r_{L^r(0,T;L_x^p)}.$$

We then apply the Strichartz estimate for the convolution, yielding

$$\begin{aligned}|\Phi_f|^{\gamma'}_{L^r(0,T;W_x^{k,p})} &\overset{(2.20)}{\lesssim} \left(\sum_{\alpha=0}^k |\partial_x^\alpha f|^r_{L^{\gamma'}(0,T;L_x^{q'})}\right)^{\gamma'/r} \overset{(2.43)}{\lesssim} \sum_{\alpha=0}^k |\partial_x^\alpha f|^{\gamma'}_{L^{\gamma'}(0,T;L_x^{q'})} = \int_0^T \sum_{\alpha=0}^k |\partial_x^\alpha f|^{\gamma'}_{L_x^{q'}}\,\mathrm{d}t \\ &\overset{(2.43)}{\lesssim} \int_0^T |\partial_x^\alpha f|^{\gamma'}_{W_x^{k,q'}}\,\mathrm{d}t = |f|^{\gamma'}_{L^{\gamma'}(0,T;W_x^{k,q'})},\end{aligned}$$

which shows (2.44). For (2.45), we write

$$\begin{aligned}|S(\cdot)\phi|^r_{L^r(0,T;W_x^{k,p})} &= \int_0^T |S(t)\phi|^r_{W_x^{k,p}}\,\mathrm{d}t = \int_0^T \left(\sum_{\alpha=0}^k |\partial_x^\alpha S(t)\phi|_{L_x^p}\right)^r \mathrm{d}t = \int_0^T \left(\sum_{\alpha=0}^k |S(t)\partial_x^\alpha \phi|_{L_x^p}\right)^r \mathrm{d}t \\ &\overset{(2.42)}{\leq} \sum_{\alpha=0}^k \int_0^T |S(t)\partial_x^\alpha \phi|^r_{L_x^p}\,\mathrm{d}t = \sum_{\alpha=0}^k |S(\cdot)\partial_x^\alpha \phi|^r_{L^r(0,T;L_x^p)} \overset{(2.21)}{\lesssim} \sum_{\alpha=0}^k |\partial_x^\alpha \phi|^r_{L_x^2},\end{aligned}$$

and (2.45) follows. □

3

A stochastic PFNLS equation

In this chapter, we analyze the parametrically-forced Schrödinger equation with multiplicative noise, Equation (4). In particular, we prove the existence and uniqueness of global mild solutions to (4). The proof forms a combined exposition of the existence and uniqueness proofs due to de Bouard and Debussche in [3, 4], that treat the nonlinear Schrödinger equation with multiplicative noise and initial data in L^2 and H^1, respectively. Minor changes to the proofs ensure that the results apply to the parametrically-forced equation. We begin in Section 3.1 by giving a precise formulation of the setting and the notion of a mild solution to (4), and we introduce various spaces that will be used in the analysis. The proof is based on a fixed-point argument, for which we will need estimates on the stochastic convolution with the PFS semigroup. We prove these estimates in Section 3.2. In Section 3.3 we then present the fixed-point argument, which applies to a version of (4) in which the nonlinear term is truncated. This truncation is necessary as the nonlinearity is not Lipschitz continuous. We proceed by constructing a local solution to the original problem based on the global solutions to the truncated problem in Section 3.4. We also show that a finite existence time can only occur in case of blow-up of the L^2-norm or H^1-norm, for initial data in L^2 and H^1, respectively. We conclude the proof of global existence in Section 3.5, by showing that blow-up does not occur. We therefore analyze the evolution of the L^2- and H^1-norm using Itô's formula (Theorem 1.3.18). A technical difficulty arises due to the fact that Itô's formula applies, in the context of SPDEs, to strong solutions instead of mild solutions. We overcome this complication via a regularization procedure.

3.1. Preliminaries

For the remainder of this chapter we fix a stochastic basis, i.e. a filtered probability space

$$(\Omega, \mathscr{F}, (\mathscr{F}(t))_{t\in[0,T_0]}, \mathbb{P}),$$

where $(\mathscr{F}(t))_{t\in[0,T]}$ is a complete and right-continuous filtration, and $T_0 > 0$. We also denote by W_H an $L^2(\mathbb{R},\mathbb{R})$-cylindrical Brownian motion on $[0, T_0]$, cf. Definition 1.3.11. We then consider the stochastic partial differential equation

$$\mathrm{d}z = (i\Delta z - i\nu z - \epsilon(\gamma z - \mu\overline{z}))\,\mathrm{d}t - \frac{1}{2}zF_\Phi\,\mathrm{d}t + 4i|z|^2 z\,\mathrm{d}t - iz\Phi\,\mathrm{d}W_H \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}^+, \tag{3.1}$$

where z is a complex-valued process defined on $\mathbb{R}\times\mathbb{R}^+$. The formulation above no longer contains the Stratonovitch product, which is based on an alternative definition of the stochastic integral. Instead, the stochastic differential in this equation should be interpreted in the Itô sense, cf. Subsection 1.3.4. Equation (3.1) is equivalent to (4), and the newly introduced term F_Φ serves as a correction to the difference between the Stratonovitch and Itô products. The function F_Φ, called the Îto correction or Itô drift, is defined as

$$F_\Phi(x) = \sum_{k=0}^{\infty} (\Phi e_k(x))^2 \quad \text{for } x \in \mathbb{R},$$

with $(e_k)_{k\in\mathbb{N}}$ and orthonormal basis of $L^2(\mathbb{R},\mathbb{R})$. One can verify that this definition of F_Φ does not depend on the choice of $(e_k)_{k\in\mathbb{N}}$. We will need to assume that the operator Φ maps the cylindrical Brownian motion into a space of more regular functions, as the noise is otherwise too irregular to work with. We make this assumption precise in Subsection 3.1.1 and furthermore denote the more regular Brownian motion by $\mathrm{d}W := \Phi\,\mathrm{d}W_H$.

We impose on (3.1) the initial condition $z(0) = z_0$, and we consider both the cases $z_0 \in L^r(\Omega; L^2(\mathbb{R}))$ and $z_0 \in L^r(\Omega; H^1(\mathbb{R}))$ for some suitably chosen constant r. As in Chapter 2, we write $\{S(t)\}_{t\in\mathbb{R}}$ for the C_0-group associated to the linear parametrically-forced Schrödinger equation. Mild solutions to (3.1) should then, by definition, satisfy

$$z(t) = S(t)z_0 + 4i\int_0^t S(t-s)(|z(s)|^2 z(s))\,\mathrm{d}s - i\int_0^t S(t-s)(z(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_0^t S(t-s)(z(s)F_\Phi)\,\mathrm{d}s, \tag{3.2}$$

for each $t \in [0, T_0]$ and $\mathbb{P}$-almost surely.

3.1.1. Assumptions

In order to obtain a solution to (3.2), we will perform a fixed-point argument in the Banach space $L^r(\Omega; X_T)$, where X_T is the space

$$X_T := C([0,T]; H_x^{\mathfrak{s}}) \cap L^r(0,T; W_x^{\mathfrak{s},p}), \quad T > 0. \tag{3.3}$$

We choose the differentiability constant $\mathfrak{s}$ appropriately as $\mathfrak{s} = 0$ or $\mathfrak{s} = 1$ to prove existence and uniqueness results in L_x^2 and H_x^1 respectively. The constants r and p are suitably chosen depending on the regularity of the noise, as well as constants γ and q which will serve as an admissible pair for the Strichartz estimates. We now present the conditions on the exponents and the regularity condition on the noise $W = \Phi W_H$ needed for the fixed-point argument.

Assumptions 3.1.1
We assume that the operator Φ is:

- *Hilbert-Schmidt on $L^2(\mathbb{R},\mathbb{R})$, cf. Definition 1.3.4. That is, $\Phi \in \mathscr{L}_2(L^2(\mathbb{R},\mathbb{R}))$.*
- *γ-radonifying from $L^2(\mathbb{R};\mathbb{R})$ to $W^{\mathfrak{s},2+\delta}(\mathbb{R};\mathbb{C})$ for some fixed $\delta > 0$ and $\mathfrak{s} \in \{0,1\}$, Cf. Definition 1.3.5.*

We denote $\mathscr{L}_2(L^2(\mathbb{R},\mathbb{R}))$ by $\mathscr{L}_2$ to lighten notation. By setting

$$\|\Phi\|_\delta := \|\Phi\|_{\mathscr{L}_2} + \|\Phi\|_{\gamma(L^2(\mathbb{R};\mathbb{R}), W^{\mathfrak{s},2+\delta}(\mathbb{R};\mathbb{C}))},$$

the assumptions on the noise amount to assuming $\|\Phi\|_\delta < \infty$. We then choose the constants r, p, γ, and q such that they satisfy:

$$p \in [4,\infty); \tag{3.4}$$

$$p \geq \frac{2(2+\delta)}{\delta}; \tag{3.5}$$

$$\frac{2}{r} = \frac{1}{2} - \frac{1}{p}; \tag{3.6}$$

$$q = 4; \tag{3.7}$$

$$\gamma = 8. \tag{3.8}$$

We now collect some identities that follow from Assumption 3.1.1 which will serve as exponents for Hölder's inequality in subsequent proofs.

Remark 3.1.2. Under Assumption 3.1.1, we have

$$\frac{1}{p'} = \frac{1}{2} + \frac{2}{r}; \tag{3.9}$$

$$\frac{1}{p'} = \frac{1}{p} + \frac{4}{r}; \tag{3.10}$$

$$\frac{1}{2} = \frac{1}{p} + \frac{2}{r}; \tag{3.11}$$

$$\frac{1}{3\gamma'} = \frac{1}{\gamma} + \frac{1}{6}; \tag{3.12}$$

$$\frac{1}{q'} = \frac{1}{q} + \frac{1}{q} + + \frac{1}{q}; \tag{3.13}$$

$$\frac{1}{\gamma'} = \frac{2}{\gamma} + \frac{1}{\gamma} + \frac{1}{2}. \tag{3.14}$$

We also collect the following useful properties of function spaces with exponents that satisfy Assumption 3.1.1.

Proposition 3.1.3
Under Assumption 3.1.1, we have that

(i) (r,p) *and* (γ,q) *are admissible pairs (cf. Definition 2.4.1);*

(ii) If $T>0$ *and* $k\in\mathbb{N}$*, we have the inclusion*

$$C([0,T];H_x^k)\cap L^r(0,T;W_x^{k,p})\subseteq L^\gamma(0,T;W_x^{k,q});$$

(iii) If $\mathfrak{s}\in\{0,1\}$ *and* $f\in H^{\mathfrak{s}}(\mathbb{R})\cap W^{\mathfrak{s},2+\delta}(\mathbb{R})$*, then*

$$|f|_{W_x^{\mathfrak{s},r/2}}\le |f|_{W_x^{\mathfrak{s},2}}+|f|_{W_x^{\mathfrak{s},2+\delta}}; \tag{3.15}$$

(iv) If $\mathfrak{s}\in\{0,1\}$*, then*

$$\|\Phi\|^2_{\gamma(L^2(\mathbb{R},\mathbb{R}),W_x^{\mathfrak{s},r/2})}\le C\|\Phi\|^2_{\delta}.$$

Proof.

(i) Follows immediately.

(ii) By inequality (2.39) (with the roles of the admissible pairs (r,p) and (γ,q) reversed, since in this case $p\ge q$) we find

$$|z|_{L^\gamma(0,T;L_x^q)}\overset{(2.39)}{\le}|z|^\theta_{L^r(0,T;L_x^p)}|z|^{1-\theta}_{L^\infty(0,T;L_x^2)}, \tag{3.16}$$

for all $z\in C([0,T];L_x^2)\cap L^r(0,T;L_x^p)$, with $\theta\in[0,1]$. We can then write for $z\in C([0,T];H_x^k)\cap L^r(0,T;W_x^{k,p})$

$$\begin{aligned}|z|^\gamma_{L^\gamma(0,T;W_x^{k,q})}&=\int_0^T|z(t)|^\gamma_{W_x^{k,q}}\,\mathrm{d}t\le C\int_0^T\sum_{\alpha=0}^k|\partial_x^\alpha z(t)|^\gamma_{L_x^q}\,\mathrm{d}t=C\sum_{\alpha=0}^k|\partial_x^\alpha z|^\gamma_{L^\gamma(0,T;L_x^q)}\\&\overset{(3.16)}{\le}C\sum_{\alpha=0}^k|\partial_x^\alpha z|^{\theta\gamma}_{L^r(0,T;L_x^p)}|\partial_x^\alpha z|^{(1-\theta)\gamma}_{L^\infty(0,T;L_x^2)}\end{aligned}$$

Then, by Young's inequality

$$\begin{aligned}|z|_{L^\gamma(0,T;W_x^{k,q})}&\lesssim\left(\sum_{\alpha=0}^k\theta|\partial_x^\alpha z|^\gamma_{L^r(0,T;L_x^p)}+(1-\theta)|\partial_x^\alpha z|^\gamma_{L^\infty(0,T;L_x^2)}\right)^{1/\gamma}\\&\lesssim\left(\sum_{\alpha=0}^k|\partial_x^\alpha z|^r_{L^r(0,T;L_x^p)}\right)^{1/r}+\sum_{\alpha=0}^k|\partial_x^\alpha z|_{L^\infty(0,T;L_x^2)}\\&\lesssim|z|^r_{L^r(0,T;W_x^{k,p})}+|z|_{L^\infty(0,T;H_x^k)},\end{aligned} \tag{3.17}$$

as desired.

(iii) Note that

$$2 \le \frac{r}{2} \le 2+\delta,$$

by Assumptions 3.5 and 3.6. Therefore,

$$|f|_{W_x^{\mathfrak{s},r/2}} = \sum_{\alpha=0}^{\mathfrak{s}} |\partial_x^\alpha f|_{L_x^{r/2}} \le \sum_{\alpha=0}^{\mathfrak{s}} (|\partial_x^\alpha f|_{L_x^2} + |\partial_x^\alpha f|_{L_x^{2+\delta}}) = |f|_{W_x^{\mathfrak{s},2}} + |f|_{W_x^{\mathfrak{s},2+\delta}}. \tag{3.18}$$

(iv) With help of (3.15),

$$\begin{aligned}\|\Phi\|^2_{\gamma(L^2(\mathbb{R},\mathbb{R}),W_x^{\mathfrak{s},r/2})} &\overset{(1.9)}{=} \mathbb{E}\left|\sum_{k\in\mathbb{N}}\gamma_k\Phi e_k\right|^2_{W_x^{\mathfrak{s},r/2}} \overset{(3.15)}{\le} \mathbb{E}\left(\left|\sum_{k\in\mathbb{N}}\gamma_k\Phi e_k\right|_{W_x^{\mathfrak{s},2}} + \left|\sum_{k\in\mathbb{N}}\gamma_k\Phi e_k\right|_{W_x^{\mathfrak{s},2+\delta}}\right)^2 \\ &\le 2(\|\Phi\|^2_{\mathscr{L}_2} + \|\Phi\|^2_{\gamma(L_x^2,W_x^{\mathfrak{s},2+\delta})}) \le 2C\|\Phi\|^2_\delta. \end{aligned} \tag{3.19}$$

□

3.2. Stochastic convolution

We introduce the following notation for the stochastic convolution appearing in the mild formulation (3.2)

$$Jz(t) = \int_0^t S(t-s)(z(s)\,\mathrm{d}W(s)) \quad \text{for } t \in [0,T_0], \tag{3.20}$$

and set

$$Iz(t_0,t) = \int_0^{t_0} S(t-s)(z(s)\,\mathrm{d}W(s)) \quad \text{for } t_0, t \in [0,T_0], \tag{3.21}$$

so that $Jz(t) = Iz(t,t)$. We show that stochastic convolution can be estimated in the space $L^r(\Omega; X_T)$, starting with an estimate on the $L^r(\Omega; L^r_t(W^{\mathfrak{s},p}_x))$-part of X_T.

Lemma 3.2.1
Let p and r satisfy the conditions in Assumption 3.1.1 and let $T>0$. If $\mathfrak{s} \in \{0,1\}$ and $z \in L^r(\Omega; L^\infty(0,T;H^{\mathfrak{s}}_x))$ is an $(\mathscr{F}(t))_{t\in[0,T]}$-adapted process, then for any $\alpha>0$ and any stopping time τ with $\tau \le T$ almost surely the stochastic convolution satisfies the estimate

$$\mathbb{E}\Big(\sup_{0\le t_0\le\tau} |Iz(t_0,\cdot)|^r_{L^r(0,\tau;W^{\mathfrak{s},p}_x)}\Big) \le Ce^{r\epsilon(\gamma+\alpha)T}\|\Phi\|^r_\delta \mathbb{E}\Big(|z|^r_{L^\infty(0,\tau;H^{\mathfrak{s}}_x)}\Big). \tag{3.22}$$

Proof. Starting from the left-hand side of (3.22), we bring the supremum and expectation into the integral to find

$$\mathbb{E}\Big(\sup_{0\le t_0\le\tau} |Iz(t_0,\cdot)|^r_{L^r(0,\tau;W^{\mathfrak{s},p}_x)}\Big) = \mathbb{E}\sup_{0\le t_0\le\tau}\Big(\int_0^\tau |Iz(t_0,t)|^r_{W^{\mathfrak{s},p}_x}\,\mathrm{d}t\Big) \le \int_0^\tau \mathbb{E}\Big(\sup_{0\le t_0\le\tau}|Iz(t_0,t)|^r_{W^{\mathfrak{s},p}_x}\Big)\mathrm{d}t. \tag{3.23}$$

Since $p\ge 2$, $W^{\mathfrak{s},p}(\mathbb{R})$ is a UMD and type 2 Banach space by [21, Prop. 4.2.15] and Proposition 1.3.3. Therefore, we can apply the Burkholder inequality (Theorem 1.3.15) to the integrand of the previous expression, which gives

$$\mathbb{E}\Big(\sup_{0\le t_0\le\tau}|Iz(t_0,t)|^r_{W^{\mathfrak{s},p}_x}\Big) \le C\mathbb{E}\Big(\int_0^\tau \|S(t-s)(z(s)\Phi)\|^2_{\gamma(L^2_x;W^{\mathfrak{s},p}_x)}\,\mathrm{d}s\Big)^{r/2}.$$

By applying Lemma 1.3.6, with $K=\Phi$ and $L: u\mapsto S(t-s)(z(s)u)$, we obtain

$$\|S(t-s)(z(s)\Phi)\|_{\gamma(L^2_x;W^{\mathfrak{s},p}_x)} \le \|L\|_{\mathscr{L}(W^{\mathfrak{s},r/2}_x,W^{\mathfrak{s},p}_x)}\|\Phi\|_{\gamma(L^2_x;W^{\mathfrak{s},r/2}_x)} \le C\|L\|_{\mathscr{L}(W^{\mathfrak{s},r/2}_x,W^{\mathfrak{s},p}_x)}\|\Phi\|_\delta,$$

where the last inequality follows from Proposition 3.1.3 (iv). We can estimate the operator norm using Theorem 2.3.4, in which we have to include a factor $e^{\epsilon(\gamma+\alpha)T}$ with $\alpha>0$ to take into account that $t-s\in[-T,T]$ takes negative values. It follows that

$$|Lu|_{W^{\mathfrak{s},p}_x} = |S(t-s)z(s)u|_{W^{\mathfrak{s},p}_x} \overset{(2.17)}{\le} Ce^{\epsilon(\gamma+\alpha)T}|t-s|^{-(\frac12-\frac1p)}|z(s)u|_{W^{\mathfrak{s},p'}_x} \overset{(3.6)}{=} Ce^{\epsilon(\gamma+\alpha)T}|t-s|^{-\frac2r}|z(s)u|_{W^{\mathfrak{s},p'}_x},$$

and with help Hölder's inequality,

$$|Lu|_{W^{\mathfrak{s},p}_x} \overset{(3.9)}{\le} Ce^{\epsilon(\gamma+\alpha)T}|t-s|^{-\frac2r}|z(s)|_{H^{\mathfrak{s}}_x}|u|_{W^{\mathfrak{s},r/2}_x}.$$

Then,

$$\|S(t-s)(z(s)\Phi)\|_{\gamma(L^2_x;W^{\mathfrak{s},p}_x)} \le Ce^{\epsilon(\gamma+\alpha)T}|t-s|^{-\frac2r}|z(s)|_{H^{\mathfrak{s}}_x}\|\Phi\|_\delta,$$

and it follows that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t_0\le\tau}|Iz(t_0,t)|^r_{W^{\mathfrak{s},p}_x}\Big) &\le Ce^{r\epsilon(\gamma+\alpha)T}\|\Phi\|^r_\delta\mathbb{E}\Big(\int_0^\tau |t-s|^{-\frac4r}|z(s)|^2_{H^{\mathfrak{s}}_x}\,\mathrm{d}s\Big)^{r/2}\\ &\le Ce^{r\epsilon(\gamma+\alpha)T}\|\Phi\|^r_\delta(T^{1-\frac4r})^{\frac r2}\mathbb{E}\Big(|z|^r_{L^\infty(0,\tau;H^{\mathfrak{s}}_x)}\Big),\end{aligned} \tag{3.24}$$

where in the last step we have used that $|z(s)|_{H_x^{\mathfrak{s}}} \leq |z|_{L^\infty(0,\tau;H_x^{\mathfrak{s}})}$ and

$$\int_0^\tau |t-s|^{-\frac{4}{r}}\,\mathrm{d}s = \int_0^t (t-s)^{-\frac{4}{r}}\,\mathrm{d}s + \int_t^\tau (s-t)^{-\frac{4}{r}}\,\mathrm{d}s = \frac{r}{r-4}\big(t^{1-\frac{4}{r}} + (\tau-t)^{1-\frac{4}{r}}\big) \leq \frac{2r}{r-4}\tau^{1-\frac{4}{r}}.$$

The bound in (3.24) does not depend on t, and we conclude that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t_0\leq\tau} |Iz(t_0,\cdot)|^r_{L^r(0,\tau;W_x^{\mathfrak{s},p})}\right) &\overset{(3.23)}{\leq} \int_0^\tau \mathbb{E}\left(\sup_{0\leq t_0\leq\tau} |Iz(t_0,\cdot)|^r_{W_x^{\mathfrak{s},p}}\right)\mathrm{d}t \\ &\overset{(3.24)}{\leq} Ce^{r\epsilon(\gamma+\alpha)T}\|\Phi\|_\delta^r (T^{1-\frac{4}{r}})^{r/2}\mathbb{E}\left(|z|^r_{L^\infty(0,\tau;H_x^{\mathfrak{s}})}\right) \\ &\leq Ce^{r\epsilon(\gamma+\alpha)T}\|\Phi\|_\delta^r T^{\frac{r}{2}-2}\mathbb{E}\left(|z|^r_{L^\infty(0,\tau;H_x^{\mathfrak{s}})}\right).\end{aligned}$$

The result follows by absorbing the factor $T^{\frac{r}{2}-2}$ into the exponential and the constant. □

By considering $t_0 = t$, we obtain the following Corollary.

Corollary 3.2.2
Let p and r satisfy the conditions in Assumption 3.1.1 and let $T > 0$. If $\mathfrak{s} \in \{0,1\}$ and $z \in L^r(\Omega; L^\infty(0,T;H_x^{\mathfrak{s}}))$ is an $(\mathscr{F}(t))_{t\in[0,T]}$-adapted process, then for any $\alpha > 0$ and any stopping time τ with $\tau \leq T$ almost surely the stochastic convolution satisfies the estimate

$$\mathbb{E}\left(|Jz(\cdot)|^r_{L^r(0,\tau;W_x^{\mathfrak{s},p})}\right) \leq Ce^{r\epsilon(\gamma+\alpha)T}\|\Phi\|_\delta^r\mathbb{E}\left(|z|^r_{L^\infty(0,\tau;H_x^{\mathfrak{s}})}\right). \tag{3.25}$$

Proof. We use that $|Jz(t)|_{W_x^{\mathfrak{s},p}} = |Iz(t,t)|_{W_x^{\mathfrak{s},p}} \leq \sup_{0\leq t_0\leq\tau}|Iz(t_0,t)|_{W_x^{\mathfrak{s},p}}$ to find

$$\mathbb{E}\left(|Jz|^r_{L^r(0,\tau;W_x^{\mathfrak{s},p})}\right) = \mathbb{E}\left(\int_0^\tau |Jz(t)|^r_{W_x^{\mathfrak{s},p}}\,\mathrm{d}t\right) \leq \mathbb{E}\left(\int_0^\tau \sup_{0\leq t_0\leq\tau}|Iz(t_0,t)|^r_{W_x^{\mathfrak{s},p}}\,\mathrm{d}t\right).$$

We then bring the supremum outside of the integral, yielding

$$\mathbb{E}\left(|Jz|^r_{L^r(0,\tau;W_x^{\mathfrak{s},p})}\right) \leq \mathbb{E}\left(\sup_{0\leq t_0\leq\tau}|Iz(t_0,\cdot)|^r_{L^r(0,\tau;W_x^{\mathfrak{s},p}}\right),$$

and the conclusion follows from Lemma 3.2.1. □

In addition to Corollary 3.2.2, we will also need an estimate of the $L^r(\Omega; L_t^\infty(H_x^{\mathfrak{s}}))$-norm of Jz. The estimate we prove slightly differs from the corresponding estimates in [3, Lemma 3.2] and [4, Lemma 4.2], as the semi-group is not contractive if $\mu > \gamma$. This leads to an additional factor, which is exponential in T.

Lemma 3.2.3
Let p and r satisfy the conditions in Assumption 3.1.1 and let $T > 0$. If $\mathfrak{s} \in \{0,1\}$ and $z \in L^r(\Omega; L_t^r(L_x^p))$ is $(\mathscr{F}(t))_{t\in[0,T]}$-adapted, then $Jz \in L^r(\Omega; C_t(H_x^{\mathfrak{s}}))$ and the stochastic convolution satisfies for any stopping time τ with $\tau \leq T$ almost surely the estimate

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}|Jz(t)|^r_{H_x^{\mathfrak{s}}}\right) \leq Ce^{r\epsilon(\mu-\gamma)T}\|\Phi\|_\delta^r T^{\frac{r}{2}-1}\mathbb{E}\left(|z|^r_{L^r(0,\tau;W_x^{\mathfrak{s},p})}\right). \tag{3.26}$$

Proof. We first rewrite the stochastic integral in the case that $\mathfrak{s} = 1$. Denote by Ψ an isometry from L_x^2 to $H_x^{\mathfrak{s}}$, and its inverse by Ψ^{-1}. In case $\mathfrak{s} = 0$ we can simply take Ψ equal to the identity, and in case $\mathfrak{s} = 1$ we can take for instance the Fourier multiplier with symbol $\xi \mapsto (1+\xi^2)^{-1}$. Then $\tilde{W} := \Psi W$ is a Brownian motion on $H_x^{\mathfrak{s}}$, and $\tilde{\Phi} := \Phi\Psi^{-1}$ is γ-radonifying from $H_x^{\mathfrak{s}}$ to $W_x^{\mathfrak{s},2+\delta}$. This follows from the following computation of its norm

$$\|\tilde{\Phi}\|^2_{\gamma(H_x^{\mathfrak{s}};W_x^{\mathfrak{s},2+\delta})} = \|\Phi\circ\Psi^{-1}\|^2_{\gamma(H_x^{\mathfrak{s}};W_x^{\mathfrak{s},2+\delta})} = \mathbb{E}\left|\sum_{k\in\mathbb{N}}\gamma_k\Phi(\Psi^{-1}e_k)\right|^2_{W_x^{\mathfrak{s},2+\delta}} = \|\Phi\|^2_{\gamma(L_x^2;W_x^{\mathfrak{s},2+\delta})},$$

where $(e_k)_{k\geq 1}$ is an orthonormal basis of $H^{\mathfrak{s}}_x$, $(\gamma_k)_{k\geq 1}$ is a sequence of standard normal random variables and we have used that $(\Psi^{-1}e_k)_{k\geq 1}$ is an orthonormal basis of L^2_x. We now apply Lemma 1.3.16 to the contractive semigroup $(e^{-\epsilon(\mu-\gamma)t}S(t))_{t\geq 0}$ on $H^{\mathfrak{s}}_x$ and find

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left|\int_0^t e^{-\epsilon(\mu-\gamma)(t-s)}S(t-s)z(s)\Phi\,\mathrm{d}W(s)\right|^r_{H^{\mathfrak{s}}_x}\right) &= \mathbb{E}\left(\sup_{0\leq t\leq\tau}\left|\int_0^t e^{-\epsilon(\mu-\gamma)(t-s)}S(t-s)z(s)\tilde{\Phi}\,\mathrm{d}\tilde{W}(s)\right|^r_{H^{\mathfrak{s}}_x}\right)\\ &\leq C\mathbb{E}\left(\int_0^\tau \|z(s)\tilde{\Phi}\|^2_{\mathscr{L}_2(H^{\mathfrak{s}}_x)}\,\mathrm{d}s\right)^{r/2}.\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left|\int_0^t e^{-\epsilon(\mu-\gamma)(t-s)}S(t-s)z(s)\Phi\,\mathrm{d}W(s)\right|^r_{H^{\mathfrak{s}}_x}\right) &= \mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{-r\epsilon(\mu-\gamma)t}\left|\int_0^t e^{\epsilon(\mu-\gamma)s}S(t-s)z(s)\Phi\,\mathrm{d}W(s)\right|^r_{H^{\mathfrak{s}}_x}\right)\\ &\geq e^{-r\epsilon(\mu-\gamma)\tau}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left|\int_0^t S(t-s)z(s)\Phi\,\mathrm{d}W(s)\right|^r_{H^{\mathfrak{s}}_x}\right),\end{aligned}$$

so that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\tau}|Jz(t)|^r_{H^{\mathfrak{s}}_x}\right) &\overset{(3.20)}{=} \mathbb{E}\left(\sup_{0\leq t\leq\tau}\left|\int_0^t S(t-s)z(s)\Phi\,\mathrm{d}W(s)\right|^r_{H^{\mathfrak{s}}_x}\right)\\ &\leq Ce^{r\epsilon(\mu-\gamma)\tau}\mathbb{E}\left(\int_0^\tau \|z(s)\tilde{\Phi}\|^2_{\mathscr{L}_2(H^{\mathfrak{s}}_x)}\,\mathrm{d}s\right)^{r/2}.\end{aligned}$$

We then apply Lemma 1.3.6 to split up the Hilbert-Schmidt norm as

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\tau}|Jz(t)|^r_{H^{\mathfrak{s}}_x}\right) &\overset{(1.10)}{\leq} Ce^{r\epsilon(\mu-\gamma)\tau}\|\tilde{\Phi}\|^r_{\gamma(H^{\mathfrak{s}}_x;W^{\mathfrak{s},r/2}_x)}\mathbb{E}\left(\int_0^\tau \|z(s)\|^2_{\mathscr{L}(W^{\mathfrak{s},r/2}_x,H^{\mathfrak{s}}_x)}\,\mathrm{d}s\right)^{r/2}\\ &\overset{(3.11)}{\leq} Ce^{r\epsilon(\mu-\gamma)\tau}\|\tilde{\Phi}\|^r_{\gamma(H^{\mathfrak{s}}_x;W^{\mathfrak{s},r/2}_x)}\mathbb{E}\left(\int_0^\tau |z(s)|^2_{W^{\mathfrak{s},p}_x}\,\mathrm{d}s\right)^{r/2}.\end{aligned}$$

Here we have used that $\|z(s)\|_{\mathscr{L}(W^{\mathfrak{s},r/2}_x,H^{\mathfrak{s}}_x)} \leq |z(s)|_{W^{\mathfrak{s},p}_x}$ via Hölder's inequality with exponents as in (3.11), where we interpret $z(s)$ as a multiplier. Via another application of Hölder's inequality with exponents $\frac{1}{2} = \frac{1}{2} - \frac{1}{r} + \frac{1}{r}$ and Proposition 3.1.3 (iv) we have

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\tau}|Jz(t)|^r_{H^{\mathfrak{s}}_x}\right) &\leq Ce^{r\epsilon(\mu-\gamma)\tau}\|\tilde{\Phi}\|^r_{\gamma(H^{\mathfrak{s}}_x;W^{\mathfrak{s},r/2}_x)}T^{r(\frac{1}{2}-\frac{1}{r})}\mathbb{E}\left(|z(s)|^r_{L^r(0,\tau;W^{\mathfrak{s},p}_x)}\right)\\ &= Ce^{r\epsilon(\mu-\gamma)\tau}\|\Phi\|^r_{\gamma(L^2_x;W^{\mathfrak{s},r/2}_x)}T^{\frac{r}{2}-1}\mathbb{E}\left(|z(s)|^r_{L^r(0,\tau;W^{\mathfrak{s},p}_x)}\right)\\ &\overset{(3.19)}{\leq} 2Ce^{r\epsilon(\mu-\gamma)T}\|\Phi\|^r_\delta T^{\frac{r}{2}-1}\mathbb{E}\left(|z(s)|^r_{L^r(0,\tau;W^{\mathfrak{s},p}_x)}\right).\end{aligned}$$

This concludes the proof. □

We may combine the results of Corollary 3.2.2 and Lemma 3.2.3 to obtain the following estimate in the space $L^r(\Omega;X_T)$, where we recall that X_T is defined as in (3.3).

Corollary 3.2.4
Let p and r satisfy the conditions in Assumption 3.1.1 and let $0 < T \leq T_0$ for some $T_0 > 0$. If $\mathfrak{s} \in \{0,1\}$ and $z \in L^r(\Omega;X_T)$ is an $(\mathscr{F}(t))_{t\in[0,T]}$-adapted process, then for any $\alpha > 0$ and any stopping time τ with $\tau \leq T$ almost surely the stochastic convolution satisfies the estimate

$$\mathbb{E}\left(|Jz(\cdot)|^r_{X_\tau}\right) \leq Ce^{r\epsilon(\mu+\gamma+\alpha)T}\|\Phi\|^r_\delta\mathbb{E}\left(|z|^r_{X_\tau}\right). \tag{3.27}$$

Proof. We write

$$\mathbb{E}\left(|Jz(\cdot)|_{X_\tau}^r\right) = \mathbb{E}\left(\Big(\sup_{0\le t\le\tau} |Jz(t)|_{H_x^{\mathfrak{s}}} + |Jz(\cdot)|_{L^r(0,\tau;W_x^{\mathfrak{s},p})}\Big)^r\right)$$
$$\le C\mathbb{E}\left(\Big(\sup_{0\le t\le\tau} |Jz(t)|_{H_x^{\mathfrak{s}}}^r\right) + \mathbb{E}\left(|Jz(\cdot)|_{L^r(0,\tau;W_x^{\mathfrak{s},p})}^r\right).$$

Applying Corollary 3.2.2 and Lemma 3.2.3 gives

$$\mathbb{E}\left(|Jz(\cdot)|_{X_\tau}^r\right) \le Ce^{r\epsilon(\mu-\gamma)T}\|\Phi\|_\delta^r T^{\frac{r}{2}-1}\mathbb{E}\left(|z|_{L^r(0,\tau;W_x^{\mathfrak{s},p})}^r\right) + Ce^{r\epsilon(\gamma+\alpha)T}\|\Phi\|_\delta^r\mathbb{E}\left(|z|_{L^\infty(0,\tau;H_x^{\mathfrak{s}})}^r\right)$$
$$\le \tilde{C}e^{r\epsilon(\mu+\gamma+\alpha)T}\|\Phi\|_\delta^r\mathbb{E}\left(|z|_{X_\tau}^r\right),$$

where we have absorbed the factor $T^{\frac{r}{2}-1}$ into the exponential and the constant in the last step. □

3.3. A truncated equation

In this section, we prove existence and uniqueness for solutions of a stochastic PFNLS equation in the space X_{T_0}, where we recall that

$$X_T := C([0,T]; H_x^{\mathfrak{s}}) \cap L^r(0,T; W_x^{\mathfrak{s},p}), \quad T > 0.$$

As the nonlinear term $|z|^2 z$ is not Lipschitz continuous, we consider a truncated version of (3.2) in which the nonlinearity is truncated in the space X_T. To this end, let $\theta_R \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function with inner radius R and outer radius $2R$. That is, $\operatorname{supp}\theta \subseteq (-2R, 2R)$, $\theta(x) = 1$ for $x \in [-R, R]$ and $0 \le \theta(x) \le 1$ for $x \in \mathbb{R}$. The truncated mild equation then takes the form

$$\begin{aligned} z(t) =& S(t)z_0 + 4i \int_0^t S(t-s)(\theta_R(|z|_{X_s})|z(s)|^2 z(s))\, \mathrm{d}s \\ &- i\int_0^t S(t-s)(z(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_0^t S(t-s)(z(s)F_\Phi)\,\mathrm{d}s, \end{aligned} \tag{3.28}$$

for each $t \in [0, T_0]$ and $\mathbb{P}$-almost surely. The following proposition asserts that Equation (3.28) has a unique solution for initial data $z_0 \in L^r(\Omega; H_x^{\mathfrak{s}})$.

Proposition 3.3.1 (Global existence and uniqueness for the truncated equation)
Let p and r satisfy the conditions in Assumption 3.1.1, and let $0 < T < T_0$. Then, if $\mathfrak{s} \in \{0,1\}$ and $z_0 \in L^r(\Omega; H_x^{\mathfrak{s}})$ is $\mathscr{F}(0)$-measurable, there exists up to T_0 a unique solution $z \in L^r(\Omega; X_{T_0})$ to Equation (3.28).

We first prove the following lemma, which is useful for estimating the nonlinear term in (3.28).

Lemma 3.3.2 (Estimate of the nonlinear term)
Let r, p, γ, and q satisfy the conditions in Assumption 3.1.1 and let $T > 0$. If $\mathfrak{s} \in \{0,1\}$ and $f, g \in L_t^\gamma(W_x^{\mathfrak{s},q})$, then

$$\||f|^2 g|_{L_t^{\gamma'}(W_x^{\mathfrak{s},q'})} \le T^{1/2} |f|^2_{L_t^\gamma(W_x^{\mathfrak{s},q})} |g|_{L_t^\gamma(W_x^{\mathfrak{s},q})}. \tag{3.29}$$

Proof. We start by calculating the Sobolev norm of $|f|^2 g$ as

$$\begin{aligned} \||f|^2 g|_{W_x^{\mathfrak{s},q'}} &= \||f|^2 g|_{L_x^{q'}} + \mathfrak{s}\||f|^2 \partial_x g|_{L_x^{q'}} + \mathfrak{s}|\partial_x(|f|^2) g|_{L_x^{q'}} \\ &\le \||f|^2 g|_{L_x^{q'}} + \mathfrak{s}\||f|^2 \partial_x g|_{L_x^{q'}} + \mathfrak{s}|\partial_x f|f|g|_{L_x^{q'}}, \end{aligned}$$

and we apply Hölder's inequality with exponents as in (3.13)

$$\begin{aligned} \||f|^2 g|_{W_x^{\mathfrak{s},q'}} &\le |f|^2_{L_x^q}|g|_{L_x^q} + \mathfrak{s}|f|^2_{L_x^q}|\partial_x g|_{L_x^q} + \mathfrak{s}|f|_{L_x^q}|g|_{L_x^q}|\partial_x f|_{L_x^q} \\ &\le |f|^2_{W_x^{\mathfrak{s},q}}|g|_{W_x^{\mathfrak{s},q}}. \end{aligned} \tag{3.30}$$

We then find

$$\||f|^2 g|_{L_t^{\gamma'}(W_x^{\mathfrak{s},q'})} = \left(\int_0^T \||f(t)|^2 g(t)|^{\gamma'}_{W_x^{\mathfrak{s},q'}}\,\mathrm{d}t\right)^{1/\gamma'} \overset{(3.30)}{\le} \left(\int_0^T |f(t)|^{2\gamma'}_{W_x^{\mathfrak{s},q}} |g(t)|^{\gamma'}_{W_x^{\mathfrak{s},q}}\,\mathrm{d}t\right)^{1/\gamma'},$$

and upon applying Hölder's inequality with exponents as in (3.14), we obtain

$$\||f|^2 g|_{L_t^{\gamma'}(W_x^{\mathfrak{s},q'})} \le T^{1/2}|f|^2_{L_t^\gamma(W_x^{\mathfrak{s},q})}|g|_{L_t^\gamma(W_x^{\mathfrak{s},q})},$$

which shows the result. □

Proof of Proposition 3.3.1. In order to show existence and uniqueness, we will perform a fixed-point argument in the Banach space $L^r(\Omega; X_T)$ with $T < T_0$ chosen suitably small. Therefore we define the operator $\mathcal{T}$ as

$$\begin{aligned}\mathcal{T} z(t) =& S(t)z_0 + 4i\int_0^t S(t-s)(\theta_R(|z|_{X_s})|z(s)|^2 z(s))\,\mathrm{d}s \\ &- i\int_0^t S(t-s)(z(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_0^t S(t-s)(z(s)F_\Phi)\,\mathrm{d}s \quad \text{for } z \in L^r(\Omega; X_T).\end{aligned} \tag{3.31}$$

Step 1.
We show that $\mathcal{T}$ defines a contraction in the Banach space $L^r(\Omega; X_T)$, if T is sufficiently small. To this end, let $z_1, z_2 \in L^r(\Omega; X_T)$ be $(\mathcal{F}(t))_{t\in[0,T]}$-adapted processes. Recall that $\mathcal{T}$ is a contraction on $L^r(\Omega; X_T)$, if

$$\mathbb{E}\left(|\mathcal{T} z_1 - \mathcal{T} z_2|^r_{X_T}\right) \le \kappa\mathbb{E}\left(|z_1 - z_2|^r_{X_T}\right) \tag{3.32}$$

holds for all $z_1, z_2 \in L^r(\Omega; X_T)$ and some $\kappa \in [0,1)$. We first treat the deterministic part of $\mathcal{T} z_1 - \mathcal{T} z_2$, for which the desired estimate (3.32) reduces to

$$|\mathcal{T} z_1 - \mathcal{T} z_2|_{X_T} \le \kappa|z_1 - z_2|_{X_T}. \tag{3.33}$$

We apply the Strichartz estimate (Theorem 2.4.2) to the deterministic integrals, and we find for the difference of $\mathcal{T} z_1$ and $\mathcal{T} z_2$

$$\begin{aligned}|\mathcal{T} z_1 - \mathcal{T} z_2|_{X_T} =& |\mathcal{T} z_1 - \mathcal{T} z_2|_{L^\infty(0,T;H^s_x)} + |\mathcal{T} z_1 - \mathcal{T} z_2|_{L^r(0,T;W^{s,p}_x)} \\ \overset{(2.44)}{\le}& C|\theta_R(|z_1|_{X_\cdot})|z_1|^2 z_1 - \theta_R(|z_2|_{X_\cdot})|z_2|^2 z_2|_{L^{\gamma'}_t(W^{s,q'}_x)} \\ &+ \left|\int_0^t S(t-s)((z_1(s) - z_2(s))\,\mathrm{d}W(s))\right|_{X_T} \\ &+ C|(z_1 - z_2)F_\Phi|_{L^{r'}_t(W^{s,p'}_x)} \\ \le:& I + II + III,\end{aligned} \tag{3.34}$$

which holds $\mathbb{P}$-almost surely. Here we have used that $(\infty, 2), (r, p)$ and (γ, q) are admissible pairs (Proposition 3.1.3 (i)), and we recall from Assumption 3.1.1 that $\gamma = 8$ and $q = 4$. To further estimate the term I, we introduce for $i = 1,2$ the times

$$t_i^R := \sup\{t \le T, |z_i|_{X_t} \le 2R\}, \tag{3.35}$$

i.e. t_i^R is the first time that we have $\theta_R(|z_i|_{X_t}) = 0$. We may assume without loss of generality that $t_1^R \le t_2^R$, so that we can write $[0,T] = [0,t_1^R] \cup [t_1^R, t_2^R] \cup [t_2^R, T]$. We then split up the integrals as

$$\begin{aligned}I \le& C|\theta_R(|z_1|_{X_\cdot})|z_1|^2 z_1 - \theta_R(|z_2|_{X_\cdot})|z_2|^2 z_2|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \\ &+ C|\theta_R(|z_2|_{X_\cdot})|z_2|^2 z_2|_{L^{\gamma'}(t_1^R,t_2^R;W^{s,q'}_x)} \\ =& C|(\theta_R(|z_1|_{X_\cdot}) - \theta_R(|z_2|_{X_\cdot}))|z_1|^2 z_1 + \theta_R(|z_2|_{X_\cdot})(|z_1|^2 z_1 - |z_2|^2 z_2)|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \\ &+ C|\theta_R(|z_2|_{X_\cdot})|z_2|^2 z_2|_{L^{\gamma'}(t_1^R,t_2^R;W^{s,q'}_x)},\end{aligned}$$

and apply the triangle inequality to find

$$\begin{aligned}I \le& |(\theta_R(|z_1|_{X_\cdot}) - \theta_R(|z_2|_{X_\cdot}))|z_1|^2 z_1|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \\ &+ |\theta_R(|z_2|_{X_\cdot})(|z_1|^2 z_1 - |z_2|^2 z_2)|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \\ &+ C|\theta_R(|z_2|_{X_\cdot})|z_2|^2 z_2|_{L^{\gamma'}(t_1^R,t_2^R;W^{s,q'}_x)} \\ =:& I_1 + I_2 + I_3.\end{aligned} \tag{3.36}$$

We now show that each of the deterministic terms I_1, I_2, I_3 and III can be controlled by

$$CT^{\nu}|z_1 - z_2|_{X_T},$$

with $\nu > 0$, so that (3.33) indeed holds if T is sufficiently small.

In order to further estimate the terms I_1 and I_3, we make use of the following auxiliary inequality

$$\begin{aligned}
|(\theta_R(|z_1|_{X_\cdot}) - \theta_R(|z_2|_{X_\cdot}))z_3|^{\gamma'}_{L^{\gamma'}(t_1,t_2;W^{s,q'}_x)} &= \int_{t_1}^{t_2} |(\theta_R(|z_1|_{X_t}) - \theta_R(|z_2|_{X_t}))z_3(t)|^{\gamma'}_{W^{s,q'}_x}\,\mathrm{d}t \\
&\le \left(\frac{|\theta'_R|_{L^\infty}}{R}\right)^{\gamma'} \int_{t_1}^{t_2} \big||z_1|_{X_t} - |z_2|_{X_t}\big|^{\gamma'} |z_3(t)|^{\gamma'}_{W^{s,q'}_x}\,\mathrm{d}t \\
&\le C_R \int_{t_1}^{t_2} |z_1 - z_2|^{\gamma'}_{X_t} |z_3(t)|^{\gamma'}_{W^{s,q'}_x}\,\mathrm{d}t \\
&\le C_R |z_1 - z_2|^{\gamma'}_{X_T} |z_3|^{\gamma'}_{L^{\gamma'}(t_1,t_2;W^{s,q'}_x)},
\end{aligned} \tag{3.37}$$

which holds for $z_1, z_2 \in X_T$ and $z_3 \in L^{\gamma'}(t_1, t_2; W^{s,q'}_x)$ with $0 \le t_1 \le t_2 \le T$.

Step 1.1 (Estimating the term I_1)
Inequality (3.37) applied to the term I_1 with $t_1 = 0$ and $t_2 = t_1^R$ gives

$$I_1 \overset{(3.37)}{\le} C|z_1 - z_2|_{X_T} \big||z_1|^2 z_1\big|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)}. \tag{3.38}$$

We now apply Lemma 3.3.2 to the nonlinear term to find

$$I_1 \le CT^{1/2}|z_1 - z_2|_{X_T} |z_1|^3_{L^{\gamma}(0,t_1^R;W^{s,q}_x)} \le CT^{1/2}|z_1 - z_2|_{X_T} |z_1|^3_{X_{t_1^R}} \tag{3.39}$$

where the last inequality follows from Proposition 3.1.3 (ii). Then, we use that by definition of t_1^R we have $|z_1|_{X_{t_1^R}} \le 2R$, and therefore

$$I_1 \overset{(3.35)}{\le} \tilde{C}R^3 T^{1/2}|z_1 - z_2|_{X_T}. \tag{3.40}$$

Step 1.2 (Estimating the term I_2)
In order to estimate the term I_2 we use that θ_R is bounded and apply the mean value theorem to the remaining difference, yielding

$$\begin{aligned}
I_2 &= |\theta_R(|z_2|_{X_\cdot})(|z_1|^2 z_1 - |z_2|^2 z_2)|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \\
&\le \big||z_1|^2 z_1 - |z_2|^2 z_2\big|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \qquad (3.41)\\
&\le \big|(|z_1|^2 + |z_2|^2)|z_1 - z_2|\big|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)}. \qquad (3.42)
\end{aligned}$$

We now note that elements of the sum in (3.42) can be bounded via Lemma 3.3.2 as

$$\big||z_i|^2|z_1 - z_2|\big|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \le T^{1/2}|z_i|^2_{L^{\gamma}(0,t_1^R;W^{s,q}_x)} |z_1 - z_2|_{L^{\gamma}_t(W^{s,q}_x)},$$

where $i = 1, 2$. Via Proposition 3.1.3 (ii), we find

$$\big|(|z_1|^2 + |z_2|^2)|z_1 - z_2|\big|_{L^{\gamma'}(0,t_1^R;W^{s,q'}_x)} \le T^{1/2}(|z_1|^2_{X_{t_1^R}} + |z_2|^2_{X_{t_1^R}})|z_1 - z_2|_{X_T}. \tag{3.43}$$

We now use again that $|z_i|_{X_{t_1^R}} \le 2R$, and we conclude that

$$I_2 \overset{(3.35)}{\le} 8T^{1/2}R^2|z_1 - z_2|_{X_T}. \tag{3.44}$$

Step 1.3 (Estimating the term I_3)
We may also apply inequality (3.37) to the term I_3, since $\theta_R(|z_1|_{X_t}) = 0$ if $t_1^R \leq t \leq t_2^R$. This gives

$$\begin{aligned} I_3 &\overset{(3.36)}{=} C|\theta_R(|z_2|_{X_\cdot})|z_2|^2 z_2|_{L^{\gamma'}(t_1^R, t_2^R; W_x^{\mathfrak{s},q'})} \\ &\overset{(3.37)}{\leq} C|z_1 - z_2|_{X_T} ||z_2|^2 z_2|_{L^{\gamma'}(t_1^R, t_2^R; W_x^{\mathfrak{s},q'})}, \end{aligned}$$

which is of the same form as estimate (3.38) of term I_1. In the same way we obtain

$$I_3 \leq CT^{1/2} R^3 |z_1 - z_2|_{X_T}. \tag{3.45}$$

Collecting inequalities (3.40), (3.44) and (3.45), we have shown that

$$I \leq C_R T^{1/2} |z_1 - z_2|_{X_T}. \tag{3.46}$$

Step 1.4 (Estimating the term III)
By writing out the Sobolev norm as

$$|(z_1 - z_2)F_\Phi|_{W_x^{\mathfrak{s},p'}} = |(z_1 - z_2)F_\Phi|_{L_x^{p'}} + \mathfrak{s}|\partial_x(z_1 - z_2)F_\Phi|_{L_x^{p'}} + \mathfrak{s}|(z_1 - z_2)\partial_x F_\Phi|_{L_x^{p'}},$$

and upon applying Hölder's inequality with $\frac{1}{p'} = \frac{1}{p} + \frac{4}{r}$, we find

$$\begin{aligned} |(z_1 - z_2)F_\Phi|_{W_x^{\mathfrak{s},p'}} &\overset{(3.10)}{\leq} |z_1 - z_2|_{L_x^p} |F_\Phi|_{L_x^{r/4}} + \mathfrak{s}|\partial_x(z_1 - z_2)|_{L_x^p} |F_\Phi|_{L_x^{r/4}} + \mathfrak{s}|z_1 - z_2|_{L_x^p} |\partial_x F_\Phi|_{L_x^{r/4}} \\ &\leq |z_1 - z_2|_{W_x^{\mathfrak{s},p}} |F_\Phi|_{W_x^{\mathfrak{s},r/4}}. \end{aligned}$$

Therefore,

$$III = C|(z_1 - z_2)F_\Phi|_{L^{r'}(0,T; W_x^{\mathfrak{s},p'})} \leq CT^{1-\frac{2}{r}} |F_\Phi|_{W_x^{\mathfrak{s},r/4}} |z_1 - z_2|_{L^r(0,T; W_x^{\mathfrak{s},p})}. \tag{3.47}$$

We now show that the $W_x^{\mathfrak{s},r/4}$-norm of the Itô drift term F_Φ is controlled by $\|\Phi\|_\delta$. Let therefore $(\gamma_k)_{k\geq 0}$ be a sequence of independent normal real valued random variables and recall that $F_\Phi(x) = \sum_{k=0}^\infty (\Phi e_k)^2(x)$, where $(e_k)_{k\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R}, \mathbb{R})$. We can then write, by independence of the normal random variables

$$F_\Phi = \mathbb{E}\left(\sum_{k=0}^\infty \gamma_k \Phi e_k\right)^2,$$

and

$$\partial_x F_\Phi = \partial_x \mathbb{E}\left(\sum_{k=0}^\infty \gamma_k \Phi e_k\right)^2 = \mathbb{E}\left[2\left(\sum_{k=0}^\infty \gamma_k \Phi e_k\right)\left(\sum_{k=0}^\infty \gamma_k \partial_x \Phi e_k\right)\right] \leq \mathbb{E}\left(\sum_{k=0}^\infty \gamma_k \Phi e_k\right)^2 + \mathbb{E}\left(\sum_{k=0}^\infty \gamma_k \partial_x \Phi e_k\right)^2, \tag{3.48}$$

where the last inequality is an application of Young's inequality. It follows that

$$|F_\Phi|_{W_x^{\mathfrak{s},r/4}} = \sum_{\alpha=0}^{\mathfrak{s}} \left(\int_{\mathbb{R}} (\partial_x^\alpha F_\Phi)^{\frac{r}{4}} \,\mathrm{d}x\right)^{\frac{4}{r}} \overset{(3.48)}{\leq} C \sum_{\alpha=0}^{\mathfrak{s}} \left(\int_{\mathbb{R}} \left(\mathbb{E}\left(\sum_{k=0}^\infty \gamma_k \partial_x^\alpha \Phi e_k(x)\right)^2\right)^{\frac{r}{4}} \mathrm{d}x\right)^{\frac{4}{r}}.$$

By applying Hölder's inequality in the probability space, and by exchanging expectation and integration, we obtain

$$|F_\Phi|_{W_x^{\mathfrak{s},r/4}} \leq C \sum_{\alpha=0}^{\mathfrak{s}} \left(\int_{\mathbb{R}} \mathbb{E}\left|\sum_{k=0}^\infty \gamma_k \partial_x^\alpha \Phi e_k(x)\right|^{\frac{r}{2}} \mathrm{d}x\right)^{\frac{4}{r}} = C \sum_{\alpha=0}^{\mathfrak{s}} \left(\mathbb{E}\left|\sum_{k=0}^\infty \gamma_k \partial_x^\alpha \Phi e_k\right|_{L_x^{r/2}}^{\frac{r}{2}}\right)^{\frac{4}{r}}.$$

We conclude by an application of the Kahane-Khintchine inequality (Theorem 1.3.2), yielding

$$|F_\Phi|_{W_x^{\mathfrak{s},r/4}} \leq C \sum_{\alpha=0}^{\mathfrak{s}} \mathbb{E}\left|\sum_{k=0}^\infty \gamma_k \partial_x^\alpha \Phi e_k\right|_{L_x^{r/2}}^2 \leq C\|\Phi\|^2_{\gamma\left(L_x^2; W_x^{\mathfrak{s},r/2}\right)}.$$

Then, via Proposition 3.1.3 (iv) we get combined with (3.47)

$$III \le CT^{1-\frac{2}{r}} \|\Phi\|_\delta^2 |z_1 - z_2|_{X_T}. \tag{3.49}$$

Step 1.5 (Collecting the estimates)
By substituting the results of (3.46) and (3.49) into the starting point (3.34), we find

$$\begin{aligned} |\mathscr{T} z_1 - \mathscr{T} z_2|_{X_T} \le\, & CT^{1/2}|z_1 - z_2|_{X_T} \\ & + \left| \int_0^t S(t-s)((z_1(s) - z_2(s))\,\mathrm{d}W(s)) \right|_{X_T} \\ & + CT^{1-\frac{2}{r}} \|\Phi\|_\delta |z_1 - z_2|_{X_T}. \end{aligned}$$

By taking the $L^r(\Omega)$-norm of this equation we obtain with help of Corollary 3.2.4

$$\begin{aligned} \mathbb{E}\left(|\mathscr{T} z_1 - \mathscr{T} z_2|_{X_T}^r \right) \le\, & CT^{1/2}\mathbb{E}(|z_1 - z_2|_{X_T}^r) \\ & + Ce^{r\epsilon(\mu+\gamma+\alpha)T} \|\Phi\|_\delta^r \mathbb{E}(|z_1 - z_2|_{X_T}^r) \\ & + CT^{1-\frac{2}{r}} \|\Phi\|_\delta \mathbb{E}(|z_1 - z_2|_{X_T}^r), \end{aligned}$$

where $\alpha > 0$. By setting $\mu := \min\left\{\frac{1}{2r}, \frac{1}{r} - \frac{2}{r^2}\right\} > 0$ we combine the constants for $T \le T_0$ as

$$|\mathscr{T} z_1 - \mathscr{T} z_2|_{L^r(\Omega;X_T)} \le \tilde{C} e^{\epsilon(\mu+\gamma+\alpha)T} T^\mu |z_1 - z_2|_{L^r(\Omega;X_T)}, \tag{3.50}$$

where $\tilde{C}$ depends on R, Φ and T_0. We then see that upon choosing T small enough, $\mathscr{T}$ is a contraction mapping in $L^r(\Omega; X_T)$.

Step 2.
In order to complete the fixed-point argument, we now show that $\mathscr{T}$ maps $L^r(\Omega; X_T)$ into itself. From the contraction estimate (3.50) it is evident that the operator $\mathscr{T}$ maps differences of elements in $L^r(\Omega; X_T)$ into $L^r(\Omega; X_T)$. We now consider an arbitrary element $z \in L^r(\Omega; X_T)$ and write

$$|\mathscr{T} z|_{L^r(\Omega;X_T)} \le |\mathscr{T} z - \mathscr{T} 0|_{L^r(\Omega;X_T)} + |\mathscr{T} 0|_{L^r(\Omega;X_T)}.$$

The first term is finite by the contraction estimate (3.50), and it, therefore, suffices to show that $\mathscr{T}0$ is an element of the space $L^r(\Omega; X_T)$. We find

$$|\mathscr{T} 0|_{X_T} \overset{(3.31)}{=} |S(t)z_0|_{L_t^\infty(H_x^s)} + |S(t)z_0|_{L_t^r(W_x^{s,p})},$$

where we have omitted the norm on the probability space Ω, since $S(t)z_0$ does not depend on the path. It follows from Corollary 2.3.5 that $|S(t)z_0|_{L_t^\infty(H_x^s)} \le C|z_0|_{H_x^s} < \infty$. The second term is finite via the homogeneous Strichartz estimate (2.45). We conclude that $\mathscr{T}$ maps $L^r(\Omega; X_T)$ into itself.

By steps 1 and 2, it now follows from the Banach fixed-point theorem that there exists a unique solution to (3.28) in $L^r(\Omega; X_T)$. To extend our solution to $[0, T_0]$, we note that at $t \le T$ we are also in the position to apply the same contraction argument with initial condition $z(T)$, since our choice of T only depends on R, T_0 and Φ. Therefore, we write $[0, T_0]$ as a union of intervals of size at most T, and define the global solution as a concatenation of the local solutions. □

3.4. Local existence and blow-up criterion

This section is devoted to the following theorem.

Theorem 3.4.1 (Local existence & blow-up criterion)
Let p and r satisfy the conditions in Assumption 3.1.1, and let $0 < T < T_0$. Then, if $\mathfrak{s} \in \{0,1\}$ and $z_0 \in L^r(\Omega; H_x^{\mathfrak{s}})$ is $\mathscr{F}(0)$-measurable, there exists a stopping time $\tau^*(z_0)$ and a unique solution z on $[0,\tau^*)$ to Equation (3.2) that is in $L^r(\Omega; X_\tau)$ for any $\tau < \tau^*$. Furthermore, we have $\mathbb{P}$-almost surely

$$\tau^*(z_0) = \infty \quad \text{or} \quad \lim_{t \nearrow \tau^*} \sup_{s \leq t} |z(t)|_{H_x^{\mathfrak{s}}} = \infty.$$

For the proof of this theorem we will need some preliminaries. Denote by z_R the unique solution of the truncated equation (3.28) with radius R. We define for $R > 0$ the stopping times

$$\tau_R := \sup\{t \in [0, T_0], |z_R|_{X_t} \leq R\},$$

which is the first time $|z_R|_{X_t}$ reaches R, and before this time no truncation takes place. Until this point, two solutions z_R and $z_{R'}$ with $R' > R$ should therefore coincide on $[0, \min\{\tau_R, \tau_{R'}\}]$. This is stated in the following lemma.

Lemma 3.4.2
If $R' > R > 0$, then for each $t \in [0, \min\{\tau_R, \tau_{R'}\}]$, we have $\mathbb{P}$-almost surely

$$z_R(t) = z_{R'}(t).$$

Proof of Lemma 3.4.2. Let $R' > R > 0$, write $\tau = \min\{\tau_R, \tau_{R'}\}$, and let $T > 0$. In case $\tau < T_0$, we define z_R on $[\tau, T_0]$ as the solution to the linearized equation

$$\mathrm{d}y = (i\Delta y - i\nu y - \epsilon(\gamma y - \mu y^*))\,\mathrm{d}t - iy\,\mathrm{d}W - \frac{1}{2} y F_\Phi\,\mathrm{d}t,$$

with initial condition $y_R(\tau) = z_R(\tau)$. In mild formulation, the linearized equation reads

$$y_R(t) = S(t-\tau)z_R(\tau) - i\int_\tau^t S(t-s)(y_R(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_\tau^t S(t-s)(y_R(s)F_\Phi)\,\mathrm{d}s, \tag{3.51}$$

for each $t \in [\tau, T_0]$, $\mathbb{P}$-almost surely. Equation (3.51) has a unique solution

$$y_R \in L^r(\Omega; C([\tau, T_0]; H_x^{\mathfrak{s}}) \cap (L^r(\tau, T; W_x^{\mathfrak{s},p})),$$

via Proposition 3.3.1, since the same proof can be applied to Equation (3.51) by leaving out the nonlinear term. We denote the extension as

$$\tilde{z}_R(t) = \begin{cases} z_R(t), & \text{if } t \in [0,\tau] \\ y_R(t), & \text{if } t \in [\tau, T_0], \end{cases}$$

and we define $\tilde{z}_{R'}(t)$ on $[0, T_0]$ in the same way. We first show that $\tilde{z}_R(t) = \tilde{z}_{R'}(t)$ on $[0, T]$ for small T, and equality on $[0, T_0]$ will follow by reiteration. On $[0, T]$ we have:

$$\begin{aligned} \tilde{z}_{R'}(t) - \tilde{z}_R(t) =& 4i\int_0^{t\wedge\tau} S(t-s)(|\tilde{z}_{R'}(s)|^2\tilde{z}_{R'}(s) - |\tilde{z}_R(s)|^2\tilde{z}_R(s))\,\mathrm{d}s \\ &- i\int_0^t S(t-s)((\tilde{z}_{R'}(s) - \tilde{z}_R(s))\,\mathrm{d}W(s)) \\ &- \frac{1}{2}\int_0^t S(t-s)\,((\tilde{z}_{R'}(s) - \tilde{z}_R(s))\,F_\Phi)\,\mathrm{d}s \\ &=: I_1(\omega,t,x) + I_2(\omega,t,x) + I_3(\omega,t,x), \end{aligned} \tag{3.52}$$

since we can omit the truncating function θ if $t \leq \tau$. Using the Strichartz inequality (Theorem 2.4.2), we can estimate the term I_1 $\mathbb{P}$-almost surely as

$$\begin{aligned}|I_1|_{X_T} &= 4\left|\int_0^t S(t-s)\chi_{[0,\tau]}(s)(|\tilde{z}_{R'}(s)|^2\tilde{z}_{R'}(s) - |\tilde{z}_R(s)|^2)\tilde{z}_R(s))\,\mathrm{d}s\right|_{X_T} \\ &\overset{(2.44)}{\leq} C\left|\chi_{[0,\tau]}(|\tilde{z}_{R'}|^2\tilde{z}_{R'} - |\tilde{z}_R|^2\tilde{z}_R)\right|_{L_t^{\gamma'}(W_x^{s,q'})} \\ &= C\left||\tilde{z}_{R'}|^2\tilde{z}_{R'} - |\tilde{z}_R|^2\tilde{z}_R\right|_{L^{\gamma'}(0,\tau\wedge T;W_x^{s,q'})},\end{aligned}$$

where we have used that $(\infty, 2), (r, p)$ and (γ, q) are admissible pairs (Proposition 3.1.3 (i)), and we recall from Assumption 3.1.1 that $\gamma = 8$ and $q = 4$. This last nonlinear term $\left||\tilde{z}_{R'}|^2\tilde{z}_{R'} - |\tilde{z}_R|^2\tilde{z}_R\right|_{L^{\gamma'}(0,\tau\wedge T;W_x^{s,q'})}$ has the same form as (3.41) in the proof of Proposition 3.3.1, and we treat it similarly. We estimate as in step 1.2 of the proof of Proposition 3.3.1

$$\begin{aligned}|I_1|_{X_T} &\overset{(3.43)}{\leq} CT^{1/2}(|\tilde{z}_{R'}|^2_{X_\tau} + |\tilde{z}_R|^2_{X_\tau})|\tilde{z}_{R'} - \tilde{z}_R|_{X_T} \\ &\leq CT^{1/2}(R'^2 + R^2)|\tilde{z}_{R'} - \tilde{z}_R|_{X_T}.\end{aligned}$$

By using (3.49) for term I_3, we may write

$$\begin{aligned}|\tilde{z}_{R'} - \tilde{z}_R|_{X_T} \leq C_R T^{1/2}|\tilde{z}_{R'} - \tilde{z}_R|_{X_T} \\ + \left|\int_0^t S(t-s)((\tilde{z}_{R'}(s) - \tilde{z}_R(s))\,\mathrm{d}W(s))\right|_{X_T} \\ + CT^{1-\frac{2}{r}}\|\Phi\|_\delta|\tilde{z}_{R'} - \tilde{z}_R|_{X_T}.\end{aligned}$$

We take the $L^r(\Omega)$-norm of this equation and via Corollary 3.2.4 we obtain

$$\begin{aligned}\mathbb{E}\left(|\tilde{z}_{R'} - \tilde{z}_R|^r_{X_T}\right) \leq C_R T^{1/2}\mathbb{E}(|\tilde{z}_{R'} - \tilde{z}_R|^r_{X_T}) \\ + Ce^{r\epsilon(\mu+\gamma+\alpha)T}\|\Phi\|^r_\delta\mathbb{E}(|\tilde{z}_{R'} - \tilde{z}_R|^r_{X_T}) \\ + CT^{1-\frac{2}{r}}\|\Phi\|_\delta\mathbb{E}(|\tilde{z}_{R'} - \tilde{z}_R|^r_{X_T}).\end{aligned}$$

By setting $\mu := \min\{\frac{1}{2r}, \frac{1}{r} - \frac{2}{r^2}\} > 0$ we combine the constants for $T \leq T_0$ as

$$|\tilde{z}_{R'} - \tilde{z}_R|_{L^r(\Omega;X_T)} \leq \tilde{C}e^{r\epsilon(\mu+\gamma+\alpha)T}T^\mu|\tilde{z}_{R'} - \tilde{z}_R|_{L^r(\Omega;X_T)}, \tag{3.53}$$

where $\tilde{C}$ depends on R, R', Φ, and T_0. Hence, for T sufficiently small, both sides of (3.53) are zero, and we have $\tilde{z}_{R'} = \tilde{z}_R$ on $[0, T]$ $\mathbb{P}$-almost surely. We can repeat this procedure to get equality of $\tilde{z}_{R'}$ and $\tilde{z}_R$ on $[0, T_0]$, $\mathbb{P}$-almost surely. $\mathbb{P}$-almost sure equality of $z_{R'}$ and z_R on $[0, \tau]$ then follows. □

From the previous lemma, we can deduce an important property of the stopping times τ_R.

Corollary 3.4.3
τ_R is $\mathbb{P}$-almost surely non-decreasing with R.

Proof. By Lemma 3.4.2, we have

$$|z_R|_{X_\tau} = |z_{R'}|_{X_\tau}, \tag{3.54}$$

$\mathbb{P}$-almost surely, where we recall that $\tau = \min\{\tau_R, \tau_{R'}\}$. Suppose, for sake of contradiction, that $\tau_{R'} < \tau_R$ on some set of positive probability. Then on that set $\tau = \tau_{R'}$, so that

$$|z_R|_{X_\tau} \overset{(3.54)}{=} |z_{R'}|_{X_{\tau_{R'}}} > R.$$

Therefore $\tau > \tau_R$, which contradicts the definition of τ. We conclude that $\tau_{R'} \geq \tau_R$, $\mathbb{P}$-almost surely. □

We can now give the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1. We denote again by z_R the unique solution of the truncated equation (3.28) with radius R given by Proposition 3.3.1, and denote

$$\tau_R(z_0) := \sup\{t \in [0, T_0], |z_R|_{X_t} \leq R\}.$$

By Corollary 3.4.3, we may define $\tau^*(z_0) := \lim_{R\nearrow\infty} \tau_R(z_0)$. We then set $z(t) := z_R(t)$ on $[0, \tau_R]$, for each $R > 0$. z is then well-defined on $[0, \tau^*)$ via Corollary 3.4.3, and solves (3.2) with $z \in L^r(\Omega; X_\tau)$ for each $\tau < \tau^*$. This proves the existence part of the theorem.

We now prove the blow-up criterion. Note that if $\tau^* = \lim_{R\nearrow\infty} \tau_R < \infty$, then

$$\lim_{t\nearrow\tau^*} |z|_{X_t} = \infty, \tag{3.55}$$

meaning that a finite stopping time τ^* is due to the blow-up of the X_t-norm. In the remainder of the proof, we show that the $L^r(0, t; W_x^{\mathfrak{s},p})$ part of the X_t-norm does not blow up, so that the $H_x^{\mathfrak{s}}$ part must be the cause. We set

$$\tilde{\tau}_R = \inf\{t \in [0, \tau^*), |z(t)|_{H_x^{\mathfrak{s}}} \geq R\},$$

i.e. $\tilde{\tau}_R$ is the first time $|z(t)|_{H_x^{\mathfrak{s}}}$ reaches R. With help of Corollary 3.4.5 (which we prove later) applied to this stopping time, we obtain that

$$\mathbb{E}\left(|z|_{L^r(0,\tilde{\tau}_R,W_x^{\mathfrak{s},p})}\right) < \infty,$$

so that $|z|_{L^r(0,\tilde{\tau}_R,W_x^{\mathfrak{s},p})}$ is $\mathbb{P}$-almost surely finite. Assume, for sake of contradiction, that

$$\mathbb{P}\left(\sup_{s\leq\tau^*} |z(s)|_{H_x^{\mathfrak{s}}} < \infty \quad \text{and} \quad \tau^* < \infty\right) > 0,$$

and denote this event by $A \subseteq \Omega$. Denote furthermore $M(\omega) := \sup_{s\leq\tau^*} |z(s)|_{H_x^{\mathfrak{s}}}$. We now take R large enough so that $R > M(\omega)$ on some subset $\tilde{A} \subseteq A$ of positive probability. On $\tilde{A}$ we then have $\sup_{s\leq\tau^*} |z(s)|_{H_x^{\mathfrak{s}}} < R$, which together with the $\mathbb{P}$-almost sure finiteness of $|z|_{L^r(0,\tilde{\tau}_R,W_x^{\mathfrak{s},p})}$ contradicts (3.55). We conclude that we must have

$$\tau^* = \infty \quad \text{or} \quad \lim_{t\nearrow\tau^*} \sup_{s\leq t} |z(t)|_{H_x^{\mathfrak{s}}} = \infty,$$

$\mathbb{P}$-almost surely. □

The remainder of the section is devoted to the proofs of Corollary 3.4.5 and the following Lemma.

Lemma 3.4.4
Let z be the solution to (3.2) on $[0, \tau^*)$ given by Theorem 3.4.1. If τ is a stopping time that $\mathbb{P}$-almost surely satisfies $\tau < \tau*$, then we have the bound

$$|z|_{L^r(0,\tau,W_x^{\mathfrak{s},p})} \leq C(T_0) K(\omega)^5,$$

where

$$K(\omega) := C(T_0)(1 + \sup_{0\leq t_0\leq\tau} |z(t_0)|^3_{H_x^{\mathfrak{s}}} + |Jz|_{L^r(0,\tau;W_x^{\mathfrak{s},p})} + \sup_{0\leq t_0\leq\tau} |Iz(t_0,\cdot)|_{L^r(0,\tau;W_x^{1,p})}).$$

Proof. By applying the Strichartz estimates (Theorem 2.4.2) to the deterministic integrals in (3.2), we obtain for $T > 0$

$$|z|_{L_t^r(W_x^{\mathfrak{s},p})} \overset{(2.44)}{\leq} C|z_0|_{H_x^{\mathfrak{s}}} + C||z|^2 z|_{L_t^{\gamma'}(W_x^{\mathfrak{s},q'})} + |Jz|_{L_t^r(W_x^{\mathfrak{s},p})} + C|zF_\Phi|_{L_t^{r'}(W_x^{\mathfrak{s},p'})}.$$

To estimate the term $||z|^2 z|_{L_t^{\gamma'}(W_x^{\mathfrak{s},q'})}$, we apply Lemma 3.3.2, which gives

$$||z|^2 z|_{L_t^{\gamma'}(W_x^{1,q'})} \leq T^{1/2}|z|^3_{L_t^{\gamma}(W_x^{\mathfrak{s},q})} \leq CT^{1/2}(|z|^3_{L_t^\infty(H_x^{\mathfrak{s}})} + |z|^3_{L_t^r(W_x^{\mathfrak{s},p})}),$$

where the last inequality follows from Proposition 3.1.3 (ii). By substituting this result, and estimating the Itô drift term as in (3.47) we obtain

$$|z|_{L_t^r(W_x^{s,p})} \le C|z_0|_{H_x^s} + CT^{1/2}(|z|^3_{L_t^\infty(H_x^s)} + |z|^3_{L_t^r(W_x^{s,p})}) + |Jz|_{L_t^r(W_x^{s,p})} + CT^{1-\frac{2}{r}}\|\Phi\|_\delta^2 |z|_{L_t^r(W_x^{s,p})}.$$

We now pick T small enough so that

$$CT^{1-\frac{2}{r}}\|\Phi\|_\delta^2 |z|_{L_t^r(W_x^{s,p})} \le \frac{1}{2},$$

and bring this term to the left-hand side to find

$$|z|_{L^r(0,T\wedge\tau;W_x^{s,p})} \le 2C|z_0|_{H_x^s} + 2CT^{1/2}(|z|^3_{L^\infty(0,T\wedge\tau;H_x^s)} + |z|^3_{L^r(0,T\wedge\tau;W_x^{s,p})}) + 2|Jz|_{L^r(0,T\wedge\tau;W_x^{s,p})},$$

where we have changed the time interval to $[0,t\wedge\tau]$. We rewrite this estimate by bounding $|z_0|_{H_x^s}$ and $T^{1/2}$ by some constant depending on T_0, and add a term $\sup_{0\le t_0\le\tau}|Iz(t_0,\cdot)|_{L^r(0,\tau;W_x^{s,p})}$ for later use. This gives

$$\begin{aligned}|z|_{L^r(0,T\wedge\tau;W_x^{s,p})} \le& C(T_0)(1+|z|^3_{L^\infty(0,T\wedge\tau;H_x^s)}) + 2|Jz|_{L^r(0,T\wedge\tau;W_x^{s,p})} + \sup_{0\le t_0\le\tau}|Iz(t_0,\cdot)|_{L^r(0,\tau;W_x^{s,p})} \\ &+ 2CT^{1/2}|z|^3_{L^r(0,T\wedge\tau;W_x^{s,p})} \\ \le& K(\omega) + 2CT^{1/2}|z|^3_{L^r(0,T\wedge\tau;W_x^{s,p})}.\end{aligned}$$

If we choose T as $T(\omega) := \min\{K^{-4}(\omega)/64C^2, \tau\}$, then it follows that

$$|z|_{L^r(0,T;W_x^{s,p})} \le 2K(\omega).$$

Indeed, if not, then we would have $|z|_{L^r(0,T;W_x^{s,p})} = 2K(\omega)$ for some ω and $T' \le T(\omega)$. This leads to a contradiction

$$\begin{aligned}2K(\omega) = |z|_{L^r(0,T';W_x^{s,p})} &\le K(\omega) + 2CT'^{1/2}|z|^3_{L^r(0,T\wedge\tau;W_x^{s,p})} \\ &\le K(\omega) + 2C(K^{-4}(\omega)/64C^2)^{1/2}8K^3(\omega) = \frac{3}{2}K(\omega).\end{aligned}$$

In case that $T(\omega)\le\tau$, we reiterate the previous process on intervals of the form $[jT,(j+1)T]\subseteq[0,T_0\wedge\tau]$, on which z satisfies

$$z(t) = S(t-jT)z(jT) + 4i\int_{jT}^t S(t-s)(|z(s)|^2 z(s))\,\mathrm{d}s - i\int_{jT}^t S(t-s)(z(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_{jT}^t S(t-s)(z(s)F_\Phi)\,\mathrm{d}s.$$

By estimating as before, we obtain

$$\begin{aligned}|z|_{L^r(jT,(j+1)T;W_x^{s,p})} \le& 2C|z(jT)|_{H_x^s} + 2CT^{1/2}(|z|^3_{L^\infty(jT,(j+1)T;H_x^s)} + |z|^3_{L^r(jT,(j+1)T;W_x^{s,p})}) \\ &+ 2\left|\int_{jT}^t S(t-s)(z(s)\,\mathrm{d}W(s))\right|_{L^r(jT,(j+1)T;W_x^{s,p})}.\end{aligned}$$

We estimate $|z(jT)|_{H_x^s} \le |z|_{L^\infty(jT,(j+1)T;H_x^s)}$, and for the stochastic convolution we note that

$$\begin{aligned}\left|\int_{jT}^t S(t-s)(z(s)\,\mathrm{d}W(s))\right|_{L^r(jT,(j+1)T;W_x^{s,p})} &\le \left|\int_0^t S(t-s)(z(s)\,\mathrm{d}W(s))\right|_{L^r(jT,(j+1)T;W_x^{s,p})} \\ &\quad+ \left|\int_0^{jT} S(t-s)(z(s)\,\mathrm{d}W(s))\right|_{L^r(jT,(j+1)T;W_x^{s,p})} \\ &\le |Jz|_{L^r(jT,(j+1)T;W_x^{s,p})} + \sup_{0\le t_0\le\tau}|Iz(t_0,\cdot)|_{L^r(0,\tau;W_x^{1,p})}.\end{aligned}$$

It follows that

$$|z|_{L^r(jT,(j+1)T\wedge\tau;W_x^{s,p})} \le 2K(\omega),$$

whenever $jT\le T_0\wedge\tau$. We sum these estimates to find

$$|z|_{L^r(0,\tau;W_x^{s,p})} \le \sum_{j=0}^{(T_0\wedge\tau)/T}|z|_{L^r(jT,(j+1)T\wedge\tau;W_x^{s,p})} \le 2\frac{T_0}{T}K(\omega) = 128CT_0K^4(\omega)K(\omega) \le C(T_0)K^5(\omega),$$

as desired. □

By applying the previous lemma to the stopping time

$$\tilde{\tau}_R = \inf\{t \in [0,\tau^*), |z(t)|_{H^{\mathfrak{s}}_x} \geq R\}, \tag{3.56}$$

we obtain the following corollary.

Corollary 3.4.5
Let z be the solution to (3.2) on $[0,\tau^*)$ given by Theorem 3.4.1, and let $\tilde{\tau}_R$ be the stopping time defined in (3.56). Then

$$\mathbb{E}\left(|z|_{L^r(0,\tilde{\tau}_R,W^{\mathfrak{s},p}_x)}\right) < C(T_0)(1+R^3+CR)^5.$$

Proof. By applying Lemma 3.4.4 to the stopping time $\tilde{\tau}_R$, we obtain

$$|z|_{L^r(0,\tilde{\tau}_R,W^{\mathfrak{s},p}_x)} \leq C(T_0)\left(1+\sup_{0\leq t_0\leq\tilde{\tau}_R}|z(t_0)|^{15}_{H^{\mathfrak{s}}_x}+|Jz|^5_{L^r(0,\tilde{\tau}_R;W^{\mathfrak{s},p}_x)}+\sup_{0\leq t_0\leq\tilde{\tau}_R}|Iz(t_0,\cdot)|^5_{L^r(0,\tilde{\tau}_R;W^{\mathfrak{s},p}_x)}\right).$$

We have upon taking expectations and applying Hölder's inequality

$$\mathbb{E}\left(|z|_{L^r(0,\tilde{\tau}_R,W^{\mathfrak{s},p}_x)}\right) \leq C(T_0)\left(1+\mathbb{E}\left(\sup_{0\leq t_0\leq\tilde{\tau}_R}|z(t_0)|^{15}_{H^{\mathfrak{s}}_x}\right)+\mathbb{E}\left(|Jz|^r_{L^r(0,\tilde{\tau}_R;W^{\mathfrak{s},p}_x)}\right)^{5/r}+\mathbb{E}\left(\sup_{0\leq t_0\leq\tilde{\tau}_R}|Iz(t_0,\cdot)|^r_{L^r(0,\tilde{\tau}_R;W^{\mathfrak{s},p}_x)}\right)^{5/r}\right).$$

Now, by Lemma 3.2.1 and Corollary 3.2.2,

$$\mathbb{E}\left(|z|_{L^r(0,\tilde{\tau}_R,W^{\mathfrak{s},p}_x)}\right) \leq C(T_0)\left(1+\mathbb{E}\left(\sup_{0\leq t_0\leq\tilde{\tau}_R}|z(t_0)|^{15}_{H^{\mathfrak{s}}_x}\right)+\frac{1}{2}C\mathbb{E}\left(|z|^r_{L^\infty(0,\tilde{\tau}_R;H^{\mathfrak{s}}_x)}\right)^{5/r}+\frac{1}{2}C\mathbb{E}\left(|z|^r_{L^\infty(0,\tilde{\tau}_R;H^{\mathfrak{s}}_x)}\right)^{5/r}\right).$$

Note that

$$\sup_{0\leq t_0\leq\tilde{\tau}_R}|z(t_0)|_{H^{\mathfrak{s}}_x} < R,$$

by definition of $\tilde{\tau}_R$, so that

$$\mathbb{E}\left(|z|_{L^r(0,\tilde{\tau}_R,W^{\mathfrak{s},p}_x)}\right) \leq C(T_0)(1+R^{15}+CR^3) \leq C(T_0)(1+R^3+CR)^5. \qquad \square$$

3.5. Global existence and uniqueness

This section is devoted to the main result of the chapter: global existence and uniqueness of solutions to the stochastic parametrically-forced NLS equation (3.1).

Theorem 3.5.1 (Global existence and uniqueness)
Let p and r satisfy the conditions in Assumption 3.1.1, and let $T_0 > 0$. Then, if $z_0 \in L^r(\Omega; H_x^{\mathfrak{s}})$ is $\mathscr{F}(0)$-measurable, there exists up to T_0 a unique solution $z \in L^r(\Omega; C([0,T_0]; H_x^{\mathfrak{s}}) \cap L^r(0,T_0; W_x^{\mathfrak{s},p}))$ to Equation (3.2).

By Theorem 3.4.1, it suffices to prove that the $H_x^{\mathfrak{s}}$-norm of the local solution remains finite. The following proposition describes the evolution of the $H_x^{\mathfrak{s}}$-norm of the local solution.

Proposition 3.5.2
Let r satisfy the conditions in Assumption 3.1.1 and let $z_0 \in L^r(\Omega; H_x^{\mathfrak{s}})$ be $\mathscr{F}(0)$-measurable. Let z be the local solution to (3.2) on $[0,\tau^*)$ given by Theorem 3.4.1. Then, we have, $\mathbb{P}$-almost surely, for $\tau \le \tau^*$:

$$\begin{aligned}|z(\tau)|^2_{H_x^{\mathfrak{s}}} =& |z_0|^2_{H_x^{\mathfrak{s}}} - 2\epsilon\int_0^\tau (\gamma|z(s)|^2_{H_x^{\mathfrak{s}}} + \mu|\operatorname{Im}(z(s))|^2_{H_x^{\mathfrak{s}}})\,\mathrm{d}s + 2\epsilon\mu\int_0^\tau |\operatorname{Re}(z(s))|^2_{H_x^{\mathfrak{s}}}\,\mathrm{d}s \\ &- 2\mathfrak{s}\int_0^\tau \langle \partial_x z(s), z(s)(\partial_x \mathrm{d}W(s))\rangle_{L_x^2} + \mathfrak{s}\sum_{k=0}^\infty \int_0^\tau |z(s)\partial_x(\Phi e_k)|^2_{L_x^2}\,\mathrm{d}s. \end{aligned} \tag{3.57}$$

In order to prove this formula for the evolution of the $H_x^{\mathfrak{s}}$-norm, we will use Itô's formula, also known as the stochastic chain rule. In particular, we apply the version of Itô's formula presented in Chapter 1: Theorem 1.3.18. It applies to strong solutions of stochastic partial differential equations, as opposed to the mild solutions that are available via Theorem 3.4.1. Recall that by Theorem 3.4.1, there exists a stopping time τ^* and an $(\mathscr{F}(t))_{t\in[0,T]}$-adapted process $z \in L^r(\Omega; C([0,\tau^*]; H_x^{\mathfrak{s}}) \cap L^r(0,\tau^*; W_x^{\mathfrak{s},p}))$ such that (3.2) holds $\mathbb{P}$-almost surely for $\tau \le \tau^*$. By applying the regulizing operator

$$I_\epsilon := (I - \epsilon\Delta)^{-1},$$

of Lemma 1.2.15 to the mild equation, we wish to obtain an equation of the form

$$\tilde{z}(t) = \tilde{z}_0 + \int_0^t \Psi_1(s)\,\mathrm{d}s + \int_0^t \Psi_2(s)\,\mathrm{d}W(s),$$

to which we can apply Itô's formula. We show that this is the case in the following lemma. In order to avoid the complication of stopping times in subsequent calculations, we will work with the global solution of the truncated equation. Since the results we obtain do not depend on the truncation radius, they also apply to the local solution of the original equation.

Lemma 3.5.3
Let r satisfy the conditions in Assumption 3.1.1 and let $z_0 \in L^r(\Omega; H_x^{\mathfrak{s}})$ be $\mathscr{F}(0)$-measurable. Let z_R be the unique solution of the truncated equation (3.28) with radius $R > 0$ given by Proposition 3.3.1. Then, we have, $\mathbb{P}$-almost surely, for $t \ge 0$:

$$\begin{aligned}I_\epsilon z_R(t) =& I_\epsilon z_0 + \int_0^t A I_\epsilon z_R(s)\,\mathrm{d}s + 4i\int_0^t \theta_R(|z_R|_{X_s}) I_\epsilon(|z_R(s)|^2 z_R(s))\,\mathrm{d}s \\ &- i\int_0^t I_\epsilon(z_R(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_0^t I_\epsilon(z_R(s)F_\Phi)\,\mathrm{d}s. \end{aligned} \tag{3.58}$$

Proof. We apply AI_ϵ to the mild equation (3.28) to find

$$\begin{aligned}AI_\epsilon z(t) =& AS(t)I_\epsilon z_0 + 4i\int_0^t \theta_R(|z|_{X_s}) AS(t-s) I_\epsilon(|z(s)|^2 z(s))\,\mathrm{d}s \\ &- i\int_0^t AS(t-s)I_\epsilon(z(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_0^t AS(t-s)I_\epsilon(z(s)F_\Phi)\,\mathrm{d}s,\end{aligned}$$

which holds $\mathbb{P}$-almost surely, for $t \geq 0$. Here, we have used that the regulizer I_ϵ commutes with the C_0-group $\{S(t)\}_{t\in\mathbb{R}}$, since the symbol of I_ϵ commutes with the symbol of $S(t)$. Indeed the symbol of I_ϵ is a multiple of the 2×2 identity matrix which commutes with the symbol in (2.4). We have also omitted the subscript R on the solution to lighten notation. We now integrate the equation to find

$$\begin{aligned}\int_0^t AI_\epsilon z(s)\,\mathrm{d}s = &\int_0^t AS(s)I_\epsilon z_0\,\mathrm{d}s + 4i\int_0^t\int_0^{\mathfrak{s}} \theta_R(|z|_{X_r})AS(s-r)I_\epsilon(|z(r)|^2 z(r))\,\mathrm{d}r\,\mathrm{d}s\\ &- i\int_0^t\int_0^{\mathfrak{s}} AS(s-r)I_\epsilon(z(r)\,\mathrm{d}W(r))\,\mathrm{d}s - \frac{1}{2}\int_0^t\int_0^{\mathfrak{s}} AS(s-r)I_\epsilon(z(r)F_\Phi)\,\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

The first integral can be computed by using that

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = AS(t) \quad \text{on } D(A) = H^2(\mathbb{R}), \tag{3.59}$$

(Proposition 1.1.6). For the double integrals, we apply Fubini's theorem and its stochastic version (Theorem 1.3.17) to change the order of integration.

$$\begin{aligned}\int_0^t AI_\epsilon z(s)\,\mathrm{d}s \overset{(3.59)}{=} &S(t)I_\epsilon z_0 - I_\epsilon z_0 + 4i\int_0^t\int_0^t \chi_{[0,s]}(r)\theta_R(|z|_{X_r})AS(s-r)I_\epsilon(|z(r)|^2 z(r))\,\mathrm{d}r\,\mathrm{d}s\\ &- i\int_0^t\int_0^t \chi_{[0,s]}(r)AS(s-r)I_\epsilon(z(r)\,\mathrm{d}W(r))\,\mathrm{d}s - \frac{1}{2}\int_0^t\int_0^t \chi_{[0,s]}(r)AS(s-r)I_\epsilon(z(r)F_\Phi)\,\mathrm{d}r\,\mathrm{d}s\\ =&S(t)I_\epsilon z_0 - I_\epsilon z_0 + 4i\int_0^t\int_0^t \chi_{[r,t]}(s)\theta_R(|z|_{X_r})AS(s-r)I_\epsilon(|z(r)|^2 z(r))\,\mathrm{d}s\,\mathrm{d}r\\ &- i\int_0^t\int_0^t \chi_{[r,t]}(s)AS(s-r)I_\epsilon(z(r)\cdot)\,\mathrm{d}s\,\mathrm{d}W(r) - \frac{1}{2}\int_0^t\int_0^t \chi_{[r,t]}(s)AS(s-r)I_\epsilon(z(r)F_\Phi)\,\mathrm{d}s\,\mathrm{d}r,\end{aligned} \tag{3.60}$$

where in the last step we have used that $\chi_{[0,s]}(r) = \chi_{[r,t]}(s)$ for all $(s,r)\in[0,t]^2$. Theorem 1.3.17 holds under the condition that

$$\int_0^t \left(\mathbb{E}\left(\int_0^{\mathfrak{s}} \|AS(s-r)I_\epsilon(z(r)\Phi)\|^2_{\mathscr{L}_2(L^2_x)}\,\mathrm{d}r\right)\right)^{1/2}\mathrm{d}s < \infty.$$

We can verify that this condition holds by splitting up the Hilbert-Schmidt norm, using Lemma 1.3.6 as

$$\|AS(s-r)I_\epsilon(z(r)\Phi)\|_{\mathscr{L}_2(L^2_x)} \overset{(1.10)}{\leq} \|S(s-r)AI_\epsilon(z(r)\cdot)\|_{\mathscr{L}(W_x^{\mathfrak{s},r/2};L^2_x)}\|\Phi\|_{\gamma(L^2_x;W_x^{\mathfrak{s},r/2})}.$$

We can further estimate the operator norm using Lemma 1.2.15 as

$$|S(s-r)AI_\epsilon(z(r)u)|_{L^2_x} \overset{(2.15)}{\leq} C|AI_\epsilon(z(r)u)|_{L^2_x} \leq C_\epsilon|z(r)u|_{L^2_x},$$

and Hölder's inequality

$$|S(s-r)AI_\epsilon(z(r)u)|_{L^2_x} \overset{(3.11)}{\leq} C_\epsilon|z(r)|_{L^p_x}|u|_{L^{r/2}_x} \leq C_\epsilon|z(r)|_{W_x^{\mathfrak{s},p}}|u|_{W_x^{\mathfrak{s},r/2}},$$

so that

$$\|AS(s-r)I_\epsilon(z(r)\Phi)\|_{\mathscr{L}_2(L^2_x)} \leq C_\epsilon\|\Phi\|_\delta|z(r)|_{W_x^{\mathfrak{s},p}},$$

where the last inequality follows via Proposition 3.1.3 (iv). We conclude that

$$\int_0^t \left(\mathbb{E}\left(\int_0^{\mathfrak{s}} \|AS(s-r)I_\epsilon(z(r)\Phi(\cdot))\|^2_{\mathscr{L}_2(L^2_x)}\,\mathrm{d}r\right)\right)^{1/2}\mathrm{d}s \leq \tilde{C}\left(\mathbb{E}\left(\int_0^t |z(r)|^2_{W_x^{\mathfrak{s},p}}\,\mathrm{d}r\right)\right)^{1/2}.$$

Therefore, the condition is satisfied if $z\in L^1(\Omega;L^2_t(W_x^{\mathfrak{s},p}))$, which is indeed the case since $z\in L^r(\Omega;L^r_t(W_x^{\mathfrak{s},p}))$

with $r \geq 2$ via (3.6). We proceed from (3.60) by simplifying the inner integrals using (3.59) as

$$\begin{aligned}
\int_0^t AI_\epsilon z(s)\,\mathrm{d}s =& S(t)I_\epsilon z_0 - I_\epsilon z_0 + 4i\int_0^t\int_r^t \theta_R(|z|_{X_r})AS(s-r)I_\epsilon(|z(r)|^2 z(r))\,\mathrm{d}s\,\mathrm{d}r\\
&- i\int_0^t\int_r^t AS(s-r)I_\epsilon(z(r)\cdot)\,\mathrm{d}s\,\mathrm{d}W(r) - \frac{1}{2}\int_0^t\int_r^t AS(s-r)I_\epsilon(z(r)F_\Phi)\,\mathrm{d}s\,\mathrm{d}r\\
=& S(t)I_\epsilon z_0 - I_\epsilon z_0 + 4i\int_0^t\int_r^t AS(s-r)\,\mathrm{d}s\,\theta_R(|z|_{X_r})I_\epsilon(|z(r)|^2 z(r))\,\mathrm{d}r\\
&- i\int_0^t\int_r^t AS(s-r)\,\mathrm{d}s\,I_\epsilon(z(r)\,\mathrm{d}W(r)) - \frac{1}{2}\int_0^t\int_r^t AS(s-r)\,\mathrm{d}s\,I_\epsilon(z(r)F_\Phi)\,\mathrm{d}r\\
=& S(t)I_\epsilon z_0 - I_\epsilon z_0 + 4i\int_0^t (S(t-s)-I)\,\theta_R(|z|_{X_s})I_\epsilon(|z(s)|^2 z(s))\,\mathrm{d}s\\
&- i\int_0^t (S(t-s)-I)\,I_\epsilon(z(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_0^t (S(t-s)-I)\,I_\epsilon(z(s)F_\Phi)\,\mathrm{d}s\\
=& I_\epsilon S(t)z_0 - I_\epsilon z_0 + 4iI_\epsilon\int_0^t S(t-s)\theta_R(|z|_{X_s})(|z(s)|^2 z(s))\,\mathrm{d}s - 4i\int_0^t \theta_R(|z|_{X_s})I_\epsilon(|z(s)|^2 z(s))\,\mathrm{d}s\\
&- iI_\epsilon\int_0^t S(t-s)(z(s)\,\mathrm{d}W(s)) + i\int_0^t I_\epsilon(z(s)\,\mathrm{d}W(s))\\
&- \frac{1}{2}I_\epsilon\int_0^t S(t-s)(z(s)F_\Phi)\,\mathrm{d}r + \frac{1}{2}\int_0^t I_\epsilon(z(s)F_\Phi)\,\mathrm{d}s.
\end{aligned}$$

In this last expression, we recognize in the terms containing the semigroup the mild form (3.28) with the operator I_ϵ applied on the left, so that we can write

$$\begin{aligned}
\int_0^t AI_\epsilon z(s)\,\mathrm{d}s =& I_\epsilon z(t) - I_\epsilon z_0 - 4i\int_0^t \theta_R(|z|_{X_s})I_\epsilon(|z(s)|^2 z(s))\,\mathrm{d}s\\
&+ i\int_0^t I_\epsilon(z(s)\,\mathrm{d}W(s)) + \frac{1}{2}\int_0^t I_\epsilon(z(s)F_\Phi)\,\mathrm{d}s,
\end{aligned}$$

and we conclude that

$$I_\epsilon z(t) = I_\epsilon z_0 + \int_0^t AI_\epsilon z(s)\,\mathrm{d}s + 4i\int_0^t \theta_R(|z|_{X_s})I_\epsilon(|z(s)|^2 z(s))\,\mathrm{d}s - i\int_0^t I_\epsilon(z(s)\,\mathrm{d}W(s)) - \frac{1}{2}\int_0^t I_\epsilon(z(s)F_\Phi)\,\mathrm{d}s,$$

holds $\mathbb{P}$-almost surely as desired. □

We now turn to the proof of Proposition 3.5.2, in which we apply the Itô formula to the stochastic process obtained in the previous lemma.

Proof of Proposition 3.5.2. Recall from Chapter 1 that Itô's formula provides an identity for the differential of functionals of a stochastic process. Here we consider the functional $|\cdot|^2_{H^{\mathfrak{s}}_x}$ on the stochastic process in (3.58), which may alternatively be written as

$$|z|^2_{H^{\mathfrak{s}}_x} = b_{\mathfrak{s}}(z,\overline{z}),$$

for $z \in H^{\mathfrak{s}}_x$, where $b_{\mathfrak{s}}$ denotes the bilinear map

$$b_{\mathfrak{s}}(f,g) = \int_{\mathbb{R}} fg\,\mathrm{d}x + \mathfrak{s}\int_{\mathbb{R}}(\partial_x f)(\partial_x g)\,\mathrm{d}x.$$

This map is well-defined if the product fg is an element of $W^{\mathfrak{s},1}_x$. Note that for this bilinear map we have by symmetry

$$b_{\mathfrak{s}}(f,\overline{g}) + b_{\mathfrak{s}}(g,\overline{f}) = b_{\mathfrak{s}}(f,\overline{g}) + \overline{b_{\mathfrak{s}}(\overline{g},f)} = b_{\mathfrak{s}}(f,\overline{g}) + \overline{b_{\mathfrak{s}}(f,\overline{g})} = 2\,\mathrm{Re}\,b_{\mathfrak{s}}(f,\overline{g}).$$

We also introduce the notation

$$\langle f,g\rangle_{L^{u'}_x\times L^u_x} := \int_{\mathbb{R}} f\overline{g}\,\mathrm{d}x,$$

for the dual pairing between $f \in L_x^{u'}$ and $g \in L_x^u$, so that for such f and g we may write

$$b_{\mathfrak{s}}(f,\overline{g}) + b_{\mathfrak{s}}(g,\overline{f}) = 2\,\mathrm{Re}\langle f, g\rangle_{L_x^{u'}\times L_x^u} + 2\mathfrak{s}\,\mathrm{Re}\langle \partial_x f, \partial_x g\rangle_{L_x^{u'}\times L_x^u}.$$

We now apply Itô's formula for bilinear maps (Theorem 1.3.18) to the map $b_{\mathfrak{s}}$ and the processes

$$\begin{aligned}
\xi_1 &= I_\epsilon z_0\\
\xi_2 &= I_\epsilon \overline{z_0}\\
\psi_1(t) &= A I_\epsilon z_R(t) + 4i\theta_R(|z_R|_{X_t}) I_\epsilon(|z_R(t)|^2 z_R(t)) - \tfrac{1}{2} I_\epsilon(z_R(t)F_\Phi)\\
\psi_2(t) &= A I_\epsilon \overline{z_R}(t) - 4i\theta_R(|\overline{z_R}|_{X_t}) I_\epsilon(|\overline{z_R}(t)|^2 \overline{z_R}(t)) - \tfrac{1}{2} I_\epsilon(\overline{z_R}(t)F_\Phi)\\
\Theta_1(t) &= -i I_\epsilon(z_R(t)\cdot)\\
\Theta_2(t) &= i I_\epsilon(\overline{z_R}(t)\cdot),
\end{aligned}$$

where we have used the same notation as in Theorem 1.3.18. By using (3.58) we then note that $\zeta_1(t) = I_\epsilon z_R(t)$ and $\zeta_2(t) = I_\epsilon \overline{z_R}(t)$, and via Itô's formula we obtain that $\mathbb{P}$-almost surely,

$$|I_\epsilon z(t)|^2_{H_x^{\mathfrak{s}}} = |I_\epsilon z_0|^2_{H_x^{\mathfrak{s}}} + J_1^\epsilon(t) + J_2^\epsilon(t) + J_3^\epsilon(t) + J_4^\epsilon(t) + J_5^\epsilon(t), \tag{3.61}$$

where the terms $J_1^\epsilon, \dots, J_5^\epsilon$ are defined as

$$\begin{aligned}
J_1^\epsilon(t) &:= 2\,\mathrm{Re}\int_0^t \langle A I_\epsilon z(s), I_\epsilon z(s)\rangle_{H_x^{\mathfrak{s}}}\,\mathrm{d}s;\\
J_2^\epsilon(t) &:= 8\,\mathrm{Im}\int_0^t \theta_R(|z|_{X_s})\langle I_\epsilon(|z(s)|^2 z(s)), I_\epsilon z(s)\rangle_{L_x^{q'}\times L_x^q}\,\mathrm{d}s\\
&\quad + 24\mathfrak{s}\,\mathrm{Im}\int_0^t \theta_R(|z|_{X_s})\langle I_\epsilon(|z(s)|^2\partial_x z(s)), I_\epsilon \partial_x z(s)\rangle_{L_x^{q'}\times L_x^q}\,\mathrm{d}s;\\
J_3^\epsilon(t) &:= -2\,\mathrm{Im}\int_0^t \langle I_\epsilon z(s), I_\epsilon(z(s)\,\mathrm{d}W(s))\rangle_{H_x^{\mathfrak{s}}};\\
J_4^\epsilon(t) &:= -\mathrm{Re}\int_0^t \langle I_\epsilon(z(s)F_\Phi), I_\epsilon z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s\\
&\quad - \mathfrak{s}\,\mathrm{Re}\int_0^t \langle I_\epsilon \partial_x(z(s)F_\Phi), I_\epsilon \partial_x z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s;\\
J_5^\epsilon(t) &:= \int_0^t \sum_{k=0}^\infty |I_\epsilon(z(s)\Phi e_k)|^2_{H_x^{\mathfrak{s}}}\,\mathrm{d}s.
\end{aligned}$$

For the term J_2^ϵ we have used that $\partial_x(|z(s)|^2 z(s)) = 3|z(s)|^2\partial_x z(s)$. We have also omitted the subscript R again to lighten notation. We now show that each of the integrals $J_1^\epsilon(t), \dots, J_5^\epsilon(t)$ converges as $\epsilon \downarrow 0$ and we identify the limits.

Step 1. (convergence of the integral $J_1^\epsilon(t)$)
We can further compute the integrand of $J_1^\epsilon(t)$ as

$$\mathrm{Re}\langle A I_\epsilon z(s), I_\epsilon z(s)\rangle_{H_x^{\mathfrak{s}}} \overset{(2.11)}{=} -\epsilon\gamma |I_\epsilon z(s)|^2_{H_x^{\mathfrak{s}}} + \epsilon\mu\Big(|\mathrm{Re}(I_\epsilon z(s))|^2_{H_x^{\mathfrak{s}}} - |\mathrm{Im}(I_\epsilon z(s))|^2_{H_x^{\mathfrak{s}}}\Big),$$

since $I_\epsilon z(s) \in H_x^{\mathfrak{s}+2}$. It follows by Lemma 1.2.15 and continuity of the squared norm that

$$\mathrm{Re}\langle A I_\epsilon z(s), I_\epsilon z(s)\rangle_{H_x^{\mathfrak{s}}} \to -\epsilon\gamma|z(s)|^2_{H_x^{\mathfrak{s}}} + \epsilon\mu(|\mathrm{Re}(z(s))|^2_{H_x^{\mathfrak{s}}} - |\mathrm{Im}(z(s))|^2_{H_x^{\mathfrak{s}}}) \quad \text{as } \epsilon \downarrow 0. \tag{3.62}$$

Since $\|I_\epsilon\|_{\mathscr{L}(H_x^{\mathfrak{s}})} \le C$ with C independent of ϵ (Equation (1.5)), the integrand is dominated by this constant times the right-hand side of (3.62). This dominating function of s is furthermore integrable on $[0,t]$, as $z \in C_t(H_x^{\mathfrak{s}})$. We conclude by the dominated convergence theorem that

$$J_1^\epsilon(t) \to -2\epsilon\int_0^t (\gamma|z(s)|^2_{H_x^{\mathfrak{s}}} + \mu|\mathrm{Im}(z(s))|^2_{H_x^{\mathfrak{s}}})\,\mathrm{d}s + 2\epsilon\mu\int_0^t |\mathrm{Re}(z(s))|^2_{H_x^{\mathfrak{s}}}\,\mathrm{d}s \quad \text{as } \epsilon \downarrow 0.$$

Step 2. (convergence of the integrals $J_2^\epsilon(t)$ and $J_4^\epsilon(t)$)
The integrands in the terms $J_2^\epsilon(t)$ and $J_4^\epsilon(t)$ are all dual pairings of the form

$$\langle I_\epsilon f(s), I_\epsilon g(s)\rangle_{L_x^{u'}\times L_x^u},$$

where $f(s)\in L_x^{u'}$ and $g(s)\in L_x^u$ for almost all $s\in[0,t]$, and some constant $b\geq 1$. In case of the first integrand in term $J_2^\epsilon(t)$ we have $f(s)=|z(s)|^2 z(s)\in L_x^{q'}$ and $g(s)=z(s)\in L_x^q$. In the second integrand we have $f(s)=|z(s)|^2\partial_x z(s)\in L_x^{q'}$ and $g(s)=\partial_x z(s)\in L_x^q$. For the first integrand in term $J_4^\epsilon(t)$, we can take $f(s)=z(s)F_\Phi\in L_x^{\frac{4+2\delta}{4+\delta}}$ and $g(s)=z(s)\in L_x^{\frac{2(2+\delta)}{\delta}}$. Lastly, for the second integrand in the term $J_4^\epsilon(t)$ we have $f(s)=\partial_x(z(s)F_\Phi)\in L_x^{\frac{4+2\delta}{4+\delta}}$ and $g(s)=\partial_x z(s)\in L_x^{\frac{2(2+\delta)}{\delta}}$. We show that

$$\int_0^t \langle I_\epsilon f(s), I_\epsilon g(s)\rangle_{L_x^{u'}\times L_x^u}\,\mathrm{d}s \to \int_0^t \langle f(s), g(s)\rangle_{L_x^{u'}\times L_x^u}\,\mathrm{d}s,$$

for these choices of f,g and u.

By Lemma 1.2.15, we know that $I_\epsilon f(s)\to f(s)$ in $L^{u'}$ and $I_\epsilon g(s)\to g(s)$ in L^u as $\epsilon\downarrow 0$. We then have $I_\epsilon f(s)\overline{I_\epsilon g(s)}\to f(s)\overline{g(s)}$ in L_x^1 by the following inequality

$$\begin{aligned}|I_\epsilon f(s)\overline{I_\epsilon g(s)} - f(s)\overline{g(s)}|_{L_x^1} &\leq |(I_\epsilon f(s)-f(s))\overline{I_\epsilon g(s)}|_{L_x^1} + |f(s)(\overline{I_\epsilon g(s)}-\overline{g(s)})|_{L_x^1}\\ &\leq |I_\epsilon f(s)-f(s)|_{L_x^{u'}}|I_\epsilon g(s)|_{L_x^u} + |f(s)|_{L_x^{u'}}|I_\epsilon g(s)-g(s)|_{L_x^u},\end{aligned}$$

and letting $\epsilon\downarrow 0$. It follows that

$$\langle I_\epsilon f(s), I_\epsilon g(s)\rangle_{L_x^{u'}\times L_x^u} \to \langle f(s), g(s)\rangle_{L_x^{u'}\times L_x^u},\quad \text{as } \epsilon\downarrow 0,$$

since

$$\begin{aligned}\left|\langle f(s), g(s)\rangle_{L_x^{u'}\times L_x^u} - \langle I_\epsilon f(s), I_\epsilon g(s)\rangle_{L_x^{u'}\times L_x^u}\right| &= \left|\int_{\mathbb{R}} f(x,s)\overline{g(x,s)} - I_\epsilon f(x,s)\overline{I_\epsilon g(x,s)}\,\mathrm{d}x\right|\\ &\leq \left|f(s)\overline{g(s)} - I_\epsilon f(s)\overline{I_\epsilon g(s)}\right|_{L_x^1}\to 0\end{aligned}$$

as $\epsilon\downarrow 0$. We proceed by showing that $\langle I_\epsilon f(s), I_\epsilon g(s)\rangle_{L_x^{u'}\times L_x^u}$ can be dominated using Hölder's inequality as

$$\langle I_\epsilon f(s), I_\epsilon g(s)\rangle_{L_x^{u'}\times L_x^u} \leq |I_\epsilon f(s)|_{L_x^{u'}}|I_\epsilon g(s)|_{L_x^u} \leq |f(s)|_{L_x^{u'}}|g(s)|_{L_x^u},$$

where we have used that the operator norms of the operators I_ϵ are uniformly bounded in ϵ on both $L_x^{u'}$ and L_x^u (Lemma 1.2.15). In order to apply the dominated convergence theorem, it remains to be shown that the dominating function $|f(s)|_{L_x^{u'}}|g(s)|_{L_x^u}$ is integrable for the various choices of f,g and u. For the integrands in the term $J_2^\epsilon(t)$ this is indeed the case, since

$$\begin{aligned}\int_0^t ||z(s)|^2 z(s)|_{L_x^{q'}}|z(s)|_{L_x^q} + \mathfrak{s}||z(s)|^2\partial_x z(s)|_{L_x^{q'}}|\partial_x z(s)|_{L_x^q}\,\mathrm{d}s &\leq \int_0^t ||z(s)|^2 z(s)|_{W_x^{\mathfrak{s},q'}}|z(s)|_{W_x^{\mathfrak{s},q}}\,\mathrm{d}s\\ &\overset{(3.30)}{\leq} \int_0^t |z(s)|^4_{W_x^{\mathfrak{s},q}}\,\mathrm{d}s \leq t^{1/8}|z|_{L^\gamma(0,t;W_x^{\mathfrak{s},q})},\end{aligned}$$

where the last inequality is an application of Hölder's inequality with exponents $\frac{1}{4}=\frac{1}{8}+\frac{1}{8}$. Integrability follows via Proposition 3.1.3 (ii), since $z\in X_t$. For the integrands in the term $J_4^\epsilon(t)$, we also have integrability via

$$\begin{aligned}&\int_0^t |z(s)F_\Phi|_{L_x^{\frac{4+2\delta}{4+\delta}}}|z(s)|_{L_x^{\frac{2(2+\delta)}{\delta}}} + \mathfrak{s}|\partial_x(z(s)F_\Phi)|_{L_x^{\frac{4+2\delta}{4+\delta}}}|\partial_x z(s)|_{L_x^{\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s\\ \leq&\int_0^t |z(s)F_\Phi|_{W_x^{\mathfrak{s},\frac{4+2\delta}{4+\delta}}}|z(s)|_{W_x^{\mathfrak{s},\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s \leq \int_0^t |F_\Phi|_{W_x^{\mathfrak{s},1+\delta/2}}|z(s)|^2_{W_x^{\mathfrak{s},\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s\\ \leq& C\int_0^t |z(s)|^2_{W_x^{\mathfrak{s},\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s \leq C\int_0^t (|z(s)|_{H_x^{\mathfrak{s}}} + |z(s)|_{W_x^{\mathfrak{s},p}})^2\,\mathrm{d}s,\end{aligned}\tag{3.63}$$

where in the last two inequalities we have used that $F_\Phi \in W_x^{1,\frac{4+2\delta}{4+\delta}}$ and $2 < \frac{2(2+\delta)}{\delta} \le p$. Integrability follows, as $z \in L^r(0,t;W_x^{\mathfrak{s},p}) \cap C([0,t];H_x^{\mathfrak{s}})$ with $r > 2$. We then conclude by the dominated convergence theorem that

$$J_2^\epsilon(t) \to 8\,\mathrm{Im}\int_0^t \theta_R(|z|_{X_s})\langle |z(s)|^2 z(s), z(s)\rangle_{L_x^{q'}\times L_x^q}\,\mathrm{d}s + 24\mathfrak{s}\,\mathrm{Im}\int_0^t \theta_R(|z|_{X_s})\langle |z(s)|^2\partial_x z(s), \partial_x z(s)\rangle_{L_x^{q'}\times L_x^q}\,\mathrm{d}s,$$

and

$$J_4^\epsilon(t) \to -\,\mathrm{Re}\int_0^t \langle z(s)F_\Phi, z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s - \mathfrak{s}\,\mathrm{Re}\int_0^t \langle \partial_x(z(s)F_\Phi), \partial_x z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s,$$

as $\epsilon \downarrow 0$. Note that

$$\langle |z(s)|^2 z(s), z(s)\rangle_{L_x^{q'}\times L_x^q} = \langle |z(s)|z(s), |z(s)|z(s)\rangle_{L_x^2} = |z(s)|_{L_x^4}^4,$$

so that

$$\mathrm{Im}\int_0^t \theta_R(|z|_{X_s})\langle |z(s)|^2 z(s), z(s)\rangle_{L_x^{q'}\times L_x^q}\,\mathrm{d}s = \mathrm{Im}\int_0^t \theta_R(|z|_{X_s})|z(s)|_{L_x^4}^4\,\mathrm{d}s = 0.$$

Similarly,

$$\mathrm{Im}\int_0^t \theta_R(|z|_{X_s})\langle |z(s)|^2\partial_x z(s), \partial_x z(s)\rangle_{L_x^{q'}\times L_x^q}\,\mathrm{d}s = 0,$$

and we find that $J_2^\epsilon(t) \to 0$ as $\epsilon \downarrow 0$.

Step 3. (convergence of the integral J_5)
By Lemma 1.2.15 and continuity of the squared norm, we then have

$$|I_\epsilon(z(s)\Phi e_k)|_{H_x^{\mathfrak{s}}}^2 \to |z(s)\Phi e_k|_{H_x^{\mathfrak{s}}}^2 \quad \text{as } \epsilon \downarrow 0.$$

We now use that $|I_\epsilon(z(s)\Phi e_k)|_{H_x^{\mathfrak{s}}}^2 \le C|z(s)\Phi e_k|_{H_x^{\mathfrak{s}}}^2$ with C independent of ϵ (Lemma 1.2.15), which serves as a dominating function of the variables s and k. In order to see that this dominating function is furthermore integrable, we note that

$$\sum_{k=0}^\infty |z(s)\Phi e_k|_{H_x^{\mathfrak{s}}}^2 = \sum_{k=0}^\infty \langle z(s)\Phi e_k, z(s)\Phi e_k\rangle_{H_x^{\mathfrak{s}}} = \sum_{k=0}^\infty \langle z(s)\Phi e_k, z(s)\Phi e_k\rangle_{L_x^2} + \mathfrak{s}\sum_{k=0}^\infty \langle \partial_x(z(s)\Phi e_k), \partial_x(z(s)\Phi e_k)\rangle_{L_x^2},$$

and by using that Φe_k is real-valued for all k,

$$\begin{aligned}\sum_{k=0}^\infty |z(s)\Phi e_k|_{H_x^{\mathfrak{s}}}^2 &= \sum_{k=0}^\infty \langle z(s)(\Phi e_k)^2, z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}} + \mathfrak{s}\sum_{k=0}^\infty \langle (\partial_x z(s))\Phi e_k, (\partial_x z(s))\Phi e_k\rangle_{L_x^2}\\ &\quad + 2\mathfrak{s}\sum_{k=0}^\infty \langle (\partial_x z(s))\Phi e_k, z(s)(\partial_x\Phi e_k)\rangle_{L_x^2} + \mathfrak{s}\sum_{k=0}^\infty \langle z(s)(\partial_x\Phi e_k), z(s)(\partial_x\Phi e_k)\rangle_{L_x^2}\\ &= \langle z(s)F_\Phi, z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}} + \mathfrak{s}\langle (\partial_x z(s))F_\Phi, \partial_x z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}}\\ &\quad + \mathfrak{s}\langle \partial_x z(s), z(s)(\partial_x F_\Phi)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}} + \mathfrak{s}\sum_{k=0}^\infty |z(s)(\partial_x\Phi e_k)|_{L_x^2}^2.\end{aligned}$$

It follows as in (3.63) that this function is integrable. We then conclude by the dominated convergence theorem that

$$\begin{aligned}J_5^\epsilon(t) \to \sum_{k=0}^\infty |z(s)\Phi e_k|_{H_x^{\mathfrak{s}}}^2 &= \langle z(s)F_\Phi, z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}} + \mathfrak{s}\langle \partial_x(z(s)F_\Phi), \partial_x z(s)\rangle_{L_x^{\frac{4+2\delta}{4+\delta}}\times L_x^{\frac{2(2+\delta)}{\delta}}}\\ &\quad + \mathfrak{s}\sum_{k=0}^\infty |z(s)(\partial_x\Phi e_k)|_{L_x^2}^2. \end{aligned} \tag{3.64}$$

Step 4. (convergence of the integral J_3)
Finally we show that

$$J_3^\epsilon(t) \to -2\,\mathrm{Im}\int_0^t \langle z(s), z(s)\,\mathrm{d}W(s)\rangle_{H_x^{\mathfrak{s}}}, \quad \text{as } \epsilon \downarrow 0,$$

in $L^2(\Omega)$. We therefore apply Itô's isometry (Theorem 1.3.14) to

$$\mathbb{E}\left|\int_0^t \langle I_\epsilon z(s), I_\epsilon(z(s)\,\mathrm{d}W(s))\rangle_{H^{\mathfrak{s}}_x} - \int_0^t \langle z(s), z(s)\,\mathrm{d}W(s)\rangle_{H^{\mathfrak{s}}_x}\right|^2 \leq \mathbb{E}\left|\int_0^t \langle (I_\epsilon - I)z(s), I_\epsilon(z(s)\,\mathrm{d}W(s))\rangle_{H^{\mathfrak{s}}_x}\right|^2 + \mathbb{E}\left|\int_0^t \langle z(s), (I_\epsilon - I)(z(s)\,\mathrm{d}W(s))\rangle_{H^{\mathfrak{s}}_x}\right|^2,$$

which gives

$$\mathbb{E}\left|\int_0^t \langle z(s), z(s)\,\mathrm{d}W(s)\rangle_{H^{\mathfrak{s}}_x} - \int_0^t \langle I_\epsilon z(s), I_\epsilon(z(s)\,\mathrm{d}W(s))\rangle_{H^{\mathfrak{s}}_x}\right|^2 \leq \mathbb{E}\int_0^t \|\Gamma_1^\epsilon(s)\|^2_{\gamma(L^2_x;\mathbb{C})}\,\mathrm{d}s + \mathbb{E}\int_0^t \|\Gamma_2^\epsilon(s)\|^2_{\gamma(L^2_x;\mathbb{C})}\,\mathrm{d}s,$$

where $\Gamma_1^\epsilon(s)$ and $\Gamma_2^\epsilon(s)$ denote the functionals

$$u \mapsto \langle (I_\epsilon - I)z(s), I_\epsilon(z(s)\Phi u)\rangle_{H^{\mathfrak{s}}_x} \quad \text{and} \quad u \mapsto \langle z(s), (I_\epsilon - I)(z(s)\Phi u)\rangle_{H^{\mathfrak{s}}_x},$$

respectively. For the second term, we proceed by splitting up the $\gamma(L^2_x;\mathbb{C})$-norm using Lemma 1.3.6, yielding

$$\|\langle z(s), (I_\epsilon - I)z(s)\Phi\rangle_{H^{\mathfrak{s}}_x}\|_{\gamma(L^2_x;\mathbb{C})} \leq \|\Phi\|_{\gamma(L^2_x;W^{\mathfrak{s},r/2}_x)}\|\langle z(s), (I_\epsilon - I)z(s)\cdot\rangle_{H^{\mathfrak{s}}_x}\|_{\mathscr{L}(W^{\mathfrak{s},r/2}_x;\mathbb{C})}.$$

We calculate the operator norm by writing

$$|\langle z(s), (I_\epsilon - I)z(s)u\rangle_{H^{\mathfrak{s}}_x}| \leq |(I_\epsilon - I)z(s)|_{H^{\mathfrak{s}}_x}|z(s)u|_{H^{\mathfrak{s}}_x}$$

and by Lemma 1.2.15 we obtain that the expression above converges to 0 as $\epsilon \downarrow 0$. With the aim of applying the dominated convergence theorem to the integrals in (28), we further estimate the expression using Lemma 1.2.15

$$|\langle z(s), (I_\epsilon - I)z(s)u\rangle_{H^{\mathfrak{s}}_x}| \leq |z(s)|_{H^{\mathfrak{s}}_x}|z(s)u|_{H^{\mathfrak{s}}_x},$$

and use Hölder's inequality, which gives

$$|\langle z(s), (I_\epsilon - I)z(s)u\rangle_{H^{\mathfrak{s}}_x}| \overset{(3.11)}{\leq} |z(s)|_{H^{\mathfrak{s}}_x}|z(s)|_{W^{\mathfrak{s},p}_x}|u|_{W^{\mathfrak{s},r/2}_x}.$$

We conclude via Proposition 3.1.3 (iv) that

$$\|\langle z(s), (I_\epsilon - I)z(s)\Phi\rangle_{L^2_x}\|_{\gamma(L^2_x;\mathbb{C})} \leq \|\Phi\|_\delta |z(s)|_{H^{\mathfrak{s}}_x}|z(s)|_{W^{\mathfrak{s},p}_x},$$

and we can similarly estimate the term involving $\Gamma_1^\epsilon(s)$. The resulting dominating function $s \mapsto |z(s)|_{H^{\mathfrak{s}}_x}|z(s)|_{W^{1,p}_x}$ is indeed an element of $L^1(\Omega; L^2_t)$ since $z \in L^r(\Omega; C_t(H^{\mathfrak{s}}_x) \cap L^r_t(W^{\mathfrak{s},p}_x))$ with $r \geq 2$. It follows by the dominated convergence theorem that

$$J_3^\epsilon(t) \to -2\,\mathrm{Im}\int_0^t \langle z(s), z(s)\,\mathrm{d}W(s)\rangle_{H^{\mathfrak{s}}_x}, \quad \text{as } \epsilon \downarrow 0,$$

in $L^2(\Omega)$. The convergence is then also $\mathbb{P}$-almost sure. Note that

$$\langle z(s), z(s)dW(s)\rangle_{H^{\mathfrak{s}}_x} = \langle z(s), z(s)dW(s)\rangle_{L^2_x} + \mathfrak{s}\langle \partial_x z(s), (\partial_x z(s))dW(s)\rangle_{L^2_x} + \mathfrak{s}\langle \partial_x z(s), z(s)(\partial_x dW(s))\rangle_{L^2_x}.$$

Since the noise is real-valued, the first two inner products vanish, and we are left with

$$J_3^\epsilon(t) \to -2\mathfrak{s}\int_0^t \langle \partial_x z(s), z(s)(\partial_x \mathrm{d}W(s))\rangle_{L^2_x} \quad \text{as } \epsilon \downarrow 0.$$

Step 5. (Collecting the results)
We conclude from (3.61) and steps 1-4 that

$$\begin{aligned} |z_R(t)|^2_{H^{\mathfrak{s}}_x} = &|z_0|^2_{H^{\mathfrak{s}}_x} - 2\epsilon\int_0^t (\gamma|z_R(s)|^2_{H^{\mathfrak{s}}_x} + \mu|\mathrm{Im}(z_R(s))|^2_{H^{\mathfrak{s}}_x})\,\mathrm{d}s + 2\epsilon\mu\int_0^t |\mathrm{Re}(z_R(s))|^2_{H^{\mathfrak{s}}_x}\,\mathrm{d}s \\ &- 2\mathfrak{s}\,\mathrm{Im}\int_0^t \langle \partial_x z_R(s), z_R(s)(\partial_x \mathrm{d}W(s))\rangle_{L^2_x} + \mathfrak{s}\sum_{k=0}^\infty \int_0^t |z_R(s)\partial_x(\Phi e_k)|^2_{L^2_x}\,\mathrm{d}s, \end{aligned}$$

holds $\mathbb{P}$-almost surely for all $t \geq 0$ and $R > 0$. Since we found in step 2 that $J_2(t) = 0$, the formula does not depend on the truncation radius R and it holds also for the local solution z. The result follows by setting t equal to $\tau \leq \tau^*$. □

With help of the previous formula for the evolution of the $H^{\mathfrak{s}}_x$-norm of solutions and the blow-up criterion provided by Theorem 3.4.1, we can now give a proof of the global existence result.

Proof of Theorem 3.5.1. By Theorem 3.4.1, it suffices to show that we $\mathbb{P}$-almost surely have

$$\sup_{s\le\tau}|z(\tau)|_{H^{\mathfrak{s}}_x}<\infty,$$

for $\tau\le\min\{\tau^*,T_0\}$ with $T_0>0$. By estimating

$$-|\operatorname{Im}(z(s))|^2_{H^{\mathfrak{s}}_x}+|\operatorname{Re}(z(s))|^2_{H^{\mathfrak{s}}_x}\le|z(s)|^2_{H^{\mathfrak{s}}_x}$$

in (3.57), we obtain

$$\begin{aligned}|z(\tau)|^2_{H^{\mathfrak{s}}_x}\le&|z_0|^2_{H^{\mathfrak{s}}_x}+2\epsilon(\mu-\gamma)\int_0^\tau|z(s)|^2_{H^{\mathfrak{s}}_x}\,\mathrm{d}s-2\mathfrak{s}\int_0^\tau\langle\partial_x z(s),z(s)(\partial_x\,\mathrm{d}W(s))\rangle_{L^2_x}\\&+\mathfrak{s}\sum_{k=0}^\infty\int_0^\tau|z(s)\partial_x(\Phi e_k)|^2_{L^2_x}\,\mathrm{d}s,\end{aligned}$$

$\mathbb{P}$-almost surely. Upon taking the square, the supremum and the expectation of the equation above, we get

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\le\tau}|z(t)|^4_{H^{\mathfrak{s}}_x}\Big)\le&2\mathbb{E}\Big(|z_0|^4_{H^{\mathfrak{s}}_x}\Big)+8\epsilon^2(\mu-\gamma)^2\mathbb{E}\Big(\sup_{t\le\tau}\Big|\int_0^t|z(s)|^2_{H^{\mathfrak{s}}_x}\,\mathrm{d}s\Big|^2\Big)+8\mathfrak{s}\mathbb{E}\Big(\sup_{t\le\tau}\Big|\int_0^t\langle\partial_x z(s),z(s)(\partial_x\,\mathrm{d}W(s))\rangle_{L^2_x}\Big|^2\Big)\\&+2\mathfrak{s}\mathbb{E}\Big(\sup_{t\le\tau}\Big|\sum_{k=0}^\infty\int_0^\tau|z(s)\partial_x(\Phi e_k)|^2_{L^2_x}\,\mathrm{d}s\Big|^2\Big).\end{aligned}$$

For the last term, we note that by Hölder's inequality

$$\begin{aligned}\sum_{k=0}^\infty\int_0^\tau|z(s)\partial_x(\Phi e_k)|^2_{L^2_x}\,\mathrm{d}s\le&\sum_{k=0}^\infty\int_0^\tau|z(s)|^2_{L_x^{\frac{2(2+\delta)}{\delta}}}|\partial_x(\Phi e_k)|^2_{L^{2+\delta}_x}\,\mathrm{d}s\\&=\sum_{k=0}^\infty|\partial_x(\Phi e_k)|^2_{L^{2+\delta}_x}\int_0^\tau|z(s)|^2_{L_x^{\frac{2(2+\delta)}{\delta}}}\,\mathrm{d}s\le C\int_0^\tau|z(s)|^2_{H^1_x}\,\mathrm{d}s,\end{aligned}\tag{3.65}$$

where in the final step we have used that $\Phi e_k\in W^{\mathfrak{s},2+\delta}_x$, and the embedding $H^1_x\hookrightarrow L_x^{\frac{2(2+\delta)}{\delta}}$ (Proposition 1.2.16). For the expectation of the stochastic integral, we apply the Burkholder inequality (Theorem 1.3.15) as

$$\mathbb{E}\Big(\sup_{t\le\tau}\Big|\int_0^t\langle\partial_x z(s),z(s)(\partial_x\,\mathrm{d}W(s))\rangle_{L^2_x}\Big|^2\Big)\overset{(1.11)}{\le}C\mathbb{E}\Big(\int_0^\tau\|\langle\partial_x z(s),z(s)(\partial_x\Phi)\rangle_{L^2_x}\|^2_{\gamma(L^2_x;\mathbb{C})}\,\mathrm{d}s\Big),$$

and we proceed by splitting up the $\gamma(L^2_x;\mathbb{C})$-norm using Lemma 1.3.6, yielding

$$\|\langle\partial_x z(s),z(s)(\partial_x\Phi)\rangle_{L^2_x}\|_{\gamma(L^2_x;\mathbb{C})}\le\|\Phi\|_{\gamma(L^2_x;W^{1,r/2}_x)}\|\langle\partial_x z(s),z(s)(\partial_x\cdot)\rangle_{L^2_x}\|_{\mathscr{L}(W^{1,r/2}_x;\mathbb{C})}.$$

We calculate the operator norm by writing

$$|\langle\partial_x z(s),z(s)(\partial_x u)\rangle_{L^2_x}|\le|\partial_x z(s)|_{L^2_x}|z(s)(\partial_x u)|_{L^2_x}.$$

We then use Hölder's inequality, which gives

$$|\langle\partial_x z(s),z(s)(\partial_x u)\rangle_{L^2_x}|\overset{(3.11)}{\le}|\partial_x z(s)|_{L^2_x}|z(s)|_{L^p_x}|\partial_x u|_{L^{r/2}_x}\le|z(s)|^2_{H^1_x}|u|_{W^{1,r/2}_x},$$

where the last inequality follows from the embedding $H^1_x\hookrightarrow L^p_x$ (Proposition 1.2.16). We conclude via Proposition 3.1.3 (iv) that

$$\|\langle\partial_x z(s),z(s)(\partial_x\Phi)\rangle_{L^2_x}\|_{\gamma(L^2_x;\mathbb{C})}\le\|\Phi\|_\delta|z(s)|^2_{H^1_x},$$

and

$$\mathbb{E}\left(\sup_{t\leq\tau}\left|\int_0^t \langle\partial_x z(s), z(s)(\partial_x \,\mathrm{d}W(s))\rangle_{L_x^2}\right|^2\right) \leq C\|\Phi\|_\delta^2 \mathbb{E}\left(\int_0^\tau |z(s)|_{H_x^1}^4 \,\mathrm{d}s\right). \tag{3.66}$$

By collecting the results of (3.65) and (3.66), we have shown that

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq\tau}|z(t)|_{H_x^s}^4\right) &\leq 2\mathbb{E}\left(|z_0|_{H_x^s}^4\right) + C_1\mathbb{E}\left(\sup_{t\leq\tau}\left|\int_0^t |z(s)|_{H_x^s}^2 \,\mathrm{d}s\right|^2\right) + C_2\mathbb{E}\left(\int_0^\tau |z(s)|_{H_x^1}^4 \,\mathrm{d}s\right)\\ &\leq 2\mathbb{E}\left(|z_0|_{H_x^s}^4\right) + C_3\int_0^\tau \mathbb{E}\left(\sup_{s\leq t}|z(s)|_{H_x^s}^4\right)\mathrm{d}t.\end{aligned}$$

By Grönwall's lemma, it then follows that

$$\mathbb{E}\left(\sup_{t\leq\tau}|z(t)|_{H_x^s}^4\right) \leq 2\mathbb{E}\left(|z_0|_{H_x^s}^4\right)e^{C_3\tau}.$$

Since $\tau < T_0$, we conclude that $\mathbb{E}\left(\sup_{t\leq\tau}|z(t)|_{H_x^s}^4\right) < \infty$ and it follows that $\sup_{t\leq\tau}|z(t)|_{H_x^s} < \infty$, $\mathbb{P}$-almost surely. This shows the result. $\square$

4

A glimpse on the stability of solitons

In this chapter, we discuss the stability of solitary standing wave solutions in the (stochastic) PFNLS equation. We start by deriving a solitary standing wave solution to the deterministic PFNLS equation, and we present a stability result due to Kapitula and Sandstede [23]. We proceed by considering an approach for tracking the position of the soliton, if subject to the stochastic PFNLS equation of the previous chapter, in Section 4.2. This position correction is defined in terms of the solution to (3.1), and is itself a stochastic process. We analyze the leading-order behavior of this process in Section 4.3. Finally, we briefly discuss directions for future research in Section 4.4. The computations in this section are informal and are meant to give a general idea of topics that could be explored.

4.1. Solitary standing wave solution

Consider the deterministic PFNLS equation

$$\mathrm{d}z = (i\Delta z - i\omega z - \epsilon(\gamma z - \mu\bar{z}))\,\mathrm{d}t + 4i|z|^2 z\,\mathrm{d}t. \tag{4.1}$$

This deterministic equation on the real line supports a solitary-wave solution to this equation of the form

$$\zeta(x) = re^{i\theta}\operatorname{sech}(cx),$$

with $r, c > 0$ and $\theta \in [0, 2\pi)$. Here, sech denotes the hyperbolic secant function, defined as

$$\operatorname{sech}(x) = \frac{2}{e^{-x} + e^{x}}.$$

Since ζ is not time-dependent, it should satisfy the solitary-wave equation

$$\Delta\zeta - \omega\zeta + 4|\zeta|^2\zeta + i\epsilon(\gamma\zeta - \mu\bar{\zeta}) = 0. \tag{4.2}$$

Using that

$$\Delta\operatorname{sech}(cx) = c^2(\operatorname{sech}(cx) - 2\operatorname{sech}^3(cx)),$$

we substitute the ansatz into the solitary-wave equation and find that we must have

$$re^{i\theta}c^2\left(\operatorname{sech}(cx) - 2\operatorname{sech}^3(cx)\right) - \omega re^{i\theta}\operatorname{sech}(cx) + 4r^3e^{i\theta}\operatorname{sech}^3(cx) + i\epsilon r(\gamma e^{i\theta} - \mu e^{-i\theta})\operatorname{sech}(cx) = 0.$$

By linear independence of $\operatorname{sech}(cx)$ and $\operatorname{sech}^3(cx)$, we find that both

$$4r^3e^{i\theta} - 2c^2re^{i\theta} = 0,$$

or equivalently $\sqrt{2}r = c$, and

$$re^{i\theta}c^2 = \omega re^{i\theta} - i\epsilon r(\gamma e^{i\theta} - \mu e^{-i\theta}),$$

must hold. The latter implies that $c^2 = \omega - i\epsilon(\gamma - \mu e^{-2i\theta})$. This can only hold (for $c > 0$) if $\gamma - \mu e^{-2i\theta}$ is purely imaginary, leading to the requirement that $\mathrm{Re}(\mu e^{-2i\theta}) = \gamma$, or equivalently $\cos(2\theta) = \frac{\gamma}{\mu}$. Then,

$$\gamma - \mu e^{-2i\theta} = -\mathrm{Im}(\mu e^{-2i\theta}) = -i\mu\sin(-2\theta) = i\mu\sin(2\theta),$$

and we find that

$$c = \sqrt{\omega + \epsilon\mu\sin(2\theta)}.$$

Note that the condition on θ can only hold if $\mu \geq \gamma$, i.e. the phase-sensitive amplification constant must be larger than the dissipation constant in order for the compensation to be successful. Note also that if θ satisfies $\cos(2\theta) = \gamma/\mu$, so does $\theta + \pi$. Thus the sign of the sine term in the soliton can be chosen positive or negative as we wish. In conclusion, we find two solitons

$$\zeta_j(x) = e^{i\theta_j}\sqrt{\frac{\omega + \epsilon\mu\sin(2\theta_j)}{2}}\,\mathrm{sech}\big(\sqrt{\omega + \epsilon\mu\sin(2\theta_j)}\,x\big), \tag{4.3}$$

for $j = 1, 2$, where $\theta_1, \theta_2 \in [0, 2\pi)$ are the two (not necessarily distinct) solutions to $\cos(2\theta_j) = \gamma/\mu$. As the equation is translation invariant, we may also shift the soliton by an arbitrary constant $a \in \mathbb{R}$. With this alteration, the soliton $\zeta_j(\cdot + a)$ remains a solution to (4.1). In [23], it was shown that for the θ_j which satisfies $\sin(2\theta_j) > 0$, the corresponding soliton ζ_j is exponentially stable. More precisely, the authors prove the following result.

Theorem 4.1.1 (Kapitula and Sandstede, [23])
Consider the PFNLS equation with initial condition $z(0) = z_0$. If ϵ is sufficiently small and $\sin(2\theta_j) > 0$, then the wave ζ is orbitally exponentially stable, i.e., if $\|z_0 - \zeta\|$ is sufficiently small, then there exists a constant $b > 0$ and a constant $a \in \mathbb{R}$ such that

$$|z(t,\cdot) - \zeta(\cdot + a)|_{L^2_x} \leq Ce^{-bt} \quad \text{for } t \geq 0.$$

Intuitively, this result shows that the soliton remains relatively unaffected by small perturbations. In the case of the stochastic PFNLS equation, the soliton is stochastically perturbed by the noise, and it would be interesting to investigate whether the soliton ζ is stable under these conditions.

4.2. Position correction

In order to gain a better understanding of the effect of the noise on the soliton ζ, we formulate an approach to track the displacement of the soliton due to the stochastic perturbation, following [15], where the stability of traveling pulses in the FitzHugh-Nagumo system is studied. More precisely, we consider the solution $z(t,\cdot)$ to the stochastic PFNLS equation with initial condition $z(0,x) = z_0$, i.e.

$$\begin{aligned} \mathrm{d}z &= (i\Delta z - i\nu z - \epsilon(\gamma z - \mu\bar{z}))\,\mathrm{d}t - \frac{1}{2}F_\Phi z\,\mathrm{d}t + 4i|z|^2 z\,\mathrm{d}t + z\,\mathrm{d}W \\ &= Az\,\mathrm{d}t + F_\Phi z\,\mathrm{d}t + 4i|z|^2 z\,\mathrm{d}t + z\,\mathrm{d}W, \end{aligned} \tag{4.4}$$

and we set

$$\tilde{z}_a(t,\cdot) := z(t,\cdot) - \zeta(\cdot + a(t)),$$

for some sufficiently smooth real-valued function a. Here, ζ is the soliton defined in (4.3) with the θ parameter satisfying both $\cos(2\theta) = \gamma/\mu$ and $\sin(2\theta) > 0$. $\tilde{z}_a$ is then the difference between the solution and a translated soliton. By linearizing (4.4) around the translated soliton, we find that it must satisfy

$$\begin{aligned} \mathrm{d}\tilde{z}_a(t,\cdot) =& (\mathrm{d}z(t,\cdot) - \dot{a}(t)\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a(t)))\,\mathrm{d}t \\ =& (Az(t,\cdot) - \frac{1}{2}F_\Phi z(t,\cdot) + 4i|z(t,\cdot)|^2 z(t,\cdot))\,\mathrm{d}t + z(t,\cdot)\,\mathrm{d}W(t,\cdot) - \dot{a}(t)\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a(t))\,\mathrm{d}t \\ =& (Az(t,\cdot) - \frac{1}{2}F_\Phi z(t,\cdot) + 4i|\zeta(\cdot + a(t))|^2\zeta(\cdot + a(t)) + 12i|\zeta(\cdot + a(t))|^2\tilde{z}_a(t,\cdot) + R_a(\tilde{z}_a(t,\cdot))\,\mathrm{d}t \\ &+ z(t,\cdot)\,\mathrm{d}W(t,\cdot) - \dot{a}(t)\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a(t)\,\mathrm{d}t, \end{aligned}$$

where R_a denotes the nonlinear term

$$R_a(z(t,\cdot)) := 4i(|z(t,\cdot)|^2 z(t,\cdot) - |\zeta(\cdot + a(t))|^2\zeta(\cdot + a(t))) - 12i|\zeta(\cdot + a(t))|^2\tilde{z}_a(t,\cdot).$$

By using that the soliton solves the time-independent version of the SPDE, i.e.

$$A\zeta(\cdot + a(t)) + 4i|\zeta(\cdot + a(t))|^2\zeta(\cdot + a(t)) = 0,$$

we obtain the simplification

$$\begin{aligned} \mathrm{d}\tilde{z}_a(t,\cdot) =& (\mathscr{L}_a\tilde{z}_a(t,\cdot) - \frac{1}{2}F_\Phi\tilde{z}_a(t,\cdot) - \frac{1}{2}F_\Phi\zeta(\cdot + a(t)) + R_a(\tilde{z}_a(t,\cdot)))\,\mathrm{d}t \\ &+ \tilde{z}_a(t,\cdot)\,\mathrm{d}W(t,\cdot) + \zeta(\cdot + a(t))\,\mathrm{d}W(t,\cdot) - \dot{a}(t)\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a(t))\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

Here, the operator $\mathscr{L}_a$ denotes the linearization of the PFNLS equation around the translated soliton $\zeta(\cdot + a(t))$, that is

$$\mathscr{L}_a z := Az + 12i|\zeta(\cdot + a(t))|^2 z.$$

In [23], it was shown that $\mathscr{L}_0$ generates a C_0-semigroup on $L^2(\mathbb{R})$, and the spectrum of $\mathscr{L}_0$, i.e., the linearization around ζ, was analyzed. We summarize the findings in Figure 4.1.

In particular, we see that 0 is an eigenvalue of $\mathscr{L}_0$. Indeed, by differentiating (4.2), we obtain

$$A\frac{\mathrm{d}\zeta}{\mathrm{d}x} + 12i|\zeta|^2\frac{\mathrm{d}\zeta}{\mathrm{d}x} = \mathscr{L}_0\frac{\mathrm{d}\zeta}{\mathrm{d}x} = 0,$$

which shows that $\frac{\mathrm{d}\zeta}{\mathrm{d}x}$ is an eigenvector of $\mathscr{L}_0$ with eigenvalue 0. This allows us to define a Riesz projection in $L^2(\mathbb{R})$ onto the corresponding eigenspace as

$$\Pi := \frac{1}{2\pi i}\int_\Gamma (\lambda I - \mathscr{L}_0)^{-1}\,d\lambda,$$

where Γ is a simple contour enclosing only the eigenvalue 0, and I is the identity on $L^2(\mathbb{R})$. We furthermore define a translation operator $\mathscr{T}_a$ on $L^2(\mathbb{R})$ for $a \in \mathbb{R}$ as

$$\mathscr{T}_a z = z(\cdot + a), \quad z \in L^2(\mathbb{R}).$$

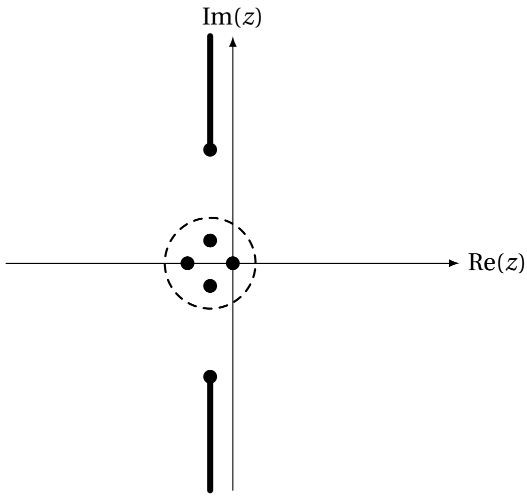

Figure 4.1: Spectrum of linearization of the PFNLS equation about the soliton. Four eigenvalues are an $O(\epsilon)$ distance removed from the origin. Two eigenvalues lie $O(\epsilon^2)$ close to the points $-\epsilon\gamma \pm i\nu$ on the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$. The spectrum is symmetric with respect to the reflection across the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$. This figure is adapted from [23, Figure 2]

and set

$$\Pi_a := \mathcal{T}_a \Pi \mathcal{T}_{-a}, \quad a \in \mathbb{R}.$$

With these definitions in place, we can formulate the approach to define a position correction process $a(t)$ that tracks the position of the soliton as it is perturbed by the noise. Ideally, the difference between the solution and the translated soliton should be minimal. We formulate this as

$$a(t) \in \mathrm{argmin}_{a\in\mathbb{R}} |\Pi_a[\tilde{z}_\phi(t,\cdot)]|_{L^2_x}.$$

A unique minimizer to this problem is, however, not ensured. Instead, we define approximations $a^m(t)$ via a random ODE, using the partial derivative of the norm that should be minimized w.r.t. the translation parameter. To this end, we compute the derivative of $\|\Pi_a[\tilde{z}_\phi(t,\cdot)]\|^2_{L^2_x}$ as

$$\begin{aligned}
\partial_a|\Pi_a[z(t,\cdot)-\zeta(\cdot+a)]|^2_{L^2_x} &= \partial_a|\Pi\,[z(t,\cdot-a)-\zeta]|^2_{L^2_x} \\
&= 2\mathrm{Re}\langle \Pi\,[z(t,\cdot-a)-\zeta]\,, \Pi\partial_a\,[z(t,\cdot-a)-\zeta]\rangle_{L^2_x} \\
&= -2\mathrm{Re}\langle \Pi\,[z(t,\cdot-a)-\zeta]\,, \Pi\,[\partial_x(z(t,\cdot-a)-\zeta)]\rangle - 2\mathrm{Re}\left\langle \Pi\,[z(t,\cdot-a)-\zeta]\,, \Pi\left[\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right]\right\rangle_{L^2_x} \\
&= -\int_{\mathbb{R}} \partial_x|\Pi\,[z(t,\cdot-a)-\zeta]\,(x)|^2\,\mathrm{d}x - 2\mathrm{Re}\left\langle \Pi\,[z(t,\cdot-a)-\zeta]\,, \frac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x} \\
&= -2\mathrm{Re}\left\langle \Pi\,[z(t,\cdot-a)-\zeta]\,, \frac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}.
\end{aligned}$$

We then define the approximations $a^m(t)$ for $m \gg 1$ via the following random ODE.

$$\dot{a}^m(t) = m\mathrm{Re}\left\langle \Pi_{a^m(t)}\left[\tilde{z}_{a^m(t)}(t,\cdot)\right], \frac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}, \tag{4.6}$$

with $a^m(0) = 0$, and thus $\dot{a}^m(0) = m\mathrm{Re}\left\langle \Pi\,[\tilde{z}_0]\,, \frac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}$. The definition of $a^m(t)$ via the random ODE (4.6) is motivated by the gradient descent method for finding local minima, and we expect that the translated soliton remains closer to the solution as m increases.

4.3. Leading-order dynamics

With the aim of studying the leading-order dynamics of the position correction defined in (4.6), we compute the differential of $\dot{a}^m(t)$. Starting from

$$\begin{aligned}\dot{a}^m(t) &= m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\tilde{z}_{a^m(t)}(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\\ &= m\operatorname{Re}\left\langle \Pi\left[\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x},\end{aligned}$$

we take the differential

$$\begin{aligned}\mathrm{d}\dot{a}^m(t) =& m\operatorname{Re}\left\langle \Pi\left[\mathrm{d}\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}\\ &- \dot{a}^m(t) m\operatorname{Re}\left\langle \Pi\left[\partial_x\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}\mathrm{d}t\\ =& m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[d\tilde{z}_{a^m(t)}(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\\ &- \dot{a}^m(t) m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\partial_x\tilde{z}_{a^m(t)}(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\mathrm{d}t,\end{aligned}$$

and we substitute (4.5), yielding

$$\begin{aligned}\mathrm{d}\dot{a}^m(t) =& m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[(\mathscr{L}_{a^m(t)}\tilde{z}_{a^m(t)}(t,\cdot) - \tfrac{1}{2}F_\Phi\tilde{z}_{a^m(t)}(t,\cdot) - \tfrac{1}{2}F_\Phi\zeta(\cdot + a^m(t)) + R_{a^m(t)}(\tilde{z}_{a^m(t)}(t,\cdot))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\mathrm{d}t\\ &+ m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\tilde{z}_{a^m(t)}(t,\cdot)\,\mathrm{d}W(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\\ &+ m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\zeta(\cdot + a^m(t))\,\mathrm{d}W(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\\ &- \dot{a}^m(t) m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\mathrm{d}t\\ &- \dot{a}^m(t) m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\partial_x\tilde{z}_{a^m(t)}(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\mathrm{d}t.\end{aligned}$$

Note that the fourth inner product can be computed as

$$\left\langle \Pi_{a^m(t)}\left[\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x} = \left\langle \Pi\left[\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x} = \left|\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right|^2_{L^2_x},$$

and we denote this constant by c. The term containing the linearization $\mathscr{L}_{a^m(t)}$ can also be simplified. To this end, we write

$$\left\langle \Pi_{a^m(t)}\left[\mathscr{L}_{a^m(t)}\tilde{z}_{a^m(t)}(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x} = \left\langle \mathscr{L}_0\Pi\left[\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x},$$

and use that Π projects onto the span of $\frac{\mathrm{d}\zeta}{\mathrm{d}x}$, so that we have

$$\Pi\left[\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right] = \left\langle \Pi\left[\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}\tfrac{\mathrm{d}\zeta}{\mathrm{d}x},$$

which gives

$$\begin{aligned}\left\langle \Pi_{a^m(t)}\left[\mathscr{L}_{a^m(t)}\tilde{z}_{a^m(t)}(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x} &= \left\langle \mathscr{L}_0\left\langle \Pi\left[\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}, \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}\\ &= \left\langle \Pi\left[\tilde{z}_{a^m(t)}(t,\cdot - a^m(t))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x}\left\langle \mathscr{L}_0\tfrac{\mathrm{d}\zeta}{\mathrm{d}x}, \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L^2_x} = 0.\end{aligned}$$

For the last equality we have used that $\frac{\mathrm{d}\zeta}{\mathrm{d}x}$ is an eigenvector with eigenvalue 0 of $\mathscr{L}_0$. The expression for the differential of $\dot{a}^m(t)$ now simplifies to

$$\begin{aligned}\mathrm{d}\dot{a}^m(t) =& m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[(-\tfrac{1}{2}F_\Phi\tilde{z}_{a^m(t)}(t,\cdot) - \tfrac{1}{2}F_\Phi\zeta(\cdot + a^m(t)) + R_{a^m(t)}(\tilde{z}_{a^m(t)}(t,\cdot))\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\mathrm{d}t\\ &+ m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\tilde{z}_{a^m(t)}(t,\cdot)\,\mathrm{d}W(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\\ &+ m\operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\zeta(\cdot + a^m(t))\,\mathrm{d}W(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\\ &- \dot{a}^m(t) m\left(c + \operatorname{Re}\left\langle \Pi_{a^m(t)}\left[\partial_x\tilde{z}_{a^m(t)}(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}(\cdot + a^m(t))\right\rangle_{L^2_x}\right)\mathrm{d}t.\end{aligned}$$

To gain a better understanding of the dynamics of the position correction, it is interesting to consider a simplification of the previous equation. In case the noise term is small, we expect that the soliton is not perturbed much, and we expect $\tilde{z}$ and $a^m(t)$ to be small. We therefore consider the following equation, describing only the leading-order dynamics

$$\mathrm{d}\dot{a}_0^m(t) = -m\,\mathrm{Re}\left\langle \Pi\left[\tfrac{1}{2}F_\Phi\zeta\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L_x^2}\mathrm{d}t + m\,\mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(t,\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L_x^2} - cm\dot{a}_0^m(t)\,\mathrm{d}t, \tag{4.7}$$

with $a_0^m(0) = 0$, and thus $\dot{a}_0^m(0) = m\mathrm{Re}\left\langle \Pi\left[\tilde{z}_0\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L_x^2}$. This linear SDE is solved by the mild solution formula, yielding

$$\dot{a}_0^m(t) = \dot{a}_0^m(0)e^{-cmt} - \frac{\alpha}{c}(1-e^{-cmt}) + m\int_0^t e^{-cm(t-\tilde{t})}\,\mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(\tilde{t},\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L_x^2},$$

where

$$\alpha = \mathrm{Re}\left\langle \Pi\left[\tfrac{1}{2}F_\Phi\zeta\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L_x^2}.$$

Upon integrating, and using that $a^m(0) = 0$, we obtain the following equation for the leading-order position correction $a_0^m(t)$

$$a_0^m(t) = \frac{\dot{a}_0^m(0) + \alpha/c}{cm}(1-e^{-cmt}) - \frac{\alpha t}{c} + m\int_0^t\int_0^{t'} e^{-cm(t'-t'')}\,\mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(t'',\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle \mathrm{d}t'.$$

We now use that

$$\begin{aligned} &m\int_0^t\int_0^{t'} e^{-cm(t'-t'')}\,\mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(t'',\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle \mathrm{d}t' \\ &= \int_0^t\int_0^{t-t''} me^{-cmt'''}\,\mathrm{d}t'''\,\mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(t'',\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle \\ &= \tfrac{1}{c}\int_0^t (1-e^{-cm(t-t'')})\,\mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(t'',\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle, \end{aligned}$$

and we conclude, using $\dot{a}^m(0) = m\,\mathrm{Re}\left\langle \Pi\left[\tilde{z}_0\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle$, that

$$a_0^m(t) = \left(\frac{1}{c}\mathrm{Re}\left\langle \Pi\left[\tilde{z}_0\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle + \frac{\alpha}{c^2 m}\right)(1-e^{-cmt}) - \frac{\alpha t}{c} + \frac{1}{c}\int_0^t (1-e^{-cm(t-t')})\,\mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(t',\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle.$$

Upon taking the limit $m \to \infty$, corresponding to an immediate position correction, we find

$$a_0^\infty(t) = \frac{1}{c}\mathrm{Re}\left\langle \Pi\left[\tilde{z}_0\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle - \frac{1}{c}\mathrm{Re}\left\langle \Pi\left[\tfrac{1}{2}F_\Phi\zeta\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle_{L_x^2} t + \frac{1}{c}\int_0^t \mathrm{Re}\left\langle \Pi\left[\zeta\,\mathrm{d}W(t',\cdot)\right], \tfrac{\mathrm{d}\zeta}{\mathrm{d}x}\right\rangle.$$

This shows that, in leading-order, and in the limit of immediate correction, the position correction consists of a primary part due to the initial difference with the soliton. The position furthermore shifts constantly in the direction of the Itô drift term $-\frac{1}{2}F_\Phi$ and experiences stochastic fluctuations.

4.4. Outlook

So far we have been rather imprecise regarding for example the meaning of the 'reduced' equation, and the conclusions that we can draw from the computations in the previous sections. In future research, it would be interesting to formalize the previous computations into rigorous results. For instance, one question would be whether the RODE definition of the position correction, Equation (4.6), effectively tracks the position of the perturbed soliton. A way to do so would be to prove a second moment estimate on the difference $\tilde{z}_{a^m(t)}$ between the solution and the translated soliton. This could also provide an answer to the question of whether the soliton is (exponentially) stable under the influence of stochastic perturbations. Indeed, an estimate controlling the second moment of $\tilde{z}_{a^m(t)}$ could be interpreted as a generalization of Theorem 4.1.1 to the stochastic setting.

If successful, such a result would, like [15], set another example of a deterministic stability theorem that is lifted to the stochastic setting. It seems also that the approach to the calculations set out in the previous sections is not specific to the equation that is studied. As can be seen in Section 4.3, the computations deal with properties of the soliton and linearized operator that fall into a more general framework of traveling waves in nonlinear equations. It could be interesting to study the stability of (standing) waves or other patterns in stochastic equations in such a more general framework.

Conclusions

In this thesis, the following variation on the nonlinear Schrödinger equation with multiplicative noise was analyzed:

$$\mathrm{d}z = (i\Delta z - i\nu z - \epsilon(\gamma z - \mu\overline{z}))\,\mathrm{d}t + 4i|z|^2 z\,\mathrm{d}t - i(z \circ \mathrm{d}W) \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}^+.$$

This parametrically-forced NLS equation includes linear loss in the modeling of pulse propagation through optical fibers, which is compensated via phase-sensitive amplification. In systems of fiber-optic communication, this phase-sensitive amplification serves to prevent signal loss in long-distance communication. The multiplicative noise models random perturbation of the electric field in the optical fiber.

It was shown in this thesis that the stochastic PFNLS equation admits global unique mild solutions for initial data in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ that take values in the spaces

$$L^r\left(\Omega; C([0,T]; L^2(\mathbb{R})) \cap L^r(0,T; L^p(\mathbb{R}))\right),$$

and

$$L^r\left(\Omega; C([0,T]; H^1(\mathbb{R})) \cap L^r(0,T; W^{1,p}(\mathbb{R}))\right),$$

for initial data in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ respectively. The constants r and p are suitably chosen depending on the regularity of the noise.

The proof of the existence and uniqueness results forms a combined exposition of works on the nonlinear Schrödinger equation with multiplicative noise by de Bouard and Debussche [3, 4]. Minor adaptations were required to fit the arguments to the parametrically-forced setting. The proofs are detailed, and we thus afford a more accessible presentation of the works by de Bouard and Debussche. In particular, we demonstrate the technical details behind a regularization procedure used to justify an application of Itô's formula to mild solutions of the stochastic PFNLS equation.

The proof of the existence and uniqueness results is based on a fixed-point argument, for which control of the convolution with the semigroup associated with the linear parametrically-forced Schrödinger equation is essential. In particular, the fixed-point argument uses space-time estimates on the semigroup, called Strichartz estimates, which hold for equations that satisfy a suitable dispersive estimate. These dispersive properties of the semigroup, which we proved via its Fourier representation, constitute the main contribution of this thesis.

The existence and uniqueness results hold under the assumption that the noise is suitably regular. For applications, it would be interesting to investigate if this regularity assumption could be relaxed. It seems, however, that this would require a different proof strategy than the fixed-point argument employed in this thesis. Another question that is natural to ask is whether the results for initial data in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ can be interpolated to obtain existence and uniqueness for initial data in the interpolation space $H^{\mathfrak{s}}(\mathbb{R})$, with $\mathfrak{s} \in (0,1)$. Further investigation might also consider initial data in the higher-order spaces $H^{\mathfrak{s}}(\mathbb{R})$, for $\mathfrak{s} = 2, 3, 4, \ldots$.

In the final part of this thesis, we discussed the stability of solitary standing wave solutions of the PFNLS equation under the influence of multiplicative noise. We considered an approach to track the displacement of a soliton affected by stochastic forcing. A more formal treatment of the problem is required to turn the presented informal computations into rigorous results. In particular, it would be interesting to obtain an estimate demonstrating the validity of the position correction. Future research might also explore whether stochastic perturbation of patterns in nonlinear equations can be analyzed in a more general framework.

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A

Technical lemmas

Lemma A.0.1
If $\alpha_\xi \in (0, \frac{\epsilon^2\mu^2}{(\xi^2+\nu)^2})$ *and* $\alpha_{\xi,t} \in \left((\xi^2+\nu)t - \frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)\sqrt{1-\alpha_\xi}}, (\xi^2+\nu)t\right)$, *then*

1. $\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2 t}{2(\xi^2+\nu)}\frac{\sin(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} \le C|t|$;
2. $\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2 t^2}{2}\frac{\text{sinc}'(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} \le Ct^2$;
3. $\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2 t^2}{2}\frac{\text{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} \le C|t| + Ct^2$.

Proof. Note that $0 < \alpha_\xi < \frac{\epsilon^2\mu^2}{\nu^2} < 1$, so that $\sqrt{1-\frac{\epsilon^2\mu^2}{\nu}} < \sqrt{1-\alpha_\xi} < 1$.

1.

$$\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2|t|}{2(\xi^2+\nu)}\frac{\sin(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} = \frac{\epsilon^2\mu^2|t|}{2}\left|\mathscr{F}^{-1}\left\{\frac{\sin(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} \le C|t|\left|\frac{\sin(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right|_{L_\xi^1}$$
$$\le C|t|\left|\frac{1}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right|_{L_\xi^1} \le C|t|\left|\frac{1}{(\xi^2+\nu)\sqrt{1-\frac{\epsilon^2\mu^2}{\nu}}}\right|_{L_\xi^1} \le C|t|.$$

2.

$$\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2 t^2}{2}\frac{\text{sinc}'(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} = \frac{\epsilon^2\mu^2 t^2}{2}\left|\mathscr{F}^{-1}\left\{\frac{\text{sinc}'(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty}$$
$$\le Ct^2\left|\frac{\text{sinc}'(\alpha_{\xi,t})}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right|_{L_\xi^1} \le Ct^2\left|\frac{1}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}}\right|_{L_\xi^1}$$
$$\le Ct^2\left|\frac{1}{(\xi^2+\nu)\sqrt{1-\frac{\epsilon^2\mu^2}{\nu}}}\right|_{L_\xi^1} \le Ct^2\left|\frac{1}{(\xi^2+\nu)\sqrt{1-\frac{\epsilon^2\mu^2}{\nu}}}\right|_{L_\xi^1}$$
$$\le Ct^2.$$

3.

$$\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2t^2}{2}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} = \frac{\epsilon^2\mu^2t^2}{2}\left|\mathscr{F}^{-1}\left\{\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} \le Ct^2\left|\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right|_{L_\xi^1}$$
$$= Ct^2\int_{\mathbb{R}}\frac{|\operatorname{sinc}'(\alpha_{\xi,t})|}{\sqrt{1-\alpha_\xi}}\,\mathrm{d}\xi.$$

To control the $|\operatorname{sinc}'(\alpha_{\xi,t})|$ term, we will use that $|\operatorname{sinc}'(x)| < \frac{2}{|x|}$. But to avoid problems around $\alpha_{\xi,t} = 0$, we split the integral into two parts and use that $|\operatorname{sinc}'(x)| < 1$ around $x = 0$. Note that as $|\xi|$ tends to infinity, the lower bound on $\alpha_{\xi,t}$ tends to infinity as well. In particular, there exists a constant $M > 0$, independent of t, so that $\alpha_{\xi,t} \ge |t|$ if $|\xi| > M$. Additionally, we choose M large enough so that $(\xi^2 + \nu)\sqrt{1-\frac{\epsilon^2\mu^2}{\nu^2}} > \frac{\epsilon^2\mu^2}{2(\xi^2+\nu)}$, which we need later on. We then split up the integral as follows

$$\begin{aligned}
\left|\mathscr{F}^{-1}\left\{\frac{\epsilon^2\mu^2t^2}{2}\frac{\operatorname{sinc}'(\alpha_{\xi,t})}{\sqrt{1-\alpha_\xi}}\right\}\right|_{L_x^\infty} &\le Ct^2\int_{-M}^{M}\frac{|\operatorname{sinc}'(\alpha_{\xi,t})|}{\sqrt{1-\alpha_\xi}}\,\mathrm{d}\xi + Ct^2\int_{\mathbb{R}\setminus[-M,M]}\frac{|\operatorname{sinc}'(\alpha_{\xi,t})|}{\sqrt{1-\alpha_\xi}}\,\mathrm{d}\xi\\
&\le Ct^2\int_{-M}^{M}\frac{1}{\sqrt{1-\alpha_\xi}}\,\mathrm{d}\xi + Ct^2\int_{\mathbb{R}\setminus[-M,M]}\frac{1}{\alpha_{\xi,t}\sqrt{1-\alpha_\xi}}\,\mathrm{d}\xi\\
&\le Ct^2\int_{-M}^{M}\frac{1}{\sqrt{1-\frac{\epsilon^2\mu^2}{\nu^2}}}\,\mathrm{d}\xi\\
&\quad + C|t|\int_{\mathbb{R}\setminus[-M,M]}\frac{1}{(\xi^2+\nu-\frac{\epsilon^2\mu^2}{2(\xi^2+\nu)\sqrt{1-\alpha_\xi}})\sqrt{1-\alpha_\xi}}\,\mathrm{d}\xi\\
&\le Ct^2 + C|t|\int_{\mathbb{R}\setminus[-M,M]}\frac{1}{(\xi^2+\nu)\sqrt{1-\alpha_\xi}-\frac{\epsilon^2\mu^2}{2(\xi^2+\nu)}}\,\mathrm{d}\xi\\
&\le Ct^2 + C|t|\int_{\mathbb{R}\setminus[-M,M]}\frac{1}{(\xi^2+\nu)\sqrt{1-\frac{\epsilon^2\mu^2}{\nu^2}}-\frac{\epsilon^2\mu^2}{2(\xi^2+\nu)}}\,\mathrm{d}\xi \le C|t| + Ct^2. \quad\square
\end{aligned}$$

Lemma A.0.2
Let $\{S(t)\}_{t\ge0}$ be the semigroup associated to the forced Schrödinger equation. Then:

$$|S^*(t_1)S(t_2)z_0|_{L_x^p} \le C|t_2-t_1|^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L_x^{p'}}, \quad \textit{for all } z_0 \in L^{p'} \textit{ and } t_1, t_2 > 0,$$

and

$$|S(t_1)S^*(t_2)z_0|_{L_x^p} \le C|t_2-t_1|^{-(\frac{1}{2}-\frac{1}{p})}|z_0|_{L_x^{p'}}, \quad \textit{for all } z_0 \in L^{p'} \textit{ and } t_1, t_2 > 0.$$

Proof. Using Parseval's theorem and the Fourier representation of the semigroup of Theorem 2.1.1 we can write for $f, g \in L^2(\mathbb{R})$

$$\langle S(t)f, g\rangle_{L_x^2} = \left\langle e^{t\hat{A}(\xi)}\begin{bmatrix}\hat{f}_1(\xi)\\ \hat{f}_2(\xi)\end{bmatrix}, \begin{bmatrix}\hat{g}_1(\xi)\\ \hat{g}_2(\xi)\end{bmatrix}\right\rangle_{L_\xi^2} = \left\langle\begin{bmatrix}\hat{f}_1(\xi)\\ \hat{f}_2(\xi)\end{bmatrix}, (e^{t\hat{A}(\xi)})^T\begin{bmatrix}\hat{g}_1(\xi)\\ \hat{g}_2(\xi)\end{bmatrix}\right\rangle_{L_\xi^2} = \langle f, S(t)^*g\rangle_{L_x^2},$$

which shows that the adjoint $S^*(t)$ acts as the matrix $(e^{t\hat{A}(\xi)})^T$ in the Fourier space. Therefore, $S^*(t_1)S(t_2)$ and $S(t_1)S^*(t_2)$ act in the Fourier space as $(e^{t_1\hat{A}(\xi)})^Te^{t_2\hat{A}(\xi)}$ and $e^{t_1\hat{A}(\xi)}(e^{t_2\hat{A}(\xi)})^T$ respectively. One can verify by direct computation that

$$e^{\epsilon\gamma(t_1+t_2)}(e^{t_1\hat{A}(\xi)})^T e^{t_2\hat{A}(\xi)} = \cos(\phi(\xi)(t_2-t_1))\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{A.1}$$

$$+(t_2-t_1)(\xi^2+\nu)\,\mathrm{sinc}(\phi(\xi)(t_2-t_1))\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \tag{A.2}$$

$$+\epsilon\mu(t_1+t_2)\,\mathrm{sinc}(\phi(\xi)(t_1+t_2))\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{A.3}$$

$$+\epsilon\mu\frac{(\xi^2+\nu)(\cos(\phi(\xi)(t_2-t_1))-\cos(\phi(\xi)(t_1+t_2)))}{(\xi^2+\nu)^2-\epsilon^2\mu^2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \tag{A.4}$$

$$+\epsilon^2\mu^2\frac{\cos(\phi(\xi)(t_2-t_1))-\cos(\phi(\xi)(t_1+t_2))}{(\xi^2+\nu)^2-\epsilon^2\mu^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{A.5}$$

And

$$e^{\epsilon\gamma(t_1+t_2)}e^{t_1\hat{A}(\xi)}(e^{t_2\hat{A}(\xi)})^T = \cos(\phi(\xi)(t_2-t_1))\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$+(t_2-t_1)(\xi^2+\nu)\,\mathrm{sinc}(\phi(\xi)(t_2-t_1))\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$+\epsilon\mu(t_1+t_2)\,\mathrm{sinc}(\phi(\xi)(t_1+t_2))\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$-\epsilon\mu\frac{(\xi^2+\nu)(\cos(\phi(\xi)(t_2-t_1))-\cos(\phi(\xi)(t_1+t_2)))}{(\xi^2+\nu)^2-\epsilon^2\mu^2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$+\epsilon^2\mu^2\frac{\cos(\phi(\xi)(t_2-t_1))-\cos(\phi(\xi)(t_1+t_2))}{(\xi^2+\nu)^2-\epsilon^2\mu^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

We prove the estimate for $S^*(t_1)S(t_2)$, the other is then analogous. The matrix entries in (A.1), (A.2) and (A.3) are also found in $e^{t\hat{A}(\xi)}$, only with t replaced by t_2-t_1 or t_1+t_2. The entries of the remaining matrices in (A.4) and (A.5) are both in $L^1_\xi \cap L^\infty_\xi$. By applying Lemma 2.3.1 to these terms, we obtain the following L^∞-estimate

$$|\mathscr{F}^{-1}\{((e^{t_1\hat{A}(\xi)})^T e^{t_2\hat{A}(\xi)})_{ij}\}|_{L^\infty_x} \le Ce^{-\epsilon\gamma(t_1+t_2)}(1/\sqrt{|t_2-t_1|}+1+|t_2-t_1|+|t_2-t_1|^2+|t_1+t_2|^2),$$

with $i,j\in\{1,2\}$. We treat the powers of $|t_2-t_1|$ first, and distinguish three cases. First assume that $|t_2-t_1|\le 1$, then

$$e^{-\epsilon\gamma(t_1+t_2)}(1/\sqrt{|t_2-t_1|}+1+|t_2-t_1|+|t_2-t_1|^2)\le 4/\sqrt{|t_2-t_1|}.$$

If $t_2-t_1>1$, then

$$e^{-\epsilon\gamma(t_1+t_2)}(1/\sqrt{|t_2-t_1|}+1+|t_2-t_1|+|t_2-t_1|^2)\le e^{-2\epsilon\gamma t_1}e^{-\epsilon\gamma(t_2-t_1)}4|t_2-t_1|^2\le 4e^{-\epsilon\gamma(t_2-t_1)}|t_2-t_1|^2,$$

which we can bound by $\tilde{C}/\sqrt{|t_2-t_1|}$, upon choosing $\tilde{C}$ large enough. Similarly, when $t_1-t_2>1$:

$$e^{-\epsilon\gamma(t_1+t_2)}(1/\sqrt{|t_2-t_1|}+1+|t_2-t_1|+|t_2-t_1|^2)\le e^{-2\epsilon\gamma t_2}e^{-\epsilon\gamma(t_1-t_2)}4|t_2-t_1|^2\le 4e^{-\epsilon\gamma(t_1-t_2)}|t_2-t_1|^2,$$

which again is bounded by $\tilde{C}/\sqrt{|t_2-t_1|}$. We apply the same reasoning to the remaining term

$$e^{-\epsilon\gamma(t_1+t_2)}|t_1+t_2|^2\le\tilde{C}/\sqrt{|t_1+t_2|}.$$

Since $t_1,t_2\ge 0$, we have $|t_1+t_2|=t_1+t_2\ge|t_2-t_1|$ and it follows that

$$e^{-\epsilon\gamma(t_1+t_2)}|t_1+t_2|^2\le\tilde{C}/\sqrt{|t_2-t_1|},$$

so that we finally obtain the L^∞-estimate

$$|\mathscr{F}^{-1}\{((e^{t_1\hat{A}(\xi)})^T e^{t_2\hat{A}(\xi)})_{ij}\}|_{L^\infty_x}\le C/\sqrt{|t_2-t_1|}.$$

We can apply the same reasoning as in the proof of Proposition 2.3.3 to show that the operator norm of $S^*(t_1)S(t_2)$ on L^2 is uniformly bounded w.r.t. t_1 and t_2. The desired L^p-estimate then follows by interpolation, as in the proof of Theorem 2.3.4. □

Proof of Lemma 1.2.15.

1. Let $\epsilon > 0$. We start by writing
$$|(I-\epsilon\Delta)^{-1}f|_{W_x^{k+2,p}} = |(I-\epsilon\Delta)^{-1}f|_{W_x^{k,p}} + |\partial_x(I-\epsilon\Delta)^{-1}f|_{W_x^{k,p}} + |\partial_x^2(I-\epsilon\Delta)^{-1}f|_{W_x^{k,p}},$$
and we treat the terms via their associated multiplier symbol. These are
$$m_1(\xi) := \frac{1}{1+\epsilon\xi^2},$$
$$m_2(\xi) := \frac{i\xi}{1+\epsilon\xi^2},$$
and
$$m_3(\xi) := \frac{-\xi^2}{1+\epsilon\xi^2},$$
for $(I-\epsilon\Delta)^{-1}$, $\partial_x(I-\epsilon\Delta)^{-1}$ and $\partial_x^2(I-\epsilon\Delta)^{-1}$ respectively. The symbols $m_1(\xi)$ and $m_2(\xi)$ are both smooth and uniformly bounded in ξ. Their derivatives are
$$\frac{\mathrm{d}m_1}{\mathrm{d}\xi}(\xi) := 2\epsilon\frac{\xi}{(1+\epsilon\xi^2)^2},$$
$$\frac{\mathrm{d}m_2}{\mathrm{d}\xi}(\xi) := i\frac{1-\epsilon\xi^2}{(1+\epsilon\xi^2)^2},$$
and
$$\frac{\mathrm{d}m_3}{\mathrm{d}\xi}(\xi) := -2\frac{\xi}{(1+\epsilon\xi^2)^2},$$
from which it follows that the functions $\xi \to |\xi|\frac{\mathrm{d}m_1}{\mathrm{d}\xi}(\xi)$, $\xi \to |\xi|\frac{\mathrm{d}m_2}{\mathrm{d}\xi}(\xi)$ and $\xi \to |\xi|\frac{\mathrm{d}m_3}{\mathrm{d}\xi}(\xi)$ are also smooth and uniformly bounded in ξ. It then follows from the Mikhlin multiplier theorem (Theorem 1.2.14) that m_1, m_2 and m_3 are L^p multipliers. As a consequence, the multipliers T_{m_1}, T_{m_2} and T_{m_3} are bounded on $W_x^{k,p}$ (Remark 1.2.10) and we find
$$|(I-\epsilon\Delta)^{-1}f|_{W_x^{k+2,p}} \le C_{\epsilon,p}|f|_{W_x^{k,p}},$$
as desired.

2. In the previous part of the proof it was shown that $(I-\epsilon\Delta)^{-1}$ is an L^p multiplier. We now show that its operator norm is uniformly bounded in ϵ. To this end, we note that
$$|m_1(\xi)| \le \frac{1}{1+\epsilon\xi^2} \le 1,$$
and
$$|\xi|\left|\frac{\mathrm{d}m_1}{\mathrm{d}\xi}(\xi)\right| \le 2\frac{\epsilon\xi^2}{(1+\epsilon\xi^2)^2} \le 2.$$
Hence, both the symbol and $|\xi|\left|\frac{\mathrm{d}m_1}{\mathrm{d}\xi}(\xi)\right|$ are uniformly bounded in ϵ. It then follows from the Mikhlin multiplier theorem that operator norm of $(I-\epsilon\Delta)^{-1}$ is uniformly bounded in ϵ.

3. We show that $((I-\epsilon\Delta)^{-1}-I)f \to 0$ in L^p, the result then follows by considering the derivatives of f up to order k. Let therefore $\delta > 0$. By density of the Schwartz space in L^p, we can choose $\phi \in \mathscr{S}(\mathbb{R})$ with $|f-\phi|_{L^p} < \delta/2$, and we can write
$$\begin{aligned}|((I-\epsilon\Delta)^{-1}-I)f|_{L_x^p} &\le |((I-\epsilon\Delta)^{-1}-I)(f-\phi)|_{L_x^p} + |((I-\epsilon\Delta)^{-1}-I)\phi|_{L_x^p}\\ &\le C_p|f-\phi|_{L_x^p} + |((I-\epsilon\Delta)^{-1}-I)\phi|_{L_x^p} < \delta/2 + |((I-\epsilon\Delta)^{-1}-I)\phi|_{L_x^p}.\end{aligned}$$
In order to deal with the second term, we write $(I-\epsilon\Delta)^{-1}-I = \mathscr{F}_\xi^{-1}\frac{\epsilon\xi^2}{1+\epsilon\xi^2}\mathscr{F}_\xi$, and find
$$g_\epsilon(x) := (((I-\epsilon\Delta)^{-1}-I)\phi)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{\epsilon\xi^2}{1+\epsilon\xi^2}\hat{\phi}(\xi)\,\mathrm{d}\xi,$$

with $\hat{\phi} \in \mathscr{S}(\mathbb{R})$. We now estimate

$$|g_\epsilon|^p_{L^p_x} = \int_{\mathbb{R}} |g_\epsilon(x)|^p \,\mathrm{d}x = \int_{\mathbb{R}} |g_\epsilon(x)|^{p-1} |g_\epsilon(x)| \,\mathrm{d}x \le |g_\epsilon|^{p-1}_{L^\infty_x} |g_\epsilon|_{L^1_x} \le \left| \frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right|_{L^1_\xi} |g_\epsilon|_{L^1_x}.$$

We can dominate the expression in the L^1_ξ-norm above as

$$\left| \frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right| \le |\hat{\phi}(\xi)|,$$

for all $\epsilon > 0$ and $\xi \in \mathbb{R}$. We also have point-wise convergence

$$\left| \frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right| \to 0$$

as $\epsilon \downarrow 0$ for all $\xi \in \mathbb{R}$. Furthermore, $\hat{\phi} \in L^1(\mathbb{R})$, since $\mathscr{S}(\mathbb{R}) \subseteq L^p(\mathbb{R})$ for all $p \ge 1$. Therefore, the dominating function $\xi \mapsto |\hat{\phi}(\xi)|$ is integrable and we conclude by the dominated convergence theorem that

$$\left| \frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right|_{L^1_\xi} \to 0 \quad \text{as} \quad \epsilon \downarrow 0. \tag{A.6}$$

To estimate the remaining factor $|g_\epsilon|_{L^1_x}$, we rewrite g_ϵ as

$$g_\epsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \,\mathrm{d}\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \frac{1}{1+x^2} (1 - \frac{\mathrm{d}}{\mathrm{d}\xi}) e^{ix\xi} \,\mathrm{d}\xi,$$

and integrate by parts, which gives

$$g_\epsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ix\xi}}{1+x^2} \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - 1 \right) \left(\frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right) \mathrm{d}\xi.$$

Note that there is no boundary term, since $\hat{\phi} \in \mathscr{S}(\mathbb{R})$. We proceed by writing

$$\begin{aligned}
|g_\epsilon|_{L^1_x} &= \int_{\mathbb{R}} \left| \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ix\xi}}{1+x^2} \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - 1 \right) \left(\frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right) \mathrm{d}\xi \right| \mathrm{d}x \\
&\le \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+x^2} \int_{\mathbb{R}} \left| e^{ix\xi} \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - 1 \right) \left(\frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right) \right| \mathrm{d}\xi \,\mathrm{d}x \\
&\le \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+x^2} \,\mathrm{d}x \int_{\mathbb{R}} \left| \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - 1 \right) \left(\frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right) \right| \mathrm{d}\xi \\
&= \frac{1}{2} \int_{\mathbb{R}} \left| \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - 1 \right) \left(\frac{\epsilon\xi^2}{1+\epsilon\xi^2} \hat{\phi}(\xi) \right) \right| \mathrm{d}\xi \\
&\le \frac{1}{2} \int_{\mathbb{R}} \left| \frac{\epsilon\xi^2}{1+\epsilon\xi^2} (\hat{\phi}(\xi) - \frac{\mathrm{d}\hat{\phi}}{\mathrm{d}\xi}(\xi)) \right| \mathrm{d}\xi + \frac{1}{2} \int_{\mathbb{R}} \left| \frac{2\epsilon\xi}{(1+\epsilon\xi^2)^2} \hat{\phi}(\xi) \right| \mathrm{d}\xi.
\end{aligned}$$

It follows from (A.6) and the fact that $\hat{\phi} - \frac{\mathrm{d}\hat{\phi}}{\mathrm{d}\xi} \in \mathscr{S}(\mathbb{R})$ that the first integral in the remaining expression converges to 0 as $\epsilon \downarrow 0$. The same follows for the second integral via the same reasoning and the bound

$$\left| \frac{2\epsilon\xi}{(1+\epsilon\xi^2)^2} \hat{\phi}(\xi) \right| \le |\hat{\phi}(\xi)|.$$

We conclude that $|g_\epsilon|_{L^p_x} = |((I - \epsilon\Delta)^{-1} - I)\phi|_{L^p_x} \to 0$ as $\epsilon \downarrow 0$. We then choose ϵ small enough so that

$$|((I - \epsilon\Delta)^{-1} - I)\phi|_{L^p_x} \le \delta/2,$$

and we find that

$$|((I - \epsilon\Delta)^{-1} - I)f|_{L^p_x} < \delta.$$

Since δ was arbitrary, the result follows. □

B

Fourier transforms

Lemma B.0.1
For $t \geq 0$, we have

$$\mathscr{F}^{-1}\{\cos((\xi^2+\nu)t)\}(x) = \frac{1}{2\sqrt{t}}(\cos(x^2/4t+\nu t)+\sin(x^2/4t+\nu t))$$

Proof.

$$\begin{aligned}
\mathscr{F}^{-1}\{\cos((\xi^2+\nu)t)\}(x) &= \mathscr{F}^{-1}\left\{\frac{1}{2}e^{i(\xi^2+\nu)t}+\frac{1}{2}e^{-i(\xi^2+\nu)t}\right\}(x)\\
&= \frac{1}{2}\mathscr{F}^{-1}\left\{e^{-2i\sqrt{\nu}t\xi}e^{i(\xi+\sqrt{\nu})^2 t}\right\}(x)+\frac{1}{2}\mathscr{F}^{-1}\left\{e^{2i\sqrt{\nu}t\xi}e^{-i(\xi+\sqrt{\nu})^2 t}\right\}(x)\\
&= \frac{1}{2}\mathscr{F}^{-1}\left\{e^{i(\xi+\sqrt{\nu})^2 t}\right\}(x-2\sqrt{\nu}t)+\frac{1}{2}\mathscr{F}^{-1}\left\{e^{-i(\xi+\sqrt{\nu})^2 t}\right\}(x+2\sqrt{\nu}t)\\
&= \frac{1}{2}e^{-i\sqrt{\nu}x}\mathscr{F}^{-1}\left\{e^{i\xi^2 t}\right\}(x-2\sqrt{\nu}t)+\frac{1}{2}e^{-i\sqrt{\nu}x}\mathscr{F}^{-1}\left\{e^{-i\xi^2 t}\right\}(x+2\sqrt{\nu}t)\\
&= \frac{1}{2}e^{-i\sqrt{\nu}x}(\frac{1}{2}+\frac{i}{2})\frac{1}{\sqrt{t}}e^{-i(x-2\sqrt{\nu}t)^2/4t}+\frac{1}{2}e^{-i\sqrt{\nu}x}(\frac{1}{2}-\frac{i}{2})\frac{1}{\sqrt{t}}e^{i(x+2\sqrt{\nu}t)^2/4t}\\
&= \frac{1}{\sqrt{t}}(\frac{1}{4}+\frac{i}{4})e^{-i(x^2/4t+\nu t)}+\frac{1}{\sqrt{t}}(\frac{1}{4}-\frac{i}{4})e^{i(x^2/4t+\nu t)}\\
&= \frac{1}{2\sqrt{t}}(\cos(x^2/4t+\nu t)+\sin(x^2/4t+\nu t))
\end{aligned}$$

□

Lemma B.0.2
For $t \geq 0$, we have

$$\mathscr{F}^{-1}\{\sin((\xi^2+\nu)t)\}(x) = \frac{1}{2\sqrt{t}}(\cos(x^2/4t+\nu t)-\sin(x^2/4t+\nu t))$$

Proof.

$$
\begin{aligned}
\mathscr{F}^{-1}\{\sin((\xi^2+\nu)t)\}(x) &= \mathscr{F}^{-1}\left\{\frac{1}{2i}e^{i(\xi^2+\nu)t} - \frac{1}{2i}e^{-i(\xi^2+\nu)t}\right\}(x) \\
&= \frac{1}{2i}\mathscr{F}^{-1}\left\{e^{-2i\sqrt{\nu}t\xi}e^{i(\xi+\sqrt{\nu})^2t}\right\}(x) - \frac{1}{2i}\mathscr{F}^{-1}\left\{e^{2i\sqrt{\nu}t\xi}e^{-i(\xi+\sqrt{\nu})^2t}\right\}(x) \\
&= \frac{1}{2i}\mathscr{F}^{-1}\left\{e^{i(\xi+\sqrt{\nu})^2t}\right\}(x-2\sqrt{\nu}t) - \frac{1}{2i}\mathscr{F}^{-1}\left\{e^{-i(\xi+\sqrt{\nu})^2t}\right\}(x+2\sqrt{\nu}t) \\
&= \frac{1}{2i}e^{-i\sqrt{\nu}x}\mathscr{F}^{-1}\left\{e^{i\xi^2t}\right\}(x-2\sqrt{\nu}t) - \frac{1}{2i}e^{-i\sqrt{\nu}x}\mathscr{F}^{-1}\left\{e^{-i\xi^2t}\right\}(x+2\sqrt{\nu}t) \\
&= \frac{1}{2i}e^{-i\sqrt{\nu}x}(\frac{1}{2}+\frac{i}{2})\frac{1}{\sqrt{t}}e^{-i(x-2\sqrt{\nu}t)^2/4t} - \frac{1}{2i}e^{-i\sqrt{\nu}x}(\frac{1}{2}-\frac{i}{2})\frac{1}{\sqrt{t}}e^{i(x+2\sqrt{\nu}t)^2/4t} \\
&= \frac{1}{\sqrt{t}}(\frac{1}{4}-\frac{i}{4})e^{-i(x^2/4t+\nu t)} + \frac{1}{\sqrt{t}}(\frac{1}{4}+\frac{i}{4})e^{i(x^2/4t+\nu t)} \\
&= \frac{1}{2\sqrt{t}}(\cos(x^2/4t+\nu t) - \sin(x^2/4t+\nu t))
\end{aligned}
$$

□

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