On Unitary Positive Energy and KMS Representations of Some Infinite-Dimensional Lie Groups

Niestijl, M.

DOI
10.4233/uuid:b0405182-6bef-47c7-9f61-84d0e29c70bc

Publication date
2023

Document Version
Final published version

Citation (APA)
Niestijl, M. (2023). *On Unitary Positive Energy and KMS Representations of Some Infinite-Dimensional Lie Groups*. [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:b0405182-6bef-47c7-9f61-84d0e29c70bc

On Unitary Positive Energy and KMS Representations of Some Infinite-Dimensional Lie Groups

Dissertation

For the purpose of obtaining the degree of doctor
at Delft University of Technology
by the authority of the Rector Magnificus Prof. dr. ir. T.H.J.J. van der Hagen,
Chair of the Board for Doctorates,
to be defended publicly on
Friday, 8 December 2023 at 12:30 o'clock

by

Milan NIESTIJL,

Master of Science in Applied Mathematics
Delft University of Technology, the Netherlands,
born in Delft, the Netherlands.

This dissertation has been approved by the promotor.

Composition of the doctoral committee:

Rector Magnificus,	chairperson
Prof. dr. J.M.A.M. van Neerven,	Delft University of Technology, promotor
Dr. B. Janssens,	Delft University of Technology, copromotor

Independent members:

Dr. D.C. Beltita,	IMAR Simion Stoilow Institute, Romania
Prof. Dr. H. Glöckner	Paderborn University, Germany
Prof. dr. K.-H. Neeb,	Friedrich-Alexander University, Germany
Prof. dr. E.P. van den Ban,	Utrecht Univeristy, the Netherlands
Prof. dr. F.H.J. Redig,	Delft University of Technology
Prof. dr. D.C. Gijswijt	Delft University of Technology, reserve member

The research in this dissertation was funded financially by the Dutch Research Council (NWO)

ISBN: 978-94-6473-297-9

Contents

Summary

In this dissertation, we study (projective) unitary representations of possibly infinite-dimensional locally convex Lie groups, in the sense of Bastiani, that either satisfy a positive energy condition, or a KMS(Kubo-Martin-Schwinger) condition. Both of these are motivated by physics. The main purpose of this thesis is to gain general understanding for these classes of representations, and more specifically to develop general tools by which they can be studied in systematic fashion. These tools are consequently applied to specific cases of interest, demonstrating that these conditions are typically extremely restrictive, that the classification of these classes of representations is feasible in various cases, and that these tools can be effectively applied towards achieving such a classification.

We begin in Chapter 1 with a general introduction to the topic, placing it in context and clarifying its relation to other aspects of mathematics and mathematical physics. This also motivates the study of these classes of representations. Finally, we explain how the various chapters fit into a larger story.

In Chapter 2, we introduce the positive energy condition and the class of semibounded representations. These play a central role in the thesis. We also fix our conventions and gather some preliminary definitions that are common to all subsequent chapters. In detail, we introduce continuous Lie algebra cohomology and recall its relation with projective unitary representations and central $\mathbb{T}$-extensions.

In Chapter 3, we define the notion of a KMS-representation, which plays an important role throughout the dissertation. We study its basic properties and provide various interesting examples. We also introduce the so-called generalized positive energy condition, which, as the name suggests, relaxes the positive energy condition. We show, perhaps surprisingly, that a KMS-representation naturally gives rise to a representation that satisfies the generalized positive energy condition and that carries a substantial amount of information about the original representation. This observation plays a crucial role in the remainder of the dissertation, because it effectively allows the classes of KMS- and positive energy representations to be treated simultaneously. It also shows that these two classes, which on the face of it appear to be unrelated, are actually quite similar in certain respects. For projective unitary representations satisfying a generalized positive energy condition it is also shown that important information is carried by the class in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ associated to it. This observation is exploited extensively in Chapter 4 and Chapter 5.

In Chapter 4, we study projective unitary generalized positive energy representations of the group $J_0^\infty(V,K)$ of ∞-jets relative to the action of another Lie group P, where V is a finite-dimensional real vector space and K is a compact simple Lie group. This contributes to the understanding of such representations for the gauge group associated to a principal K-bundle, by discussing those representations that factor through the ∞-jets at a single point. We determine sufficient conditions for such representations of $J_0^\infty(V,K)$ to factor through $J_0^2(V,K)$, or even through K, in terms of natural non-resonance conditions. Additionally, we obtain normal form results for the $\mathfrak{p}$-action on $J_0^\infty(V,K)$, where $\mathfrak{p}$ is the Lie algebra of P.

In Chapter 5, we consider projective unitary representations of the Lie group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms on a smooth manifold M. Provided that M is connected and that $\dim(M) > 1$, we show that such a representation is necessarily trivial on the identity component $\mathrm{Diff}_c(M)_0$ if it is of generalized positive energy with respect to the $\mathbb{R}$-action on $\mathrm{Diff}_c(M)$ induced by a non-zero and complete vector field v on M. In order to establish this result, we first determine the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M),\mathbb{R})$ of the Lie algebra of compactly supported vector fields on M, equipped with the natural LF-topology.

In the context of possibly infinite-dimensional Lie groups, we extend in Chapter 6 the theory of holomorphic induction of unitary representations, by relaxing the extremely restrictive assumption that the representation being induced is continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. We extend important known results regarding holomorphic induction to this more general setting. In particular, we establish uniqueness of the holomorphically induced representation. Under mild conditions, we also obtain an isomorphism of von Neumann algebras between the commutant of the induced representation, and that of the representation from which it is holomorphically induced. We also show that holomorphic induction is closely related to the positive energy condition, clarifying their precise relationship. These results make the powerful tool that is holomorphic induction available in a much more general context.

Samenvatting

In dit proefschrift bestuderen we (projectieve) unitaire representaties van een mogelijk oneindig-dimensionale lokaal convexe Lie-groep, in de zin van Bastiani, die ofwel aan een positieve energievoorwaarde, of aan een KMS(Kubo-Martin-Schwinger)-voorwaarde voldoen. Deze zijn beide gemotiveerd vanuit de natuurkunde. Het hoofddoel van dit proefschrift is om beter algemeen begrip te ontwikkelen over deze klassen van representaties, en om algemene methodes te ontwikkelen waarmee ze systematisch bestudeerd kunnen worden. Deze worden vervolgens toegepast op specifieke gevallen. Hiermee illustreren we dat deze voorwaarden typisch extreem restrictief zijn, dat de classificatie van deze klassen van representaties in verschillende gevallen mogelijk is en dat deze methodes effectief kunnen worden toegepast om dergelijke classificaties te behalen.

We beginnen in hoofdstuk 1 met een algemene introductie van het onderwerp. We verduidelijken de relatie met andere gebieden binnen de wiskunde en mathematische fysica en plaatsen het hiermee in context. Dit verheldert ook de motivatie voor het bestuderen van deze klassen van representaties. We leggen daarnaast uit hoe de verschillende hoofdstukken in een groter geheel passen.

In hoofdstuk 2 introduceren we de positive energievoorwaarde en de klasse van half-begrensde representaties. Deze spelen een centrale rol in het proefschrift. Verder specificeren we onze conventies en bespreken we enkele standaardbegrippen die in alle daaropvolgende hoofdstukken terug komen. Concreet introduceren we continue Lie-algebra cohomologie en haar relatie met projective unitaire representaties en centrale $\mathbb{T}$-uitbreidingen.

In hoofdstuk 3 definiëren we de zogenoemde KMS-representaties, die een een belangrijke rol spelen in dit proefschrift. We bestuderen hun basiseigenschappen en bespreken verschillende interessante voorbeelden. Verder introduceren we de gegeneraliseerde positieve energievoorwaarde. We laten daarbij zien dat men aan elke KMS-representatie op een natuurlijke wijze een representatie kan associëren die aan deze gegeneraliseerde positieve energievoorwaarde voldoet, en die veel informatie bevat over de oorspronkelijke representatie. Deze observatie speelt een cruciale rol in de rest van dit proefschrift, omdat het mogelijk wordt om de klassen van KMS- en positieve energierepresentaties tegelijk te behandelen. Het laat ook zien dat deze twee klassen, die in eerste instantie ongerelateerd lijken, zich in bepaalde opzichten vergelijkbaar gedragen. Voor projectieve unitaire representaties die aan

een gegeneraliseerde positieve energievoorwaarde voldoen laten we verder zien dat belangrijke informatie herleid kan worden uit de bijbehorende klasse in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$. Deze observatie speelt een belangrijke rol in de hoofdstukken 4 en 5.

In hoofdstuk 4 bestuderen we projectieve unitaire gegeneraliseerde positieve energierepresentaties van de groep $J^\infty_0(V, K)$ van ∞-jets ten opzichte van de werking van een andere Lie-groep P, waarbij V een eindig-dimensionale reële vectorruimte en K een compacte enkelvoudige Lie-groep is. Dit draagt bij aan het begrip van dergelijke representaties voor de ijkgroep geassocieerd aan een K-hoofdvezelbundel, door díé representaties van de ijkgroep te beschouwen die door de ∞-jets in een enkel punt van de basisruimte factoriseren. We bespreken voldoende voorwaarden waaronder zulke representaties van $J^\infty_0(V, K)$ door $J^2_0(V, K)$ of zelfs door K factoriseren. Deze voorwaarden stellen dat bepaalde resonanties ontbreken. Verder verkrijgen we normaalvormresultaten voor de $\mathfrak{p}$-werking op $J^\infty_0(V, K)$, waarbij $\mathfrak{p}$ de Lie-algebra is van P.

In hoofdstuk 5 beschouwen we projectieve unitaire representaties van de Lie-group $\mathrm{Diff}_c(M)$ van compact gedragen diffeomorfismen op een differentiëerbare variëteit M. Op voorwaarde dat M samenhangend is en dat $\dim(M) > 1$, laten we zien dat een dergelijke representatie noodzakelijkerwijs triviaal is op de identiteitscomponent $\mathrm{Diff}_c(M)_0$ als deze van gegeneraliseerde positieve energie is ten opzichte van de $\mathbb{R}$-werking op $\mathrm{Diff}_c(M)$ geassocieerd aan een volledig en niet-triviaal vectorveld v op M. Als tussenstap naar dit resultaat bepalen we de continue tweede Lie-algebra cohomologie $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ van de Lie-algebra van compact gedragen vectorvelden op M, uitgerust met haar natuurlijke LF-topologie.

In de context van mogelijk oneindig-dimensionale Lie-groepen breiden we in hoofdstuk 6 de theorie over holomorfe inductie van unitaire representaties uit, naar het geval waarin de representatie die moet worden geïnduceerd niet langer continu hoeft te zijn ten opzichte van de normtopologie op $\mathrm{U}(\mathcal{H})$, een uiterst beperkende veronderstelling die voorheen altijd vereist werd. We breiden belangrijke resultaten over holomorfe inductie uit naar deze algemenere situatie. In het bijzonder stellen we de uniciteit van de holomorf geïnduceerde representatie vast. Onder milde voorwaarden tonen we ook een isomorfisme van von Neumann-algebra's aan tussen de commutator van de geïnduceerde representatie en die van waaruit wordt geïnduceerd. We tonen ook aan dat holomorfe inductie nauw verbonden is met de positieve energievoorwaarde, waarbij hun precieze relatie wordt verduidelijkt. Deze resultaten stellen het nuttige hulpmiddel holomorfe inductie beschikbaar in een aanzienlijk algemenere context.

Acknowledgements

Before getting into the subject matter of the thesis, I would like to express my gratitude to all the people that have assisted me in the PhD process in any form or shape.

Firstly, I am grateful to be given the opportunity to pursue a PhD in the first place. I would therefore like to express sincere gratitude to my supervisor Bas Janssens, who had sufficient confidence in my mathematical ability to offer me this possibility. He has subsequently taught me a great deal and has properly guided me through the process of maturing as a mathematician.

Let me next express my gratitude to all members of the doctoral committee, for agreeing to be a part thereof, and for their time and effort.

I would moreover like to express sincere gratitude to Karl-Hermann Neeb, who has spent a considerable amount of time and effort in assisting me in various ways. The numerous valuable conversations and his detailed feedback on my work have undoubtedly improved the quality of this dissertation considerably. He has also provided valuable assistance in my pursuit for a career in mathematics. On that note, I am also very grateful to Helge Glöckner, who has granted me the possibility to continue doing mathematics in the near future.

I am also grateful to my fellow PhD candidate colleagues, as it is meaningful to have others around that are going through a similar endeavor, and to all other colleagues, each of which has contributed to a pleasant environment in some way. I am particularly thankful to Tobias Diez and Lukas Miaskiwskyi, who have played an important role during my PhD.

Lastly, I would like to thank those dear to me, whose importance can not be underestimated. In particular, Anne Hoekstra has been very meaningful to me, who also provided some much-needed balance by regularly pulling me back from the abstract realms of mathematics to the present-day environment. Good friends have similarly offered plenty much-appreciated and pleasant distractions. Finally, I would like to express my deepest gratitude to my wonderful parents and brother.

Chapter 1

Introduction

1.1 General introduction

Symmetries are mathematically described by (Lie) groups and their actions. As symmetries are ubiquitous in mathematics and physics, their importance is unsurprising. It is substantiated by Emmy Noether's celebrated theorem, which establishes a close correspondence between symmetries and conserved quantities in physics, such as energy and momentum. Symmetries also play an essential role in gauge theory, the mathematical framework underpinning the standard model of elementary particle physics.

In the early to mid-20th century, it was realized that Lie groups acting on the phase space of a classical system can also play a vital role in the task of quantizing it, which is to say, describing it in a manner that is compatible with the laws enforced upon us by quantum physics. As the state space of the latter is usually taken to be a projective Hilbert space, symmetries are in the quantum setting understood in terms of *projective* unitary representations of an appropriate Lie group G. For example, the notion of *spin* in quantum mechanics is understood using the projective unitary representation theory of the group $\mathrm{SO}(3)$ of rotations on $\mathbb{R}^3$, and the quantum harmonic oscillator is fully described using that of the group $\mathbb{R}^2 \rtimes \mathbb{T}$.

A remarkable amount of information of a physical system can be extracted from a good understanding of the irreducible projective unitary representations of the corresponding symmetry group G. A good example is provided by a result of Wigner [Wig39], who classified the projective unitary representations of the Poincaré group, the symmetry group of special relativity, and showed that physical quantities such as *proper mass*, *spin* and *helicity* admit a clear representation-theoretic interpretation. They turn out to precisely label the irreducible projective unitary representations of the relevant symmetry group, the Poincaré group. Bargmann and Wigner further showed that the so-called *Dirac equation*, which describes the time evolution of relativistic free fermions with non-zero proper mass, is fundamentally related to the Poincaré group and its unitary representations, in a precise way [BW48]. In fact, the Dirac equation can be recovered from the representation theory of the Poincaré group alone, without any additional physical considerations. It is therefore fundamentally tied to the Poincaré group and its representation theory.

The study of representations is also a tremendously interesting and useful endeavor from a purely mathematical point of view. As symmetries are omnipresent, one finds that most parts of mathematics have some relation with Lie groups, Lie algebras and their actions. For example, they play a key role in the notion of a principal bundle and their associated vector bundles, which in turn are important concepts in differential geometry, particularly in gauge theory. Intriguing connections with complex geometry can be established through results like the Borel-Weil Theorem (cf. [DK00, Sec. 4.12], [CG97, Nee00]). Actions of Lie groups play a role in symplectic and Poisson geometry via momentum maps, symplectic reduction and Kirillov's orbit method [Kir04]. In Riemannian geometry they appear via groups of isometries or conformal transformations, spin structures and associated spinor bundles. Lie groups are directly related to operator algebras via group C^*-algebras, which also connects their respective representation theories (cf. [Dix77, Wil07]). Representations also play a central role in the vast and rich field of abstract harmonic analysis, which generalizes Fourier theory. It studies the decomposition of group representations into irreducible pieces and is fundamentally related to analysis on homogeneous and (locally) symmetric spaces. The theory of special functions is also closely tied to Lie groups and their representations. Finally, quantum groups arise as suitable deformations of (semisimple) Lie algebras. The above list of examples is nowhere near exhaustive, but let us stop here, having hopefully made clear the ubiquity of symmetries throughout mathematics.

In both physics and mathematics, one frequently encounters symmetry groups that are very large, in the sense that they are *infinite-dimensional* Lie groups. For example, gauge theory admits the Lie group of compactly supported gauge transformations as infinite-dimensional symmetry group, because the Yang-Mills functional, from which the Yang-Mills equations are derived, is invariant under the natural action of this group on the space of connections on a principal bundle. As another example, groups of diffeomorphisms appear in both general relativity and in fluid dynamics. In the former, one observes that Einstein's field equations are invariant under the group of diffeomorphisms. In fluid dynamics, a diffeomorphism is interpreted as describing the displacement of some initial configuration of a fluid, so that the time evolution of a fluid can be interpreted as a path on the diffeomorphism group. Groups of unitary elements in a unital Banach $*$-algebra also provide natural examples of infinite-dimensional Lie groups. In particular, the unitary group $\mathrm{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is a Banach-Lie group. Infinite-dimensional Lie groups also occur in a wide variety of other contexts, some of which are described in [KW09, AK98, Ott95, Sch04].

The preceding, and plain mathematical curiosity, lead us to consider the unitary representation theory of infinite-dimensional Lie groups. This perspective, where the symmetry group is placed in the center of study, may then lead to a better understanding of related physical systems, and their possible quantizations. In particular, if we are to be optimistic, the study of projective unitary representations of gauge groups may lead to an increased understanding of quantum analogues of gauge theory, whereas a consideration of groups of diffeomorphisms might provide an illuminating perspective on some aspects of quantum gravity. Let us also mention that unitary representations of groups of diffeomorphisms have been studied in relation to quantum- and statistical physics in [GMS80, GMS81, GMS83, Gol04]. Representations of infinite-dimensional Lie algebras also occur in the context of (geometric) quantization, where one is interested in representations of (Lie subalgebras of) the Lie algebra $(C^\infty(M), \{-,-\})$ associated to a symplectic manifold.

It should be mentioned at this stage that differential geometry in infinite dimensions comes with numerous hurdles and subtleties. For example, Lie algebras need not integrate to a Lie group [vEK64] (cf. [Nee06, Thm. VI.2.5 and VI.2.7]), and an exponential map might not exist [Nee06, Ex. II.5.5]. When it does, it could be neither locally injective, nor locally surjective ([Mil84, Warning 1.6] and [PS86, Prop. 3.3.1]). Beyond the context of Banach spaces, the Implicit and Inverse Function Theorems are also no longer available [Nee06, Rem. I.2.6]. As a consequence of such difficulties, the infinite-dimensional context requires additional caution, and one typically makes suitable assumptions to rule out certain unwanted phenomena. We will in particular oftentimes assume that the Lie group G is *regular* in the sense of Milnor [Mil84] (cf. [Nee06, Def. II.5.2]). Another difference with the finite-dimensional setting originates from the observation that infinite-dimensional Lie groups are not locally compact, and therefore do not admit a left-Haar measure. Consequently, there is no natural unitary regular representation, and one can no longer associate a group C^*-algebra to the group under consideration in a natural way. This entails also that many standard techniques from the finite-dimensional context break down, and that the rich results from operator algebras are not as directly accessible.

One is also quickly faced with the realization that for most infinite-dimensional Lie groups, a full classification of its irreducible (projective) unitary representations is utterly intractable. It is therefore necessary to isolate a class of representations that are both physically relevant and more susceptible to systematic study. This thesis is mostly concerned with two such classes, namely *positive energy*- and *KMS*-representations.

To describe these, let G be a regular locally convex Lie group with Lie algebra $\mathfrak{g}$. We say that a unitary representation ρ of G on the Hilbert space $\mathcal{H}_\rho$ is *smooth* if it admits a dense set of smooth vectors, that is, vectors $\psi \in \mathcal{H}_\rho$ for which the orbit map $g \mapsto \rho(g)\psi$ is smooth $G \to \mathcal{H}_\rho$. Such a smooth unitary representation is said to be of positive energy at $\xi \in \mathfrak{g}$ if $\langle \psi, -id\rho(\xi)\psi \rangle \geq 0$ for all smooth vectors $\psi \in \mathcal{H}_\rho^\infty$. If we think of the unitary 1-parameter group $t \mapsto \rho(e^{t\xi})$ as representing time translation, then its generator would be the Hamiltonian, whose spectrum admits an interpretation in terms of energy levels. As the Hamiltonian is in quantum physics nearly always assumed to be a self-adjoint operator with non-negative spectrum, physically relevant representations typically reside in this class. It is then no surprise that positive energy representations of Lie groups are abundant in mathematical physics literature [SW64, Bor87, Bor66, Haa92, LM75, Ol'81, PS86, Seg81], thereby motivating their study. As a consequence, they were subject to a substantial amount of research [NR22, Was98, Tan11, Nee01a, Nee14a, JN21, NS15, Nee17, TL99]. The positive energy condition is typically also very restrictive, making classification results feasible.

A different way of isolating physically relevant representations resides in the operator-algebraic approach to quantum statistical mechanics, where an equilibrium state of the system is described by a state ϕ on a von Neumann algebra $\mathcal{M}$ that satisfies a so-called *Kubo-Martin-Schwinger* (KMS) condition, relative to a specified group of automorphisms on $\mathcal{M}$. This condition appeared in [HHW67] and is named after the authors of the papers [Kub57, KYN57, MS59]. Such a state may be thought of as a generalization of the *Gibbs states* $\phi(x) = \frac{1}{\mathrm{Tr}(e^{-\beta H})}\mathrm{Tr}(e^{-\beta H}x)$ on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and H is a self-adjoint operator for which $e^{-\beta H}$ is trace-class, to the setting where $e^{-\beta H}$ is no longer required to be trace-class. These Gibbs states play a fundamental role in statistical physics. In particular, the map $\beta \mapsto \mathrm{Tr}(e^{-\beta H})$ is known as the partition function, whenever it makes sense. In this context, the parameter $\beta > 0$ admits the interpretation of an inverse-temperature. The aforementioned second class of representations are in a suitable sense compatible with a KMS-state on the von Neumann algebra generated by the representation.

It is the purpose of this thesis to develop further understanding of the restrictions imposed on a (projective) unitary representation of a possibly infinite-dimensional Lie group G by either of these two conditions, the positive energy or KMS condition. To understand these, we will occasionally take either a specialized or a general perspective. On the one hand, we consider these classes for specific infinite-dimensional Lie groups, such as gauge and diffeomorphism groups. On the other hand, we collect and discern general facts regarding these representations, and contribute to their theory in a more fundamental sense. In particular, an already well-established and appealing relation between the positive energy condition and holomorphic induction is extended to a more general setting than was previously available. It is moreover shown that these two seemingly very different classes of representations exhibit similar behavior in certain respects.

1.2 Outline of this thesis

1.2.1 Generalized positive energy and KMS representations

After introducing in Chapter 2 some preliminaries that are common to all other chapters, we proceed in Section 3.1 with the study of KMS-representations. An observation that is central to this chapter is that both the positive energy and the KMS condition are encapsulated by a third one, the so-called *generalized positive energy condition*, so that they can be studied using the same techniques. This allows us in subsequent chapters to study these classes simultaneously.

Briefly, a smooth unitary representation ρ of the possibly infinite-dimensional Lie group G with Lie algebra $\mathfrak{g}$ is said to be of *generalized positive energy* at $\xi \in \mathfrak{g}$ if there exists a linear subspace $\mathcal{D}_\xi \subseteq \mathcal{H}_\rho^\infty$ that is dense in $\mathcal{H}_\rho$ such that for every $\psi \in \mathcal{D}_\xi$ we have

$$\inf_{g \in G_0} \langle \psi, -id\rho(\mathrm{Ad}_g(\xi))\psi \rangle > -\infty, \tag{1.2.1}$$

where G_0 denotes the identity component of G. It is apparent that this relaxes the positive energy condition. Provided that (1.2.1) is valid, a simple observation is made in Section 3.1 that can be used to identify elements $\eta \in \mathfrak{g}$ in the kernel of the derived representations $d\rho$ on $\mathcal{H}_\rho^\infty$. This is used in later chapters for specific Lie groups G to determine very large ideals in $\mathfrak{g}$ on which $d\rho$ must vanish.

Let us now introduce the KMS condition in a bit more detail. We consider it from a few different perspectives, to clarify its role in various parts of mathematics and mathematical physics. This should then serve as orientation and motivation for the study of KMS representations, whose definition is presented shortly thereafter.

First, we recall the notion of a KMS state on a von Neumann algebra. We refer to [Tak03a, Ch. VIII] and [BR97, Ch. 5.3] for more details. If $\mathcal{M}$ is a von Neumann algebra, let $\mathcal{S}(\mathcal{M})$ denote the set of normal states on $\mathcal{M}$. Define $\mathrm{St} := \{ z \,:\, z \in \mathbb{C},\ 0 < \mathrm{Im}(z) < 1 \}$. There are many equivalent formulations of the KMS-condition, that we shall not restate, the most commonly encountered one being the following ([HHW67]):

Definition 1.2.1. Let $\phi \in \mathcal{S}(\mathcal{M})$ be a normal state. Let $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ be a one-parameter group of automorphisms of $\mathcal{M}$.

— ϕ is said to satisfy the *modular condition* for σ if the following two conditions are satisfied:

1. $\phi = \phi \circ \sigma_t$ for all $t \in \mathbb{R}$.
2. For every $x, y \in \mathcal{M}$, there exists a bounded and continuous function $F_{x,y} : \overline{\mathrm{St}} \to \mathbb{C}$ which is holomorphic on St and s.t. for every $t \in \mathbb{R}$:

$$F_{x,y}(t) = \phi(\sigma_t(x)y),$$
$$F_{x,y}(t+i) = \phi(y\sigma_t(x)).$$

— ϕ is said to be *KMS* w.r.t. σ at inverse temperature $\beta > 0$ if it satisfies the modular condition for $t \mapsto \sigma_{-\beta t}$. In that case, we also say that ϕ is σ-KMS at inverse-temperature β. If $\beta = 1$ we simply say that ϕ is a σ-KMS state.

Although it is not at all obvious from this definition, the KMS-condition is considered to be a well-founded characterization of thermodynamical equilibrium states in quantum statistical physics. In this regard, let us mention that if $\beta > 0$ and H is a self-adjoint operator on the Hilbert space $\mathcal{H}$ for which $e^{-\beta H}$ is trace class, then the unique normal state ϕ on $\mathcal{B}(\mathcal{H})$ that is KMS w.r.t. the automorphism group $\sigma_t(x) := e^{itH}xe^{-itH}$ at inverse temperature $\beta > 0$ is precisely the corresponding Gibbs state $\phi(x) = \frac{1}{\mathrm{Tr}(e^{-\beta H})}\mathrm{Tr}(e^{-\beta H}x)$, which is characterized by the fact that it satisfies a minimum free energy principle, also referred to as the maximum entropy principle at fixed energy ([BR97, Sec. 6.2.3] and [Haa92, Sec. V.1.3], cf. [AS77a, AS77b, Ara74]). It was moreover shown in [PW78] that a σ-KMS state satisfies a condition that is suggested by the second law of thermodynamics, called *passivity*. Suitable converse statements are also shown, under stronger assumptions. Characterizations of equilibrium states in terms of certain stability properties were considered in [HKTP74], which further motivate the KMS condition.

KMS states are also important in the modular theory of von Neumann algebras, also known as Tomita-Takesaki theory. A detailed account of this can be found in [Tak02, Tak03a, Tak03b]. The central result is the following. Suppose that $\Omega \in \mathcal{H}$ is a vector which is both cyclic and separating for the von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, meaning that $\overline{\mathcal{M}\Omega} = \mathcal{H}$ and $x\Omega = 0$ implies $x = 0$ for all $x \in \mathcal{M}$. Then there is a self-adjoint positive and invertible operator Δ, and a conjugate linear involution J on $\mathcal{H}$ that fix the vector Ω and satisfy the following properties:

1. $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ for all $t \in \mathbb{R}$.
2. $J\Delta J = \Delta^{-1}$.
3. $J\mathcal{M}J = \mathcal{M}'$.

In particular, one obtains the one-parameter group

$$\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M}),\ \sigma_t(x) = \Delta^{it}x\Delta^{-it} \tag{1.2.2}$$

and a linear isomorphism $j : \mathcal{M} \to \mathcal{M}'$ defined by $j(x) := Jx^*J$. Importantly, the vector state $\langle \Omega, \bullet\, \Omega \rangle$ on $\mathcal{M}$ then satisfies the KMS-condition w.r.t. the automorphism group σ from (1.2.2) at inverse temperature $\beta = 1$, and σ is uniquely determined by this condition. As was observed by Alain Connes, the image of σ in the quotient $\mathrm{Out}(\mathcal{M}) := \mathrm{Aut}(\mathcal{M})/\mathrm{Inn}(\mathcal{M})$ of outer automorphisms does not depend on Ω, which leads to the observation that any σ-finite von Neumann algebra comes with a *canonical* homomorphism $\mathbb{R} \to \mathrm{Out}(\mathcal{M})$. In this sense, von Neumann algebras are therefore *intrinsically dynamical* objects. This modular theory also leads to two invariants of von Neumann algebras, which were essential for the study of type III factors [Con73]. Modular theory also has other important consequences, such as the existence of an essentially unique standard form of a von Neumann algebra $\mathcal{M}$, and a corresponding unitary representation of the automorphism group $\mathrm{Aut}(\mathcal{M})$ [Haa75].

It is an absolutely stunning, and a somewhat mysterious fact that the KMS condition, which historically originates from quantum statistical mechanics, simultaneously plays a fundamental role in this seemingly unrelated abstract modular theory of von Neumann algebras.

There is yet a third role to be played by the KMS-condition and by modular operators, namely in certain aspects of the operator algebraic approach to quantum field theory, in the sense of Haag-Kastler [HK64, Haa92]. Here, one considers von

Neumann algebras $\mathcal{M}(\mathcal{O}) \subseteq \mathcal{B}(\mathcal{H})$ that are associated to certain regions $\mathcal{O} \subseteq M$ of some Lorentzian manifold M, and the assignment $\mathcal{O} \to \mathcal{M}(\mathcal{O})$, also called a net of von Neumann algebras is required to satisfy a particular list of axioms that we shall not repeat here [HK64]. Suppose that G is a Lie group acting on M, with Lie algebra $\mathfrak{g}$. Suppose further that $U : G \to \mathrm{U}(\mathcal{H})$ is a unitary G-representation on $\mathcal{H}$, that $\Omega \in \mathcal{H}$ is a G-invariant vector, and that the net is *G-covariant* in the sense that $U(g)\mathcal{M}(\mathcal{O})U(g)^{-1} = \mathcal{M}(g.\mathcal{O})$ for all $g \in G$ and all regions $\mathcal{O}$. If Ω is additionally both cyclic and separating for every $\mathcal{M}(\mathcal{O})$, then we obtain the associated modular unitary 1-parameter groups $\Delta_{\mathcal{O}}^{it}$ on $\mathcal{H}$. It is now an interesting matter whether or not these modular operators are *geometric*, in the sense that for every region $\mathcal{O}$ we have $\Delta_{\mathcal{O}}^{-it} = U(e^{t\xi_{\mathcal{O}}})$ for some $\xi_{\mathcal{O}} \in \mathfrak{g}$ and all $t \in \mathbb{R}$. This is usually referred to as the *Bisognano-Wichmann property* of the net $\mathcal{O} \mapsto \mathcal{M}(\mathcal{O})$. The study of such nets is an active subject of research [Bor00, Bor92, KW01, Mun01, BGL93, BDFS00, BY99, GL95]. To simplify the study of such nets tremendously, one may forget most of the structure involved, remembering only the real subspaces $V_{\mathcal{O}} := \overline{\mathcal{M}(\mathcal{O})_{\mathrm{sa}}\Omega} \subseteq \mathcal{H}$. As it turns out, these still provide enough information to define the modular operators $\Delta_{\mathcal{O}}$. For this reason, so-called *nets of standard subspaces* are frequently studied as an intermediate step [BGL02, NO21, Mor18, Lon08, MN21, LL15, NO17, MNO23, NÓ22, FNÓ23].

The preceding motivates study of the unitary representations of Lie groups that are compatible with a KMS condition in a suitable sense, to be defined shortly. It also provides context to some of the examples that are given in Section 3.2.2 below, where it is also shown that in many cases, there is actually a smooth unitary representation of a possibly infinite-dimensional Lie group that governs the modular structure involved, in the sense that the Lie group representation generates the respective von Neumann algebra and implements the modular flow.

Now that we have established some context for the KMS-condition, let us provide the definition of a KMS-representation. For this introductory text, we will provide a slightly simplified version (cf. Definition 3.2.6 and Remark 3.2.7 below). Let $\alpha : \mathbb{R} \to \mathrm{Aut}(G)$ be a smooth $\mathbb{R}$-action on the possibly infinite-dimensional Lie group G. A unitary representation ρ of $G \rtimes_\alpha \mathbb{R}$ is called *smoothly-KMS* if there is a state ϕ on the von Neumann algebra $\mathcal{M} := \rho(G)''$ that is KMS with respect to the automorphism group $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$, $t \mapsto \mathrm{Ad}(\rho(t))$ and for which the following map is smooth:

$$\widehat{\phi} : G \to \mathbb{C}, \qquad \widehat{\phi}(g) := \phi(\rho(g)),$$

where we have considered G and $\mathbb{R}$ as subgroups of $G \rtimes_\alpha \mathbb{R}$ in the obvious fashion.

It is the purpose of Chapter 3 to establish a basic study of such representations. We present various illustrative and non-trivial examples, and investigate the basic properties of such representations. In particular, we relate KMS-representations to the generalized positive energy condition.

To describe this relation, suppose that $\phi \in \mathcal{S}(\mathcal{M})$ is a σ-KMS state. We then naturally obtain a unitary representation ρ_ϕ of $G \rtimes_\alpha \mathbb{R}$ on the corresponding GNS-

Hilbert space $\mathcal{H}_\phi$ according to $\rho_\phi(g,t) = \pi_\phi(\rho(g))\Delta_\phi^{-it}$, where $\pi_\phi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\phi)$ is the GNS-representation of $\mathcal{M}$ associated to ϕ, and where Δ_ϕ is the modular operator defined using the canonical cyclic and separating vector $\Omega \in \mathcal{H}_\phi$ for $\mathcal{M}_\phi := \rho_\phi(G)''$. The following now relates the KMS- and the generalized positive energy conditions:

Theorem 1.2.2. *Let ρ be a unitary representation of $G \rtimes_\alpha \mathbb{R}$ that is smoothly-KMS. Let $\phi \in \mathcal{S}(\mathcal{M})$ be a σ-KMS state for which $\widehat{\phi}$ is smooth. Then the associated representation ρ_ϕ of $G \rtimes_\alpha \mathbb{R}$ on the GNS-Hilbert space $\mathcal{H}_\phi$ is smooth and of generalized positive energy at $(0,1) \in \mathfrak{g} \rtimes_D \mathbb{R}$.*

1.2.2 Generalized positive energy representations of groups of jets

Now that we have established that the positive energy and KMS representations may be simultaneously studied using the generalized positive energy condition, we proceed in Chapter 4 by studying the latter class of representations in the projective context for a specific infinite-dimensional Lie group, namely the group $G = J_0^\infty(V, K)$ of ∞-jets of smooth maps $V \to K$, where V is a finite-dimensional real vector space, and K is some simple 1-connected compact Lie group. Before diving into the results of this chapter, let us explain the motivation for considering this group in the first place.

Namely, representations of this group G fit into a much larger program, which aims to study gauge theory from a representation-theoretic perspective. Gauge fields are typically described by a connection on a principal K-bundle $\mathcal{K}$ over some manifold M. The set of all connections on $\mathcal{K}$ carries a natural action of the group $\mathcal{G} = \mathrm{Gau}_c(\mathcal{K}) := \Gamma_c(\mathrm{Ad}(\mathcal{K}))$ of compactly supported gauge transformations of the principal bundle. If P is a finite-dimensional Lie group with Lie algebra $\mathfrak{p}$, acting smoothly on the principal bundle $\mathcal{K}$, there is a corresponding smooth action α of P on the gauge group $\mathcal{G}$, and we can consider projective smooth unitary representations of the Lie group $\mathcal{G} \rtimes_\alpha P$ that are of positive energy at all elements in some P-invariant cone $\mathcal{C} \subseteq \mathfrak{p}$.

These were studied in [JN21], where the case in which the P-action on M has no fixed points was essentially fully solved (cf. [JN21, Theorem 7.19]). It therefore remains to understand what happens if α *does* have fixed points. In Chapter 4, the latter setting is studied by considering those projective unitary representations that are entirely localized at a *single* fixed point $a \in M$, in the sense that they factor through the germs at the fixed point. Choosing local coordinates near $a \in M$, this leads one to consider the projective unitary representations of the group $G = J_0^\infty(V, K)$, where $V = T_a(M)$.

Briefly, it is shown in Chapter 4 that the generalized positive energy condition imposes severe restrictions on the derived Lie-algebra representation $d\overline{\rho}$ of a projective unitary representation $\overline{\rho}$ of $G \rtimes P$, leading in particular to sufficient conditions for $\overline{\rho}|_G$ to factor through the space $J_0^2(V, K)$ of 2-jets, or even through K. This reduces the classification problem to much simpler, well-understood finite-dimensional Lie groups. The main results, Theorem 4.4.1, Theorem 4.4.3 and Theorem 4.4.6 are formulated in terms of natural non-resonance conditions for the $\mathfrak{p}$-action on $\mathfrak{g}$. We also obtain certain normal form results for this $\mathfrak{p}$-action (Theorem 4.3.12 and Theorem 4.3.13).

1.2.3 Central extensions and generalized positive energy representations of the group of compactly supported diffeomorphisms

Let $v \in \mathcal{X}(M)$ be a complete and non-zero vector field on a smooth manifold M with flow $h : \mathbb{R} \to \mathrm{Diff}(M)$. Let $\mathrm{Diff}_c(M) \rtimes_v \mathbb{R}$ be the semidirect product of $\mathrm{Diff}_c(M)$ and $\mathbb{R}$ relative to the $\mathbb{R}$-action on $\mathrm{Diff}_c(M)$ defined by $\alpha_t(f) := h_t \circ f \circ h_t^{-1}$ for $t \in \mathbb{R}$ and $f \in \mathrm{Diff}_c(M)$. Its Lie algebra is $\mathcal{X}_c(M) \rtimes \mathbb{R}v$, where v acts on $\mathcal{X}_c(M)$ by the derivation $[v, -]$. Aiming to study the extent to which the classical symmetry group of general relativity can be implemented as symmetries of a quantum system, we consider the projective unitary representations of $\mathrm{Diff}_c(M) \rtimes_v \mathbb{R}$ that are of generalized positive energy at $v \in \mathcal{X}_c(M) \rtimes \mathbb{R}v$.

Assuming that M is connected and that $\dim(M) > 1$, the main result of this chapter (Theorem 5.3.2) asserts that any such representation is necessarily trivial on the identity component $\mathrm{Diff}_c(M)_0$. This naturally leads to *asymptotic* symmetry groups, for if G is any Lie group of diffeomorphisms of M containing $\mathrm{Diff}_c(M)_0$ as Lie subgroup, then any such projective unitary G-representation necessarily factors through the quotient $G/\mathrm{Diff}_c(M)_0$, and is in this sense 'localized at infinity'. It now becomes an interesting matter to determine this class of representations for groups of diffeomorphisms having certain specified behavior at infinity. In particular, one might wonder whether or not this class of representations, for suitable groups G, naturally leads to the asymptotic symmetry groups that appear in general relativity in the context of asymptotically flat spacetimes [Pen64, Ash15, Wal84], such as the BMS group (Bondi-Metzner-Sachs) [BvdBM62, Sac62, AE18, PS22, AS81], or extensions thereof [NU62, Ruz20].

A noteworthy corollary of the aforementioned result is that any smooth representation $\overline{\rho} : \mathrm{Diff}_c(M) \to \mathrm{PU}(\mathcal{H}_\rho)$ that is continuous w.r.t. the norm-topology on $\mathrm{PU}(\mathcal{H}_\rho)$ is necessarily trivial on the identity component $\mathrm{Diff}_c(M)_0$ (Corollary 5.3.4).

As an intermediate step towards this result, we determine for an arbitrary manifold M the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ with coefficients in the trivial representation, where $\mathcal{X}_c(M)$ denotes the Lie algebra of compactly supported vector fields on M equipped with its natural locally convex LF-topology. A crucial observation is that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \{0\}$ for $\dim(M) > 1$. We also obtain that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R})$ is one-dimensional and is spanned by an analogue of the well-known Virasoro 2-cocycle on $\mathcal{X}(S^1)$. Finally, we consider the relationship between $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$

of the Lie algebra $\mathcal{X}(M)$ of *all* smooth vector fields on M, equipped with its natural Fréchet topology. It should be remarked at this point that the Lie algebra cohomology $H^\bullet_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ is also known as Gelfand-Fuks cohomology in reference to authors of the papers [GF68, GF69, GF70b, GF70c], and is well-studied [Los98, Gui73, BS77, Hae76, Fuk86, Bot73, Mia22].

1.2.4 Holomorphic induction beyond the norm-continuous setting, with applications to positive energy representations

In Chapter 6, we consider the theory of holomorphic induction, a well-known concept in representation theory which relates certain unitary G-representations to those of a subgroup H of G, in such a way that properties such as irreducibility are preserved. As it turns out, this concept is closely related to the positive energy condition. It has proven to be a very powerful tool, in particular for obtaining classification results of positive energy representations. The availability of holomorphic induction is an important reason that the class of positive energy representations is susceptible to systematic study. In virtually all cases where a full classification of irreducible positive energy representations was obtained, this was done using holomorphic induction in a crucial way ([Nee14a, Nee12, Nee01a, PS86]). It is therefore vital to develop full understanding and applicability of this tool. Holomorphic induction is already well-established, even in the infinite-dimensional context [Nee13, Nee14a]. However, the H-representation $\sigma : H \to \mathrm{U}(V_\sigma)$ being induced is so far always required to be continuous w.r.t. the norm-topology on $\mathrm{U}(V_\sigma)$. This is an extremely restrictive assumption which greatly limits the applicability of holomorphic induction. It is the purpose of Chapter 6 to remove this assumption.

Roughly speaking, holomorphic induction attempts to generalize the picture portrayed by the Borel-Weil Theorem, which describes the irreducible unitary representations of a compact simple Lie group K. Letting $T \subseteq K$ be a maximal torus with Lie algebra $\mathfrak{t}$, such a representation is fully characterized by the corresponding lowest weight $\lambda \in i\mathfrak{t}^*$, and can be realized on the space of holomorphic sections of an associated line bundle over the corresponding coadjoint orbit. Moreover, to each $\lambda \in i\mathfrak{t}^*$, there exists up to unitary equivalence at most *one* irreducible K-representation having λ as its lowest weight. The unitary dual $\widehat{K}$ is therefore understood once it has been determined which weights are inducible to a unitary K-representation, which is then necessarily irreducible and unique up to unitary equivalence.

This can be generalized vastly, whilst retaining a similar picture. This was exploited by Pressley and Segal in [Seg81, PS86] (cf. [Nee01a]) to obtain a classification of the positive energy representations of loop groups, and was further developed in the infinite-dimensional context by K.-H. Neeb [Nee13, Nee14a]. In particular, returning to the infinite-dimensional Lie group G, the subgroup T is replaced by one that is no longer required to be Abelian, say $H \subseteq G$. In all previous results, it was required that the H-representation $\sigma : H \to \mathcal{B}(V_\sigma)$ being induced, which takes the role of the lowest weight λ, is continuous with respect to the norm topology on $\mathcal{B}(V_\sigma)$.

A central component to the conventional approach towards holomorphic induction is to equip the G-homogeneous vector bundle $\mathbb{E}_\sigma := G \times_H V_\sigma$ over G/H with a G-invariant complex-analytic bundle structure (cf. [Nee13, Thm. 2.6]). This construction fails when σ is not norm-continuous, so that a new approach is required in order to go beyond the norm-continuous setting. This difficulty is overcome in Chapter 6 by considering a suitable subspace of real-analytic functions $G \to V_\sigma$ which corresponds to the space $\mathcal{O}(\mathbb{E}_\sigma)$ of holomorphic sections of $\mathbb{E} \to G/H$ whenever the latter makes sense. It is the main point of Chapter 6 that the theory of holomorphic induction can be developed in its entirety without the need for an explicit complex bundle structure on $\mathbb{E}_\sigma$. This allows the theory to be formulated in a significantly more general context, at the cost of a clear complex-geometric interpretation.

Assuming that G is connected, two key results in Chapter 6 clarify the potential of holomorphic induction. Firstly, there is up to unitary equivalence at most one unitary G-representation ρ that is holomorphically induced from σ (Theorem 6.4.21). Secondly, under a mild assumption there is in that case an isomorphism of von Neumann algebras $\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(V_\sigma)^H$ between the corresponding commutants (Theorem 6.4.29 and Theorem 6.4.30). In particular, ρ is irreducible if and only if σ is so. The relation with the positive energy condition is furthermore clarified, where we assume that G is regular. In particular, it is shown that if ρ is a unitary positive energy representation of $G \rtimes_\alpha \mathbb{R}$ for which the "energy-zero" subspace $\mathcal{H}_\rho(0) := \ker(d\rho(0,1)) \subseteq \mathcal{H}_\rho$ is cyclic for G and admits a dense set of G-analytic vectors, then $\rho|_G$ is holomorphically induced from the unitary H-representation σ on $\mathcal{H}_\rho(0)$, where $H := (G^\alpha)_0$ is identity component of the group G^α of α-fixed points. As a consequence, we obtain an isomorphism $\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(\mathcal{H}_\rho(0))^H$ between the corresponding commutants. In particular, ρ is irreducible of and only if σ is so. To classify such G-representations ρ, one would need to determine which unitary H-representations are holomorphically inducible up to G, although it should be mentioned that this task can be extremely difficult in general.

1.3 Further relevant literature

In this final section of the introduction, some additional related literature is gathered for the interested reader.

Extensive surveys on infinite-dimensional Lie theory, in the sense of Bastiani, can be found in [Mil84, Nee06, GN]. Various applications are discussed in [KW09, Ott95, Sch04]. A survey regarding infinite-dimensional Lie groups and their representations is given in [Nee04].

A class of infinite-dimensional Lie groups whose unitary representation theory was studied in detail involves various variants of infinite-dimensional unitary groups on a separable Hilbert space $\mathcal{H}$. The smallest of these is the direct-limit Lie group $\mathrm{U}(\infty) = \bigcup_{n\in\mathbb{N}} \mathrm{U}(n)$, defined w.r.t. natural inclusions. Another option is the Banach-Lie group $\mathrm{U}(\mathcal{H})$, which carries the operator-norm topology. There is a range of intermediate Banach-Lie groups $\mathrm{U}_p(\mathcal{H})$, indexed by $p \in [1, \infty]$. These are defined as $\mathrm{U}_p(\mathcal{H}) := \mathrm{U}(\mathcal{H}) \cap (1 + \mathcal{B}_p(\mathcal{H}))$, where the Banach space $\mathcal{B}_p(\mathcal{H})$ is the Schatten p-class. In particular, $\mathcal{B}_\infty(\mathcal{H})$ consists of the compact operators. Finally, one can consider $\mathrm{U}(\mathcal{H})$ as a topological group with the strong topology, denoted $\mathrm{U}(\mathcal{H})_s$. These are all related by the continuous inclusions

$$\mathrm{U}(\infty) \hookrightarrow \mathrm{U}_1(\mathcal{H}) \hookrightarrow \cdots \mathrm{U}_p(\mathcal{H}) \cdots \hookrightarrow \mathrm{U}_\infty(\mathcal{H}) \hookrightarrow \mathrm{U}(\mathcal{H}) \hookrightarrow \mathrm{U}(\mathcal{H})_s.$$

The unitary representations of these groups, and of Banach analogs of certain symmetric pairs (G, K), was studied in [Seg57, Pic88, Pic90, Kir73, Ol'84, Ol'78, Boy88, Boy80, Boy93, SV75]. A streamlined survey of these results can be found in [Nee14b]. The paper [SV78] also considers KMS states on the group $\mathrm{U}(\infty)$. The papers [Wol05, Wol14, Wol13] moreover discuss analogues of principal series representations of classical direct-limit Lie groups. Projective unitary representations of $\mathrm{U}_2(\mathcal{H})$ are considered in [Car84], and infinite-dimensional analogues of the metaplectic and spin representations are considered in [SS65, Nee10b, Ott95] (cf. [PR94]).

With regards to holomorphic induction, it should be mentioned that various infinite-dimensional analogues of the Borel-Weil Theorem have been achieved in [NRCW01, NS11, Nee01a, MNS10, PS86]. The paper [Nee13] later established holomorphic induction in the context of of bounded representations of Banach-Lie groups, which in turn was generalized to certain Fréchet-Lie groups in [Nee14a, Appendix C]. It was applied for various infinite-dimensional Lie groups in [Nee14a, Nee12, Nee01a, PS86, NS15, Nee04], in order to obtain classification results for positive energy or semibounded unitary representations.

Holomorphic induction and the positive energy condition are both closely related to highest weight representations (cf. [Nee00, Thm. X.3.9]). The latter have been considered in great detail for various infinite-dimensional Lie groups and algebras. For example, [Kac90, KR87] consider these in the context of Kac-Moody Lie algebras (cf. [Nee10c]), whereas [Nat94, DP99] do so for direct-limit Lie groups and Lie algebras. The papers [Nee98a, NØ98] study highest weight representations of infinite-dimensional classical Lie algebras and their relation with holomorphic representations of infinite-dimensional complex classical groups (cf. [MN16]).

Let us also mention some appearances of the KMS-condition in relation to unitary representations of Lie groups. The paper [Sim23] fully characterizes KMS representations of finite-dimensional Lie groups that generate a factor of type I. Certain projective unitary KMS representations of groups of $\mathrm{U}(N)$-valued functions on S^1 and $\mathbb{R}$ were considered in [CH92, CH87, BMT88]. Projective unitary positive energy representations of loop groups have been studied in detail in [PS86, Seg57, CR87, Nee01a, Nee14a, TL99]. Other unitary representations of groups of Lie-group valued maps were also considered in [GGV77, GG68, Ism76,

AHKM+93, AHK78, AHKT81, PS76].

With regards the central extensions and projective unitary representations of diffeomorphism groups, the Lie group $\mathrm{Diff}(S^1)$ has been subject to detailed study [PS86, Seg81, GW84, GW85, CDVIT21, NS15, JP04, Wit88, Kir81, LP76, AM22]. A survey regarding unitary representations of more general groups of diffeomorphism and currents be found in [Ism96]. In [Boy03], certain factor representations of the Lie group $\mathrm{Diff}_c(\mathbb{R}^n)$ are constructed using inductive limits of finite-dimensional unitary groups.

Chapter 2

Common notation and preliminaries

In this brief chapter, we establish some notation and preliminary definitions that are common to all chapters in the thesis.

2.1 Locally convex Lie theory

Let us first clarify how differential geometry can be extended to the infinite-dimensional setting, in the sense of Bastiani [Bas64], where manifolds are considered that are modeled on a possibly infinite-dimensional locally convex real vector space, rather than on $\mathbb{R}^n$ for some $n \in \mathbb{N}_{\geq 0}$. For a more elaborate exposition than the one presented below, we refer to [Bas64, Mil84, Nee06, Glö02b, GN].

In the following, all locally convex vector spaces are assumed to be Hausdorff without further mention. The definition of a smooth manifold is readily established once the notion of smoothness is clarified for a map between two locally convex vector spaces. In the Bastiani approach, one considers the following definition ([Mil84, Sec. 3]):

Definition 2.1.1 ([Nee06, Def. I.2.1])**.**
Let E, F be two locally convex vector spaces over $\mathbb{R}$. Let $U \subseteq E$ be open and let $f : U \to F$ be a function. Then *the derivative of f at $x \in U$ in the direction of $v \in E$* is defined as

$$d_x(f)(v) := (D_v f)(x) := \frac{d}{dt}\bigg|_{t=0} f(x + tv) = \lim_{t \to 0} \frac{1}{t}(f(x + tv) - f(x)),$$

whenever it exists. The function f is called *differentiable* at x if $d_x(f)(v)$ exists for all $v \in E$. It is called *continuously differentiable* if it is differentiable at every $x \in U$, and the map

$$df : U \times E \to F, \quad df(x, v) := d_x(f)(v)$$

is continuous. We say that f is C^k for $k \in \mathbb{N} \cup \{\infty\}$ if the iterated directional derivatives

$$d_x^n(f)(v_1, \cdots, v_n) := (D_{v_1} \cdots D_{v_n} f)(x)$$

exist for all integers $n \leq k$, $x \in U$ and $v_1, \cdots, v_n \in E$, and the map

$$d^n(f) : U \times E^n \to F, \quad d^n(f)(x; v_1, \cdots, v_n) := d^n_x(f)(v_1, \cdots, v_n)$$

is continuous for every integer $n \leq k$. We call f *smooth* if it is C^∞, and we write $C^\infty(U, F)$ for the space of all smooth maps $U \to F$.

With this notion of smooth maps between locally convex vector spaces, the chain rule holds as expected [Nee06, Prop. I.2.3]. Considering Hausdorff topological spaces that are locally homeomorphic to an open subset of a locally convex vector space, this allows the category of locally convex smooth manifold and smooth maps between them to be defined in the usual way, in complete analogy with the finite-dimensional case (cf. [Nee06, Def. I.3.1]). It is then also straightforward to define the notion of a locally convex Lie group, which is defined to be a group G which is also a locally convex smooth manifold, such that multiplication $G \times G \to G, (x, y) \mapsto xy$ and the inversion $G \to G, x \mapsto x^{-1}$ are both smooth maps.

Defining the tangent bundle $T(M)$ of a locally convex manifold M in terms of equivalence classes of smooth curves, as usual, one also readily defines the tangent functor analogously to the finite-dimensional case (cf. [Nee06, Def. I.3.3]). The Lie algebra $\mathfrak{g}$ of a locally convex Lie group G is then defined as the tangent space $\mathfrak{g} := T_e(G)$ of G at the identity element $e \in G$, which naturally comes equipped with a Lie bracket (cf. [Nee06, Def. II.1.5]). We also write $\mathbf{L}(G) := T_e(G)$ for the Lie algebra of G, and we write $\mathbf{L}(f) := T_e(f)$ when $f : G \to H$ is a homomorphism of Lie groups. In this setting, $\mathbf{L}(f) : \mathbf{L}(G) \to \mathbf{L}(H)$ is a continuous homomorphism of Lie algebras, and we call $\mathbf{L}$ the *Lie functor*.

Remark 2.1.2. An important reason for requiring local convexity of the model space of a (possibly infinite-dimensional) smooth manifold is that the continuous dual space E' of a locally convex vector space E separates the points of E [Rud91, p. 60]. This ensures in particular that $df = 0$ implies that f is locally constant [Nee06, Prop. I.2.3], where U, E and F are as in Definition 2.1.1 and where $f \in C^1(U, F)$. This implication would generally not hold true if E is required to be a metric topological vector space, for example, as [Nee06, Ex. I.2.5] shows.

Let $I := [0, 1]$. If $\gamma \in C^\infty(I, G)$, we write $\delta(\gamma) : I \to \mathfrak{g}$ for its *left logarithmic derivative*, defined by $\delta(\gamma)(t) := \frac{d}{ds}\big|_{s=t} \gamma(t)^{-1}\gamma(s) \in \mathfrak{g}$. An important concept in locally convex Lie theory, especially in the context of integrability questions, is Milnor's notion of *regular* Lie groups [Mil84, Def. 7.6]:

Definition 2.1.3 ([Nee06, Def. II.5.2]). A Lie group G with Lie algebra $\mathfrak{g}$ is said to be *regular* if the initial value problem

$$\begin{aligned} \delta(\gamma) &= \xi, \\ \gamma(0) &= e, \end{aligned} \tag{2.1.1}$$

has a solution $\gamma_\xi \in C^\infty(I, G)$ for every $\xi \in C^\infty(I, \mathfrak{g})$, and the corresponding evolution map

$$\mathrm{evol}_G : C^\infty(I, \mathfrak{g}) \to G, \qquad \xi \mapsto \gamma_\xi(1)$$

is smooth.

Some remarks are in order:

Remark 2.1.4.

1. Any Banach-Lie group is regular [Nee06, Rem. II.5.4].

2. A regular Lie group G admits a smooth exponential map $\exp_G : \mathfrak{g} \to G$, defined by $\exp_G(x) := \mathrm{evol}_G(\xi_x)$, where $\xi_x(t) := tx$ for $x \in \mathfrak{g}$ and $t \in I$. In this regard, it is important to mention that beyond the Banach setting, the exponential map $\exp_G$ need not be a local diffeomorphism in zero ([Mil84, Warning 1.6] and [PS86, Prop. 3.3.1]). Also, a general locally convex Lie group may not admit a smooth exponential function at all [Nee06, Ex. II.5.5].

3. If G and H are Lie groups, with G 1-connected and H regular, then any continuous homomorphism $\phi : \mathbf{L}(G) \to \mathbf{L}(H)$ of Lie algebras integrates to a unique smooth homomorphism $f : G \to H$ of Lie groups satisfying $\mathbf{L}(f) = \phi$ [Mil84, Thm. 8.1].

4. If G_1 and G_2 are two 1-connected regular Lie groups with $\mathbf{L}(G_1) \cong \mathbf{L}(G_1)$ as topological Lie algebras, then $G_1 \cong G_2$ as Lie groups [Mil84, Cor. 8.2].

As illustrated by Remark 2.1.4, the concept of regularity saves various results that are familiar from the finite-dimensional context, but which are not necessarily true for general locally convex Lie groups.

In Remark 2.1.4, we noticed that any Banach-Lie group is a regular locally convex Lie group. Some further illustrative examples are given below. Throughout this thesis, we will also encounter various other examples.

Example 2.1.5.

1. If M is a finite-dimensional compact smooth manifold, then the group $\mathrm{Diff}(M)$ of diffeomorphisms on M is a regular Lie group with Lie algebra $\mathcal{X}(M)$, the Fréchet space of smooth vector fields on M [Mil84, Sec. 6 and 7] (cf. [Nee06, Ex. II.3.14]).

2. If M is a finite-dimensional smooth manifold and K is a finite-dimensional Lie group with Lie algebra $\mathfrak{k}$, then the group $C_c^\infty(M, K)$ of compactly supported smooth functions $M \to K$, equipped with the pointwise product, is a regular Lie group with Lie algebra $C_c^\infty(M, \mathfrak{k})$ [Nee06, Thm. IV.1.12].

3. If $\mathcal{A}$ is a unital C^*-algebra, then the group $\mathcal{A}^\times$ of invertible elements is a regular Banach-Lie group with Lie algebra $(\mathcal{A}, [-,-])$, equipped with the usual Lie bracket $[a, b] = ab - ba$ $(a, b \in \mathcal{A})$ [Nee06, Thm. IV.1.11]. Similarly, it follows using [Nee06, Thm. IV.3.3] that group $\mathcal{U}(\mathcal{A}) = \{ u \in \mathcal{A} : u^* = u^{-1} \}$ of unitary elements is a Banach-Lie group whose Lie algebra $\mathfrak{u}(\mathcal{A})$ is the Lie subalgebra $\mathfrak{u}(\mathcal{A}) = \{ x \in \mathcal{A} : x + x^* = 0 \}$ of $(\mathcal{A}, [-,-])$, cf. example 6.8.7 below.

2.2 Continuous Lie algebra cohomology

We now briefly define continuous Lie algebra cohomology (cf. [Nee06, Def. V.2.2]):

Definition 2.2.1 (Continuous Lie algebra cohomology)**.**
Let $\mathfrak{g}$ be a locally convex Lie algebra and let E be a topological $\mathfrak{g}$-module. The continuous Lie algebra cohomology $H^\bullet_{\mathrm{ct}}(\mathfrak{g}, E)$ with coefficients in the $\mathfrak{g}$-module E is the cohomology of the complex $C^\bullet_{\mathrm{ct}}(\mathfrak{g}, E)$, where for $q \in \mathbb{N}_{\geq 0}$, the vector space $C^q_{\mathrm{ct}}(\mathfrak{g}, E)$ consists of continuous alternating multi-linear maps $\mathfrak{g}^q \to E$. The differential $d_\mathfrak{g} : C^\bullet_{\mathrm{ct}}(\mathfrak{g}, E) \to C^{\bullet+1}_{\mathrm{ct}}(\mathfrak{g}, E)$ of this complex is given by

$$\begin{aligned} d_\mathfrak{g}\omega(\xi_0, \dots, \xi_q) :=& \sum_{j=0}^{q} (-1)^j \xi_j \cdot \omega(\xi_0, \dots, \widehat{\xi_j}, \dots, \xi_q) \\ &+ \sum_{0 \leq i < j \leq q} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_q). \end{aligned} \tag{2.2.1}$$

As usual, the arguments in (2.2.1) with a caret are to be omitted. Unless mentioned otherwise, the vector space $\mathbb{R}$ is considered as trivial $\mathfrak{g}$-module.

2.3 Representations and central extensions

Throughout this section, let G denote a locally convex Lie group that admits an exponential map. Let $\mathfrak{g}$ be the Lie algebra of G.

Definition 2.3.1.

- — A unitary representation of G is a homomorphism $\rho : G \to \mathrm{U}(\mathcal{H}_\rho)$, where $\mathcal{H}_\rho$ is a Hilbert space and $\mathrm{U}(\mathcal{H}_\rho)$ denotes the group of unitary operators on $\mathcal{H}_\rho$. Similarly, a projective unitary representation of G is a homomorphism $\overline{\rho} : G \to \mathrm{PU}(\mathcal{H}_\rho) := \mathrm{U}(\mathcal{H}_\rho)/\mathbb{T}\mathrm{id}_{\mathcal{H}_\rho}$. Such a (projective) unitary representation of G is also denoted as a pair $(\rho, \mathcal{H}_\rho)$.
- — A (projective) unitary representation ρ of G on $\mathcal{H}_\rho$ is said to be continuous if it is so w.r.t. the strong topology on $\mathrm{U}(\mathcal{H}_\rho)$.
- — Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of G on $\mathcal{H}_\rho$. A vector $\psi \in \mathcal{H}_\rho$ is called *smooth* if the orbit map $G \to \mathcal{H}_\rho,\ g \mapsto \rho(g)\psi$ is smooth. We denote by $\mathcal{H}_\rho^\infty \subseteq \mathcal{H}_\rho$ the subspace of smooth vectors. The representation ρ is called smooth if $\mathcal{H}_\rho^\infty$ is dense in $\mathcal{H}_\rho$.
- — Let $(\overline{\rho}, \mathcal{H}_\rho)$ be a projective unitary representation of G on $\mathcal{H}_\rho$. We say that a ray $[\psi] \in \mathrm{P}(\mathcal{H}_\rho)$ is *smooth* if the orbit map $G \to \mathrm{P}(\mathcal{H}_\rho),\ g \mapsto \overline{\rho}(g)[\psi]$ is smooth. Denote by $\mathrm{P}(\mathcal{H}_\rho)^\infty$ the subspace of smooth rays. The projective representation $\overline{\rho}$ is called smooth if $\mathrm{P}(\mathcal{H}_\rho)^\infty$ is dense in $\mathrm{P}(\mathcal{H}_\rho)$.

Definition 2.3.2. If $\mathcal{D}$ is a complex vector space, denote by $\mathcal{L}(\mathcal{D})$ the algebra of linear operators on $\mathcal{D}$. If $\mathcal{D}$ is a pre-Hilbert space, we also define the algebra

$$\mathcal{L}^\dagger(\mathcal{D}) := \left\{ X \in \mathcal{L}(\mathcal{D}) \,:\, \exists X^\dagger \in \mathcal{L}(\mathcal{D}) \,:\, \forall \psi, \eta \in \mathcal{D} : \langle X^\dagger \psi, \eta \rangle = \langle \psi, X\eta \rangle \right\}.$$

The element $X^\dagger$ corresponding to $X \in \mathcal{L}^\dagger(\mathcal{D})$ is unique, again an element of $\mathcal{L}^\dagger(\mathcal{D})$ and satisfies $(X^\dagger)^\dagger = X$, so $(-)^\dagger$ endows $\mathcal{L}^\dagger(\mathcal{D})$ with an involution. We also define the Lie algebra

$$\mathfrak{u}(\mathcal{D}) := \{\, X \in \mathcal{L}^\dagger(\mathcal{D}) \,:\, X^\dagger + X = 0 \,\}.$$

Definition 2.3.3. Let $\mathcal{D}$ be a complex pre-Hilbert space.

— A *unitary representation* of the locally convex Lie algebra $\mathfrak{g}$ on $\mathcal{D}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{u}(\mathcal{D})$. A *projective unitary representation* is a Lie algebra homomorphism $\overline{\pi} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{D}) := \mathfrak{u}(\mathcal{D})/i\mathbb{R}I$.

— A unitary representation π of $\mathfrak{g}$ is called continuous if $\xi \mapsto \pi(\xi)\psi$ is continuous for any $\psi \in \mathcal{D}$. Similarly, a projective unitary representation $\overline{\pi}$ is continuous if $\xi \mapsto \pi(\xi)[\psi]$ is continuous for every $[\psi] \in \mathrm{P}(\mathcal{D})$.

Remark 2.3.4. Any unitary G-representation on $\mathcal{H}_\rho$ defines a unitary $\mathfrak{g}$-representation $d\rho : \mathfrak{g} \to \mathfrak{u}(\mathcal{H}_\rho^\infty)$ on $\mathcal{H}_\rho^\infty$ by $d\rho(\xi)\psi := \frac{d}{dt}\big|_{t=0}\rho(e^{t\xi})\psi$. We will always consider elements of $d\rho(\mathfrak{g})$ as unbounded operators defined on the invariant domain $\mathcal{H}_\rho^\infty$. If G is finite-dimensional, then $\mathcal{H}_\rho^\infty$ is dense in $\mathcal{H}_\rho$ for any continuous unitary representation ρ of G, by a result of Gårding [Går47] (cf. [War72, Prop. 4.4.1.1]). The analogous statement is generally false for infinite-dimensional Lie groups [BN08a].

A continuous projective unitary representation $\overline{\rho} : G \to \mathrm{PU}(\mathcal{H}_\rho)$ is equivalently given by a continuous central $\mathbb{T}$-extension $\mathring{G}$ together with a unitary representation $\rho : \mathring{G} \to \mathrm{U}(\mathcal{H}_\rho)$ which satisfies $\rho(z) = zI$ for z in the central $\mathbb{T}$ component. Of course, $\mathring{G}$ is the pull-back of the central $\mathbb{T}$-extension $\mathrm{U}(\mathcal{H}_\rho) \to \mathrm{PU}(\mathcal{H}_\rho)$ along $\overline{\rho}$. We say that ρ *lifts* $\overline{\rho}$. Suppose $\overline{\rho}_1$ and $\overline{\rho}_2$ are two projective unitary representations, inducing by pull-back the lifts $\rho_1 : \mathring{G}_1 \to \mathrm{U}(\mathcal{H}_{\rho_1})$ and $\rho_2 : \mathring{G}_2 \to \mathrm{U}(\mathcal{H}_{\rho_1})$ of $\overline{\rho}_1$ and $\overline{\rho}_2$, respectively. Then $\overline{\rho}_1$ and $\overline{\rho}_2$ are unitarily equivalent if and only if there is an isomorphism $\Phi : \mathring{G}_1 \to \mathring{G}_2$ of central G-extensions and a unitary $U : \mathcal{H}_{\rho_1} \to \mathcal{H}_{\rho_2}$ such that $\rho_2(\Phi(x)) = U\rho_1(x)U^{-1}$ for all $x \in \mathring{G}_1$. Analogously, any projective unitary $\mathfrak{g}$-representation $\overline{\pi}$ with domain $\mathcal{D}$ can be lifted to a unitary representation $\pi : \mathring{\mathfrak{g}} \to \mathfrak{u}(\mathcal{D})$ of some central $\mathbb{R}$-extension $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$. The continuous central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are up to isomorphism classified by $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$, the continuous second Lie algebra cohomology with trivial coefficients [JN19, Def. 6.2, Prop. 6.3]. To the 2-cocycle $\omega \in C^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ is associated the Lie algebra $\mathbb{R} \oplus_\omega \mathfrak{g}$, which as a vector space is the direct sum $\mathbb{R} \oplus \mathfrak{g}$, and whose Lie bracket is given by

$$[(a, \xi), (b, \eta)]_{\mathbb{R}\oplus_\omega \mathfrak{g}} := (\omega(\xi, \eta), [\xi, \eta]_{\mathfrak{g}}), \qquad a, b \in \mathbb{R},\ \xi, \eta \in \mathfrak{g}.$$

Thus, to study the projective unitary representations of $\mathfrak{g}$ up to equivalence, one may first determine $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$, choose for each class $[\omega] \in H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ a representative ω and then proceed to determine the equivalence classes of unitary representations π of $\mathbb{R} \oplus_\omega \mathfrak{g}$ satisfying $\pi(1, 0) = iI$. We will also write $\boldsymbol{c} := (1, 0) \in \mathbb{R} \oplus_\omega \mathfrak{g}$ for the central generator.

Remark 2.3.5. In the literature, one encounters the notion of the *level* of a unitary representation π of $\mathbb{R}\boldsymbol{c} \oplus_\omega \mathfrak{g}$, which is the number $l \in \mathbb{R}$ such that $\pi(\boldsymbol{c}) = ilI$ (see e.g.

[PS86, sec. 9.3]). Let us briefly clarify how such representations are included in the program outlined above, even though $\pi(\boldsymbol{c}) = iI$ is always assumed. Simply notice that such a representation of level l factors through the map $\mathbb{R}\boldsymbol{c} \oplus_\omega \mathfrak{g} \to \mathbb{R}\boldsymbol{c} \oplus_{l\cdot\omega} \mathfrak{g}$ induced by multiplication by l on the central factor. The corresponding representation π_2 of $\mathbb{R}\boldsymbol{c} \oplus_{l\cdot\omega} \mathfrak{g}$ satisfies $\pi_2(\boldsymbol{c}) = iI$. Notice that $\mathbb{R}\boldsymbol{c} \oplus_\omega \mathfrak{g} \to \mathbb{R} \oplus_{l\cdot\omega} \mathfrak{g}$ is an isomorphism of Lie algebras whenever $l \neq 0$, but *not* of central extensions unless $l = 1$, because a morphism of central extensions is required to be the identity on the central component. For $1 \neq l \in \mathbb{R}$, the cocycles ω and $l \cdot \omega$ are *not* equivalent unless $[\omega] = 0$ in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$.

Remark 2.3.6. If a projective unitary representation $\overline{\rho}$ of G is smooth, then the corresponding central $\mathbb{T}$-extension $\mathring{G}$ is again a locally convex Lie group [JN19, Thm. 4.3]. Moreover, there is a similar correspondence between smooth projective unitary representations $\overline{\rho}$ of G and their lifts $\rho : \mathring{G} \to \mathrm{U}(\mathcal{H}_\rho)$, which are then again smooth [JN19, Cor. 4.5, Thm. 7.3]. We furthermore have $\mathrm{P}(\mathcal{H}_\rho)^\infty = \mathrm{P}(\mathcal{H}_\rho^\infty)$ by [JN19, Thm. 4.3].

2.4 Positive energy representations

Let us now introduce the class of positive energy representations. Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$. We assume that G admits an exponential map. If $c \in \mathbb{R}$, $\mathcal{D}$ is a complex pre-Hilbert space and $X \in \mathcal{L}^\dagger(\mathcal{D})$, we write $X \geq c$ if $X^\dagger = X$ and $\langle \psi, X\psi \rangle \geq c\|\psi\|^2$ for every $\psi \in \mathcal{D}$.

Definition 2.4.1. Let $\mathcal{D}$ be a pre-Hilbert space.

— A continuous unitary representation π of the locally convex Lie algebra $\mathfrak{g}$ on $\mathcal{D}$ is said to be of *positive energy* (p.e.) at $\xi \in \mathfrak{g}$ if $-i\pi(\xi) \geq 0$.

— Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g}$ on $\mathcal{D}$ with lift $\pi : \mathring{\mathfrak{g}} \to \mathfrak{u}(\mathcal{D})$. Then $\overline{\pi}$ is of p.e. at ξ if π is of p.e. at some $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covering ξ.

— A smooth (projective) unitary representation of G on $\mathcal{H}_\rho$ is said to be of p.e. at $\xi \in \mathfrak{g}$ if the corresponding derived (projective) unitary representation of $\mathfrak{g}$ on $\mathcal{H}_\rho^\infty$ is so.

— We say that a (projective) unitary representation of G or $\mathfrak{g}$ is of positive energy at the convex cone $\mathcal{C} \subseteq \mathfrak{g}$ if it so at every $\xi \in \mathcal{C}$.

Remark 2.4.2. Let ρ be a smooth unitary representation of G. Then the set $\mathcal{C} := \mathcal{C}(d\rho) := \{\, \xi \in \mathfrak{g} \,:\, d\rho \text{ is of p.e. at } \xi \,\}$ is always a closed, G-invariant convex cone. Consequently, $\mathcal{C} \cap -\mathcal{C}$ and $\mathcal{C} - \mathcal{C}$ are ideals in $\mathfrak{g}$, called the *edge* and *span* of $\mathcal{C}$, respectively. If $\xi \in \mathcal{C} \cap -\mathcal{C}$ then $\xi \in \ker d\rho$, so by passing to the quotient $\mathfrak{g}/\ker d\rho$ one may always achieve that $\mathcal{C}$ is pointed.

2.5 Semibounded representations

We now introduce the class of semibounded representations, a 'stable' analogue of the positive energy condition. This class has been extensively studied in [Nee00, Nee17, Nee10b], and we collect some relevant results associated to such representations. Throughout the section, G continues to denote a locally convex Lie group with Lie algebra $\mathfrak{g}$, and which admits an exponential map.

Definition 2.5.1. A smooth unitary G-representation ρ is said to be *semibounded* if the set

$$W_\rho := \left\{ \xi \in \mathfrak{g} \,:\, \inf \operatorname{Spec}(-i\overline{d\rho(\xi)}) > -\infty \right\}$$

contains an interior point.

Remark 2.5.2. For finite-dimensional Lie groups, the class of semibounded representations has been subject to detailed study in [Nee00]. In particular, they are highest weight representations [Nee00, Def. X.2.9, Thm. X.3.9]. For a consideration of semibounded representations in the context of infinite-dimensional Lie groups, we refer to [Nee17] and [Nee10b].

In the finite-dimensional context, the semiboundedness condition turns out to be extremely restrictive, which in turn has consequences for arbitrary positive energy representations. The following result, Theorem 2.5.3, is based on the results in the monograph [Nee00]. We say that G is locally exponential if the restriction of its exponential map to a small-enough open 0-neighborhood in $\mathfrak{g}$ is a diffeomorphism onto an open 1-neighborhood of G.

Theorem 2.5.3. *Assume that G is connected and locally exponential. Take $\boldsymbol{d} \in \mathfrak{g}$ and let $\mathfrak{a} = \langle \boldsymbol{d} \rangle \lhd \mathfrak{g}$ be the closed ideal in $\mathfrak{g}$ generated by $\boldsymbol{d}$. Assume that $\dim(\mathfrak{a}) < \infty$ and that $\mathfrak{a}$ is stable, in the sense that $Ad_G(\mathfrak{a}) \subseteq \mathfrak{a}$. Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary G-representation which is of p.e. at $\boldsymbol{d} \in \mathfrak{g}$. Define $\mathfrak{h} := \mathfrak{a}/\ker d\rho$. The following assertions are valid:*

1. *$\mathfrak{a} = \mathcal{C} - \mathcal{C}$, where $\mathcal{C} \subseteq \mathfrak{g}$ is the closed G-invariant convex cone in $\mathfrak{g}$ generated by $\boldsymbol{d}$.*
2. *The closure of $\mathcal{C} + \ker d\rho$ in $\mathfrak{h}$ is a pointed, generating and G-invariant convex cone. Thus $\mathcal{C} \cap -\mathcal{C} \subseteq \ker d\rho$.*
3. *Let $A \lhd G$ be a connected normal Lie subgroup integrating $\mathfrak{a}$. Then $\rho|_A$ is semibounded.*
4. *Let $\mathfrak{h}_n$ denote the maximal nilpotent ideal of $\mathfrak{h}$. Then $[\mathfrak{h}_n, \mathfrak{h}_n] \subseteq \mathfrak{z}(\mathfrak{h})$. Moreover, there exists a reductive Lie algebra l such that $\mathfrak{h} \cong \mathfrak{h}_n \rtimes l$.*
5. *Let $\mathfrak{a}_n$ denote the maximal nilpotent ideal of $\mathfrak{a}$. Then $[\mathfrak{a}, [\mathfrak{a}_n, \mathfrak{a}_n]] \subseteq \ker d\rho$.*

Proof. For the first point, let $\mathfrak{a}'$ be the closure of $\mathcal{C} - \mathcal{C}$ in $\mathfrak{g}$. As $\mathfrak{a}'$ is a closed ideal in $\mathfrak{g}$ containing $\boldsymbol{d}$, we have $\mathfrak{a} \subseteq \mathfrak{a}'$. On the other hand, we know that $\mathrm{Ad}_G(\boldsymbol{d}) \subseteq \mathfrak{a}$ because $\mathfrak{a}$ is stable. Thus $\mathcal{C} \subseteq \mathfrak{a}$ and hence $\mathfrak{a}' \subseteq \mathfrak{a}$. So $\mathfrak{a}' = \mathfrak{a}$. In particular $\dim(\mathfrak{a}') < \infty$ and so $\mathcal{C} - \mathcal{C} = \mathfrak{a}' = \mathfrak{a}$. Next we prove the second statement. Take

$\xi \in (\overline{\mathcal{C} + \ker d\rho}) \cap -(\overline{\mathcal{C} + \ker d\rho})$. Then $d\rho(\xi) \geq 0$ and $d\rho(\xi) \leq 0$, in view of Remark 2.4.2, and hence $\mathrm{Spec}(\overline{d\rho(\xi)}) = \{0\}$. As $d\rho(\xi)$ is essentially skew-adjoint, it follows that $\xi \in \ker d\rho$. Thus $\overline{\mathcal{C} + \ker d\rho}$ is pointed in $\mathfrak{h}$. As $\mathcal{C}$ is G-invariant and convex, it is clear that the same holds for the closure of $\mathcal{C} + \ker d\rho$ in $\mathfrak{h}$. The latter is also generating in $\mathfrak{h}$ because $\mathfrak{a} = \mathcal{C} - \mathcal{C}$. Next we show that $\rho|_A$ is semibounded, where $A \triangleleft G$ is a connected normal Lie subgroup integrating $\mathfrak{a} \triangleleft \mathfrak{g}$. As $\mathfrak{a}$ is spanned by $\mathcal{C}$ and $\dim \mathfrak{a} < \infty$, it follows that $\mathcal{C} \subseteq \mathfrak{a}$ has interior points. As $\mathcal{C} \subseteq W_\rho$, this implies that W_ρ has interior points. Hence $\rho|_A$ is semibounded. For the remaining points, we use the results in [Nee00]. We first show that $\mathfrak{h}$ is *admissible*, in the sense of [Nee00, Def. VII.3.2]. Using the second point, the convex cone $(\overline{\mathcal{C} + \ker d\rho}) \oplus \mathbb{R}_{\geq 0}$ in $\mathfrak{h} \oplus \mathbb{R}$ is closed, pointed, generating and $\mathrm{Inn}(\mathfrak{h})$-invariant. By [Nee00, Lem. VII.3.1, Def. VII.3.2] this implies that $\mathfrak{h}$ is admissible. By [Nee00, Thm. VII.3.10], it follows that $[\mathfrak{h}_n, \mathfrak{h}_n] \subseteq \mathfrak{z}(\mathfrak{h})$ and that $\mathfrak{h}$ contains a compactly embedded Cartan subalgebra $\mathfrak{t}$ (where as in [Nee00, Def. VII.1.1], a subalgebra $\mathfrak{t} \subseteq \mathfrak{h}$ is called *compactly embedded* if $\overline{\langle e^{\mathrm{ad}(\mathfrak{t})} \rangle}$ is compact in $\mathrm{Aut}(\mathfrak{h})$). Using [Nee00, Lem. VII.2.26(iv)], we obtain that there exists some reductive Lie algebra l with $\mathfrak{h} \cong \mathfrak{h}_n \rtimes l$. Since $[\mathfrak{h}, [\mathfrak{h}_n, \mathfrak{h}_n]] = 0$ and $\mathfrak{h} = \mathfrak{a}/\ker d\rho$, it follows in particular that $[\mathfrak{a}, [\mathfrak{a}_n, \mathfrak{a}_n]] \subseteq \ker d\rho$. ☐

For projective p.e. representations, this leads to:

Corollary 2.5.4. *Let G, $\boldsymbol{d}, \mathfrak{a}$ and $\mathfrak{a}_n$ be as Theorem 2.5.3. Let $(\overline{\rho}, \mathcal{H}_\rho)$ be a smooth projective unitary representation of G. Suppose that $\overline{\rho}$ is of p.e. at $\boldsymbol{d} \in \mathfrak{g}$. Then $[\mathfrak{a}, [\mathfrak{a}_n, \mathfrak{a}_n]] \subseteq \ker d\overline{\rho}$.*

Proof. Let $\rho : \mathring{G} \to \mathrm{U}(\mathcal{H}_\rho)$ be the lift of $\overline{\rho}$ to a central $\mathbb{T}$-extension $\mathring{G}$ of G. Let $\mathring{\mathfrak{g}} := \mathrm{Lie}(\mathring{G})$. There exists some $\mathring{\boldsymbol{d}} \in \mathring{\mathfrak{g}}$ s.t. $d\rho$ is of p.e. at $\mathring{\boldsymbol{d}} \in \mathring{\mathfrak{g}}$. Let $\mathring{\mathfrak{a}}$ denote the ideal in $\mathring{\mathfrak{g}}$ generated by $\mathring{\boldsymbol{d}}$ and let $\mathring{\mathfrak{a}}_n$ denote the maximal nilpotent ideal in $\mathring{\mathfrak{a}}$. Then $d\rho([\,\mathring{\mathfrak{a}}, [\,\mathring{\mathfrak{a}}_n, \mathring{\mathfrak{a}}_n\,]]) = \{0\}$ by Theorem 2.5.3. Thus $d\overline{\rho}([\mathfrak{a}, [\mathfrak{a}_n, \mathfrak{a}_n]]) = \{0\}$, where we used that the quotient map $\mathring{\mathfrak{g}} \to \mathfrak{g}$ projects $\mathring{\mathfrak{a}}$ and $\mathring{\mathfrak{a}}_n$ onto $\mathfrak{a}$ and $\mathfrak{a}_n$, respectively. ☐

The following simple lemma will also be useful.

Lemma 2.5.5. *Assume that $\dim(G) < \infty$. Let $\overline{\rho} : G \to \mathrm{PU}(\mathcal{H}_\rho)$ be a continuous projective unitary representation of G which is of p.e. at every element of $\mathfrak{g}$. Then $\overline{\rho}$ is continuous w.r.t. the norm-topology on $\mathrm{U}(\mathcal{H}_\rho)$.*

Proof. Let $d\rho : \mathring{\mathfrak{g}} \to \mathfrak{u}(\mathcal{H}_\rho^\infty)$ be the lift of $d\overline{\rho}$. Identify $\mathring{\mathfrak{g}} \cong \mathbb{R} \oplus_\omega \mathfrak{g}$ for some 2-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. The assumptions imply that for every $\xi \in \mathfrak{g}$ there exists $E_\xi \in \mathbb{R}$ s.t. $-id\rho(\xi) \geq E_\xi$. As this holds in particular for both ξ and $-\xi$, $d\rho(\xi)$ is a bounded operator for any $\xi \in \mathfrak{g}$. As $\dim(\mathfrak{g}) < \infty$, one finds by choosing a basis (e_μ) of $\mathfrak{g}$ that there exists $C > 0$ s.t. $\|d\rho(\xi)\| \leq C\|\xi\|$ where $\|\xi\| := \sup_\mu |\xi_\mu|$ if $\xi = \sum_\mu \xi_\mu e_\mu$. Thus $\xi \mapsto d\rho(\xi)$ is norm-continuous. This implies norm-continuity of $\overline{\rho}$ because $\mathcal{B}(\mathcal{H}_\rho) \to \mathcal{B}(\mathcal{H}_\rho), T \mapsto e^T$ is norm-continuous and $\overline{\rho}(\exp_G(\xi)) = [e^{d\rho(\xi)}] \in \mathrm{PU}(\mathcal{H}_\rho)$ for $\xi \in \mathfrak{g}$. ☐

Chapter 3

Generalized positive energy and KMS representations

As explained in Chapter 1, the main purpose of this thesis is to develop further understanding of positive energy and KMS representations. These two seemingly different classes exhibit similar behavior in certain respects. In particular, we will see that both give rise to so-called generalized positive energy representations, a notion that is introduced in Section 3.1 below. Its utility lies primarily in the fact that it unifies the positive energy and KMS conditions to some extent, thereby allowing their simultaneous study. In Section 3.2, we briefly recall the modular theory of von Neumann algebras and then proceed to define KMS representations. We then consider a number of examples and discuss some of their properties, in particular making the important observation that KMS representations give rise to generalized positive energy ones.

This chapter is based on [Nie23a, Part I].

3.1 Generalized positive energy representations

Let G denote a regular locally convex Lie group with Lie algebra $\mathfrak{g}$. The class of positive energy representations can be generalized by relaxing the condition $-id\rho(\xi) \geq 0$ in Definition 2.4.1. We define a suitable relaxed notion, the *generalized positive energy condition*, and show that it can still be very restrictive. In particular, we show that for a projective unitary representation which of is generalized positive energy, its kernel is related to a particular quadratic form canonically associated to the corresponding class in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$. This observation will play a key role in Chapter 4 and Chapter 5.

Definition 3.1.1. Let $\mathcal{D}$ be a complex pre-Hilbert space with Hilbert space completion $\mathcal{H}$. Let $\mathfrak{h}$ be a locally convex topological Lie algebra.

— A continuous unitary representation $\pi : \mathfrak{h} \to \mathfrak{u}(\mathcal{D})$ is of *generalized positive energy* (g.p.e.) at $\xi \in \mathfrak{h}$ if there exists a 1-connected regular Lie group H

with Lie algebra $\mathfrak{h}$ and a dense linear subspace $\mathcal{D}_\xi \subseteq \mathcal{D}$ such that

$$\forall \psi \in \mathcal{D}_\xi \;:\; E_\psi(\pi, \xi) := \inf_{h \in H} \langle \psi, -i\pi(\mathrm{Ad}_h(\xi))\psi \rangle > -\infty. \tag{3.1.1}$$

— Let $\overline{\pi} : \mathfrak{h} \to \mathfrak{pu}(\mathcal{D})$ be a continuous projective unitary representation of $\mathfrak{h}$ on $\mathcal{D}$ with lift $\pi : \mathring{\mathfrak{h}} \to \mathfrak{u}(\mathcal{D})$. Then $\overline{\pi}$ is said to be of generalized positive energy at $\xi \in \mathfrak{h}$ if there is some $\mathring{\xi} \in \mathring{\mathfrak{h}}$ covering ξ such that π is of g.p.e. at $\mathring{\xi}$.

— Let $\rho : G \to \mathrm{U}(\mathcal{H}_\rho)$ be a smooth unitary representation of G. Then ρ is said to be of g.p.e. at $\xi \in \mathfrak{g}$ if its derived representation $d\rho$ on $\mathcal{H}_\rho^\infty$ is so.

— Let $\overline{\rho} : G \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation of G with lift $\rho : \mathring{G} \to \mathrm{U}(\mathcal{H}_\rho)$. Let $\mathring{\mathfrak{g}}$ be the Lie algebra of $\mathring{G}$. Then $\overline{\rho}$ is of g.p.e. at $\xi \in \mathfrak{g}$ if ρ is of g.p.e. at some $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covering ξ.

Remark 3.1.2. If $(\rho, \mathcal{H}_\rho)$ is a smooth unitary G-representation, then

$$\langle \psi, d\rho(\mathrm{Ad}_g^{-1}(\xi))\psi \rangle = \langle \rho(g)\psi, d\rho(\xi)\rho(g)\psi \rangle, \qquad \forall g \in G,\ \xi \in \mathfrak{g},\ \psi \in \mathcal{H}_\rho^\infty.$$

This implies that any (projective) unitary representation of G that is of positive energy at $\xi \in \mathfrak{g}$ is also of generalized positive energy at ξ.

Remark 3.1.3. A unitary representation $(\rho, \mathcal{H}_\rho)$ of G is of generalized positive energy at $\xi \in \mathfrak{g}$ if and only if

$$\inf_{g \in G_0} \langle \rho(g)\psi, -id\rho(\xi)\rho(g)\psi \rangle > -\infty$$

for all ψ in a linear subspace $\mathcal{D}_\xi \subseteq \mathcal{H}_\rho^\infty$ which is dense in $\mathcal{H}_\rho$. A projective unitary G-representation $\overline{\rho}$ with lift ρ is of g.p.e. at $\xi \in \mathfrak{g}$ if and only if for some (and hence any) $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covering ξ, the function

$$\mu : \mathrm{P}(\mathcal{H}_\rho^\infty) \to \mathbb{R}, \quad \mu([\psi]) := \frac{1}{\|\psi\|^2}\langle \psi, -id\rho(\mathring{\xi})\psi \rangle$$

is bounded below on the G_0-orbit $\mathcal{O}_{[\psi]}$ for all ψ in a dense linear subspace $\mathcal{D}_\xi \subseteq \mathcal{H}_\rho^\infty$.

Remark 3.1.4. If π is a continuous (projective) unitary representation of $\mathfrak{g}$, then the set

$$\mathfrak{C}(\pi) := \{\, \xi \in \mathfrak{g} \,:\, \pi \text{ is of g.p.e. at } \xi \,\} \subseteq \mathfrak{g}$$

is always an Ad_{G_0}-invariant cone in $\mathfrak{g}$.

An important observation for the class of g.p.e. representations is the following one:

Lemma 3.1.5. *Let $\pi : \mathfrak{g} \to \mathfrak{u}(\mathcal{D})$ be a continuous unitary representation of $\mathfrak{g}$ on the pre-Hilbert space $\mathcal{D}$ that is of g.p.e. at $\xi \in \mathfrak{g}$. Suppose that $\eta \in \mathfrak{g}$ satisfies $[[\xi, \eta], \eta] \in \mathfrak{z}(\mathfrak{g})$. Then for every ψ in a dense linear subspace $\mathcal{D}_\xi \subseteq \mathcal{D}$ we have:*

$$\begin{aligned} 0 &\leq \langle \psi, -i\pi([[\xi, \eta], \eta])\psi \rangle, \\ \langle \psi, -i\pi([\xi, \eta])\psi \rangle^2 &\leq 2\langle \psi, -i\pi([[\xi, \eta], \eta])\psi \rangle \Big(\langle \psi, -i\pi(\xi)\psi \rangle - E_\psi(\pi, \xi) \Big). \end{aligned} \tag{3.1.2}$$

In particular, if $[[\xi, \eta], \eta] = 0$ then $\pi([\xi, \eta]) = 0$.

Proof. Let $\mathcal{D}_\xi \subseteq \mathcal{D}$ be a dense linear subspace for which (3.1.1) is valid. Let $\psi \in \mathcal{D}_\xi$. Then $\langle \psi, -i\pi(e^{t\mathrm{ad}_\eta}\xi)\psi\rangle \geq E_\psi(\pi, \xi)$ for all $t \in \mathbb{R}$. As $[[\xi,\eta],\eta] \in \mathfrak{Z}(\mathfrak{g})$, the third derivative $\gamma^{(3)} : \mathbb{R} \to \mathfrak{g}$ of the smooth path $\gamma : \mathbb{R} \to \mathfrak{g}$, $t \mapsto e^{t\mathrm{ad}_\eta}\xi$ vanishes. From Taylor's formula (which holds for smooth maps between locally convex vector spaces by [Nee06, Prop. I.2.3]), it follows that $e^{t\mathrm{ad}_\eta}\xi = \xi + t[\eta,\xi] + \frac{t^2}{2}[[\xi,\eta],\eta]$ for all $t \in \mathbb{R}$. Thus

$$\langle \psi, -i\pi(\xi)\psi\rangle + t\langle \psi, -i\pi([\eta,\xi])\psi\rangle + \frac{t^2}{2}\langle \psi, -i\pi([[\xi,\eta],\eta])\psi\rangle \geq E_\psi(\pi,\xi), \qquad \forall t \in \mathbb{R}$$

The equations (3.1.2) follows from the fact that $at^2 + bt + c \geq 0$ for all $t \in \mathbb{R}$ if and only if $a, c \geq 0$ and $b^2 \leq 4ac$. So if $[[\xi,\eta],\eta] = 0$, then $\langle \psi, -i\pi([\xi,\eta])\psi\rangle = 0$ for all $\psi \in \mathcal{D}_\xi$. As $\mathcal{D}_\xi$ is a complex vector space, this implies by the polarization identity that $\langle \psi_1, -i\pi([\xi,\eta])\psi_2\rangle = 0$ for all $\psi_1, \psi_2 \in \mathcal{D}_\xi$. As $\mathcal{D}_\xi$ is dense and $\pi([\xi,\eta])^\dagger = -\pi([\xi,\eta])$, it follows that $\pi([\xi,\eta])\psi = 0$ for all $\psi \in \mathcal{D}$. □

In the projective context, this sets up a relation between $\ker \overline{\pi}$ and the class $[\omega] \in H^2_{\mathrm{ct}}(\mathfrak{g};\mathbb{R})$ defined by the corresponding central $\mathbb{R}$-extension $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$. This is exploited in Section 4.4 and Chapter 5 below.

Proposition 3.1.6. *Let $\overline{\pi}$ be a continuous projective unitary $\mathfrak{g}$-representation on the pre-Hilbert space $\mathcal{D}$ with lift $\pi : \mathring{\mathfrak{g}} \to \mathfrak{u}(\mathcal{D})$ for some continuous central $\mathbb{R}$-extension $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$. Let ω represent the corresponding class in $H^2_{\mathrm{ct}}(\mathfrak{g},\mathbb{R})$. Assume that $\overline{\pi}$ is of g.p.e. at $\xi \in \mathfrak{g}$. Suppose that $\eta \in \mathfrak{g}$ satisfies $[[\xi,\eta],\eta] = 0$. Then $\omega([\xi,\eta],\eta) \geq 0$ and*

$$\omega([\xi,\eta],\eta) = 0 \iff \overline{\pi}([\xi,\eta]) = 0.$$

Proof. Identify $\mathring{\mathfrak{g}}$ with $\mathbb{R} \oplus_\omega \mathfrak{g}$. Let $\mathring{\xi} \in \mathfrak{C}(\pi)$ and $\mathring{\eta} \in \mathring{\mathfrak{g}}$ be lifts of ξ and η, respectively. We have that $[[\mathring{\xi},\mathring{\eta}],\mathring{\eta}] = \omega([\xi,\eta],\eta) \in \mathfrak{Z}(\mathring{\mathfrak{g}})$, because $[[\xi,\eta],\eta] = 0$. Using Lemma 3.1.5 it follows that $\omega([\xi,\eta],\eta) \geq 0$. If $\omega([\xi,\eta],\eta) = 0$, then $[[\mathring{\xi},\mathring{\eta}],\mathring{\eta}] = 0$ and so Lemma 3.1.5 implies that $\pi([\mathring{\xi},\mathring{\eta}]) = 0$. Hence $\overline{\pi}([\xi,\eta]) = 0$. Conversely, if $\overline{\pi}([\xi,\eta]) = 0$, then $i\omega([\xi,\eta],\eta) = [\pi([\xi,\eta]),\pi(\eta)] - \pi([[\xi,\eta],\eta]) = 0$, because $[[\xi,\eta],\eta] = 0$. □

Remark 3.1.7. Notice in Proposition 3.1.6 that whenever $[[\xi,\eta],\eta] = 0$, the value of $\omega([\xi,\eta],\eta)$ does not depend on the choice of representative ω of $[\omega] \in H^2_{\mathrm{ct}}(\mathfrak{g},\mathbb{R})$.

The following special case of Proposition 3.1.6 will be particularly useful:

Corollary 3.1.8. *Let $\mathfrak{p}$ and $\mathfrak{g}$ be locally convex Lie algebras. Let $D : \mathfrak{p} \to \mathrm{der}(\mathfrak{g})$ be a homomorphism for which the corresponding action $\mathfrak{p} \times \mathfrak{g} \to \mathfrak{g}$ is continuous. Let $\mathcal{D}$ be a complex pre-Hilbert space and let $\overline{\pi} : \mathfrak{g} \rtimes_D \mathfrak{p} \to \mathfrak{pu}(\mathcal{D})$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$ on $\mathcal{D}$ that is of g.p.e. at $p \in \mathfrak{p}$. Let $[\omega] \in H^2_{\mathrm{ct}}(\mathfrak{g} \rtimes_D \mathfrak{p};\mathbb{R})$ be the corresponding class in $H^2_{\mathrm{ct}}(\mathfrak{g} \rtimes_D \mathfrak{p};\mathbb{R})$. Let $\eta \in \mathfrak{g}$ satisfy $[D(p)\eta,\eta] = 0$. Then $\omega(D(p)\eta,\eta) \geq 0$ and*

$$\omega(D(p)\eta,\eta) = 0 \iff \overline{\pi}(D(p)\eta) = 0.$$

3.2 KMS representations

In the following, we introduce the class of KMS representations. We will see in particular that these give rise to generalized positive energy representations. Its definition makes use of the modular theory of von Neumann algebras, which we recall first.

3.2.1 Modular theory of von Neumann algebras

We recall the modular condition and the notion of a KMS state on a von Neumann algebra $\mathcal{M}$, whilst fixing our conventions and notation. We refer to [Tak03a, Ch. VIII], [BR87, Ch. 2.5] and [BR97, Ch. 5.3] for a detailed consideration of the modular theory of von Neumann algebras and of KMS states.

If $\mathcal{M}$ is a von Neumann algebra, we write $\mathcal{M}_*$ for its pre-dual, equipped with the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology. We write $\mathcal{S}(\mathcal{M}) \subseteq \mathcal{M}_*$ for the set of normal states on $\mathcal{M}$. Further, if $\phi \in \mathcal{S}(\mathcal{M})$, we write $\pi_\phi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\phi)$ for the GNS-representation of $\mathcal{M}$ relative to ϕ and we define $\mathcal{M}_\phi := \pi_\phi(\mathcal{M})''$. Let $\Omega_\phi \in \mathcal{H}_\phi$ denote the canonical cyclic vector satisfying $\phi(x) = \langle \Omega_\phi, \pi_\phi(x)\Omega_\phi \rangle$ for all $x \in \mathcal{M}$. Whenever Ω_ϕ is separating for $\mathcal{M}_\phi$, let S_ϕ denote the unique closed conjugate-linear operator satisfying $S_\phi x \Omega_\phi = x^* \Omega_\phi$ for all $x \in \mathcal{M}_\phi$. Let $S_\phi = J_\phi \Delta_\phi^{\frac{1}{2}}$ be its polar decomposition, where the operators Δ_ϕ and J_ϕ are positive and anti-unitary, respectively.

Definition 3.2.1. A map $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ is said to be $\sigma(\mathcal{M}_*, \mathcal{M})$-continuous if for every $x \in \mathcal{M}$, the map $\mathbb{R} \to \mathcal{M},\ t \mapsto \sigma_t(x)$ is continuous w.r.t. the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology on $\mathcal{M}$.

Definition 3.2.2. Let $\phi \in \mathcal{S}(\mathcal{M})$ be a normal state. Let $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ be a one-parameter group of automorphisms of $\mathcal{M}$. Define $\mathrm{St} := \{\, z \,:\, z \in \mathbb{C},\ 0 < \mathrm{Im}(z) < 1 \,\}$.

— ϕ is said to satisfy the *modular condition* for σ if the following two conditions are satisfied:

1. $\phi = \phi \circ \sigma_t$ for all $t \in \mathbb{R}$.
2. For every $x, y \in \mathcal{M}$, there exists a bounded continuous function $F_{x,y} : \overline{\mathrm{St}} \to \mathbb{C}$ which is holomorphic on St and s.t. for every $t \in \mathbb{R}$:

$$F_{x,y}(t) = \phi(\sigma_t(x)y),$$
$$F_{x,y}(t+i) = \phi(y\sigma_t(x)).$$

— ϕ is said to be *KMS* w.r.t. σ at inverse temperature $\beta > 0$ if it satisfies the modular condition for $t \mapsto \sigma_{-\beta t}$. In that case, we also say that ϕ is σ-KMS at inverse-temperature β. If $\beta = 1$ we simply say that ϕ is a σ-KMS state.

Remark 3.2.3.

1. Suppose that $\phi \in \mathcal{S}(\mathcal{M})$ is faithful. Then there exists a unique automorphism group $\sigma^\phi : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ for which ϕ satisfies the modular condition

[Tak03a, Thm. VIII.1.2], [BR87, Thm. 2.5.14]. The automorphism group σ^ϕ is $\sigma(\mathcal{M}_*, \mathcal{M})$-continuous. As ϕ is faithful, $\pi_\phi : \mathcal{M} \to \mathcal{M}_\phi$ is injective and hence a $*$-isomorphism between $\mathcal{M}$ and $\mathcal{M}_\phi$ [BR87, Thm. 2.4.24]. Thus one may identify $\mathcal{M}$ with $\mathcal{M}_\phi$ via $\pi_\phi : \mathcal{M} \to \mathcal{M}_\phi \subseteq \mathcal{B}(\mathcal{H}_\phi)$. Finally, there is a unique conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{M}^\mathbb{R}$ s.t. $\phi = \phi_0 \circ \mathcal{E}$, where $\mathcal{M}^\mathbb{R} := \left\{ x \in \mathcal{M} : \sigma_t^\phi(x) = x \quad \forall t \in \mathbb{R} \right\}$ and where $\phi_0 := \phi|_{\mathcal{M}^\mathbb{R}}$ [Tak03a, Thm. IX.4.2].

2. If ϕ is not necessarily faithful, then ϕ satisfies the modular condition for some $\sigma(\mathcal{M}_*, \mathcal{M})$-continuous 1-parameter group $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ of $*$-automorphisms of $\mathcal{M}$ if and only if $\Omega_\phi \in \mathcal{H}_\phi$ is separating for $\mathcal{M}_\phi$. In that case, there is a central projection $p \in \mathcal{Z}(\mathcal{M})$ such that $\phi(1-p) = 0$ and ϕ is faithful on $\mathcal{M}p$. Moreover, $\sigma_t(p) = p$ for all $t \in \mathbb{R}$ and $\sigma|_{\mathcal{M}p}$ is uniquely determined by the modular condition for ϕ [BR97, Thm. 5.3.10].

3. In particular, if $\mathcal{M}$ is a factor and ϕ is KMS w.r.t. $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$, then necessarily $p = I$ and whence ϕ must be faithful. Consequently $\sigma = \sigma_t^\phi$ is necessary.

4. In the converse direction, given a $\sigma(\mathcal{M}_*, \mathcal{M})$-continuous automorphism group $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$, there may be no, precisely one, or multiple states in $\mathcal{S}(\mathcal{M})$ that are KMS w.r.t. σ. The set of σ-KMS states in $\mathcal{S}(\mathcal{M})$ is considered in [BR97, Ch. 5.3.2]. In particular, if $\phi \in \mathcal{S}(\mathcal{M})$ is a faithful σ-KMS state and $\psi \in \mathcal{S}(\mathcal{M})$, then ψ is σ-KMS if and only if there is a (necessarily unique) positive operator T affiliated to $\mathcal{Z}(\mathcal{M})$ such that $\psi(x) = \phi(T^{\frac{1}{2}} x T^{\frac{1}{2}})$ for all $x \in \mathcal{M}$ [BR97, Prop. 5.3.29]. In [BEK80] and [BEK86], the set K_β of normal σ-KMS states at inverse temperature β is studied in the setting of C^*-dynamical systems.

5. As a consequence of the previous items, if $\mathcal{M}$ is a factor and $\phi, \psi \in \mathcal{S}(\mathcal{M})$ are both σ-KMS, then $\phi = \psi$, so that two distinct normal states can not share the same modular automorphism group.

Remark 3.2.4. Suppose $\phi \in \mathcal{S}(\mathcal{M})$ is KMS w.r.t. $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$. Let σ^ϕ denote the modular automorphism group of $\mathcal{M}_\phi$ defined by the faithful state $\langle \Omega_\phi, \cdot\, \Omega_\phi \rangle$ on $\mathcal{M}_\phi$. It then holds true that $\sigma_t^\phi(\pi_\phi(x)) = \pi_\phi(\sigma_{-t}(x))$ for any $x \in \mathcal{M}$ and $t \in \mathbb{R}$. Indeed, by [BR97, cor. 5.3.4], the state $\langle \Omega_\phi, \cdot\, \Omega_\phi \rangle$ on $\mathcal{M}_\phi$ is KMS w.r.t. the unique automorphism group $\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{M}_\phi)$ satisfying $\tau_t(\pi_\phi(x))\Omega_\phi = \pi_\phi(\sigma_t(x))\Omega_\phi$ for all $t \in \mathbb{R}$. Then $\sigma_t^\phi = \tau_{-t}$ by the uniqueness of the modular automorphism group (and the minus sign in the definition of KMS states). As Ω_ϕ is separating for $\mathcal{M}_\phi$, it follows that $\sigma_t^\phi(\pi_\phi(x)) = \pi_\phi(\sigma_{-t}(x))$.

Example 3.2.5 (Gibbs States)**.** Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\sigma_t(x) = e^{itH} x e^{-itH}$ for some self-adjoint operator H satisfying $Z_\beta := \mathrm{Tr}(e^{-\beta H}) < \infty$ for some $\beta > 0$. Consider the normal state $\phi(x) = \frac{1}{Z_\beta} \mathrm{Tr}(e^{-\beta H} x)$ on $\mathcal{M}$. The modular automorphism group corresponding to ϕ is given by $\sigma_t^\phi(x) = e^{-i\beta t H} x e^{i\beta t H} = \sigma_{-\beta t}(x)$ [BR87, Example 2.5.16]. Thus ϕ satisfies the modular condition for $\sigma_{-\beta t}$ and is therefore KMS at inverse temperature β w.r.t. σ_t.

Gibbs states $\phi(x) = \frac{1}{Z_\beta}\operatorname{Tr}(e^{-\beta H}x)$ constitute the simplest class of examples of KMS states. We will encounter a variety of different KMS states in Section 3.2.2 below.

3.2.2 KMS representations

In the following, let G be a regular locally convex Lie group with Lie algebra $\mathfrak{g}$. Let $N \subseteq G$ be an embedded Lie subgroup.

Definition 3.2.6. Let $(\rho, \mathcal{H}_\rho)$ be a unitary G-representation. Let $\mathcal{N} := \rho(N)'' \subseteq \mathcal{B}(\mathcal{H}_\rho)$ be the von Neumann-algebra generated by $\rho(N)$. For $\phi \in \mathcal{N}_*$, define the function

$$\widehat{\phi} : N \to \mathbb{C}, \qquad \widehat{\phi}(n) := \phi(\rho(n)).$$

Define $\mathcal{N}_*^\infty := \left\{ \phi \in \mathcal{N}_* \,:\, \widehat{\phi} \in C^\infty(N;\mathbb{C}) \right\}$ and set $\mathcal{S}(\mathcal{N})^\infty := \mathcal{S}(\mathcal{N}) \cap \mathcal{N}_*^\infty$.

- Let $\xi \in \mathfrak{g}$ and $\phi \in \mathcal{S}(\mathcal{N})$. We say that ϕ is *KMS-compatible* with (ρ, ξ, N) if $e^{t\xi}Ne^{-t\xi} \subseteq N$ for all $t \in \mathbb{R}$ and ϕ is KMS w.r.t. the automorphism group $\mathbb{R} \to \operatorname{Aut}(\mathcal{N})$ defined by $t \mapsto \operatorname{Ad}(\rho(e^{t\xi}))$.
- Define $\operatorname{KMS}(\rho, \xi, N) := \{ \phi \in \mathcal{S}(\mathcal{N}) \,:\, \phi \text{ is KMS-compatible with } (\rho, \xi, N) \}$. Similarly, let $\operatorname{KMS}(\rho, \xi, N)^\infty := \operatorname{KMS}(\rho, \xi, N) \cap \mathcal{S}(\mathcal{N})^\infty$.
- ρ is said to be *KMS at $\xi \in \mathfrak{g}$ relative to N* if $\operatorname{KMS}(\rho, \xi, N) \neq \emptyset$. It is called *smoothly-KMS at ξ relative to N* if $\operatorname{KMS}(\rho, \xi, N)^\infty \neq \emptyset$.

If the subgroup N is clear from the context, we drop N from the notation and simply write $\operatorname{KMS}(\rho, \xi)$ and $\operatorname{KMS}(\rho, \xi)^\infty$. We then also say that ρ is KMS at ξ if it is so relative to N.

Remark 3.2.7. For any fixed $\xi \in \mathfrak{g}$ satisfying $\operatorname{Ad}(e^{t\xi})N \subseteq N$ for all $t \in \mathbb{R}$, one may as well consider the semidirect product $N \rtimes_\alpha \mathbb{R}$, where $\alpha : \mathbb{R} \to \operatorname{Aut}(N)$ is defined by $\alpha_t := \operatorname{Ad}(e^{t\xi})\big|_N$. Definition 3.2.6 additionally allows for the situation where ρ is KMS at multiple $\xi_I \in \mathfrak{g}$, relative to possibly distinct subgroups $N_I \subseteq G$, where $I \in \mathcal{I}$ for some indexing set $\mathcal{I}$. We will see an example of this situation in Example 3.2.22 below.

In the projective context, we make the following definition:

Definition 3.2.8. Let $\overline{\rho} : G \to \operatorname{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation with lift $\rho : \mathring{G} \to \operatorname{U}(\mathcal{H}_\rho)$. Let $\mathring{\mathfrak{g}}$ be the Lie algebra of $\mathring{G}$ and let $\mathring{N} \subseteq \mathring{G}$ be the Lie subgroup covering N. We say that $\overline{\rho}$ is smoothly-KMS at $\xi \in \mathfrak{g}$ relative to N if there exists $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covering ξ such that ρ is smoothly-KMS at $\mathring{\xi}$ relative to $\mathring{N}$.

In the following, let $(\rho, \mathcal{H}_\rho)$ be a unitary G-representation let $\mathcal{N} := \rho(N)''$ be the von Neumann-algebra generated by $\rho(N)$. We denote by $\pi_\phi : \mathcal{N} \to \mathcal{B}(\mathcal{H}_\phi)$ the GNS-representation of $\mathcal{N}$ relative to $\phi \in \mathcal{S}(\mathcal{N})$. We also denote by $\Omega_\phi \in \mathcal{H}_\phi$ the canonical $\mathcal{N}$-cyclic vector satisfying $\phi(x) = \langle \Omega_\phi, \pi_\phi(x)\Omega_\phi \rangle$ for all $x \in \mathcal{N}$, and we write $\rho_\phi := \pi_\phi \circ \rho : N \to \mathrm{U}(\mathcal{H}_\phi)$ for the unitary N-representation on $\mathcal{H}_\phi$. Finally, define $\mathcal{N}_\phi := \rho_\phi(N)'' \subseteq \mathcal{B}(\mathcal{H}_\phi)$.

Lemma 3.2.9. *Let $\phi \in \mathcal{S}(\mathcal{N})$. Then $\widehat{\phi}$ is smooth on N if and only if $\Omega_\phi \in \mathcal{H}^\infty_{\rho_\phi}$. In this case, $\mathcal{H}^\infty_{\rho_\phi}$ is dense, so ρ_ϕ is smooth.*

Proof. Assume that $\widehat{\phi}$ is smooth. Then $n \mapsto \langle \Omega_\phi, \rho_\phi(n)\Omega_\phi \rangle$ is smooth. By [Nee10a, Thm. 7.2], it follows $n \mapsto \rho_\phi(n)\Omega_\phi$ is smooth $N \to \mathcal{H}_\phi$. The converse direction is trivial. Assume that $\Omega_\phi \in \mathcal{H}^\infty_{\rho_\phi}$. As $\mathcal{H}^\infty_{\rho_\phi}$ is N-invariant and Ω_ϕ is cyclic for N, it follows that $\mathcal{H}^\infty_{\rho_\phi}$ is dense in $\mathcal{H}_\phi$. □

Consider the left action of G on $\mathcal{S}(\mathcal{N})$ defined by

$$(g.\phi)(x) := \phi(\rho(g)^{-1} x \rho(g)), \qquad x \in \mathcal{N},\ \phi \in \mathcal{S}(\mathcal{N}).$$

Notice that this action leaves $\mathcal{S}(\mathcal{N})^\infty$ invariant.

Lemma 3.2.10. *Let $g \in G$ and $\xi \in \mathfrak{g}$. Then*

$$\phi \in \mathit{KMS}(\rho, \xi, N) \iff g.\phi \in \mathit{KMS}(\rho, \mathit{Ad}_g(\xi), gNg^{-1}).$$

Proof. Write $\mathcal{N}_g := \rho(g)\mathcal{N}\rho(g)^{-1}$. Let $\phi \in \mathrm{KMS}(\rho, \xi, N)$. As $e^{t\xi}Ne^{-t\xi} \subseteq N$ it follows that $e^{t\mathrm{Ad}_g(\xi)}$ normalizes gNg^{-1} for every $t \in \mathbb{R}$. Define the following automorphism groups:

$$\begin{aligned} \sigma^\xi &: \mathbb{R} \to \mathrm{Aut}(\mathcal{N}), & \sigma^\xi &:= \mathrm{Ad}(\rho(e^{t\xi})), \\ \eta^\xi &: \mathbb{R} \to \mathrm{Aut}(\mathcal{N}_g), & \eta^\xi &:= \mathrm{Ad}(\rho(e^{t\mathrm{Ad}_g\xi})). \end{aligned}$$

In order to show $g.\phi \in \mathrm{KMS}(\rho, \mathrm{Ad}_g(\xi), gNg^{-1})$, we must verify that $g.\phi$ satisfies the modular condition for the automorphism group η^ξ_{-t} of $\mathcal{N}_g$. Notice that as isomorphisms $\mathcal{N}_g \to \mathcal{N}$ we have:

$$\sigma^\xi_t \circ \mathrm{Ad}(\rho(g)^{-1}) = \mathrm{Ad}(\rho(g)^{-1}) \circ \eta^\xi_t, \qquad \forall t \in \mathbb{R}. \tag{3.2.1}$$

As $\phi \in \mathrm{KMS}(\rho, \xi, N)$, we know that $\phi \circ \sigma^\xi_t = \phi$ for all $t \in \mathbb{R}$. It then follows immediately from (3.2.1) that

$$(g.\phi)\circ\eta^\xi_t = \phi\circ\mathrm{Ad}(\rho(g)^{-1})\circ\eta^\xi_t = \phi\circ\sigma^\xi_t\circ\mathrm{Ad}(\rho(g)^{-1}) = \phi\circ\mathrm{Ad}(\rho(g)^{-1}) = g.\phi, \quad \forall t \in \mathbb{R}.$$

Next, take $x, y \in \mathcal{N}_g$. Then $x = \rho(g)x'\rho(g)^{-1}$ and $y = \rho(g)y'\rho(g)^{-1}$ for some $x', y' \in \mathcal{N}$. Let the function $F_{x',y'} : \overline{\mathrm{St}} \to \mathbb{C}$ be continuous and bounded, holomorphic on St and satisfy $F_{x',y'}(t) = \phi(\sigma^\xi_{-t}(x')y')$ and $F_{x',y'}(t+i) = \phi(y'\sigma^\xi_{-t}(x'))$ for all $t \in \mathbb{R}$. Define $\widetilde{F}_{x,y} : \overline{\mathrm{St}} \to \mathbb{C}$ by $\widetilde{F}_{x,y}(z) := F_{x',y'}(z)$. Then $\widetilde{F}_{x,y}$ satisfies the conditions of Definition 3.2.2 for η^ξ_{-t}. Indeed, notice using Equation (3.2.1) that $\sigma^\xi_t(x') = \rho(g)^{-1}\eta^\xi_t(x)\rho(g)$. Thus

$$\widetilde{F}_{x,y}(t) = F_{x',y'}(t) = \phi(\sigma^\xi_{-t}(x')y') = \phi\Big(\rho(g)^{-1}\eta^\xi_{-t}(x)y\rho(g)\Big) = (g.\phi)(\eta^\xi_{-t}(x)y),$$

$$\widetilde{F}_{x,y}(t+i) = F_{x',y'}(t+i) = \phi(y'\sigma^\xi_{-t}(x')) = \phi\Big(\rho(g)^{-1}y\eta^\xi_{-t}(x)\rho(g)\Big) = (g.\phi)(y\eta^\xi_{-t}(x)).$$

Thus $g.\phi \in \mathrm{KMS}(\rho, \mathrm{Ad}_g(\xi), gNg^{-1})$. □

Let $\phi \in \mathrm{KMS}(\rho, \xi, N)$. Let α denote the smooth $\mathbb{R}$-action on N defined by

$$\alpha_t(n) := e^{t\xi} n e^{-t\xi}, \qquad \text{for } t \in \mathbb{R} \text{ and } n \in N.$$

We extend ρ_ϕ to $N \rtimes_\alpha \mathbb{R}$ by setting $\rho_\phi(n,t) = \rho_\phi(n)\Delta_\phi^{-it}$. Define

$$\begin{aligned} \mathcal{N}^{\infty,\phi} &:= \{\, x \in \mathcal{N} \,:\, (n,t) \mapsto \rho_\phi(n,t)\pi_\phi(x)\Omega_\phi \text{ is smooth } N \rtimes_\alpha \mathbb{R} \to \mathcal{H}_\phi \,\}, \\ \mathcal{D}_\phi &:= \pi_\phi(\mathcal{N}^{\infty,\phi})\Omega_\phi \subseteq \mathcal{H}^\infty_{\rho_\phi}. \end{aligned} \tag{3.2.2}$$

Notice that $\mathcal{N}^{\infty,\phi}$ and $\mathcal{D}_\phi$ are invariant under the left N- and $N \rtimes_\alpha \mathbb{R}$-actions, respectively.

Lemma 3.2.11. *If $\phi \in KMS(\rho, \xi, N)^\infty$, then $\mathcal{N}^{\infty,\phi}$ is dense in $\mathcal{N}$ w.r.t. the strong operator topology, and $\mathcal{D}_\phi$ is dense in $\mathcal{H}_\phi$. In particular, ρ_ϕ is smooth when considered as representation of $N \rtimes_\alpha \mathbb{R}$.*

Proof. Since $\phi \in \mathcal{S}(\mathcal{N})^\infty$, the vector Ω_ϕ is smooth for the N-action ρ_ϕ by Lemma 3.2.9. Let $m \in N$. Then for every $n \in N$ ant $t \in \mathbb{R}$ we have:

$$\rho_\phi(n,t)\rho_\phi(m)\Omega_\phi = \rho_\phi(n)\Delta_\phi^{-it}\rho_\phi(m)\Delta_\phi^{it}\Omega_\phi = \rho_\phi(n)\sigma^\phi_{-t}(\rho_\phi(m))\Omega_\phi = \rho_\phi(ne^{t\xi}me^{-t\xi})\Omega_\phi.$$

where the last equality follows by Remark 3.2.4. Thus $(n,t) \mapsto \rho_\phi(n,t)\rho_\phi(m)\Omega_\phi$ is smooth $N \rtimes_\alpha \mathbb{R} \to \mathcal{H}_\phi$ and so $\rho(m) \in \mathcal{N}^{\infty,\phi}$. Thus $\rho(N) \subseteq \mathcal{N}^{\infty,\phi}$ and $\rho_\phi(N)\Omega_\phi \subseteq \mathcal{D}_\phi$. Since $\rho(N)'' = \mathcal{N}$ and $\rho_\phi(N)\Omega_\phi$ is total for $\mathcal{H}_\phi$, it follows that $\mathcal{N}^{\infty,\phi}$ is SOT-dense in $\mathcal{N}$ and that $\mathcal{D}_\phi$ is dense in $\mathcal{H}_\phi$. As $\mathcal{D}_\phi$ is contained in the set of $N \rtimes_\alpha \mathbb{R}$-smooth vectors by definition, the final observation follows. □

Restrictions imposed by the KMS condition

Let us next determine some consequences of the KMS condition. Most notably, we will show that representations ρ which are smoothly-KMS give rise to generalized positive energy representations ρ_ϕ on the GNS-Hilbert space $\mathcal{H}_\phi$ of the corresponding state ϕ.

We continue in the notation of Section 3.2.2. Fixing a Lie subgroup $N \subseteq G$ and some element $\xi \in \mathfrak{g}$ satisfying $\mathrm{Ad}(e^{t\xi})N \subseteq N$ for all $t \in \mathbb{R}$, we may as well suppose that $G = N \rtimes_\alpha \mathbb{R}$ for some smooth $\mathbb{R}$-action α on N by automorphisms. Let $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{n} := \mathrm{Lie}(N)$ and write $D \in \mathrm{der}(\mathfrak{n})$ for the derivation on $\mathfrak{n}$ corresponding to α. Thus $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}\boldsymbol{d}$, where $\boldsymbol{d} := 1 \in \mathbb{R}$ denotes the standard basis element. Assume that ρ is KMS at $\boldsymbol{d}$ relative to N, and let $\phi \in \mathrm{KMS}(\rho, \boldsymbol{d}, N)$. We extend the N-representation $\rho_\phi = \pi_\phi \circ \rho$ on the GNS-Hilbert space $\mathcal{H}_\phi$ to $G = N \rtimes_\alpha \mathbb{R}$ by setting $\rho_\phi(n,t) = \rho_\phi(n)\Delta_\phi^{-it}$. Define further $H_\phi := -\log \Delta_\phi = -i\overline{d\rho_\phi(\boldsymbol{d})}$.

A first observation is the following:

Proposition 3.2.12. *Let A be an Abelian Lie subgroup of N such that $\alpha_t(A) \subseteq A$ for all $t \in \mathbb{R}$. Then $\rho_\phi(\alpha_t(a)) = \rho_\phi(a)$ for every $t \in \mathbb{R}$ and $a \in A$. In particular, if $\mathcal{N}$ is a factor then $\rho(\alpha_t(a)) = \rho(a)$ for every $t \in \mathbb{R}$ and $a \in A$.*

Proof. Let $\mathcal{A}_\phi := \rho_\phi(A)''$. Write again ϕ for the vector state $\langle \Omega_\phi, \bullet\, \Omega_\phi \rangle$ on $\mathcal{N}_\phi$. Let $\psi := \phi|_{\mathcal{A}_\phi}$ denote its restriction to $\mathcal{A}_\phi$. As A is $\mathbb{R}$-invariant, so is $\mathcal{A}_\phi \subseteq \mathcal{N}_\phi$. Thus, the modular automorphism group σ^ϕ of $\mathcal{N}_\phi$ leaves $\mathcal{A}_\phi$ invariant. As ϕ satisfies the modular condition for σ^ϕ, so does ψ for the automorphism group $t \mapsto \sigma_t^\phi\big|_{\mathcal{A}_\phi}$. Recall from Remark 3.2.3(2) that Ω_ϕ is separating for $\mathcal{N}_\phi$. Hence it is so for $\mathcal{A}_\phi$. In view of Remark 3.2.3(1) this implies that the modular automorphism group σ^ψ on $\mathcal{A}_\phi$ is uniquely determined by the modular condition. Thus $\sigma_t^\psi = \sigma_t^\phi\big|_{\mathcal{A}_\phi}$ for all $t \in \mathbb{R}$. As $\mathcal{A}_\phi$ is Abelian, we know by [BR97, Prop. 5.3.28] that $\sigma_t^\psi = \mathrm{id}_{\mathcal{A}_\phi}$. Thus $\sigma_t^\phi\big|_{\mathcal{A}_\phi} = \mathrm{id}_{\mathcal{A}_\phi}$. We know from Remark 3.2.4 that $\rho_\phi \circ \alpha_{-t} = \sigma_t^\phi \circ \rho_\phi$. Thus $\rho_\phi(\alpha_t(a)) = \rho_\phi(a)$ for all $a \in \mathcal{A}$ and $t \in \mathbb{R}$. If $\mathcal{N}$ is a factor, then ϕ is faithful and π_ϕ is injective by Remark 3.2.3(1,3). Thus $\rho(\alpha_t(a)) = \rho(a)$ follows from $\rho_\phi(\alpha_t(a)) = \rho_\phi(a)$. ☐

Let us illustrate Proposition 3.2.12 with the following noteworthy consequence for loop groups:

Corollary 3.2.13. *Let K be a compact 1-connected simple Lie group with Lie algebra $\mathfrak{k}$. Define $LK := C^\infty(S^1; K)$ and $L\mathfrak{k} := C^\infty(S^1; \mathfrak{k})$. Let α denote the $\mathbb{T}$-action on LK by rotations, with corresponding derivation $D := \frac{d}{d\theta}$ on $L\mathfrak{k}$. Consider the Lie group $G := LK \rtimes_\alpha \mathbb{T}$ with Lie algebra $\mathfrak{g} := L\mathfrak{k} \rtimes_D \mathbb{R}\boldsymbol{d}$, where $\boldsymbol{d} := 1 \in \mathbb{R}$. Suppose that the smooth unitary G-representation ρ is KMS at $\boldsymbol{d} \in \mathfrak{g}$ relative to LK. Assume that $\rho(LK)''$ is a factor. Then $LK \subseteq \ker \rho$.*

Proof. Suppose $T \subseteq K$ is a maximal torus with Lie algebra $\mathfrak{t}$. Then $LT \subseteq LK$ is an Abelian α-invariant subgroup. By Proposition 3.2.12 it follows that $d\rho(DL\mathfrak{t}) = \{0\}$. As any $X \in \mathfrak{k}$ is contained in a maximal torus, it follows that $d\rho(\frac{df}{d\theta} \otimes X) = 0$ for any $f \in C^\infty(S^1)$ and $X \in \mathfrak{k}$. Consequently $d\rho(D\mathfrak{g}) = \{0\}$ and hence $D\mathfrak{g}_\mathbb{C} \subseteq \ker d\rho$, where we have extended $d\rho : \mathfrak{g} \to \mathcal{L}^\dagger(\mathcal{H}_\rho^\infty)$ $\mathbb{C}$-linearly to the complexification $\mathfrak{g}_\mathbb{C}$. As $\ker d\rho$ is an ideal in $\mathfrak{g}_\mathbb{C}$ and $L\mathfrak{k}_\mathbb{C} = DL\mathfrak{k}_\mathbb{C} + [DL\mathfrak{k}_\mathbb{C}, DL\mathfrak{k}_\mathbb{C}]$, it follows that $L\mathfrak{k}_\mathbb{C} \subseteq \ker d\rho$. Notice that LK is connected because K is 1-connected. It is also locally exponential by [Nee01b, Thm. II.1]. It follows that $LK \subseteq \ker \rho$. ☐

Thus, one necessarily has to pass to a non-trivial central $\mathbb{T}$-extension $\mathring{LK}$ of $LK \rtimes_\alpha \mathbb{T}$ to allow for interesting KMS-representations of $\mathring{LK}$ that are smoothly-KMS at some $\mathring{\boldsymbol{d}} \in \mathring{L\mathfrak{k}}$ covering $\boldsymbol{d} \in L\mathfrak{k} \rtimes_D \mathbb{R}\boldsymbol{d}$, as one may have expected from the positive energy analogue (which follows from [PS86, Thm 9.3.5]).

We now proceed with the observation that KMS representations give rise to generalized positive energy representations on the GNS-Hilbert space corresponding to the KMS state:

Theorem 3.2.14. *Let $\phi \in KMS(\rho, \boldsymbol{d}, N)^{\infty}$. Let $x \in \mathcal{N}^{\phi,\infty}$ and assume that $\psi := \pi_\phi(x)\Omega_\phi \in \mathcal{D}_\phi$ has unit norm. Then*

$$\langle \pi_\phi(x)\Omega_\phi, -id\rho_\phi(Ad_n(\boldsymbol{d}))\pi_\phi(x)\Omega_\phi\rangle \geq -\log\left(\|\pi_\phi(x)\|^2\right) \qquad \forall n \in N. \tag{3.2.3}$$

In particular the representation ρ_ϕ of $N \rtimes_\alpha \mathbb{R}$ on $\mathcal{H}_\phi$ is of generalized positive energy at $\boldsymbol{d} \in \mathfrak{n} \rtimes_D \mathbb{R}\boldsymbol{d}$.

Lemma 3.2.15. *Let $x \in \mathcal{N}$ be such that $0 \neq \psi := \pi_\phi(x)\Omega_\phi \in \mathrm{dom}(H_\phi)$. Then*

$$\frac{\langle \psi, H_\phi \psi\rangle}{\|\psi\|^2} \geq -\log\left(\frac{\|S_\phi\psi\|^2}{\|\psi\|^2}\right). \tag{3.2.4}$$

Proof. In view of the correlation lower bounds satisfied by KMS states, see e.g. [BR97, Thm. 5.3.15 (1) $\implies$ (2)] or [FV77, Thm. II.4, (i) $\implies$ (iii)], we have:

$$\langle \pi_\phi(x)\Omega_\phi, [H_\phi, \pi_\phi(x)]\Omega_\phi\rangle \geq -\|\pi_\phi(x)\Omega_\phi\|^2 \log\left(\frac{\|\pi_\phi(x)^*\Omega_\phi\|^2}{\|\pi_\phi(x)\Omega_\phi\|^2}\right).$$

Since $H_\phi\Omega_\phi = 0$, it follows that $\langle \pi_\phi(x)\Omega_\phi, [H_\phi, \pi_\phi(x)]\Omega_\Phi\rangle = \langle \pi_\phi(x)\Omega_\phi, H_\phi\pi_\phi(x)\Omega_\phi\rangle$. The assertion follows. □

Proof of Theorem 3.2.14:
Recall that $\mathcal{D}_\phi \subseteq \mathcal{H}^\infty_{\rho_\phi}$ and that $\mathcal{D}_\phi \subseteq \mathrm{dom}(S_\phi)$, because the stronger condition $\mathcal{N}_\phi\Omega_\phi \subseteq \mathrm{dom}(S_\phi)$ is satisfied. Let $n \in N$. Notice that $\|S_\phi\rho_\phi(n)\psi\| = \|\pi_\phi(x^*)\rho_\phi(n)^{-1}\Omega_\phi\| \leq \|\pi_\phi(x)\|$. Recalling that $\mathcal{D}_\phi$ is N-invariant, we can apply equation (3.2.4) to the vector $\rho_\phi(n)\psi$. Using $-i\overline{d\rho_\phi(\boldsymbol{d})} = -\log\Delta_\phi = H_\phi$ it follows that

$$\begin{aligned}\langle \psi, -id\rho_\phi(\mathrm{Ad}_{n^{-1}}(\boldsymbol{d}))\psi\rangle &= \langle \rho_\phi(n)\psi, -id\rho_\phi(\boldsymbol{d})\rho_\phi(n)\psi\rangle \\ &\geq -\log\left(\|S_\phi\rho_\phi(n)\psi\|^2\right) \\ &\geq -\log\left(\|\pi_\phi(x)\|^2\right).\end{aligned}$$
□

As a consequence of Theorem 3.2.14, we find that the observations of Section 3.1 impose restrictions on KMS representations. Let us illustrate this with the following immediate consequence:

Corollary 3.2.16. *Let $\overline{\rho}$ be a smooth projective unitary representation of G on $\mathcal{H}_\rho$. Assume that $\mathcal{N} := \overline{\rho}(N)''$ is a factor. Let $\rho : \mathring{G} \to \mathrm{U}(\mathcal{H}_\rho)$ be the lift of $\overline{\rho}$, for some central $\mathbb{T}$-extension $\mathring{G}$ of G with Lie algebra $\mathring{\mathfrak{g}}$. Let $\mathring{N} \subseteq \mathring{G}$ cover N. Let ω represent the class in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ corresponding to $\mathring{\mathfrak{g}}$. Let $\xi \in \mathfrak{g}$ and suppose $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covers ξ. Let $\phi \in KMS(\rho, \mathring{\xi}, \mathring{N})^{\infty}$. Assume that $\eta \in \mathfrak{n}$ satisfies $[[\xi, \eta], \eta] = 0$. Then $\omega([\xi, \eta], \eta) \geq 0$ and*

$$\omega([\xi, \eta], \eta) = 0 \iff d\overline{\rho}([\xi, \eta]) = 0.$$

Proof. Consider the representation ρ_ϕ of $\mathring{N} \rtimes \mathbb{R}$ on the GNS Hilbert space $\mathcal{H}_\phi$, where $\mathbb{R}$ acts on $\mathring{N}$ by $\mathrm{Ad}(e^{t\mathring{\xi}})\Big|_{\mathring{N}}$ and where $\rho_\phi(1, t) = \Delta_\phi^{-it}$ for $t \in \mathbb{R}$. Let $\overline{\rho}_\phi$

be the corresponding projective unitary representation of $N \rtimes \mathbb{R}$ on $\mathcal{H}_\phi$, where $\mathbb{R}$ acts on N by $\mathrm{Ad}(e^{t\xi})\big|_N$. By Theorem 3.2.14, ρ_ϕ is of g.p.e. at $\boldsymbol{d} \in \mathring{\mathfrak{n}} \rtimes \mathbb{R}\boldsymbol{d}$ and so $\overline{\rho}_\phi$ is of g.p.e. at $\boldsymbol{d}$. It follows from Proposition 3.1.6 that $\omega([\xi,\eta],\eta) \geq 0$ and $\omega([\xi,\eta],\eta) = 0 \iff d\overline{\rho}_\phi([\xi,\eta]) = 0$. As $\mathcal{N}$ is a factor, the KMS state $\phi \in \mathcal{S}(\mathcal{N})$ is faithful and the corresponding GNS-representation $\pi_\phi : \mathcal{N} \to \mathcal{B}(\mathcal{H}_\phi)$ is injective, by Remark 3.2.3(1,3). This implies that $\ker d\rho_\phi = \ker d\rho$. Thus $\omega([\xi,\eta],\eta) = 0 \iff d\overline{\rho}([\xi,\eta]) = 0$. □

Remark 3.2.17. A related notation is that of a *passive state*, which is usually considered in the context of a C^*-dynamical system $(\mathcal{A},\sigma)$, where $\mathcal{A}$ is a unital C^*-algebra and $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ is a strongly continuous homomorphism. If δ is the generator of σ with domain $\mathcal{D}(\delta) \subseteq \mathcal{A}$, a state ϕ on $\mathcal{A}$ is said to be *passive* if

$$-i\phi(u^*\delta(u)) \geq 0, \qquad \forall u \in \mathrm{U}_0(\mathcal{A}) \cap \mathcal{D}(\delta), \tag{3.2.5}$$

where $\mathrm{U}_0(\mathcal{A})$ denotes the identity component of the group $\mathrm{U}(\mathcal{A})$ of unitary elements in $\mathcal{A}$. In this case, ϕ is necessarily σ-invariant [PW78, Thm. 1.1], so that σ is canonically implemented by a strongly-continuous unitary 1-parameter group $t \mapsto e^{itH_\phi}$ on the GNS-Hilbert space $\mathcal{H}_\phi$. Let $\pi_\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$ be the GNS-representation of $\mathcal{A}$ associated to ϕ and let $\Omega_\phi \in \mathcal{H}_\phi$ be the corresponding cyclic vector. Then (3.2.5) becomes

$$-i\langle \Omega_\phi, \pi_\phi(u)^{-1} H_\phi \pi_\phi(u)\Omega_\phi\rangle \geq 0, \qquad \forall u \in \mathrm{U}_0(\mathcal{A}) \cap \mathcal{D}(\delta),$$

which is similar to equation (3.1.1). It was moreover shown in [PW78] that any ground- or σ-KMS state is necessarily passive (cf. [BR97, Thm. 5.3.22]), which is analogous to the observation that both positive energy and KMS representations provide examples of generalized positive energy ones, in view of Theorem 3.2.14. We refer to [PW78] and [BR97] for more information on (completely) passive states.

Some examples of KMS representations

Let us consider a variety of examples of KMS representations, thereby showing in various situations that a well-known σ-KMS state ϕ on a von Neumann algebra $\mathcal{N}$ admits some underlying smooth structure. More precisely, we construct a continuous unitary representation ρ of a (typically infinite-dimensional) Lie group G such that $\mathcal{N} = \rho(N)''$, $\phi \in \mathrm{KMS}(\rho,\xi,N)^\infty$ and $\sigma_t = \mathrm{Ad}(\rho(e^{t\xi}))$ for some $\xi \in \mathfrak{g}$ and Lie subgroup N of G. In particular, in this case the 1-parameter group σ on $\mathcal{N}$ implements the $\mathbb{R}$-action $t \mapsto \mathrm{Ad}(e^{t\xi})\big|_N$ on the Lie subgroup N of G.

Let us begin with the simplest class of examples, which correspond to Gibbs states, as in Example 3.2.5:

Example 3.2.18. Take for N simply $N = G$. Let $(\rho,\mathcal{H})$ be a continuous irreducible unitary G-representation. Then $\mathcal{N} = \mathcal{B}(\mathcal{H})$. Let $\xi \in \mathfrak{g}$ and define the self-adjoint operator $H := -i\,\frac{d}{dt}\big|_{t=0}\rho(e^{t\xi})$. Let $\beta > 0$ and assume that $Z_\beta := \mathrm{Tr}(e^{-\beta H}) < \infty$. Define the Gibbs state $\phi(x) := \frac{1}{Z_\beta}\mathrm{Tr}(e^{-\beta H}x)$ for $x \in \mathcal{N}$. As in Example 3.2.5, we have $\sigma^\phi_{-t}(x) = e^{it\beta H}xe^{-it\beta H} = \rho(e^{t\beta\xi})x\rho(e^{-t\beta\xi})$ for any

$x \in \mathcal{N}$. Consequently, $\phi \in \mathrm{KMS}(\rho, \beta\xi)$ and so ρ is a KMS representation at $\beta\xi \in \mathfrak{g}$. If in addition $\widehat{\phi} : G \to \mathbb{C}$ is smooth, then ρ is smoothly-KMS at $\beta\xi \in \mathfrak{g}$. By Lemma 3.2.10, ρ is also KMS at any element in the adjoint orbit of $\beta\xi$. In view of Example 3.2.5, the representation ρ_ϕ of $G \rtimes \mathbb{R}$ on $\mathcal{H}_\phi := \overline{\mathcal{B}(\mathcal{H}_\rho)}^{\langle -,- \rangle_\phi}$ is given by $\rho_\phi(g,t)x\Omega_\phi = \rho(g)\rho(e^{t\beta\xi})x\rho(e^{-t\beta\xi})\Omega_\phi = \rho(g)\sigma^\phi_{-t}(x)\Omega_\phi$, where $\Omega_\phi := I \in \mathcal{B}(\mathcal{H}_\rho) \subseteq \mathcal{H}_\phi$ denotes the cyclic vector.

In fact, Proposition 3.2.19 below entails that any KMS representation ρ for which $\mathcal{N}$ is a factor of type I is of the form described in Example 3.2.18. Moreover a complete characterization of such representations was very recently obtained in the context where N is a finite-dimensional Lie group [Sim23].

Proposition 3.2.19. *Let $\xi \in \mathfrak{g}$ and $\beta > 0$. Suppose that $\rho|_N$ is irreducible and that $\phi \in KMS(\rho, \beta\xi, N)$. Let $H := -i \frac{d}{dt}\big|_{t=0} \rho(e^{t\xi})$. Then $Z_\beta := \mathrm{Tr}(e^{-\beta H}) < \infty$ and $\phi(x) = \frac{1}{Z_\beta}\mathrm{Tr}(e^{-\beta H}x)$.*

Proof. As $\rho|_N$ is irreducible, it follows that $\mathcal{N} = \mathcal{B}(\mathcal{H}_\rho)$. Thus $\phi(x) = \mathrm{Tr}(\delta x)$ for some $\delta \in L^1(\mathcal{H}_\rho)_+$ satisfying $\mathrm{Tr}(\delta) = 1$, where $L^1(\mathcal{H}_\rho)$ denotes Banach space of trace-class operators on $\mathcal{H}_\rho$. Moreover, in view of Remark 3.2.3(3), we know that ϕ is faithful on $\mathcal{N}$. By assumption, ϕ satisfies the modular condition for the automorphism group $t \mapsto \mathrm{Ad}(\rho(e^{-t\beta\xi})) =: \sigma_{-\beta t}$. On the other hand, as ϕ is faithful, there exists by Remark 3.2.3(1) a *unique* automorphism group σ^ϕ_t of $\mathcal{N}$ for which ϕ satisfies the modular condition. It follows that $\sigma_{-\beta t} = \sigma^\phi_t$. When $\mathcal{N} = \mathcal{B}(\mathcal{H}_\rho)$ and $\phi(x) = \mathrm{Tr}(\delta x)$, the modular automorphism group σ^ϕ_t corresponding to ϕ is $\sigma^\phi_t(x) = \delta^{it}x\delta^{-it}$. In view of $\sigma^\phi_t = \sigma_{-t\beta}$, it follows that $\delta^{it}x\delta^{-it} = \rho(e^{-t\beta\xi})x\rho(e^{t\beta\xi})$ for every $x \in \mathcal{N}$. As $\mathcal{Z}(\mathcal{N}) = \mathbb{C}I$ and both $t \mapsto \delta^{it}$ and $t \mapsto \rho(e^{t\beta\xi})$ are strongly continuous unitary 1-parameter groups, it follows that there is some continuous homomorphism $c : \mathbb{R} \to \mathbb{T}$ such that $\delta^{it} = c(t)\rho(e^{-t\beta\xi}) = c(t)e^{-it\beta H}$ for all $t \in \mathbb{R}$. Thus there exists $\mu \in \mathbb{R}$ such that $\delta^{it} = e^{-it(\beta H + \mu I)}$ for all $t \in \mathbb{R}$. So $\log\delta = -(\beta H + \mu I)$. Since $\mathrm{Tr}(\delta) = 1$, we have $Z_\beta = \mathrm{Tr}(e^{-\beta H}) = \mathrm{Tr}(e^{-(\beta H+\mu)}e^\mu 1) = \mathrm{Tr}(\delta e^\mu 1) = e^\mu\phi(1) = e^\mu < \infty$. It follows for every $x \in \mathcal{B}(\mathcal{H})$ that

$$\frac{1}{Z_\beta}\mathrm{Tr}(e^{-\beta H}x) = e^{-\mu}\,\mathrm{Tr}(e^{-\beta H}x) = \mathrm{Tr}(e^{-(\beta H+\mu)}x) = \mathrm{Tr}(\delta x) = \phi(x). \quad \square$$

For more interesting examples, one has to consider a Lie subgroup N of G which is not of type I, so that the von Neumann algebra $\mathcal{N}$ need not be type I.

Example 3.2.20 (Powers' factors)**.** Define $G_n := \prod_{k=1}^n \mathrm{SU}(2)$ and let the inclusion $\eta_n : G_n \hookrightarrow G_{n+1}$ be defined by

$$\eta_n : G_n \xrightarrow{\mathrm{id}\times 1} G_n \times \mathrm{SU}(2) = G_{n+1}.$$

Write $\mathfrak{g}_n := \mathrm{Lie}(G_n)$ and $L(\eta_n) := \mathrm{Lie}(\eta_n)$. The direct limit $G := \varinjlim_n (G_n, \eta_n)$ consists of sequences (u_k) in $\mathrm{SU}(2)$ with $u_k = 1$ for all but finitely many values of k. It can be equipped with the structure of a regular Lie group that is modeled on the locally convex inductive limit $\mathfrak{g} := \varinjlim_n (\mathfrak{g}_n, L(\eta_n))$ [Glö05, Thm. 4.3] and has the exponential map $\exp_G = \varinjlim_n \exp_{G_n}$ [Glö05, Prop. 4.6]. Let $H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\xi := iH \in \mathfrak{su}(2)$. Consider the following $\mathbb{R}$-action α on G defined by $(\alpha_t(u))_k := e^{t\xi} u_k e^{-t\xi}$ for $u \in G$. The corresponding action $\mathbb{R} \times G \to G$ is smooth. Indeed, the restriction of α to $\mathbb{R} \times G_n$ yields a smooth action $\alpha^{(n)} : \mathbb{R} \times G_n \to G_n$ for every $n \in \mathbb{N}$. It follows from [Glö05, Thm. 3.1] that $\varinjlim_n \alpha^{(n)} : \varinjlim_n (\mathbb{R} \times G_n) \to G$ is smooth. By [Glö05, Prop. 3.7] we further have $\varinjlim_n (\mathbb{R} \times G_n) = \mathbb{R} \times G$ as smooth manifolds. This shows that $\alpha : \mathbb{R} \times G \to G$ is smooth. Consider the Lie group $G^\sharp := G \rtimes_\alpha \mathbb{R}$ with Lie algebra $\mathfrak{g}^\sharp := \mathfrak{g} \rtimes_D \mathbb{R}\boldsymbol{d}$, where $\boldsymbol{d} := (0,1)$. Using the so-called Powers' factors, we define unitary representations ρ of $G^\sharp$ which are smoothly-KMS at $\boldsymbol{d}$ relative to $G \triangleleft G^\sharp$ and for which $\rho(G)''$ is a factor of type III_λ for arbitrary $\lambda \in (0,1)$.

Define the finite-dimensional C^*-algebra $\mathcal{M}_n := \bigotimes_{k=1}^n \mathcal{B}(\mathbb{C}^2)$ for every $n \in \mathbb{N}$. Let $\beta > 0$. Define the state $\phi(x) := \frac{1}{Z} \operatorname{Tr}(e^{-\beta H} x)$ on $\mathcal{B}(\mathbb{C}^2)$, where $Z := \operatorname{Tr}(e^{-\beta H}) = 2\cosh(\beta)$. Let ϕ_n be the state on $\mathcal{M}_n$ defined by

$$\phi_n(x_1 \otimes \cdots \otimes x_n) = \prod_{k=1}^n \phi(x_k), \qquad x_k \in \mathcal{B}(\mathbb{C}^2).$$

The GNS-representation of $\mathcal{B}(\mathbb{C}^2)$ defined by ϕ is $\mathcal{H}_\phi := \mathcal{B}(\mathbb{C}^2)$ equipped with left $\mathcal{B}(\mathbb{C}^2)$-action and the inner product $\langle a, b \rangle := \frac{1}{Z} \operatorname{Tr}(e^{-\beta H} a^* b)$. Similarly the GNS-representation of $\mathcal{M}_n$ corresponding to ϕ_n is $\mathcal{H}_{\phi_n} := \bigotimes_{k=1}^n \mathcal{H}_\phi$. The isometric inclusions

$$\mathcal{H}_{\phi_n} \hookrightarrow \mathcal{H}_{\phi_{n+1}}, \ x \mapsto x \otimes 1$$

define a directed system of Hilbert spaces, and the algebraic direct limit $\varinjlim_n \mathcal{H}_{\phi_n}$ becomes naturally a pre-Hilbert space. Let $\mathcal{H}$ denote its Hilbert space completion. Let $\iota_n : \mathcal{H}_{\phi_n} \hookrightarrow \mathcal{H}$ denote the canonical inclusion. For every $n \in \mathbb{N}$, there is a $*$-representation π_n of $\mathcal{M}_n$ on $\mathcal{H}$ defined for $x = x_1 \otimes \cdots \otimes x_n \in \mathcal{M}_n$ by

$$\pi_n(x)\iota_m(\psi_1 \otimes \cdots \otimes \psi_m) := \iota_m(x_1\psi_1 \otimes \cdots x_n\psi_n \otimes \psi_{n+1} \otimes \cdots \otimes \psi_m), \qquad m \geq n.$$

Let $\mathcal{M}_\infty := \left(\bigcup_{n\in\mathbb{N}} \pi_n(\mathcal{M}_n) \right)''$. The vector $\Omega := 1 \otimes 1 \otimes \cdots \in \mathcal{H}$ is cyclic and separating for $\mathcal{M}_\infty$ [Tak03b, XIV, Prop. 1.11], so $\mathcal{H}$ may be identified with the GNS-representation of $\mathcal{M}_\infty$ w.r.t. the state $\phi_\infty := \langle \Omega, \bullet\, \Omega \rangle$ on $\mathcal{M}_\infty$. Observe that ϕ_∞ satisfies $\phi_\infty(\iota_n(x)) = \phi_n(x)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{M}_n$. The von Neumann algebra $(\mathcal{M}_\infty, \phi_\infty) =: \bigotimes_{k=1}^\infty (\mathcal{B}(\mathbb{C}^2), \phi)$ is the so-called Powers' factor with parameter $a := \frac{e^{-\beta}}{2\cosh(\beta)} \in (0, \frac{1}{2})$, which is a factor of type III_λ with $\lambda = e^{-2\beta} = \frac{a}{1-a} \in (0,1)$ [Tak03b, XVIII, Theorem 1.1]. The modular automorphism group $\sigma_t^{\phi_\infty}$ on $\mathcal{M}_\infty$ defined by ϕ_∞ is given by $\sigma_t^{\phi_\infty} = \bigotimes_{k=1}^\infty \operatorname{Ad}(e^{-\beta t \xi})$ [Tak03b, XIV, Prop. 1.11], where $\bigotimes_{k=1}^\infty \operatorname{Ad}(e^{\beta t \xi}) \in \operatorname{Aut}(\mathcal{M}_\infty)$ satisfies $\bigotimes_{k=1}^\infty \operatorname{Ad}(e^{\beta t \xi}) \circ \iota_n = \iota_n \circ \bigotimes_{k=1}^n \operatorname{Ad}(e^{\beta t \xi})$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$ and is defined from this condition by continuity, where we used [Tak03b, XIV, Thm. 1.13] and that $\phi \circ \operatorname{Ad}(e^{\beta t \xi}) = \phi$ for all $t \in \mathbb{R}$. Consider the unitary representation $\rho : G \rtimes_\alpha \mathbb{R} \to \mathrm{U}(\mathcal{H})$ defined by

$$\rho(u, \beta t) := \left(\bigotimes_{k=1}^{\infty} u_k \right) \circ \Delta_{\phi_\infty}^{-it}, \qquad u \in G, t \in \mathbb{R}$$

which is well-defined because $u = (u_k) \in G$ is a sequence in SU(2) with $u_k = 1$ for all k sufficiently large. Since $\rho(\beta t) = \Delta_{\phi_\infty}^{-it}$ and $\rho(G)'' = \mathcal{M}_\infty$, it follows that ρ is KMS at $\beta \boldsymbol{d} \in \mathfrak{g}^\sharp$ relative to $G \triangleleft G^\sharp$. To see that $\widehat{\phi_\infty} : G \to \mathbb{C}$ is smooth, it suffices to show that its restriction to G_n is smooth for every $n \in \mathbb{N}$, using the universal property of the smooth manifold structure on $G = \varinjlim_n G_n$ [Glö05, Thm. 3.1]. This is the case, as $\langle \Omega, \rho(u)\Omega \rangle = \prod_{k=1}^n \phi(u_k)$ for any $u \in G_n$, which is smooth $G_n \to \mathbb{C}$. Thus $\Omega \in \mathcal{H}_\rho^\infty$ and so ρ is smoothly-KMS at $\beta \boldsymbol{d} \in \mathfrak{g}^\sharp$ relative to $G \triangleleft G^\sharp$.

Example 3.2.21 (Standard real subspaces and Heisenberg representations)**.** Let $\mathcal{H}$ be a complex Hilbert space. Consider the real Heisenberg group $G := \mathrm{H}(\mathcal{H}, \omega)$, where $\omega(v, w) = \mathrm{Im}\langle v, w \rangle$. An $\mathbb{R}$-linear closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is called *cyclic* if $\mathcal{K} + i\mathcal{K}$ is dense in $\mathcal{H}$. It is called *separating* if $\mathcal{K} \cap i\mathcal{K} = \{0\}$. A *standard subspace* is a closed $\mathbb{R}$-linear subspace $\mathcal{K} \subseteq \mathcal{H}$ which is both cyclic and separating. We show that any standard real subspace gives rise to a smooth KMS representation. Let $\mathcal{K} \subseteq \mathcal{H}$ be a standard real subspace. Write $\delta_{\mathcal{K}}$ for the corresponding modular operator on $\mathcal{H}$, which is generally unbounded, positive and self-adjoint, see e.g. [NO17, Sec. 3]. Then $t \mapsto \delta_{\mathcal{K}}^{it}$ is a strongly-continuous unitary 1-parameter group on $\mathcal{H}$ satisfying in particular $\delta_{\mathcal{K}}^{it}\mathcal{K} \subseteq \mathcal{K}$. We first pass to the $\mathbb{R}$-smooth vectors $\mathcal{K}^\infty$ to obtain a regular Lie group $\mathrm{H}(\mathcal{K}^\infty, \omega) \rtimes \mathbb{R}$. We then construct a KMS representation thereof using second-quantization. The details are given below.

Let $\mathcal{K}^\infty$ denote the set $\mathbb{R}$-smooth vectors in $\mathcal{K}$. Then $\mathcal{K}^\infty$ is dense in $\mathcal{K}$ and $\mathbb{R}$-invariant. It moreover carries a Fréchet topology which is finer than the one inherited as a subspace of $\mathcal{K}$ and for which the action $\mathbb{R} \times \mathcal{K}^\infty \to \mathcal{K}^\infty$ is smooth [Nee10a, Thm. 4.4, Lem. 5.2]. As $\omega : \mathcal{K}^\infty \times \mathcal{K}^\infty \to \mathbb{R}$ is bilinear and continuous w.r.t. this topology, it is smooth. Thus the generalized Heisenberg group $N := \mathrm{H}(\mathcal{K}^\infty, \omega)$ is a Lie group. (Notice that $\omega|_{\mathcal{K}^\infty}$ may be degenerate.) It is as a subgroup of G generated by $\mathcal{K}^\infty$. As $\mathcal{K}^\infty$ is a Frèchet space, it is Mackey complete by [KM97, Thm. I.4.11], which implies using [Nee06, Thm. V.1.8] that N is regular. Write $\mathfrak{n} := \mathrm{Lie}(N)$. By construction $\mathbb{R}$ acts smoothly on N by $\delta_{\mathcal{K}}^{it}$, so that $N^\sharp := N \rtimes \mathbb{R}$ is a regular Lie group. Let $\mathfrak{n}^\sharp := \mathfrak{n} \rtimes \mathbb{R}\boldsymbol{d}$ denote its Lie algebra. We construct a representation of $N^\sharp$ which is smoothly KMS at $\boldsymbol{d} \in \mathfrak{n}^\sharp$ relative to $N \triangleleft N^\sharp$. Let us recall the standard representation of $\mathrm{H}(\mathcal{H}, \omega)$ on the Bosonic Fock space $\mathcal{F}(\mathcal{H})$. Equip the symmetric algebra $S^\bullet(\mathcal{H})$ with the inner product

$$\langle v_1 \cdots v_n, w_1 \cdots w_n \rangle = \sum_{\sigma \in S_n} \prod_{j=1}^n \langle v_j, w_{\sigma_j} \rangle. \tag{3.2.6}$$

Let $\mathcal{F}(\mathcal{H})$ denote the Hilbert space completion of $S^\bullet(\mathcal{H})$ and let $\Omega := 1 \in \mathcal{H}$ denote the vacuum vector. Then $\mathcal{H}$ contains (and is generated by) the vectors $e^v := \sum_{n=0}^\infty \frac{1}{n!} v^n \in \mathcal{H}$ for $v \in \mathcal{H}$. There is a continuous irreducible unitary representation

W of $\mathrm{H}(\mathcal{H},\omega)$ on $\mathcal{F}(\mathcal{H})$ satisfying $W(z,v)e^w = ze^{-\frac{1}{2}\|v\|^2-\langle v,w\rangle}e^{v+w}$ for $v,w\in\mathcal{H}$ and $z\in\mathbb{T}$ [PS86, Sec. 9.5]. Moreover, any unitary $u\in\mathrm{U}(\mathcal{H})$ extends canonically to a unitary $\mathcal{F}(u)\in\mathrm{U}(\mathcal{F}(\mathcal{H}))$. We further have:

$$W(uv) = \mathcal{F}(u)W(v)\mathcal{F}(u)^{-1}, \qquad \forall u\in\mathrm{U}(\mathcal{H}), v\in\mathcal{H} \tag{3.2.7}$$

In view of (3.2.7), W and $\mathcal{F}$ together define a representation ρ of the Lie group $N^\sharp$ by $\rho(n,t) := W(n)\mathcal{F}(\delta_{\mathcal{K}}^{it})$. Let $\mathcal{N} := W(N)''$. As $\mathcal{K}$ is a standard real subspace and $\mathcal{K}^\infty$ is dense in $\mathcal{K}$, it follows that Ω is cyclic and separating for $\mathcal{N}$ [NO17, Lem. 6.2]. Let ϕ denote the faithful vector state on $\mathcal{N}$ defined by $\phi(x) = \langle\Omega, x\Omega\rangle$. Using [NO17, Prop. 6.10] we have $\Delta_\phi^{it} = \mathcal{F}(\delta_{\mathcal{K}}^{it})$ for all $t\in\mathbb{R}$. Consequently ρ is KMS at $-\boldsymbol{d}\in\mathfrak{n}^\sharp$ relative to $N\lhd N^\sharp$ (notice the minus sign in Definition 3.2.2). To see it is smoothly KMS, observe that $\widehat{\phi}: N\to\mathbb{C}$ is smooth because it is given by

$$\widehat{\phi}(z,v) = \langle\Omega, W(z,v)\Omega\rangle = ze^{-\frac{1}{2}\|v\|^2}. \tag{3.2.8}$$

The following provides an example where ρ is smoothly-KMS at various $\xi_I\in\mathfrak{g}$, relative to distinct subgroups $N_I\subseteq G$, where $I\in\mathcal{I}$ for some indexing set $\mathcal{I}$:

Example 3.2.22 (Bisognano-Wichmann and $\mathrm{SU}(1,1)$-covariant nets)**.**
Recall that $\mathrm{SU}(1,1)$ acts on S^1. Explicitly, for $g = \begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix}\in\mathrm{SU}(1,1)$ with $\alpha,\beta\in\mathbb{C}$ satisfying $|\alpha|^2-|\beta|^2=1$, define $g(z) := \frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$ for $z\in\mathbb{C}$ with $|z|=1$. With g as above, define the unitary action of $\mathrm{SU}(1,1)$ on the complex Hilbert space $L^2(S^1,\mathbb{C})$ by $(u(g)f)(z) := (\alpha-\overline{\beta}z)^{-1}f(g^{-1}(z))$ for $f\in L^2(S^1,\mathbb{C})$. Let $H^2_+(S^1,\mathbb{C})$ be the closed subspace of $L^2(S^1,\mathbb{C})$ spanned by the non-negative Fourier modes. Let $H^2_-(S^1,\mathbb{C})$ be its orthogonal complement in $L^2(S^1,\mathbb{C})$. Notice that $\mathrm{SU}(1,1)$ leaves these subspaces invariant. Consider the complex Hilbert space $V := H^2_+(S^1,\mathbb{C})\oplus\overline{H^2_-(S^1,\mathbb{C})}$, where $\overline{H^2_-(S^1,\mathbb{C})}$ denotes the Hilbert space complex-conjugate to $H^2_-(S^1,\mathbb{C})$. Let $V_\mathbb{R} = L^2(S^1,\mathbb{C})$ denote the real vector space underlying V. Define the real Fréchet space $V_\mathbb{R}^\infty := C^\infty(S^1,\mathbb{C})$ and consider the symplectic vector space $(V_\mathbb{R}^\infty,\omega)$, where $\omega(v,w) := \mathrm{Im}\langle v,w\rangle_V$ for $v,w\in V_\mathbb{R}^\infty$. Let $\mathrm{H}(V_\mathbb{R}^\infty,\omega)$ be the corresponding real Heisenberg group. Consider the regular Fréchet-Lie group $G := \mathrm{H}(V_\mathbb{R}^\infty,\omega)\rtimes\mathrm{SU}(1,1)$. Let $\boldsymbol{r} := \frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\boldsymbol{d} := \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ denote the generators in $\mathfrak{su}(1,1)$ of the rotation and the dilation subgroups in $\mathrm{SU}(1,1)$, respectively. By an interval of S^1, we mean a connected, open, non-empty and non-dense subset of S^1. Write $\mathcal{I}$ for the set of intervals of S^1, on which $\mathrm{SU}(1,1)$ acts naturally, and let $I_0\in\mathcal{I}$ denote the upper-semicircle. For $I\in\mathcal{I}$, define $\xi_I\in\mathfrak{su}(1,1)$ by $\xi_I := \mathrm{Ad}_g(\boldsymbol{d})$, where $g\in\mathrm{SU}(1,1)$ is any element satisfying $g.I_0 = I$. Notice that ξ_I is well-defined. Define further the closed real subspace $V_I := L^2(I,\mathbb{C})$ of $V_\mathbb{R}$ and set $V_I^\infty := V_I\cap V_\mathbb{R}^\infty$. Let $N_I := \mathrm{H}(V_I^\infty,\omega)\subseteq G$ be the corresponding closed subgroup of G. We construct a unitary representation ρ of G which is of p.e. at $\boldsymbol{r}\in\mathfrak{su}(1,1)$ and which is KMS at $\xi_I\in\mathfrak{su}(1,1)$ relative to N_I, for every $I\in\mathcal{I}$. The details are given below.

As the $\mathrm{SU}(1,1)$-action u on $L^2(S^1,\mathbb{C})$ leaves both $H^2_+(S^1,\mathbb{C})$ and $H^2_-(S^1,\mathbb{C})$ invariant, we obtain a unitary representation $\widetilde{u}$ of $\mathrm{SU}(1,1)$ on $V = H^2_+(S^1,\mathbb{C})\oplus\overline{H^2_-(S^1,\mathbb{C})}$

which is by construction of p.e. at $\boldsymbol{r} \in \mathfrak{su}(1,1)$. As in Example 3.2.21, let W denote the standard representation of the real Heisenberg group $\mathrm{H}(V, \mathrm{Im}\langle -, - \rangle)$ on the Fock space $\mathcal{F}(V)$. Letting $\mathrm{SU}(1,1)$ act on $\mathcal{F}(V)$ by second quantization, we obtain a smooth unitary representation ρ of G on $\mathcal{F}(V)$ which is of p.e. at $\boldsymbol{r} \in \mathfrak{su}(1,1)$. Explicitly, ρ is given by $\rho(v, g) = W(v)\mathcal{F}(\widetilde{u}(g))$ for $v \in \mathrm{H}(V_{\mathbb{R}}^{\infty}, \omega)$ and $g \in \mathrm{SU}(1,1)$. It follows from [Was98, Sec. II.14] that $V_I \subseteq V$ is a standard real subspace for any interval $I \in \mathcal{I}$. Let δ_I^{it} denote the corresponding modular 1-parameter group, as in Example 3.2.21. The assignment $I \mapsto V_I$, called a *net of standard subspaces*, satisfies $I_1 \subseteq I_2 \implies V_{I_1} \subseteq V_{I_2}$ (*isotony*), $V_{g.I} = \widetilde{u}(g)V_I$ for $g \in \mathrm{SU}(1,1)$ ($\mathrm{SU}(1,1)$-*covariance*) and $I_1 \cap I_2 = \emptyset \implies V_{I_2} \subseteq V_{I_1}^{\perp_\omega}$ (*locality*). It moreover follows from [Was98, Sec. II.14] that $\delta_I^{it} = \widetilde{u}(e^{-2\pi t \xi_I})$ for all $t \in \mathbb{R}$ and $I \in \mathcal{I}$ (cf. [Lon08, Thm. 3.3.1] and [Bor92, Thm. II.9]). Passing to the second quantization, let $\mathcal{N}_I := \rho(N_I)'' = W(V_I^{\infty})''$ denote the von Neumann algebra generated by $W(V_I^{\infty})$ for $I \in \mathcal{I}$. By Example 3.2.21, we obtain that

$$\rho(e^{-2\pi t \xi_I}) = \mathcal{F}(\widetilde{u}(e^{-2\pi t \xi_I})) = \mathcal{F}(\delta_I^{it}) = \Delta_I^{it},$$

where Δ_I denotes the modular operator on $\mathcal{F}(V)$ defined from $\mathcal{N}_I$ using the cyclic and separating vector $\Omega := 1 \in \mathcal{F}(V)$. Let $\phi = \langle \Omega, \bullet\, \Omega \rangle$ be the corresponding state on $\mathcal{N}_I$. Then (3.2.8) shows that $\widehat{\phi} : N_I \to \mathbb{C}$ is smooth. Thus ρ is smoothly-KMS at $2\pi\xi_I \in \mathfrak{su}(1,1)$ relative to $N_I \subseteq G$, for any $I \in \mathcal{I}$. For more details on the Bisognano-Wichmann property and nets of standard subspaces, see e.g. [Mor18] or [Mun01].

Chapter 4

Generalized positive energy representations of groups of jets

Abstract

Let V be a finite-dimensional real vector space and K a 1-connected compact simple Lie group with Lie algebra $\mathfrak{k}$. Consider the Fréchet-Lie group $G := J_0^\infty(V, K)$ of ∞-jets at $0 \in V$ of smooth maps $V \to K$, with Lie algebra $\mathfrak{g} = J_0^\infty(V, \mathfrak{k})$. Let P be a Lie group and write $\mathfrak{p} := \mathrm{Lie}(P)$. Let α be a smooth P-action on G. We study smooth projective unitary representations $\overline{\rho}$ of $G \rtimes_\alpha P$ that satisfy a generalized positive energy condition. We show that this condition imposes severe restrictions on the derived representation $d\overline{\rho}$ of $\mathfrak{g} \rtimes \mathfrak{p}$, leading in particular to sufficient conditions for $\overline{\rho}|_G$ to factor through $J_0^2(V, K)$, or even through K.

This chapter is based on [Nie23a, Part II].

4.1 Introduction

This chapter is concerned with projective representations of groups and Lie algebras of jets. Let K be a 1-connected compact simple Lie group with Lie algebra $\mathfrak{k}$ and let V be a finite-dimensional real vector space. Then we consider the Fréchet-Lie group $G := J_0^\infty(V, K)$ with Lie algebra $\mathfrak{g} := J_0^\infty(V, \mathfrak{k}) \cong \mathbb{R}[\![V^*]\!] \otimes \mathfrak{k}$. These consist of ∞-jets at $0 \in V$ of smooth K- and $\mathfrak{k}$-valued functions, respectively. Let P be a finite-dimensional Lie group with Lie algebra $\mathfrak{p}$, and assume that there is a smooth action α of P on G. We are interested in smooth projective unitary representations of $G \rtimes_\alpha P$ that are of generalized positive energy at all elements p in some cone $\mathcal{C} \subseteq \mathfrak{p}$. Recall from Chapter 3 that both positive energy- and KMS representations give rise to ones that satisfy the generalized positive energy condition.

The motivation for looking at the (generalized) positive energy representations of the group $G \rtimes_\alpha P$ originates in prior work by B. Janssens and K.H. Neeb, who studied in [JN21] a class of projective unitary representations of the group of compactly supported gauge transformations $\mathcal{G} := \Gamma_c(M, \mathrm{Ad}(\mathcal{K}))$ of a principal K-bundle $\mathcal{K}$ over M, where $\mathrm{Ad}(\mathcal{K})$ denotes the corresponding adjoint bundle. Suppose that the Lie group P acts smoothly on $\mathcal{K}$ by automorphisms of the principal bundle $\mathcal{K}$. This induces a smooth action of P on the infinite-dimensional Lie group $\mathcal{G}$. Their main result is:

Theorem 4.1.1. *([JN21, Theorem 7.19]):*
Let $(\overline{\rho}, \mathcal{H})$ be a projective unitary representation of $\mathcal{G} \rtimes P$ which has a dense set of smooth rays and is of positive energy at the cone $\mathcal{C} \subseteq \mathfrak{p}$. If the cone $\mathcal{C}$ has no fixed points in M, then there exists a 1-dimensional P-equivariantly embedded submanifold $S \subseteq M$ s.t. on the connected component $\mathcal{G}_0$ of the identity, the projective representation $\overline{\rho}$ factors through the restriction map $r : \mathcal{G}_0 \to \Gamma_c(S, Ad(\mathcal{K}))$.

Thus, *if* there are no fixed points in M for $\mathcal{C}$, then the problem of classifying the projective unitary positive energy representations of $\mathcal{G} \rtimes P$ is essentially reduced to the one-dimensional case, which has been extensively studied, see for example [PS86, Was98, Tan11, Kac90, KR87, GW84, TL99]. Moreover, if there are no one-dimensional P-equivariantly embedded submanifolds in M, one is essentially reduced to the case where $\overline{\rho}$ factors through the germs at the fixed point set $\Sigma \subseteq M$ of the cone $\mathcal{C} \subseteq \mathfrak{p}$. In this chapter, we address the setting where fixed points *do* exist and where $\overline{\rho}$ actually factors through the germs at a *single* fixed point.

Thus, let $a \in M$ be a fixed point of the P-action on M and let $V := T_a(M)$. If a smooth projective unitary representation $\overline{\rho}$ of $\mathcal{G}$ factors through the germs at $a \in M$, then the continuity of $\overline{\rho}$ implies that it must further factor through the ∞-jets $J^\infty_a(\mathrm{Ad}(\mathcal{K})) \cong J^\infty_0(V, K) = G$ at $a \in M$, as is shown in Section 4.5.1 of the appendix. The brings us to groups of jets and motivates the study of smooth projective unitary representations of $G \rtimes_\alpha P$. Clearly, any smooth projective unitary representation of $G \rtimes_\alpha P$ defines one of $\mathcal{G} \rtimes P$ via the jet homomorphism $j^\infty_a : \mathcal{G} \to J^\infty_a(\mathrm{Ad}(\mathcal{K})) \cong G$. In this way, the results of this chapter contribute to the understanding of positive energy and KMS-representations of gauge groups.

In [Sim23], KMS-representations were very recently studied in the context of finite-dimensional Lie groups, leading to full characterization of such representations that generate a factor of type I. In relation to the unitary representation theory of gauge groups, let us also mention the papers [GGV77, AHK78, PS76] and [Ism76], in which unitary representations of gauge groups $C^\infty_c(M, K)$ are constructed which are non-local in the sense that they do not factor through the restriction map $C^\infty_c(M, K) \to C^\infty_c(N, K)$ for some proper submanifold $N \subseteq M$. When $\dim(M) \geq 3$, these are irreducible ([Wal87] and [AHKT81]). They are also considered in [ADGV16] and [AHKM$^+$93]. Unitary representations of groups of jets have also been considered in [GG68] and [AT94].

Structure of the chapter

After fixing our notation in Section 4.2, we discuss in Section 4.3 a normal-form problem for the $\mathfrak{p}$-action on $\mathfrak{g} = J^\infty_a(V, \mathfrak{k})$. Using the observations made in Section 2.5 together with the normal-form results obtained in Section 4.3, we proceed in Section 4.4 with the study of (generalized) positive energy representations of the Lie algebra $\mathfrak{g} \rtimes_D \mathfrak{p}$, where $D : \mathfrak{p} \to \mathrm{der}(\mathfrak{g})$ is the $\mathfrak{p}$-action on $\mathfrak{g}$ corresponding to α.

Overview of main results

To describe the main results of Section 4.4, we first need to introduce some notation. We write $R := \mathbb{R}[\![V^*]\!] := \prod_{n=0}^{\infty} P^n(V)$ for the ring of formal power series on V, where $P^n(V)$ denotes the set of degree-n homogeneous polynomials on V. Recall that $\mathfrak{g} := J_0^\infty(V, \mathfrak{k}) \cong R \otimes \mathfrak{k}$. We write $\mathcal{X}_I$ for the Lie algebra of formal vector fields on V vanishing at the origin. The $\mathfrak{p}$-action D splits into a horizontal and a vertical part according to $D(p) = -\mathcal{L}_{\boldsymbol{v}(p)} + \mathrm{ad}_{\sigma(p)}$, for some Lie algebra homomorphism $\boldsymbol{v} : \mathfrak{p} \to \mathcal{X}_I^{op}$ and a linear map $\sigma : \mathfrak{p} \to \mathfrak{g}$ satisfying the Maurer-Cartan equation

$$-\mathcal{L}_{\boldsymbol{v}(p_1)}\sigma(p_2) + \mathcal{L}_{\boldsymbol{v}(p_2)}\sigma(p_1) - \sigma([p_1, p_2]) + [\sigma(p_1), \sigma(p_2)] = 0, \qquad \forall p_1, p_2 \in \mathfrak{p}.$$

For any $p \in \mathfrak{p}$, the formal vector field $\boldsymbol{v}(p)$ splits further as

$$\boldsymbol{v}(p) = \boldsymbol{v}_{\mathrm{l}}(p) + \boldsymbol{v}_{\mathrm{ho}}(p),$$

into its linearization $\boldsymbol{v}_{\mathrm{l}}(p)$, a linear vector field on V, and its higher order part $\boldsymbol{v}_{\mathrm{ho}}(p)$, a formal vector field on V vanishing up to first order at the origin. Let $\sigma_0(p) \in \mathfrak{k}$ be the constant part of the formal power series $\sigma(p) \in R \otimes \mathfrak{k}$.

Let $\Sigma_p \subseteq \mathbb{C}$ denote the additive subsemigroup of $\mathbb{C}$ generated by $\mathrm{Spec}(\boldsymbol{v}_{\mathrm{l}}(p))$. Let $V_c^{\mathbb{C}}(p)$ denote the span in $V_{\mathbb{C}}$ of all generalized eigenspaces of $\boldsymbol{v}_{\mathrm{l}}(p)$ corresponding to eigenvalues with zero real part, and define $V_c(p) := V_c^{\mathbb{C}}(p) \cap V$. If $\mathfrak{C} \subseteq \mathfrak{p}$ is a subset, define $V_c(\mathfrak{C}) := \bigcap_{p \in \mathfrak{C}} V_c(p)$, which we call the 'center subspace of V associated to $\mathfrak{C}$', in analogy with the center manifold of a fixed point of a dynamical system. Let $V_c(\mathfrak{C})^\perp \subseteq V^*$ be its annihilator in V^*.

If $\overline{\pi}$ is a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$, we write

$$\mathfrak{C}(\overline{\pi}) := \{\, p \in \mathfrak{p} \,:\, \overline{\pi} \text{ is of generalized positive energy at } p \,\}.$$

The first main result concerns positive energy representations. It states that unless the spectrum of $\boldsymbol{v}_{\mathrm{l}}(p)$ and $\sigma_0(p)$ happens to intersect non-trivially, any smooth projective unitary representation $\overline{\rho}$ of $G \rtimes_\alpha P$ which is of positive energy at $p \in \mathfrak{p}$ factors through the 2-jets $J_0^2(V, K) \rtimes_\alpha P$:

Theorem 4.4.1. *Let $\overline{\rho}$ be a smooth projective unitary representation of $G \rtimes_\alpha P$ which is of p.e. at $p \in \mathfrak{p}$. Assume that $\mathrm{Spec}(ad_{\sigma_0(p)}) \cap \mathrm{Spec}(\boldsymbol{v}_{\mathrm{l}}(p)) = \emptyset$. Then $\overline{\rho}$ factors through $J_0^2(V, K) \rtimes_\alpha P$. Moreover the image of $-\mathcal{L}_{\boldsymbol{v}_{\mathrm{l}}(p)} + ad_{\sigma_0(p)}$ in $P^2(V) \otimes \mathfrak{k} \subseteq J_0^2(V, K)$ is contained in $\ker \overline{\rho}$.*

The second main result determines restrictions imposed by the generalized positive energy condition. If $p \in \mathfrak{C}(\overline{\pi})$, then unless possibly when the "non-resonance condition" $\mathrm{Spec}(\mathrm{ad}_{\sigma_0(p)}) \cap \Sigma_p = \emptyset$ is violated, it suffices to consider the case where all eigenvalues of $\boldsymbol{v}_{\mathrm{l}}(p)$ are purely imaginary. The precise statement is:

Theorem 4.4.3. *Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$. Assume that* $\mathrm{Spec}(ad_{\sigma_0(p)}) \cap \Sigma_p = \emptyset$ *for all $p \in \mathfrak{C}$.*
Then $RV_c(\mathfrak{C})^\perp \otimes \mathfrak{k} \subseteq \ker \overline{\pi}$ and hence $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c(\mathfrak{C})^*]\!] \otimes \mathfrak{k}$.

Since $\mathbb{R}[\![V_c(\mathfrak{C})^*]\!] \otimes \mathfrak{k} = \mathfrak{k}$ whenever $V_c(\mathfrak{C}) = \{0\}$, Theorem 4.4.3 in particular gives sufficient conditions for $\overline{\pi}$ to factor through $\mathfrak{k}$, that depend only on the spectrum of σ_0 and $\boldsymbol{v}_1(p)$.

For the third main result, we consider the special case where $\mathfrak{p}$ is non-compact and simple:

Theorem 4.4.6. *Assume that $\mathfrak{p}$ is non-compact and simple. Suppose that $\boldsymbol{v}_1$ defines a non-trivial irreducible $\mathfrak{p}$-representation on V. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$ be a P-invariant convex cone. Either $\mathfrak{C}$ is pointed or $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathfrak{k}$.*

Remark 4.1.2. If $\overline{\rho}$ is a smooth projective unitary representation of $G \rtimes_\alpha P$ which is of generalized positive energy at the cone $\mathcal{C} \subseteq \mathfrak{p}$, then its derived representation $d\overline{\rho}$ on the space of smooth vectors $\mathcal{H}_\rho^\infty$ is so, too. Moreover, as we shall see in Lemma 4.2.3 below, the exponential map of $G = J_0^\infty(V, K)$ restricts to a diffeomorphism from the pro-nilpotent ideal $\ker\big(\mathrm{ev}_0 : J_0^\infty(V, \mathfrak{k}) \to \mathfrak{k}\big)$ onto $\ker\big(\mathrm{ev}_0 : J_0^\infty(V, K) \to K\big)$. Thus, the above results all have immediate analogous consequences on the group level.

4.2 Notation

Let V be a finite-dimensional real vector space and K a 1-connected compact simple Lie group with Lie algebra $\mathfrak{k}$. For any $n \in \mathbb{N}_{\geq 0}$, we denote by $P^n(V) \subseteq R$ the space of homogeneous polynomials on V of degree n. Let $R := \mathbb{R}[\![V^*]\!] := \prod_{n=0}^\infty P^n(V)$ denote the ring of formal power series on V with coefficients in $\mathbb{R}$, equipped with the direct product topology. Let $I = (V^*)$ be the maximal ideal of R, containing those elements with vanishing constant term. We write $\mathrm{ev}_0 : R \to \mathbb{R} \cong R/I$ for the corresponding quotient map. Let $\mathfrak{g}$ be the R-module $\mathfrak{g} := R \otimes \mathfrak{k}$ of formal power series on V with coefficients in $\mathfrak{k}$. Then $\mathfrak{g}$ is a topological Lie algebra with the Lie bracket defined by

$$[f \otimes X, g \otimes Y] := fg \otimes [X, Y], \qquad f, g \in R, \quad X, Y \in \mathfrak{k}.$$

We also write fX instead of $f \otimes X$ for $f \in R$ and $X \in \mathfrak{k}$. Define $R_k := R/I^{k+1}$, $I_k := I/I^{k+1}$ and $\mathfrak{g}_k := \mathfrak{g}/(I^{k+1}.\mathfrak{g})$ for $k \in \mathbb{N}_{\geq 0}$. Then $R = \varprojlim R_k$ and $\mathfrak{g} = \varprojlim \mathfrak{g}_k$ as topological vector spaces and Lie algebras, respectively. Let $G_k = J_0^k(V, K)$ be the unique 1-connected Lie group integrating the finite-dimensional Lie algebra $\mathfrak{g}_k$ for any $k \in \mathbb{N}_{\geq 0}$. Let $G := J_0^\infty(V, K) := \varprojlim G_k$ be the corresponding projective limit, which is a pro-Lie group with topological Lie algebra $\mathfrak{g} = \varprojlim_k \mathfrak{g}_k$. (See e.g. [HM07] for a detailed consideration of pro-Lie groups). We write $\mathcal{X}_I$ for the Lie algebra of formal vector fields on V vanishing at the origin. Identify $\mathcal{X}_I \cong \mathrm{der}(I)$ using the Lie derivative $\boldsymbol{v} \mapsto \mathcal{L}_{\boldsymbol{v}}$. Notice further that $\mathrm{der}(I) \cong I \otimes V$. Define similarly $\mathcal{X}_{I_k} := \mathcal{X}_I/(I^{k+1}\mathcal{X}_I) \cong \mathrm{der}(I_k)$.

Let $\mathfrak{p}$ be a finite-dimensional Lie algebra acting on $\mathfrak{g}$ through the homomorphism $D : \mathfrak{p} \to \mathrm{der}(\mathfrak{g})$. Using the fact that all derivations of $\mathfrak{k}$ are inner, by Whitehead's first Lemma [Jac79, III.7. Lem. 3], it follows from [Kac90, Ex. 7.4] that $D(p)$ splits into a horizontal and vertical part according to

$$D(p) = -\mathcal{L}_{\boldsymbol{v}(p)} + \mathrm{ad}_{\sigma(p)} \qquad \text{for } p \in \mathfrak{p},$$

where $\boldsymbol{v} : \mathfrak{p} \to \mathcal{X}_I^{op}$ is a homomorphism of Lie algebras and where $\sigma : \mathfrak{p} \to \mathfrak{g}$ is a linear map that necessarily satisfies the following Maurer-Cartan equation:

$$-\mathcal{L}_{\boldsymbol{v}(p_1)}\sigma(p_2) + \mathcal{L}_{\boldsymbol{v}(p_2)}\sigma(p_1) - \sigma([p_1, p_2]) + [\sigma(p_1), \sigma(p_2)] = 0, \quad \forall p_1, p_2 \in \mathfrak{p}. \quad (4.2.1)$$

Remark 4.2.1. As we shall see in Section 4.3.3 below, Equation (4.2.1) can be written as $\delta\sigma + \frac{1}{2}[\sigma, \sigma] = 0$ in the differential graded Lie algebra $(\bigwedge^\bullet \mathfrak{p}^*) \otimes \mathfrak{g}$, whose differential is that of the Chevalley-Eilenberg complex, where $\mathfrak{g}$ is considered as $\mathfrak{p}$-module according to $p \mapsto -\mathcal{L}_{\boldsymbol{v}(p)}$.

We will refer to D as a *lift* of the $\mathfrak{p}$-action on R to $\mathfrak{g}$, and we call σ the *vertical twist* of the lift D. We remark also that $D(p)$ satisfies the following Leibniz rule:

$$D(p)(f\xi) = -\mathcal{L}_{\boldsymbol{v}(p)}(f)\xi + fD(p)\xi, \qquad f \in R, \xi \in \mathfrak{g}, \qquad \forall p \in \mathfrak{p}. \qquad (4.2.2)$$

We will denote by j^k various k-jet projections $R \to R_k$, $\mathfrak{g} \to \mathfrak{g}_k$ and $\mathcal{X}_I \to \mathcal{X}_{I_k}$. It should be clear from the context which map is being used. Also, we will freely identify the quotient $\mathfrak{g}_0 \cong \mathfrak{k}$ with the Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ of formal power series having only a non-trivial constant term. Similarly, we identify $j^1\mathcal{X}_I = \mathcal{X}_{I_1} \cong \mathfrak{gl}(V)$ with the subalgebra $\mathfrak{gl}(V) \subseteq \mathcal{X}_I$ of linear vector fields on V.

A first observation is the fact that $G = J_0^\infty(V, K)$ is not just a pro-Lie group, but actually a regular Lie group modeled on the Fréchet space $\mathfrak{g} = J_0^\infty(V, \mathfrak{k})$:

Proposition 4.2.2. *Both G and $G \rtimes_\alpha P$ are regular Fréchet-Lie groups.*

Proof. It is clear that the Lie algebras $\mathfrak{g}$ and $\mathfrak{g} \rtimes_D \mathfrak{p}$ are Fréchet. Notice for $n \in \mathbb{N}$ that the map $G_n \to G_{n-1}$ defines a fiber bundle whose typical fiber $P^n(V) \otimes \mathfrak{k}$ is contractible. It follows by considering the associated long exact sequence of homotopy groups that $\pi_k(G_n) \cong \pi_k(G_{n-1})$ for every $k \in \mathbb{N}_{\geq 0}$, and hence $\pi_k(G_n) \cong \pi_k(K)$ by induction on n. In particular, the Lie group $G_n = J_0^n(V, K)$ is 1-connected for every n, because K is so. Then also G is 1-connected, since $\pi_k(G) = \pi_k\big(\varprojlim_n G_n\big) = \varprojlim_n \pi_k(G_n)$ for every $k \in \mathbb{N}$, by [Hat02, Prop. 4.67]. Thus G is the unique 1-connected pro-Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, which is locally contractible by [HN09, Theorem 1.2]. Then [HN09, Theorem 1.3, Prop. 5.7] entails that G is a regular Lie group. As the action $\alpha : P \times G \to G$ is smooth, also $G \rtimes_\alpha P$ is a Lie group and it is regular by [Nee06, Thm. V.I.8], because both G and P are so. □

Moreover, we have the following useful fact:

Lemma 4.2.3. *The exponential map* $\exp_G : \mathfrak{g} \to G$ *restricts to a diffeomorphism from* $I \otimes \mathfrak{k} = \ker\big(\mathrm{ev}_0 : \mathfrak{g} \to \mathfrak{k}\big)$ *onto* $\ker\big(\mathrm{ev}_0 : G \to K\big)$.

Proof. For $k \in \mathbb{N}_{\geq 0}$, let $H_k := \ker\big(\mathrm{ev}_0 : G_k \to K\big) \lhd G_k$ be the maximal nilpotent normal subgroup of G_k. Then $\mathfrak{h}_k := \mathrm{Lie}(H_k) = \ker\big(\mathrm{ev}_0 : \mathfrak{g}_k \to \mathfrak{k}\big) = (I_k \otimes \mathfrak{k}) \lhd \mathfrak{g}_k$. Write $H := \varprojlim_k H_k$ for the corresponding normal subgroup of G and $\mathfrak{h} = \varprojlim \mathfrak{h}_k$ for its Lie algebra. Let $k \in \mathbb{N}$. Notice that H_k is nilpotent and 1-connected. Consequently, its exponential map is a diffeomorphism $\exp_{H_k} : \mathfrak{h}_k \to H_k$ [CG90, Thm. 1.2.1]. Write $\log_{H_k} : H_k \to \mathfrak{h}_k$ for its inverse. If $m \geq k$, then $\exp_{H_k} \circ j^k = j^k \circ \exp_{H_m} : \mathfrak{h}_m \to H_k$ and consequently $\log_{H_k} \circ j^k = j^k \circ \log_{H_m} : H_m \to \mathfrak{h}_k$. Passing to the projective limit, we obtain the inverse $\log_H := \varprojlim_k \log_{H_k}$ of $\exp_H$. It is smooth because $H = \varprojlim_k H_k$ carries the projective limit topology and $\log_{H_k}$ is smooth for every $k \in \mathbb{N}$. Thus $\exp_H : \mathfrak{h} \to H$ is a global diffeomorphism. □

4.3 Normal form results

By choosing suitable local coordinates, one may attempt to simplify the vector fields $\boldsymbol{v}(p)$ and the vertical twist $\sigma(p)$ of the lift $D(p) = -\mathcal{L}_{\boldsymbol{v}(p)} + \mathrm{ad}_{\sigma(p)}$ simultaneously. One might for example try to show that there are local coordinates in which the formal vector fields $\boldsymbol{v}(p)$ are linear for every $p \in \mathfrak{p}$ simultaneously, thereby linearizing the formal $\mathfrak{p}$-action. Similarly, one might aim to show that in suitable coordinates, $\sigma(p) \in \mathfrak{k} \subseteq R \otimes \mathfrak{k}$ is constant for all $p \in \mathfrak{p}$, so that σ is a Lie algebra homomorphism $\mathfrak{p} \to \mathfrak{k}$. In the following, this 'normal form problem' is considered. The results of Section 4.4 will depend on the availability of suitable normal forms, whose existence we study in the present section.

In Section 4.3.1, we briefly recall the transformation behavior of $\boldsymbol{v}$ and σ under suitable automorphisms of $\mathfrak{g}$. We proceed in Section 4.3.2 to recollect some known results regarding normal forms for Lie algebras of vector fields with a common fixed point. Finally, we consider in Section 4.3.3 the vertical twist σ.

4.3.1 Transformation behavior

Definition 4.3.1.

— A *formal diffeomorphism of* V is an automorphism h of R. An automorphism of $\mathfrak{g}$ is said to be *horizontal* if it is of the form $h \otimes \mathrm{id}_{\mathfrak{k}}$ for some $h \in \mathrm{Aut}(R)$. We write $h.\xi$ or $h(\xi)$ instead of $(h \otimes \mathrm{id}_{\mathfrak{k}})(\xi)$ for $\xi \in \mathfrak{g}$.

— A *gauge transformation* is an automorphism of $\mathfrak{g}$ of the form e^{ad_ξ} for some $\xi \in \mathfrak{g}$.

Remark 4.3.2. Any formal diffeomorphism $h \in \mathrm{Aut}(R)$ preserves the maximal proper ideal I and is determined by its restriction $h|_{V^*}$, which can be regarded as an element $\widetilde{h}$ of $I \otimes V$ for which $j^1\widetilde{h} \in V^* \otimes V \cong \mathfrak{gl}(V)$ is invertible. It is then a consequence of Borel's Lemma [Hör03, Thm. 1.2.6] and the Inverse Function Theorem that for any automorphism h of R, there exist 0-neighborhoods $U, U' \subseteq V$ and a diffeomorphism $h_0 : U \to U'$ satisfying $h_0(0) = 0$ such that $h(j_0^\infty(f)) = j_0^\infty(f \circ h_0^{-1})$. Similarly, for $\xi \in \mathfrak{g}$ there exists $\eta \in C_c^\infty(V, \mathfrak{k})$ s.t. $j_0^\infty(\eta) = \xi$, where we have identified $\mathfrak{g} \cong J_0^\infty(V, \mathfrak{k})$. We then have $e^{\mathrm{ad}_\xi} \circ j_0^\infty = j_0^\infty \circ e^{\mathrm{ad}_\eta}$.

To determine the transformation behavior of $D : \mathfrak{p} \to \mathrm{der}(\mathfrak{g})$, we have to consider the adjoint action of $\mathrm{Aut}(\mathfrak{g})$ on $\mathrm{der}(\mathfrak{g})$. Instead of considering arbitrary automorphisms of $\mathfrak{g}$, we will specialize to horizontal ones and to gauge transformations. For $h \in \mathrm{Aut}(R)$ and $v \in \mathcal{X}_I^{op}$, we write $h.v$ for the action of $\mathrm{Aut}(R)$ on $\mathcal{X}_I^{op}$ obtained from the adjoint action of $\mathrm{Aut}(R)$ on $\mathrm{der}(R) \cong \mathcal{X}_I \cong \mathcal{X}_I^{op}$. The following two proofs are due to K.H. Neeb and B. Janssens. They appear in the presently unpublished article [JN].

Lemma 4.3.3 ([JN]). *Let $D \in der(\mathfrak{g})$ and $\xi \in \mathfrak{g}$. Then*

$$e^{ad_\xi} \circ D \circ e^{-ad_\xi} = D + ad\big(F(ad_\xi)D\xi\big),$$

where $F(w) = -\int_0^1 e^{tw}dt = -\sum_{n=0}^\infty \frac{1}{(n+1)!}w^n$.

Proof. Let $k \in \mathbb{N}$ be arbitrary. Consider the continuous path $\gamma : I \to \mathrm{der}(\mathfrak{g})$ defined by $\gamma(t) = e^{t\mathrm{ad}_\xi} D e^{-t\mathrm{ad}_\xi}$. Notice that $j^k \circ \gamma : I \to \mathrm{der}(\mathfrak{g}_k)$ is smooth for all k and consequently so is γ. Moreover

$$\gamma'(t) = e^{t\mathrm{ad}_\xi}[\mathrm{ad}_\xi, D]e^{-t\mathrm{ad}_\xi} = -e^{t\mathrm{ad}_\xi}\mathrm{ad}_{D\xi}e^{-t\mathrm{ad}_\xi} = -\mathrm{ad}\big(e^{t\mathrm{ad}_\xi}D\xi\big),$$

where the last step uses that $\alpha \circ \mathrm{ad}_\eta = \mathrm{ad}_{\alpha(\eta)} \circ \alpha$ for any $\alpha \in \mathrm{Aut}(\mathfrak{g})$. Thus

$$\begin{aligned} e^{\mathrm{ad}_\xi} \circ D \circ e^{-\mathrm{ad}_\xi} - D &= \int_0^1 \gamma'(t)dt = -\int_0^1 \mathrm{ad}\big(e^{t\mathrm{ad}_\xi}D\xi\big)dt \\ &= -\mathrm{ad}\left(\int_0^1 e^{t\mathrm{ad}_\xi}dt\right)(D\xi) = \mathrm{ad}\big(F(\mathrm{ad}_\xi)D\xi\big). \end{aligned}$$

□

Proposition 4.3.4 ([JN]). *Let $h \in Aut(R) \subseteq Aut(\mathfrak{g})$, $\sigma, \xi \in \mathfrak{g}$ and $v \in \mathcal{X}_I$. Consider the derivation $D := -\mathcal{L}_v + ad_\sigma \in der(\mathfrak{g})$. Then*

$$\begin{aligned} h \circ D \circ h^{-1} &= -\mathcal{L}_{h.v} + ad(h.\sigma), \\ e^{ad_\xi} \circ D \circ e^{-ad_\xi} &= -\mathcal{L}_v + ad\left(e^{ad_\xi}\sigma + F(ad_\xi)(-\mathcal{L}_v\xi)\right). \end{aligned} \tag{4.3.1}$$

Proof. It is trivial that $h \circ \mathcal{L}_v \circ h^{-1} = \mathcal{L}_{h.v}$. Moreover, $h \circ \mathrm{ad}_\sigma \circ h^{-1} = \mathrm{ad}_{h.\sigma}$ is valid because $\alpha \circ \mathrm{ad}_\sigma = \mathrm{ad}_{\alpha(\sigma)} \circ \alpha$ for any $\alpha \in \mathrm{Aut}(\mathfrak{g})$. Notice next that $F(\mathrm{ad}_\xi)([\sigma, \xi]) = \sum_{n=1}^\infty \frac{1}{n!}{\mathrm{ad}_\xi}^n\sigma = e^{\mathrm{ad}_\xi}\sigma - \sigma$. It follows using Lemma 4.3.3 that

$$\begin{aligned} e^{\mathrm{ad}_\xi} \circ D \circ e^{-\mathrm{ad}_\xi} &= -\mathcal{L}_v + \mathrm{ad}_\sigma + \mathrm{ad}\big(F(\mathrm{ad}_\xi)(-\mathcal{L}_v\xi)\big) + \mathrm{ad}\big(F(\mathrm{ad}_\xi)[\sigma, \xi]\big) \\ &= -\mathcal{L}_v + \mathrm{ad}\left(e^{\mathrm{ad}_\xi}\sigma + F(\mathrm{ad}_\xi)(-\mathcal{L}_v\xi)\right). \end{aligned}$$

□

Definition 4.3.5.

— Two homomorphisms $\boldsymbol{v}, \boldsymbol{w} : \mathfrak{p} \to \mathcal{X}_I^{op}$ are said to be *formally-equivalent* if there is a formal diffeomorphism $h \in \mathrm{Aut}(R)$ such that $h.\boldsymbol{v}(p) = \boldsymbol{w}(p)$ for all $p \in \mathfrak{p}$.

— Two linear maps $\sigma, \eta : \mathfrak{p} \to R \otimes \mathfrak{k}$ satisfying the Maurer-Cartan equation (4.2.1) are called *gauge-equivalent* if there is some $\xi \in \mathfrak{g}$ such that

$$\eta(p) = e^{\mathrm{ad}_\xi}\sigma(p) + F(\mathrm{ad}_\xi)(-\mathcal{L}_{\boldsymbol{v}(p)}\xi), \qquad \forall p \in \mathfrak{p}. \tag{4.3.2}$$

In this case, we write $\sigma \sim \eta$ and say that σ and η are related by the gauge transformation e^{ad_ξ}.

4.3.2 Lie algebras of formal vector fields with a common fixed point

The normal form problem for vector fields near a fixed point has been subject to extensive study. Let us first gather some relevant known results.

The case of a single vector field

Naturally, the special case which has been considered most is the case where $\mathfrak{p}$ is simply $\mathbb{R}$, in which case one is looking for normal forms of dynamical systems near a fixed point, in the formal context. This case is already quite interesting. Let us recollect some relevant results. For more information, we refer to [Arn88].

Let $\boldsymbol{v}$ be a vector field on V. Write $\boldsymbol{v} = \boldsymbol{v}_{\mathrm{l}} + \boldsymbol{v}_{\mathrm{ho}}$, where $\boldsymbol{v}_{\mathrm{l}} = j_0^1(\boldsymbol{v}) \in \mathfrak{gl}(V) \subseteq \mathcal{X}_I$ is the linearization of $\boldsymbol{v}$ at $0 \in V$ and $\boldsymbol{v}_{\mathrm{ho}} \in \mathcal{X}_{I^2}$ is a vector field vanishing up to first order at $0 \in V$. Let $\boldsymbol{v}_{\mathrm{l}} = \boldsymbol{v}_{\mathrm{l,s}} + \boldsymbol{v}_{\mathrm{l,n}}$ be the Jordan decomposition of $\boldsymbol{v}_{\mathrm{l}}$ over $\mathbb{C}$, where $\boldsymbol{v}_{\mathrm{l,s}}$ is semisimple and $\boldsymbol{v}_{\mathrm{l,n}}$ is nilpotent. Write $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Let $(e_j)_{j=1}^d$ be a basis of eigenvectors of $\boldsymbol{v}_{\mathrm{l,s}}$ in $V_{\mathbb{C}}$ with dual basis $(x_j)_{j=1}^d$ of V^*. Let $(\mu_j)_{j=1}^d$ denote the corresponding eigenvalues.

Definition 4.3.6. Let $\boldsymbol{n} \in \mathbb{N}_{\geq 0}^d$ be a multi-index. A monomial vector field $x^{\boldsymbol{n}}\partial_{x_j}$ with $|\boldsymbol{n}| \geq 2$ is called *resonant* if $\langle \boldsymbol{n}, \boldsymbol{\mu} \rangle = \mu_j$, where $\langle \boldsymbol{n}, \boldsymbol{\mu} \rangle := \sum_{i=1}^d n_i \mu_i$. Identifying $\boldsymbol{v}_{\mathrm{l,s}}$ with the linear vector field $\sum_{j=1}^d \mu_j x_j \partial_{x_j}$ on $\mathbb{C}^d$, this is equivalent to $[\,\boldsymbol{v}_{\mathrm{l,s}},\, x^{\boldsymbol{n}}\partial_{x_j}\,] = 0$.

Theorem 4.3.7 (Poincaré-Dulac Theorem [Dul04] [Arn88, ch.5])**.**
There exists $\boldsymbol{w}_{\mathrm{ho}} \in \mathcal{X}_{I^2}$ which is a $\mathbb{C}$-linear combination of resonant monomials s.t. $\boldsymbol{v}$ is formally equivalent to $\boldsymbol{w} = \boldsymbol{v}_{\mathrm{l}} + \boldsymbol{w}_{\mathrm{ho}} \in \mathcal{X}_I$. In particular $[\boldsymbol{v}_{\mathrm{l,s}}, \boldsymbol{w}_{\mathrm{ho}}] = 0$ in $\mathcal{X}_I$.

Corollary 4.3.8 (Poincaré [Poi79])**.** *If there are no resonances, meaning that $\langle \boldsymbol{n}, \boldsymbol{\mu} \rangle \neq \mu_j$ for all $\boldsymbol{n} \in \mathbb{N}_{\geq 0}^d$ with $|\boldsymbol{n}| \geq 2$ and $j \in \{1, \ldots, d\}$, then the vector field $\boldsymbol{v}$ can be formally linearized, so that $\boldsymbol{v}$ is formally equivalent to the linear vector field $\boldsymbol{v}_{\mathrm{l}}$.*

The case of actions by a compact Lie group

For actions of compact Lie groups there is the following well-known result, see also [DK00, Ch. 2.2]:

Theorem 4.3.9 (Bochner's Linearization Theorem [Boc45])**.**
Let $G \times M \to M$ be a smooth action of compact Lie group on a smooth manifold which has a fixed point $a \in M$. Then, in suitably chosen smooth local coordinates around the fixed point, the action is linear.

The case of actions by semisimple Lie algebras

Next, we move to Lie algebra representations by formal vector fields of semisimple Lie algebras. As nicely explained in [FM04] and first observed by Hermann in [Her68], in the formal setting the obstructions to being able to linearize a Lie algebra of vector fields simultaneously lie in various first Lie algebra cohomology groups $H^1(\mathfrak{p}, W)$ for suitable finite-dimensional $\mathfrak{p}$-modules W. In view of Whitehead's First Lemma [Jac79, III.7. Lem. 3], this results in:

Theorem 4.3.10 ([Her68])**.** *Let $\mathfrak{p}$ be a semisimple Lie algebra and $\boldsymbol{v} : \mathfrak{p} \to \mathcal{X}_I^{op}$ be a Lie algebra homomorphism. Then $\boldsymbol{v}$ is formally equivalent to its linearization $j^1\boldsymbol{v} : \mathfrak{p} \to \mathfrak{gl}(V) \subseteq \mathcal{X}_I^{op}$ around the origin.*

Remark 4.3.11. Corresponding statements of Theorem 4.3.10 in the setting of germs of smooth/analytic vector fields and diffeomorphisms have been proven in [GS68] and [FM04] under additional assumptions. They are false in general without suitable extra conditions, as was shown in [GS68].

4.3.3 Normal form results for the vertical twist

Let us next consider the vertical twist $\sigma : \mathfrak{p} \to \mathfrak{g}$ of the lift $D(p) = -\mathcal{L}_{\boldsymbol{v}(p)} + \mathrm{ad}_{\sigma(p)}$ to $\mathfrak{g}$ of the $\mathfrak{p}$-action $-\mathcal{L}_{\boldsymbol{v}(p)}$ on R, which has to satisfy the Maurer Cartan equation (4.2.1). We fix the horizontal part $\boldsymbol{v} : \mathfrak{p} \to \mathcal{X}_I^{op}$ and act by gauge transformations. The main results of this section are the following two theorems, whose proof comprises the remainder of the section. The reader who is eager to consider the projective unitary g.p.e. representations of $\mathfrak{g}$ can proceed to Section 4.4 after reading Theorem 4.3.12 and Theorem 4.3.13 below.

Let us also remark that the methods used in this section to prove Theorem 4.3.12 and Theorem 4.3.13 were communicated to the author by B. Janssens and K.H. Neeb and appear in similar form in their presently unpublished work [JN], albeit in a more specific context. The author has placed their approach in a more general context and extracted the two theorems below.

Theorem 4.3.12. *Assume that $\mathfrak{p}$ is semisimple. Let the linear map $\sigma : \mathfrak{p} \to \mathfrak{g}$ satisfy the Maurer-Cartan equation* (4.2.1)*. Then σ is gauge-equivalent to $\sigma_0 := \mathrm{ev}_0 \circ \sigma : \mathfrak{p} \to \mathfrak{k}$. If $\mathfrak{p}$ has no non-trivial compact ideals, then σ is gauge-equivalent to* 0.

The next result concerns the case $\mathfrak{p} = \mathbb{R}$, in which case we identify $\boldsymbol{v}$ with $\boldsymbol{v}(1) \in \mathcal{X}_I$ and σ with $\sigma(1) \in \mathfrak{g}$. In this case, the Maurer-Cartan equation (4.2.1) is trivially satisfied for any $\sigma \in \mathfrak{g}$ and $\boldsymbol{v} \in \mathcal{X}_I$.

Theorem 4.3.13. *Assume that* $\mathfrak{p} = \mathbb{R}$. *Let* $\sigma \in \mathfrak{g}$ *and* $\boldsymbol{v} \in \mathcal{X}_I$. *Let* $\boldsymbol{v}_{\mathrm{l}} := j^1\boldsymbol{v} \in \mathfrak{gl}(V)$ *be the linearization of* $\boldsymbol{v}$ *at* $0 \in V$. *Assume w.l.o.g. that* $\sigma_0 := \mathrm{ev}_0(\sigma) \in \mathfrak{t}$ *for some maximal torus* $\mathfrak{t} \subseteq \mathfrak{k}$. *The following assertions hold true:*

1. *Assume that* $\langle \boldsymbol{n}, \boldsymbol{\mu} \rangle \neq \alpha(\sigma_0)$ *for any root* $\alpha \in i\mathfrak{t}^*$ *of* $\mathfrak{k}$ *and* $\boldsymbol{n} \in \mathbb{N}^d_{\geq 0}$ *with* $|\boldsymbol{n}| \geq 1$.
   Then σ *is gauge-equivalent to some* $\sigma' \in R \otimes \mathfrak{t}$.
2. *If* $\boldsymbol{v}_{\mathrm{l}}$ *is semisimple, then* σ *is gauge-equivalent to some* $\nu \in R \otimes \mathfrak{k}$ *satisfying* $-\mathcal{L}_{\boldsymbol{v}_{\mathrm{l}}}\nu + [\sigma_0, \nu] = 0$.
3. *Suppose that* $\boldsymbol{v} = \boldsymbol{v}_{\mathrm{l}}$ *is linear. Assume that* $D = -\mathcal{L}_{\boldsymbol{v}} + ad_\sigma$ *integrates to a continuous* $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$*-action on* $\mathfrak{g}$. *Then* σ *is gauge-equivalent to* $\sigma_0 \in \mathfrak{t}$. *Moreover* $\mathrm{Spec}(\boldsymbol{v}_{\mathrm{l}}) \cup \mathrm{Spec}(ad_\sigma) \subseteq 2\pi i\mathbb{Z}$.

Remark 4.3.14. Suppose that $\mathcal{K} \to M$ is a principal fiber bundle with compact simple structure group K. Let $\alpha : \mathbb{T} \to \mathrm{Aut}(\mathcal{K})$ be a smooth action on $\mathcal{K}$ by bundle automorphisms. Suppose that $a \in M$ is a fixed point of the induced $\mathbb{T}$-action on M and set $V := T_a(M)$. By Theorem 4.3.9, the $\mathbb{T}$-action on M is linear in suitable local coordinates around $a \in M$. Passing to $J^\infty_a(M) \cong R$ and $J^\infty_a(\mathrm{Ad}(\mathfrak{K})) \cong \mathfrak{g}$, one obtains a $\mathbb{T}$-action on both R and $\mathfrak{g}$. The corresponding derivations are given by $-\mathcal{L}_{\boldsymbol{v}}$ and $D = -\mathcal{L}_{\boldsymbol{v}} + \mathrm{ad}_\sigma$ respectively, for some linear semisimple vector field $\boldsymbol{v}$ on V and some $\sigma \in \mathfrak{g}$. This is the setting of the third item in Theorem 4.3.13, according to which we may further assume that $\sigma \in \mathfrak{t}$, where $\mathfrak{t} \subseteq \mathfrak{k}$ is a maximal torus, by acting with gauge transformations.

The remainder of this section is devoted to the proof of Theorem 4.3.12 and Theorem 4.3.13.

Reformulation using differential graded Lie algebras

In order to classify the equivalence classes of vertical twists $\sigma : \mathfrak{p} \to \mathfrak{g}$, we interpret equation (4.2.1) as the Maurer Cartan equation in the differential graded Lie algebra (DGLA) $L := L_R := (\bigwedge^\bullet \mathfrak{p}^*) \otimes \mathfrak{g}$. As a cochain complex, L is the Chevalley-Eilenberg complex associated to the $\mathfrak{p}$-module $\mathfrak{g}$, where $\mathfrak{p}$ acts on $\mathfrak{g}$ by $p.\psi = -\mathcal{L}_{\boldsymbol{v}(p)}\psi$. Explicitly, the differential δ is given by

$$\begin{aligned}\delta(\alpha)(p_1, \ldots, p_{k+1}) = &\sum_i (-1)^{i+1} p_i.\alpha(p_1, \ldots, \widehat{p_i}, \ldots, p_{k+1})\\ &+ \sum_{i<j} (-1)^{i+j} \alpha([p_i, p_j], p_1, \ldots, \widehat{p_i}, \ldots, \widehat{p_j}, \ldots, p_{k+1}),\end{aligned}$$

where as usual, the arguments with a caret are to be omitted. The graded Lie bracket on L is the unique bilinear map $[-,-] : L \times L \to L$ satisfying

$$[\alpha \otimes \sigma, \beta \otimes \psi] := (\alpha \wedge \beta) \otimes [\sigma, \psi], \qquad \forall \alpha, \beta \in \textstyle\bigwedge^\bullet \mathfrak{p}^*, \quad \sigma, \psi \in \mathfrak{g}.$$

We write $L^k := \left(\bigwedge^k \mathfrak{p}^*\right) \otimes \mathfrak{g}$ for the degree k-elements in L. Interpreting σ as a degree-1 element of L, equation (4.2.1) can now equivalently be written as the usual MC-equation $\delta\sigma + \frac{1}{2}[\sigma,\sigma] = 0$ in L.

Let us next reformulate the gauge-action (4.3.2) of $\mathfrak{g}$ on the set of vertical twists, using the DGLA L. Consider the extended DGLA $L \rtimes \mathbb{R}D$, where D is a degree-1 element satisfying $[D,\sigma] = \delta(\sigma)$ for any $\sigma \in L$. Notice for $\xi \in \mathfrak{g} = L^0$ that $\delta(\xi)(p) = -\mathcal{L}_{\boldsymbol{v}(p)}\xi$. We define the gauge-action of $L^0 = \mathfrak{g}$ on L by

$$\xi.\sigma = e^{\mathrm{ad}(\xi)}(D+\sigma) - D = e^{\mathrm{ad}_\xi}(\sigma) + F(\mathrm{ad}_\xi)(\delta(\xi)), \qquad \xi \in \mathfrak{g}, \tag{4.3.3}$$

considered as an expression in $L \rtimes \mathbb{R}D$, where $F(w) = -\sum_{n=0}^{\infty} \frac{1}{(n+1)!} w^n = -\int_0^1 e^{tw} dt$. Let us check that the above is indeed well-defined, even though L is not a nilpotent DGLA. Since $G = \varprojlim_k G_k$ is a Lie group, it has an exponential map and so the automorphism $e^{\mathrm{ad}_\xi} := \mathrm{Ad}(e^\xi)$ on $\mathfrak{g}$ is defined. Consequently, so is

$$F(\mathrm{ad}_\xi)(-\mathcal{L}_{\boldsymbol{v}(p)}\xi) = \int_0^1 e^{t\mathrm{ad}_\xi}(\mathcal{L}_{\boldsymbol{v}(p)}\xi)dt$$

for any $p \in \mathfrak{p}$. Thus the expression in equation (4.3.3) makes sense. Notice further that for $\sigma \in L^1$, the above reduces precisely to the transformation behavior (4.3.2) of the vertical twist. In accordance with Definition 4.3.5, we say that the MC-elements $\sigma, \sigma' \in L^1$ are gauge-equivalent if they satisfy $\sigma' = \xi.\sigma$ for some $\xi \in L^0$, in which case we write $\sigma \sim \sigma'$. Our goal is to study the MC-elements in L^1 up to gauge-equivalence.

Let $n \in \mathbb{N}_{\geq 0}$. Define analogously the following DGLAs, where we consider $P^n(V)$ as $\mathfrak{p}$-module by identifying $P^n(V)$ with I^n/I^{n+1} for $n \in \mathbb{N}_{\geq 0}$, so that $p.f = -\mathcal{L}_{\boldsymbol{v}_1(p)}f$ for $p \in \mathfrak{p}$ and $f \in P^n(V)$:

$$L_I := \bigwedge\nolimits^\bullet \mathfrak{p}^* \otimes (I \otimes \mathfrak{k}), \qquad L_{R_n} := \bigwedge\nolimits^\bullet \mathfrak{p}^* \otimes (R_n \otimes \mathfrak{k}),$$
$$L_{I_n} := \bigwedge\nolimits^\bullet \mathfrak{p}^* \otimes (I_n \otimes \mathfrak{k}), \qquad L_{P^n} := \bigwedge\nolimits^\bullet \mathfrak{p}^* \otimes (P^n(V) \otimes \mathfrak{k}).$$

Shifted DGLAs

It will be beneficial to split off the constants terms of the $\mathfrak{k}$-valued formal power series, because contrary to L_R, L_I is a projective limit of *nilpotent* DGLAs. We discuss next how this can be done.

For any MC-element $\chi \in L^1_{R_0} = \mathfrak{p}^* \otimes \mathfrak{k} \subseteq L_R$ of degree 1, define the "shifted" DGLA L_R^χ, which agrees with L_R as a graded Lie algebra but has a shifted differential given by $\delta_\chi(\sigma) := \delta(\sigma) + [\chi,\sigma]$. The differential δ_χ agrees with the Chevalley-Eilenberg differential of $\left(\bigwedge^\bullet \mathfrak{p}^*\right) \otimes \mathfrak{g}$ if $\mathfrak{g}$ is considered as $\mathfrak{p}$-module with the twisted action $p.\sigma := -\mathcal{L}_{\boldsymbol{v}(p)} + [\chi,-]$. In particular, $\delta_\chi^2 = 0$. Let us write $R \otimes_\chi \mathfrak{k}$ for this module structure to distinguish it form the usual one on $\mathfrak{g} = R \otimes \mathfrak{k}$, which was given by $p.\xi = -\mathcal{L}_{\boldsymbol{v}(p)}\xi$. Define also the extended DGLA $L_R^\chi \rtimes \mathbb{R}D_\chi$, where $[D_\chi,\sigma] = \delta_\chi(\sigma)$. Define in analogous fashion $L_{I_n}^\chi$, L_I^χ and $L_{P^n}^\chi$, where we have used that the $\mathfrak{p}$-action on $R \otimes_\chi \mathfrak{k}$ leaves $I_n \otimes \mathfrak{k}$ invariant for every n, so that $P^n(V) \otimes \mathfrak{k} \cong (I_n \otimes \mathfrak{k})/(I_{n+1} \otimes \mathfrak{k})$ is naturally a $\mathfrak{p}$-module. The following is a standard result:

Lemma 4.3.15.

1. *Let $\sigma \in L^1_R$. Then $\chi + \sigma$ is a MC-element in L_R if and only if σ is a MC-element in L^χ_R.*
2. *Let $\sigma, \sigma' \in L^\chi_R$ be degree-1 MC-elements. Then $\chi + \sigma \sim \chi + \sigma'$ in L_R if and only if $\sigma \sim \sigma'$ in L^χ_R.*
3. *Let $\psi \in L^1_R$ be a MC-element. Then $\psi = \chi + \sigma$ for some degree-1 MC-elements $\sigma \in L^\chi_I$ and $\chi \in L_{R_0}$.*

Proof.

1. As χ is a MC-element and $[\sigma, \chi] = [\chi, \sigma]$ we have

$$\delta(\chi + \sigma) + \frac{1}{2}[\chi + \sigma, \chi + \sigma] = \delta(\sigma) + \frac{1}{2}[\sigma, \sigma] + [\chi, \sigma] = \delta_\chi(\sigma) + \frac{1}{2}[\sigma, \sigma].$$

2. Observe that $-F(\mathrm{ad}_\xi)([\xi, \chi]) = e^{\mathrm{ad}_\xi}(\chi) - \chi$. Consequently

$$F(\mathrm{ad}_\xi)(\delta_\chi(\xi)) = F(\mathrm{ad}_\xi)(\delta(\xi)) + F(\mathrm{ad}_\xi)([\chi, \xi]) = F(\mathrm{ad}_\xi)(\delta(\xi)) + e^{\mathrm{ad}_\xi}(\chi) - \chi.$$

Thus, for any $\xi \in \mathfrak{g}$ we have

$$e^{\mathrm{ad}_\xi}(\chi + \sigma) + F(\mathrm{ad}_\xi)(\delta(\xi)) = \chi + \Big(e^{\mathrm{ad}_\xi}(\sigma) + F(\mathrm{ad}_\xi)(\delta_\chi(\xi))\Big).$$

3. Since $R = R_0 \oplus I$ as a vector space, we can write $\psi = \chi + \sigma$, where $\chi = j^0(\psi) \in L_{R_0}$ and $\sigma \in L^\chi_I$. As j^0 is a morphism of DGLAs, it is clear that $\chi = j^0(\psi)$ is a MC-element in $L_{R_0} \subseteq L_R$. By the first point it follows that σ is a MC-element in $L^\chi_I \subseteq L^\chi_R$. □

Study of MC-elements

In view of Lemma 4.3.15, let us first study the classification problem of gauge-orbits of MC-elements in $L^1_{R_0}$ and then, for each MC-element $\chi \in L^1_{R_0}$ consider the orbits in L^χ_I under the gauge-action.

Lemma 4.3.16.

1. *Let $\chi : \mathfrak{p} \to \mathfrak{k}$ be linear. Then χ is a MC-element in $L^1_{R_0}$ if and only if it is a Lie algebra homomorphism. Thus if there are no homomorphisms $\mathfrak{p} \to \mathfrak{k}$, then any MC-element $\chi \in L^1_{R_0} = \mathfrak{p}^* \otimes \mathfrak{k}$ is trivial.*
2. *The gauge-action of $X \in \mathfrak{k} = L^0_{R_0}$ on L_{R_0} is given by $X.\chi = e^X \chi$.*

Proof. Notice that $\delta(X) = 0$ for any $X \in \mathfrak{k} \subseteq \mathfrak{g}$, because $-\mathcal{L}_{\boldsymbol{v}(p)} X = 0$ for any $p \in \mathfrak{p}$. So $\mathfrak{p}$ acts trivially on $\mathfrak{k} = \mathfrak{g}_0 = j^0 \mathfrak{g}$. Thus the Maurer-Cartan condition reads simply $\chi([p_1, p_2]) - [\chi(p_1), \chi(p_2)] = 0$ for all $p_1, p_2 \in \mathfrak{p}$, proving the first statement. The second statement follows at once from the definition (4.3.3), using once more that the $\mathfrak{p}$-action on $\mathfrak{k}$ is trivial. □

Next, we fix a homomorphism $\chi : \mathfrak{p} \to \mathfrak{k}$ and turn to the MC-elements of the twisted DGLAs L_I^χ. Consider the following diagram of DGLAs:

$$\begin{array}{ccccccccc} & & L_I^\chi & & & & & & \\ & & \downarrow & \searrow & & & & \searrow & \\ \cdots & \longrightarrow & L_{I_{k+1}}^\chi & \longrightarrow & L_{I_k}^\chi & \longrightarrow & \cdots & \longrightarrow & L_{I_0}^\chi = \{0\} \end{array}$$

Any MC-element in L_I^χ projects to one in $L_{I_k}^\chi$ for any $k \in \mathbb{N}$, and all maps in the above diagram are equivariant w.r.t. the gauge-actions. Notice further that each $L_{I_k}^\chi$ is nilpotent. To study the MC-elements in L_I^χ, we consider lifts of MC-elements from $L_{I_k}^\chi$ to $L_{I_{k+1}}^\chi$, so as to solve the problem step-by-step. This can be done using the following central extension of nilpotent DGLAs, where $L_{I_0}^\chi = \{0\}$ is trivial:

$$0 \to L_{P^k}^\chi \to L_{I_k}^\chi \to L_{I_{k-1}}^\chi \to 0, \qquad k \in \mathbb{N} \tag{4.3.4}$$

in combination with the following known result from deformation theory (cf. [Man04, Sec. V.6]):

Lemma 4.3.17. *Let $0 \to K \to L \to M \to 0$ be a central extension of nilpotent DGLAs. Let $\sigma_M \in M^1$ be a MC-element.*

1. *Suppose that $\sigma_L \in L$ projects to σ_M. Then $h := \delta\sigma_L + \frac{1}{2}[\sigma_L, \sigma_L] \in K^2$ is closed and $[h] \in H^2(K)$ is independent of the lift σ_L of σ_M. Moreover, there is some $\eta \in K^1$ such that $\sigma_L + \eta$ is a MC-element in L^1 if and only if $[h] = 0$ in $H^2(K)$.*

2. *If σ_L and σ_L' are two lifts of σ_M that are both MC-elements in L^1, then $\Delta := \sigma_L' - \sigma_L \in K^1$ is closed. Conversely, if $\Delta \in K^1$ is closed and σ_L is a lift of σ_M which is a MC-element, then $\sigma_L' := \sigma_L + \Delta$ is also a lift of σ_M which is a MC-element. Moreover, the class $[\Delta] \in H^1(K)$ vanishes if and only if σ_L and $\sigma_{L'}$ are related by a gauge transformation of some element $\xi \in K^0$.*

3. *If σ_L is any lift of σ_M which is a MC-element, then the map $\Delta \mapsto \sigma_L + \Delta$ induces a bijection between $H^1(K)$ and K^0-orbits of MC-elements in L^1 lifting σ_M.*

Proof.

1. It is clear that $h \in K^2$ as it projects to zero in M^2. Since $\delta h = [\delta\sigma_L, \sigma_L]$ (by the graded Leibniz rule), we find using $\delta\sigma_L = h - \frac{1}{2}[\sigma_L, \sigma_L]$ that

$$\delta h = [h, \sigma_L] - \frac{1}{2}[[\sigma_L, \sigma_L], \sigma_L] = 0,$$

where the second term vanishes by the graded Jacobi identity and the first term vanishes because $h \in K$ is central. Thus h is closed. Suppose that σ_L' is some other lift of σ_M and define $h' := \delta\sigma_L' + \frac{1}{2}[\sigma_L', \sigma_L'] \in K^2$. Then $\Delta := \sigma_L' - \sigma_L \in K$ lies in the center, so that $h' = h + \delta\Delta$. It follows that

$[h] \in H^2(K)$ does not depend on the lift. If there is some $\eta \in K^1$ such that $\sigma_L + \eta$ is a MC-element in L^1, then

$$0 = \delta(\sigma_L + \eta) + \frac{1}{2}[\sigma_L + \eta, \sigma_L + \eta] = \delta\eta + h.$$

Hence $[h] = 0$. Conversely, if $[h] = 0 \in H^2(K)$, then there exists $\eta \in K^1$ such that $h + \delta\eta = 0$. Then $\sigma_L + \eta$ is a MC-element, by the same computation.

2. Let σ'_L and σ_L be MC-elements in L^1 lifting σ_M. We have already noticed that $h' = h + \delta\Delta$, where $\Delta := \sigma'_L - \sigma_L \in K^1$. Since $h = h' = 0$ by assumption, it follows that $\delta\Delta = 0$. Conversely, suppose $\Delta \in K^1$ is closed and that σ_L is a MC-element projecting to σ_M. Then $\sigma'_L := \sigma_L + \Delta$ projects to σ_M as well. Also, σ'_L is a MC-element, because $\delta\sigma'_L + \frac{1}{2}[\sigma'_L, \sigma'_L] = \delta\sigma_L + \frac{1}{2}[\sigma_L, \sigma_L] + \delta\Delta = 0$. To see that $[\Delta] = 0$ in $H^1(K)$ if and only if σ_L and $\sigma'_L = \sigma_L + \Delta$ are related by a gauge transformation by some element $\xi \in K^0$, observation that if $\xi \in K^0$, then as ξ is central we have

$$\xi.\sigma_L = e^{\mathrm{ad}_\xi}(\sigma_L + D) - D = \sigma_L - [D, \xi] = \sigma_L - \delta\xi.$$

3. This is immediate from the previous point. □

Next, we apply Lemma 4.3.17 to the exact sequences (4.3.4).

Lemma 4.3.18. *For every sequence $(\xi_k)_{k\in\mathbb{N}}$ of degree-0 elements in L_I^χ with $\xi_k \in P^k(V) \otimes_\chi \mathfrak{k}$ for every $k \in \mathbb{N}$, there exists $\eta \in I \otimes_\chi \mathfrak{k}$ such that $j^n(\eta.\sigma) = \xi_n.\xi_{n-1}.\cdots.\xi_1.\sigma$ for every $\sigma \in L_I^\chi$ and $n \in \mathbb{N}$.*

Proof. Consider the Lie subgroup $H := \ker\big(\mathrm{ev}_0 : G \to K\big) \triangleleft G$ with Lie algebra $\mathfrak{h} := \ker\big(\mathrm{ev}_0 : \mathfrak{g} \to \mathfrak{k}\big) = I \otimes \mathfrak{k}$. Similarly, for $n \in \mathbb{N}$ let $H_n := \ker\big(\mathrm{ev}_0 : G_n \to K\big)$ and $\mathfrak{h}_n := \mathrm{Lie}(H_n)$. Recall that the exponential map $\exp : \mathfrak{h} \to H$ is a global diffeomorphism, by Lemma 4.2.3. Write $\log : H \to \mathfrak{h}$ for its inverse. From $j^n \circ \exp = \exp \circ j^n : \mathfrak{h} \to H_n$ we obtain that $\log \circ j^n = j^n \circ \log : H \to \mathfrak{h}_n$ for any $n \in \mathbb{N}$. As $[\xi_k, I \otimes \mathfrak{k}] \subseteq I^{k+1} \otimes \mathfrak{k}$ and the ξ_k are of increasing order, we claim that the limit $\eta := \lim_{N\to\infty} \log\Big(\prod_{k=1}^N e^{\xi_k}\Big)$ exists in $I \otimes \mathfrak{k}$ w.r.t. the projective limit-topology, where k increases from *right to left* in the expression. Indeed, to see this it suffices to show that for each $n \in \mathbb{N}$ the sequence $(j^n\eta_N)_{N=1}^\infty$ stabilizes for large enough values of N, where $\eta_N := \log\Big(\prod_{k=1}^N e^{\xi_k}\Big)$. This is the case because for $N \geq n$ we have:

$$j^n\eta_N = j^n \log\left(\prod_{k=1}^N e^{\xi_k}\right) = \log\left(\prod_{k=1}^N e^{j^n(\xi_k)}\right) = \log\left(\prod_{k=1}^n e^{j^n(\xi_k)}\right) = j^n\eta_n$$

where it was used that $j^n(\xi_k) = 0$ for all $k > n$, because $\xi_k \in I^k$. Thus $\eta = \lim_N \eta_N$ is well-defined and satisfies $j^n\eta = j^n\eta_n$ for all $n \in \mathbb{N}$. Let $\sigma \in L_I^\chi$. Using the fact that

$$\begin{aligned}\xi_{n+1}.\eta_n.\sigma &= e^{\mathrm{ad}_{\xi_{n+1}}}(\eta_n.\sigma + D_\chi) - D_\chi \\ &= e^{\mathrm{ad}_{\xi_{n+1}}} e^{\mathrm{ad}_{\eta_n}}(D_\chi + \sigma) - D_\chi = e^{\mathrm{ad}_{\eta_{n+1}}}(D_\chi + \sigma) - D_\chi = \eta_{n+1}.\sigma,\end{aligned}$$

it follows by induction that for any $n \in \mathbb{N}$, the equality $\eta_n.\sigma = \xi_n.\xi_{n-1} \cdots \xi_1.\sigma$ is valid. We thus get:

$$j^n(\eta.\sigma) = j^n(\eta_n.\sigma) = j^n(\xi_n.\xi_{n-1} \cdots \xi_1.\sigma), \qquad \forall n \in \mathbb{N}. \quad \square$$

Proposition 4.3.19. *Assume that $H^1(\mathfrak{p}, P^k(V) \otimes_\chi \mathfrak{k}) = 0$ for every $k \in \mathbb{N}$. Then every degree-1 MC-element in L_I^χ is gauge-equivalent to 0 in L_I^χ.*

Proof. Fix a MC-element $\sigma \in L_I^\chi$. Recall that $j^n(\zeta.\sigma)$ is again a MC-element in $L_{I_n}^\chi$ for any $n \in \mathbb{N}$ and $\zeta \in L_I^\chi$ of degree 0. Notice also that $j^0\sigma = 0$. As $H^1(\mathfrak{p}, P^k(V) \otimes_\chi \mathfrak{k}) = 0$ for every $k \in \mathbb{N}$, it follows using Lemma 4.3.17 and the exact sequences (4.3.4), by induction on $n \in \mathbb{N}$, that we can find a sequence of degree-0 elements $(\xi_k)_{k \in \mathbb{N}}$ in L_I^χ with $\xi_k \in P^k(V) \otimes \mathfrak{k}$ such that $j^n(\xi_n.\xi_{n-1} \cdots \xi_1.\sigma) = 0$ for every $n \in \mathbb{N}$. It follows from Lemma 4.3.18 that there is some $\eta \in I \otimes_\chi \mathfrak{k}$ such that $j^n(\eta.\sigma) = \xi_n.\xi_{n-1}. \cdots .\xi_1.\sigma = 0$ in L_I^χ for all $n \in \mathbb{N}$. Thus $\sigma \sim 0$. □

Lemma 4.3.20. *Let $\mathfrak{p}$ be a semisimple Lie algebra with no nontrivial compact ideals. If $\mathfrak{k}$ is a compact semisimple Lie algebra, then there are no non-trivial homomorphisms $\mathfrak{p} \to \mathfrak{k}$.*

Proof. Let $\chi : \mathfrak{p} \to \mathfrak{k}$ be a homomorphism. Then $\mathfrak{p}/\ker\chi$ is isomorphic to a subalgebra of $\mathfrak{k}$ and is therefore compact. As $\mathfrak{p}$ is semisimple and has no nontrivial compact ideals, it also has no non-trivial compact quotients. Thus $\mathfrak{p}/\ker\chi = \{0\}$ or equivalently $\mathfrak{p} = \ker\chi$, so χ is trivial. □

Proposition 4.3.21. *Assume that $\mathfrak{p}$ is semisimple. Let $\chi : \mathfrak{p} \to \mathfrak{k}$ be a homomorphism and let $\sigma' \in L_I^\chi$. Suppose that $\sigma := \chi + \sigma'$ is a degree-1 MC-element in L_R. Then σ is gauge-equivalent to χ in L_R.*

Proof. Since $H^1(\mathfrak{p}, P^k(V) \otimes_\chi \mathfrak{k}) = 0$ for all $k \in \mathbb{N}_{\geq 0}$ by Whitehead's Lemma [Jac79, III.7. Lem. 3], Proposition 4.3.19 implies that σ is equivalent to 0 in L_I^χ. Equivalently $\chi + \sigma$ is equivalent to χ in L_R. □

Theorem 4.3.12. *Assume that $\mathfrak{p}$ is semisimple. Let the linear map $\sigma : \mathfrak{p} \to \mathfrak{g}$ satisfy the Maurer-Cartan equation (4.2.1). Then σ is gauge-equivalent to $\sigma_0 := \mathrm{ev}_0 \circ \sigma : \mathfrak{p} \to \mathfrak{k}$. If $\mathfrak{p}$ has no non-trivial compact ideals, then σ is gauge-equivalent to 0.*

Proof. Since $\sigma \in \mathfrak{p}^* \otimes \mathfrak{g}$ is a MC-element in L_R, there is some degree-1 MC-element $\sigma' \in L_I^{\sigma_0}$ such that $\sigma = \sigma_0 + \sigma'$, by Lemma 4.3.15. Then Proposition 4.3.21 implies that σ is gauge-equivalent to σ_0. By Lemma 4.3.16 we further know that $\sigma_0 : \mathfrak{p} \to \mathfrak{k}$ is a homomorphism of Lie algebras. Thus, if $\mathfrak{p}$ has no non-compact ideals then σ_0 is trivial by Lemma 4.3.20. □

Remark 4.3.22. Alternatively, Theorem 4.3.12 also follows from the structure theory of pro-Lie algebras, developed in [HM07]. To see this, assume that $\mathfrak{p}$ is semisimple. Consider the pro-Lie algebra $\mathfrak{h} \rtimes_{D_0} \mathfrak{p}$, were $\mathfrak{h} := I \otimes \mathfrak{k} \subseteq \mathfrak{g}$ and where $D_0 : \mathfrak{p} \to \mathrm{der}(\mathfrak{h})$ is given by $D_0(p) = -\mathcal{L}_{\boldsymbol{v}(p)} + \mathrm{ad}_{\sigma_0(p)}$ for $p \in \mathfrak{p}$. Since $\mathfrak{p}$ is semisimple, the radical and Levi-factor of $\mathfrak{h} \rtimes_{D_0} \mathfrak{p}$ are $\mathfrak{h}$ and $\mathfrak{p}$, respectively. A Levi subalgebra of $(\mathfrak{g} \rtimes_{D_0} \mathfrak{p})$ is equivalently given by a splitting of the exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{h} \rtimes_{D_0} \mathfrak{p} \to \mathfrak{p} \to 0, \tag{4.3.5}$$

which in turn is equivalently given by a linear map $\sigma' : \mathfrak{p} \to \mathfrak{h}$ satisfying the Maurer-Cartan equation

$$\sigma'([p_1, p_2]) = [\sigma'(p_1), \sigma'(p_2)] + D_0(p_1)\sigma'(p_2) - D_0(p_2)\sigma'(p_1), \qquad \forall p_1, p_2 \in \mathfrak{p}.$$

That is, by a degree-1 MC-element σ' in the DGLA $L_I^{\sigma_0}$. The splitting $s_{\sigma'}$ and Levi subalgebra $\mathfrak{l}_{\sigma'}$ corresponding to σ' are given by $s_{\sigma'} : \mathfrak{p} \to \mathfrak{h} \rtimes_{D_0} \mathfrak{p}$, $s_{\sigma'}(p) := (\sigma'(p), p)$, and $\mathfrak{l}_{\sigma'} := \{ (\sigma'(p), p) : p \in \mathfrak{p} \} \subseteq \mathfrak{h} \rtimes_{D_0} \mathfrak{p}$, respectively. Any two Levi subalgebras in $\mathfrak{h} \rtimes_{D_0} \mathfrak{p}$ are conjugate by an automorphism of the form e^{ad_ξ} for some $\xi \in \mathfrak{h}$, by [HM07, Thm. 7.77(i)]. So if $\sigma' \in L_I^{\sigma_0}$ is a degree-1 MC-element, there exists $\xi \in \mathfrak{h}$ such that $e^{\mathrm{ad}_\xi}(\sigma'(p), p) = (0, p)$ for all $p \in \mathfrak{p}$. Notice for $p \in \mathfrak{p}$ that $e^{\mathrm{ad}_\xi}(\sigma'(p), p) = ((\xi.\sigma')(p), p)$, where

$$(\xi.\sigma')(p) = e^{\mathrm{ad}_\xi}\sigma'(p) + F(\mathrm{ad}_\xi)(D_0(p)\xi) = e^{\mathrm{ad}_\xi}\sigma'(p) + F(\mathrm{ad}_\xi)(\delta_{\sigma_0}(\xi)(p))$$

is precisely the gauge action of the degree-zero elements $(L_I^{\sigma_0})^0 = \mathfrak{h}$ on $L_I^{\sigma_0}$. We thus find that $\xi.\sigma' = 0$, so $\sigma' \sim 0$ in $L_I^{\sigma_0}$. By Lemma 4.3.16, this is equivalent with $\sigma \sim \sigma_0$ in L_R.

We now prove Theorem 4.3.13.

Theorem 4.3.13. *Assume that $\mathfrak{p} = \mathbb{R}$. Let $\sigma \in \mathfrak{g}$ and $\boldsymbol{v} \in \mathcal{X}_I$. Let $\boldsymbol{v}_\mathrm{l} := j^1\boldsymbol{v} \in \mathfrak{gl}(V)$ be the linearization of $\boldsymbol{v}$ at $0 \in V$. Assume w.l.o.g. that $\sigma_0 := \mathrm{ev}_0(\sigma) \in \mathfrak{t}$ for some maximal torus $\mathfrak{t} \subseteq \mathfrak{k}$. The following assertions hold true:*

1. *Assume that $\langle \boldsymbol{n}, \boldsymbol{\mu} \rangle \neq \alpha(\sigma_0)$ for any root $\alpha \in i\mathfrak{t}^*$ of $\mathfrak{k}$ and $\boldsymbol{n} \in \mathbb{N}_{\geq 0}^d$ with $|\boldsymbol{n}| \geq 1$.*
   Then σ is gauge-equivalent to some $\sigma' \in R \otimes \mathfrak{t}$.

2. *If $\boldsymbol{v}_\mathrm{l}$ is semisimple, then σ is gauge-equivalent to some $\nu \in R \otimes \mathfrak{k}$ satisfying $-\mathcal{L}_{\boldsymbol{v}_\mathrm{l}}\nu + [\sigma_0, \nu] = 0$.*

3. *Suppose that $\boldsymbol{v} = \boldsymbol{v}_\mathrm{l}$ is linear. Assume that $D = -\mathcal{L}_{\boldsymbol{v}} + ad_\sigma$ integrates to a continuous $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$-action on $\mathfrak{g}$. Then σ is gauge-equivalent to $\sigma_0 \in \mathfrak{t}$. Moreover $\mathrm{Spec}(\boldsymbol{v}_\mathrm{l}) \cup \mathrm{Spec}(ad_\sigma) \subseteq 2\pi i\mathbb{Z}$.*

Proof.

1. Using Lemma 4.3.15, write $\sigma = \sigma_0 + \sigma'$, where $\sigma' \in L_I^{\sigma_0}$ is a degree-1 MC-element in the shifted DGLA $L_I^{\sigma_0}$. Passing to the complexification, observe for $n \in \mathbb{N}$ that

$$P^n(V_\mathbb{C}) \otimes_{\sigma_0} (\mathfrak{k}/\mathfrak{t})_\mathbb{C} \cong \bigoplus_\alpha P^n(V_\mathbb{C}) \otimes_{\sigma_0} (\mathfrak{k}_\mathbb{C})_\alpha$$

as $\mathfrak{p}$-modules. The eigenvalues of $-\mathcal{L}_{\boldsymbol{v}_1} + \mathrm{ad}_{\sigma_0}$ acting on $P^n(V) \otimes_{\sigma_0} (\mathfrak{k}_{\mathbb{C}})_\alpha$ are given by $\alpha(\sigma_0) - \langle \boldsymbol{n}, \boldsymbol{\mu} \rangle$, as $\boldsymbol{n}$ ranges over the multi-indices $\boldsymbol{n} \in \mathbb{N}^d_{\geq 0}$ with $|\boldsymbol{n}| = n$, and α over the roots of $\mathfrak{k}$. Thus $-\mathcal{L}_{\boldsymbol{v}_1} + \mathrm{ad}_{\sigma_0}$ is invertible on $P^n(V) \otimes_{\sigma_0} (\mathfrak{k}_{\mathbb{C}})_\alpha$. Consequently

$$H^0(\mathfrak{p}, P^n(V_{\mathbb{C}}) \otimes_{\sigma_0} (\mathfrak{k}_{\mathbb{C}})_\alpha) = H^1(\mathfrak{p}, P^n(V_{\mathbb{C}}) \otimes_{\sigma_0} (\mathfrak{k}_{\mathbb{C}})_\alpha) = 0$$

for any $n \in \mathbb{N}$ and root α, which in turn implies that

$$H^0(\mathfrak{p}, P^n(V) \otimes_{\sigma_0} \mathfrak{k}/\mathfrak{t}) = H^1(\mathfrak{p}, P^n(V) \otimes_{\sigma_0} \mathfrak{k}/\mathfrak{t}) = 0.$$

By the long exact sequence of cohomology groups associated to the short exact sequence

$$0 \to \left(\bigwedge\nolimits^\bullet \mathfrak{p}^*\right) \otimes P^n(V) \otimes \mathfrak{t} \to \left(\bigwedge\nolimits^\bullet \mathfrak{p}^*\right) \otimes P^n(V) \otimes_{\sigma_0} \mathfrak{k} \to \left(\bigwedge\nolimits^\bullet \mathfrak{p}^*\right) \otimes P^n(V) \otimes_{\sigma_0} \mathfrak{k}/\mathfrak{t} \to 0,$$

it follows that for every $n \in \mathbb{N}$, the inclusion $\mathfrak{p}^* \otimes P^n(V) \otimes \mathfrak{t} \hookrightarrow \mathfrak{p}^* \otimes P^n(V) \otimes_{\sigma_0} \mathfrak{k}$ induces an isomorphism $H^1(\mathfrak{p}, P^n(V) \otimes \mathfrak{t}) \cong H^1(\mathfrak{p}, P^n(V) \otimes_{\sigma_0} \mathfrak{k})$. It then follows using Lemma 4.3.17 by induction on $n \in \mathbb{N}$ that we can find elements $\xi_k \in P^k(V) \otimes_{\sigma_0} \mathfrak{k}$ s.t. $j^n(\xi_n.\xi_{n-1} \cdots \xi_1.\sigma') \in I_n \otimes \mathfrak{t}$ for every $n \in \mathbb{N}$, the gauge action taking place in $L_I^{\sigma_0}$. By Lemma 4.3.18 there exists $\eta \in I \otimes_{\sigma_0} \mathfrak{k}$ s.t. $j^n(\eta.\sigma') = j^n(\xi_n.\xi_{n-1}.\cdots.\xi_1.\sigma') \in I_n \otimes \mathfrak{t}$ for every $n \in \mathbb{N}$. Hence $\zeta := \eta.\sigma' \in I \otimes \mathfrak{t}$ and σ' is gauge-equivalent to ζ in $L_I^{\sigma_0}$. By Lemma 4.3.15 it follows that $\sigma = \sigma_0 + \sigma'$ is gauge-equivalent to $\sigma_0 + \zeta \in R \otimes \mathfrak{t}$.

2. As before, decompose $\sigma = \sigma_0 + \sigma'$ using Lemma 4.3.15, where $\sigma' \in L_I^{\sigma_0}$ is a degree-1 MC-element in the shifted DGLA $L_I^{\sigma_0}$. Let $n \in \mathbb{N}$. Identify $\mathfrak{p}^* \otimes P^n(V) \otimes_{\sigma_0} \mathfrak{k}$ with $P^n(V) \otimes_{\sigma_0} \mathfrak{k}$ by evaluating elements of $\mathfrak{p}^*$ at $1 \in \mathfrak{p} = \mathbb{R}$. This induces an isomorphism

$$H^1(\mathfrak{p}, P^n(V) \otimes_{\sigma_0} \mathfrak{k}) \cong (P^n(V) \otimes_{\sigma_0} \mathfrak{k}) / \operatorname{Im}(-\mathcal{L}_{\boldsymbol{v}_1} + \mathrm{ad}_{\sigma_0})$$

Since $\boldsymbol{v}_1$ is semisimple, so is $-\mathcal{L}_{\boldsymbol{v}_1} + \mathrm{ad}_{\sigma_0}$ as operator on $P^n(V) \otimes_{\sigma_0} \mathfrak{k}$. Consequently, the inclusion $(P^n(V) \otimes_{\sigma_0} \mathfrak{k})^{\mathfrak{p}} \hookrightarrow P^n(V) \otimes_{\sigma_0} \mathfrak{k}$ induces an isomorphism

$$(P^n(V) \otimes_{\sigma_0} \mathfrak{k})^{\mathfrak{p}} \cong (P^n(V) \otimes_{\sigma_0} \mathfrak{k}) / \operatorname{Im}(-\mathcal{L}_{\boldsymbol{v}_1} + \mathrm{ad}_{\sigma_0}).$$

So every element of $H^1(\mathfrak{p}, P^n(V) \otimes_{\sigma_0} \mathfrak{k})$ admits a representative in $\mathfrak{p}^* \otimes (P^n(V) \otimes_{\sigma_0} \mathfrak{k})^{\mathfrak{p}}$, for any $n \in \mathbb{N}$. By a similar argument as in the previous item, it follows that σ' is gauge-equivalent to some $\zeta \in I \otimes_{\sigma_0} \mathfrak{k}$ in $L_I^{\sigma_0}$ that satisfies $-\mathcal{L}_{\boldsymbol{v}_1}(\zeta) + [\sigma_0, \zeta] = 0$. By Lemma 4.3.15 it follows that $\sigma_0 + \sigma' \sim \sigma_0 + \zeta$ in L_R. Notice that $\nu := \sigma_0 + \zeta$ satisfies $-\mathcal{L}_{\boldsymbol{v}_1}\nu + [\sigma_0, \nu] = 0$.

3. Observe first that any derivation $D' \in \mathrm{der}(\mathfrak{g})$ satisfying $D'(I \otimes \mathfrak{k}) \subseteq (I \otimes \mathfrak{k})$ integrates to a unique 1-parameter group $t \mapsto e^{tD'}$ of automorphisms on $\mathfrak{g}$ that leave the ideal $I \otimes \mathfrak{k} \subseteq \mathfrak{g}$ invariant. Indeed, this follows from fact that the corresponding statement is true for the finite-dimensional Lie algebra $\mathrm{der}(\mathfrak{g}_n)$ for every $n \in \mathbb{N}$, where we use that $\mathfrak{g}$ is the projective limit $\mathfrak{g} = \varprojlim_n \mathfrak{g}_n$. In particular this applies to the $\mathbb{T}$-action e^{tD} on $\mathfrak{g}$, which therefore leaves $I \otimes \mathfrak{k}$

invariant. It thus induces a continuous $\mathbb{T}$-action on $V^* \otimes \mathfrak{k} \cong (I \otimes \mathfrak{k})/(I^2 \otimes \mathfrak{k})$, which integrates the linear operator $-\mathcal{L}_{\boldsymbol{v}_1} \otimes 1 + 1 \otimes \mathrm{ad}_{\sigma_0}$ on $V^* \otimes \mathfrak{k}$. This implies that $\boldsymbol{v}_1 \in \mathfrak{gl}(V)$ and $\mathrm{ad}_{\sigma_0} \in \mathrm{der}(\mathfrak{k})$ integrate to continuous $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$-actions on V and $\mathfrak{k}$, respectively. As $\mathbb{T}$ is compact it follows in particular that $\boldsymbol{v}_1$ is semisimple and that $\mathrm{Spec}(\boldsymbol{v}_1) \cup \mathrm{Spec}(\mathrm{ad}_{\sigma_0}) \subseteq 2\pi i\mathbb{Z}$. By Lemma 4.3.15 we know that there is some degree-1 MC-element $\sigma' \in L_I^{\sigma_0}$ such that $\sigma = \sigma_0 + \sigma'$. By the previous item, it follows that we may assume that σ' satisfies $-\mathcal{L}_{\boldsymbol{v}_1}\sigma' + [\sigma_0, \sigma'] = 0$, by acting with gauge transformations in $L_I^{\sigma_0}$ if necessary. Let $n \in \mathbb{N}$. The $\mathbb{T}$-action on $\mathfrak{g}_n = R_n \otimes \mathfrak{k}$ must be unitarizable because $\mathbb{T}$ is compact, so that its generator $D_n := -\mathcal{L}_{\boldsymbol{v}_1} + [j^n\sigma, -] \in \mathrm{der}(\mathfrak{g}_n)$ must be semisimple. Notice further that $-\mathcal{L}_{\boldsymbol{v}_1} + [\sigma_0, -]$ is semisimple on $\mathfrak{g}_n$ whereas $[j^n\sigma', -]$ is nilpotent. Since $-\mathcal{L}_{\boldsymbol{v}_1}\sigma' + [\sigma_0, \sigma'] = 0$, the operators $-\mathcal{L}_{\boldsymbol{v}_1} + [\sigma_0,$-$-]$ and $[j^n\sigma', -]$ on $\mathfrak{g}_n$ commute. Thus $D_n = \big(-\mathcal{L}_{\boldsymbol{v}_1} + [\sigma_0, -]\big) + [j^n\sigma', -]$ is the Jordan decomposition of D_n. As D_n is semisimple, this implies that $[j^n\sigma', -] = 0$. Thus $j^n\sigma' \in \mathfrak{z}(\mathfrak{g}_n)$, where $\mathfrak{z}(\mathfrak{g}_n)$ denotes the center of $\mathfrak{g}_n$. As $\mathfrak{k}$ is simple, we know that $\mathfrak{z}(\mathfrak{g}_n) = P^n(V) \otimes \mathfrak{k} \subseteq \mathfrak{g}_n$. Thus $\sigma' \in I^{n-1} \otimes \mathfrak{k}$ for every $n \in \mathbb{N}$, where $I^0 := R$. As $\bigcap_{n\in\mathbb{N}}(I^{n-1} \otimes \mathfrak{k}) = \{0\}$, it follows that $\sigma' = 0$. Hence $\sigma = \sigma_0 \in \mathfrak{t}$. □

4.4 Projective unitary generalized positive energy representations

Having obtained the normal form results Theorem 4.3.12 and Theorem 4.3.13, we now proceed with the study of continuous projective unitary representation of jet Lie groups and algebras that are of generalized positive energy.

Let us begin by briefly recalling the setting and our notation. We have that V is a finite-dimensional real vector space, $R = \mathbb{R}[\![V^*]\!] := \prod_{n=0}^\infty P^k(V)$ is the ring of formal power series on V with coefficients in $\mathbb{R}$ and equipped with the direct product topology. Moreover, $\mathfrak{g}$ denotes the topological Lie algebra $\mathfrak{g} = R \otimes \mathfrak{k}$, where $\mathfrak{k}$ is a compact simple Lie algebra and $\mathfrak{p}$ is a finite-dimensional real Lie algebra acting on $\mathfrak{g}$ by the homomorphism $D : \mathfrak{p} \to \mathrm{der}(\mathfrak{g})$, which splits into a horizontal and a vertical part according to $D(p) = -\mathcal{L}_{\boldsymbol{v}(p)} + \mathrm{ad}_{\sigma(p)}$, where $\boldsymbol{v} : \mathfrak{p} \to \mathcal{X}_I^{op}$ is a homomorphism and where the linear map $\sigma : \mathfrak{p} \to \mathfrak{g}$ satisfies the Maurer Cartan equation (4.2.1). Let P and K be the 1-connected Lie groups integrating $\mathfrak{p}$ and K, respectively. For $n \in \mathbb{N}$ write $G_n := J_0^n(V, K)$, $G_n^\sharp := J_0^n(V, K) \rtimes P$ and $\mathfrak{g}_n^\sharp := \mathfrak{g}_n \rtimes_D \mathfrak{p}$. Define further $G := J_0^\infty(V, K) := \varprojlim_n G_n$, $G^\sharp := G \rtimes_\alpha P$ and $\mathfrak{g}^\sharp := \mathfrak{g} \rtimes_D \mathfrak{p} = \varprojlim_n \mathfrak{g}_n^\sharp$.

In the following, we are interested in understanding the extent to which the linearization $\boldsymbol{v}_1$ of $\boldsymbol{v}$ and the values of $\sigma(p)$ at the origin already determine properties of the class of representations which are of g.p.e. at a given cone $\mathfrak{C} \subseteq \mathfrak{p}$. To describe the main results, we first have to introduce some more notation.

Define $\sigma_0 := \mathrm{ev}_0 \circ \sigma : \mathfrak{p} \to \mathfrak{k}$ and let $\boldsymbol{v}_1 = j^1\boldsymbol{v} : \mathfrak{p} \to \mathfrak{gl}(V)$ be the linearization of $\boldsymbol{v}$ at the origin. For $p \in \mathfrak{p}$, the vector fields $\boldsymbol{v}(p)$ splits as $\boldsymbol{v}(p) = \boldsymbol{v}_1(p) + \boldsymbol{v}_{\mathrm{ho}}(p)$

for some formal vector field $\boldsymbol{v}_{\mathrm{ho}}(p) \in \mathcal{X}_{I^2}$ vanishing up to first order at the origin. Let $\boldsymbol{v}_{\mathrm{l}}(p) = \boldsymbol{v}_{\mathrm{l}}(p)_{\mathrm{s}} + \boldsymbol{v}_{\mathrm{l}}(p)_{\mathrm{n}}$ be the Jordan decomposition of $\boldsymbol{v}_{\mathrm{l}}(p)$ over $\mathbb{C}$. Let $V_c^{\mathbb{C}}(p)$ denote the span in $V_{\mathbb{C}}$ of all generalized eigenspaces of $\boldsymbol{v}_{\mathrm{l}}(p)$ corresponding to eigenvalues with zero real part. Set $V_c(p) := V_c^{\mathbb{C}}(p) \cap V$. If $\mathfrak{C} \subseteq \mathfrak{p}$ is a subset, define $V_c(\mathfrak{C}) := \bigcap_{p\in\mathfrak{C}} V_c(p)$. We call $V_c(\mathfrak{C})$ the 'center subspace associated to $\mathfrak{C}$', in analogy with the center manifold of a fixed point of a dynamical system. Let $V_c(\mathfrak{C})^\perp \subseteq V^*$ denote the annihilator of $V_c(\mathfrak{C})$ in V^*. For any $p \in \mathfrak{p}$, let $\Sigma_p \subseteq \mathbb{C}$ denote the additive subsemigroup of $\mathbb{C}$ generated by $\mathrm{Spec}(\boldsymbol{v}_{\mathrm{l}}(p))$. For any continuous projective unitary representation $\overline{\pi}$ of $\mathfrak{g} \rtimes_D \mathfrak{p}$, we define

$$\mathfrak{C}(\overline{\pi}) := \{\, p \in \mathfrak{p} \,:\, \overline{\pi} \text{ is of generalized positive energy at } p \,\}.$$

Let us describe the main results of this section. In the context of positive energy representations, we have:

Theorem 4.4.1. *Let $\overline{\rho}$ be a smooth projective unitary representation of $G \rtimes_\alpha P$ which is of p.e. at $p \in \mathfrak{p}$. Assume that $\mathrm{Spec}(ad_{\sigma_0(p)}) \cap \mathrm{Spec}(\boldsymbol{v}_{\mathrm{l}}(p)) = \emptyset$. Then $\overline{\rho}$ factors through $J_0^2(V,K) \rtimes_\alpha P$. Moreover the image of $-\mathcal{L}_{\boldsymbol{v}_{\mathrm{l}}(p)} + ad_{\sigma_0(p)}$ in $P^2(V) \otimes \mathfrak{k} \subseteq J_0^2(V,K)$ is contained in $\ker \overline{\rho}$.*

Remark 4.4.2. Notice that $\mathrm{Spec}(\mathrm{ad}_{\sigma_0(p)}) = \{\, \alpha(\sigma_0(p)) \,:\, \alpha \in \Delta \,\} \cup \{0\}$ is a finite subset of $i\mathbb{R}$. In particular, the condition $\mathrm{Spec}(\boldsymbol{v}_{\mathrm{l}}(p)) \cap \mathrm{Spec}(\mathrm{ad}_{\sigma_0(p)}) = \emptyset$ is satisfied if $\boldsymbol{v}_{\mathrm{l}}$ has no purely imaginary eigenvalues.

This is complemented by the following results, which in particular give sufficient conditions for $\overline{\pi}|_{\mathfrak{g}}$ to factor through $\mathfrak{k}$, as $\mathbb{R}[\![V_c(\mathfrak{C})^*]\!] \otimes \mathfrak{k} \cong \mathfrak{k}$ whenever $V_c(\mathfrak{C}) = \{0\}$.

Theorem 4.4.3. *Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$. Assume that $\mathrm{Spec}(ad_{\sigma_0(p)}) \cap \Sigma_p = \emptyset$ for all $p \in \mathfrak{C}$.*
Then $RV_c(\mathfrak{C})^\perp \otimes \mathfrak{k} \subseteq \ker \overline{\pi}$ and hence $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c(\mathfrak{C})^*]\!] \otimes \mathfrak{k}$.

Theorem 4.4.4. *Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a maximal Abelian subalgebra. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g}^\sharp$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$ and assume that $\sigma(p) \in R \otimes \mathfrak{t}$ and $[\, \boldsymbol{v}_{\mathrm{l}}(p)_{\mathrm{s}}, \boldsymbol{v}_{\mathrm{ho}}(p) \,] = 0$ for every $p \in \mathfrak{C}$. Then $RV_c(\mathfrak{C})^\perp \otimes \mathfrak{k} \subseteq \ker \overline{\pi}$ and hence $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c(\mathfrak{C})^*]\!] \otimes \mathfrak{k}$.*

To prove these results, we consider in Section 4.4.1 the second continuous Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathfrak{g} \rtimes_D \mathfrak{p}, \mathbb{R})$ so as to obtain particular representatives ω of cohomology classes therein. We proceed in Section 4.4.2 to show that any irreducible smooth projective unitary representation $G \rtimes_\alpha P$ factors through the finite-dimensional $G_n \rtimes P$ for some $n \in \mathbb{N}$. This gives us access to techniques that are available for finite-dimensional Lie groups, and in particular to Corollary 2.5.4, which leads to Theorem 4.4.1. In Section 4.4.3 and Section 4.4.4 we study the kernel of the quadratic form $\xi \mapsto \omega(D(p)\xi, \xi)$. Recalling from Corollary 3.1.8 that

$$[D(p)\eta, \eta] = 0 \implies \Big(\omega(D(p)\eta, \eta) = 0 \iff \overline{\pi}(D(p)\eta) = 0\Big), \qquad \forall \eta \in \mathfrak{g},$$

this leads to an ideal in $\mathfrak{g}$ contained in $\ker \overline{\pi}$, and to the proof of Theorem 4.4.3 and Theorem 4.4.4. These results are supplemented in Section 4.4.5 by a consideration of the special case where $\mathfrak{p}$ is a simple non-compact Lie algebra, in which case Theorem 4.3.12 is available. This leads to the following:

Theorem 4.4.5. *Assume that $\mathfrak{p}$ is non-compact and simple. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Then $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c(\mathfrak{C}(\overline{\pi}))^*]\!] \otimes \mathfrak{k}$.*

Theorem 4.4.6. *Assume that $\mathfrak{p}$ is non-compact and simple. Suppose that $\boldsymbol{v}_1$ defines a non-trivial irreducible $\mathfrak{p}$-representation on V. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$ be a P-invariant convex cone. Either $\mathfrak{C}$ is pointed or $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathfrak{k}$.*

4.4.1 The second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathfrak{g} \rtimes_D \mathfrak{p}, \mathbb{R})$.

We next determine suitable representatives of classes in the second continuous Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathfrak{g} \rtimes_D \mathfrak{p}, \mathbb{R})$, which classifies the continuous $\mathbb{R}$-central extensions of $\mathfrak{g} \rtimes \mathfrak{p} =: \mathfrak{g}^\sharp$ up to equivalence. As an intermediate step we first consider $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$, which is completely understood.

Define $\Omega^k_R := R \otimes \bigwedge^k V^*$, equipped with projective limit topology obtained from $\Omega^k_R = \varprojlim_n \Omega^k_{R_n}$, where $\Omega^k_{R_n} := R_n \otimes \bigwedge^k V^*$. This makes Ω^k_R into a Fréchet space. In particular $\Omega^0_R = R$. Since $J^\infty_0(\Omega^k(V)) \cong \Omega^k_R$, we can define a continuous differential $d : \Omega^k_R \to \Omega^{k+1}_R$ by $dj^\infty_0 \alpha := j^\infty_0 d\alpha \in J^\infty_0(\Omega^{n+1}(V)) \cong \Omega^{k+1}_R$, which is indeed well-defined. Choosing a basis $(e_\mu)_{\mu=1}^d$ of V with dual basis $(dx_\mu)_{\mu=1}^d$ of V^*, the above differential d is on R given by $df = \sum_\mu (\partial_\mu f) \otimes dx_\mu$ for $f \in R$.

Lemma 4.4.7. *Let E be a topological R-module and let $D : R \to E$ be a continuous derivation. Then there exists a unique continuous R-linear map $\overline{D} : \Omega^1_R \to E$ such that $D = \overline{D} \circ d$.*

Proof. Let $\mathbb{R}[V^*]$ denote the ring of polynomial functions on V. As $\mathbb{R}[V^*] \subseteq R$, E is also a $\mathbb{R}[V^*]$-module and $D|_{\mathbb{R}[V^*]} : \mathbb{R}[V^*] \to E$ is a derivation. Using the universal property of the Kähler differential forms $\Omega^1_{\mathbb{R}[V^*]} \cong \mathbb{R}[V^*] \otimes V^*$, there is a unique $\mathbb{R}[V^*]$-linear map $\overline{D} : \Omega^1_{\mathbb{R}[V^*]} \to E$ such that $\overline{D} \circ d|_{\mathbb{R}[V^*]} = D|_{\mathbb{R}[V^*]}$. As $\Omega^1_{\mathbb{R}[V^*]}$ is dense in Ω^1_R, it remains to extend $\overline{D}$ continuously to the Fréchet space Ω^1_R. Let $\alpha \in \Omega^1_R$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\Omega^1_{\mathbb{R}[V^*]}$ s.t. $\alpha_n \to \alpha$ in Ω^1_R. We show that $D\alpha := \lim_{n\to\infty} \overline{D}\alpha_n$ exists and is independent of the approximating sequence (α_n). Choose a basis $(dx_\mu)_{\mu=1}^d$ of V^*. Write $\alpha_n = \sum_{\mu=1}^d f^{(n)}_\mu dx_\mu$ and $\alpha = \sum_{\mu=1}^d f_\mu dx_\mu$ for some unique $f^{(n)}_\mu \in \mathbb{R}[V^*]$ and $f_\mu \in R$. Then $f^{(n)}_\mu \to f_\mu$ in R for every μ and hence $\lim_{n\to\infty} \overline{D}\alpha_n = \sum_{\mu=1}^d \lim_{n\to\infty} f^{(n)}_\mu Dx_\mu = \sum_{\mu=1}^d f_\mu Dx_\mu$, which is independent of the approximating sequence (α_n). It follows that $\overline{D}$ extends to a continuous R-linear map $D : \Omega^1_R \to E$, which satisfies $\overline{D} \circ d = D$ by construction. It is unique with these properties because its restriction to the dense subspace $\mathbb{R}[V^*] \subseteq R$ is so. ☐

Remark 4.4.8. Lemma 4.4.7 entails that $d : R \to \Omega^1_R$ is the universal differential module R in the category of complete locally convex $\mathbb{R}$-modules, in the sense of [Mai02, Thm. 6]

Proposition 4.4.9. *Any class $[\omega] \in H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ has a unique representative of the form $\omega(\xi, \eta) = \lambda(\kappa(\xi, d\eta))$, where $\lambda \in {\Omega^1_R}^*$ is closed and continuous functional on Ω^1_R and κ is the Killing form on $\mathfrak{k}$. (Closed meaning that $\lambda(dR) = 0$.) Conversely, any such λ defines a 2-cocycle representing some non-zero class in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$. Consequently, the center of the universal central extension of $\mathfrak{g}$ is (Ω^1_R/dR).*

Proof. This is a special case of [Mai02, Theorem 16], seeing as $\mathfrak{g} = R \otimes \mathfrak{k}$, where R is a unital, associative and commutative Fréchet algebra and $\mathfrak{k}$ is a simple Lie algebra. □

Lemma 4.4.10. *Let ω be an extension of the 2-cocycle $\lambda(\kappa(\xi, d\eta))$ on $\mathfrak{g}$ to a 2-cocycle on $\mathfrak{g}^\sharp = \mathfrak{g} \rtimes \mathfrak{p}$. Then λ is $\mathfrak{p}$-invariant in the sense that $\lambda(\mathcal{L}_{\boldsymbol{v}(p)}\Omega^1_R) = 0$ for every $p \in \mathfrak{p}$.*

Proof. Take $\xi = f \otimes X$ and $\eta = g \otimes X$ for $f, g \in R$ and $0 \neq X \in \mathfrak{k}$. Notice that $[\xi, \eta] = 0$ in $R \otimes \mathfrak{k}$. Using the cocycle identity, this implies

$$\omega(D(p)\xi, \eta) + \omega(\xi, D(p)\eta) = \omega(D(p), [\xi, \eta]) = 0$$
$$\omega([\sigma(p), \xi], \eta) + \omega(\xi, [\sigma(p), \eta]) = \omega(\sigma(p), [\xi, \eta]) = 0.$$

Using $D(p)\xi = -\mathcal{L}_{\boldsymbol{v}(p)}\xi + [\sigma(p), \xi]$ it follows that

$$0 = \omega(\mathcal{L}_{\boldsymbol{v}(p)}\xi, \eta) + \omega(\xi, \mathcal{L}_{\boldsymbol{v}(p)}\eta) = \lambda(\mathcal{L}_{\boldsymbol{v}(p)} f dg)\kappa(X, X).$$

As $\kappa(X, X) \neq 0$ and $RdR = \Omega^1_R$, this shows the claim. □

Lemma 4.4.11. *Let $\omega : \mathfrak{g}^\sharp \times \mathfrak{g}^\sharp \to \mathbb{R}$ be a continuous 2-cocycle on $\mathfrak{g}^\sharp = \mathfrak{g} \rtimes_D \mathfrak{p}$. Then there exists $n \in \mathbb{N}$ and a 2-cocycle ω_n on $\mathfrak{g}^\sharp_n$ such that $\omega(\xi, \eta) = \omega_n(j^n\xi, j^n\eta)$ for all $\xi, \eta \in \mathfrak{g}^\sharp$.*

Proof. Let $\omega : \mathfrak{g}^\sharp \times \mathfrak{g}^\sharp \to \mathbb{R}$ be a continuous 2-cocycle on $\mathfrak{g}^\sharp$. Choose norms $\| - \|_n$ on the finite-dimensional Lie algebras $\mathfrak{g}^\sharp_n$ s.t. the quotient maps $j^n : \mathfrak{g}^\sharp_m \to \mathfrak{g}^\sharp_n$ are contractive for any $n, m \in \mathbb{N}$ with $n \leq m$. The topology on $\mathfrak{g}^\sharp = \varprojlim \mathfrak{g}^\sharp_n$ is specified by the seminorms $\xi \mapsto \|j^n\xi\|_n$ for $n \in \mathbb{N}$ and $\xi \in \mathfrak{g}^\sharp$. As ω is continuous and the maps $\mathfrak{g}^\sharp_m \to \mathfrak{g}^\sharp_n$ are contractive for $n \leq m$, there exist $n \in \mathbb{N}$ such that $|\omega(\xi, \eta)| \leq \|j^n\xi\|_n \|j^n\eta\|_n$ for all $\xi, \eta \in \mathfrak{g}^\sharp$ (using e.g. [Tre67, Prop. 43.1 and Prop. 43.4]). As $j^n : \mathfrak{g}^\sharp \to \mathfrak{g}^\sharp_n$ is surjective, it follows that $\omega(\xi, \eta) = \omega_n(j^n\xi, j^n\eta)$ for a unique 2-cocycle ω_n on $\mathfrak{g}^\sharp_n$. □

4.4.2 Factorization through finite jets

In the context of smooth projective unitary representations $\overline{\rho}$ of the Lie group $G^\sharp$, it is no loss of generality to consider the case where $\overline{\rho}$ factors through the finite-dimensional Lie group $G^\sharp_n$ for some $n \in \mathbb{N}$:

Theorem 4.4.12. *Let $\overline{\rho}$ be a smooth projective unitary representation of $G^\sharp$ with lift $\rho : \mathring{G} \to \mathrm{U}(\mathcal{H}_\rho)$ for some central $\mathbb{T}$-extension $\mathring{G}$ of $G^\sharp$. Then ρ decomposes as a (possibly uncountable) direct sum $\rho = \bigoplus_{i\in\mathcal{I}} \rho_i$ s.t. for every $i \in \mathcal{I}$ there exists $n \in \mathbb{N}$ s.t. the projective unitary representations $\overline{\rho}_i$ associated to ρ_i factors through $G^\sharp_n$. In particular, if $\overline{\rho}$ is irreducible then it factors through $G^\sharp_n$ for some $n \in \mathbb{N}$.*

Proof. Define

$$N^\sharp_m := \ker\left(j^m : G^\sharp \to G^\sharp_m\right) \quad \text{and}$$
$$\mathfrak{n}^\sharp_m := \ker\left(j^m : \mathfrak{g}^\sharp \to \mathfrak{g}^\sharp_m\right)$$

for any $m \in \mathbb{N}_{\geq 0}$, so that $G^\sharp_m \cong G^\sharp/N^\sharp_m$. Notice that $N^\sharp_m \subseteq N^\sharp_n$ whenever $n \leq m$. Since $\overline{\rho}$ is a smooth projective representation, it follows from [JN19, Thm. 4.3] that $\mathring{G}$ is a Lie group. It is moreover regular by [Nee06, Thm. V.I.8], because both $G^\sharp$ and $\mathbb{T}$ are so. Let $\mathring{\mathfrak{g}} := \mathrm{Lie}(\mathring{G})$. Then $\mathring{\mathfrak{g}}$ is a central $\mathbb{R}$-extension of $\mathfrak{g}^\sharp$ in the category of locally convex Lie algebras. Let the continuous 2-cocycle $\omega : \mathfrak{g}^\sharp \times \mathfrak{g}^\sharp \to \mathbb{R}$ represent the corresponding class in $H^2_{\mathrm{ct}}(\mathfrak{g}^\sharp, \mathbb{R})$. By Lemma 4.4.11, there is some $n \in \mathbb{N}$ such that for all $\xi \in \mathfrak{g}^\sharp$ we have $j^n\xi = 0 \implies \omega(\xi, \eta) = 0$ for all $\eta \in \mathfrak{g}^\sharp$. Let $\mathring{N}_n$ be the closed normal subgroup of $\mathring{G}$ covering $N^\sharp_n$ and let $\mathring{\mathfrak{n}}_n$ be its Lie algebra. Then $\mathring{N}_n$ is a central $\mathbb{T}$-extension of $N^\sharp_n$ integrating $\mathring{\mathfrak{n}}_n$. Since $\omega|_{\mathfrak{n}^\sharp_n \times \mathfrak{n}^\sharp_n} = 0$, the central $\mathbb{R}$-extension $\mathring{\mathfrak{n}}_n$ is trivial. Hence $\mathring{\mathfrak{n}}_n \cong \mathbb{R} \oplus \mathfrak{n}^\sharp_n$ as central $\mathbb{R}$-extensions of $\mathfrak{n}^\sharp_n$. As $N^\sharp_n$ is regular and 1-connected, it follows from [Nee06, Thm. III.1.5] that there is a commutative diagram

$$\begin{array}{ccccc}
\mathbb{R} & \longrightarrow & \mathbb{R} \times N^\sharp_n & \longrightarrow & N^\sharp_n \\
\big\downarrow{\scriptstyle e^{2\pi i\,\cdot}} & & \big\downarrow{\scriptstyle \widetilde{\phi}} & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\mathbb{T} & \longrightarrow & \mathring{N}_n & \longrightarrow & N^\sharp_n
\end{array}$$

of locally convex regular Lie groups. Observe that $\widetilde{\phi}$ is surjective and that $\ker\widetilde{\phi} = \mathbb{Z}$. Thus $\mathring{N}_n \cong \mathbb{T} \times N^\sharp_n$ as central $\mathbb{T}$-extension of $N^\sharp_n$. Let $\phi : \mathbb{T} \times N^\sharp_n \to \mathring{N}$ realize the isomorphism. For any integer $m \geq n$, let $\mathcal{N}_m := \phi(\{1\} \times N^\sharp_m) \subseteq \mathring{N}_n \subseteq \mathring{G}$, which is a closed normal subgroup of $\mathring{G}$ isomorphic to and covering $N^\sharp_m$. Then $\mathcal{N} := \{\mathcal{N}_m\}_{m\geq n}$ is a filter basis of (decreasing) closed normal subgroups of $\mathring{G}$ satisfying $\varinjlim \mathcal{N} = \{1\}$, in the sense that for any 1-neighborhood U of $\mathring{G}$ there exists $m \geq n$ such that $\mathcal{N}_m \subseteq U$. Indeed, since $G^\sharp = \varprojlim_m G^\sharp_m$ carries the projective limit topology and $\mathring{G}$ is a locally trivial principal $\mathbb{T}$-bundle over $G^\sharp$ [JN19, Thm. 4.3], it follows that any 1-neighborhood $U \subseteq \mathring{G}$ contains $\phi(I \times N^\sharp_m)$ for large enough m and some open 1-neighborhood $I \subseteq \mathbb{T}$. It now follows from [Nee10a, Thm.12.2] that ρ decomposes as a possibly uncountable direct sum $\rho \cong \bigoplus_{i\in\mathcal{I}} \rho_i$ such that for every $i \in \mathcal{I}$ there exists some $m \geq n$ with $\rho_i(\mathcal{N}_m) = \{1\}$, which implies that $\overline{\rho}_i(N^\sharp_m) = \{1\}$. □

Theorem 4.4.12 gives us access to techniques that are available for finite-dimensional Lie groups, and in particular to Corollary 2.5.4. This can be used to prove Theorem 4.4.1.

Theorem 4.4.1. *Let $\overline{\rho}$ be a smooth projective unitary representation of $G \rtimes_\alpha P$ which is of p.e. at $p \in \mathfrak{p}$. Assume that* $\mathrm{Spec}(ad_{\sigma_0(p)}) \cap \mathrm{Spec}(\boldsymbol{v}_1(p)) = \emptyset$*. Then $\overline{\rho}$ factors through $J_0^2(V,K) \rtimes_\alpha P$. Moreover the image of $-\mathcal{L}_{\boldsymbol{v}_1(p)} + ad_{\sigma_0(p)}$ in $P^2(V) \otimes \mathfrak{k} \subseteq J_0^2(V,K)$ is contained in* $\ker \overline{\rho}$.

To prove Theorem 4.4.1, it suffices by Theorem 4.4.12 to consider the case where $\overline{\rho}$ factors through the finite-dimensional Lie group $G_k^\sharp$ for some $k \in \mathbb{N}$, which we thus assume. Write $\mathfrak{a}$ for the ideal in $\mathfrak{g}_k^\sharp$ generated by $p \in \mathfrak{p}$. Let $\mathfrak{a}_n \subseteq \mathfrak{a}$ denote the maximal nilpotent ideal in $\mathfrak{a}$. According to Corollary 2.5.4 we have $[\mathfrak{a}, [\mathfrak{a}_n, \mathfrak{a}_n]] \subseteq \ker d\overline{\rho}$. Recall that $I_k := I/I^{k+1}$.

Lemma 4.4.13. *Suppose that $V^* \otimes \mathfrak{k} \subseteq j^1\mathfrak{a}_n$. Then $I_k \otimes \mathfrak{k} \subseteq \mathfrak{a}_n$.*

Proof. By assumption $V^* \otimes \mathfrak{k} \subseteq \mathfrak{a}_n + I_k^2 \otimes \mathfrak{k}$. As $\mathfrak{k}$ is perfect it follows that

$$I_k^{l+1} \otimes \mathfrak{k} = [V^* \otimes \mathfrak{k}, I_k^l \otimes \mathfrak{k}] \subseteq \mathfrak{a}_n + [I_k^2 \otimes \mathfrak{k}, I_k^l \otimes \mathfrak{k}] = \mathfrak{a}_n + I_k^{l+2} \otimes \mathfrak{k}, \qquad \forall l \in \mathbb{N}.$$

Thus it follows by induction that $V^* \otimes \mathfrak{k} \subseteq \mathfrak{a}_n + I_k^{l+1} \otimes \mathfrak{k}$ for all $l \in \mathbb{N}$. As $\bigcap_l (\mathfrak{a}_n + I_k^{l+1} \otimes \mathfrak{k}) = \mathfrak{a}_n$, it follows that $V^* \otimes \mathfrak{k} \subseteq \mathfrak{a}_n$ and hence $I_k \otimes \mathfrak{k} \subseteq \mathfrak{a}_n$. ☐

Proof of Theorem 4.4.1. We may assume that $\overline{\rho}$ factors through $G_k^\sharp$ for some $k \in \mathbb{N}$. It suffices to show that $d\overline{\rho}$ factors through $\mathfrak{g}_2^\sharp$ and that the image of $-\mathcal{L}_{\boldsymbol{v}_1(p)} + \mathrm{ad}_{\sigma_0(p)}$ in $P^2(V) \otimes \mathfrak{k} \subseteq \mathfrak{g}_2$ is contained in $\ker d\overline{\rho}$. By Corollary 2.5.4 we know that $\big[\mathfrak{a}, [\mathfrak{a}_n, \mathfrak{a}_n]\big] \subseteq \ker d\overline{\rho}$. Moreover $I_k^3 \otimes \mathfrak{k} = [I_k \otimes \mathfrak{k}, [I_k \otimes \mathfrak{k}, I_k \otimes \mathfrak{k}]]$, because $\mathfrak{k}$ is perfect. To see that $d\overline{\rho}$ factors through $\mathfrak{g}_2^\sharp$ it thus suffices to show that $I_k \otimes \mathfrak{k} \subseteq \mathfrak{a}_n$. By Lemma 4.4.13 it is further sufficient to show that $V^* \otimes \mathfrak{k} \subseteq j^1(\mathfrak{a}_n)$. Write $D_1(p) := -\mathcal{L}_{\boldsymbol{v}_1(p)} + [\sigma_0(p), -]$. Notice that $j^1(D(p)\xi) = D_1(p)\xi$ for $\xi \in V^* \otimes \mathfrak{k}$. The assumption $\mathrm{Spec}(\boldsymbol{v}_1(p)) \cap \mathrm{Spec}(\mathrm{ad}_{\sigma_0(p)}) = \emptyset$ implies that $D_1(p)$ is invertible on $V^* \otimes \mathfrak{k}$. Thus if $\eta \in V^* \otimes \mathfrak{k}$ is arbitrary, there exists $\xi \in V^* \otimes \mathfrak{k}$ such that $\eta = D_1(p)\xi$. Then

$$\eta = D_1(p)\xi = j^1(D(p)\xi) = j^1([p,\xi]) \in j^1(\mathfrak{a}_n).$$

Thus $V^* \otimes \mathfrak{k} \subseteq j^1(\mathfrak{a}_n)$. We obtain that $I_k \otimes \mathfrak{k} \subseteq \mathfrak{a}_n$ and $I_k^3 \otimes \mathfrak{k} \subseteq \ker d\overline{\rho}$, so $d\overline{\rho}$ factors through $\mathfrak{g}_2^\sharp$. We may thus assume that $k = 2$. We then obtain

$$D_1(p)(P^2(V) \otimes \mathfrak{k}) = D(p)(I_k^2 \otimes \mathfrak{k}) = [p, [I_k \otimes \mathfrak{k}, I_k \otimes \mathfrak{k}]] \subseteq [\mathfrak{a}, [\mathfrak{a}_n, \mathfrak{a}_n] \subseteq \ker d\overline{\rho}.$$ ☐

4.4.3 The case where $\mathfrak{p} = \mathbb{R}$

We proceed with the study of projective unitary representations $\overline{\pi}$ of $\mathfrak{g}^\sharp = \mathfrak{g} \rtimes_D \mathfrak{p}$ which are of generalized positive energy. We first specialize to the case where $\mathfrak{p} = \mathbb{R}$, aiming to consider its consequences for the general case afterwards.

As $\mathfrak{p} = \mathbb{R}$, we may as well identify $\boldsymbol{v}$ with $\boldsymbol{v}(1) \in \mathcal{X}_I$, D with $D(1)$ and σ with $\sigma(1) \in \mathfrak{g}$. Recall that the derivation D is given by $D = -\mathcal{L}_{\boldsymbol{v}} + [\sigma, -]$. Write

$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{l}} + \boldsymbol{v}_{\mathrm{ho}}$, where $\boldsymbol{v}_{\mathrm{l}} := j^1\boldsymbol{v} \in \mathfrak{gl}(V)$ is the linearization of $\boldsymbol{v}$ at $0 \in V$ and where $\boldsymbol{v}_{\mathrm{ho}} \in \mathcal{X}_{I^2}$. Let $\boldsymbol{v}_{\mathrm{l}} = \boldsymbol{v}_{\mathrm{l,s}} + \boldsymbol{v}_{\mathrm{l,n}}$ denote the Jordan decomposition of $\boldsymbol{v}_{\mathrm{l}}$ over $\mathbb{C}$. We write $V_c^{\mathbb{C}}$ for the span of the eigenspaces of $\boldsymbol{v}_{\mathrm{l}}$ whose corresponding eigenvalue has zero real part. Set $V_c := V_c^{\mathbb{C}} \cap V$. Write $\boldsymbol{d} := (0,1) \in \mathfrak{g}^\sharp = \mathfrak{g} \rtimes_D \mathbb{R}$. Let $V_c^\perp \subseteq V^*$ denote the annihilator of V_c in V^*, so $V_c^\perp \cong (V/V_c)^*$.

Theorem 4.4.14. *Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a maximal Abelian subalgebra. Assume that $\sigma \in R \otimes \mathfrak{t} \subseteq \mathfrak{g}$ and $[\boldsymbol{v}_{\mathrm{l,s}}, \boldsymbol{v}_{\mathrm{ho}}] = 0$ in $\mathcal{X}_I$. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g}^\sharp$ on the pre-Hilbert space $\mathcal{D}$. Assume that $\overline{\pi}$ is of g.p.e. at $\boldsymbol{d} \in \mathfrak{g} \rtimes_D \mathbb{R}\boldsymbol{d}$. Then $RV_c^\perp \subseteq \ker\overline{\pi}$. Consequently, $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c^*]\!] \otimes \mathfrak{k}$. In particular, if $V_c = \{0\}$ then $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathfrak{k}$.*

Remark 4.4.15. By acting with formal diffeomorphisms if necessary, one may by Theorem 4.3.7 always bring $\boldsymbol{v}$ into a normal form, in the sense that $[\boldsymbol{v}_{\mathrm{l,s}}, \boldsymbol{v}_{\mathrm{ho}}] = 0$ in $\mathcal{X}_I$. Moreover, Theorem 4.3.13 provides sufficient conditions guaranteeing that σ is gauge equivalent to some element in $R \otimes \mathfrak{t}$.

Proof of Theorem 4.4.14

Let ω be a continuous 2-cocycle on $\mathfrak{g}^\sharp$ that represents the class in $H^2_{\mathrm{ct}}(\mathfrak{g}^\sharp, \mathbb{R})$ corresponding to the central $\mathbb{R}$-extension of $\mathfrak{g}^\sharp$ obtained from $\overline{\pi}$ by pulling back $\mathfrak{u}(\mathcal{D}) \to \mathfrak{pu}(\mathcal{D})$ along $\overline{\pi}$. In view of Proposition 4.4.9 and Lemma 4.4.10, we may and do assume that ω satisfies $\omega(\xi,\eta) = \lambda(\kappa(\xi, d\eta))$ for any $\xi,\eta \in \mathfrak{g}$, where $\lambda : \Omega_R \to \mathbb{R}$ is continuous, $\mathfrak{p}$-invariant and closed. We write fX instead of $f \otimes X$ for $f \in R_{\mathbb{C}}$ and $X \in \mathfrak{k}_{\mathbb{C}}$. Let $\Delta \subseteq i\mathfrak{t}^*$ denote the set of roots of $\mathfrak{k}$. Finally, write $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{k}_{\mathbb{C}}$. Recall from Corollary 3.1.8 that

$$[D\eta, \eta] = 0 \implies \Big(\omega(D\eta,\eta) = 0 \iff \overline{\pi}(D\eta) = 0\Big), \qquad \forall \eta \in \mathfrak{g}. \tag{4.4.1}$$

Moreover, $\omega(D\eta,\eta) \geq 0$ whenever $[D\eta,\eta] = 0$. In the present setting, this yields:

Proposition 4.4.16. *Fix $f \in R$. Then $\overline{\pi}(R\mathcal{L}_{\boldsymbol{v}}f \otimes \mathfrak{k}) = \{0\} \iff \lambda(fd\mathcal{L}_{\boldsymbol{v}}f) = 0$.*

Proof. For any $H \in \mathfrak{t}$, observe that $DfH = -\mathcal{L}_{\boldsymbol{v}}fH$ because $\sigma \in R \otimes \mathfrak{t}$, so $[DfH, fH] = -[\mathcal{L}_{\boldsymbol{v}}fH, fH] = 0$. Using (4.4.1) we obtain that

$$\kappa(H,H)\lambda(\mathcal{L}_{\boldsymbol{v}}(f)df) = 0 \iff \overline{\pi}(\mathcal{L}_{\boldsymbol{v}}fH) = 0, \qquad \forall H \in \mathfrak{t}. \tag{4.4.2}$$

Assume that $\overline{\pi}(R\mathcal{L}_{\boldsymbol{v}}f \otimes \mathfrak{k}) = \{0\}$. Then $\overline{\pi}(\mathcal{L}_{\boldsymbol{v}}fH) = 0$ for any $H \in \mathfrak{t}$, so $\lambda(\mathcal{L}_{\boldsymbol{v}}(f)df) = 0$ by (4.4.2). Conversely, suppose that $\lambda(\mathcal{L}_{\boldsymbol{v}}(f)df) = 0$. Then $\overline{\pi}(\mathcal{L}_{\boldsymbol{v}}fH) = 0$ for all $H \in \mathfrak{t}$, by (4.4.2). Taking the commutator with $\overline{\pi}(gX_\alpha)$, where $g \in R$, $\alpha \in \Delta$ is a root and $X_\alpha \in (\mathfrak{k}_{\mathbb{C}})_\alpha$ is a corresponding root vector, it follows that

$$\overline{\pi}(g\mathcal{L}_{\boldsymbol{v}}fX_\alpha) = 0 \qquad \forall X_\alpha \in (\mathfrak{k}_{\mathbb{C}})_\alpha,\ g \in R. \tag{4.4.3}$$

Take $X_\alpha \in (\mathfrak{k}_{\mathbb{C}})_\alpha$ and $Y_{-\alpha} \in (\mathfrak{k}_{\mathbb{C}})_{-\alpha}$. Write $H_\alpha = [X_\alpha, Y_{-\alpha}]$. By taking commutators with $\overline{\pi}(1 \otimes Y_{-\alpha})$ in equation (4.4.3) we find that $\overline{\pi}(g\mathcal{L}_{\boldsymbol{v}}fH_\alpha) = 0$. As $\mathfrak{h} = \sum_\alpha [(\mathfrak{k}_{\mathbb{C}})_\alpha, (\mathfrak{k}_{\mathbb{C}})_{-\alpha}]$, this shows by linearity together with equation (4.4.3) and the root space decomposition that $\overline{\pi}(R\mathcal{L}_{\boldsymbol{v}}f \otimes \mathfrak{k}) = \{0\}$. ☐

Define the quadratic form $q(f) := \lambda(\mathcal{L}_{\boldsymbol{v}}(f)df) = -\lambda(fd\mathcal{L}_{\boldsymbol{v}}f)$ on R. Let $\mathcal{N} := \ker q$ denote its kernel. By Proposition 4.4.16, $\mathcal{N}$ generates an ideal $J \otimes \mathfrak{k}$ on which $\overline{\pi}$ vanishes, where $J := R\mathcal{L}_{\boldsymbol{v}}\mathcal{N}$.

Corollary 4.4.17. *Set $J := R\mathcal{L}_{\boldsymbol{v}}\mathcal{N}$. Then $J \otimes \mathfrak{k} \subseteq \ker(\overline{\pi})$.*

Together with the fact that λ vanishes on exact forms and is $\mathcal{L}_{\boldsymbol{v}}$-invariant, this puts severe restrictions on the representation $\overline{\pi}$ and leads to Theorem 4.4.14. Let us also remark the following:

Lemma 4.4.18. *The bilinear form $\beta_q(f,g) := \lambda(\mathcal{L}_{\boldsymbol{v}}(f)dg)$ on R associated to q is symmetric, the quadratic form q is positive semi-definite and*

$$\mathcal{N} = \{\, f \in R \,:\, \beta_q(f,g) = 0 \;\; \forall g \in R \,\}.$$

Proof. As λ is closed and $\mathcal{L}_{\boldsymbol{v}}$-invariant, it follows that β is symmetric. To see that q is positive semi-definite, let $f \in R$ and $0 \neq H \in \mathfrak{t}$. Write $\eta := fH$ and notice that $[D\eta, \eta] = 0$. By Corollary 3.1.8 we have $-\kappa(H,H)\lambda(\mathcal{L}_{\boldsymbol{v}}(f)df) = \omega(D\eta, \eta) \geq 0$. As κ is negative definite on $\mathfrak{k}$ we obtain that q is positive semi-definite. It follows that $|\beta_q(f,g)|^2 \leq q(f)q(g)$, which implies $\mathcal{N} = \{\, f \in R \,:\, \beta_q(f,g) = 0 \quad \forall g \in R \,\}$. □

The following observation is also noteworthy, although it will not be used:

Lemma 4.4.19. *$\mathcal{N} \subseteq R$ is a subalgebra.*

Proof. Let $f, g \in \mathcal{N}$. Then using the Leibniz rule and Proposition 4.4.16 we obtain

$$\overline{\pi}(R\mathcal{L}_{\boldsymbol{v}}(fg) \otimes \mathfrak{k}) \subseteq \overline{\pi}(fR\mathcal{L}_{\boldsymbol{v}}g \otimes \mathfrak{k}) + \overline{\pi}(gR\mathcal{L}_{\boldsymbol{v}}f \otimes \mathfrak{k}) \subseteq \{0\},$$

Applying Proposition 4.4.16 once more, we conclude that $fg \in \mathcal{N}$. □

Lemma 4.4.20. $\lambda \circ \mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}} = 0$.

Proof. As $\lambda : \Omega^1_R \to \mathbb{R}$ is continuous, it factors through the finite-dimensional space $\Omega^1_{R_k} = R_k \otimes V^*$ for some $k \in \mathbb{N}$. Notice that both $\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,n}}}$ and $\mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}$ are nilpotent on $\Omega^1_{R_k} \otimes_{\mathbb{R}} \mathbb{C}$, whereas $\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}}$ is semisimple on it. Also $[\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}}, \mathcal{L}_{\boldsymbol{v}_{\mathrm{l,n}}} + \mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}] = 0$ because $[\,\boldsymbol{v}_{\mathrm{l,s}}, \boldsymbol{v}_{\mathrm{ho}}\,] = [\boldsymbol{v}_{\mathrm{l,s}}, \boldsymbol{v}_{\mathrm{l,n}}] = 0$. Thus $\mathcal{L}_{\boldsymbol{v}} = \mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}} + \big(\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,n}}} + \mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}\big)$ is the Jordan decomposition of $\mathcal{L}_{\boldsymbol{v}}$ acting on $\Omega^1_{R_k} \otimes_{\mathbb{R}} \mathbb{C}$. Thus $\mathrm{Im}(\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}}) \subseteq \mathrm{Im}(\mathcal{L}_{\boldsymbol{v}})$ when $\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}}$ and $\mathcal{L}_{\boldsymbol{v}}$ are considered as operators on $\Omega^1_{R_k} \otimes_{\mathbb{R}} \mathbb{C}$. As λ is $\mathcal{L}_{\boldsymbol{v}}$-invariant, we know $\lambda \circ \mathcal{L}_{\boldsymbol{v}} = 0$. Thus $\lambda \circ \mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}} = 0$. □

In particular, λ vanishes on the eigenspaces in $\Omega^1_{R_{\mathbb{C}}}$ of $\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}}$ corresponding to non-zero eigenvalues. We introduce some more notation. Let $E_{\mathbb{C}}$ denote the span of all eigenspaces in $R_{\mathbb{C}}$ of $\mathcal{L}_{\boldsymbol{v}_{\mathrm{l,s}}}$ corresponding to eigenvalues with non-zero real part. Define $E := E_{\mathbb{C}} \cap R$ and $E^n := E \cap I^n$.

Lemma 4.4.21. $E \subseteq \mathcal{N}$.

Proof. Let $\mu \in \operatorname{Spec}(\mathcal{L}_{\boldsymbol{v}_{l,s}})$ with $\operatorname{Re}(\mu) \neq 0$. Set $E_\mu := \ker(\mathcal{L}_{\boldsymbol{v}_{l,s}} - \mu I) \subseteq R_{\mathbb{C}}$. Suppose first that $\mu \in \mathbb{R}$. If $\psi \in E_\mu \cap R$ then because $\mathcal{L}_{\boldsymbol{v}}$ leaves the eigenspaces of $\mathcal{L}_{\boldsymbol{v}_{l,s}}$ invariant, the 1-form $\psi d\mathcal{L}_{\boldsymbol{v}}\psi$ is an eigenvector of $\mathcal{L}_{\boldsymbol{v}_{l,s}}$ with non-zero eigenvalue 2μ. By Lemma 4.4.20 it follows that $q(\psi) = 0$ and hence $\psi \in \mathcal{N}$. Thus $E_\mu \subseteq \mathcal{N}$. Next, suppose that μ is not real. Then also $\overline{\mu}$ is an eigenvalue of $\mathcal{L}_{\boldsymbol{v}_{l,s}}$. Write $W_{\mathbb{C}} := E_\mu \oplus E_{\overline{\mu}}$ and $W := W_{\mathbb{C}} \cap R$. Take $\psi \in W$ arbitrary. Then $\psi = \eta + \overline{\eta}$ for some $\eta \in E_\mu$ (and hence $\overline{\eta} \in E_{\overline{\mu}}$). As $\mu + \overline{\mu} = 2\operatorname{Re}(\mu) \neq 0$ and $\mathcal{L}_{\boldsymbol{v}}$ leaves the eigenspaces of $\mathcal{L}_{\boldsymbol{v}_{l,s}}$ invariant, each of the 1-forms $\eta d\mathcal{L}_{\boldsymbol{v}}\eta$, $\eta d\mathcal{L}_{\boldsymbol{v}}\overline{\eta}$, $\overline{\eta} d\mathcal{L}_{\boldsymbol{v}}\eta$ and $\overline{\eta} d\mathcal{L}_{\boldsymbol{v}}\overline{\eta}$ are eigenvectors of $\mathcal{L}_{\boldsymbol{v}_{l,s}}$ with non-zero eigenvalue. Using Lemma 4.4.20 it follows that $q(\psi) = 0$ and hence $\psi \in \mathcal{N}$. Thus $W \subseteq \mathcal{N}$. As $\mathcal{N}$ is a linear subspace, we have shown $E \subseteq \mathcal{N}$. □

Lemma 4.4.22. *$RE \subseteq R\mathcal{L}_{\boldsymbol{v}}E$.*

Proof. Write $J := R\mathcal{L}_{\boldsymbol{v}}E$. As J is an ideal in R it suffices to show $E \subseteq J$. We claim that $E^n \subseteq J + E^{n+1}$ for every $n \in \mathbb{N}_{\geq 0}$. Indeed, take $\psi \in E^n$. As $\mathcal{L}_{\boldsymbol{v}_l}$ is invertible on E^n (which is true because $\mathcal{L}_{\boldsymbol{v}_l}$ is invertible on every finite-dimensional and $\mathcal{L}_{\boldsymbol{v}_l}$-invariant subspace $E \cap P^k(V) \subseteq E$), there exists some $\eta \in E^n$ s.t. $\mathcal{L}_{\boldsymbol{v}_l}\eta = \psi$. Observe that $\mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}E \subseteq E$ because $[\mathcal{L}_{\boldsymbol{v}_{l,s}}, \mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}] = 0$. Also $\mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}I^n \subseteq I^{n+1}$, since $\boldsymbol{v}_{\mathrm{ho}} \in \mathcal{X}_{I^2}$. Thus $\mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}E^n \subseteq E^{n+1}$. In particular $\mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}\eta \in E^{n+1}$. Then

$$\psi = \mathcal{L}_{\boldsymbol{v}_l}\eta = \mathcal{L}_{\boldsymbol{v}}\eta - \mathcal{L}_{\boldsymbol{v}_{\mathrm{ho}}}\eta \in J + E^{n+1},$$

as required. By induction it follows that $E = E^0 \subseteq J + E^n$ for every $n \in \mathbb{N}$. As $\bigcap_{n\in\mathbb{N}}(J + E^n) = J$, this implies $E \subseteq J$. □

Proof of Theorem 4.4.14:
Using Lemma 4.4.21 we obtain $E \subseteq \mathcal{N}$. By Corollary 4.4.17, this implies $J \otimes \mathfrak{k} \subseteq \ker\overline{\pi}$, where $J = R\mathcal{L}_{\boldsymbol{v}}E$. By Lemma 4.4.22, we know $RE \subseteq J$. Notice that $E \cap V^* = V_c^\perp$, so in particular $RV_c^\perp \subseteq J$. Thus $RV_c^\perp \otimes \mathfrak{k} \subseteq \ker\overline{\pi}$. Notice that $R/(RV_c^\perp) \cong \mathbb{R}[\![V_c^*]\!]$, because $V_c^* = V^*/V_c^\perp$. We conclude that $\overline{\pi}$ factors through the quotient $(R \otimes \mathfrak{k})/(RV_c^\perp \otimes \mathfrak{k}) \cong (\mathbb{R}[\![V_c^*]\!] \otimes \mathfrak{k})$. □

4.4.4 The case of general $\mathfrak{p}$

Let us return to the case where P is a 1-connected finite-dimensional Lie group with Lie algebra $\mathfrak{p}$. Let us recall some of the notation introduced earlier in Section 4.4.

Define $\sigma_0 := \mathrm{ev}_0 \circ \sigma : \mathfrak{p} \to \mathfrak{k}$ and let $\boldsymbol{v}_l = j^1\boldsymbol{v} : \mathfrak{p} \to \mathfrak{gl}(V)$ be the linearization of $\boldsymbol{v}$ at $0 \in V$. For $p \in \mathfrak{p}$, the vector fields $\boldsymbol{v}(p)$ splits as $\boldsymbol{v}(p) = \boldsymbol{v}_l(p) + \boldsymbol{v}_{\mathrm{ho}}(p)$ for some formal vector field $\boldsymbol{v}_{\mathrm{ho}}(p) \in \mathcal{X}_{I^2}$ vanishing up to first order at the origin. Let $\boldsymbol{v}_l(p) = \boldsymbol{v}_l(p)_{\mathrm{s}} + \boldsymbol{v}_l(p)_{\mathrm{n}}$ be the Jordan decomposition of $\boldsymbol{v}_l(p)$ over $\mathbb{C}$. Let $V_c^{\mathbb{C}}(p)$ denote the span in $V_{\mathbb{C}}$ of all generalized eigenspaces of $\boldsymbol{v}_l(p)$ corresponding to eigenvalues with zero real part. Set $V_c(p) := V_c^{\mathbb{C}}(p) \cap V$. If $\mathfrak{C} \subseteq \mathfrak{p}$ is a subset, define $V_c(\mathfrak{C}) := \bigcap_{p\in\mathfrak{C}} V_c(p)$. Let $V_c(\mathfrak{C})^\perp \subseteq V^*$ denote the annihilator of $V_c(\mathfrak{C})$ in V^*. Let $\Sigma_p \subseteq \mathbb{C}$ denote the additive subsemigroup of $\mathbb{C}$ generated by $\operatorname{Spec}(\boldsymbol{v}_l(p))$. For any continuous projective unitary representation $\overline{\pi}$ of $\mathfrak{g} \rtimes_D \mathfrak{p}$, let $\mathfrak{C}(\overline{\pi})$ denote the set of all points $p \in \mathfrak{p}$ for which $\overline{\pi}$ is of generalized positive energy at p.

We use Theorem 4.4.14 combined with suitable normal form results to prove Theorem 4.4.3 and Theorem 4.4.4.

Lemma 4.4.23. *Let W be a finite-dimensional real vector space and let $W_i \subseteq W$ be a collection of linear subspaces, where $i \in \mathcal{I}$ for some indexing set $\mathcal{I}$. Then $\left(\bigcap_{i\in\mathcal{I}} W_i\right)^\perp = Span_{i\in\mathcal{I}} W_i^\perp$.*

Proof. Notice first that $\bigcap_{i\in\mathcal{I}} W_i^\perp = \left[\mathrm{Span}_{i\in\mathcal{I}} W_i\right]^\perp$. Applying this observation to the subspaces $W_i^\perp \subseteq W^*$, we obtain that $\bigcap_{i\in\mathcal{I}} W_i = \bigcap_{i\in\mathcal{I}} (W_i^\perp)^\perp = \left[\mathrm{Span}_{i\in\mathcal{I}} W_i^\perp\right]^\perp$, where we also used that $(W_i^\perp)^\perp \cong W_i$ for any $i \in \mathcal{I}$. Taking annihilators, this implies $\left(\bigcap_{i\in\mathcal{I}} W_i\right)^\perp = \mathrm{Span}_{i\in\mathcal{I}} W_i^\perp$. □

Recall that $\mathfrak{C}(\overline{\pi})$ denotes the set of all points $p \in \mathfrak{p}$ s.t. $\overline{\pi}$ is of generalized positive energy at p.

Theorem 4.4.3. *Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$. Assume that $\mathrm{Spec}(ad_{\sigma_0(p)}) \cap \Sigma_p = \emptyset$ for all $p \in \mathfrak{C}$.
Then $RV_c(\mathfrak{C})^\perp \otimes \mathfrak{k} \subseteq \ker \overline{\pi}$ and hence $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c(\mathfrak{C})^*]\!] \otimes \mathfrak{k}$.*

Proof. Let $p \in \mathfrak{C}$. By Theorem 4.3.7, there is a formal vector field $\boldsymbol{w}_{\mathrm{ho}}(p) \in \mathcal{X}_{I^2}$ satisfying $[\boldsymbol{v}_{\mathrm{l}}(p)_{\mathrm{s}}, \boldsymbol{w}_{\mathrm{ho}}(p)] = 0$ s.t. $\boldsymbol{v}(p)$ is formally equivalent to $\boldsymbol{w}(p) := \boldsymbol{v}_{\mathrm{l}}(p) + \boldsymbol{w}_{\mathrm{ho}}(p)$. If $h \in \mathrm{Aut}(R) \subseteq \mathrm{Aut}(\mathfrak{g})$ is a formal diffeomorphism s.t. $\boldsymbol{w}(p) = h.\boldsymbol{v}(p)$, then h leaves the constant part $\sigma_0(p)$ of $\sigma(p)$ fixed, so $\mathrm{ev}_0(h.\sigma(p)) = \mathrm{ev}_0\, \sigma(p) = \sigma_0(p)$. Thus, we may assume that $[\boldsymbol{v}_{\mathrm{l}}(p)_{\mathrm{s}}, \boldsymbol{v}_{\mathrm{ho}}(p)] = 0$ and $\mathrm{Spec}(\mathrm{ad}_{\sigma_0(p)}) \cap \Sigma_p = \emptyset$. By acting with gauge transformations, we may by Theorem 4.3.13 further assume that $\sigma \in R \otimes \mathfrak{t}$, where $\mathfrak{t}$ is a maximal torus containing $\sigma_0(p)$. By Theorem 4.4.14, it follows that $RV_c(p)^\perp \subseteq \ker \overline{\pi}$. The above holds for all $p \in \mathfrak{C}$, so $\mathrm{Span}_{p\in\mathfrak{C}} RV_c(p)^\perp \subseteq \ker \overline{\pi}$. By Lemma 4.4.23 we know $\mathrm{Span}_{p\in\mathfrak{C}}\left(V_c(p)^\perp\right) = V_c(\mathfrak{C})^\perp$, so that $R/(\mathrm{Span}_{p\in\mathfrak{C}} RV_c(p)^\perp) \cong \mathbb{R}[\![V_c(\mathfrak{C})^*]\!]$. □

Theorem 4.4.4. *Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a maximal Abelian subalgebra. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g}^\sharp$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$ and assume that $\sigma(p) \in R \otimes \mathfrak{t}$ and $[\,\boldsymbol{v}_{\mathrm{l}}(p)_{\mathrm{s}}, \boldsymbol{v}_{\mathrm{ho}}(p)\,] = 0$ for every $p \in \mathfrak{C}$. Then $RV_c(\mathfrak{C})^\perp \otimes \mathfrak{k} \subseteq \ker \overline{\pi}$ and hence $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c(\mathfrak{C})^*]\!] \otimes \mathfrak{k}$.*

Proof. By Theorem 4.4.14 it follows that $\mathrm{Span}_{p\in\mathfrak{C}} RV_c(p)^\perp = RV_c(\mathfrak{C})^\perp \subseteq \ker \overline{\pi}$. □

4.4.5 The case where $\mathfrak{p}$ is simple

Let us consider the special case where $\mathfrak{p}$ is simple. Let P be a 1-connected Lie group with $\mathrm{Lie}(P) = \mathfrak{p}$. In this case, suitable normal form theorems for $\boldsymbol{v}$ and σ are available (see Theorem 4.3.10 and Theorem 4.3.12). We consequently know that $\boldsymbol{v} : \mathfrak{p} \to \mathcal{X}_I^{op}$ is always formally equivalent to its linearization $\boldsymbol{v}_{\mathrm{l}}$ at $0 \in V$. Similarly the vertical twist $\sigma : \mathfrak{p} \to \mathfrak{g}$ is gauge-equivalent to some Lie algebra homomorphism $\sigma_0 : \mathfrak{p} \to \mathfrak{k}$, by Theorem 4.3.12. In particular, if $\mathfrak{p}$ is not compact then we may and do assume that $\sigma = 0$ by acting with gauge transformations if necessary, for in that case there are no homomorphisms $\mathfrak{p} \to \mathfrak{k}$ (because $\mathfrak{k}$ is compact, see Lemma 4.3.20). Combined with Theorem 4.4.4 we immediately obtain Theorem 4.4.5 below, where $V_c(\mathfrak{C}) := \bigcap_{p\in\mathfrak{C}} V_c(p)$ for a subset $\mathfrak{C} \subseteq \mathfrak{p}$. Recall also that $\mathfrak{C}(\overline{\pi})$ denotes the set of all points $p \in \mathfrak{p}$ for which $\overline{\pi}$ is of generalized positive energy at p.

Theorem 4.4.5. *Assume that $\mathfrak{p}$ is non-compact and simple. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Then $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathbb{R}[\![V_c(\mathfrak{C}(\overline{\pi}))^*]\!] \otimes \mathfrak{k}$.*

Let $\mathfrak{p} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{p}$, so that $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are the $+1$ and -1 eigenspaces of a Cartan-involution θ, respectively [Kna96, cor. 6.18]. Let $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ be a maximal Abelian subalgebra of $\mathfrak{p}_0$. According to the Iwasawa decomposition [Kna96, Prop. 6.43], $\mathfrak{p}$ decomposes as $\mathfrak{p} \cong \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$, where $\mathfrak{n}_0 \subseteq \mathfrak{p}$ is nilpotent. For $p \in \mathfrak{p}$ we write $p = p_e + p_h + p_n$ for the corresponding decomposition of p, where $p_e \in \mathfrak{k}_0, p_h \in \mathfrak{a}_0$ and $p_n \in \mathfrak{n}_0$. Then $\mathrm{Spec}(\mathrm{ad}_{p_e}) \subseteq i\mathbb{R}$, $\mathrm{Spec}(\mathrm{ad}_{p_h}) \subseteq \mathbb{R}$ and ad_{p_n} is nilpotent [Kna96, Lem. 6.45]. Moreover, $\mathfrak{a}_0$ is contained in a Cartan subalgebra of $\mathfrak{p}_0$ [Kna96, cor. 6.47].

Proposition 4.4.24. *Suppose that $\mathfrak{p}$ is simple and that the $\mathfrak{p}$-representation $\boldsymbol{v}_{\mathrm{l}}$ on V is non-trivial and irreducible. Let $\mathfrak{C} \subseteq \mathfrak{p}$ be an Ad_P-invariant convex cone and let $V_c(\mathfrak{C}) := \bigcap_{p\in\mathfrak{C}} V_c(p)$. Assume that $\mathfrak{C}$ contains some non-zero $p_h \in \mathfrak{a}_0$. Then $V_c(\mathfrak{C}) = \{0\}$.*

Proof. Notice first that as P is 1-connected, the $\mathfrak{p}$-action $\boldsymbol{v}_{\mathrm{l}} : \mathfrak{p} \to \mathfrak{gl}(V)$ integrates to a continuous representation of P on V. As $\mathfrak{C}$ is Ad_P-invariant, the subspace $V_c(\mathfrak{C})$ is P-invariant. Thus either $V_c(\mathfrak{C}) = \{0\}$ or $V_c(\mathfrak{C}) = V$, so it suffices to show $V_c(\mathfrak{C}) \neq V$. By assumption $p_h \neq 0$. In view of Cartan's unitary trick, see e.g. [Kna01, V. Prop. 5.3], the image of elements in $\mathfrak{a}_0$ in any finite-dimensional representation are semisimple and have real spectrum. Thus $\mathrm{Spec}(\boldsymbol{v}_{\mathrm{l}}(p_h)) \subseteq \mathbb{R}$. As $\mathfrak{p}$ is simple and $\boldsymbol{v}_{\mathrm{l}}$ is a non-trivial $\mathfrak{p}$-representation by assumption, it follows that $\boldsymbol{v}_{\mathrm{l}}$ is injective. As $\boldsymbol{v}_{\mathrm{l}}(p_h) \in \mathfrak{gl}(V)$ is semisimple, there exists $0 \neq v \in V$ s.t. $\boldsymbol{v}_{\mathrm{l}}(p_h)v = \mu v$ for some $0 \neq \mu \in \mathbb{R}$. Thus $0 \neq v \notin V_c(\mathfrak{C})$. Hence $V_c(\mathfrak{C}) \neq V$ and so $V_c(\mathfrak{C}) = \{0\}$. □

Theorem 4.4.6. *Assume that $\mathfrak{p}$ is non-compact and simple. Suppose that $\boldsymbol{v}_{\mathrm{l}}$ defines a non-trivial irreducible $\mathfrak{p}$-representation on V. Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{g} \rtimes_D \mathfrak{p}$. Let $\mathfrak{C} \subseteq \mathfrak{C}(\overline{\pi})$ be a P-invariant convex cone. Either $\mathfrak{C}$ is pointed or $\overline{\pi}|_{\mathfrak{g}}$ factors through $\mathfrak{k}$.*

Remark 4.4.25. Notice that if $\mathfrak{p}$ is simple and $\mathcal{C}$ is a closed Ad_P-invariant convex cone which is not pointed, then $\mathcal{C} \cap -\mathcal{C} = \mathfrak{p}$ and hence $\mathcal{C} = \mathfrak{p}$.

Proof of Theorem 4.4.6: The edge $\mathfrak{e} := \overline{\mathfrak{C}} \cap -\overline{\mathfrak{C}}$ of the closure $\overline{\mathfrak{C}}$ of $\mathfrak{C}$ is an ideal in $\mathfrak{p}$. Assume that $\mathfrak{C}$ is not pointed. Then neither is $\overline{\mathfrak{C}}$. As $\mathfrak{p}$ is simple, it follows that $\mathfrak{e} = \mathfrak{p}$ and hence $\overline{\mathfrak{C}} = \mathfrak{p}$. Thus $\mathfrak{C}$ is a dense convex cone in the finite-dimensional real vector space $\mathfrak{p}$, which implies that $\mathfrak{C} = \mathfrak{p}$. As $\mathfrak{p}$ is non-compact, it contains some hyperbolic element. Thus, so does $\mathfrak{C}$. By Proposition 4.4.24 it follows that $V_c(\mathfrak{C}) = \{0\}$ and hence Theorem 4.4.5 implies that $\overline{\pi}$ factors through $\mathfrak{k}$. □

Thus if $\mathcal{C}$ is an Ad_P-invariant convex cone which is not pointed, then $\mathfrak{g}$ admits no continuous projective unitary representations which are of g.p.e. at $\mathcal{C} \subseteq \mathfrak{p}$ other than those which factor through $\mathfrak{k}$. On the other hand, we know by [Pan81, cor. 2.3] that if $\mathfrak{p}$ is simple, then a non-trivial pointed closed and P-invariant convex cone exists in $\mathfrak{p}$ if and only if $\mathfrak{p}$ is of *hermitian type*, meaning that $\dim(\mathfrak{z}(\mathfrak{k}_0)) = 1$, where $\mathfrak{p} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{p}$ and where $\mathfrak{k}_0$ is the Lie algebra of a compact Lie group.

Let us shift our attention to positive energy representations, in which case a different argument is available.

Lemma 4.4.26. *Suppose that P is a non-compact simple connected Lie group. If $(\sigma, \mathcal{H}_\sigma)$ is a unitary P-representation that is norm-continuous, then σ is trivial.*

Proof. As $\mathfrak{p}$ is simple, $d\sigma$ is either injective or trivial. Assume that $d\sigma$ is not trivial. Let $\mathfrak{p} = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ be the Iwasawa decomposition of $\mathfrak{p}$. Take $x \in \mathfrak{a}_0$. Then ad_x is semisimple and $\mathrm{Spec}(\mathrm{ad}_x) \subseteq \mathbb{R}$. As σ is unitary and $d\sigma$ is injective, $z \mapsto \|d\sigma(z)\|_{\mathcal{B}(\mathcal{H})}$ defines a P-invariant norm on $\mathfrak{p}$. With respect to this norm, $e^{t\mathrm{ad}_x}$ is an isometry on $\mathfrak{p}$ for every $t \in \mathbb{R}$. As ad_x is semisimple, it follows that $\mathrm{Spec}(\mathrm{ad}_x) \subseteq i\mathbb{R}$. So $\mathrm{Spec}(\mathrm{ad}_x) \subseteq \mathbb{R} \cap i\mathbb{R} = \{0\}$ and hence $\mathrm{ad}_x = 0$. Since $\mathfrak{p}$ has trivial center it follows that $x = 0$. So $\mathfrak{a}_0 = \{0\}$ and hence $\mathfrak{p}$ is compact. But P is non-compact by assumption. So $d\sigma$ must be trivial. As P is connected, it follows that σ is trivial. □

Proposition 4.4.27. *Suppose that P is a non-compact 1-connected simple Lie group. Assume that the P-action on V is irreducible and non-trivial. Let $\overline{\rho}$ be a continuous projective unitary representation of G which is of positive energy at $\mathcal{C} := \mathfrak{p}$. Then $\overline{\rho}|_G$ factors through K.*

Proof. By Theorem 4.4.12 it suffices to consider the case where $\overline{\rho}$ factors through G_k for some $k \in \mathbb{N}$. From Whitehead's Second Lemma, [Jac79, III.9. Lem. 6], we know that $H^2(\mathfrak{p}, \mathbb{R}) = \{0\}$. Using in addition that P is simply connected, it follows that $\overline{\rho}|_P$ lifts to a continuous unitary representation $\sigma : P \to \mathrm{U}(\mathcal{H}_\rho)$ of P, so that $\overline{\rho}(p) = [\sigma(p)]$ in $\mathrm{PU}(\mathcal{H}_\rho)$ for all $p \in P$. By Lemma 2.5.5, the fact that $\overline{\rho}|_P$ is of p.e. at $\mathcal{C} = \mathfrak{p}$ implies that σ is norm-continuous. It follows from Lemma 4.4.26 that σ is trivial. Thus $\overline{\rho}(\alpha_p(g)) = \overline{\rho}(g)$ for all $g \in G$ and $p \in P$. It follows that $d\overline{\rho}$ vanishes on $D(\mathfrak{p})\mathfrak{g}$. As $\mathfrak{p}$ acts irreducibly and non-trivially on V, it follows that the ideal in $\mathfrak{g}$ generated by $D(\mathfrak{p})\mathfrak{g}$ is $I \otimes \mathfrak{k}$. Thus $I \otimes \mathfrak{k} \subseteq \ker d\overline{\rho}$. This implies that $\overline{\rho}|_{G_k}$ factors through K. □

The following provides a simple example of a projective p.e. representation $\overline{\rho}$ of $G_1 \rtimes P$ s.t. $\overline{\rho}|_{G_1}$ does not factor through K.

Example 4.4.28. Let $P = \mathrm{Mp}(2, \mathbb{R})$ be the double-cover of $\mathrm{SL}(2, \mathbb{R})$. Let P act on $V := \mathbb{R}^2$ via the defining action of $\mathrm{SL}(2, \mathbb{R})$. We consider a trivial vertical twist, so that the $\mathfrak{p}$-action on $\mathfrak{g} = R \otimes \mathfrak{k}$ is given by $D(p) = -\mathcal{L}_{\boldsymbol{v}(p)}$. In this case the generator p_0 of rotations generates the unique (up to a sign) pointed, closed and P-invariant convex cone $\mathcal{C}$ in $\mathfrak{p}$. Explicitly, $\boldsymbol{v}(p_0) = y\partial_x - x\partial_y$. Let us construct a non-trivial continuous projective unitary representation of $G_1 \rtimes P \cong (V^* \otimes \mathfrak{k}) \rtimes (K \times P)$ that is of p.e. at the cone $\mathcal{C} \subseteq \mathfrak{p}$. Write $W := V^* \otimes \mathfrak{k}$.

We begin by specifying a suitable 2-cocycle on $V^* \otimes \mathfrak{k} \subseteq \mathfrak{g}_1$. Notice that $\left(\bigwedge^2 V\right)^{\mathfrak{p}} \cong \mathbb{R}$ is one-dimensional. Let $0 \neq \lambda \in \left(\bigwedge^2 V\right)^{\mathfrak{p}}$ and consider it as a $\mathfrak{p}$-invariant bilinear map $V^* \times V^* \to \mathbb{R}$. To be consistent with Proposition 4.4.9, let us write $\lambda(fdg)$ instead of $\lambda(f, g)$ for $f, g \in V^*$. Let $x, y \in V^*$ be the usual basis of V^*. Then λ is fully specified by the number $\lambda(ydx)$. If $\lambda(ydx) > 0$, then the quadratic form $q(v) := \lambda(\mathcal{L}_{\boldsymbol{v}(p_0)} v dv)$ is positive-definite, because $q(ax + by) = (a^2 + b^2)\lambda(ydx)$ for $a, b \in \mathbb{R}$. Let ω be the unique symplectic form on W satisfying $\omega(vX, wY) := \lambda(vdw)\kappa(X, Y)$ for $X, Y \in \mathfrak{k}$ and $v, w \in V^*$. Then $\omega(D(p_0)\xi, \xi) \geq 0$ for every $\xi \in W$ (recalling that κ is *negative* definite). Let $\mathrm{H}(W, \omega)$ be the corresponding Heisenberg group. Let $L_\pm$ be the $\pm i$-eigenspaces in $W_\mathbb{C}$ of the complex structure $\mathcal{J} := D(p_0)$ on $W_\mathbb{C}$, so that $W_\mathbb{C} = L_- \oplus L_+$. The $\mathfrak{p}$-invariance of λ ensures that $\mathcal{J}^*\omega = \omega$. Indeed, extend ω $\mathbb{C}$-bilinearly to $W_\mathbb{C}$. As λ is $\mathfrak{p}$-invariant, it follows that $\omega(\mathcal{J}\xi, \eta) + \omega(\xi, \mathcal{J}\eta) = 0$ for all $\xi, \eta \in W_\mathbb{C}$, which implies that $L_\pm \subseteq W_\mathbb{C}$ are $\mathcal{J}$-invariant Lagrangian subspaces for ω. Then $\mathcal{J}^*\omega = \omega$ follows from $\mathcal{J}^*\omega(w_+, w_-) = \omega(iw_+, -iw_-) = \omega(w_+, w_-)$ for $w_\pm \in L_\pm$. Notice further that $\omega(\mathcal{J}\xi, \xi) \geq 0$ holds for all $\xi \in W$, by construction. Equip L_+ with the positive definite hermitian form defined by $\langle v, w\rangle_{L_+} := -2i\omega(\overline{v}, w)$ for $v, w \in L_+$. For each $n \in \mathbb{N}$, equip the symmetric algebra $S^n(L_+)$ with the inner product satisfying

$$\langle v_1 \cdots v_n, w_1 \cdots w_n\rangle := \sum_{\sigma \in S_n} \prod_{k=1}^{n} \langle v_{\sigma_k}, w_k\rangle_{L_+}, \qquad v_k, w_k \in L_+.$$

Let $\mathcal{F} := \overline{S^\bullet(L_+)}$ be the Hilbert space completion, where $S^\bullet(L_+) = \bigoplus_{n=0}^\infty S^n(L_+)$. The metaplectic representation ρ of $\mathrm{H}(W, \omega) \rtimes \mathrm{Mp}(W, \omega)$, with $\rho(z) = zI$ on the central $\mathbb{T}$ component, can be realized on the Fock space $\mathcal{F}$, where $\mathrm{Mp}(W, \omega)$ denotes the metaplectic group [Nee00, Thm X.3.3]. Notice that $\mathrm{SL}(2, \mathbb{R}) \hookrightarrow \mathrm{Sp}(W, \omega)$ because λ is $\mathfrak{p}$-invariant. By pulling back the metaplectic representation we obtain a continuous unitary representation of $\mathrm{H}(W, \omega) \rtimes P$ which is of p.e. at $\mathcal{C}$ and does not factor through K.

4.5 Appendix

4.5.1 From germs to jets

Let $\mathcal{K} \to M$ be locally trivial bundle of Lie groups with typical fiber a finite-dimensional Lie group G with Lie algebra $\mathfrak{g}$. Write $\mathfrak{K} \to M$ for the corresponding Lie algebra bundle. The following justifies the claim made in Section 4.1 that any continuous projective unitary representation of $\Gamma_c(\mathcal{K})$ which factors through the germs at a point $a \in M$ actually factors through the ∞-jets $J^\infty_a(\mathcal{K})$ at $a \in M$. The group $\Gamma_c(\mathcal{K})$ is a locally exponential Lie group modeled on the LF-Lie algebra $\Gamma_c(\mathfrak{K})$ [JN17, Prop. 2.3].

Let $U \subseteq \mathbb{R}^d$ be an open neighborhood of the origin. Let $C^\infty_{\text{flat}}(U)$ denote the kernel of the ∞-jet projection

$$j_0^\infty : C_c^\infty(U) \to J_0^\infty(C_c^\infty(U)) \cong \mathbb{R}[\![x_1, \dots, x_d]\!]$$

at $0 \in U$. In the following we show known fact that the closure $C_c^\infty(U \setminus \{0\})$ in $C^\infty(U)$ is $C^\infty_{\text{flat}}(U)$. As a consequence, we deduce that if a continuous projective unitary representation of the Lie algebra $\Gamma_c(\mathfrak{K})$ factors through the germs at a point $a \in M$, then it factors through the ∞-jets $J_a^\infty(\mathfrak{K})$ at $a \in M$. In turn, this implies a group level-analogue.

If $K \subset U$ is a compact set, write $C_K^\infty(U)$ for the subspace of $C^\infty(U)$ consisting of functions on U with support in K. Then $C_K^\infty(U)$ is the projective limit $C_K^\infty(U) = \varprojlim_n C_K^n(U)$ of the Banach spaces $C_K^n(U)$, which we equip with the norm $\|f\|_{C_K^n(U)} := \sup_{|k| \leq n} \|D^\alpha f\|_{C_K(U)}$, where the supremum runs over all multi-indices $k \in \mathbb{N}_{\geq 0}^d$ with $|k| \leq n$. Then $C_c^\infty(U) := \varinjlim C_K^\infty(U)$ is the corresponding locally convex inductive limit. See e.g. [Rud91, Theorem 6.5] for a description of this topology. For $r > 0$, write $B_r := \{ x \in \mathbb{R}^d : \|x\| \leq r \}$ for the closed ball centered at $0 \in \mathbb{R}^d$ with radius r.

Lemma 4.5.1. *The closure of $C_c^\infty(U \setminus \{0\})$ in $C_c^\infty(U)$ is $C^\infty_{\mathit{flat}}(U)$.*

Proof. As $C^\infty_{\text{flat}}(U) \subseteq C_c^\infty(U)$ is closed and $C_c^\infty(U \setminus \{0\}) \subseteq C^\infty_{\text{flat}}(U)$, it follows that $\overline{C_c^\infty(U \setminus \{0\})} \subseteq C^\infty_{\text{flat}}(U)$. It remains to show the reverse inclusion. Fix $f \in C^\infty_{\text{flat}}(U) \subseteq C_c^\infty(U)$. We show $f \in \overline{C_c^\infty(U \setminus \{0\})}$. Let $K_0 \subseteq M$ be a relatively compact open subset such that $\operatorname{supp} f \subseteq K_0$. Set $K := \overline{K_0}$. We may assume that $0 \in K_0$, for otherwise $f \in C_c^\infty(U \setminus \{0\})$ and we are done. By [Mal67, Lem. I.4.2], we can find constants $C_k > 0$ for $k \in \mathbb{N}_{\geq 0}^d$, depending only on k, such that for any $0 < r < 1$ with $B_{2r} \subseteq K_0$, there exists a smooth function $\psi_r \in C^\infty(\mathbb{R}^d)$ s.t. $\psi_r \geq 0$, $\psi_r|_{B_r} = 0$, $\psi_r|_{(\mathbb{R}^d \setminus B_{2r})} = 1$ and $\sup_{x \in \mathbb{R}^d} |D^k \psi_r(x)| \leq C_k r^{-|k|}$ for every $k \in \mathbb{N}_{\geq 0}^d$. In particular $f\psi_r \in C_c^\infty(U \setminus \{0\})$ and $\operatorname{supp} f\psi_r \subseteq K$. Moreover observe that $\operatorname{supp}(1 - \psi_r) \subseteq B_{2r}$ and $\|(1 - \psi_r)\|_{C^n_{B_{2r}}(\mathbb{R}^d)} \lesssim r^{-n}$ for some constant depending on $n \in \mathbb{N}_{\geq 0}$, where we used that $0 < r < 1$. On the other hand, suppose that $\alpha \in \mathbb{N}_{\geq 0}^d$ is a multi-index. Since $j_0^\infty(D^\alpha f) = 0$, it follows from Taylor's Theorem that $\|D^\alpha f\|_{C(B_{2r})} \lesssim r^l$ for arbitrary $l \in \mathbb{N}_{\geq 0}$, with a constant depending on f, α and l but not on r. Thus $\|f\|_{C^n(B_{2r})} \lesssim r^l$ for arbitrary $n, l \in \mathbb{N}_{\geq 0}$. In particular $\|f\|_{C^n(B_{2r})} \lesssim r^{n+1}$. Combining the previous observations, we obtain that

$$\begin{aligned}\|f - f\psi_r\|_{C^n(K)} &= \|f(1 - \psi_r)\|_{C^n(K)} = \|f(1 - \psi_r)\|_{C^n(B_{2r})} \\ &\lesssim \|f\|_{C^n(B_{2r})} \|(1 - \psi_r)\|_{C^n(B_{2r})} \lesssim r,\end{aligned}$$

the constants depending only on f and n but not on r. This shows that $f\psi_r \to f$ in $C_K^\infty(U)$ as $r \to 0$. Thus $f\psi_r \to f$ in $C_c^\infty(U)$. Since $\psi_r f \in C_c^\infty(U \setminus \{0\})$ for every r, we conclude that $f \in \overline{C_c^\infty(U \setminus \{0\})}$. ☐

If $a \in M$, define the spaces of smooth section of $\mathcal{K}$ and $\mathfrak{K}$ which are flat at $a \in M$:

$$\Gamma_{\mathrm{flat}(a)}(\mathcal{K}) := \ker\left(j_a^\infty : \Gamma_c(\mathcal{K}) \to J_a^\infty(\mathcal{K})\right),$$

$$\Gamma_{\mathrm{flat}(a)}(\mathfrak{K}) := \ker\left(j_a^\infty : \Gamma_c(\mathfrak{K}) \to J_a^\infty(\mathfrak{K})\right).$$

Proposition 4.5.2 below clarifies the apparent ambiguity in the topology on $J_a^\infty(\mathfrak{K})$, for which two candidates are available.

Proposition 4.5.2. *Let $a \in M$. The projective limit topology on $J_a^\infty(\mathfrak{K}) := \varprojlim_k J^k(\mathfrak{K})$ coincides with the quotient topology obtained from $J_a^\infty(\mathfrak{K}) \cong \Gamma_c(\mathfrak{K})/\Gamma_{\mathit{flat}(a)}(\mathfrak{K})$.*

Proof. The continuous k-jet projections $j_a^k : \Gamma_c(\mathfrak{K}) \to J_a^k(\mathfrak{K})$ at $a \in M$ all descend to continuous maps $\Gamma_c(\mathfrak{K})/\Gamma_{\mathrm{flat}(a)}(\mathfrak{K}) \to J_a^k(\mathfrak{K})$. By the universal property of the projective limit, they induce a continuous map $\Phi : \Gamma_c(\mathfrak{K})/\Gamma_{\mathrm{flat}(a)}(\mathfrak{K}) \to J_a^\infty(\mathfrak{K})$. Using Borel's Lemma [Hör03, Thm. 1.2.6], it is not hard to check that this map is bijective. It remains to show it is an open map, which follows immediately from the Open Mapping Theorem [Rud91, cor. 2.12] because $\Gamma_c(\mathfrak{K})/\Gamma_{\mathrm{flat}(a)}(\mathfrak{K})$ and $J_a^\infty(\mathfrak{K})$ are both Fréchet spaces and Φ is bijective and continuous. □

Proposition 4.5.3. *Let $a \in M$.*

- *The closure of $\Gamma_c(M \setminus \{a\}, \mathfrak{K})$ in $\Gamma_c(M, \mathfrak{K})$ is $\Gamma_{\mathit{flat}(a)}(\mathfrak{K})$.*
- *The closure of $\Gamma_c(M \setminus \{a\}, \mathcal{K})$ in $\Gamma_c(M, \mathcal{K})$ is $\Gamma_{\mathit{flat}(a)}(\mathcal{K})$.*

Proof. By a partition of unity argument, we may assume that the bundle $\mathfrak{K} \to M$ is trivial, that $M \subseteq \mathbb{R}^d$ is open neighborhood of $0 \in \mathbb{R}^d$ and that $a = 0$. Then $\Gamma_c(M, \mathfrak{K}) \cong C_c^\infty(M, \mathfrak{k})$. The claim now follows from Lemma 4.5.1. Notice for the second assertion that $\Gamma_{\mathrm{flat}(a)}(M, \mathcal{K})$ is a locally exponential, being an embedded closed Lie subgroup of the locally exponential Lie group $\Gamma_c(M, \mathcal{K})$. The result is then immediate from the previous point. □

Proposition 4.5.4. *Let $a \in M$.*

1. *Let $\overline{\pi} : \Gamma_c(M, \mathfrak{K}) \to \mathcal{L}^\dagger(\mathcal{D})$ be a continuous projective unitary representation on the pre-Hilbert space $\mathcal{D}$. Assume that $\overline{\pi}$ vanishes on $\Gamma_c(M \setminus \{a\}, \mathfrak{K})$. Then $\overline{\pi}$ factors continuously through $J_a^\infty(\mathfrak{K})$.*
2. *Let $\overline{\rho} : \Gamma_c(M, \mathcal{K}) \to \mathrm{PU}(\mathcal{H})$ be a continuous projective unitary representation of $\Gamma_c(M, \mathcal{K})$. Assume that $\overline{\rho}$ vanishes on $\Gamma_c(M \setminus \{a\}, \mathcal{K})$. Then $\overline{\rho}$ factors through $\Gamma_{\mathit{flat}(a)}(M, \mathcal{K})$.*

Proof. For the first point, notice by continuity that $\overline{\pi}$ must also vanish on the closure of $\Gamma_c(M \setminus \{a\}, \mathfrak{K})$ in $\Gamma_c(M, \mathfrak{K})$, which by Proposition 4.5.3 equals $\Gamma_{\mathrm{flat}(a)}(M, \mathfrak{K})$. Thus $\Gamma_c(M, \mathfrak{K})$ factors continuously through the quotient space

$$J_a^\infty(\mathfrak{K}) \cong \Gamma_c(M, \mathfrak{K})/\Gamma_{\mathrm{flat}(a)}(M, \mathfrak{K}),$$

where Proposition 4.5.2 was used. The second point is proven similarly using Proposition 4.5.3. □

Chapter 5

Central extensions and generalized positive energy representations of the group of compactly supported diffeomorphisms

Abstract

We consider the projective unitary representations $\overline{\rho}$ of the Lie group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms of a smooth manifold M that admit a dense set of smooth rays and satisfy a the generalized positive energy condition. The latter in particular captures representations that are in a suitable sense compatible with a KMS state on the von Neumann algebra generated by $\overline{\rho}$. We show that if M is connected and $\dim(M) > 1$, then any such representation $\overline{\rho}$ is necessarily trivial on the identity component $\mathrm{Diff}_c(M)_0$. As an intermediate step towards this result, we consider the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ of the Lie algebra of compactly supported vector fields with trivial coefficients, showing in particular that it is trivial if $\dim(M) > 1$, and comparing it to the second Gelfand-Fuks cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$.

This chapter is based on [JN23], which is joint work with B. Janssens.

5.1 Introduction

Motivated by the aspiration for a suitable theory of general relativity that is compatible with the postulates of quantum physics, we study the extent to which the classical symmetry group can be implemented as symmetries of a quantum system. Classically, the symmetry group of general relativity contains the Lie group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms of a smooth manifold M, because the Einstein-Hilbert action is invariant under diffeomorphisms. On the other hand, the state space of quantum physics is commonly taken to be a projective Hilbert space. So we are interested in the study of projective unitary representations of the Lie group $\mathrm{Diff}_c(M)$.

Suppose that $\upsilon \in \mathcal{X}(M)$ is a complete and non-zero vector field on a smooth manifold M with flow $h : \mathbb{R} \to \mathrm{Diff}(M)$. Let $\mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R}$ be the semidirect product of $\mathrm{Diff}_c(M)$ and $\mathbb{R}$ relative to the $\mathbb{R}$-action on $\mathrm{Diff}_c(M)$ defined by $\alpha_t(f) := h_t \circ f \circ h_t^{-1}$ for $t \in \mathbb{R}$ and $f \in \mathrm{Diff}_c(M)$. Its Lie algebra is $\mathcal{X}_c(M) \rtimes \mathbb{R}\upsilon$, where υ acts on $\mathcal{X}_c(M)$ by the derivation $[\upsilon, -]$. We consider those projective unitary representation of $\mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R}$ that are of generalized positive energy at υ. The following is our main result:

Theorem 5.3.2. *Suppose that M is connected and that* $\dim(M) > 1$*. Consider a complete vector field $\upsilon \in \mathcal{X}(M) \setminus \{0\}$ on M. Let $\overline{\rho} : \mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R} \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation that is of generalized positive energy at υ. Then* $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

This has the following consequence in the KMS-setting:

Corollary 5.3.3. *Suppose that M is connected and that* $\dim(M) > 1$*. Consider a complete vector field $\upsilon \in \mathcal{X}(M) \setminus \{0\}$ on M. Let $\overline{\rho} : \mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R} \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation that is smoothly-KMS at υ relative to* $\mathrm{Diff}_c(M)$*. Assume that the von Neumann algebra $\rho(\mathrm{Diff}_c(M))''$ is a factor. Then* $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

Finally, Theorem 5.3.2 has the following consequence for norm-continuous projective unitary representations:

Corollary 5.3.4. *Suppose that* $\dim(M) > 1$*. Let $\overline{\rho} : \mathrm{Diff}_c(M) \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation that is bounded, i.e., continuous w.r.t. the norm topology on* $\mathrm{PU}(\mathcal{H}_\rho)$*. Then* $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

It is important to mention that these results naturally lead to *asymptotic* symmetry groups, for if G is any Lie group of diffeomorphisms of M containing $\mathrm{Diff}_c(M)_0$ as Lie subgroup, then any such projective unitary G-representation necessarily factors through the quotient $G/\mathrm{Diff}_c(M)_0$, and is in this sense 'localized at infinity'. It now becomes an interesting matter to determine this class of representations for groups of diffeomorphisms having certain specified behavior at infinity. In particular, one might wonder whether or not this class of representations, for suitable groups G, naturally leads to the asymptotic symmetry groups that appear in general relativity in the context of asymptotically flat spacetimes [Pen64, Ash15, Wal84], such as the BMS group (Bondi-Metzner-Sachs) [BvdBM62, Sac62, AE18, PS22, AS81], or extensions thereof [NU62, Ruz20]. In this context, the group G could for example reflect the freedom in the choice of a conformal completion of an asymptotically flat spacetime (cf. [Ash15, Def. 1 and section B]). This line of reasoning, for which Theorem 5.3.2 provides a key first ingredient, constitutes the main motivation of this chapter.

Although interesting in its own right from a mathematical perspective, one might wonder about the physical relevance of the generalized positive energy condition, as it singles out a distinguished vector field υ on M that is to serve as the generator of time translation, which seems opposed to the spirit of general relativity. Notice in this regard that the (generalized) positive energy and the KMS conditions are invariant under the adjoint action of $\mathrm{Diff}_c(M)$ on $\mathcal{X}_c(M) \rtimes \mathbb{R}v$, in the sense that if $\overline{\rho}$ satisfies any of these conditions relative to the vector field v, then it also does so relative to $\mathrm{Ad}_f(v) := T(f) \circ v \circ f^{-1}$ for any $f \in \mathrm{Diff}_c(M)$. So the choice of v is only significant up to the action of $\mathrm{Diff}_c(M)$. In general, these conditions depend only on the choice of an adjoint orbit (cf. [Nie23a, Lem. 5.9] for the KMS case), so if $\overline{\rho}$ extends to a smooth representation of a larger Lie group $G \rtimes \mathbb{R}$, the choice of v is only significant up to the adjoint action of G.

As an intermediate step towards Theorem 5.3.2, we consider the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ with coefficients in the trivial representation, where $\mathcal{X}_c(M)$ denotes the Lie algebra of compactly supported vector fields on M, equipped with its natural locally convex LF-topology. The cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ labels the equivalence classes of continuous central extensions of $\mathcal{X}_c(M)$ by $\mathbb{R}$. Our main result in this regard is Theorem 5.2.1 below:

Theorem 5.2.1. *Let M be a smooth manifold.*

1. *If $\dim(M) > 1$, then $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \{0\}$.*
2. *If $\dim(M) = 1$, then $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) \cong H^0_{\mathrm{dR}}(M)$ is the de Rham cohomology of M in degree 0.*

We also consider the relationship between $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ of the Lie algebra $\mathcal{X}(M)$ of *all* vector fields on M, equipped with its natural Fréchet topology. In particular, we show that the canonical linear map $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is injective (Proposition 5.2.11). Combined with Theorem 5.2.1, it follows at once that $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = \{0\}$ whenever $\dim(M) > 1$.

For $M = \mathbb{R}$ we find that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R}) \cong \mathbb{R}$ is spanned by the class of the Virasoro 2-cocycle

$$\psi_{\mathrm{vir}}(f\partial_x, g\partial_x) = \int_{\mathbb{R}} f'''(x)g(x)dx, \qquad f, g \in C^\infty_c(\mathbb{R}).$$

This is analogous to the well-known Virasoro 2-cocycle on $\mathcal{X}(S^1)$, whose class spans $H^2_{\mathrm{ct}}(\mathcal{X}(S^1), \mathbb{R})$ [KW09, Prop. 2.3]. It is also noteworthy that $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R}) = 0$, unlike its compactly supported sibling $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R}) \cong \mathbb{R}$ (Lemma 5.2.9 and Corollary 5.2.12). This shows that the cohomology theories $H^\bullet_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and $H^\bullet_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ differ for non-compact manifolds.

Let us mention here that the cohomology $H^\bullet_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ of the Lie algebra of all smooth vector fields, also known as Gelfand-Fuks cohomology in reference to authors of the papers [GF68, GF69, GF70b, GF70c], has been extensively studied [Los98, Gui73, BS77, Hae76]. This is in contrast to its compactly supported sibling. The latter was considered in [Shn76], where a certain spectral sequence is associated to the cohomology $H^\bullet_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}^n), \mathbb{R})$. Some expositions of Gelfand-Fuks cohomology can be found in [Fuk86, Bot73, Mia22]. In particular, the vanishing of $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ is well-known for all compact manifolds with $\dim(M) > 1$ ([Mia22, Thm. 4.13 and Cor. 4.25]). Nevertheless, Theorem 5.2.1 seems to be a new result if

M is a non-compact manifold. Its proof is moreover quite brief, in contrast to the more demanding methods commonly employed in Gelfand-Fuks cohomology, and is therefore of independent interest.

For some further related literature, we mention the paper [Sim23], which fully characterizes KMS representations of finite-dimensional Lie groups that generate a factor of type I. Certain projective unitary KMS representations of groups of $\mathrm{U}(N)$-valued maps on S^1 and $\mathbb{R}$ were considered in [CH92, CH87, BMT88]. The unitary representations of the BMS group were studied in [McC72, McC73, MC73, McC78, Pia77a] (cf. [Pia77b]). Gelfand-Fuks cohomologies with non-trivial coefficient modules have been considered in [Tsu81, GF70a, BN08b, LO00, AAL04]. In [JV16], the continuous central $\mathbb{R}$-extensions are classified for the Poisson Lie algebras $C^\infty(M)$ and $C^\infty_c(M)$, associated to a connected symplectic manifold (M, ω), and for the Lie algebra $\mathrm{Sp}(M, \omega)$ of symplectic vector fields. Related integrability questions are addressed in [JV19, Viz09] (cf. [NV03] and [DJNV21]).

In Section 5.2, we consider in detail the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$, in particular proving Theorem 5.2.1. We also consider its relation with the second Gelfand-Fuks cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$. We proceed in Section 5.3 with the proofs of Theorem 5.3.2 and Corollary 5.3.3.

5.2 The second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$

In the following, we determine the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ for manifolds of positive dimension. This classification will play a crucial role in Section 5.3 below. The main results of this section are summarized in Theorem 5.2.1 below.

Theorem 5.2.1. *Let M be a smooth manifold.*

1. *If $\dim(M) > 1$, then $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \{0\}$.*
2. *If $\dim(M) = 1$, then $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) \cong H^0_{\mathrm{dR}}(M)$ is the de Rham cohomology of M in degree 0.*

In Section 5.2.1 and Section 5.2.2, we consider the proof of Theorem 5.2.1 for the cases $\dim(M) > 1$ and $\dim(M) = 1$ separately. We will see in particular that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R}) \cong \mathbb{R}$ is spanned by the class of the Virasoro cocycle

$$\psi_{\mathrm{vir}}(f\partial_x, g\partial_x) = \int_{\mathbb{R}} f'''(x)g(x)dx, \qquad \forall f, g \in C^\infty_c(\mathbb{R}),$$

which is analogous to the well-known Virasoro 2-cocycle on $\mathcal{X}(S^1)$ (cf. [KW09, Prop. 2.3]). In Section 5.2.3, we clarify the relationship between $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and the second Gelfand-Fuks cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$, in particular showing that the canonical map $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is injective (Proposition 5.2.11). It then follows from Theorem 5.2.1 that $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = \{0\}$ whenever $\dim(M) > 1$. We will also show that $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R}) = \{0\}$ (cf. Corollary 5.2.12), contrary to its compactly supported counterpart.

We now proceed with the proof of Theorem 5.2.1. After making some general observations, we will consider the two cases $\dim(M) > 1$ and $\dim(M) = 1$ separately. In the following, we will make use of the Einstein summation convention, so that repeated indices are summed over.

Definition 5.2.2. A 2-cochain $\psi\colon \mathcal{X}_c(M) \times \mathcal{X}_c(M) \to \mathbb{R}$ is called *diagonal* if

$$\mathrm{supp}(v) \cap \mathrm{supp}(w) = \emptyset \quad \Longrightarrow \quad \psi(v, w) = 0, \qquad \forall v, w \in \mathcal{X}_c(M).$$

Lemma 5.2.3. *Every* 2*-cocycle on* $\mathcal{X}_c(M)$ *is diagonal.*

Proof. Let $v, w \in \mathcal{X}_c(M)$ have disjoint support. Then we can find open subsets $U_1, U_2 \subseteq M$ with $U_1 \cap U_2 = \emptyset$ such that $\mathrm{supp}(v) \subseteq U_1$ and $\mathrm{supp}(w) \subseteq U_2$. Since the Lie algebra $\mathcal{X}_c(U_1)$ is perfect ([Ban97, Thm. 1.4.3] or [Jan16, Cor. 1]), there exist $v_1^i, v_2^i \in \mathcal{X}_c(U_1)$ s.t. $v = \sum_{i=1}^N [v_1^i, v_2^i]$. Since $\psi\colon \mathcal{X}_c(M) \times \mathcal{X}_c(M) \to \mathbb{R}$ satisfies the cocycle identity

$$\psi([u, v], w) = \psi([u, w], v) + \psi(u, [v, w]), \tag{5.2.1}$$

and since w has support disjoint from that of v_1^i and v_2^i, we have

$$\psi(v, w) = \sum_{i=1}^N \psi([v_1^i, v_2^i], w) = \sum_{i=1}^N c([v_1^i, w], v_2^i) + c(v_1^i, [v_2^i, w]) = 0.$$

□

If $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M))$ is a 2-cochain on $\mathcal{X}_c(M)$, then the functional $\hat{\psi}(v)\colon \mathcal{X}_c(M) \to \mathbb{R}$ defined by $\hat{\psi}(v)(w) := \psi(v, w)$ is continuous for any $v \in \mathcal{X}_c(M)$. The corresponding linear map $\hat{\psi}\colon \mathcal{X}_c(M) \to \mathcal{X}_c(M)'$ is continuous w.r.t. the strong dual topology on $\mathcal{X}_c(M)'$. Equation (5.2.1) is moreover equivalent to the cocycle equation

$$\hat{\psi}([v, w]) = v \cdot \hat{\psi}(w) - w \cdot \hat{\psi}(v) \tag{5.2.2}$$

for 1-cochains in the Chevalley-Eilenberg complex $C^\bullet_{\mathrm{ct}}(\mathcal{X}_c(M), \mathcal{X}_c(M)')$ with coefficients in the continuous coadjoint representation $\mathcal{X}_c(M)'$, so $v \cdot \hat{\psi}(w) = \hat{\psi}([-v, w])$. Thus, if ψ is a 2-cocycle, then $\hat{\psi}$ is a 1-cocycle in $C^1_{\mathrm{ct}}(\mathcal{X}_c(M), \mathcal{X}_c(M)')$. If $K \subseteq M$ is a compact subset, let $\mathcal{X}_K(M)$ denote the set of smooth vector fields v on M with $\mathrm{supp}(v) \subseteq K$.

Proposition 5.2.4. *Let* $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M))$ *be a* 2*-cocycle. Then* $\hat{\psi} \in C^1_{\mathrm{ct}}(\mathcal{X}_c(M), \mathcal{X}_c(M)')$ *extends to a continuous* 1*-cocycle* $\hat{\psi} \in C^1_{\mathrm{ct}}(\mathcal{X}(M), \mathcal{X}_c(M)')$.

Proof. Let K_i be an exhaustion of M by compact subsets. Let $v \in \mathcal{X}(M)$. Define $\hat{\psi}_i(v) \in \mathcal{X}_{K_i}(M)'$ by $\hat{\psi}_i(v)(w) := \psi(f_{K_i} v, w)$ for an arbitrary $f_{K_i} \in C^\infty_c(M)$ that satisfies $f_{K_i}(x) = 1$ for all x in some open neighborhood U_i of K_i. This is independent of f_{K_i} by Lemma 5.2.3. The various $\hat{\psi}_i(v)$ define an element $\hat{\psi}(v)$ of $\mathcal{X}_c(M)'$, because $\mathcal{X}_c(M)$ is the locally convex inductive limit $\mathcal{X}_c(M) = \varinjlim_i X_{K_i}(M)$ and the various $\hat{\psi}_i(v)$ are compatible in the sense that $\hat{\psi}_j(v)\Big|_{\mathcal{X}_{K_i}(M)} = \hat{\psi}_i(v)$ whenever $K_i \subseteq K_j$. The linear map $\hat{\psi} : \mathcal{X}(M) \to \mathcal{X}_c(M)'$ obtained in this way clearly extends the original map $\hat{\psi} : \mathcal{X}_c(M) \to \mathcal{X}_c(M)'$ and because ψ is diagonal, it satisfies the cocycle identity

$$\hat{\psi}([v, w]) = v \cdot \hat{\psi}(w) - w \cdot \hat{\psi}(v) \tag{5.2.3}$$

for the action of $\mathcal{X}(M)$ on $\mathcal{X}_c(M)'$ by $(v \cdot \phi)(u) = \phi([-v, u])$. It remains to show that $\hat{\psi} : \mathcal{X}(M) \to \mathcal{X}_c(M)'$ is continuous w.r.t. the strong dual topology on $\mathcal{X}_c(M)'$. To see this, observe that $\mathcal{X}_c(M) = \varinjlim \mathcal{X}_{K_i}(M)$ is the *strict* inductive limit of the Fréchet spaces $\mathcal{X}_{K_i}(M)$ (cf. [Rud91, Thm. 6.5]). So any bounded set of $\mathcal{X}_c(M)$ is contained in some $\mathcal{X}_{K_i}(M)$. This implies that $\mathcal{X}_c(M)' = \varprojlim_i \mathcal{X}_{K_i}(M)'$ as topological vector spaces. It therefore suffices to show that the composition of $\hat{\psi}$ with the projection $\mathcal{X}_c(M)' \to \mathcal{X}_{K_i}(M)'$ is continuous. Choosing f_{K_i} with support in the interior of K_{i+1}, we notice that the latter composition factors through a linear map $\mathcal{X}_{K_{i+1}}(M) \to \mathcal{X}_{K_i}(M)'$ that is continuous because the original 1-cocycle $\hat{\psi} \in C^1_{\mathrm{ct}}(\mathcal{X}_c(M), \mathcal{X}_c(M)')$ is so. □

Remark 5.2.5. The map $\hat{\psi} : \mathcal{X}(M) \to \mathcal{X}_c(M)'$ from Proposition 5.2.4, associated to the 2-cocycle $\psi \in C^2_{\mathrm{ct}}(X_c(M))$, is *support decreasing* in the sense that $\mathrm{supp}(\hat{\psi}(v)) \subseteq \mathrm{supp}(v)$ for any $v \in \mathcal{X}(M)$, because the original 2-cocycle ψ is diagonal (Lemma 5.2.3). This implies by Peetre's Theorem that $\hat{\psi} : \mathcal{X}(M) \to \mathcal{X}_c(M)'$ is a differential operator of locally finite degree [Pee60].

5.2.1 Manifolds M of dimension $\dim(M) > 1$

We now proceed with the proof of Theorem 5.2.1 for manifolds of dimension $\dim(M) > 1$.

The local setting

We begin with the case where $M = \mathbb{R}^n$ for some integer $n > 1$. This should be regarded as the local analog of Theorem 5.2.1 for $\dim(M) > 1$. The following is an adaptation of a result in [JRV23], and we thank Cornelia Vizman and Leonid Ryvkin for illuminating discussions on this topic. The analogous statement in Gelfand-Fuks cohomology for $n > 1$ follows e.g. from [Mia22, Thm. 3.12].

Proposition 5.2.6. *Let $n > 1$ be an integer. Then $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}^n), \mathbb{R}) = \{0\}$.*

Before proceeding with the proof of Proposition 5.2.6, we make some preliminary observations. The Lie algebra $W_n \subseteq \mathcal{X}(\mathbb{R}^n)$ of vector fields with polynomial coefficients is $\mathbb{Z}$-graded, with W_n^k being the vector fields with homogeneous polynomial coefficients of degree $k+1$. Since $[W_n^k, W_n^l] \subseteq W_n^{k+l}$, the constant vector fields W_n^{-1} decrease the degree by 1. Also, every W_n^k is a representation of the Lie algebra W_n^0 of linear vector fields, which we identify with $\mathfrak{gl}(n, \mathbb{R})$ via the isomorphism $\mathfrak{gl}(n, \mathbb{R}) \to W_n^0$ that maps $(A^\mu_\nu)^n_{\mu,\nu=1}$ to the linear vector field $a^\mu_\nu x^\nu \partial_\mu$ with constant coefficients $a^\mu_\nu = -A^\mu_\nu$. Under this identification, we have $W_n^k \cong S^{k+1}(\mathbb{R}^d)^* \otimes \mathbb{R}^d$ as $\mathfrak{gl}(n, \mathbb{R})$-representation for every $k \in \mathbb{N}_{\geq 0}$, where $S^{k+1}(\mathbb{R}^d)^*$ denotes the space of homogeneous polynomials on $\mathbb{R}^n$ of degree $k+1$. The Euler vector field $E = x^\mu \partial_\mu$ acts on $v \in W_n^k$ by $[E, v] = kv$.

Lemma 5.2.7. *The translation-invariant subspace of $\mathcal{X}_c(\mathbb{R}^n)'$ is equivalent to $(\mathbb{R}^n)^* \otimes \wedge^n(\mathbb{R}^n)^*$ as representation of $\mathfrak{gl}(n, \mathbb{R})$. In particular, the Euler vector field $E \in W_n^0$ acts on a translation-invariant $\phi \in \mathcal{X}_c(\mathbb{R}^n)'$ by*

$$E \cdot \phi = (n+1)\phi. \tag{5.2.4}$$

Proof. The linear vector field $a^\mu_\nu x^\nu \partial_\mu$ corresponding to $A = (A^\mu_\nu)^n_{\mu,\nu=1}$ acts on $\phi \in \mathcal{X}_c(\mathbb{R}^n)'$ according to

$$\begin{aligned}(a^\mu_\nu x^\nu \partial_\mu \cdot \phi)(u^\sigma \partial_\sigma) &= \phi(-a^\mu_\nu [x^\nu \partial_\mu, u^\sigma \partial_\sigma]) \\ &= -\phi(a^\mu_\nu x^\nu (\partial_\mu u^\sigma)\partial_\sigma) + \phi(a^\mu_\sigma u^\sigma \partial_\mu) \\ &= -\mathrm{tr}(A)\phi(u^\sigma \partial_\sigma) + \phi(a^\mu_\sigma u^\sigma \partial_\mu) + (\partial_\mu \cdot \phi)(a^\mu_\nu x^\nu u^\sigma \partial_\sigma),\end{aligned} \tag{5.2.5}$$

where in the last equality we used that

$$(\partial_\mu \cdot \phi)(a^\mu_\nu x^\nu u^\sigma \partial_\sigma) = -\phi(a^\mu_\nu u^\sigma \partial_\sigma)\delta^\nu_\mu - \phi(a^\mu_\nu x^\nu (\partial_\mu u^\sigma)\partial_\sigma).$$

Assume now that ϕ is translation-invariant. Then $(\partial_\mu \cdot \phi)(a^\mu_\nu x^\nu u^\sigma \partial_\sigma) = 0$ and $\phi(u^\sigma \partial_\sigma) = b_\sigma I(u^\sigma)$ for some vector $b = (b_\sigma)^n_{\sigma=1} \in \mathbb{R}^n$, where $I(f) := \int_{\mathbb{R}^n} f dx$ for $f \in C^\infty_c(\mathbb{R}^n)$. It follows using (5.2.5) that

$$(a^\mu_\nu x^\nu \partial_\mu \cdot \phi)(u^\sigma \partial_\sigma) = \big(-\mathrm{Tr}(A) b_\sigma + a^\mu_\sigma b_\mu \big) I(u^\sigma) = b'_\sigma I(u^\sigma),$$

where $b' := -\mathrm{Tr}(A^T)b - A^T b$. This corresponds to the natural action of $\mathfrak{gl}(n, \mathbb{R})$ on $(\mathbb{R}^n)^* \otimes \wedge^n(\mathbb{R}^n)^*$ under the isomorphism $\mathfrak{gl}(n, \mathbb{R}) \cong W_n^0$ specified above, so the assertion follows. □

Proof of Proposition 5.2.6.
Let ψ be a continuous 2-cocycle on $\mathcal{X}_c(\mathbb{R}^n)$, and let $\hat{\psi} \in C^1_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}^n), \mathcal{X}_c(\mathbb{R}^n)')$ be the corresponding 1-cocycle obtained using Proposition 5.2.4. By Remark 5.2.5, we can expand $\hat{\psi}$ into a locally finite sum as

$$\hat{\psi}(v) = \sum_{\vec{\sigma} \in \mathbb{N}^n_{\geq 0}} \left(\frac{\partial^{|\vec{\sigma}|}}{\partial x^{\vec{\sigma}}} v^\mu \right) \phi^{\vec{\sigma}}_\mu, \tag{5.2.6}$$

where $\phi^{\vec{\sigma}}_\mu \in \mathcal{X}_c(\mathbb{R}^n)'$. We show for any integer $k \geq -1$ that $\hat{\psi}$ is cohomologous to a 1-cocycle that vanishes on the subspace $W_n^{\leq k}$ of vector fields with polynomial coefficients of degree at most $k+1$.

The case $k = -1$. The cocycle identity (5.2.3) for constant vector fields $v = \partial_\mu$ and $w = \partial_\nu$ yields

$$\partial_\mu \cdot \phi^{\vec{0}}_\nu - \partial_\nu \cdot \phi^{\vec{0}}_\mu = 0. \tag{5.2.7}$$

We identify $\mathcal{X}_c(\mathbb{R}^n)' \simeq \mathcal{D}'(\mathbb{R}^n) \otimes (\mathbb{R}^n)^*$ with n copies of the distributions $\mathcal{D}'(\mathbb{R}^n)$ by setting

$$(\zeta_\sigma \otimes dx^\sigma)(v^\mu \partial_\mu) := \zeta_\sigma(v^\sigma), \qquad \text{for } \zeta_\sigma \in \mathcal{D}'(\mathbb{R}^n) \text{ and } v^\mu \in C^\infty_c(\mathbb{R}^n).$$

The action of ∂_μ on $\mathcal{X}_c(\mathbb{R}^n)'$ is then simply given by differentiating the components in $\mathcal{D}'(\mathbb{R}^n)$, so that for $\phi^{\vec{0}}_\nu = \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma$ we have $\partial_\mu \cdot \phi^{\vec{0}}_\nu = \partial_\mu \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma$. Indeed, we compute that

$$\begin{aligned}(\partial_\mu \cdot \phi^{\vec{0}}_\nu)(X^\tau \partial_\tau) &= \phi^{\vec{0}}_\nu(-[\partial_\mu, X^\tau\partial_\tau]) = \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma(-(\partial_\mu X^\tau)\partial_\tau) \\ &= \phi^{\vec{0}}_{\nu\sigma}(-(\partial_\mu X^\sigma)) = (\partial_\mu \phi^{\vec{0}}_{\nu\sigma})(X^\sigma) = (\partial_\mu \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma)(X^\tau \partial_\tau).\end{aligned}$$

Equation (5.2.7) therefore yields for each σ that $\partial_\mu \cdot \phi^{\vec{0}}_{\nu\sigma} - \partial_\nu \cdot \phi^{\vec{0}}_{\mu\sigma} = 0$ for all integers $1 \leq \mu, \nu \leq n$. With respect to the differential

$$d : \Omega^{n-p}_c(\mathbb{R}^n)' \to \Omega^{n-(p+1)}_c(\mathbb{R}^n)', \ \langle dT, \quad \alpha\rangle := (-1)^{p+1}\langle T, d\alpha\rangle$$

on the space of currents (cf. [dR84, III§11], this means precisely that the current $c_\sigma := \phi^{\vec{0}}_{\mu\sigma} dx^\mu \in \Omega^{n-1}_c(\mathbb{R}^n)'$ is closed, because

$$dc_\sigma = \sum_{1\leq\mu<\nu\leq n} (\partial_\mu \cdot \phi^{\vec{0}}_{\nu\sigma} - \partial_\nu \cdot \phi^{\vec{0}}_{\mu\sigma}) dx^\mu \wedge dx^\nu = 0.$$

(Identifying $\mathcal{D}'(\mathbb{R}^n) \cong \Omega^n_c(\mathbb{R}^n)'$ using the volume form $dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$, for any $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\alpha \in \Omega^p(\mathbb{R}^n)$, we interpret $T\alpha$ as element of $\Omega^{n-p}_c(\mathbb{R}^n)'$ via the pairing $\langle T\alpha, \beta\rangle := T(\alpha \wedge \beta)$ for $\beta \in \Omega^{n-p}_c(\mathbb{R}^n)$, cf. [dR84, p. 36].) By the Poincaré Lemma for currents [dR84, IV§19], it follows that there exist distributions $\eta_\sigma \in \mathcal{D}'(\mathbb{R}^n)$ with $\partial_\mu \eta_\sigma = \phi^{\vec{0}}_{\mu\sigma}$ for all integers $1 \leq \mu, \sigma \leq n$. The 1-coboundary $d_{\mathfrak{g}}(\eta_\sigma \otimes dx^\sigma)$ in $C^1_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}^n), \mathcal{X}_c(\mathbb{R}^n)')$ thus agrees with $\hat{\psi}$ on $\partial_\mu \in W^{-1}_n$:

$$d_{\mathfrak{g}}(\eta_\sigma \otimes dx^\sigma)(\partial_\mu) = \partial_\mu \cdot (\eta_\sigma \otimes dx^\sigma) = \partial_\mu \eta_\sigma \otimes dx^\sigma = \phi^{\vec{0}}_{\mu\sigma} \otimes dx^\sigma = \phi^{\vec{0}}_\mu = \hat{\psi}(\partial_\mu)$$

Replacing $\hat{\psi}$ by the 1-cocycle $\hat{\psi} - d_{\mathfrak{g}}(\eta_\sigma \otimes dx^\sigma)$, we assume from now on that $\hat{\psi}$ vanishes on W^{-1}_n.

The case $0 \leq k \leq n$**.** Suppose that $\hat{\psi}$ vanishes on $W^{\leq(k-1)}_n$. Let $v \in W^k_n$. Since $[\partial_\mu, v] \in W^{k-1}_n$, the cocycle identity (5.2.3) yields $\partial_\mu \cdot \hat{\psi}(v) = 0$ for all μ, so that $\hat{\psi}(v)$ is translation invariant. So $E \cdot \hat{\psi}(v) = (n+1)\hat{\psi}(v)$, in view of Lemma 5.2.7. From the cocycle identity $\hat{\psi}([E, v]) = E \cdot \hat{\psi}(v) - v \cdot \hat{\psi}(E)$, we find for any $v \in W^k_n$ that

$$(n + 1 - k)\hat{\psi}(v) = v \cdot \hat{\psi}(E). \tag{5.2.8}$$

We consider separately the cases $k = 0$ and $0 < k \leq n$. Suppose that $k = 0$. The preceding then shows that $\hat{\psi}(v) = \frac{1}{n+1} v \cdot \hat{\psi}(E)$ for all $v \in W^0_n$. The 0-cochain $\eta = \frac{1}{n+1}\hat{\psi}(E)$ therefore satisfies $(d_{\mathfrak{g}}\eta)(v) = \hat{\psi}(v)$ for any $v \in W^0_n$. Since $E \in W^0_n$, we know that $\hat{\psi}(E)$ is translation-invariant, so we also have $(d_{\mathfrak{g}}\eta)(v) = \frac{1}{n+1} v \cdot \hat{\psi}(E) = 0$ for $v \in W^{-1}_n$. Replacing $\hat{\psi}$ by the cohomologous cocycle $\hat{\psi} - d_{\mathfrak{g}}\eta$ if necessary, we may assume that $\hat{\psi}$ vanishes on $W^{\leq 0}_n$. Suppose next that $0 < k \leq n$. Then $E \in W^{\leq(k-1)}_n$, so $\hat{\psi}(E) = 0$. Consequently, (5.2.8) implies that $\hat{\psi}(v) = 0$ for any $v \in W^k_n$ and hence $\hat{\psi}$ vanishes on $W^{\leq k}_n$. Inductively, we thus find that $\hat{\psi}$ vanishes on $W^{\leq n}_n$, and that $\hat{\psi}(v)$ is translation invariant for any $v \in W^{n+1}_n$.

The case $k = n+1$. The cocycle identity (5.2.3) for $A \in W_n^0$ and $v \in W_n^{n+1}$ reads $\hat{\psi}([A, v]) = A \cdot \hat{\psi}(v)$, because $\hat{\psi}(A) = 0$. Since $\hat{\psi}(v)$ is translation invariant for any $v \in W_n^{n+1}$, we conclude using Lemma 5.2.7 that the linear map

$$\hat{\psi}\Big|_{W_n^{n+1}} : W_n^{n+1} \to (\mathbb{R}^n)^* \otimes \wedge^n(\mathbb{R}^n)^* \subseteq \mathcal{X}_c(\mathbb{R}^n)'$$

is an intertwiner of $\mathfrak{gl}(n, \mathbb{R})$-representations. The action of $\mathfrak{sl}(n, \mathbb{R})$ on $\wedge^n(\mathbb{R}^n)^*$ is trivial, and we notice that

$$\mathrm{Hom}_{\mathfrak{sl}(n,\mathbb{R})}\left(W_n^{n+1}, (\mathbb{R}^n)^*\right) \cong \mathrm{Hom}_{\mathfrak{sl}(n,\mathbb{R})}\left(S^{n+2}(\mathbb{R}^n)^*, (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*\right) = 0,$$

because $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \cong S^2(\mathbb{R}^n)^* \oplus \bigwedge^2(\mathbb{R}^n)^*$ does not contain the irreducible $\mathfrak{sl}(n, \mathbb{R})$-representation on $S^{n+2}(\mathbb{R}^n)^*$ (cf. [FH91, Prop. 15.15]). So $\hat{\psi}(v) = 0$ for any $v \in W_n^{n+1}$.

The case $k > n+1$. Suppose that $\hat{\psi}$ vanishes on $W_n^{\leq(k-1)}$ for $k > n+1$. Then (5.2.8) implies that $\hat{\psi}(v) = 0$ for any $v \in W_n^k$, so $\hat{\psi}$ vanishes on $W_n^{\leq k}$. Inductively, we thus find that $\hat{\psi}$ vanishes on $W_n^{\leq k}$ for any integer $k \geq -1$. This implies that all the coefficients $\phi_\mu^{\vec{\sigma}}$ in equation (5.2.6) are zero, so $\hat{\psi} = 0$ and hence $\psi = 0$. □

A local-to-global argument

Having established that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}^n), \mathbb{R}) = \{0\}$, we employ a local-to-global argument to determine $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ for general manifolds M with $\dim(M) > 1$. We let $\mathcal{X}_c'$ denote the presheaf defined by $U \mapsto \mathcal{X}_c(U)'$ and the natural restriction maps. This is in fact an acyclic sheaf by Proposition 5.4.1.

Theorem 5.2.8. *Assume that* $\dim(M) > 1$*. Then* $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \{0\}$*.*

Proof. Let $n := \dim(M)$. The continuous Chevalley-Eilenberg cochains define a presheaf $U \mapsto C^m_{\mathrm{ct}}(\mathcal{X}_c(U), \mathbb{R})$ for any $m \in \mathbb{N}$, that we denote by $C^m_{\mathrm{ct}}(\mathcal{X}_c)$. We denote by $Z^m_{\mathrm{ct}}(\mathcal{X}_c) \subseteq C^m_{\mathrm{ct}}(\mathcal{X}_c)$ its sub-presheaf consisting of cocycles. Let $\mathcal{U} = \{U_i : i \in S\}$ be an open cover of M such that every U_i is diffeomorphic to $\mathbb{R}^n$, and consider the (augmented) double complex $\check{C}^\bullet(\mathcal{U}, C^\bullet_{\mathrm{ct}}(\mathcal{X}_c))$ for the Čech-cohomology with coefficients in the presheaf $C^\bullet_{\mathrm{ct}}(\mathcal{X}_c)$. Restricted to cocycles in Chevalley-Eilenberg degree 2, the left lower portion looks as follows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & Z^2_{\mathrm{ct}}(\mathcal{X}_c(M)) & \xrightarrow{\check{\delta}} & \prod_{i\in S} Z^2_{\mathrm{ct}}(\mathcal{X}_c(U_i)) & \xrightarrow{\check{\delta}} & \prod_{i,j\in S} Z^2_{\mathrm{ct}}(\mathcal{X}_c(U_i \cap U_j)) \\
 & & {\scriptstyle d_{\mathfrak{g}}}\uparrow & & {\scriptstyle d_{\mathfrak{g}}}\uparrow & & {\scriptstyle d_{\mathfrak{g}}}\uparrow \\
0 & \longrightarrow & C^1_{\mathrm{ct}}(\mathcal{X}_c(M)) & \xrightarrow{\check{\delta}} & \prod_{i\in S} C^1_{\mathrm{ct}}(\mathcal{X}_c(U_i)) & \xrightarrow{\check{\delta}} & \prod_{i,j\in S} C^1_{\mathrm{ct}}(\mathcal{X}_c(U_i \cap U_j)) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0.
\end{array}$$

The middle column is exact by Proposition 5.2.6, as every $U_i \in \mathcal{U}$ is diffeomorphic to $\mathbb{R}^n$, and the column on the right is exact at $\prod_{i,j\in S} C^1_{\mathrm{ct}}(\mathcal{X}_c(U_i \cap U_j))$ because $\mathcal{X}_c(U_i \cap U_j)$ is perfect for any $i, j \in S$ [Ban97, Thm. 1.4.3]. The bottom row is exact because $C^1_{\mathrm{ct}}(\mathcal{X}_c) = \mathcal{X}'_c$ is an acyclic sheaf, by Proposition 5.4.1. Lemma 5.2.3 further guarantees that the map $\check{\delta}\colon Z^2_{\mathrm{ct}}(\mathcal{X}_c(M)) \to \prod_{i\in S} C^2_{\mathrm{ct}}(\mathcal{X}_c(U_i))$ is injective. Indeed, suppose that $\psi(\mathcal{X}_c(U_i), \mathcal{X}_c(U_i)) = \{0\}$ for all $i \in S$. Then $\psi(X_c(U_i), \mathcal{X}_c(M)) = \{0\}$ for any $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M))$ and $i \in S$, because ψ is diagonal. So $\psi = 0$, by a partition of unity argument. A straightforward diagram chase now shows that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ vanishes. $\square$

5.2.2 Manifolds M of dimension one

We proceed with the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ for 1-dimensional manifolds. In the connected case, M must be diffeomorphic to either $\mathbb{R}$ or S^1. It is well-known that $H^2_{\mathrm{ct}}(\mathcal{X}(S^1), \mathbb{R}) = \mathbb{R}$ is spanned by the class of the Virasoro cocycle (cf. [KW09, Prop. 2.3]):

$$\psi_{\mathrm{vir}}(f\partial_\theta, g\partial_\theta) = \int_{S^1} f'''(\theta)g(\theta)d\theta, \qquad f, g \in C^\infty(S^1).$$

A slight adaptation of the proof of Proposition 5.2.6 allows us to prove the analogous result on the real line:

Lemma 5.2.9. *The second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R})$ is 1-dimensional. It is spanned by the class of the Virasoro cocycle*

$$\psi_{\mathrm{vir}}(f\partial_x, g\partial_x) = \int_{\mathbb{R}} f'''(x)g(x)dx, \qquad f, g \in C^\infty_c(\mathbb{R}). \tag{5.2.9}$$

Proof. Let us first observe that the cocycle

$$\psi_{\mathrm{vir}}(f\partial_x, g\partial_x) = \int_{\mathbb{R}} f'''(x)g(x)dx$$

is not a coboundary. Indeed, if $\eta \in C^1_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R})$ is a 1-cochain, then the map $\widehat{d_{\mathfrak{g}}\eta} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}_c(\mathbb{R})'$ obtained using Proposition 5.2.4 is the first-order differential operator $\widehat{d_{\mathfrak{g}}\eta}(f\partial_x) = f(\partial_x \cdot \eta) + 2f'\eta$. Indeed, this follows from the calculation

$$\begin{aligned}\widehat{d_{\mathfrak{g}}\eta}(f\partial_x)(g\partial_x) &= -\eta([f\partial_x, g\partial_x]) = \eta(f'g\partial_x - fg'\partial_x) = 2\eta(f'g\partial_x) - \eta((fg)'\partial_x) \\ &= (2f'\eta + f(\partial_x \cdot \eta))(g\partial_x)\end{aligned}$$

for $f \in C^\infty(\mathbb{R})$ and $g \in C^\infty_c(\mathbb{R})$. On the other hand, $\hat{\psi}_{\mathrm{vir}}$ is the third-order differential operator $\hat{\psi}_{\mathrm{vir}}(f\partial_x) = f'''I$, where $I \in \mathcal{X}_c(\mathbb{R})'$ is defined by $I(f\partial_x) = \int_{\mathbb{R}} f(x)dx$. So ψ_{vir} can not be a coboundary.

Let ψ be a continuous 2-cocycle. Let $\hat{\psi} \in C^1_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ be the corresponding 1-cocycle obtained using Proposition 5.2.4. We show that $\hat{\psi}$ is cohomologous to a 1-cocycle in $C^1_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ that vanishes on the subspace $W_1^{-1} = \mathbb{R}\partial_x$. Choose $\chi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \chi(x)dx = 1$. For any $f \in C_c^\infty(\mathbb{R})$, the smooth function $P(f)(x) := \int_{-\infty}^x f(s) - I(f\partial_x)\chi(s)ds$ is smooth and compactly supported. We moreover have $P(f') = f$, because $I(f'\partial_x) = 0$. Observe that the 0-cochain $\eta \in \mathcal{X}_c(\mathbb{R})' = C^0_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ defined by $\eta(f\partial_x) := \hat{\psi}(\partial_x)(P(f)\partial_x)$ satisfies $\hat{\psi}(\partial_x) + (d_{\mathfrak{g}}\eta)(\partial_x) = 0$, because

$$(d_{\mathfrak{g}}\eta)(\partial_x)(f\partial_x) = -\eta(f'\partial_x) = -\hat{\psi}(\partial_x)(P(f')\partial_x) = -\hat{\psi}(\partial_x)(f\partial_x), \qquad \forall f \in C_c^\infty(\mathbb{R}).$$

Replacing $\hat{\psi}$ by $\hat{\psi} + d_{\mathfrak{g}}\eta$, we assume from now on that $\hat{\psi}$ vanishes on $W_1^{-1} = \mathbb{R}\partial_x$. Following the case $0 \leq k \leq n$ in the proof of Proposition 5.2.6, we may then further assume that $\hat{\psi}$ vanishes on $W_1^{\leq 1}$ and that $\hat{\psi}(x^3\partial_x) \in \mathcal{X}_c(\mathbb{R})'$ is translation invariant. The latter implies that $\hat{\psi}(x^3\partial_x) = cI$ for some constant $c \in \mathbb{R}$. It follows that the 1-cocycle $\hat{\psi} - c\hat{\psi}_{\mathrm{vir}} \in C^1_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ vanishes on $W_1^{\leq 2}$. Following the case $k > n+1$ in the proof of Proposition 5.2.6, this implies that $\hat{\psi} - c\hat{\psi}_{\mathrm{vir}}$ vanishes on $W_1^{\leq k}$ for any integer $k \geq -1$ and therefore that $\hat{\psi} = c\hat{\psi}_{\mathrm{vir}}$. So $\psi = c\psi_{\mathrm{vir}}$. □

We have thus shown that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) \cong \mathbb{R}$ for any connected 1-dimensional manifold. Combined with Proposition 5.2.6, the following now completes the proof of Theorem 5.2.1.

Theorem 5.2.10. *Let M be a smooth manifold of dimension 1. Then*

$$H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = H^0_{\mathrm{dR}}(M).$$

Proof. Let $\{M_\alpha\}_{\alpha\in\mathcal{I}}$ be the set of connected components of M, where $\mathcal{I}$ is some countable indexing set. (Here we used that M is second-countable.) As the support of a compactly supported vector field on M intersects only finitely many M_α nontrivially, $\mathcal{X}_c(M)$ is isomorphic to the locally convex direct sum

$$\mathcal{X}_c(M) \cong \bigoplus_{\alpha\in\mathcal{I}} \mathcal{X}_c(M_\alpha).$$

So $\mathcal{X}_c(M)' \cong \prod_{\alpha\in\mathcal{I}} \mathcal{X}_c(M_\alpha)'$. Furthermore, any 2-cocycle $\psi : \mathcal{X}_c(M) \times \mathcal{X}_c(M) \to \mathbb{R}$ is diagonal by Lemma 5.2.3, and therefore decomposes as $\psi = \sum_\alpha \psi_\alpha$ for some 2-cochains ψ_α on $\mathcal{X}_c(M_\alpha)$. Moreover, ψ is a cocycle, resp. a coboundary, if and only if every ψ_α is so. It follows that

$$H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \prod_{\alpha\in\mathcal{I}} H^2_{\mathrm{ct}}(\mathcal{X}_c(M_\alpha), \mathbb{R}) = \prod_{\alpha\in\mathcal{I}} \mathbb{R} \cong H^0_{\mathrm{dR}}(M).$$

□

5.2.3 Relation between $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and the second Gelfand-Fuks cohomology

Finally, let us consider the relationship between $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and the Gelfand-Fuks cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$. The continuous injection $\mathcal{X}_c(M) \hookrightarrow \mathcal{X}(M)$ induces a natural morphism $C^\bullet_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to C^\bullet_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ of cochain complexes, which descends to a linear map $H^\bullet_{\text{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^\bullet_{\text{ct}}(\mathcal{X}_c(M), \mathbb{R})$ on cohomology. If $\psi \in C^2_{\text{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is a diagonal 2-cochain, its *support* $\text{supp}(\psi)$ is the set of points $x \in M$ with the property that for any neighborhood U of x, there exist $v, w \in \mathcal{X}_c(U)$ with $\psi(v, w) \neq 0$. If $x \notin \text{supp}(\psi)$ and U is a neighborhood with $\psi(\mathcal{X}_c(U), \mathcal{X}_c(U)) = \{0\}$, then $\psi(\mathcal{X}_c(U), \mathcal{X}_c(M)) = \{0\}$, because ψ is diagonal. The following is a straightforward adaptation of [JV16, Lem. 4.19] to the present setting:

Proposition 5.2.11.

1. *A continuous* 2*-cocycle* $\psi \in C^2_{\text{ct}}(\mathcal{X}_c(M), \mathbb{R})$ *extends to a continuous* 2*-cocycle on* $\mathcal{X}(M)$ *if and only if it has compact support.*

2. *Assume that the* 2*-cocycle* $\psi \in C^2_{\text{ct}}(\mathcal{X}_c(M), \mathbb{R})$ *has compact support and satisfies* $\psi = d_{\mathfrak{g}}\eta$ *for some* $\eta \in \mathcal{X}_c(M)'$*. Then* $\text{supp}(\eta) = \text{supp}(\psi)$ *and* η *extends to a continuous linear map* $\mathcal{X}(M) \to \mathbb{R}$*.*

3. *The canonical linear map* $H^2_{\text{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^2_{\text{ct}}(\mathcal{X}_c(M), \mathbb{R})$ *is injective.*

Proof.

1. Assume that ψ has compact support, say $\text{supp}(\psi) = K$. Consider the 1-cocycle $\hat{\psi} \in C^1_{\text{ct}}(\mathcal{X}(M), \mathcal{X}_c(M)')$ obtained from Proposition 5.2.4. Let $\chi \in C^\infty_c(M)$ satisfy $\chi|_U = 1$ for some open neighborhood U of K. Define the bilinear map

$$\widetilde{\psi} : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}, \qquad \widetilde{\psi}(v, w) := \hat{\psi}(v)(\chi w),$$

which extends ψ and is independent of the choice of χ, because ψ has support K. It is moreover continuous, in view of the continuity of both $\hat{\psi}$ and the map $\mathcal{X}(M) \to \mathcal{X}_c(M), w \mapsto \chi w$. We next show that $\widetilde{\psi}$ is a 2-cocycle. Observe that $\hat{\psi}(u)(\chi[v, w]) = \hat{\psi}(u)([v, \chi w])$ for any $u, v, w \in \mathcal{X}(M)$, because $\mathcal{L}_v(\chi)w \in \mathcal{X}_c(M)$ vanishes on a neighborhood of K, so that $\hat{\psi}(u)(\mathcal{L}_v(\chi)w) = 0$. Using (5.2.3), we therefore have

$$\begin{aligned}
\widetilde{\psi}([u, v], w) + \widetilde{\psi}(v, [u, w]) &= \hat{\psi}([u, v])(\chi w) + \hat{\psi}(v)(\chi[u, w]) \\
&= \hat{\psi}(u)([v, \chi w]) - \hat{\psi}(v)([u, \chi w]) + \hat{\psi}(v)(\chi[u, w]) \\
&= \hat{\psi}(u)(\chi[v, w]) \\
&= \widetilde{\psi}(u, [v, w]).
\end{aligned}$$

Conversely, assume that ψ extends to a continuous 2-cocycle on $\mathcal{X}(M)$, again denoted ψ. Suppose that $K := \text{supp}(\psi)$ is not compact. Then we can find a countably infinite sequence $(x_i)_{i \in \mathbb{N}}$ in K of distinct points which has no convergent subsequence. Let $\{U_i\}_{i \in \mathbb{N}}$ be a collection of pairwise disjoint open subsets of M so that $x_i \in U_i$ for all $i \in \mathbb{N}$. Since $x_i \in K$, there exist for every $i \in \mathbb{N}$ some $v_i, w_i \in \mathcal{X}_c(U_i)$ satisfying $\psi(v_i, w_i) = 1$. Notice that $v := \sum_{i=1}^\infty v_i$ and $w := \sum_{i=1}^\infty w_i$ are well-defined smooth vector fields on M, because the open sets U_i are pairwise disjoint. Since $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ is diagonal and continuous, we obtain the evident contradiction that

$$\lim_{N\to\infty} N = \lim_{N\to\infty} \sum_{i=1}^{N} \psi(v_i, w_i) = \lim_{N\to\infty} \psi\left(\sum_{i=1}^{N} v_i, \sum_{i=1}^{N} w_i\right) = \psi(v, w) < \infty.$$

So $\mathrm{supp}(\psi)$ must be compact.

2. Let $x \notin \mathrm{supp}(\psi)$. Then there exists an open neighborhood U of x s.t. $\psi(\mathcal{X}_c(U), \mathcal{X}_c(U)) = \{0\}$. Let $u \in \mathcal{X}_c(U)$. Since $\mathcal{X}_c(U)$ is perfect ([Ban97, Thm. 1.4.3]), there exist $N \in \mathbb{N}$ and $v_i, w_i \in \mathcal{X}_c(U)$ for $i \in \{1, \ldots, N\}$ s.t. $u = \sum_{i=1}^{N} [v_i, w_i]$. Then $\eta(u) = \sum_{i=1}^{N} \eta([v_i, w_i]) = -\sum_{i=1}^{N} \psi(v_i, w_i) = 0$. So η vanishes on $\mathcal{X}_c(U)$. So $x \notin \mathrm{supp}(\eta)$. It follows that $\mathrm{supp}(\eta) \subseteq \mathrm{supp}(\psi)$. Conversely, suppose that $x \notin \mathrm{supp}(\eta)$. Then there exists an open neighborhood U of x such that $\eta(\mathcal{X}_c(U)) = \{0\}$. Then $\psi = d_{\mathfrak{g}}\eta$ implies that $\psi(\mathcal{X}_c(U), \mathcal{X}_c(U)) = \{0\}$ as well. So $x \notin \mathrm{supp}(\psi)$. So $\mathrm{supp}(\eta) = \mathrm{supp}(\psi) =: K$. As η has compact support K, it admits a continuous linear extension $\widetilde{\eta}$ to $\mathcal{X}(M)$ by setting $\widetilde{\eta}(v) := \eta(\chi v)$ for any $\chi \in C^\infty_c(M)$ satisfying $\chi|_V = 1$ for some open neighborhood V of K. Notice that $\widetilde{\eta}$ is indeed well-defined and continuous.

3. Let $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ be a 2-cocycle and assume that $\eta \in \mathcal{X}_c(M)'$ satisfies $\psi(v, w) = -\eta([v, w])$ for all $v, w \in \mathcal{X}_c(M)$. The previous items ensure that η extends to a continuous functional on $\mathcal{X}(M)$. As $\mathcal{X}_c(M)$ is dense in $\mathcal{X}(M)$ and ψ is continuous on $\mathcal{X}(M) \times \mathcal{X}(M)$, it follows that $\psi(v, w) = -\eta([v, w])$ for all $v, w \in \mathcal{X}(M)$. So $\psi = d_{\mathfrak{g}}\eta$. Hence $[\psi] = 0$ in $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$. ☐

Proposition 5.2.11, Theorem 5.2.8 and Lemma 5.2.9 have the following consequence for Gelfand-Fuks cohomology:

Corollary 5.2.12.

1. *Assume that* $\dim(M) > 1$. *Then* $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = 0$.

2. *Assume that* $\dim(M) = 1$. *Then* $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = H^0_{dR,c}(M)$ *is the compactly supported de Rham cohomology of* M *in degree 0. In particular, we have* $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R}) = 0$.

Proof.

1. Assume that $\dim(M) > 1$. Then $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = 0$ by Theorem 5.2.1. Since linear map $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is injective, by Proposition 5.2.11, it follows that $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = 0$.

2. By reasoning similar to that in the proof of Theorem 5.2.10, it suffices to consider the case where M is connected, so that M is either S^1 or $\mathbb{R}$. Since $H^2_{\mathrm{ct}}(\mathcal{X}(S^1), \mathbb{R}) \cong \mathbb{R}$, it remains to show that $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R}) = 0$. By Lemma 5.2.9 we know that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R}) \cong \mathbb{R}$, which by Proposition 5.2.11 implies that $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R})$ is at most one-dimensional. The non-trivial class

in $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}),\mathbb{R})$ is spanned by the cocycle ψ_{vir}, defined by (5.2.9). Assume that $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}),\mathbb{R})$ is a 2-cocycle on $\mathcal{X}(\mathbb{R})$ whose restriction $r(\psi)$ to $\mathcal{X}_c(M) \times \mathcal{X}_c(M)$ is cohomologous to ψ_{vir}. Then $r(\psi) = \psi_{\mathrm{vir}} + d_{\mathfrak{g}}\eta$ for some $\eta \in \mathcal{X}_c(\mathbb{R})'$. By Proposition 5.2.11, we know that $r(\psi)$ has compact support. Consider the associated map $\widehat{r(\psi)} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}_c(\mathbb{R})'$. We saw in the proof of Lemma 5.2.9 that $\widehat{d_{\mathfrak{g}}\eta}(f\partial_x) = f(\partial_x \cdot \eta) + 2f'\eta$, and that $\hat{\psi}_{\mathrm{vir}}(f\partial_x) = f'''I$, where $I(f\partial_x) := \int_{\mathbb{R}} f(x)dx$. So $\widehat{r(\psi)}$ is the differential operator given by

$$\widehat{r(\psi)}(f\partial_x) = f'''I + f(\partial_x \cdot \eta) + 2f'\eta. \tag{5.2.10}$$

Since $\widehat{r(\psi)}(f\partial_x) \in \mathcal{X}_c(\mathbb{R})'$ has compact support for any $f \in C^\infty(\mathbb{R})$, we obtain by taking $f = 1$ in (5.2.10) that $\partial_x \cdot \eta$ has compact support. Choosing subsequently $f(x) = x$ in (5.2.10), it follows that η has compact support, and hence so does $\hat{\psi}_{\mathrm{vir}} = \widehat{r(\psi)} - \widehat{d_{\mathfrak{g}}\eta}$. But the support of $\hat{\psi}_{\mathrm{vir}}$ is all of $\mathbb{R}$, which is not compact, a clear contradiction. ☐

5.3 Generalized positive energy representations

Let M be a smooth manifold of dimension $\dim(M) > 1$. If $v \in \mathcal{X}(M)$ is a complete vector field on M with flow $h : \mathbb{R} \to \mathrm{Diff}(M)$, we write $\mathrm{Diff}_c(M) \rtimes_v \mathbb{R}$ for the semidirect product of $\mathrm{Diff}_c(M)$ and $\mathbb{R}$ relative to the smooth $\mathbb{R}$-action on $\mathrm{Diff}_c(M)$ defined by $\alpha_s(f) = h_s \circ f \circ h_s^{-1}$ for $s \in \mathbb{R}$ and $f \in \mathrm{Diff}_c(M)$. The corresponding Lie algebra is $\mathcal{X}_c(M) \rtimes \mathbb{R}v$, where v acts on $\mathcal{X}_c(M)$ by the derivation $Dw := [v, w]$.

The fact that $H^2_{\mathrm{ct}}(X_c(M),\mathbb{R})$ is trivial for $\dim(M) > 1$ puts severe restrictions on the class of projective unitary representations of $\mathcal{X}_c(M) \rtimes \mathbb{R}v$ that are of generalized positive energy at v. The following result is the crux of the matter, from which group-level analogues immediately follow:

Theorem 5.3.1. *Suppose that* $\dim(M) > 1$. *Let* $\overline{\pi} : \mathcal{X}_c(M) \to \mathfrak{pu}(\mathcal{D})$ *be a continuous projective unitary representation of* $\mathcal{X}_c(M)$ *on the complex pre-Hilbert space* $\mathcal{D}$. *Let* $\mathcal{C} \subseteq \mathcal{X}(M)$ *be a cone of complete vector fields on* M, *and define the open set*

$$U := \bigcup_{v \in \mathcal{C}} \{\, p \in M \;:\; v(p) \neq 0 \,\}.$$

Suppose that for every $v \in \mathcal{C}$ *the representation* $\overline{\pi}$ *extends to a continuous projective unitary representation of* $\mathcal{X}_c(M) \rtimes \mathbb{R}v$ *that is of generalized positive energy at* v. *Then* $\mathcal{X}_c(U) \subseteq \ker \overline{\pi}$.

Before proving Theorem 5.3.1, let us mention some important consequences:

Theorem 5.3.2. *Suppose that* M *is connected and that* $\dim(M) > 1$. *Consider a complete vector field* $v \in \mathcal{X}(M) \setminus \{0\}$ *on* M. *Let* $\overline{\rho} : \mathrm{Diff}_c(M) \rtimes_v \mathbb{R} \to \mathrm{PU}(\mathcal{H}_\rho)$ *be a smooth projective unitary representation that is of generalized positive energy at* v. *Then* $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

Proof. Since the derived representation $\overline{d\rho} : \mathcal{X}_c(M) \rtimes \mathbb{R}v \to \mathfrak{pu}(\mathcal{H}_\rho^\infty)$ is of generalized positive energy at v, it is so at every v' in the cone $\mathcal{C}$ generated by the adjoint orbit of v in $\mathcal{X}_c(M) \rtimes \mathbb{R}v$. Since v is non-zero, there exists some open subset $U_0 \subseteq M$ on which v is non-vanishing. Then $\mathrm{Ad}_f(v)$ is non-zero on $f(U_0)$ for every $f \in \mathrm{Diff}_c(M)$. Since M is connected, $\mathrm{Diff}_c(M)$ acts transitively on M (because all orbit are open, and therefore also closed). Hence $\bigcup_{v' \in \mathcal{C}} \{ p \in M \, : \, v'(p) \neq 0 \} = M$. We obtain using Theorem 5.3.1 that $\mathcal{X}_c(M) \subseteq \ker \overline{d\rho}$. Corollary 5.4.4 now implies that $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$. ☐

Corollary 5.3.3. *Suppose that M is connected and that $\dim(M) > 1$. Consider a complete vector field $v \in \mathcal{X}(M) \setminus \{0\}$ on M. Let $\overline{\rho} : \mathrm{Diff}_c(M) \rtimes_v \mathbb{R} \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation that is smoothly-KMS at v relative to $\mathrm{Diff}_c(M)$. Assume that the von Neumann algebra $\rho(\mathrm{Diff}_c(M))''$ is a factor. Then $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.*

Proof. Let $\rho : G \to \mathrm{U}(\mathcal{H}_\rho)$ be the lift of $\overline{\rho}$, where the Lie group G is a central $\mathbb{T}$-extension of $\mathrm{Diff}_c(M) \rtimes_v \mathbb{R}$. Let $H \subseteq G$ be the Lie subgroup covering $\mathrm{Diff}_c(M)$. Let $\mathfrak{h}$ and $\mathfrak{g}$ denote the Lie algebras of H and G, respectively. Let $\mathcal{N} := \rho(H)''$ be the von Neumann algebra generated by $\rho(H)$. As $\overline{\rho}$ is smoothly-KMS at v relative to $\mathrm{Diff}_c(M)$, there is some $\xi \in \mathfrak{g}$ covering v s.t. ρ is smoothly-KMS at $\xi \in \mathfrak{g}$ relative to H. Let $\phi \in \mathrm{KMS}(\rho, \xi, H)^\infty$ and let $\rho_\phi : H \rtimes \mathbb{R} \to \mathrm{U}(\mathcal{H}_\phi)$ be the associated unitary representation of $H \rtimes \mathbb{R}$ on the GNS-Hilbert space $\mathcal{H}_\phi$. According to Theorem 3.2.14, the representation ρ_ϕ on $\mathcal{H}_\phi$ is smooth and of generalized positive energy at $(0,1) \in \mathfrak{h} \rtimes \mathbb{R}$. It follows from Theorem 5.3.2 that $\rho_\phi(H_0) \subseteq \mathbb{T}\mathrm{id}_{\mathcal{H}_\phi}$, where H_0 denotes the identity component of H. Because the von Neumann algebra $\mathcal{N}$ is a factor, the GNS-representation $\mathcal{N} \to \mathcal{B}(\mathcal{H}_\phi)$ is injective (see e.g. [Nie23a, Rem. 5.3 items 1 and 3]). It follows that $\rho(H_0) \subseteq \mathbb{T}\mathrm{id}_{\mathcal{H}_\rho}$. Since H_0 covers $\mathrm{Diff}_c(M)_0$, this implies that $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$. ☐

Corollary 5.3.4. *Suppose that $\dim(M) > 1$. Let $\overline{\rho} : \mathrm{Diff}_c(M) \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation that is bounded, i.e., continuous w.r.t. the norm topology on $\mathrm{PU}(\mathcal{H}_\rho)$. Then $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.*

Proof. Let $\rho : G \to \mathrm{U}(\mathcal{H}_\rho)$ be the lift of $\overline{\rho}$, where G is a central $\mathbb{T}$-extension of $\mathrm{Diff}_c(M)$ with Lie algebra $\mathfrak{g}$. Let $p \in M$. Take $v \in \mathcal{X}_c(M)$ with $v(p) \neq 0$ and let $\xi \in \mathfrak{g}$ cover v. Notice that ρ is continuous w.r.t. the norm-topology on $\mathrm{U}(\mathcal{H}_\rho)$, so that the self-adjoint operator $-i \left.\frac{d}{dt}\right|_{t=0} \rho(\exp_G(t\xi))$ is bounded. It follows that $\overline{\rho}$ is of (generalized) positive energy at $v \in \mathcal{X}_c(M)$. Using Theorem 5.3.2, this implies that $v' \in \ker \overline{d\rho}$ for any $v' \in \mathcal{X}_c(M)$ for which $\mathrm{supp}(v')$ is contained in the connected component of p in M. As $p \in M$ was arbitrary, we find that $\mathcal{X}_c(M) \subseteq \ker \overline{d\rho}$. We conclude using Corollary 5.4.4 that $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$. ☐

We now proceed with the proof of Theorem 5.3.1.

5.3.1 Ideals of the Lie algebra of compactly supported vector fields

Let M be a smooth manifold. For any $x \in M$, let $I_x \subseteq \mathcal{X}_c(M)$ denote the closed ideal of vector fields that are flat at x. So $v \in I_x \iff j^\infty_x(v) = 0$ for $v \in \mathcal{X}_c(M)$. The proof of Theorem 5.3.1 uses a particular observation concerning ideals in $\mathcal{X}_c(M)$, namely Proposition 5.3.7 below.

Definition 5.3.5. If $J \subseteq \mathcal{X}_c(M)$ is an ideal, define its *hull* by

$$h(J) := \{ x \in M \, : \, v(x) = 0 \text{ for all } v \in J \} .$$

Remark 5.3.6. The set of maximal ideals in $\mathcal{X}_c(M)$ is given by $\{I_x : x \in M\}$ [SP54, Thm. 1] (cf. [Ban97, prop. 7.2.2] or [Jan16, Prop. 1]). Moreover, if $x \in M$ and $J \subseteq \mathcal{X}_c(M)$ is an ideal, then $x \in h(J)$ if and only if $j^\infty_x(v) = 0$ for all $v \in J$. Indeed, if $x \in h(J)$, then for any $w_1, \ldots, w_m \in \mathcal{X}_c(M)$ and $v \in J$ we have $\mathcal{L}_{w_1} \cdots \mathcal{L}_{w_m} v \in J$, as J is an ideal, and so $\big(\mathcal{L}_{w_1} \cdots \mathcal{L}_{w_m} v\big)(x) = 0$. Consequently $j^\infty_x(v) = 0$. We thus see that $h(J) = \{x \in M : J \subseteq I_x\}$ corresponds to the set of maximal ideals of $\mathcal{X}_c(M)$ containing J.

Proposition 5.3.7. *Let $J \subseteq \mathcal{X}_c(M)$ be an ideal and let $x \in M$. Then either $x \in h(J)$, or there is an open neighborhood $U \subseteq M$ of x such that*

$$\mathcal{X}_c(U) \subseteq [J, \mathcal{X}_c(M)] \subseteq J.$$

Proof. This is immediate from the proof of [Jan16, Lem. 2.1], which does not require the ideal $J \subseteq \mathcal{X}_c(M)$ to be maximal. □

Although $\mathcal{X}_c(M)$ is not simple, the following related result does hold true:

Corollary 5.3.8. *Assume that M is connected. Suppose that $J \subseteq \mathcal{X}_c(M)$ is an ideal that is stable, in the sense that $Ad_g(J) \subseteq J$ for all $g \in \mathrm{Diff}_c(M)$. Then either $J = \mathcal{X}_c(M)$ or $J = \{0\}$.*

Proof. That J is stable implies that its hull $h(J) \subseteq M$ is $\mathrm{Diff}_c(M)$-invariant. Since M is connected, $\mathrm{Diff}_c(M)$ acts transitively on M. It follows that either $h(J) = \emptyset$ or $h(J) = M$. Using a partition of unity argument, Proposition 5.3.7 implies that either $J = \mathcal{X}_c(M)$ or $J = \{0\}$. □

Remark 5.3.9. Suppose that M is connected. Let $\rho : \mathrm{Diff}_c(M) \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation. Let $\overline{d\rho} : \mathcal{X}_c(M) \to \mathfrak{pu}(\mathcal{H}^\infty_\rho)$ be its derived representation. Its kernel $J := \ker \overline{d\rho}$ is a closed ideal in $\mathcal{X}_c(M)$ that satisfies $\mathrm{Ad}_g(J) \subseteq J$ for all $g \in \mathrm{Diff}_c(M)$. So $\overline{d\rho}$ is either trivial or injective by Corollary 5.3.8.

5.3.2 The proof of Theorem 5.3.1

We now proceed with the proof of Theorem 5.3.1. Let $n := \dim(M) > 1$. We start with a lemma that concerns the local situation near a regular point of a vector field $v \in \mathcal{C}$. We thus consider the following setting:

Let $I \subseteq \mathbb{R}$ be an open interval containing zero. Let $U_0 \subseteq \mathbb{R}^{n-1}$ be an open subset that is diffeomorphic to $\mathbb{R}^{n-1}$. Define $U := I \times U_0$, which is then diffeomorphic to $\mathbb{R}^n$. We consider the locally convex Lie algebra $\mathcal{X}_c(U)$ of compactly supported smooth vector fields on U. We write $(t, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n$ for the coordinates on $\mathbb{R}^n$, and $(\partial_t, \partial_{x_1}, \ldots, \partial_{x_{n-1}})$ for the corresponding basis of $\mathcal{X}(\mathbb{R}^n)$ over $C^\infty(\mathbb{R}^n)$. Notice that the derivation $[\partial_t, -]$ on $\mathcal{X}_c(U)$ does not necessarily integrate to a 1-parameter group of automorphisms of $\mathcal{X}_c(U)$, because the open set U need not be invariant under the flow of ∂_t.

Lemma 5.3.10. *Let $\overline{\pi} : \mathcal{X}_c(U) \rtimes \mathbb{R}\partial_t \to \mathfrak{pu}(\mathcal{D})$ be a continuous projective unitary representation on the pre-Hilbert space $\mathcal{D}$. Assume that*

$$[v, Dv] = 0 \implies \overline{\pi}(Dv) = 0, \qquad \forall v \in \mathcal{X}_c(U). \tag{5.3.1}$$

Then $\mathcal{X}_c(U) \subseteq \ker \overline{\pi}$.

Proof. Let $p_0 = (t_0, x_0) \in U = I \times U_0$ be arbitrary. Let $f \in C_c^\infty(I)$ and $w \in \mathcal{X}_c(U_0)$ be s.t. $f'(t_0) \neq 0$ and $w(x_0) \neq 0$. Define $v \in \mathcal{X}_c(U)$ by $v(t,x) := f(t)w(x)$ for $t \in I$ and $x \in U_0$. Observe that $Dv(t,x) = f'(t)w(x)$. In particular, $Dv(p_0) \neq 0$ and $[v, Dv](t,x) = f(t)f'(t)[w,w](x) = 0$. It follows using (5.3.1) that $Dv \in \ker \overline{\pi}$. Let $J \subseteq \mathcal{X}_c(U)$ be the closed ideal generated by Dv. Since $\overline{\pi}$ is a strongly continuous homomorphism of Lie algebras, we have $J \subseteq \ker \overline{\pi}$. As $Dv(p_0) \neq 0$, it follows using Proposition 5.3.7 that $\mathcal{X}_c(V) \subseteq J$ for some open neighborhood $V \subseteq U$ of p_0. So we have $\mathcal{X}_c(V) \subseteq \ker \overline{\pi}$. We have thus shown that any $p \in U$ has a neighborhood $V \subseteq U$ for which $\mathcal{X}_c(V) \subseteq \ker \overline{\pi}$. Consequently, if $K \subseteq U$ is a compact subset, we can find a finite open cover $\{U_1, \ldots, U_m\}$ of K with $\mathcal{X}_c(U_k) \subseteq \ker \overline{d\rho}$ for all $k \in \{1, \ldots, m\}$. Using a partition of unity argument, it follows that $\mathcal{X}_K(U) \subseteq \ker \overline{d\rho}$ for any compact set $K \subseteq M$, so that $\mathcal{X}_c(M) \subseteq \ker \overline{d\rho}$. □

We now return to the setting of Theorem 5.3.1.

Proof of Theorem 5.3.1:
Let $p \in U$ and let $v \in \mathcal{V}$ satisfy $v(p) \neq 0$. By assumption, $\overline{\pi}$ extends to a continuous projective unitary representation of $\mathcal{X}_c(M) \rtimes \mathbb{R}v$ that is of generalized positive energy at v, again denoted $\overline{\pi}$. Since $v(p) \neq 0$, we can find an open neighborhood $U_p \subseteq M$ of p, an open interval $I \subseteq \mathbb{R}$ containing zero, an open subset $U_0 \subseteq \mathbb{R}^{n-1}$ that is diffeomorphic to $\mathbb{R}^{n-1}$, and a diffeomorphism $\phi : I \times U_0 \to U_p$ such that $\phi_*([\partial_t, w]) = [v, \phi_*(w)]$ for all $w \in \mathcal{X}_c(I \times U_0)$ [Lee13, Thm. 9.22]. So ϕ_* defines an isomorphism

$$\phi_* : \mathcal{X}_c(I \times U_0) \rtimes \mathbb{R}\partial_t \to \mathcal{X}_c(U_p) \rtimes \mathbb{R}v.$$

In view of Theorem 5.2.8, we know that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = 0$. As $\overline{\pi}$ is of generalized positive energy at v, it follows using Corollary 3.1.8 that $[w, Dw] = 0$ implies

$\overline{\pi}(Dw) = 0$ for any $w \in \mathcal{X}_c(M)$. As a consequence, the pull-back of $\overline{\pi}$ along the composition

$$\mathcal{X}_c(I \times U_0) \rtimes \mathbb{R}\partial_t \xrightarrow{\phi_*} \mathcal{X}_c(U_p) \rtimes \mathbb{R}v \hookrightarrow \mathcal{X}_c(M) \rtimes \mathbb{R}v$$

satisfies the conditions of Lemma 5.3.10, from which it subsequently follows that $\mathcal{X}_c(U_p) \subseteq \ker \overline{\pi}$. So any $p \in U$ has an open neighborhood $U_p \subseteq M$ satisfying $\mathcal{X}_c(U_p) \subseteq \ker \overline{\pi}$. This implies that $\mathcal{X}_c(U) \subseteq \ker \overline{\pi}$. □

5.4 Appendix

5.4.1 Sheaves of distributions

Let $E \to M$ be a smooth vector bundle over the smooth manifold M. If $U \subseteq M$ is an open subset, we denote by $\Gamma_c(U, E)$ the locally convex vector space of smooth compactly supported sections of $E|_U \to U$, equipped with the natural LF-topology. Let $\Gamma_c(U, E)'$ denote its continuous dual space. It is clear that the assignment $U \mapsto \Gamma_c(U, E)'$ defines a presheaf Γ_c' w.r.t. the natural restriction maps. In the following, we show that Γ_c' actually defines an acyclic sheaf. Since we are considering the continuous dual space, we have to slightly extend the usual sheaf-theoretic arguments (such as [Bre97, §V.1 Prop. 1.6 and 1.10]).

Proposition 5.4.1. *Γ_c' is an acyclic sheaf.*

Proof. Let $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of open subsets of M and define $U := \bigcup_{\alpha \in \mathcal{I}} U_\alpha$. Let $\{\chi_\alpha\}_{\alpha \in \mathcal{I}}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of U [Lee13, Thm. 2.23]. Notice for $s \in \Gamma_c(U, E)$ that $\chi_\alpha s$ is non-zero for only finitely many $\alpha \in \mathcal{I}$, because $\{\operatorname{supp} \chi_\alpha\}_{\alpha \in \mathcal{I}}$ is locally finite. To see that Γ_c' satisfies the locality axiom, suppose that $\lambda \in \Gamma_c(U, E)'$ satisfies $\lambda_\alpha := \lambda|_{\Gamma_c(U_\alpha, E)} = 0$ for all $\alpha \in \mathcal{I}$. Then $\lambda(s) = \sum_{\alpha \in \mathcal{I}} \lambda_\alpha(\chi_\alpha s) = 0$ for any $s \in \Gamma_c(U, E)$, so $\lambda = 0$. For the gluing axiom, take $\lambda_\alpha \in \Gamma_c(U_\alpha, E)'$ for all $\alpha \in \mathcal{I}$ and suppose for any $\alpha, \beta \in \mathcal{I}$ that the restrictions of λ_α and λ_β to $\Gamma_c(U_\alpha \cap U_\beta, E)$ coincide whenever $U_\alpha \cap U_\beta \neq 0$. Define $\lambda \in \Gamma_c(U, E)'$ by $\lambda(s) := \sum_{\alpha \in \mathcal{I}} \lambda_\alpha(\chi_\alpha s)$ for $s \in \Gamma_c(U, E)$. Notice that λ does indeed define a continuous functional on the LF-space $\Gamma_c(U, E)$ because $\{\operatorname{supp} \chi_\alpha\}_{\alpha \in \mathcal{I}}$ is locally finite. If $s \in \Gamma_c(U_\beta, E)$ for some $\beta \in \mathcal{I}$, then $\chi_\alpha s \in \Gamma_c(U_\alpha \cap U_\beta, E)$ and consequently $\lambda_\alpha(\chi_\alpha s) = \lambda_\beta(\chi_\alpha s)$ for any $\alpha \in \mathcal{I}$. Hence $\lambda(s) = \sum_\alpha \lambda_\alpha(\chi_\alpha s) = \sum_\alpha \lambda_\beta(\chi_\alpha s) = \lambda_\beta(s)$. So $\lambda|_{\Gamma_c(U_\beta, E)} = \lambda_\beta$ for any $\beta \in \mathcal{I}$. It follows that Γ_c' is a sheaf. We show next that it is fine (cf. [Wel80, Def. II.3.3]). Assume henceforth that $U = M$. Define for any open set $V \subseteq M$ and $\alpha \in \mathcal{I}$ the linear map $\eta_\alpha : \Gamma_c(V, E)' \to \Gamma_c(V, E)'$ by $\eta_\alpha(\lambda)(s) := \lambda(\chi_\alpha s)$. This defines a morphism $\eta_\alpha : \Gamma_c' \to \Gamma_c'$ of sheaves. Since $\sum_\alpha \eta_\alpha(\lambda)(s) = \sum_\alpha \lambda(\chi_\alpha s) = \lambda(s)$ for any $s \in \Gamma_c(V, E)'$, the sum being finite, we have $\sum_\alpha \eta_\alpha = 1$. Additionally, η_α vanishes on the stalk of the sheaf Γ_c' at x for any x in the open neighborhood $M \setminus \operatorname{supp} \chi_\alpha$ of $M \setminus U_\alpha$. So Γ_c' is fine and therefore acyclic [Wel80, II. Prop. 3.5 and Thm. 3.11]. □

5.4.2 Unitary equivalence of projective representations

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp_G : \mathfrak{g} \to G$.

Definition 5.4.2. Suppose for $k \in \{1,2\}$ that $\mathcal{D}_k$ is a complex pre-Hilbert space, and let $\overline{\pi}_k : \mathfrak{g} \to \mathfrak{pu}(\mathcal{D}_k)$ be a projective unitary representation of $\mathfrak{g}$ on $\mathcal{D}_k$. We say that $\overline{\pi}_1$ and $\overline{\pi}_2$ are unitarily equivalent if there is a unitary operator $U : \mathcal{D}_1 \to \mathcal{D}_2$ such that $\overline{\pi}_2(\xi) = \overline{U}\,\overline{\pi}_1(\xi)\,\overline{U}^{-1}$ for all $\xi \in \mathfrak{g}$, where $\overline{U} : \mathrm{P}(\mathcal{D}_1) \to \mathrm{P}(\mathcal{D}_2)$ is the descent of U to the projective spaces. In this case, we write $\overline{\pi}_1 \cong \overline{\pi}_2$.

The following is the projective analogue of [JN19, Prop. 3.4]:

Proposition 5.4.3. *Assume that G is connected. For $k \in \{1,2\}$, let $(\overline{\rho}_k, \mathcal{H}_{\rho_k})$ be a smooth projective unitary representation of G with derived representation $\overline{d\rho_k} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{H}^\infty_{\rho_k})$ on $\mathcal{H}^\infty_{\rho_k}$. Then*

$$\overline{\rho}_1 \cong \overline{\rho}_2 \qquad \Longleftrightarrow \qquad \overline{d\rho}_1 \cong \overline{d\rho}_2.$$

Proof. Passing to the universal cover of G, which is a Lie group by [Nee06, Cor. II.2.4], we may and do assume that G is 1-connected. Let $U : \mathcal{H}_{\rho_1} \to \mathcal{H}_{\rho_2}$ be a unitary map, and let $\overline{U} : \mathrm{P}(\mathcal{H}_{\rho_1}) \to \mathrm{P}(\mathcal{H}_{\rho_2})$ be its descent to the projective spaces. Assume first that $\overline{U}\overline{\rho}_1(g)\overline{U}^{-1} = \overline{\rho}_2(g)$ for all $g \in G$. Then by [JN19, Cor. 4.5], we know that $\overline{\rho}_1$ and $\overline{\rho}_2$ correspond to the same central $\mathbb{T}$-extension $\mathring{G}$ of G, up to isomorphism of central extensions. Let $\mathring{\mathfrak{g}}$ denote the Lie algebra of $\mathring{G}$. Let the smooth unitary $\mathring{G}$-representations ρ_1 and ρ_2 be lifts of $\overline{\rho}_1$ and $\overline{\rho}_2$ respectively. Then there exists a smooth character $\zeta : \mathring{G} \to \mathrm{U}(1)$ such that $\rho_2(\mathring{g}) = \zeta(\mathring{g})U\rho_1(\mathring{g})U^{-1}$ for all $\mathring{g} \in \mathring{G}$. This implies in particular that $U\mathcal{H}^\infty_{\rho_1} = \mathcal{H}^\infty_{\rho_2}$. Differentiating the preceding equation at the identity of $\mathring{G}$, it also follows that $d\rho_2(\mathring{\xi}) = Ud\rho_1(\mathring{\xi})U^{-1} + d\zeta(\mathring{\xi})I$ for all $\mathring{\xi} \in \mathring{\mathfrak{g}}$, where I denotes the identity on $\mathcal{H}^\infty_{\rho_2}$. Hence $\overline{d\rho}_2(\xi)[\psi] = \overline{U}\,\overline{d\rho}_1(\xi)\overline{U}^{-1}[\psi]$ for all $\xi \in \mathfrak{g}$ and $[\psi] \in \mathrm{P}(\mathcal{H}^\infty_{\rho_2})$. So $\overline{d\rho}_1 \cong \overline{d\rho}_2$.

Assume conversely that $U\mathcal{H}^\infty_{\rho_1} = \mathcal{H}^\infty_{\rho_2}$ and that $\overline{U}\,\overline{d\rho}_1(\xi)\,\overline{U}^{-1}[\psi] = \overline{d\rho}_2(\xi)[\psi]$ for all $\xi \in \mathfrak{g}$ and $[\psi] \in \mathrm{P}(\mathcal{H}^\infty_{\rho_2})$. This implies that $\overline{d\rho}_1$ and $\overline{d\rho}_2$ induce isomorphic central $\mathbb{R}$-extension of $\mathfrak{g}$, up to isomorphism. Since G is 1-connected, it follows using [Nee02, Cor. 7.5(i)] that $\overline{\rho}_1$ and $\overline{\rho}_2$ induce the same central $\mathbb{T}$-extension $\mathring{G}$ of G, up to isomorphism. Let the smooth unitary $\mathring{G}$-representations ρ_1 and ρ_2 once again be lifts of $\overline{\rho}_1$ and $\overline{\rho}_2$, respectively. As $\overline{U}$ is equivariant w.r.t. the projective $\mathfrak{g}$-actions $\overline{d\rho}_1$ and $\overline{d\rho}_2$, there exists a continuous linear map $\lambda : \mathring{\mathfrak{g}} \to \mathbb{R}$ such that

$$d\rho_2(\mathring{\xi}) = Ud\rho_1(\mathring{\xi})U^{-1} + i\lambda(\mathring{\xi})I, \qquad \forall \mathring{\xi} \in \mathring{\mathfrak{g}}, \tag{5.4.1}$$

where I denotes the identity on $\mathcal{H}^\infty_{\rho_2}$. Let $\psi \in \mathcal{H}^\infty_{\rho_1}$ and $\chi \perp \psi$. Let $\gamma : \mathbb{R} \to \mathring{G}$ be a smooth path in $\mathring{G}$ with $\gamma_0 = 1$ the identity of $\mathring{G}$, and let $\overline{\gamma} : \mathbb{R} \to G$ be its projection to G. Consider the smooth function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(t) := \langle U\chi, \rho_2(\gamma_t)U\rho_1(\gamma_t)^{-1}\psi\rangle.$$

Notice that $f(0) = 0$. Let $\gamma' : \mathbb{R} \to \mathring{\mathfrak{g}}$ be the left-logarithmic derivative of γ, defined by $\gamma'_s := \frac{d}{dt}\big|_{t=s} \gamma_s^{-1}\gamma_t$ for $s \in \mathbb{R}$. Observe using equation (5.4.1) that the derivative f' of f satisfies

$$\begin{aligned} f'(s) &= \langle U\chi, \rho_2(\gamma_s) d\rho_2(\gamma'_s) U \rho_1(\gamma_s)^{-1}\psi\rangle - \langle U\chi, \rho_2(\gamma_s) U d\rho_1(\gamma'_s)\rho_1(\gamma_s)^{-1}\psi\rangle \\ &= i\lambda(\gamma'_s)\langle U\chi, \rho_2(\gamma_s) U \rho_1(\gamma_s)^{-1}\psi\rangle \\ &= i\lambda(\gamma'_s) f(s), \qquad \forall s \in \mathbb{R}. \end{aligned}$$

The unique solution of the ODE $f'(s) = i\lambda(\gamma'_s)f(s)$ with initial condition $f(0) = 0$ is given by $f(t) = 0$ for all $t \in \mathbb{R}$. So $\langle U\chi, \rho_2(\gamma_t)U\rho_1(\gamma_t)^{-1}\psi\rangle = 0$ for every $\chi \perp \psi$ and $t \in \mathbb{R}$. Thus $[\rho_2(\gamma_t)U\rho_1(\gamma_t)^{-1}\psi] = [U\psi]$ for all $t \in \mathbb{R}$ and $\psi \in \mathcal{H}^\infty_{\rho_1}$. Hence $\overline{U}\,\overline{\rho}_1(\overline{\gamma}_t) = \overline{\rho}_2(\overline{\gamma}_t)\,\overline{U}$ for all $t \in \mathbb{R}$. Since $\mathring{G}$ is a path-connected principal $\mathbb{T}$-bundle over G, it follows that $\overline{U}\,\overline{\rho}_1(g) = \overline{\rho}_2(g)\,\overline{U}$ for all $g \in G$. Thus $\overline{\rho}_1 \cong \overline{\rho}_2$. ☐

Corollary 5.4.4. *Let $\overline{\rho} : G \to \mathrm{PU}(\mathcal{H}_\rho)$ be a smooth projective unitary representation with derived representation $\overline{d\rho} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{H}^\infty_\rho)$. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$, and let $f : H \to G$ be smooth homomorphism of Lie groups. If $T_e(f)(\mathfrak{h}) \subseteq \ker \overline{d\rho}$, then $f(H) \subseteq \ker \overline{\rho}$.*

Proof. By considering the pull-back of $\overline{\rho}$ along f, it suffices to consider the case where $H = G$ and $f = \mathrm{id}_G$. Thus, assume that G is connected and that $\mathfrak{g} \subseteq \ker \overline{d\rho}$. Then Proposition 5.4.3 implies that $G \subseteq \ker \overline{\rho}$. ☐

Chapter 6

Holomorphic induction beyond the norm-continuous setting, with applications to positive energy representations

Abstract

We extend the theory of holomorphic induction of unitary representations of a possibly infinite-dimensional Lie group G beyond the setting where the representation being induced is required to be norm-continuous. We allow the group G to be a connected BCH(Baker-Campbell-Hausdorff) Fréchet-Lie group. Given a smooth $\mathbb{R}$-action α on G, we proceed to show that the corresponding class of so-called positive energy representations is intimately related with holomorphic induction. Assuming that G is regular, we in particular show that if ρ is a unitary ground state representation of $G \rtimes_\alpha \mathbb{R}$ for which the energy-zero subspace $\mathcal{H}_\rho(0)$ admits a dense set of G-analytic vectors, then $\rho|_G$ is holomorphically induced from the representation of the connected subgroup $H := (G^\alpha)_0$ of α-fixed points on $\mathcal{H}_\rho(0)$. As a consequence, we obtain an isomorphism $\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(\mathcal{H}_\rho(0))^H$ between the corresponding commutants. We also find that any two such ground state representations are necessarily unitary equivalent if their energy-zero subspaces are unitarily equivalent as H-representations. These results were previously only available under the assumption of norm-continuity of the H-representation on $\mathcal{H}_\rho(0)$.

This chapter is based on [Nie23b].

6.1 Introduction

This chapter is concerned with unitary representations of a possibly infinite-dimensional connected Lie group G that is modeled on a locally convex vector space. Given a smooth action α of $\mathbb{R}$ on G, we consider those G-representations that extend to a unitary representation ρ of $G \rtimes_\alpha \mathbb{R}$ which is *smooth*, in the sense that it admits a dense set of smooth vectors, and which is of *positive energy*, meaning that the self-adjoint generator $-i \frac{d}{dt}\big|_{t=0} \rho(1_G, t)$ of the unitary 1-parameter group $t \mapsto \rho(1_G, t)$

has non-negative spectrum.

For infinite-dimensional Lie groups, a full classification of all irreducible representations is typically not tractable, and even less so for factor representations. The positive energy condition serves to isolate a class of representations that are more susceptible to systematic study. It is also quite natural from a physical perspective, because the Hamiltonian in quantum physics is nearly always required to be a positive self-adjoint operator. It is then no surprise that positive energy representations of Lie groups are abundant in physics literature [SW64, Bor87, Bor66, Haa92, LM75, Ol'81, PS86, Seg81].

Holomorphic induction has proven to be a particularly effective tool in the study of positive energy representations. Let us first describe the main idea of holomorphic induction in the case where G is finite-dimensional. Let $H := (G^{\alpha})_0$ be the connected subgroup of α-fixed points in G, with Lie algebra $\mathfrak{h} = \mathbf{L}(H)$. A unitary G-representation ρ is typically called holomorphically induced from the unitary H-representation σ on V_σ if the homogeneous Hermitian vector bundle $\mathbb{V} := G \times_H V_\sigma$ over G/H can be equipped with a G-invariant complex-analytic bundle structure, with respect to which the Hilbert space $\mathcal{H}_\rho$ can be G-equivariantly embedded into the space of holomorphic sections $\mathcal{O}(G/H, \mathbb{V})$ of $\mathbb{V}$, in such a way that the corresponding point evaluations $\mathcal{E}_x : \mathcal{H}_\rho \to \mathbb{V}_x$ are continuous and satisfy $\mathcal{E}_x\mathcal{E}_x^* = \mathrm{id}_{\mathbb{V}_x}$ for every $x \in G/H$. In particular, these conditions imply that $\mathcal{H}_\rho$ is unitarily equivalent to the G-representation on a reproducing kernel Hilbert space, and that $\mathcal{H}_\rho$ contains V_σ as an H-subrepresentation.

An important special case is obtained when V_σ is one-dimensional. If ρ is holomorphically induced from σ, we may identify V_σ with a cyclic ray $[v_0]$ in $\mathcal{H}_\rho$, whose G-orbit in the projective space $\mathrm{P}(\mathcal{H}_\rho)$ is a complex submanifold. This means that ρ is a so-called *coherent state representation* [Nee00, Def. XV.2.1]. In this case, the G-homogeneous line bundle $\mathbb{V}$ is the pull-back of the tautological line bundle over $\mathrm{P}(\mathcal{H}_\rho)$ along the map $G/H \to \mathrm{P}(\mathcal{H}_\rho)$, $gH \mapsto [\rho(g)v_0]$, and elements in the image of the corresponding map $\mathbb{V} \to \mathcal{H}_\rho$ are usually called *coherent states*. This is also the setting of the well-known Borel-Weil Theorem [DK00, Thm. 4.12.5]. Such representations have been studied extensively [Per86, Nee00, Lis95], and are known to be tightly related to highest-weight representations [Nee00, Def. X.2.9, Ch. XV]. In particular, every unitary highest weight representation of G is a coherent state representation [Nee00, Prop. XV.2.6]. The converse is not true. The Schrödinger representation of the Heisenberg group $\mathrm{Heis}(\mathbb{R}^2, \omega)$ provides a counterexample [Nee00, Ex. XV.3.5].

Holomorphic induction, defined as above, was studied in [Nee13] in the context where G is a Banach-Lie group and where σ is *bounded*, meaning that it is continuous with respect to the norm-topology on $\mathcal{B}(V_\sigma)$. Writing $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{g}_\mathbb{C}$ for its complexification, invariant complex structures on G/H correspond to closed Lie subalgebras $\mathfrak{b} \subseteq \mathfrak{g}_\mathbb{C}$ satisfying $\mathfrak{b} + \overline{\mathfrak{b}} = \mathfrak{g}_\mathbb{C}$, $\mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{h}_\mathbb{C}$ and $\mathrm{Ad}_h(\mathfrak{b}) \subseteq \mathfrak{b}$ for all $h \in H$ [Bel05, Thm. 15] (cf. [Kir76, p. 203] for the case where G is finite dimensional). The corresponding G-invariant holomorphic bundle structures on $\mathbb{V}$ then turn out to be parametrized by extensions of $d\sigma : \mathfrak{h} \to \mathcal{B}(V_\sigma)$ to a Lie algebra homomorphism $\chi : \mathfrak{b} \to \mathcal{B}(V_\sigma)$ satisfying $\chi(\mathrm{Ad}_h(\xi)) = \sigma(h)\chi(\xi)\sigma(h)^{-1}$ for all $\xi \in \mathfrak{b}$ and $h \in H$ [Nee13, Thm. 2.6], as is to be expected from the finite-dimensional setting [TW71, Thm. 3.6]. The holomorphic structure is used to relate various important properties of the G-representation ρ with those of σ. For example, [Nee13, Thm. 3.12] entails that the commutants $\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(V_\sigma)^{H,\chi}$ are isomorphic as von Neumann algebras if $V_\sigma \subseteq \mathcal{H}_\rho$ is invariant under $\mathcal{B}(\mathcal{H}_\rho)^G$, which implies in particular that ρ is irreducible, multiplicity-free or of type I, II or III if and only if this is true for σ [Nee13, Cor. 3.14]. Moreover, [Nee13, Cor. 3.16] states that there is up to unitary equivalence at most one unitary G-representation ρ that is holomorphically induced from a given pair (σ, χ). The relation between holomorphically induced representations and the positive energy condition is then explained by [Nee13, Thm. 4.12, 4.14], which essentially state that in the above context, and under suitable assumptions, holomorphically induced representations correspond to so-called *semibounded* ones, the semiboundedness condition being a 'stable' and stronger version of the positive energy condition (cf. [Nee10b]). These observations suggest that the class of holomorphically induced representations may well admit a fruitful classification theory of its factor representations. This line of reasoning was pursued in [Nee14a, Thm. 5.4, 5.10] and [Nee12, Thm. 6.1, 7.3, 8.1], resulting in a classification of the irreducible semibounded unitary representations of certain double extensions of Hilbert Loop groups and of hermitian Lie groups corresponding to infinite-dimensional irreducible symmetric spaces.

In [Nee14a, Appendix C], the theory of holomorphic induction was further developed, allowing G to be a connected BCH Fréchet-Lie group, under certain additional assumptions. (Recall that G is BCH if it is real-analytic and has an analytic exponential map which is a local diffeomorphism in $0 \in \mathfrak{g}$.) Still, σ was required to be norm-continuous. Let us mention that a particular and well-known special case of such a situation had already appeared in the study of smooth positive energy representations of loop groups. In fact, these had been completely classified using holomorphic induction [PS86] (cf. [Nee01a]).

Still, the assumption of norm-continuity of σ is too restrictive in numerous examples, some of which we encounter in Section 6.8 below. It is typically only suitable for describing the class of semibounded unitary representations of G. In order to obtain a theory that can be used to describe the possibly larger class of all positive energy representations, one must go beyond the norm-continuity of σ.

The purpose of this chapter is to remove this assumption of norm-continuity of the representation σ being induced, whilst still allowing G to be a connected BCH Fréchet-Lie group. A main difficulty in this direction is that of equipping the homogeneous vector bundle $G \times_H V_\sigma$ with a G-invariant complex-analytic bundle structure. The proof of [Nee13, Thm. 2.6] breaks down beyond the norm-continuous setting, so a new approach is required.

We provide two possible solutions to this problem. As in [Nee14a, Appendix C], we assume that $\mathfrak{g}_\mathbb{C}$ admits a triangular decomposition of the form $\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm$ and $\mathfrak{h}_\mathbb{C}$ are closed Lie subalgebras of $\mathfrak{g}_\mathbb{C}$ satisfying $\overline{\mathfrak{n}_\pm} \subseteq \mathfrak{n}_\mp$, and where $\mathfrak{b} = \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_-$. In the first, which we call the *general approach*, we avoid specifying a complex-analytic vector bundle altogether. Instead we replace the space of holomorphic sections by a suitable subspace $C^\omega(G, V_\sigma)^{H,\chi}$ of the space of real-analytic H-equivariant maps $C^\omega(G, V_\sigma)^H$, defined directly in terms of an extension $\chi : \mathfrak{b} \to \mathcal{L}(\mathcal{D})$ of $d\sigma$ to $\mathfrak{b}$ with some domain $\mathcal{D} \subseteq V_\sigma^\omega$ consisting of analytic vectors. This also avoids the need for a G-invariant complex structure on the homogeneous space G/H. In the second, which we call the *geometric approach*, we define a stronger notion of holomorphic induction. In this case, $\mathcal{H}_\rho^\infty$ actually embeds into a space of holomorphic mappings on a homogeneous vector bundle. It therefore requires complex geometry. A significant drawback of this approach is that it requires a dense set of so-called b-strongly-entire vectors, whose availability is usually not known, unless G happens to be finite-dimensional, in which case it is completely understood by the results of [Goo69] and [Pen74], see also Theorem 6.3.6 below.

Let us also mention that the results presented in this chapter do not complete the story of holomorphic induction. The developed theory still excludes Fréchet-Lie groups that are not BCH, such as the Virasoro group. Yet, it is known that holomorphic induction can be used to obtain a complete classification of the positive energy representations of the Virasoro group [NS15]. Nevertheless, the present chapter makes substantial progress towards a more complete understanding of holomorphic induction in the infinite-dimensional context. In relation to positive energy representations, progress was made in a different direction in [NR22], where the class of ground state representations is studied in the setting of topological groups.

Structure of the chapter

— In Section 6.2, we first recall some preliminaries regarding analytic functions on locally convex spaces. We proceed to define smooth, analytic and strongly-entire representations, which are increasingly regular. We also recall some important results related to positive energy and ground state representations.

— We proceed in Section 6.3 to define and study the spaces $\mathcal{H}_\rho^\mathcal{O}$ and $\mathcal{H}_\rho^{\mathcal{O}_b}$ of so-called strongly-entire and b-strongly-entire vectors, respectively. We equip these spaces with a locally convex topology, and extend the results of [Goo69] from the setting of finite-dimensional Lie groups to the present one, where G is allowed to be infinite-dimensional. In particular, if $G_\mathbb{C}$ is a complex 1-connected regular BCH Fréchet-Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$, we obtain that both $\mathcal{H}_\rho^\mathcal{O}$ and $\mathcal{H}_\rho^{\mathcal{O}_b}$ carry a representation of $G_\mathbb{C}$ that has holomorphic orbit maps. The space $\mathcal{H}_\rho^{\mathcal{O}_b}$ plays an important role in the geometric approach to holomorphic induction.

— In Section 6.4.2 we present the general approach towards holomorphic induction. After determining a useful equivalent formulation, we characterize the inducibility of pairs (σ, χ) in terms of positive definite functions on G, which leads to the uniqueness of the holomorphically induced representation up to unitary equivalence. We then proceed to show that there is an isomorphism of von Neumann algebras $\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(V_\sigma)^{H,\chi}$ between the commutants, provided that $V_\sigma \subseteq \mathcal{H}_\rho$ is invariant under $\mathcal{B}(\mathcal{H}_\rho)^G$, in complete analogy with the previously described norm-continuous setting. We also briefly discuss holomorphic induction in stages.

— After equipping the G-homogeneous vector bundle $\mathbb{V}_\sigma := G \times_H V_\sigma^{\mathcal{O}_b}$ with a complex-analytic bundle structure, using a suitable extension χ of $d\sigma$ with domain $V_\sigma^{\mathcal{O}_b}$ and under certain assumptions, we define in Section 6.5.4 the geometric notion of holomorphically induced representations. We also compare the geometric notion to the one presented in Section 6.4.2.

— In relating holomorphic induction with the positive energy condition, we shall have need for a suitably general notion of Arveson spectral subspaces. We therefore generalize in Section 6.6 the results of [NSZ15, Sec. A.3] and [Nee13, Sec. A.2] to the level of generality needed in the next section.

— In Section 6.7 we study the relation between holomorphic induction and the positive energy condition, under the additional assumption that G is regular. In particular, we show that if ρ is a unitary ground state representation of $G \rtimes_\alpha \mathbb{R}$ for which the energy-zero subspace $\mathcal{H}_\rho(0)$ admits a dense set of G-analytic vectors, then $\rho|_G$ is holomorphically induced from the H-representation on $\mathcal{H}_\rho(0)$. As a consequence, we obtain an isomorphism $\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(\mathcal{H}_\rho(0))^H$ of von Neumann algebras between the corresponding commutants. We also find that any two such ground state representations are necessarily unitary equivalent if their energy-zero subspaces are unitarily equivalent as H-representations.

— In Section 6.8, we consider numerous interesting examples of unitary representations that are holomorphically induced from representations that are not norm-continuous.

6.2 Preliminaries

6.2.1 Analytic functions on locally convex vector spaces

Let us recall some definitions and properties of analytic functions between locally convex vector spaces. The main references are [BS71b], [BS71a] and [Glö02b]. Throughout the following, fix locally convex vector spaces E and F over the field $\mathbb{K}$ that both are complete and Hausdorff, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Define

$$\Delta_k : E \to E^k, \ \Delta_k(h) = (h, \dots, h).$$

Homogeneous polynomials

Definition 6.2.1. Suppose $U \subseteq E$ is open and $f \in C^\infty(U, F)$. For any $x \in U$ and $k \in \mathbb{N}$, define $\delta_x^0(f) : E \to F$ and $\delta_x^k(f) : E \to F$ by $\delta_x^0(f)(v) := f(x)$ and $\delta_x^k(f)(v) := d^k f(x; v, \ldots, v)$.

Definition 6.2.2. Let $k \in \mathbb{N}$. A map $f : E \to F$ is called a *homogeneous polynomial of degree* k if there exists a k-linear symmetric map $\widetilde{f} : E^k \to F$ such that $f = \widetilde{f} \circ \Delta_k$. Let $P^k(E, F)$ denote the space of continuous homogeneous polynomials $E \to F$ of degree k. For $k = 0$, we set $P^0(E, F) := F$.

Set $E^0 := \mathbb{K}$. For $k \in \mathbb{N}_{\geq 0}$, we write $\mathrm{Mult}(E^k, F)$ for the space of continuous k-linear maps $E^k \to F$, equipped with the topology of uniform convergence on products of compact sets in E. For the case $k = 1$, we also write $\mathcal{B}(E, F) := \mathrm{Mult}(E, F)$. Let $\mathrm{Sym}^k(E, F) \subseteq \mathrm{Mult}(E^k, F)$ denote the closed subspace of continuous symmetric k-linear maps $E^k \to F$. Let $E \widehat{\otimes} F$ denote the completed projective tensor product of E and F [Tre67, Def. 43.2, 43.5]. Define $E^{\widehat{\otimes} k} := E \widehat{\otimes} \cdots \widehat{\otimes} E$ (k times). The topology on $E^{\widehat{\otimes} k}$ is defined by the seminorms $q_1 \otimes \cdots \otimes q_k$, where each q_i is a continuous seminorms on E, see also [Tre67, Def. 43.3]. On algebraic tensors $t \in E^{\otimes k}$, this seminorm is given by

$$(q_1 \otimes \cdots \otimes q_k)(t) := \inf \left\{ \sum_j \prod_{i=1}^k q_i(\xi_i^{(j)}) \ : \ t = \sum_j \xi_1^{(j)} \otimes \cdots \otimes \xi_k^{(j)}, \text{ with } \xi_i^{(j)} \in E \right\}. \tag{6.2.1}$$

On simple tensors we have $(q_1 \otimes \cdots \otimes q_k)(\xi_1 \otimes \cdots \otimes \xi_k) = \prod_{i=1}^k q_i(\xi_i)$, where $\xi_i \in E$ [Tre67, Prop. 43.1].

Proposition 6.2.3 ([Tre67, Prop. 43.4, Cor. 3 on p. 465])**.**
There is a canonical linear isomorphism $\mathrm{Mult}(E^k, F) \cong \mathcal{B}(E^{\widehat{\otimes} k}, F)$. *It is a homeomorphism if* E *is Fréchet.*

Equip $P^k(E, F)$ with the topology of uniform convergence on compact sets. If p is a continuous seminorm on F, $B \subseteq E$ is a subset and $f : E \to F$ is a function, we write $p_B(f) := \sup_{x \in B} p(f(x))$.

Proposition 6.2.4. *Let* $k \in \mathbb{N}_{\geq 0}$. *Then* $P^k(E, F) \cong \mathrm{Sym}^k(E, F)$ *as locally convex vector spaces.*

Proof. If $\widetilde{f} : E^k \to F$ is a symmetric k-linear map and $f = \widetilde{f} \circ \Delta_k$ is the corresponding homogeneous polynomial, then $\widetilde{f}$ can be recovered from f using the formula [BS71b, Thm. A]:

$$\widetilde{f}(x_1, \ldots, x_k) = \frac{1}{k!} \sum_{\epsilon_1, \ldots, \epsilon_k = 0}^{1} (-1)^{k - (\epsilon_1 + \cdots + \epsilon_k)} f(\epsilon_1 x_1 + \cdots + \epsilon_k x_k). \tag{6.2.2}$$

This formula moreover shows that $\widetilde{f}$ is continuous if and only if f is so, and there is a linear isomorphism $\mathrm{Sym}^k(E, F) \to P^k(E, F)$ given by $\widetilde{f} \mapsto \widetilde{f} \circ \Delta_k =: f$. It remains to show that this map is also a homeomorphism. Suppose that $f = \widetilde{f} \circ \Delta_k$ for some $\widetilde{f} \in \mathrm{Sym}^k(E,F)$. If $B \subseteq E$ is a compact subset and p is a continuous seminorm on F, then $p_B(f) \leq p_{B^k}(\widetilde{f})$. Hence $\mathrm{Sym}^k(E,F) \to P^k(E,F), \widetilde{f} \mapsto f$ is continuous. For the continuity of the inverse, we use (6.2.2), from which it follows that if $B_i \subseteq E$ are compact subsets for $i \in \mathbb{N}$ and p is a continuous seminorm on F, then

$$\sup_{x_i \in B_i} p(\widetilde{f}(x_1, \ldots, x_k)) \leq \frac{2^k}{k!} p_B(f), \tag{6.2.3}$$

where

$$B = \{\, \epsilon_1 x_1 + \cdots + \epsilon_k x_k \,:\, \epsilon_i \in \{0,1\},\ x_i \in B_i \quad \text{for } i \in \{1, \ldots, k\}\,\},$$

which is a compact subset of E. Consequently the map $f \mapsto \widetilde{f}$ is continuous $P^k(E,F) \to \mathrm{Sym}^k(E,F)$. □

Define the locally convex space $P(E,F) := \prod_{k=0}^{\infty} P^k(E,F)$, equipped with the product topology. If $F = \mathbb{K}$, we simply write $P^n(E) := P^n(E,\mathbb{K})$.

Analytic functions

Let $U \subseteq E$ be open and let $f : U \to F$ be a function.

Definition 6.2.5.

- Suppose $\mathbb{K} = \mathbb{C}$. The function $f : U \to F$ is called *complex-analytic* or *holomorphic* if it is continuous, and for every $x \in U$ there exists a 0 neighborhood V in E with $x + V \subseteq U$ and functions $f_k \in P^k(E,F)$ for $k \in \mathbb{N}_{\geq 0}$ such that:

$$f(x+h) = \sum_{k=0}^{\infty} f_k(h), \qquad \forall h \in V.$$

- Suppose $\mathbb{K} = \mathbb{R}$. The function $f : U \to F$ is called *real-analytic* if it extends to some complex-analytic map $f_{\mathbb{C}} : U_{\mathbb{C}} \to F_{\mathbb{C}}$ for some open neighborhood $U_{\mathbb{C}}$ of U in $E_{\mathbb{C}}$.

- Suppose $\mathbb{K} = \mathbb{C}$ and $U = E$. The function $f : E \to F$ is called *entire* if it is continuous and there exist functions $f_k \in P^k(E,F)$ for $k \in \mathbb{N}_{\geq 0}$ such that $f(x) = \sum_{k=0}^{\infty} f_k(x)$ for all $x \in E$.

Remark 6.2.6. The above definition of a real-analytic map differs from the one used in [BS71a], where a function $f : U \to F$ is called real-analytic if it is continuous and for every $x \in U$ there exists a 0-neighborhood V in U with $x + V \subseteq U$ and homogeneous polynomials $f_k : E \to F$ such that $f(x+h) = \sum_{k=0}^{\infty} f_k(h)$ holds for all $h \in V$. The two notions are equivalent if E and F are Fréchet spaces [Glö02b, Rem. 2.9], [BS71a, Thm. 7.1].

Proposition 6.2.7 ([BS71a, Prop. 5.1]).
Suppose $\mathbb{K} = \mathbb{C}$. Let $f_k \in P^k(E, F)$ for every $k \in \mathbb{N}_{\geq 0}$. Let $U \subseteq E$ be a 0-neighborhood s.t. $f(h) := \sum_k f_k(h)$ is convergent for every $h \in U$. Assume that $f : U \to F$ is continuous at $0 \in U$. Then, for every continuous seminorm p on F, there exists a 0-neighborhood $V \subseteq U$ such that $\sum_{k=0}^{\infty} p_V(f_k) < \infty$.

Lemma 6.2.8. *Suppose $\mathbb{K} = \mathbb{C}$. Let $f_n \in P^n(E, F)$ for every $n \in \mathbb{N}_{\geq 0}$. Consider the following assertions:*

1. *$f := \sum_{n=0}^{\infty} f_n$ defines an entire function $E \to F$.*
2. *$\sum_{n=0}^{\infty} p_B(f_n) < \infty$ for any compact subset $B \subseteq E$ and continuous seminorm p on F.*

We have that (1) $\implies$ (2). If E is a Fréchet space, then also (2) $\implies$ (1) holds true.

Proof. Assume that $f = \sum_{n=0}^{\infty} f_n$ defines an entire function $E \to F$. Let $B \subseteq E$ be a compact subset and let p be a continuous seminorm on F. We may assume that B is balanced. As f is continuous, $f(2B) \subseteq F$ is compact and hence bounded. So $M_p := p_{2B}(f) < \infty$. As f is entire, we have $f(zx) = \sum_{n=0}^{\infty} f_n(x)z^n$ for any $x \in E$ and $z \in \mathbb{C}$. Let $x \in 2B$. Then also $zx \in 2B$ for any $z \in \mathbb{C}$ with $|z| \leq 1$, as B is balanced. Applying [BS71a, Cor. 3.2] to the holomorphic map

$$g : \mathbb{C} \to F, \qquad g(z) := f(zx),$$

we find that $f_n(x) = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{z^{n+1}} dz$ and moreover that

$$p(f_n(x)) \leq \sup_{|z|=1} p(g(z)) \leq p_{2B}(f) = M_p, \qquad \forall n \in \mathbb{N}_{\geq 0}.$$

Hence $p_{2B}(f_n) \leq M_p$, so that $p_B(f_n) \leq M_p 2^{-n}$ for all $n \in \mathbb{N}_{\geq 0}$. Thus

$$\sum_{n=0}^{\infty} p_B(f_n) \leq M_p \sum_{n=0}^{\infty} 2^{-n} < \infty.$$

Suppose that E is a Fréchet space. Assume that (2) holds true. Then in particular the series $\sum_{n=0}^{\infty} f_n(x)$ is convergent for any $x \in E$. So $f := \sum_{n=0}^{\infty} f_n$ defines a function $E \to F$. To show f is entire, it remains only to show that it is continuous. The condition (2) implies that $s_N \to f$ uniformly on compact subsets, where $s_N := \sum_{n=0}^{N} f_n$ for any $N \in \mathbb{N}$. As s_N is continuous for every $N \in \mathbb{N}$ and E is Fréchet by assumption, this implies that f is continuous (by a standard 3ϵ argument). □

Proposition 6.2.9 ([Glö02b, Prop. 2.4]).
Every real- or complex-analytic map is smooth.

Proposition 6.2.10 ([BS71a, Prop. 5.5]).
Suppose $\mathbb{K} = \mathbb{C}$. If $f : U \to F$ is complex-analytic, then $f(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} \delta_x^k(f)(h)$ for all $h \in V$, where V is the maximal balanced 0-neighborhood of E such that $x + V \subseteq U$.

Proposition 6.2.11 ([Glö02b, Lem. 2.5]).
Suppose $\mathbb{K} = \mathbb{C}$. Then f is complex-analytic if and only if f is smooth and the map $\delta_x^1 = df(x; -) : E \to F$ is complex-linear for every $x \in U$.

Proposition 6.2.12 ([Glö02b, Lem. 2.6]).
Suppose $\mathbb{K} = \mathbb{C}$. If $f : U \to F$ is complex-analytic, then so is $df : U \times E \to F$.

With these definitions, the chain rule holds for both real- and complex-analytic mappings. One proceeds to define real- and complex- analytic manifolds and Lie groups, see e.g. [Mil84] and [Nee06] for more details.

Definition 6.2.13. If M is a real-analytic manifold and V is a locally convex vector space, we write $C^\omega(M, V)$ for the set of analytic functions $M \to V$. If M is a complex-analytic manifold and V is complex, we write $\mathcal{O}(M, V)$ for the space of complex-analytic mappings $M \to V$.

Proposition 6.2.14 (Identity Theorems [BS71a, Prop. 6.6]).

1. *Suppose that E and F are complex. Let $f : U \to F$ be complex-analytic and assume that U is connected. If $f(x) = 0$ for all $x \in V$ for some open and non-empty $V \subseteq U$, then $f = 0$.*

2. *Suppose that E is real and F is complex. Let $f : U_{\mathbb{C}} \to F$ be complex-analytic, where $U_{\mathbb{C}} \subseteq E_{\mathbb{C}}$ is open and connected. If $U_{\mathbb{C}}$ contains a non-empty subset $V \subseteq E$ that is open in E and $f(x) = 0$ holds for every $x \in V$, then $f = 0$.*

Proposition 6.2.15. *Let $x \in U$. The following linear map is continuous:*

$$\begin{aligned} j_x^\infty : C^\infty(U, F) &\to P(E, F), \\ f &\mapsto \sum_{k=0}^{\infty} \frac{1}{k!} \delta_x^k(f) \end{aligned}$$

If U is connected, then its restriction to $C^\omega(U, F)$ is injective.

Proof. The map j_x^∞ is linear, as each $\delta_x^k : C^\infty(U, F) \to P^k(E, F)$ is so. As $P(E, F) = \prod_{n=0}^\infty P^n(E, F)$ carries the product topology, to see j_x^∞ is continuous it suffices to show that δ_x^k is continuous for every $k \in \mathbb{N}_{\geq 0}$. This is immediate from the definition of the compact-open C^∞-topology on $C^\infty(U, F)$ [Nee06, Def. I.5.1(d)], and the topology of uniform convergence on compact subsets carried by $P^k(E, F)$. Assume that U is connected. Let $f \in C^\omega(U, F)$ and suppose that $j_x^\infty(f) = 0$. Using Proposition 6.2.10 it follows that $f(x + h) = 0$ for all h in some 0-neighborhood of E. By Proposition 6.2.14 this implies that $f = 0$. ☐

6.2.2 Smooth, analytic and strongly-entire representations

Let G be a BCH(Baker-Campbell-Hausdorff) Fréchet-Lie group with Lie algebra $\mathfrak{g}$. We write $\mathfrak{g}_{\mathbb{C}}$ for the complexification of $\mathfrak{g}$.

Let us first recall some notation, introduced earlier in Chapter 2. If $\mathcal{D}$ is a pre-Hilbert space, we write $\mathcal{L}(\mathcal{D})$ for the set of linear operators on $\mathcal{D}$. We further define the algebra

$$\mathcal{L}^{\dagger}(\mathcal{D}) := \left\{ X \in \mathcal{L}(\mathcal{D}) \,:\, \exists X^{\dagger} \in \mathcal{L}(\mathcal{D}) \,:\, \forall \psi, \eta \in \mathcal{D} : \langle X^{\dagger}\psi, \eta \rangle = \langle \psi, X\eta \rangle \right\}.$$

Then $(-)^{\dagger}$ is an involution on $\mathcal{L}^{\dagger}(\mathcal{D})$, turning it into a $*$-algebra. We will also have need for various involutions on the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$. Let $\theta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be defined by $\theta(\xi + i\eta) := \xi - i\eta$ for $\xi, \eta \in \mathfrak{g}$.

Definition 6.2.16. Extend the conjugation θ on $\mathfrak{g}_{\mathbb{C}}$ to a complex conjugate-linear automorphism of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. Let τ denote the involutive anti-automorphism of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ extending $\xi \mapsto -\xi$ on $\mathfrak{g}_{\mathbb{C}}$. Define $x^* := \tau(\theta(x))$ for $x \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. Explicitly, θ, τ and $(-)^*$ satisfy the following relations, where $\xi_j \in \mathfrak{g}_{\mathbb{C}}$ for $j \in \mathbb{N}$:

$$\begin{aligned} \theta(\xi_1 \cdots \xi_n) &= \theta(\xi_1) \cdots \theta(\xi_n), \\ \tau(\xi_1 \cdots \xi_n) &= (-1)^n \xi_n \cdots \xi_1, \\ (\xi_1 \cdots \xi_n)^* &= (-1)^n \theta(\xi_n) \cdots \theta(\xi_1). \end{aligned}$$

If $(\rho, \mathcal{H}_\rho)$ is a unitary G-representation, we say that it is continuous if it is so with respect to the strong operator topology on $\mathrm{U}(\mathcal{H}_\rho)$.

Definition 6.2.17. Let $(\rho, \mathcal{H}_\rho)$ be a continuous unitary representation of G. A vector $\psi \in \mathcal{H}_\rho$ is called *smooth*, resp. *analytic*, if the orbit map $G \to \mathcal{H}_\rho, g \mapsto \rho(g)v$ is smooth, resp. analytic. We write $\mathcal{H}_\rho^\infty$ and $\mathcal{H}_\rho^\omega$ for the linear subspaces of smooth and analytic vectors, respectively. We say that the representation ρ is *smooth* if $\mathcal{H}_\rho^\infty$ is dense in $\mathcal{H}_\rho$ and *analytic* if $\mathcal{H}_\rho^\omega$ is dense in $\mathcal{H}_\rho$.

Remark 6.2.18. If ρ is a smooth unitary representation of G, then the derived representation $d\rho$ of $\mathfrak{g}_{\mathbb{C}}$ on $\mathcal{H}_\rho^\infty$ extends to a homomorphism $d\rho : \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{L}^{\dagger}(\mathcal{H}_\rho^\infty)$ satisfying $d\rho(x)^{\dagger} = d\rho(x^*)$ for any $x \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.

Definition 6.2.19. Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of G.

— Following [JN19, Def. 3.9], we define two locally convex topologies on $\mathcal{H}_\rho^\infty$:

- The *weak topology* on $\mathcal{H}_\rho^\infty$ is defined by the seminorms

  $$p_\xi(\psi) := \|d\rho(\xi_1 \cdots \xi_n)\psi\|, \qquad \text{where } n \in \mathbb{N}_{\geq 0} \text{ and } \xi = (\xi_1, \ldots, \xi_n) \in \mathfrak{g}^n.$$

- The *strong topology* is defined by the seminorms

  $$p_B(\psi) := \sup_{\xi \in B} \|d\rho(\xi_1 \cdots \xi_n)\psi\|,$$

  where $B \subseteq \mathfrak{g}^n$ is bounded and $n \in \mathbb{N}_{\geq 0}$.

If G is regular, then the space $\mathcal{H}_\rho^\infty$ is complete w.r.t. to either of these topologies [JN19, Prop. 3.19], where we also used that G is a Fréchet-Lie group.

— A vector $\psi \in \mathcal{H}_\rho^\infty$ is called *entire* if

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sup_{\xi \in B} \|d\rho(\xi^n)\psi\| < \infty$$

for every compact $B \subseteq \mathfrak{g}_\mathbb{C}$.

— If $\psi \in \mathcal{H}_\rho^\infty$ and $B \subseteq \mathfrak{g}_\mathbb{C}$, we define

$$p_B^n(\psi) := \sup_{\xi_1, \dots, \xi_n \in B} \|d\rho(\xi_1 \cdots \xi_n)\psi\|$$

and

$$q_B(\psi) := \sum_{n=0}^{\infty} \frac{1}{n!} p_B^n(\psi).$$

— A vector $\psi \in \mathcal{H}_\rho^\infty$ is called *strongly-entire* if $q_B(\psi) < \infty$ for every compact subset $B \subseteq \mathfrak{g}_\mathbb{C}$. It is said to be *b-strongly-entire* if $q_B(\psi) < \infty$ for every bounded subset $B \subseteq \mathfrak{g}_\mathbb{C}$.

— We write $\mathcal{H}_\rho^{\mathcal{O}} \subseteq \mathcal{H}_\rho^\infty$ and $\mathcal{H}_\rho^{\mathcal{O}_b}$ for the linear subspace of strongly-entire and b-strongly-entire vectors, respectively. Equip $\mathcal{H}_\rho^{\mathcal{O}}$ (resp. $\mathcal{H}_\rho^{\mathcal{O}_b}$) with the locally convex topology defined by the seminorms q_B for compact (resp. bounded) subsets $B \subseteq \mathfrak{g}_\mathbb{C}$.

— We say that the representation ρ is *strongly-entire* if $\mathcal{H}_\rho^{\mathcal{O}}$ is dense in $\mathcal{H}_\rho$, and that it is *b-strongly-entire* if $\mathcal{H}_\rho^{\mathcal{O}_b}$ is dense in $\mathcal{H}_\rho$.

If $\psi \in \mathcal{H}_\rho^\infty$, we write $f^\psi : G \to \mathcal{H}_\rho$ for the orbit map $f^\psi(g) = \rho(g)\psi$. As f^ψ is smooth, the homogeneous polynomial $f_n^\psi(\xi) := \frac{1}{n!} d\rho(\xi^n)\psi$ is continuous as a map $\mathfrak{g}_\mathbb{C} \to \mathcal{H}_\rho$, so $f_n^\psi \in P^n(\mathfrak{g}_\mathbb{C}, \mathcal{H}_\rho)$. Notice further that $j_0^\infty(f^\psi) = \sum_{n=0}^\infty f_n^\psi \in P(\mathfrak{g}_\mathbb{C}, \mathcal{H}_\rho)$. Let β_n^ψ be the unique element of $\mathrm{Sym}^n(\mathfrak{g}_\mathbb{C}, \mathcal{H}_\rho)$ satisfying $f_n^\psi = \beta_n^\psi \circ \Delta_n$. Explicitly, $\beta_n^\psi(\xi_1, \dots, \xi_n) = \frac{1}{(n!)^2} \sum_{\sigma \in S_n} d\rho(\xi_{\sigma 1} \cdots \xi_{\sigma_n})v$.

Lemma 6.2.20. *Let $\psi \in \mathcal{H}_\rho^\infty$. Assume that $q_B(\psi) < \infty$ for every compact subset $B \subseteq \mathfrak{g}$. Then $q_B(\psi) < \infty$ for every compact subset $B \subseteq \mathfrak{g}_\mathbb{C}$.*

Proof. Let $B_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ be compact. Replacing $B_\mathbb{C}$ by its balanced hull, we may assume that $B_\mathbb{C}$ is balanced. Let $B := \{ \xi + \overline{\xi} : \xi \in B_\mathbb{C} \} \subseteq \mathfrak{g}$, which is compact in $\mathfrak{g}$. Then $B_\mathbb{C} \subseteq B + iB$ and so $q_{B_\mathbb{C}}(\psi) \leq q_{2B}(\psi) < \infty$. □

Proposition 6.2.21. *Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of G. Let $\psi \in \mathcal{H}_\rho^\infty$. The following assertions are equivalent:*

1. *$\psi \in \mathcal{H}_\rho^\omega$.*
2. *There exists a 0-neighborhood $V \subseteq \mathfrak{g}$ such that $\sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$ converges for every $\xi \in V$ and the map $V \to \mathcal{H}_\rho$, $\xi \mapsto \sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$ is continuous.*
3. *$\sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$ converges for every ξ in a 0-neighborhood $\mathfrak{g}$.*
4. *There is a 0-neighborhood $V \subseteq \mathfrak{g}$ such that $\sum_{n=0} \frac{1}{n!} p_V^n(\psi) < \infty$.*
5. *There is a 0-neighborhood $V \subseteq \mathfrak{g}$ such that $\sum_{n=0}^\infty \frac{1}{n!} \langle \psi, d\rho(\xi^n)\psi \rangle$ converges for all $\xi \in V$.*
6. *The map $G \to \mathbb{C}$, $g \mapsto \langle \psi, \rho(g)\psi \rangle$ is analytic on a neighborhood of $1 \in G$.*

Proof. Assume that $\psi \in \mathcal{H}_\rho^\omega$. Then the orbit map $f^\psi : G \to \mathcal{H}_\rho$ is real-analytic, and hence so is $f^\psi \circ \exp : \mathfrak{g} \to \mathcal{H}_\rho$. Notice that $f^\psi(e^\xi) = \rho(e^\xi)\psi$, so that $\delta_0^n(f^\psi \circ \exp) = d\rho(\xi^n)\psi$. Using Proposition 6.2.10, it follows that $f^\psi(e^\xi) = \sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$ on some balanced 0-neighborhood $V \subseteq \mathfrak{g}$. So (1) $\implies$ (2).

We show that (2) $\implies$ (1). Let $V \subseteq \mathfrak{g}$ be a 0-neighborhood such that $\sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$ converges for every $\xi \in V$ and s.t. the map $\xi \mapsto \sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$ is continuous on V. Replacing V by some smaller balanced open set, we may assume that V is balanced. Define $h^\psi(\xi) := \sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$. In view of Remark 6.2.6, the assumptions imply that h^ψ is real-analytic on V, where it was used that $\mathfrak{g}$ is Fréchet and $\mathcal{H}_\rho$ is a Hilbert space. Then h^ψ is smooth by Proposition 6.2.9. Let $\xi \in V$. We show that $h^\psi(\xi) = \rho(e^\xi)\psi$. Let $s \in I := [-1, 1]$. Then $s\xi \in V$, because V is balanced. Notice that

$$\frac{d}{dt}\bigg|_{t=s} h^\psi(t\xi)\psi = d\rho(\xi)h^\psi(s\xi), \qquad \text{and} \qquad \frac{d}{dt}\bigg|_{t=s} \rho(e^{t\xi})\psi = d\rho(\xi)\rho(e^{s\xi})\psi.$$

Let $\eta \in \mathcal{H}_\rho^\infty$. Using $d\rho(\xi)^*\eta = -d\rho(\xi)\eta$ it follows that $\frac{d}{dt}\big|_{t=s} \langle \rho(e^{t\xi})\eta, h^\psi(t\xi) \rangle = 0$ for all $s \in I$. Hence $\langle \eta, \rho(e^{-t\xi})h^\psi(t\xi) \rangle = \langle \eta, \psi \rangle$ for all $t \in I$. As this is valid for any η in the dense set $\mathcal{H}_\rho^\infty$ it follows that $\rho(e^{-t\xi})h^\psi(t\xi)\psi = \psi$ or equivalently that $h^\psi(t\xi)\psi = \rho(e^{t\xi})\psi$ for all $t \in I$. In particular, taking $t = 1$ we conclude that $h^\psi(\xi) = \rho(e^\xi)\psi$ for all $\xi \in V$. As h^ψ is real-analytic on V, so is $\xi \mapsto \rho(e^\xi)\psi$. Since G is BCH, this implies that $g \mapsto \rho(g)\psi$ is analytic on a neighborhood of $1 \in G$. In turn, this implies that it is analytic everywhere, where we have used that G is a real-analytic Lie group and that the composition of real-analytic maps is again real-analytic [Glö02b, Proposition 2.8]. Thus $\psi \in \mathcal{H}_\rho^\omega$.

The implication (2) $\implies$ (3) is trivial whereas (3) $\implies$ (4) follows from [BS71a, Prop. 5.2] because V is absorbing and $\mathfrak{g}$ is a Baire space, as it is Fréchet. To see that (4) $\implies$ (2), assume that $V \subseteq \mathfrak{g}$ is a 0-neighborhood s.t. $\sum_{n=0}^\infty \frac{1}{n!} p_V^n(\psi) < \infty$. For $\xi \in V$, we write $s_N(\xi) := \sum_{n=0}^N \frac{1}{n!} d\rho(\xi^n)\psi$ and $s(\xi) := \sum_{n=0}^\infty \frac{1}{n!} d\rho(\xi^n)\psi$. It remains only to prove that s is continuous on V. Let $\xi \in V$. Suppose that (ξ_k) is a sequence in V with $\xi_k \to \xi$. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be s.t. $\sum_{n=N+1}^{\infty} \frac{1}{n!} p_V^n(\psi) < \epsilon$. Then for any $\eta \in V$ we have $\|s(\eta) - s_N(\eta)\| \leq \sum_{n=N+1}^{\infty} \frac{1}{n!} p_V^n(\psi) < \epsilon$. Using that s_N is continuous, let $N' \in \mathbb{N}$ be s.t. $\|s_N(\xi) - s_N(\xi_k)\| < \epsilon$ and $\xi_k \in V$ for all $k \geq N'$. Then

$$\|s(\xi)-s(\xi_k)\| \leq \|s(\xi)-s_N(\xi)\|+\|s_N(\xi)-s_N(\xi_k)\|+\|s_N(\xi_k)-s(\xi_k)\| < 3\epsilon, \quad \forall k \geq N'.$$

Thus $s(\xi_k) \to s(\xi)$. Hence s is sequentially continuous at 0. As $\mathfrak{g}$ is Fréchet, this implies that s is continuous at ξ. Thus (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). It is trivial that (3) $\implies$ (5) whereas (5) $\implies$ (3) follows immediately from [Nee11, Prop. 3.4, 6.3] (by considering $\mathcal{D} := \mathcal{H}_\rho^\infty$ and $v := \psi$). Finally, (6) $\iff$ (1) is precisely [Nee11, Thm. 5.2]. This completes the proof. $\square$

Let us consider an analogous statements for entire vectors:

Proposition 6.2.22. *Let $\psi \in \mathcal{H}_\rho^\infty$. The following assertions are equivalent:*

1. *The series $\sum_{n=0}^{\infty} f_n^\psi(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} d\rho(\xi^n)\psi$ defines an entire function*

$$\mathfrak{g}_\mathbb{C} \to \mathcal{H}_\rho, \qquad \xi \mapsto \sum_{n=0}^{\infty} f_n^\psi(\xi).$$

2. *ψ is an entire vector for ρ, i.e., $\sum_{n=0}^{\infty} \frac{1}{n!} \sup_{\xi \in B} \|d\rho(\xi^n)\psi\| < \infty$ for every compact $B \subseteq \mathfrak{g}_\mathbb{C}$.*

3. *The map $\mathfrak{g} \to \mathcal{H}_\rho$, $\xi \mapsto \rho(e^\xi)\psi$ extends to an entire function $\mathfrak{g}_\mathbb{C} \to \mathcal{H}_\rho$.*

4. *$\sum_{n=0}^{\infty} \sup_{\xi_i \in B} \|\beta_n^\psi(\xi_1, \ldots, \xi_n)\| < \infty$ for every compact $B \subseteq \mathfrak{g}$.*

Proof. As $\mathfrak{g}_\mathbb{C}$ is Fréchet by assumption, we know using Lemma 6.2.8 that the series $\sum_{n=0}^{\infty} f_n^\psi(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} d\rho(\xi^n)\psi$ defines an entire function on $\mathfrak{g}_\mathbb{C}$ if and only if

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sup_{\xi \in B} \|d\rho(\xi^n)\psi\| < \infty, \qquad \forall B \subseteq \mathfrak{g}_\mathbb{C} \text{ compact}.$$

That is, if and only if (2) holds true. Thus (1) $\iff$ (2). Assume next that (2) is valid. As singletons are compact, it follows in particular that $\sum_{n=0} f_n^\psi(\xi)$ converges for every $\xi \in \mathfrak{g}_\mathbb{C}$. By Proposition 6.2.21, this implies that $\psi \in \mathcal{H}_\rho^\omega$. Hence the orbit map $f^\psi : G \to \mathcal{H}_\rho$ is real-analytic. As G is BCH, the exponential map $\exp : \mathfrak{g} \to G$ is real-analytic and hence $\xi \mapsto f^\psi(e^\xi) = \rho(e^\xi)\psi$ is a real-analytic map $\mathfrak{g} \to \mathcal{H}_\rho$. Since $\delta_0^n(f^\psi \circ \exp; \xi) = d\rho(\xi^n)\psi$ for every $n \in \mathbb{N}$, Proposition 6.2.10 implies that $f^\psi(e^\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} d\rho(\xi^n)\psi$ on some 0-neighborhood in V. As (2) and hence (1) hold by assumption, it follows that $\sum_{n=0}^{\infty} f_n^\psi$ is an entire function extending $\xi \mapsto \rho(e^\xi)\psi$. Thus (3) holds true. Suppose conversely that (3) is valid, so that $f^\psi \circ \exp$ extends to an entire function $F : \mathfrak{g}_\mathbb{C} \to \mathcal{H}_\rho$. By Proposition 6.2.10 and using that $\delta_0^n(f^\psi \circ \exp; \xi) = d\rho(\xi^n)\psi$ for $n \in \mathbb{N}$, we find that $F(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} d\rho(\xi^n)\psi$ for every $\xi \in \mathfrak{g}_\mathbb{C}$. Thus (1) holds true. We have shown (1) $\iff$ (2) $\iff$ (3).

Next we show (2) $\implies$ (4). Let $B \subseteq \mathfrak{g}_\mathbb{C}$ be compact. As $\mathfrak{g}_\mathbb{C}$ is complete, the closed convex hull of B is again compact [Tre67, p. 67]. Thus we may assume that B is convex. Replacing B further by its balanced hull, we may assume that B is balanced. Then $B + \cdots + B$ (n times) $\subseteq nB$. From equation (6.2.3) it follows that

$$\sup_{\xi_i \in B} \|\beta_n^\psi(\xi_1, \ldots, \xi_n)\| \leq \frac{2^n}{n!} \sup_{\xi \in nB} \|f_n^\psi(\xi)\| = \frac{(2n)^n}{n!} \sup_{\xi \in B} \|f_n^\psi(\xi)\|.$$

Choose some $t > 2e$. Since $\sum_{n=0}^\infty \sup_{\xi \in B} \|f_n^\psi(\xi)\| < \infty$ for every compact B, it follows (by considering tB) that there exists some $C > 0$ s.t. $\sup_{\xi \in B} \|f_n^\psi(\xi)\| \leq Ct^{-n}$ for every $n \in \mathbb{N}_{\geq 0}$. Then

$$\sum_{n=0}^\infty \sup_{\xi_i \in B} \|\beta_n^\psi(\xi_1, \ldots, \xi_n)\| \leq C \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{2n}{t}\right)^n < \infty.$$

The implication (4) $\implies$ (2) is trivial. □

Remark 6.2.23. The characterization (4) of entire vectors in Proposition 6.2.22 makes the difference between entire and strongly-entire vectors clear, namely whether one considers the symmetric n-linear maps β_n^ψ or their non-symmetric analogues $(\xi_1, \ldots, \xi_n) \mapsto \frac{1}{n!} d\rho(\xi_1 \cdots \xi_n)\psi$. Analogous to [Nee11, Rem. 3.7], it is in general not known whether or not any entire vector is in fact strongly-entire. In the case where $\mathfrak{g}$ is finite-dimensional, this follows immediately from [Pen74, Thm. I.3, Rem. I.7].

Corollary 6.2.24. *$\mathcal{H}_\rho^\mathcal{O} \subseteq \mathcal{H}_\rho^\omega \subseteq \mathcal{H}_\rho^\infty$.*

Proof. Any strongly-entire vector is entire. Consequently, the first inclusion follows by combining Proposition 6.2.22 and Proposition 6.2.21. The second one follows from the fact that if the orbit map $f^\psi : G \to \mathcal{H}_\rho$ is real-analytic, then it is smooth by Proposition 6.2.9. □

The space $\mathcal{H}_\rho^\mathcal{O}$ of strongly-entire vectors will be considered in more detail in Section 6.3 below.

6.2.3 Positive energy and ground state representations.

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$. If $\mathcal{H}$ is a Hilbert space and $S \subseteq \mathcal{H}$ is a subset, we write $[\![S]\!] \subseteq \mathcal{H}$ for the closed linear span of S.

Theorem 6.2.25 (Borchers-Arveson [BR87, Thm. 3.2.46], [BGN20, Lem. 4.17])**.** *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra on the Hilbert space $\mathcal{H}$. Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous unitary one-parameter group satisfying $U_t\mathcal{M}U_t^{-1} \subseteq \mathcal{M}$ for all $t \in \mathbb{R}$. Assume that $U_t = e^{itH}$ with $H \geq 0$. Define $\alpha : \mathbb{R} \to Aut(\mathcal{M})$ by $\alpha_t(x) := Ad_{U_t}(x) := U_t x U_t^{-1}$ for $t \in \mathbb{R}$ and $x \in \mathcal{M}$. Denote by $\mathcal{M}^\alpha(S) \subseteq \mathcal{M}$ the Arveson spectral subspace for $S \subseteq \mathbb{R}$. Then*

1. *There exists a strongly continuous unitary one-parameter group $V_t = e^{itH_0}$ in $\mathcal{M}$ with $H_0 \geq 0$ and $Ad_{V_t} = \alpha_t$ for every $t \in \mathbb{R}$.*

2. $\bigcap_{t>0} [\![\mathcal{M}^\alpha[t,\infty)\mathcal{H}]\!] = \{0\}$.

3. *V_t is uniquely determined by the additional requirement that for any other such $V_t' = e^{itH_0'}$, we have $H_0' \geq H_0$. In this case, the spectral projection P corresponding to V_t is determined uniquely by*

$$P[t,\infty)\mathcal{H} = \bigcap_{s<t} [\![\mathcal{M}^\alpha[s,\infty)\mathcal{H}]\!].$$

Definition 6.2.26. Consider the setting of Theorem 6.2.25. A unitary one-parameter group $V_t = e^{itH_0}$ satisfying the conditions of Theorem 6.2.25(1) is called a *positive inner implementation* of $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ on $\mathcal{H}$. If V_t additionally satisfies the condition in Theorem 6.2.25(3) then it is said to be the *minimal positive inner implementation* of α on $\mathcal{H}$.

Definition 6.2.27.
A smooth unitary representation $(\rho, \mathcal{H}_\rho)$ of G is of *positive energy* (p.e.) at $\xi \in \mathfrak{g}$ if $-i\,\mathrm{Spec}(\overline{d\rho(\xi)}) \geq 0$. If additionally $\mathcal{H}_\rho(0) := \ker \overline{d\rho(\xi)}$ is cyclic for G, then $(\rho, \mathcal{H}_\rho)$ is said to be *ground state* at $\xi \in \mathfrak{g}$.

Definition 6.2.28. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(G)$ be a homomorphism for which the corresponding action $\mathbb{R}\times G \to G$ is smooth. Define $G^\sharp := G \rtimes_\alpha \mathbb{R}$ and $\mathfrak{g}^\sharp := \mathbf{L}(G^\sharp) = \mathfrak{g} \rtimes_D \mathbb{R}\boldsymbol{d}$, where $\boldsymbol{d} := 1 \in \mathbb{R}$. Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of G. We write $\mathcal{M} := \rho(G)''$ for the von Neumann algebra generated by $\rho(G)$.

1. An *extension* of ρ to $G^\sharp$ is a smooth unitary representation $\widetilde{\rho}$ of $G^\sharp$ on $\mathcal{H}_\rho$ such that $\widetilde{\rho}|_G = \rho$.

2. We say that ρ is of positive energy w.r.t. α if there exists an extension $\widetilde{\rho}$ of ρ to $G^\sharp$ which is of p.e. at $\boldsymbol{d} \in \mathfrak{g}^\sharp$. In this case $\widetilde{\rho}$ is called a *positive extension* of ρ.

3. Assume that ρ is of positive energy w.r.t. α. A *minimal positive extension* $\widetilde{\rho}$ of ρ is a positive extension $\widetilde{\rho}$ of ρ to $G^\sharp$ such that $V_t := \widetilde{\rho}(e,t)$ is the minimal positive inner implementation of the automorphism group $\mathbb{R} \to \mathrm{Aut}(\mathcal{M})$, $t \mapsto \mathrm{Ad}_{V_t}$. Then in particular $V_t \in \mathcal{M}$ for every $t \in \mathbb{R}$.

4. A unitary representation ρ of G that is of p.e. w.r.t. α is said to be *ground state* if it has a minimal positive extension that is ground state at $\boldsymbol{d} \in \mathfrak{g}^\sharp$.

Definition 6.2.29. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(G)$ be an $\mathbb{R}$-action on G for which the corresponding map $\mathbb{R} \times G \to G$ is smooth. Let $\widehat{G}$ denote the set of equivalence classes of irreducible unitary representations of G that are smooth. Define

$$\widehat{G}_{\mathrm{pos}(\alpha)} := \left\{ \rho \in \widehat{G} \,:\, \rho \text{ is of p.e. w.r.t. } \alpha \right\}.$$

Proposition 6.2.30.
Consider the setting of Definition 6.2.28. Assume that ρ is of p.e. w.r.t. α.

1. *There exists a unique minimal positive extension $\widetilde{\rho}_0$ of ρ to $G^\sharp$.*

2. *If $\widetilde{\rho}$ is any other positive extension of ρ to $G^\sharp$, there exists a strongly continuous unitary 1-parameter group (U_t) in $\mathcal{M}'$ such that $\widetilde{\rho}(t) = \widetilde{\rho}_0(t)U_t$. In this case $\widetilde{\rho}_0(G^\sharp)'' = \rho(G)''$. In particular, ρ is irreducible if and only if $\widetilde{\rho}_0$ is.*

3. *Assume that $\alpha_T = id_G$ for some $T > 0$. Then $\widetilde{\rho}_0(T) = id_{\mathcal{H}_\rho}$ and ρ is ground state w.r.t α.*

4. *Let P denote the spectral measure associated to $t \mapsto \widetilde{\rho}_0(t)$. Let $\epsilon > 0$. Then the projection $P[0,\epsilon)$ has central support $1_\mathcal{M} = id_{\mathcal{H}_\rho} \in \mathcal{Z}(\mathcal{M})$. In particular $P[0,\epsilon)\mathcal{H}_\rho$ is cyclic for $\mathcal{M}$.*

Proof. The first three assertions follow by [JN21, Cor. 3.9] and the last by [BGN20, Lem. 4.17]. □

6.3 The space $\mathcal{H}_\rho^\mathcal{O}$ of strongly-entire vectors

Let G be a BCH Fréchet-Lie group with Lie algebra $\mathfrak{g}$. Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of G. In this section, we extend some results of [Goo69] concerning the space of strongly-entire vectors $\mathcal{H}_\rho^\mathcal{O}$ from the case where G is finite-dimensional to the present setting.

6.3.1 Necessary conditions for the existence of strongly-entire representations

We first show that when $\dim(\mathfrak{g}) < \infty$, the definition for $\mathcal{H}_\rho^\mathcal{O}$ (Definition 6.2.19) agrees with the one used in [Goo69, p.61]. The existence of a dense set of strongly-entire vectors is well-understood for continuous unitary representations of finite-dimensional Lie groups, yielding immediate necessary conditions for the existence of strongly-entire representations in the infinite-dimensional setting. This will turn out to be quite restrictive.

Assume that $\dim(\mathfrak{g}) < \infty$. Let us recall the definition used in [Goo69, p.61]. Let $\{e_\mu\}_{\mu=1}^d$ be a basis of $\mathfrak{g}$. For $v \in \mathcal{H}_\rho^\infty$, we define

$$E_s(v) := \sum_{n=0}^\infty \frac{s^n}{n!} \sup_{1 \leq \mu_k \leq d} \|d\rho(e_{\mu_1} \cdots e_{\mu_n})v\| \in [0, \infty].$$

Set $\mathcal{H}_\rho^{\omega_t} := \left\{ v \in \mathcal{H}_\rho^\infty : E_s(v) < \infty \text{ for all } 0 < s < t \right\}$ for $t > 0$. We now define $\mathcal{H}_\rho^{\mathcal{O}'} := \bigcap_{t>0} \mathcal{H}_\rho^{\omega_t}$. Equip $\mathcal{H}_\rho^{\mathcal{O}'}$ with the locally convex topology defined by the seminorms E_s for $s > 0$.

Lemma 6.3.1. $\mathcal{H}_\rho^{\mathcal{O}_b} = \mathcal{H}_\rho^\mathcal{O} = \mathcal{H}_\rho^{\mathcal{O}'}$ *as an equality of locally convex vector spaces.*

Proof. Define for $s > 0$ the compact subsets

$$B_s := \left\{ \sum_{\mu=1}^{d} c_\mu e_\mu \;:\; c_\mu \in \mathbb{C},\ |c_\mu| \leq s \quad \forall \mu \in \{1, \dots, d\} \right\} \subseteq \mathfrak{g}_\mathbb{C}.$$

Let $s > 0$. As $se_\mu \in B_s$ for any $\mu \in \{1, \dots, d\}$, it is immediate that $E_s(v) \leq q_{B_s}(v)$. Conversely, take $\xi_j \in B_s$ for $j \in \{1, \dots, n\}$. Then $\xi_j = \sum_{\mu_j=1}^{d} c_{\mu_j} e_{\mu_j} \in B_s$ for some $c_{\mu_j} \in \mathbb{C}$ with $|c_{\mu_j}| \leq s$. So

$$d\rho(\xi_{j_1} \cdots \xi_{j_n})v = \sum_{\mu_1, \dots \mu_n = 1}^{d} c_{\mu_1} \cdots c_{\mu_n} d\rho(e_{\mu_1} \cdots e_{\mu_n})v.$$

Consequently

$$\|d\rho(\xi_{j_1} \cdots \xi_{j_n})v\| \leq s^n \sum_{\mu_1, \dots \mu_n = 1}^{d} \|d\rho(e_{\mu_1} \cdots e_{\mu_n})v\| \leq s^n d^n \sup_{1 \leq \mu_k \leq d} \|d\rho(e_{\mu_1} \cdots e_{\mu_n})v\|.$$

Hence $E_s(v) \leq q_{B_s}(v) \leq E_{sd}(v)$ for any $s > 0$. This shows that $\mathcal{H}_\rho^{\mathcal{O}} = \mathcal{H}_\rho^{\mathcal{O}'}$ as locally convex vector spaces. Since $\dim(\mathfrak{g}_\mathbb{C}) < \infty$, it is moreover clear that $\mathcal{H}_\rho^{\mathcal{O}_b} = \mathcal{H}_\rho^{\mathcal{O}}$, because the closure of any bounded set in $\mathfrak{g}_\mathbb{C}$ is compact. ☐

Following [AM66, p. 128], [Jen73, p. 115] and [Pen74], we define:

Definition 6.3.2.

- A finite-dimensional Lie group G is said to be of *type R* if $\mathrm{Spec}(\mathrm{Ad}_g) \subseteq \mathbb{T}$ for every $g \in G$, where $\mathbb{T} \subseteq \mathbb{C}$ is the unit-circle.
- A finite-dimensional Lie algebra $\mathfrak{g}$ is said to be of *type R* if $\mathrm{Spec}(\mathrm{ad}_\xi) \subseteq i\mathbb{R}$ for every $\xi \in \mathfrak{g}$.

Remark 6.3.3. Lie algebras of type R are by some authors also called *weakly elliptic* [Nee98b, Def. II.1].

Proposition 6.3.4 ([Jen73, Prop. 1.3])**.**
Let G be a finite-dimensional connected Lie group with Lie algebra $\mathfrak{g}$. Then G is of type R if and only if $\mathfrak{g}$ is of type R.

Proposition 6.3.5 ([Pen74, Lem. on p. 120])**.**
A finite-dimensional Lie algebra $\mathfrak{g}$ is of type R if and only if it is the semi-direct product $\mathfrak{s} \rtimes \mathfrak{k}$ of a compact semisimple Lie algebra $\mathfrak{k}$ and a solvable Lie algebra $\mathfrak{s}$ that is of type R.

Theorem 6.3.6 ([Pen74, Cor. II.5])**.**
Let G be a finite-dimensional Lie group and ρ a continuous unitary representation of G. Then $\mathcal{H}_\rho^{\mathcal{O}}$ is dense if and only if ρ factors through a Lie group of type R.

In the setting where G is a possibly infinite-dimensional BCH Fréchet-Lie group, this yields:

Corollary 6.3.7. *Let G be a possibly infinite-dimensional BCH Fréchet-Lie group. Suppose that $(\rho, \mathcal{H}_\rho)$ is a strongly-entire unitary representation of G. If ρ is injective, then any finite-dimensional Lie subgroup of G is of type R.*

Proof. Let H be a finite-dimensional Lie subgroup of G. Then $\pi := \rho|_H$ is a continuous unitary H-representation on $\mathcal{H}_\pi := \mathcal{H}_\rho =: \mathcal{H}$. Since $\mathcal{H}_\rho^{\mathcal{O}} \subseteq \mathcal{H}_\pi^{\mathcal{O}}$, $\mathcal{H}_\pi^{\mathcal{O}}$ is dense in $\mathcal{H}$. As ρ is injective, it follows by Theorem 6.3.6 that H is of type R. □

As an illustration: If ρ is injective and $\mathcal{H}_\rho^{\mathcal{O}}$ is dense, then G can not contain a single copy of the $ax + b$ group. On the other hand, Theorem 6.3.6 provides ample examples of continuous representations that admit a dense set of strongly-entire vectors. Indeed, simply take any continuous unitary representation of a finite-dimensional Lie group of type R. The following examples show that also infinite-dimensional Lie groups may admit a dense set of strongly-entire vectors.

Example 6.3.8 (Norm-continuous representations)**.**
Let G be a BCH Fréchet-Lie group and let $\rho : G \to \mathrm{U}(\mathcal{H}_\rho)$ a unitary representation of G which is continuous w.r.t. the norm-topology on $\mathrm{U}(\mathcal{H}_\rho)$. Equipped with the norm topology, $\mathrm{U}(\mathcal{H}_\rho)$ is a Banach-Lie group with Lie algebra

$$\mathfrak{u}(\mathcal{H}_\rho) := \{\, T \in \mathcal{B}(\mathcal{H}_\rho) \,:\, T^* = -T \,\},$$

and the continuous homomorphism $\rho : G \to \mathrm{U}(\mathcal{H}_\rho)$ is automatically analytic by [Nee06, Thm. IV.1.18]. This implies that $\mathcal{H}_\rho^\omega = \mathcal{H}_\rho$. Let us show that we even have $\mathcal{H}_\rho^{\mathcal{O}_b} = \mathcal{H}_\rho$. As the representation $d\rho : \mathfrak{g} \to \mathfrak{u}(\mathcal{H}_\rho)$ is continuous, there exist a continuous seminorm p on $\mathfrak{g}$ s.t. $\|d\rho(\xi)\| \leq p(\xi)$ for all $\xi \in \mathfrak{g}$ [Tre67, Ch. I.7, Prop. 7.7]. So $\|d\rho(\xi_1) \cdots d\rho(\xi_n)\psi\| \leq p(\xi_1) \cdots p(\xi_n)\|\psi\|$, where $\xi_j \in \mathfrak{g}$ for $j \in \mathbb{N}$ and $\psi \in \mathcal{H}_\rho$. So if $B \subseteq \mathfrak{g}$ is bounded, then with $M := \sup p(B) < \infty$ we get that

$$q_B(\psi) := \sum_{n=0}^{\infty} \frac{1}{n!} \sup_{\xi_i \in B} \|d\rho(\xi_1) \cdots d\rho(\xi_n)\psi\| \leq \sum_{n=0}^{\infty} \frac{M^n}{n!} \|\psi\| = e^M \|\psi\| < \infty.$$

Using Lemma 6.2.20, this proves that $\mathcal{H}_\rho^{\mathcal{O}_b} = \mathcal{H}_\rho$.

Example 6.3.9 (Positive energy representations of Heisenberg groups)**.**
We recall the construction of positive energy representations of Heisenberg groups, and show that they admit a dense set of b-strongly-entire vectors. Let V be a real Fréchet space and ω a non-degenerate continuous skew bilinear form $V \times V \to \mathbb{R}$. Let $G := \mathrm{Heis}(V, \omega)$ be the corresponding Heisenberg group, so its underlying set is $\mathbb{T} \times V$ and it has multiplication $(z_1, v_1) \cdot (z_2, v_2) := (z_1 z_2 e^{-i\omega(v_1, v_2)}, v_1 + v_2)$. As V is a Frèchet space, it is Mackey complete by [KM97, Thm. I.4.11]. Using [Nee06, Thm. V.1.8], this implies that G is regular. Let $G_{\mathbb{C}} := \mathrm{Heis}(V_{\mathbb{C}}, \omega)$ be the corresponding complexification. Let $\mathcal{J}$ be a compatible positive complex structure on V, meaning that $\mathcal{J}^*\omega = \omega$ and $\omega(v, \mathcal{J}v) > 0$ for any non-zero $v \in V$. The positive-definite sesquilinear form $\langle v, w \rangle := \omega(v, \mathcal{J}w) + i\omega(v, w)$ makes V into a complex pre-Hilbert space, whose completion we denote by $V_{\mathcal{J}}$. Notice that the inclusion $V \to V_{\mathcal{J}}$ is continuous. Equip the symmetric algebra $\mathrm{S}^\bullet(V_{\mathcal{J}})$ with the

inner product satisfying

$$\langle v_1 \cdots v_n, w_1 \cdots w_n \rangle = \sum_{\sigma \in S_n} \prod_{j=1}^{n} \langle v_j, w_{\sigma_j} \rangle, \qquad \text{for } v_j, w_j \in V_{\mathcal{J}}. \tag{6.3.1}$$

Let $\mathcal{H}_\rho$ be the corresponding Hilbert space completion of $\mathrm{S}^\bullet(V_{\mathcal{J}})$. Then $\mathcal{H}_\rho$ contains and is generated by the "coherent states" $e^v := \sum_{n=0}^{\infty} \frac{1}{n!} v^n \in \mathcal{H}_\rho$ for $v \in V_{\mathcal{J}}$, and there is a unitary representation ρ of $\mathrm{Heis}(V, \omega)$ on $\mathcal{H}_\rho$ satisfying ([PS86, Sec. 9.5]):

$$\rho(z, v) e^w = z e^{-\frac{1}{2}\|v\|^2 - \langle v, w \rangle} e^{v+w}, \qquad \text{for } v, w \in V \text{ and } z \in \mathbb{T}.$$

A direct computation verifies the equation $\rho(v_1)\rho(v_2) = e^{-i\omega(v_1, v_2)} \rho(v_1 + v_2)$ for $v_1, v_2 \in V$. Let $\Omega \in \mathcal{H}_\rho$ be the vacuum vector. The map

$$G \to \mathbb{C}, \qquad (z, v) \mapsto \langle \Omega, \rho(z, v) \Omega \rangle = z e^{-\frac{1}{2}\|v\|^2}$$

is smooth, so it follows from [Nee10a, Thm. 7.2] that $\mathcal{H}_\rho^\infty$ contains the cyclic vector Ω and is therefore dense in $\mathcal{H}_\rho$. So ρ is smooth. The infinitesimal $\mathfrak{g}$-action $d\rho$ satisfies $d\rho(v)\psi = (\mathfrak{c}(v) - \mathfrak{a}(v))\psi$ for any $v, w \in V$ and $\psi \in \mathrm{S}^\bullet(V_{\mathcal{J}})$, where $\mathfrak{c}(v)\psi = v\psi$ is the creation operator with core $\mathrm{S}^\bullet(V_{\mathcal{J}})$ and $\mathfrak{a}(v) := \mathfrak{c}(v)^*$ is its adjoint, the annihilation operator. From $\mathfrak{c}(\mathcal{J}v) = i\mathfrak{c}(v)$ and $\mathfrak{a}(\mathcal{J}v) = -i\mathfrak{a}(v)$ we obtain that the $\mathbb{C}$-linear extension of $d\rho$ to $\mathfrak{g}_\mathbb{C}$ satisfies $d\rho(v + iw) = \mathfrak{c}(v + \mathcal{J}w) - \mathfrak{a}(v - \mathcal{J}w)$ for $v, w \in V$.

To see that $\mathcal{H}_\rho^{\mathcal{O}_b}$ is dense in $\mathcal{H}_\rho$, it suffices to show that it contains the cyclic vector Ω, because $\mathcal{H}_\rho^{\mathcal{O}_b}$ is G-invariant (cf. Lemma 6.3.19 below). Let B be the open unit-ball in $V_{\mathcal{J}}$. Let $K \subseteq V$ be a bounded subset of the real Fréchet space V. Then K is also bounded as subspace of $V_{\mathcal{J}}$, and is thus contained in sB for some $s > 0$. If $v \in B$, then ([BR97, p. 9])

$$\| \mathfrak{c}(v)|_{\mathrm{S}^n(V_{\mathcal{J}})} \| = \| \mathfrak{a}(v)|_{\mathrm{S}^{n+1}(V_{\mathcal{J}})} \| < \sqrt{n+1}.$$

So if $(v_j)_{j \in \mathbb{N}}$ is a sequence in B, then $\sup_{v_1, \ldots, v_n \in B} \|d\rho(v_n) \cdots d\rho(v_1)\Omega\| < 2^n \sqrt{n!}$ for any $n \in \mathbb{N}$. Consequently,

$$q_K(\Omega) \leq q_{sB}(\Omega) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \sup_{v_j \in B} \|d\rho(v_n) \cdots d\rho(v_1)\Omega\| < \sum_{n=0}^{\infty} \frac{(2s)^n}{\sqrt{n!}} < \infty.$$

It follows using Lemma 6.2.20 that $\Omega \in \mathcal{H}_\rho^{\mathcal{O}_b}$. Hence $\mathcal{H}_\rho^{\mathcal{O}_b}$ is dense in $\mathcal{H}_\rho$ and ρ is b-strongly-entire.

6.3.2 Properties of $\mathcal{H}_\rho^{\mathcal{O}}$ and holomorphic extensions

Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of G. Throughout this section, we assume in addition that G is a *regular* Lie group. Our goal is to determine various properties of the locally convex space $\mathcal{H}_\rho^{\mathcal{O}}$. These are summarized below:

Theorem 6.3.10. *The locally convex space $\mathcal{H}_\rho^{\mathcal{O}}$ has the following properties:*

1. *The inclusion $\mathcal{H}_\rho^{\mathcal{O}} \hookrightarrow \mathcal{H}_\rho^\infty$ is continuous w.r.t. the weak topology on $\mathcal{H}_\rho^\infty$.*
2. *$\mathcal{H}_\rho^{\mathcal{O}}$ is Hausdorff and complete.*
3. *$\mathcal{H}_\rho^{\mathcal{O}}$ is both G- and $\mathfrak{g}$-invariant.*
4. *The series $\sum_{m=0}^\infty \frac{1}{m!} d\rho(\eta^m)\psi$ converges in $\mathcal{H}_\rho^{\mathcal{O}}$ for every $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$ and $\eta \in \mathfrak{g}_\mathbb{C}$. The corresponding map*
$$\mathfrak{g}_\mathbb{C} \times \mathcal{H}_\rho^{\mathcal{O}} \to \mathcal{H}_\rho^{\mathcal{O}}, \qquad (\eta, \psi) \mapsto \sum_{m=0}^\infty \frac{1}{m!} d\rho(\eta^m)\psi =: \widetilde{\rho}_\mathbb{C}(\eta)\psi \tag{6.3.2}$$
is separately continuous and extends the map $\mathfrak{g} \times \mathcal{H}_\rho^{\mathcal{O}} \to \mathcal{H}_\rho^{\mathcal{O}}, (\eta, \psi) \mapsto \rho(e^\eta)\psi$. In particular, the function $\mathfrak{g}_\mathbb{C} \to \mathcal{H}_\rho^{\mathcal{O}},\ \eta \mapsto \widetilde{\rho}_\mathbb{C}(\eta)\psi$ is entire for every $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$.
5. *For any $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$, the orbit map $G \to \mathcal{H}_\rho^{\mathcal{O}},\ g \mapsto \rho(g)\psi$ is real-analytic.*

Before proceeding with the proof of Theorem 6.3.10, we first mention some important corollaries and related remarks.

Corollary 6.3.11. *Assume that $\mathcal{H}_\rho^{\mathcal{O}}$ is dense in $\mathcal{H}_\rho$. Define the map*
$$\widetilde{\rho}_\mathbb{C} : \mathfrak{g}_\mathbb{C} \to \mathcal{B}(\mathcal{H}_\rho^{\mathcal{O}}), \qquad \widetilde{\rho}_\mathbb{C}(\eta)v := \sum_{n=0}^\infty \frac{1}{n!} d\rho(\eta^n)v. \tag{6.3.3}$$
Let $U \subseteq \mathfrak{g}_\mathbb{C}$ be open and convex. Assume that $U \cap \mathfrak{g}$ is non-empty and open in $\mathfrak{g}$. Suppose that the BCH series defines a complex-analytic map $* : U \times U \to \mathfrak{g}_\mathbb{C}$. Then $\widetilde{\rho}_\mathbb{C}(\xi * \eta) = \widetilde{\rho}_\mathbb{C}(\xi)\widetilde{\rho}_\mathbb{C}(\eta)$ for any $(\xi, \eta) \in U \times U$.

Proof. Define $U_\mathbb{R} := U \cap \mathfrak{g}$. Let $v, w \in \mathcal{H}_\rho^{\mathcal{O}} \subseteq \mathcal{H}_\rho^\infty$. Recall from Remark 6.2.18 that $d\rho(x)^\dagger = d\rho(x^*)$ in $\mathcal{L}^\dagger(\mathcal{H}_\rho^\infty)$ for any $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. Using Theorem 6.3.10(4) and the fact that compositions of analytic maps are again analytic [BS71a, Thm. 6.4], it follows that the two maps $U^2 \to \mathbb{C}$ given by
$$\begin{aligned} &(\xi, \eta) \mapsto \langle v, \widetilde{\rho}_\mathbb{C}(\xi)\widetilde{\rho}_\mathbb{C}(\eta)w\rangle = \langle \widetilde{\rho}_\mathbb{C}(\xi^*)v, \widetilde{\rho}_\mathbb{C}(\eta)w\rangle, \\ &(\xi, \eta) \mapsto \langle v, \widetilde{\rho}_\mathbb{C}(\xi * \eta)w\rangle \end{aligned}$$
are both complex-analytic. They agree on the real subspace $U_\mathbb{R}^2$, on which they both equal $(\xi, \eta) \mapsto \langle v, \rho(e^\xi)\rho(e^\eta)w\rangle = \langle v, \rho(e^\xi e^\eta)w\rangle$. It follows using Proposition 6.2.14 that they must be equal everywhere. Since $\mathcal{H}_\rho^{\mathcal{O}}$ is dense in $\mathcal{H}_\rho$, we find with $\xi, \eta \in U$ that $\widetilde{\rho}_\mathbb{C}(\xi)\widetilde{\rho}_\mathbb{C}(\eta)w = \widetilde{\rho}_\mathbb{C}(\xi * \eta)w$ for every $w \in \mathcal{H}_\rho^{\mathcal{O}}$, and therefore that $\widetilde{\rho}_\mathbb{C}(\xi)\widetilde{\rho}_\mathbb{C}(\eta) = \widetilde{\rho}_\mathbb{C}(\xi * \eta)$. ☐

Corollary 6.3.12. *Let $(\rho, \mathcal{H}_\rho)$ be a strongly-entire unitary G-representation and define $\widetilde{\rho}_\mathbb{C} : \mathfrak{g}_\mathbb{C} \to \mathcal{B}(\mathcal{H}_\rho^\mathcal{O})$ by equation (6.3.3). Let $G_\mathbb{C}$ be a regular 1-connected complex BCH Fréchet-Lie group with $\mathbf{L}(G_\mathbb{C}) = \mathfrak{g}_\mathbb{C}$. Then there is a representation*

$$\rho_\mathbb{C} : G_\mathbb{C} \to \mathcal{B}(\mathcal{H}_\rho^\mathcal{O})^\times$$

of $G_\mathbb{C}$ on $\mathcal{H}_\rho^\mathcal{O}$ satisfying $\rho_\mathbb{C}(e^\xi) = \widetilde{\rho}_\mathbb{C}(\xi)$ for all $\xi \in \mathfrak{g}_\mathbb{C}$, and for which the orbit map $G_\mathbb{C} \to \mathcal{H}_\rho^\mathcal{O}$, $g \mapsto \rho_\mathbb{C}(g)\psi$ is complex-analytic for every $\psi \in \mathcal{H}_\rho^\mathcal{O}$.

Proof. As $G_\mathbb{C}$ is a complex BCH Lie group, there are open symmetric convex 0-neighborhoods $U, U' \subseteq \mathfrak{g}_\mathbb{C}$ such that $U \subseteq U'$, $U \cap \mathfrak{g}$ is open in $\mathfrak{g}$ and the BCH series $*$ defines a complex-analytic map $* : U \times U \to U' \subseteq \mathfrak{g}_\mathbb{C}$. Shrinking U and U' if necessary, we may further assume that the restriction of $\exp_{G_\mathbb{C}}$ to U' is biholomorphic onto some open 1-neighborhood V of $G_\mathbb{C}$. Define the function $f : V \to \mathcal{B}(\mathcal{H}_\rho^\mathcal{O})$ by $f(e^\xi) := \widetilde{\rho}_\mathbb{C}(\xi)$. In view of Corollary 6.3.11, f satisfies

$$f(e^\xi e^\eta) = f(e^{\xi * \eta}) = \widetilde{\rho}_\mathbb{C}(\xi * \eta) = \widetilde{\rho}_\mathbb{C}(\xi)\widetilde{\rho}_\mathbb{C}(\eta) = f(e^\xi)f(e^\eta), \qquad \forall \xi, \eta \in U, \quad (6.3.4)$$

where the first equality follows from [Nee06, Thm. IV.2.8] and Proposition 6.2.15. In particular $f(e^\xi) \in \mathcal{B}(\mathcal{H}_\rho^\mathcal{O})^\times$ and $f(e^\xi)^{-1} = f(e^{-\xi})$ for any $\xi \in U$. As $G_\mathbb{C}$ is a 1-connected topological group, (6.3.4) further implies that there is a group homomorphism $\rho_\mathbb{C} : G_\mathbb{C} \to \mathcal{B}(\mathcal{H}_\rho^\mathcal{O})^\times$ extending f (cf. [GN, Proposition C.2.1]). Let $\psi \in \mathcal{H}_\rho^\mathcal{O}$. As $\exp_{G_\mathbb{C}}$ restricts to a biholomorphic map $U' \to V$, it follows using Theorem 6.3.10(4) that the map

$$V \to \mathcal{H}_\rho^\mathcal{O}, \qquad e^\xi \mapsto \rho_\mathbb{C}(e^\xi)\psi = f(e^\xi)\psi = \widetilde{\rho}_\mathbb{C}(\xi)\psi, \qquad \xi \in U'$$

is complex-analytic. As $G_\mathbb{C}$ is a complex-analytic Lie group and $V \subseteq G_\mathbb{C}$ is an open 1-neighborhood, this implies that orbit map $g \mapsto \rho_\mathbb{C}(g)\psi$ is complex-analytic $G_\mathbb{C} \to \mathcal{H}_\rho^\mathcal{O}$. The map $\mathfrak{g}_\mathbb{C} \to \mathcal{H}_\rho^\mathcal{O}$, $\xi \mapsto \rho_\mathbb{C}(e^\xi)\psi$ is therefore complex-analytic, and it agrees by construction with the entire map $\xi \mapsto \widetilde{\rho}(\xi)\psi$ on the open 0-neighborhood U' of $\mathfrak{g}_\mathbb{C}$. It follows by Proposition 6.2.14 that they are equal everywhere, so $\rho_\mathbb{C}(e^\xi)\psi = \widetilde{\rho}(\xi)\psi$ for every $\xi \in \mathfrak{g}_\mathbb{C}$. Since this holds for all $\psi \in \mathcal{H}_\rho^\mathcal{O}$, we conclude that $\rho_\mathbb{C}(e^\xi) = \widetilde{\rho}(\xi)$ for all $\xi \in \mathfrak{g}_\mathbb{C}$. ∎

One might wonder whether or not the map in equation (6.3.2) is also jointly continuous. The following example shows that this is generally false:

Example 6.3.13. Consider the Frèchet-Lie group $G = \mathbb{R}^\mathbb{N}$, equipped with the product topology. Let $\mathfrak{g} = \mathbb{R}^\mathbb{N}$ be its Lie algebra. Notice that G is also regular and BCH. Consider the unitary representation of G on $\mathcal{H}_\rho := \ell^2(\mathbb{N}, \mathbb{C})$ defined by $(\rho(g)\psi)(k) := e^{ig(k)}\psi(k)$ for $g \in G$ and $\psi \in \ell^2(\mathbb{N}, \mathbb{C})$. Letting $\mathbb{C}^{(\mathbb{N})}$ denote the space of sequences in $\mathbb{C}$ with only finitely many non-zero components, the space $\mathcal{H}_\rho^\infty$ of smooth vectors is $\mathcal{H}_\rho^\infty = \mathbb{C}^{(\mathbb{N})}$ [Nee10a, Ex. 4.8]. Notice that $(d\rho(\xi)\psi)(k) = i\xi(k)\psi(k)$ for $\xi \in \mathfrak{g}_\mathbb{C}$ and $\psi \in \mathbb{C}^{(\mathbb{N})}$. The weak and strong topologies on $\mathcal{H}_\rho^\infty$ agree, and they both coincide with the locally convex inductive limit topology on $\mathbb{C}^{(\mathbb{N})}$ [JN19, Ex. 11]. This is the strongest locally convex topology on $\mathbb{C}^{(\mathbb{N})}$ for which the inclusion $\mathbb{C}^N \hookrightarrow \mathbb{C}^{(\mathbb{N})}$ is continuous for every $N \in \mathbb{N}$. We claim in addition that $\mathcal{H}_\rho^{\mathcal{O}_b} = \mathcal{H}_\rho^{\mathcal{O}} = \mathbb{C}^{(\mathbb{N})}$ as locally convex spaces.

Let $\pi_k : \mathbb{C}^{\mathbb{N}} \to \mathbb{C},\ \psi \mapsto \psi(k)$ be the projection onto the k-th component for $k \in \mathbb{N}$. Let $\psi \in \mathbb{C}^{(\mathbb{N})}$. Then there exists $N \in \mathbb{N}$ s.t. $\psi(k) = 0$ for all $k > N$. Let $B \subseteq \mathfrak{g}_{\mathbb{C}}$ be a compact subset, and notice that B is contained in the compact set $B' := \prod_{k=1}^\infty B_k$, where $B_k := \pi_k(B)$. Set $M := \sup\{|\xi(k)| : \xi \in B',\ 1 \leq k \leq N\} < \infty$. Then

$$\|d\rho(\xi_1 \cdots \xi_n)\psi\|_{\ell^2}^2 = \sum_{k=1}^N |\xi_1(k) \cdots \xi_n(k)\psi(k)|^2 \leq M^{2n}\|\psi\|_{\ell^2}^2, \qquad \forall \xi_1, \cdots, \xi_n \in B'.$$

We thus obtain that

$$q_B(\psi) \leq q_{B'}(\xi) = \sum_{n=0}^\infty \frac{1}{n!} \sup_{\xi_j \in B'} \|d\rho(\xi_1 \cdots \xi_n)\psi\|_{\ell^2} \leq e^M \|\psi\|_{\ell^2} < \infty. \tag{6.3.5}$$

Hence $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$. So $\mathbb{C}^{(\mathbb{N})} \subseteq \mathcal{H}_\rho^{\mathcal{O}}$. We also have $\mathcal{H}_\rho^{\mathcal{O}} \subseteq \mathcal{H}_\rho^\infty = \mathbb{C}^{(\mathbb{N})}$, so $\mathcal{H}_\rho^{\mathcal{O}} = \mathbb{C}^{(\mathbb{N})}$. Noticing that the constant M in (6.3.5) depends only on N and B, the estimate (6.3.5) moreover shows that the inclusion $\mathbb{C}^N \hookrightarrow \mathcal{H}_\rho^{\mathcal{O}}$ is continuous for every $N \in \mathbb{N}$. The locally convex inductive limit topology on $\mathbb{C}^{(\mathbb{N})}$ is therefore finer than that of $\mathcal{H}_\rho^{\mathcal{O}}$. As the inclusion $\mathcal{H}_\rho^{\mathcal{O}} \hookrightarrow \mathcal{H}_\rho^\infty = \mathbb{C}^{(\mathbb{N})}$ is moreover continuous w.r.t. the weak topology on $\mathcal{H}_\rho^\infty$, by Theorem 6.3.10(1), it follows that $\mathcal{H}_\rho^{\mathcal{O}} = \mathcal{H}_\rho^\infty = \mathbb{C}^{(\mathbb{N})}$ as locally convex vector spaces. Since $\prod_{k=1}^\infty B_k \subseteq \mathbb{R}^{\mathbb{N}}$ is bounded whenever every $B_k \subseteq \mathbb{R}$ is so, it is similarly shown that $\mathcal{H}_\rho^{\mathcal{O}_b} = \mathbb{C}^{(\mathbb{N})}$ as locally convex vector spaces.

Now, it is shown in [Nee10a, Ex. 4.8] that the action

$$\mathfrak{g} \times \mathbb{C}^{(\mathbb{N})} \to \ell^2(\mathbb{N}, \mathbb{C}), \qquad (\xi, \psi) \mapsto d\rho(\xi)\psi$$

is not jointly continuous (w.r.t. *any* locally convex topology on $\mathbb{C}^{(\mathbb{N})}$). This also implies that the map

$$\mathfrak{g}_{\mathbb{C}} \times \mathbb{C}^{(\mathbb{N})} \to \mathbb{C}^{(\mathbb{N})}, \qquad (\xi, \psi) \mapsto \sum_{k=0}^\infty \frac{1}{k!} d\rho(\xi^k)\psi$$

can not be jointly continuous.

Remark 6.3.14. Replacing 'compact' by 'bounded' and $\mathcal{H}_\rho^{\mathcal{O}}$ by $\mathcal{H}_\rho^{\mathcal{O}_b}$ in the proof of Theorem 6.3.10 (given shortly), one verifies that all statements in Theorem 6.3.10 remain true if $\mathcal{H}_\rho^{\mathcal{O}}$ is replaced by the locally convex space $\mathcal{H}_\rho^{\mathcal{O}_b}$. The inclusion $\mathcal{H}_\rho^{\mathcal{O}_b} \hookrightarrow \mathcal{H}_\rho^\infty$ is moreover evidently continuous w.r.t. the strong topology on $\mathcal{H}_\rho^\infty$.

Remark 6.3.15. It may in certain situations be desirable to consider $\mathcal{H}_\rho^{\mathcal{O}_b}$ instead of $\mathcal{H}_\rho^{\mathcal{O}}$. For example, if G is a Banach-Lie group, then the locally convex topology on $\mathcal{H}_\rho^{\mathcal{O}_b}$ is Fréchet. Since the map

$$\mathfrak{g}_{\mathbb{C}} \times \mathcal{H}_\rho^{\mathcal{O}_b} \to \mathcal{H}_\rho^{\mathcal{O}_b}, \qquad (\xi, \psi) \mapsto \sum_{m=0}^\infty \frac{1}{m!} d\rho(\eta^m)\psi = \widetilde{\rho}(\xi)\psi \tag{6.3.6}$$

is separately continuous and linear in the $\mathcal{H}_\rho^{\mathcal{O}_b}$-variable, this implies using [Nee10a, Prop. 5.1] that the function in (6.3.6) is *jointly* continuous, and is therefore entire.

In this case we consequently obtain a stronger analogue of Corollary 6.3.12. Indeed, take $U \subseteq \mathfrak{g}_\mathbb{C}$ as in Corollary 6.3.11, and suppose that the BCH series defines a complex-analytic map $* : U \times U \to \mathfrak{g}_\mathbb{C}$. The functions $(\xi,\eta) \mapsto \widetilde{\rho}_\mathbb{C}(\xi)\widetilde{\rho}_\mathbb{C}(\eta)v$ and $(\xi,\eta) \mapsto \widetilde{\rho}_\mathbb{C}(\xi * \eta)v$ are then both complex-analytic $U \times U \to \mathcal{H}_\rho^{\mathcal{O}_b}$ for every $v \in \mathcal{H}_\rho^{\mathcal{O}_b}$. As they agree on $U_\mathbb{R} \times U_\mathbb{R}$, the must be equal on $U \times U$. Consequently, with $G_\mathbb{C}$ as in Corollary 6.3.12, we obtain a representation $\rho_\mathbb{C} : G_\mathbb{C} \to \mathcal{B}(\mathcal{H}_\rho^{\mathcal{O}_b})^\times$ that satisfies $\rho_\mathbb{C}(e^\xi) = \widetilde{\rho}_\mathbb{C}(\xi)$ for all ξ in some 0-neighborhood of $\mathfrak{g}_\mathbb{C}$, and for which the corresponding action $G_\mathbb{C} \times \mathcal{H}_\rho^{\mathcal{O}_b} \to \mathcal{H}_\rho^{\mathcal{O}_b}$ is complex-analytic. We will come back to this point in Section 6.5 below. Notice also that Example 6.3.13 above shows that (6.3.6) need not be jointly continuous if G is only assumed to be Fréchet.

The proof of Theorem 6.3.10

Lemma 6.3.16. *Let $B \subseteq \mathfrak{g}_\mathbb{C}$ be compact and let $\psi \in \mathcal{H}_\rho^\mathcal{O}$. Then $\frac{1}{n!}p_B^n(\psi) \leq q_B(\psi)$ for any $n \in \mathbb{N}$. In particular, the inclusion $\mathcal{H}_\rho^\mathcal{O} \hookrightarrow \mathcal{H}_\rho^\infty$ is continuous w.r.t. the weak topology on $\mathcal{H}_\rho^\infty$.*

Proof. Let $\psi \in \mathcal{H}_\rho^\mathcal{O}$. It is trivial that $\frac{1}{n!}p_B^n(\psi) \leq q_B(\psi)$. For the final statement, consider the continuous seminorm $p_{\boldsymbol{\xi}}(\psi) := \|d\rho(\xi_1 \cdots \xi_n)\psi\|$ on $\mathcal{H}_\rho^\infty$ for some $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathfrak{g}^n$. Taking for B the finite set $B := \{\xi_1, \dots, \xi_n\} \subseteq \mathfrak{g}_\mathbb{C}$, we obtain that $\frac{1}{n!}p_{\boldsymbol{\xi}}(\psi) \leq \frac{1}{n!}p_B^n(\psi) \leq q_B(\psi)$. ☐

Lemma 6.3.17. *$\mathcal{H}_\rho^\mathcal{O}$ is both Hausdorff and complete.*

Proof. It is clear that $\mathcal{H}_\rho^\mathcal{O}$ is Hausdorff, because $\mathcal{H}_\rho^\infty$ is so. Let us show that it is complete. Let $(\psi_\alpha)_{\alpha\in I}$ be a Cauchy net in $\mathcal{H}_\rho^\mathcal{O}$. Then it is also a Cauchy net in $\mathcal{H}_\rho^\infty$. The latter is complete [JN19, Prop. 3.19], where we use that G is a regular Fréchet-Lie group. Thus $\psi_\alpha \to \psi$ in $\mathcal{H}_\rho^\infty$ for some $\psi \in \mathcal{H}_\rho^\infty$. We must show that $\psi \in \mathcal{H}_\rho^\mathcal{O}$ and $\psi_\alpha \to \psi$ in $\mathcal{H}_\rho^\mathcal{O}$. Fix a compact set $B \subseteq \mathfrak{g}$. Let $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $\epsilon_0(1+\epsilon_0) < \epsilon$. Let $t > 1$ be such that $\frac{t}{t-1} < 1 + \epsilon_0$. As $(\psi_\alpha)_{\alpha\in I}$ is a Cauchy net in $\mathcal{H}_\rho^\mathcal{O}$, there exists $\gamma \in I$ such that $q_{tB}(\psi_\alpha - \psi_\beta) < \epsilon_0$ whenever $\alpha, \beta \geq \gamma$. In particular $\frac{1}{k!}p_B^k(\psi_\alpha - \psi_\beta) < \epsilon_0 t^{-k}$ for any $\alpha, \beta \geq \gamma$ and $k \in \mathbb{N}_{\geq 0}$. Consequently, for any $\xi_i \in B$ with $i \in \{1, \dots, k\}$ we have (using that $\psi_\alpha \to \psi$ in $\mathcal{H}_\rho^\infty$):

$$\frac{1}{k!}\|d\rho(\xi_1 \cdots \xi_k)(\psi - \psi_\beta)\| = \frac{1}{k!}\lim_\alpha \|d\rho(\xi_1 \cdots \xi_k)(\psi_\alpha - \psi_\beta)\| \leq \epsilon_0 t^{-k} \qquad \text{for } \beta \geq \gamma.$$

Thus $\frac{1}{k!}p_B^k(\psi - \psi_\beta) \leq \epsilon_0 t^{-k}$ for any $\beta \geq \gamma$. Hence

$$q_B(\psi - \psi_\beta) = \sum_{k=0}^\infty \frac{1}{k!}p_B^k(\psi - \psi_\beta) \leq \epsilon_0 \sum_{k=0}^\infty t^{-k} = \frac{t}{t-1}\epsilon_0 \leq \epsilon_0(1+\epsilon_0) < \epsilon, \qquad \forall \beta \geq \gamma$$

This shows that $q_B(\psi) \leq q_B(\psi - \psi_\beta) + q_B(\psi_\beta) < \infty$ and that $q_B(\psi - \psi_\beta) < \epsilon$ for all $\beta \geq \gamma$. As B and ϵ were arbitrary, we conclude (using the proof of Lemma 6.2.20) that $\psi \in \mathcal{H}_\rho^\mathcal{O}$ and $\psi_\alpha \to \psi$ in $\mathcal{H}_\rho^\mathcal{O}$. ☐

Lemma 6.3.18. *Let $B, B_0 \subseteq \mathfrak{g}_\mathbb{C}$ be compact subsets and let $t > 1$. Then there exists a compact subset $B' \subseteq \mathfrak{g}_\mathbb{C}$ and some $C > 0$, both depending on B, B_0 and t, such that $B \subseteq B'$ and*

$$\frac{1}{m!}\sum_{n=0}^{\infty}\frac{1}{n!}\sup_{\eta_j\in B_0} p_B^n(d\rho(\eta_1\cdots\eta_m)\psi) \leq Ct^{-m}q_{B'}(\psi) \tag{6.3.7}$$

for every $m \in \mathbb{N}_{\geq 0}$ and $\psi \in \mathcal{H}_\rho^\mathcal{O}$. In particular, we have

$$\frac{1}{m!}q_B(d\rho(\eta^m)\psi) \leq Ct^{-m}q_{B'}(\psi)$$

for any $\psi \in \mathcal{H}_\rho^\mathcal{O}$, $\eta \in B_0$ and $m \in \mathbb{N}_{\geq 0}$.

Proof. We may assume that B_0 and B are both balanced. Define $B'' := B \cup B_0$, which is again compact and balanced in $\mathfrak{g}_\mathbb{C}$. For any $\eta_1, \ldots, \eta_m \in B_0$ and $\psi \in \mathcal{H}_\rho^\mathcal{O}$ we have

$$p_B^n(d\rho(\eta_1\cdots\eta_m)\psi) \leq p_{B''}^{n+m}(\psi) = t^{-(n+m)}p_{tB''}^{n+m}(\psi).$$

Hence $\sup_{\eta_j\in B_0} p_B^n(d\rho(\eta_1\cdots\eta_m)\psi) \leq t^{-(n+m)}p_{tB''}^{n+m}(\psi)$. It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n!}\sup_{\eta_j\in B_0} p_B^n(d\rho(\eta_1\cdots\eta_m)\psi) &\leq t^{-m}\sum_{n=0}^{\infty}\frac{t^{-n}}{n!}p_{tB''}^{n+m}(\psi)\\
&\leq t^{-m}\left(\sum_{n=0}^{\infty}t^{-n}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}p_{tB''}^{n+m}(\psi)\right)\\
&= \frac{t^{-m}}{1-t^{-1}}\sum_{n=0}^{\infty}\frac{1}{n!}p_{tB''}^{n+m}(\psi)
\end{aligned}$$

Let $s > 2$. Notice that $\sum_{n=0}^{\infty}\frac{(n+m)!}{n!}s^{-(n+m)} < \infty$, and that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n!}p_{tB''}^{n+m}(\psi) &= \sum_{n=0}^{\infty}\left(\frac{(n+m)!}{n!}s^{-(n+m)}\cdot\frac{1}{(n+m)!}p_{stB''}^{n+m}(\psi)\right)\\
&\leq \left(\sum_{n=0}^{\infty}\frac{(n+m)!}{n!}s^{-(n+m)}\right)\cdot q_{stB''}(\psi).
\end{aligned}$$

Consequently, with $C_m := \sum_{n=0}^{\infty}\frac{(n+m)!}{m!n!}s^{-(n+m)} > 0$ and $B' := stB''$, we have:

$$\frac{1}{m!}\sum_{n=0}^{\infty}\frac{1}{n!}\sup_{\eta_j\in B_0} p_B^n(d\rho(\eta_1\cdots\eta_m)\psi) \leq C_m\frac{t^{-m}}{1-t^{-1}}q_{B'}(\psi). \tag{6.3.8}$$

Using $\sum_{k=0}^{N}\binom{N}{k} = 2^N$, notice that $\sum_{m=0}^{\infty}C_m = \sum_{N=0}^{\infty}\left(\frac{2}{s}\right)^N < \infty$, and hence the sequence $(C_m)_{m\in\mathbb{N}_{\geq 0}}$ is bounded. So there exists $C > 0$ with $C_m \leq (1-t^{-1})C$ for all $m \in \mathbb{N}_{\geq 0}$. Now simply observe using (6.3.8) that (6.3.7) holds for this C and B'. Notice also that $B \subseteq B'$. ☐

Lemma 6.3.19. *$\mathcal{H}_\rho^\mathcal{O}$ is both G- and $\mathfrak{g}$-invariant.*

Proof. Let $\psi \in \mathcal{H}_\rho^\mathcal{O}$ and let $B \subseteq \mathfrak{g}_\mathbb{C}$ be compact. As the adjoint action of G on $\mathfrak{g}_\mathbb{C}$ is continuous, $\mathrm{Ad}_g(B)$ is again compact in $\mathfrak{g}_\mathbb{C}$ for every $g \in G$. Since $\rho(g)$ is unitary, we find that $q_B(\rho(g)\psi) = q_{\mathrm{Ad}_{g^{-1}}(B)}(\psi) < \infty$. Thus $\rho(g)\psi \in \mathcal{H}_\rho^\mathcal{O}$ and so $\mathcal{H}_\rho^\mathcal{O}$ is G-invariant. The $\mathfrak{g}$-invariance of $\mathcal{H}_\rho^\mathcal{O}$ is immediate from Lemma 6.3.18. □

Lemma 6.3.20. *Define for every $m \in \mathbb{N}_{\geq 0}$ the function*

$$f_m : \mathfrak{g}_\mathbb{C} \times \mathcal{H}_\rho^\mathcal{O} \to \mathcal{H}_\rho^\mathcal{O}, \qquad f_m(\xi, \psi) := \frac{1}{m!} d\rho(\xi^m)\psi.$$

The series $\sum_{m=0}^\infty f_m(\xi,\psi)$ converges in $\mathcal{H}_\rho^\mathcal{O}$ for every $\xi \in \mathfrak{g}_\mathbb{C}$ and $\psi \in \mathcal{H}_\rho^\mathcal{O}$. It defines a separately continuous map

$$f : \mathfrak{g}_\mathbb{C} \times \mathcal{H}_\rho^\mathcal{O} \to \mathcal{H}_\rho^\mathcal{O}, \qquad f(\xi, \psi) := \sum_{m=0}^\infty f_m(\xi,\psi). \tag{6.3.9}$$

In particular, the map $\mathfrak{g}_\mathbb{C} \to \mathcal{H}_\rho^\mathcal{O}$, $\xi \mapsto f(\xi,\psi)$ is entire for every $\psi \in \mathcal{H}_\rho^\mathcal{O}$.

Proof. Let $\xi \in \mathfrak{g}_\mathbb{C}$, $t > 1$ and let $B \subseteq \mathfrak{g}_\mathbb{C}$ be a compact subset. According to Lemma 6.3.18, there is a constant $C > 0$ and a compact subset $B' \subseteq \mathfrak{g}_\mathbb{C}$ s.t. $\frac{1}{m!} q_B(d\rho(\xi^m)\psi) \leq Ct^{-m} q_{B'}(\psi)$ for every $m \in \mathbb{N}_{\geq 0}$ and $\psi \in \mathcal{H}_\rho^\mathcal{O}$. So for $\psi \in \mathcal{H}_\rho^\mathcal{O}$ we have

$$\sum_{m=0}^\infty \frac{1}{m!} q_B(d\rho(\xi^m)\psi) \leq C\left(\sum_{m=0}^\infty t^{-m}\right) q_{B'}(\psi) = \frac{C}{1-t^{-1}} q_{B'}(\psi) < \infty.$$

We thus find that the series $\sum_{m=0}^\infty f_m(\xi,\psi)$ converges in $\mathcal{H}_\rho^\mathcal{O}$, and that

$$q_B(f(\xi,\psi)) \leq \frac{C}{1-t^{-1}} q_{B'}(\psi), \qquad \forall \psi \in \mathcal{H}_\rho^\mathcal{O}.$$

In particular, the linear map $\mathcal{H}_\rho^\mathcal{O} \to \mathcal{H}_\rho^\mathcal{O}$, $\psi \mapsto f(\xi,\psi)$ is continuous.

We now show that f is also separately continuous in the ξ-variable. Take $\psi \in \mathcal{H}_\rho^\mathcal{O}$ and define $f^\psi(\xi) := f(\xi,\psi)$ for $\xi \in \mathfrak{g}_\mathbb{C}$. Let $B_0 \subseteq \mathfrak{g}_\mathbb{C}$ be a compact subset. Consider the functions $h_m : B_0 \to \mathcal{H}_\rho^\mathcal{O}$ defined by $h_m(\xi) := \frac{1}{m!} d\rho(\xi^m)\psi$ for $m \in \mathbb{N}_{\geq 0}$. Define also $h := f^\psi\big|_{B_0}$. So $h(\xi) = \sum_{m=0}^\infty h_m(\xi) = f^\psi(\xi)$ for $\xi \in B_0$. We show that h_m is continuous for every $m \in \mathbb{N}_{\geq 0}$ and that $\sum_{m=0}^N h_m \to h$ uniformly on B_0 as $N \to \infty$. This will imply that h is continuous.

Let $m \in \mathbb{N}_{\geq 0}$. Assume that $(\eta_n)_{n\in\mathbb{N}}$ is a sequence in B_0 with $\eta_n \to \eta$ for some $\eta \in B_0$. Let $B \subseteq \mathfrak{g}_\mathbb{C}$ be a compact subset and let $t > 1$. Using Lemma 6.3.18, there exists a constant $C > 0$ and a compact subset $B' \subseteq \mathfrak{g}_\mathbb{C}$ containing B such that (6.3.7) holds true. Let $m \in \mathbb{N}_{\geq 0}$. Notice that

$$q_B(h_m(\eta_n) - h_m(\eta)) = \frac{1}{m!} q_B(d\rho(\eta_n^m - \eta^m)\psi) = \frac{1}{m!}\sum_{k=0}^\infty \frac{1}{k!} p_B^k(d\rho(\eta_n^m - \eta^m)\psi). \tag{6.3.10}$$

Since the multilinear map

$$\mathfrak{g}_{\mathbb{C}}^m \to \mathcal{H}_\rho^\infty, \qquad (\xi_1, \cdots, \xi_m) \mapsto d\rho(\xi_1, \cdots, \xi_m)\psi$$

is continuous w.r.t. the strong topology on $\mathcal{H}_\rho^\infty$ [JN19, Lem. 3.22] and p_B^k is a continuous seminorm on $\mathcal{H}_\rho^\infty$, it follows that the function

$$\tau_N : B_0 \to [0, \infty), \qquad \tau_N(\xi) := \sum_{k=0}^{N} \frac{1}{k!} p_B^k(d\rho(\xi^m - \eta^m)\psi)$$

is continuous for every $N \in \mathbb{N}$. Moreover, in view of (6.3.7) we have that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sup_{\xi \in B_0} p_B^k(d\rho(\xi^m - \eta^m)\psi) \leq 2 \sum_{k=0}^{\infty} \frac{1}{k!} \sup_{\zeta \in B_0} p_B^k(d\rho(\zeta^m)\psi) \leq 2C \frac{m!}{t^m} q_{B'}(\psi) < \infty.$$

It follows that the continuous functions τ_N converge uniformly to

$$\tau : B_0 \to [0, \infty), \qquad \tau(\xi) := \sum_{k=0}^{\infty} \frac{1}{k!} p_B^k(d\rho(\xi^m - \eta^m)\psi).$$

It follows that τ is continuous. In particular we obtain that $\tau(\eta_n) \to \tau(\eta) = 0$ as $n \to \infty$, which in view of equation (6.3.10) implies that

$$q_B(h_m(\eta_n) - h_m(\eta)) \to 0 \text{ as } n \to \infty.$$

We can thus conclude that h_m is continuous for every $m \in \mathbb{N}_{\geq 0}$.

With $t > 1$, B, B' and $C > 0$ as above, notice using Lemma 6.3.18 that

$$\sum_{m=0}^{\infty} \sup_{\xi \in B_0} q_B(h_m(\xi)) = \sum_{m=0}^{\infty} \frac{1}{m!} \sup_{\xi \in B_0} q_B(d\rho(\xi^m)\psi) \leq \frac{C}{1 - t^{-1}} q_{B'}(\psi) < \infty.$$

Hence $\sum_{m=0}^{N} h_m \to h$ uniformly on B_0 as $N \to \infty$. Thus $h = f^\psi\big|_{B_0}$ is continuous.

We have shown that $f^\psi\big|_{B_0} : B_0 \to \mathcal{H}_\rho^{\mathcal{O}}$ is continuous for every compact subset $B_0 \subseteq \mathfrak{g}_{\mathbb{C}}$. Since $\mathfrak{g}_{\mathbb{C}}$ is Fréchet and thus first-countable, it is compactly generated in the sense that a set $S \subseteq \mathfrak{g}_{\mathbb{C}}$ is open if and only if $S \cap B_0$ is open in B_0 for every compact $B_0 \subseteq \mathfrak{g}_{\mathbb{C}}$. From the fact that $f^\psi\big|_{B_0}$ is continuous for every compact subset $B_0 \subseteq \mathfrak{g}_{\mathbb{C}}$, it therefore follows that f^ψ is continuous. So f is indeed separately continuous. □

Lemma 6.3.21. *Consider the function f in* (6.3.9)*. For any $\xi \in \mathfrak{g}$ and $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$, we have $f(\xi, \psi) = \rho(e^\xi)\psi$.*

Proof. Let $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$. Then ψ is an analytic vector for ρ, by Corollary 6.2.24. In view of Lemma 6.3.20, we find that the two maps $\xi \mapsto \rho(e^\xi)\psi$ and $\xi \mapsto f(\xi, \psi)$ are both real analytic as maps $\mathfrak{g} \to \mathcal{H}_\rho$. They moreover have the same image under the jet-projection $j_0^\infty : C^\omega(\mathfrak{g}, \mathcal{H}_\rho) \to P(\mathfrak{g}, \mathcal{H}_\rho)$. Using Proposition 6.2.15, we conclude that $\rho(e^\xi)\psi = f(\xi, \psi)$ for all $\xi \in \mathfrak{g}$. □

Lemma 6.3.22. *The orbit map $G \to \mathcal{H}_\rho^{\mathcal{O}}$, $g \mapsto \rho(g)v$ is analytic for any $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$.*

Proof. Let $\psi \in \mathcal{H}_\rho^{\mathcal{O}}$. As the Lie group G is BCH, Lemma 6.3.20 and Lemma 6.3.21 together imply that the map $G \to \mathcal{H}_\rho^{\mathcal{O}}$, $g \mapsto \rho(g)\psi$ is analytic on U for some 1-neighborhood $U \subseteq G$, which implies the assertion. □

6.4 A general approach to holomorphic induction

In this section, we define and study a notion of holomorphic induction for unitary representations of Lie groups. The presented definition and results extend that of [Nee13], by removing the requirement of norm-continuity of the representation being induced. The precise setting we consider is as described below.

Let G be a connected BCH Fréchet-Lie group with Lie algebra $\mathfrak{g}$. Let $H \subseteq G$ be a connected and closed subgroup of G, and assume that H is a locally exponential Lie subgroup of G (cf. [Nee06, Def. IV.3.2]). Let $\mathfrak{h} := \mathbf{L}(H)$ be its Lie algebra, which we identify as Lie subalgebra of $\mathfrak{g}$ using the pushforward of the inclusion $H \hookrightarrow G$. We then have $\exp_G(\xi) = \exp_H(\xi)$ for all $\xi \in \mathfrak{h} \subseteq \mathfrak{g}$. We furthermore have:

Lemma 6.4.1. *There exists an open neighborhood U of $\mathfrak{g}$ s.t. $\exp_G|_U$ is an analytic diffeomorphism onto an open neighborhood in G and s.t.*

$$\exp_G(U \cap \mathfrak{h}) \;=\; \exp_G(U) \cap H.$$

In particular, H is BCH and an analytic embedded Lie subgroup of G.

Proof. Since H is a locally exponential Lie subgroup of G by assumption, it follows using [Nee06, Thm. IV.3.3] that there exists an open neighborhood $U \subseteq \mathfrak{g}$ such that $\exp_G|_U$ is a diffeomorphism onto an open 1-neighborhood in G and $\exp_G(U \cap \mathfrak{h}) = \exp_G(U) \cap H$. Consequently, H is an analytic embedded Lie subgroup of G, and the exponential map $\exp_G$ can be used to obtain analytic slice charts for H. Since G is BCH, it follows that also H is BCH. □

Let $\theta : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ be the conjugation defined by $\theta(\xi + i\eta) = \xi - i\eta$ for $\xi, \eta \in \mathfrak{g}$. We assume given a triangular decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_\pm$ and $\mathfrak{h}_\mathbb{C}$ are closed Lie subalgebras of $\mathfrak{g}_\mathbb{C}$ satisfying $\theta(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\mp$, $\theta(\mathfrak{h}_\mathbb{C}) \subseteq \mathfrak{h}_\mathbb{C}$ and $[\mathfrak{h}_\mathbb{C}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$. Set $\mathfrak{b}_\pm := \mathfrak{n}_\pm \rtimes \mathfrak{h}_\mathbb{C}$.

The structure of this chapter is as follows. In Section 6.4.1 we establish some notation and preliminary definitions, in particular specifying a certain space of functions on G that takes the role usually taken by the holomorphic sections of a complex homogeneous vector bundle over G/H. In Section 6.4.2 we present the definition of holomorphically induced representations and establish an equivalent characterization. We then proceed in Section 6.4.3, Section 6.4.4 and Section 6.4.5 to study the most important properties enjoyed by holomorphically induced representations.

As the theory of this section no longer has a clear interpretation in terms of holomorphic maps, we present in Section 6.5 a stronger notion that involves complex geometry. The approach presented there depends crucially on the availability of a dense set of b-strongly-entire vectors in the representation being induced.

6.4.1 A substitute for holomorphic sections

Let (σ, V_σ) be an analytic unitary representation of H. Let us establish some notation and preliminary definitions.

Definition 6.4.2. For $\xi \in \mathfrak{g}$, define the differential operators $\mathcal{L}_{\boldsymbol{v}(\xi)}$ and $\mathcal{L}_{\boldsymbol{v}(\xi)^r}$ on $C^\infty(G, V_\sigma)$ by

$$(\mathcal{L}_{\boldsymbol{v}(\xi)} f)(g) := \frac{d}{dt}\bigg|_{t=0} f(ge^{t\xi}),$$
$$(\mathcal{L}_{\boldsymbol{v}(\xi)^r} f)(g) := \frac{d}{dt}\bigg|_{t=0} f(e^{-t\xi}g), \qquad \forall g \in G,\ f \in C^\infty(G, V_\sigma).$$

Extend both $\xi \mapsto \mathcal{L}_{\boldsymbol{v}(\xi)}$ and $\xi \mapsto \mathcal{L}_{\boldsymbol{v}(\xi)^r}$ $\mathbb{C}$-linearly to $\mathfrak{g}_\mathbb{C}$ and further to algebra homomorphisms on $\mathcal{U}(\mathfrak{g}_\mathbb{C})$, so we have e.g. $\mathcal{L}_{\boldsymbol{v}(\xi_1\cdots\xi_n)^r} = \mathcal{L}_{\boldsymbol{v}(\xi_1)^r}\cdots\mathcal{L}_{\boldsymbol{v}(\xi_n)^r}$ for all $\xi_k \in \mathfrak{g}_\mathbb{C}$ and $k \in \{1, \dots, n\}$.

Remark 6.4.3. We thus adopt the convention that for $\xi \in \mathfrak{g}$, $\boldsymbol{v}(\xi)$ denotes the *left*-invariant vector field on G associated to $\xi \in \mathfrak{g}$ whereas $\boldsymbol{v}(\xi)^r$ denotes the *right*-invariant one.

Definition 6.4.4. Let $\mathcal{D} \subseteq V_\sigma^\omega$ be a subspace that is dense in V_σ.

- An *extension* of $d\sigma$ to $\mathfrak{b}_\pm$ with domain $\mathcal{D}$ is a Lie algebra homomorphism $\chi : \mathfrak{b}_\pm \to \mathcal{L}(\mathcal{D})$ such that $\chi(\xi) = d\sigma(\xi)|_\mathcal{D}$ for all $\xi \in \mathfrak{h}_\mathbb{C}$. We call (σ, χ) an $(H, \mathfrak{b}_-)$*-extension pair with domain* $\mathcal{D}$.

- The *trivial extension* of $d\sigma$ to $\mathfrak{b}_\pm$ with domain $\mathcal{D}$ is defined by letting $\mathfrak{n}_\pm$ act trivially on $\mathcal{D}$.

Definition 6.4.5. For $k \in \{1, 2\}$, let (σ_k, χ_k) be an $(H, \mathfrak{b}_-)$-extension pair with domain $\mathcal{D}_k$. We say that (σ_1, χ) and (σ_2, χ_2) are *unitarily equivalent* if there is a unitary isomorphism $U : V_{\sigma_1} \to V_{\sigma_2}$ of H-representation such that $U\mathcal{D}_1 = \mathcal{D}_2$ and $U\chi_1(\xi)v = \chi_2(\xi)Uv$ for all $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_1$. In this case we write $(\sigma_1, \chi_1) \cong (\sigma_2, \chi_2)$.

Definition 6.4.6. For $k \in \{1, 2\}$, let (σ_k, χ_k) be an $(H, \mathfrak{b}_-)$-extension pair with domain $\mathcal{D}_k$. Define the direct sum $(\sigma_1, \chi_1) \oplus (\sigma_2, \chi_2) := (\sigma_1 \oplus \sigma_2, \chi_1 \oplus \chi_2)$, where $\chi_1 \oplus \chi_2$ is defined by

$$\chi_1 \oplus \chi_2 : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_1 \oplus \mathcal{D}_2), \qquad (\chi_1 \oplus \chi_2)(\xi)(v_1, v_2) = (\chi_1(\xi)v_1, \chi_2(\xi)v_2).$$

Definition 6.4.7. Let (σ, χ) be an $(H, \mathfrak{b}_-)$-extension pair with domain $\mathcal{D}$. We say that (σ, χ) is *decomposable* if $(\sigma, \chi) \cong (\sigma_1, \chi_1) \oplus (\sigma_2, \chi_2)$ for some non-trivial $(H, \mathfrak{b}_-)$-extension pairs (σ_1, χ_1) and (σ_2, χ_2). We say that (σ, χ) is *indecomposable* if it is not decomposable.

Recall the definition of the involutions τ, θ and $(-)^*$ on $\mathcal{U}(\mathfrak{g}_\mathbb{C})$, specified in Definition 6.2.16. Recalling that $\theta(\xi + i\eta) = \theta(\xi - i\eta)$ for $\xi, \eta \in \mathfrak{g}$, the involutions τ and $(-)^*$ satisfy $\tau(\xi) = -\xi$ and $\xi^* = -\theta(\xi)$ for $\xi \in \mathfrak{g}_\mathbb{C}$. Extensions are used to specify a suitable G-subrepresentation of $C^\omega(G, V_\sigma)^H$:

Definition 6.4.8. Let (σ, χ) be an $(H, \mathfrak{b}_-)$-extension pair with domain $\mathcal{D}$. Define

$$\begin{aligned} C^\omega(G, V_\sigma)^H &:= \left\{ f \in C^\omega(G, V_\sigma) \,:\, f(gh) = \sigma(h)^{-1} f(g), \qquad \forall\, g \in G, h \in H \right\} \\ C^\omega(G, V_\sigma)^{H,\chi} &:= \left\{ f \in C^\omega(G, V_\sigma)^H \,:\, \langle v, \mathcal{L}_{\boldsymbol{v}(\xi)} f \rangle = -\langle \chi(\xi^*) v, f \rangle, \qquad \forall \xi \in \mathfrak{b}_+, v \in \mathcal{D} \right\}. \end{aligned}$$

Let us next record an important property regarding $C^\omega(G, V_\sigma)^{H,\chi}$:

Proposition 6.4.9. *Let (σ, χ) be an $(H, \mathfrak{b}_-)$-extension pair with domain $\mathcal{D}$. Let $f \in C^\omega(G, V_\sigma)^{H,\chi}$. Then*

$$f(g) \in \mathrm{dom}(\chi(x^*)^*) \quad \text{and} \quad (\mathcal{L}_{\boldsymbol{v}(\tau(x))} f)(g) = \chi(x^*)^* f(g), \qquad \forall x \in \mathcal{U}(\mathfrak{b}_+),\ \forall g \in G.$$

Proof. Let $v \in \mathcal{D}$. Suppose that $x = \xi_1 \cdots \xi_n$ for $n \in \mathbb{N}$ and $\xi_i \in \mathfrak{b}_+$. Observe that $f(g) \in \mathrm{dom}(\chi(\eta^*)^*)$ and $(\mathcal{L}_{\boldsymbol{v}(\eta)} f)(g) = -\chi(\eta^*)^* f(g)$ for any $g \in G$ and $\eta \in \mathfrak{b}_+$, as a consequence of Definition 6.4.8. It follows by induction on $n \in \mathbb{N}$ that $\langle v, \mathcal{L}_{\boldsymbol{v}(\xi_1 \cdots \xi_n)} f \rangle = (-1)^n \langle \chi(\xi_1^* \cdots \xi_n^*) v, f \rangle$. This implies $\langle v, \mathcal{L}_{\boldsymbol{v}(\tau(x))} f \rangle = \langle \chi(x^*) v, f \rangle$ for any $x \in \mathcal{U}(\mathfrak{b}_+)$ and $v \in \mathcal{D}$. The assertion follows. ☐

6.4.2 Holomorphically induced representations

We now define holomorphically induced representations. Fix throughout the section an $(H, \mathfrak{b}_-)$-extension pair (σ, χ) with a domain $\mathcal{D}_\chi \subseteq V_\sigma^\omega$ that is dense in V_σ. Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of G.

Remark 6.4.10. The theory of holomorphic induction presented in the upcoming section makes use of reproducing kernel Hilbert spaces. For more details thereon, one may refer e.g. to [Nee00, Ch. I-II]. The most relevant properties are recalled in Section 6.9.1 below.

Definition 6.4.11. We say that $(\rho, \mathcal{H}_\rho)$ is *holomorphically induced* from (σ, χ) if there exists a G-equivariant injective linear map $\Phi : \mathcal{H}_\rho \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ satisfying the following conditions:

1. The point evaluation $\mathcal{E}_x : \mathcal{H}_\rho \to V_\sigma$, $\mathcal{E}_x(\psi) := \Phi_\psi(x)$ is continuous for every $x \in G$.
2. $\mathcal{E}_x \mathcal{E}_x^* = \mathrm{id}_{V_\sigma}$ for every $x \in G$.
3. $\mathcal{D}_\chi = \left\{ v \in V_\sigma \,:\, \Phi(\mathcal{E}_e^* v) \in C^\omega(G, V_\sigma)^{H,\chi} \right\}$.

Remark 6.4.12. The first condition entails that $(\rho, \mathcal{H}_\rho)$ is unitarily equivalent to the natural G-representation on the reproducing kernel Hilbert space $\mathcal{H}_Q$, where $Q \in C(G \times G, \mathcal{B}(V_\sigma))^{H \times H}$ is the positive definite and G-invariant kernel defined by $Q(x,y) := \mathcal{E}_x \mathcal{E}_y^*$, cf. Theorem 6.9.3 and Proposition 6.9.5 below.

We have the following equivalent characterization, whose proof comprises the remainder of the section:

Theorem 6.4.13. *The following assertions are equivalent.*

1. *The G-representation $(\rho, \mathcal{H}_\rho)$ is holomorphically induced from the $(H, \mathfrak{b}_-)$-extension pair (σ, χ).*
2. *There is a closed H-invariant subspace $V \subseteq \mathcal{H}_\rho$ with the following properties:*
   - *(a) V is cyclic for the G-representation $\mathcal{H}_\rho$.*
   - *(b) $\mathcal{D}_{\widetilde{\chi}} := V \cap \mathcal{H}_\rho^\omega$ satisfies $d\rho(\mathfrak{n}_-)\mathcal{D}_{\widetilde{\chi}} \subseteq \mathcal{D}_{\widetilde{\chi}}$.*
   - *(c) (σ, χ) is unitarily equivalent to $(\widetilde{\sigma}, \widetilde{\chi})$, where $(\widetilde{\sigma}, \widetilde{\chi})$ is the $(H, \mathfrak{b}_-)$-extension pair defined by*

$$\begin{aligned} \widetilde{\sigma} : H \to \mathrm{U}(V), \qquad & \widetilde{\chi} : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_{\widetilde{\chi}}), \\ \widetilde{\sigma}(h) := \rho(h)|_V, \qquad & \widetilde{\chi}(\xi) := d\rho(\xi)|_{\mathcal{D}_{\widetilde{\chi}}}. \end{aligned}$$

   In particular, $\mathcal{D}_{\widetilde{\chi}}$ is dense in V.

If these equivalent assertions are satisfied, then ρ is an analytic G-representation.

We proceed with the proof of Theorem 6.4.13. We have the following simple but important observation:

Lemma 6.4.14. *Let $\Phi : \mathcal{H}_\rho \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ be a G-equivariant injective linear map. Assume that the point evaluation $\mathcal{E}_x(\psi) := \Phi_\psi(x)$ is continuous for every $x \in G$. Define $f_v := \Phi(\mathcal{E}_e^* v) \in \mathrm{Map}(G, V_\sigma)^H$ for $v \in V_\sigma$. Then:*

1. *$\mathcal{E}_g = \mathcal{E}_e \rho(g)^{-1}$ for any $g \in G$.*
2. *$\Phi_\psi(g) = \mathcal{E}_e \rho(g)^{-1}\psi$ for any $\psi \in \mathcal{H}_\rho$. In particular, $f_v(g) = \mathcal{E}_e\rho(g)^{-1}\mathcal{E}_e^* v$ for $v \in V_\sigma$.*
3. *Let $v \in V_\sigma$. Then*

$$\mathcal{E}_e^* v \in \mathcal{H}_\rho^\omega \iff f_v \in C^\omega(G, V_\sigma)^H \iff \langle v, f_v\rangle \in C^\omega(G, \mathbb{C}).$$

Proof.

1. As Φ is G-equivariant, we have $\mathcal{E}_g\psi = \Phi_\psi(g) = \Phi_{\rho(g)^{-1}\psi}(e) = \mathcal{E}_e\rho(g)^{-1}\psi$ for any $\psi \in \mathcal{H}_\rho$.
2. This is immediate from the first assertion.

3. Let $v \in V_\sigma$. If $\mathcal{E}_e^* v \in \mathcal{H}_\rho^\omega$, then the orbit map $g \mapsto \rho(g)\mathcal{E}_e^* v$ is analytic $G \to \mathcal{H}_\rho$. It follows that $f_v(g) = \mathcal{E}_e \rho(g)^{-1}\mathcal{E}_e^* v$ is analytic $G \to V_\sigma$, which in turn implies that $\langle v, f_v \rangle \in C^\omega(G, \mathbb{C})$. Assume that $\langle v, f_v\rangle \in C^\omega(G, \mathbb{C})$. Then $g \mapsto \langle \mathcal{E}_e^* v, \rho(g)\mathcal{E}_e^* v\rangle_{\mathcal{H}_\rho} = \langle v, f_v(g^{-1})\rangle_V$ is analytic. As G is a BCH Fréchet-Lie group, this implies using [Nee11, Thm. 5.2] that $\mathcal{E}_e^* v \in \mathcal{H}_\rho^\omega$. □

We first prove that (1) $\implies$ (2) in Theorem 6.4.13. Assume that ρ is holomorphically induced from (σ, χ). Let the map $\Phi : \mathcal{H}_\rho \to \mathrm{Map}(G, V_\sigma)^H$ satisfy the conditions in Definition 6.4.11. Let $\mathcal{E}_x := \mathrm{ev}_x \circ \Phi$ be the point evaluation at $x \in G$. We write $f_v := \Phi(\mathcal{E}_e^* v) \in C^\omega(G, V_\sigma)^{H,\chi}$ for $v \in \mathcal{D}_\chi$.

We show that the H-invariant subspace $W := \mathcal{E}_e^* V_\sigma \subseteq \mathcal{H}_\rho$ satisfies the conditions in Theorem 6.4.13. Define $\mathcal{D}_{\widetilde{\chi}} := \mathcal{E}_e^* \mathcal{D}_\chi \subseteq W$. By Theorem 6.9.3 we know that $\rho(G)W = \bigcup_{g\in G} \mathcal{E}_g^* V_\sigma$ is total in $\mathcal{H}_\rho$, so that W is cyclic for ρ. It is moreover immediate from Lemma 6.4.14 that $\mathcal{D}_{\widetilde{\chi}} \subseteq \mathcal{H}_\rho^\omega$. Because $\mathcal{D}_{\widetilde{\chi}}$ is dense in the cyclic subspace W and $\mathcal{H}_\rho^\omega$ is G-invariant, we obtain that $\mathcal{H}_\rho^\omega$ is dense in $\mathcal{H}_\rho$. Hence ρ is analytic.

Lemma 6.4.15. *Let $v \in \mathcal{D}_\chi$. The following assertions hold true:*

1. *$\mathcal{E}_e \rho(g)\mathcal{E}_e^* v \in \mathrm{dom}(\chi(x^*)^*)$ and $\mathcal{E}_e d\rho(x)\rho(g)\mathcal{E}_e^* v = \chi(x^*)^* \mathcal{E}_e \rho(g)\mathcal{E}_e^* v$ for any $x \in \mathcal{U}(\mathfrak{b}_+)$ and $g \in G$.*

2. *$d\rho(\mathfrak{b}_-)\mathcal{D}_{\widetilde{\chi}} \subseteq \mathcal{D}_{\widetilde{\chi}}$ and $d\rho(x)\mathcal{E}_e^* v = \mathcal{E}_e^* \chi(x) v$ for any $x \in \mathcal{U}(\mathfrak{b}_-)$.*

Proof.

1. Let $x \in \mathcal{U}(\mathfrak{b}_+)$. Since $f_v \in C^\omega(G, V_\sigma)^{H,\chi}$, we obtain from Proposition 6.4.9 that $f_v(e) \subseteq \mathrm{dom}(\chi(x^*)^*)$ and that $\mathcal{L}_{\boldsymbol{v}(\tau(x))} f_v = \chi(x^*)^* f_v$. On the other hand, notice using the formula $f_v(g) = \mathcal{E}_e \rho(g)^{-1}\mathcal{E}_e^* v$ that $(\mathcal{L}_{\boldsymbol{v}(\tau(x))} f_v)(g) = \mathcal{E}_e d\rho(x)\rho(g)^{-1}\mathcal{E}_e^* v$ holds true for any $g \in G$, say by induction on the degree of x. We thus obtain that $\mathcal{E}_e d\rho(x)\rho(g)^{-1}\mathcal{E}_e^* v = \chi(x^*)^* f_v(g) = \chi(x^*)^* \mathcal{E}_e \rho(g)^{-1}\mathcal{E}_e^* v$ for any $g \in G$.

2. Let $x \in \mathcal{U}(\mathfrak{b}_-)$. Recall from Lemma 6.4.14 that $\mathcal{D}_{\widetilde{\chi}} \subseteq \mathcal{H}_\rho^\omega$. Let $\psi \in \rho(G)\mathcal{D}_{\widetilde{\chi}}$. Using the first assertion, observe that

$$\langle \mathcal{E}_e^* \chi(x) v, \psi\rangle = \langle v, \chi(x)^* \mathcal{E}_e \psi\rangle = \langle v, \mathcal{E}_e d\rho(x^*)\psi\rangle = \langle d\rho(x)\mathcal{E}_e^* v, \psi\rangle.$$

As $\mathcal{D}_{\widetilde{\chi}}$ is cyclic for G in $\mathcal{H}_\rho$, it follows that $\mathcal{E}_e^* \chi(x) v = d\rho(x)\mathcal{E}_e^* v$. In particular $d\rho(\mathfrak{b}_-)\mathcal{D}_{\widetilde{\chi}} \subseteq \mathcal{D}_{\widetilde{\chi}}$. □

Define the unitary H-action $\widetilde{\sigma}$ on W by $\widetilde{\sigma}(h) = \rho(h)|_W$. Consider the extension $\widetilde{\chi}(\xi) := d\rho(\xi)|_{\mathcal{D}_{\widetilde{\chi}}}$ of $d\widetilde{\sigma}$ to $\mathfrak{b}_-$, whose domain is $\mathcal{D}_{\widetilde{\chi}}$. By Lemma 6.4.15, $\mathcal{E}_e^*$ defines a unitary equivalence between (σ, χ) and $(\widetilde{\sigma}, \widetilde{\chi})$. In particular, it follows that $\mathcal{D}_{\widetilde{\chi}}$ is dense in W, because $\mathcal{D}_\chi$ is dense in V_σ by assumption.

Lemma 6.4.16. $\mathcal{D}_{\widetilde{\chi}} = W \cap \mathcal{H}_\rho^\omega$.

Proof. The inclusion $\mathcal{D}_{\widetilde{\chi}} \subseteq W \cap \mathcal{H}_\rho^\omega$ follows from Lemma 6.4.14. Let $w \in W \cap \mathcal{H}_\rho^\omega$. Then $w = \mathcal{E}_e^* v$ for some $v \in V_\sigma$. We must show that $v \in \mathcal{D}_\chi$. Lemma 6.4.14 implies that $f_v \in C^\omega(G, V_\sigma)^H$. Let $v_2 \in \mathcal{D}_\chi$ and $\xi \in \mathfrak{b}_+$. Using Lemma 6.4.15 and the formula $f_v(g) = \mathcal{E}_e \rho(g)^{-1} \mathcal{E}_e^* v$ we obtain:

$$\begin{aligned}\langle v_2, (\mathcal{L}_{\boldsymbol{v}(\xi)} f_v)(g)\rangle &= -\langle d\rho(\xi^*)\mathcal{E}_e^* v_2, \rho(g)^{-1}\mathcal{E}_e^* v\rangle = -\langle \chi(\xi^*)v_2, \mathcal{E}_e \rho(g)^{-1}\mathcal{E}_e^* v\rangle \\ &= -\langle \chi(\xi^*)v_2, f_v(g)\rangle.\end{aligned}$$

It follows that $f_v \in C^\omega(G, V_\sigma)^{H,\chi}$. By the third property in Definition 6.4.11, this means that $v \in \mathcal{D}_\chi$. ☐

This completes the proof of (1) $\implies$ (2) in Theorem 6.4.13. The converse is Lemma 6.4.17 below:

Lemma 6.4.17. *Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of G. Let $V \subseteq \mathcal{H}_\rho$ be a closed H-invariant subspace. Define an H-representation σ on V by $\sigma(h) := \rho(h)|_V$. Set $\mathcal{D}_\chi = V \cap \mathcal{H}_\rho^\omega$. Assume that $\rho(G)V$ is total in $\mathcal{H}_\rho$, that $\mathcal{D}_\chi$ is dense in V_σ and that $d\rho(\mathfrak{n}_-)\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$. Define the extension $\chi(\xi)v := d\rho(\xi)v$ of $d\sigma$ to $\mathfrak{b}_-$ with domain $\mathcal{D}_\chi$, where $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_\chi$. Then ρ is holomorphically induced from (σ, χ).*

Proof. Let $p_V : \mathcal{H}_\rho \to V_\sigma$ denote the orthogonal projection onto V_σ. For $\psi \in \mathcal{H}_\rho$, define $\Phi_\psi(g) := p_V \rho(g)^{-1}\psi$. Consider the linear map

$$\Phi : \mathcal{H}_\rho \to C(G, V_\sigma)^H, \quad \psi \mapsto \Phi_\psi.$$

It is clear that Φ is G-equivariant and that the point-evaluation $\mathcal{E}_g = p_V \rho(g)^{-1}$ is continuous for any $g \in G$. The map Φ is injective because $\Phi_\psi = 0$ is equivalent to $\psi \perp \rho(G)V$ and $\rho(G)V$ is total in $\mathcal{H}_\rho$, by assumption. Notice next that $\mathcal{E}_g^* v = \rho(g)v$ for any $v \in V$ and so $\mathcal{E}_g\mathcal{E}_g^* = \mathrm{id}_V$. Define $V^0 := \{v \in V : \Phi_v \in C^\omega(G, V_\sigma)^{H,\chi}\}$. It remains to show that $\mathcal{D}_\chi = V^0$. It is immediate from the third assertion in Lemma 6.4.14 that $V^0 \subseteq \mathcal{D}_\chi$. Suppose conversely that $v \in \mathcal{D}_\chi$. By Lemma 6.4.14, we have $\Phi_v \in C^\omega(G, V_\sigma)^H$. Let $\xi \in \mathfrak{b}_+$ and $w \in \mathcal{D}_\chi$. Using $\mathcal{L}_{\boldsymbol{v}(\xi)}\Phi_v(g) = -p_V d\rho(\xi)\rho(g)^{-1}v$, we find that

$$\langle w, \mathcal{L}_{\boldsymbol{v}(\xi)}\Phi_v(g)\rangle = -\langle d\rho(\xi^*)w, \rho(g)^{-1}v\rangle = -\langle \chi(\xi^*)w, \rho(g)^{-1}v\rangle = -\langle \chi(\xi^*)w, \Phi_v(g)\rangle.$$

Thus $\Phi_v \in C^\omega(G, V_\sigma)^{H,\chi}$, which means that $v \in V^0$. Thus $V^0 = \mathcal{D}_\chi$. ☐

6.4.3 Uniqueness

In the following, we determine that there is up to unitary equivalence at most one unitary G-representation that is holomorphically induced from a given $(H, \mathfrak{b}_-)$-extension pair. Let (σ, χ) be such an $(H, \mathfrak{b}_-)$-extension pair, whose domain $\mathcal{D}_\chi \subseteq V_\sigma^\omega$ is dense in V_σ. Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of G.

Definition 6.4.18. We say that (σ, χ) is *holomorphically inducible* to G if there is a unitary G-representation which is holomorphically induced from (σ, χ).

Proposition 6.4.19. *Assume that ρ is holomorphically induced from (σ, χ). Let the map $\Phi : \mathcal{H}_\rho \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ satisfy the conditions in Definition 6.4.11 and let $\mathcal{E}_x := \mathrm{ev}_x \circ \Phi$ be the point evaluation at $x \in G$. Define:*

$$F : G \to \mathcal{B}(V_\sigma), \quad F(g) := \mathcal{E}_e \rho(g) \mathcal{E}_e^*.$$

Then F satisfies the following properties:

1. *$F(e) = id_{V_\sigma}$.*
2. *The function $Q : G \times G \to \mathcal{B}(V_\sigma)$, $Q(x, y) := F(x^{-1}y)$ is positive definite (cf. Definition 6.9.2).*
3. *$\mathcal{D}_\chi = \{ v \in V_\sigma \,:\, g \mapsto \langle v, F(g)v\rangle$ is real-analytic $G \to \mathbb{C} \}$.*
4. *For all $v, w \in \mathcal{D}_\chi$ and $g \in G$, $\xi \in \mathfrak{b}_+$ we have:*

$$\big[\mathcal{L}_{\boldsymbol{v}(\xi)^r} \langle w, Fv\rangle\big](g) = -\langle \chi(\xi^*)w, F(g)v\rangle. \tag{6.4.1}$$

Finally, ρ is unitarily equivalent to the G-representation on the reproducing kernel Hilbert space $\mathcal{H}_Q$.

Proof. Define the $\widetilde{Q} : G \times G \to \mathcal{B}(V_\sigma)$ by $\widetilde{Q}(x, y) := \mathcal{E}_x \mathcal{E}_y^*$, which is positive-definite by Theorem 6.9.3. In view of the first assertion in Lemma 6.4.14, we have $\widetilde{Q}(x, y)v = \mathcal{E}_e \rho(x^{-1}y)\mathcal{E}_e^* v = F(x^{-1}y)v = Q(x, y)v$ for any $v \in V_\sigma$. Thus $\widetilde{Q} = Q$. In particular, Q is positive definite and $F(e) = Q(e, e) = \mathrm{id}_{V_\sigma}$. Let $v \in V_\sigma$. Writing $f_v := \Phi(\mathcal{E}_e^* v)$, notice that $f_v(g) = F(g^{-1})v$ for $g \in G$. We find that $\mathcal{E}_e^* v \in \mathcal{H}_\rho^\omega$ if and only if $\langle v, Fv\rangle \in C^\omega(G, \mathbb{C})$, using Lemma 6.4.14. Then

$$D_\chi = \big\{ v \in V_\sigma \,:\, \mathcal{E}_e^* v \in \mathcal{H}_\rho^\omega \big\} = \{ v \in V_\sigma \,:\, \langle v, Fv\rangle \in C^\omega(G, \mathbb{C}) \},$$

where we used Lemma 6.4.16 in the first equality. Finally, notice that

$$\langle w, F(g)v\rangle = \langle \mathcal{E}_e^* w, \rho(g)\mathcal{E}_e^* v\rangle \quad \text{for } v, w \in \mathcal{D}_\chi \text{ and } g \in G.$$

It thus follows from Lemma 6.4.15 that F satisfies (6.4.1) for all $g \in G$ and $\xi \in \mathfrak{b}_+$. The final statement is immediate from Proposition 6.9.5. ☐

The next result, Theorem 6.4.20, gives a characterization of (σ, χ) being holomorphically inducible in terms $\mathcal{B}(V_\sigma)$-valued positive-definite functions on G.

Theorem 6.4.20. *The following assertions are equivalent:*

1. *(σ, χ) is holomorphically inducible.*
2. *There is a function $F : G \to \mathcal{B}(V_\sigma)$ satisfying the properties in Proposition 6.4.19.*

Assume that these assertions are valid. Let $F : G \to \mathcal{B}(V_\sigma)$ satisfy the conditions in Proposition 6.4.19. Then $F(g)^* = F(g^{-1})$ for all $g \in G$. Moreover, for $v \in \mathcal{D}_\chi$ and $w \in V_\sigma$ we have:

$$\big[\mathcal{L}_{\boldsymbol{v}(x_+)^r} \mathcal{L}_{\boldsymbol{v}(x_-)} \langle w, Fv\rangle\big](g) = \langle w, \chi(\tau(x_+)^*)^* F(g)\chi(x_-)v\rangle, \tag{6.4.2}$$

$$\big[\mathcal{L}_{\boldsymbol{v}(x_+ x_-)} \langle w, Fv\rangle\big](e) = \langle w, \chi(x_+^*)^* \chi(x_-)v\rangle, \tag{6.4.3}$$

for all $g \in G$ and $x_\pm \in \mathcal{U}(\mathfrak{b}_\pm)$. Finally, the function $F : G \to \mathcal{B}(V_\sigma)$ is unique.

Proof. The implication (1) $\implies$ (2) is immediate from Proposition 6.4.19. Conversely, let $F : G \to \mathcal{B}(V_\sigma)$ be a function satisfying the conditions in Proposition 6.4.19. Define $Q(x,y) := F(x^{-1}y)$ for $x, y \in G$. Let $\mathcal{H}_\rho$ be the corresponding reproducing kernel Hilbert space. Using Proposition 6.9.5 we obtain a unitary representation ρ of G on $\mathcal{H}_\rho$ and a G-equivariant injective linear map $\Phi : \mathcal{H}_\rho \to \mathrm{Map}(G, V_\sigma)^H$ for which the point evaluation $\mathcal{E}_x := \mathrm{ev}_x \circ \Phi$ is continuous and satisfies $\mathcal{E}_x = \mathcal{E}_e \rho(x)^{-1}$ for every $x \in G$. From $F(e) = \mathrm{id}_{V_\sigma}$ it follows that $Q(x,x) = \mathrm{id}_{V_\sigma}$ for every $x \in G$. Define $f_v := \Phi(\mathcal{E}_e^* v)$ for $v \in V_\sigma$.

To see that (1) holds true, it remains only to show that

$$\mathcal{D}_\chi = \left\{ v \in V_\sigma \,:\, f_v \in C^\omega(G, V_\sigma)^{H,\chi} \right\}.$$

Let $x \in G$ and $v \in V_\sigma$. From the equations $f_v(x) = \mathcal{E}_x \mathcal{E}_e^* v = Q(x,e)v = F(x^{-1})v$ and $\mathcal{E}_x \mathcal{E}_e^* v = \mathcal{E}_e \rho(x) \mathcal{E}_e^* v$, we conclude that $F(x)v = \mathcal{E}_e \rho(x) \mathcal{E}_e^* v = f_v(x^{-1})$. It follows that

$$\mathcal{D}_\chi = \{ v \in V_\sigma \,: \langle v, Fv \rangle \in C^\omega(G, \mathbb{C}) \} = \left\{ v \in V_\sigma \,:\, f_v \in C^\omega(G, V_\sigma)^H \right\},$$

where Lemma 6.4.14 was used in the second equality. Assume that $f_v \in C^\omega(G, V_\sigma)^H$. Let $w \in \mathcal{D}_\chi$ and $\xi \in \mathfrak{b}_+$. From the equation $F(g)v = f_v(g^{-1})$ we obtain that $\mathcal{L}_{\boldsymbol{v}(\xi)} f_v(g) = \left[\mathcal{L}_{\boldsymbol{v}(\xi)^r} Fv\right](g^{-1})$ for any $g \in G$. Using Equation (6.4.1) we find:

$$\begin{aligned} \left\langle w, \mathcal{L}_{\boldsymbol{v}(\xi)} f_v(g) \right\rangle &= \left[\mathcal{L}_{\boldsymbol{v}(\xi)^r} \langle w, Fv \rangle\right](g^{-1}) = -\langle \chi(\xi^*) w, F(g^{-1}) v \rangle \\ &= -\langle \chi(\xi^*) w, f_v(g) \rangle, \qquad\qquad \forall g \in G. \end{aligned}$$

Hence $f_v \in C^\omega(G, V_\sigma)^{H,\chi}$. Thus $\mathcal{D}_\chi = \left\{ v \in V_\sigma \,:\, f_v \in C^\omega(G, V_\sigma)^{H,\chi} \right\}$. We conclude that $(\rho, \mathcal{H}_\rho)$ is holomorphically induced from (σ, χ). So (1) $\iff$ (2).

Assume these equivalent assertions are satisfied. From $F(g) = \mathcal{E}_e \rho(g) \mathcal{E}_e^*$ it is immediate that $F(g^{-1}) = F(g)^*$ for all $g \in G$. We next show (6.4.2) and (6.4.3). Let $v \in \mathcal{D}_\chi$. Notice using $F(g) = \mathcal{E}_e \rho(g) \mathcal{E}_e^*$ that for any $x, y \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ we have

$$\left[\mathcal{L}_{\boldsymbol{v}(y)^r} \mathcal{L}_{\boldsymbol{v}(x)} Fv\right](g) = \mathcal{E}_e d\rho(\tau(y)) \rho(g) d\rho(x) \mathcal{E}_e^* v \tag{6.4.4}$$

for all $g \in G$. Thus, for $x_\pm \in \mathcal{U}(\mathfrak{b}_\pm)$ we obtain using (6.4.4) and Lemma 6.4.15 that

$$\begin{aligned} \left[\mathcal{L}_{\boldsymbol{v}(x_+)^r} \mathcal{L}_{\boldsymbol{v}(x_-)} Fv\right](g) &= \mathcal{E}_e d\rho(\tau(x_+)) \rho(g) d\rho(x_-) \mathcal{E}_e^* v \\ &= \chi(\tau(x_+)^*)^* \mathcal{E}_e \rho(g) \mathcal{E}_e^* \chi(x_-) v, \end{aligned} \tag{6.4.5}$$

$$\begin{aligned} \left[\mathcal{L}_{\boldsymbol{v}(x_+ x_-)} Fv\right](e) &= \mathcal{E}_e d\rho(x_+ x_-) \mathcal{E}_e^* v \\ &= \chi(x_+^*)^* \chi(x_-) v. \end{aligned} \tag{6.4.6}$$

From (6.4.5) we conclude that $\left[\mathcal{L}_{\boldsymbol{v}(x_+)^r} \mathcal{L}_{\boldsymbol{v}(x_-)} Fv\right](g) = \chi(\tau(x_+)^*)^* F(g) \chi(x_-) v$ for all $g \in G$, which implies (6.4.2). On the other hand, (6.4.3) is implied by (6.4.6). Finally, assume that F_1 and F_2 are two functions satisfying the conditions in Proposition 6.4.19. Let $v \in \mathcal{D}_\chi$. The functions $g \mapsto F_1(g)v$ and $g \mapsto F_2(g)v$

are both analytic and satisfy (6.4.6). As $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ is spanned by $\mathcal{U}(\mathfrak{n}_+)\mathcal{U}(\mathfrak{b}_-)$ by the PBW Theorem, it follows that $j_e^\infty(F_1 v) = j_e^\infty(F_2 v)$. As G is connected, it follows from Proposition 6.2.15 that $F_1(g)v = F_2(g)v$ for all $g \in G$ and $v \in \mathcal{D}_\chi$. For any fixed $g \in G$, the map $v \mapsto (F_1(g) - F_2(g))v$ is continuous and vanishes on the dense subset $\mathcal{D}_\chi \subseteq V_\sigma$. Hence $F_1 = F_2$. □

Combining Proposition 6.4.19 with the uniqueness of $F : G \to \mathcal{B}(V_\sigma)$ in Theorem 6.4.20, we obtain the desired uniqueness of the holomorphically induced representation up to unitary equivalence:

Theorem 6.4.21. *Let ρ_1 and ρ_2 be unitary G-representations which are both holomorphically induced from (σ, χ). Then $\rho_1 \cong \rho_2$ as unitary G-representations.*

Finally, we focus our attention on the important special case where χ is a trivial extension. Using the PBW Theorem, notice that we have the vector space decomposition $\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \mathcal{U}(\mathfrak{h}_\mathbb{C}) \oplus (\mathfrak{n}_+\mathcal{U}(\mathfrak{g}_\mathbb{C}) + \mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{n}_-)$.

Definition 6.4.22. Let $E_0 : \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathcal{U}(\mathfrak{h}_\mathbb{C}) \cong \mathcal{U}(\mathfrak{g}_\mathbb{C})/(\mathfrak{n}_+\mathcal{U}(\mathfrak{g}_\mathbb{C}) + \mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{n}_-)$ be the quotient map.

Lemma 6.4.23. *Assume that ρ is holomorphically induced from (σ, χ), where χ is the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $\mathcal{D} \subseteq V_\sigma$. Let $v \in \mathcal{D}$ and $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. Then $d\sigma(E_0(x^*))v = d\sigma(E_0(x))^*v$. Moreover for all $w \in V_\sigma^\infty$ we have*

$$\begin{aligned} \langle w, d\rho(x)v\rangle &= \langle w, d\sigma(E_0(x))v\rangle, \qquad \textit{and} \\ \langle d\rho(x^*)v, w\rangle &= \langle v, d\sigma(E_0(x))w\rangle. \end{aligned}$$

Proof. By Theorem 6.4.13 we may assume that $V_\sigma \subseteq \mathcal{H}_\rho$ is a closed subspace, that $\mathcal{D}_\chi = V \cap \mathcal{H}_\rho^\omega$, $\sigma(h) = \rho(h)|_{V_\sigma}$ and $\chi(\xi) = d\rho(\xi)|_{\mathcal{D}_\chi}$ for every $h \in H$ and $\xi \in \mathfrak{b}_-$. Let $p_V : \mathcal{H}_\rho \to V_\sigma$ be the orthogonal projection onto V_σ. Take $v \in \mathcal{D}_\chi$, $\xi_+ \in \mathfrak{n}_+$, $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $\xi_- \in \mathfrak{n}_-$. From Lemma 6.4.15 we obtain $p_V d\rho(x\xi_-)v = p_V d\rho(x)\chi(\xi_-)v = 0$ and $p_V d\rho(\xi_+ x)v = \chi(\xi_+^*)^* p_V d\rho(x)v = 0$. Thus

$$p_V d\rho(x)v = p_V d\rho(E_0(x))v = d\sigma(E_0(x))v, \qquad \forall x \in \mathcal{U}(\mathfrak{g}_\mathbb{C}).$$

Let $w \in \mathcal{D}_\chi$. Recall from Lemma 6.4.16 that $\mathcal{D}_\chi \subseteq \mathcal{H}_\rho^\omega$. We have:

$$\begin{aligned} \langle d\sigma(E_0(x^*))v, w\rangle &= \langle d\rho(x^*)v, w\rangle = \langle v, d\rho(x)w\rangle = \langle v, d\sigma(E_0(x))w\rangle \\ &= \langle d\sigma(E_0(x))^*v, w\rangle. \end{aligned}$$

As $\mathcal{D}_\chi$ is dense in V_σ we conclude $d\sigma(E_0(x^*))v = d\sigma(E_0(x))^*v$. Consequently, if $w \in V_\rho^\infty$ then

$$\langle d\rho(x^*)v, w\rangle = \langle d\sigma(E_0(x^*))v, w\rangle = \langle d\sigma(E_0(x))^*v, w\rangle = \langle v, d\sigma(E_0(x))w\rangle.$$

□

We complement Theorem 6.4.20 with the following result, regarding the uniqueness of the domain:

Proposition 6.4.24. *Let σ be an analytic unitary representation of H. Assume that there exists a subspace $\mathcal{D}_\chi \subseteq V_\sigma^\omega$ dense in V_σ for which (σ, χ) is holomorphically inducible, where $\chi\ \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_\chi)$ is the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $\mathcal{D}_\chi$. Then $\mathcal{D}_\chi$ is unique with this property.*

Proof. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two such domains. For $k \in \{1, 2\}$, let χ_k denote the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $\mathcal{D}_k$. By assumption (σ, χ_k) is holomorphically inducible. Let $F_k : G \to \mathcal{B}(V_\sigma)$ satisfy the conditions in Proposition 6.4.19 for (σ, χ_k). Let $v_k \in \mathcal{D}_k$. Observe using Lemma 6.4.23 that for any $k \in \{1, 2\}$, $x \in \mathcal{U}(\mathfrak{b}_-)$ and $v \in \mathcal{D}_k$ we have $\chi_k(x)v = d\sigma(E_0(x))v$ and $\chi_k(x)^*v = d\sigma(E_0(x))^*v = d\sigma(E_0(x^*))v$. Consider the functions $a, b : G \to \mathbb{C}$ defined by $a(g) := \langle v_1, F_1(g)v_2\rangle$ and $b(g) := \langle v_1, F_2(g)v_2\rangle$. Notice that both a and b are analytic, where we remark that $a(g) = \langle F_1(g^{-1})v_1, v_2\rangle$. Let $x_\pm \in \mathcal{U}(\mathfrak{b}_\pm)$. Using (6.4.3) we obtain:

$$\begin{aligned}(\mathcal{L}_{\boldsymbol{v}(x_+x_-)}b)(e) &= \langle v_1, \chi_2(x_+^*)^*\chi_2(x_-)v_2\rangle = \langle v_1, d\sigma(E_0(x_+))d\sigma(E_0(x_-))v_2\rangle \\ &= \langle d\sigma(E_0(x_+^*))v_1, d\sigma(E_0(x_-))v_2\rangle.\end{aligned}$$

We next compute $(\mathcal{L}_{\boldsymbol{v}(x_+x_-)}a)(e)$. Let $\iota : G \to G, g \mapsto g^{-1}$ denote the inversion on G and $\Sigma : \mathbb{C} \to \mathbb{C}, z \mapsto \overline{z}$ the conjugation on $\mathbb{C}$. Define

$$h : G \to \mathbb{C}, \qquad h(g) = \langle v_2, F_1(g)v_1\rangle,$$

so that $a = \Sigma \circ h \circ \iota$. For any $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $f \in C^\infty(G, \mathbb{C})$, we have

$$\begin{aligned}\left[\mathcal{L}_{\boldsymbol{v}(x)}(f \circ \iota)\right](e) &= (\mathcal{L}_{\boldsymbol{v}(\tau(x))}f)(e), \\ \left[\mathcal{L}_{\boldsymbol{v}(x)}(\Sigma \circ f)\right](e) &= \Sigma\left[\mathcal{L}_{\boldsymbol{v}(\theta(x))}f\right](e).\end{aligned}$$

Using these equations we obtain that $(\mathcal{L}_{\boldsymbol{v}(x)}a)(e) = \Sigma\left[\mathcal{L}_{\boldsymbol{v}(x^*)}h\right](e)$ for any $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. By equation (6.4.3) we have

$$\begin{aligned}\left[\mathcal{L}_{\boldsymbol{v}(x_-^*x_+^*)}h\right](e) &= \langle v_2, \chi(x_-)^*\chi(x_+^*)v_1\rangle = \langle v_2, d\sigma(E_0(x_-))^*d\sigma(E_0(x_+^*))v_1\rangle \\ &= \langle d\sigma(E_0(x_-))v_2, d\sigma(E_0(x_+^*))v_1\rangle.\end{aligned}$$

Thus

$$\begin{aligned}(\mathcal{L}_{\boldsymbol{v}(x_+x_-)}a)(e) &= \Sigma\left[\mathcal{L}_{\boldsymbol{v}(x_-^*x_+^*)}h\right](e) = \langle d\sigma(E_0(x_+^*))v_1, d\sigma(E_0(x_-))v_2\rangle \\ &= (\mathcal{L}_{\boldsymbol{v}(x_+x_-)}b)(e).\end{aligned}$$

As $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ is spanned by elements in $\mathcal{U}(\mathfrak{n}_+)\mathcal{U}(\mathfrak{b}_-)$ by the PBW Theorem, it follows that $j_e^\infty(a) = j_e^\infty(b)$. Since G is connected, it follows from Proposition 6.2.15 that $a = b$. Thus $\langle v_1, F_1(g)v_2\rangle = \langle v_1, F_2(g)v_2\rangle$ for all $g \in G$, $v_1 \in \mathcal{D}_1$ and $v_2 \in \mathcal{D}_2$. As both $\mathcal{D}_1$ and $\mathcal{D}_2$ are dense, it follows that $F_1 = F_2 =: F$. From the third property in Proposition 6.4.19, we conclude that

$$\mathcal{D}_1 = \mathcal{D}_2 = \{\, v \in V_\sigma \,:\, g \mapsto \langle v, F(g)v \in C^\omega(G, \mathbb{C}) \,\}.$$

$\square$

Theorem 6.4.21 and Proposition 6.4.24 justify the following notation:

Definition 6.4.25. We write $\rho = \mathrm{HolInd}_H^G(\sigma, \chi)$ if ρ is holomorphically induced from (σ, χ). If additionally χ is the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ on some necessarily unique domain $\mathcal{D}_\chi \subseteq V_\sigma^\omega$, we simply write $\rho = \mathrm{HolInd}_H^G(\sigma)$.

Remark 6.4.26. For $k \in \{1, 2\}$, let (σ_k, χ_k) be an $(H, \mathfrak{b}_-)$-extension pair and let ρ_k be a unitary G-representation with $\rho_k = \mathrm{HolInd}_H(\sigma_k, \chi_k)$. In view of Theorem 6.4.21, one might wonder whether or not $\rho_1 \cong \rho_2$ implies $(\sigma_1, \chi_1) \cong (\sigma_2, \chi_2)$. This turns out to be false. For an explicit and simple counterexample, consider $G = \mathrm{SU}(3)$. Let $H \subseteq G$ be the subgroup consisting of diagonal matrices and let $\mathfrak{b}_- \subseteq \mathfrak{sl}(3, \mathbb{C})$ consist of upper-triangular matrices. The defining representation ρ of G on $\mathbb{C}^3$ is holomorphically induced from the two $(H, \mathfrak{b}_-)$-extension pairs obtained by restricting $\rho|_H$ and $d\rho|_{\mathfrak{b}_-}$ to either $V_{\sigma_1} := \mathbb{C}e_1$ or $V_{\sigma_2} := \mathbb{C}e_1 \oplus \mathbb{C}e_2$, as is quickly verified using Theorem 6.4.13. These are not unitary equivalent.

6.4.4 Commutants

Suppose that $\rho = \mathrm{HolInd}_H^G(\sigma, \chi)$.

Definition 6.4.27. Let $T \in \mathcal{B}(V_\sigma)$. We say that T *commutes with* (σ, χ) if $T \in \mathcal{B}(V_\sigma)^H$, $T\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$ and $T\chi(\xi)v = \chi(\xi)Tv$ for every $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_\chi$. Define the $*$*-closed commutant* $\mathcal{B}(V_\sigma)^{H,\chi}$ *of* (σ, χ) by

$$\mathcal{B}(V_\sigma)^{H,\chi} := \left\{ T \in \mathcal{B}(V_\sigma)^H \ : \ \text{both } T \text{ and } T^* \text{ commute with } (\sigma, \chi) \right\}.$$

Remark 6.4.28. Orthogonal projections in $\mathcal{B}(V_\sigma)^{H,\chi}$ correspond to direct sum decompositions of (σ, χ). To see this, suppose $p_1 \in \mathcal{B}(V_\sigma)^{H,\chi}$ is an orthogonal projection. Let $p_2 := 1 - p_1$. For $k \in \{1, 2\}$, define $V_k := p_k V_\sigma$ and $\mathcal{D}_k := p_k \mathcal{D}_\chi \subseteq \mathcal{D}_\chi$. Define the $(H, \mathfrak{b}_-)$-extension pair (σ_k, χ_k) by $\sigma_k(h) := \sigma(h)|_{V_k}$ and $\chi_k(\xi) := \chi(\xi)|_{\mathcal{D}_k}$, where $h \in H$ and $\xi \in \mathfrak{b}_-$. Then $(\sigma, \chi) \cong (\sigma_1, \chi_1) \oplus (\sigma_2, \chi_2)$.

The main results of this section are Theorem 6.4.29 and Theorem 6.4.30 below:

Theorem 6.4.29. *Suppose that* $\rho = \mathrm{HolInd}_H^G(\sigma, \chi)$. *Let* V_σ *be a closed subspace of* $\mathcal{H}_\rho$ *satisfying the conditions in Theorem 6.4.13.2. Let* $q_V \in \mathcal{B}(\mathcal{H}_\rho)$ *be the orthogonal projection onto* V_σ. *Then*

1. $\mathcal{B}(V_\sigma)^{H,\chi}$ *is a von Neumann algebra.*

2. *Assume that* $q_V \in \rho(G)''$. *Then*

$$r : \mathcal{B}(\mathcal{H}_\rho)^G \to \mathcal{B}(V_\sigma)^{H,\chi}, \ r(T) := T|_{V_\sigma}$$

defines a $*$*-isomorphism of von Neumann algebras. In particular,* ρ *is irreducible if and only if* (σ, χ) *is indecomposable.*

Theorem 6.4.30. *Consider the setting of Theorem 6.4.29. Let $\chi : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_\chi)$ denote the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $\mathcal{D}_\chi$. The following assertions are valid:*

1. $\mathcal{B}(V_\sigma)^{H,\chi} = \mathcal{B}(V_\sigma)^H$.
2. *Assume that $q_V \in \rho(G)''$. Then $\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(V_\sigma)^H$. In particular, ρ is irreducible if and only if σ is.*

Remark 6.4.31. In the context of positive energy representations, the case where χ is a trivial extension is of central importance. In that setting we can typically guarantee that $q_V \in \rho(G)''$. The relation between positive energy representations and holomorphic induction is considered Section 6.7.

Proof of Theorem 6.4.29 and Theorem 6.4.30

Assume throughout the following that ρ is holomorphically induced from (σ, χ). In view of Theorem 6.4.13, we may and do assume that $V_\sigma \subseteq \mathcal{H}_\rho$ is a closed subspace, $\sigma(h) = \rho(h)|_{V_\sigma}$ for all $h \in H$, that $\mathcal{D}_\chi = V_\sigma \cap \mathcal{H}_\rho^\omega$, $d\rho(\mathfrak{b}_-)\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$ and that $\chi(\xi)v = d\rho(\xi)v$ for all $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_\chi$. We may further assume that the map $\Phi : \mathcal{H}_\rho \to \mathrm{Map}(G, V_\sigma)^H$ satisfying the conditions in Definition 6.4.11 is given by $\Phi_\psi(g) = p_V \rho(g)^{-1}\psi$. In particular $\mathcal{E}_e = p_V$ is the orthogonal projection $p_V : \mathcal{H}_\rho \to V_\sigma$ and $\mathcal{E}_e^* = \iota_V$ is the inclusion $\iota_V : V_\sigma \hookrightarrow \mathcal{H}_\rho$. We also have $q_V = \iota_V p_V$.

Lemma 6.4.32. *Let $T \in \mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$, $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $v, w \in \mathcal{D}_\chi$. Then*

$$\langle v, T d\rho(x) w\rangle = \langle v, d\rho(x) T w\rangle.$$

Proof. Using the PBW Theorem, it suffices to consider the case where $x = x_+ x_-$ for some $x_+ \in \mathcal{U}(\mathfrak{n}_+)$ and $x_- \in \mathcal{U}(\mathfrak{b}_-)$. In that case we obtain using Lemma 6.4.15 and the fact that $T \in \mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$:

$$\begin{aligned}\langle v, T d\rho(x) w\rangle &= \langle \chi(\xi_+^*) T^* v, \chi(x_-) w\rangle = \langle \chi(\xi_+^*) v, T\chi(x_-) w\rangle \\ &= \langle \chi(\xi_+^*) v, \chi(x_-) T w\rangle = \langle v, d\rho(x) T w\rangle.\end{aligned}$$ ☐

Lemma 6.4.33. *Let $T \in \mathcal{B}(V_\sigma)$. Assume that $\langle v, T\rho(e^\xi) w\rangle = \langle v, \rho(e^\xi) T w\rangle$ for all $v, w \in \mathcal{D}_\chi$ and all ξ in some 0-neighborhood in $\mathfrak{g}$. Then $T\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$ and*

$$\langle w, T\rho(g) v\rangle = \langle w, \rho(g) T v\rangle, \qquad \forall g \in G,\ \forall v, w \in V_\sigma. \tag{6.4.7}$$

Proof. Let $v, w \in \mathcal{D}_\chi$. Both $g \mapsto \langle w, T\rho(g)v\rangle$ and $g \mapsto \langle w, \rho(g)Tv\rangle$ are real-analytic $G \to \mathbb{C}$. As G is BCH, so in particular locally exponential, these functions agree on some 1-neighborhood in G by assumption. As G is connected, it follows from Proposition 6.2.14 that they are equal everywhere. We thus obtain that $\langle w, T\rho(g)v\rangle = \langle w, \rho(g)Tv\rangle$ for all $g \in G$. As $\mathcal{D}_\chi$ is dense, equation (6.4.7) follows. Let $v \in \mathcal{D}_\chi$. Then using (6.4.7) we find that $\langle Tv, \rho(g)Tv\rangle = \langle Tv, T\rho(g)v\rangle$ for all $g \in G$. The right-hand side defines a real-analytic function $G \to \mathbb{C}$ because $v \in \mathcal{H}_\rho^\omega$. Thus also $g \mapsto \langle Tv, \rho(g)Tv\rangle$ is real-analytic. Recalling that G is a BCH Fréchet-Lie group, we conclude using [Nee11, Thm. 5.2] that $Tv \in \mathcal{H}_\rho^\omega$. Thus $Tv \in \mathcal{H}_\rho^\omega \cap V_\sigma = \mathcal{D}_\chi$. ☐

Lemma 6.4.34. *$\mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$ is a von Neumann algebra. Moreover we have*

$$\langle w, T\rho(g)v\rangle = \langle w, \rho(g)Tv\rangle, \qquad \forall T \in \mathcal{B}(\mathcal{H}_\rho)^{H,\chi},\ \forall g \in G,\ \forall v, w \in V_\sigma. \tag{6.4.8}$$

Proof. Let $\mathcal{N} \subseteq \mathcal{B}(V_\sigma)^H$ denote the von Neumann algebra in $\mathcal{B}(V_\sigma)$ generated by $\mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$. We show $\mathcal{N} = \mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$. It only remains to show $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$. As $\mathcal{N}$ is $*$-closed, it suffices to show that $T\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$ and that $T\chi(\xi)v = \chi(\xi)Tv$ for all $T \in \mathcal{N}$, $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_\chi$. Let $T \in \mathcal{N}$. Let (T_λ) be a net in $\mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$ such that $T_\lambda \to T$ strongly. Let $v, w \in \mathcal{D}_\chi$ and $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. Using Lemma 6.4.32 we have:

$$\begin{aligned}\langle v, Td\rho(x)w\rangle &= \lim_\lambda \langle v, T_\lambda d\rho(x)w\rangle = \lim_\lambda \langle v, d\rho(x)T_\lambda w\rangle = \lim_\lambda \langle d\rho(x^*)v, T_\lambda w\rangle \\ &= \langle d\rho(x^*)v, Tw\rangle\end{aligned} \tag{6.4.9}$$

As $v, w \in \mathcal{D}_\chi \subseteq H_\rho^\omega$, the orbit maps $g \mapsto \rho(g)v$ and $g \mapsto \rho(g)w$ are both real-analytic $G \to \mathcal{H}_\rho$. We obtain using (6.4.9) for all $\xi \in \mathfrak{g}$ in a small-enough 0-neighborhood in $\mathfrak{g}$ that:

$$\begin{aligned}\langle w, T\rho(e^\xi)v\rangle &= \sum_{n=0}^\infty \frac{1}{n!}\langle w, Td\rho(\xi^n)v\rangle = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\langle d\rho(\xi^n)w, Tv\rangle = \langle \rho(e^{-\xi})w, Tv\rangle \\ &= \langle w, \rho(e^\xi)Tv\rangle.\end{aligned} \tag{6.4.10}$$

It follows from Lemma 6.4.33 that $T\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$ and that equation (6.4.7) is valid for T. Thus $\mathcal{N}\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$. Differentiating (6.4.7) at the identity $e \in G$ we find that $\langle w, Td\rho(\xi)v\rangle = \langle w, d\rho(\xi)Tv\rangle$ for all $\xi \in \mathfrak{g}_\mathbb{C}$ and $w \in \mathcal{D}_\chi$. Suppose $\xi \in \mathfrak{b}_-$. Using that $T\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$, we obtain

$$\langle w, T\chi(\xi)v\rangle = \langle w, Td\rho(\xi)v\rangle = \langle w, d\rho(\xi)Tv\rangle = \langle w, \chi(\xi)Tv\rangle, \qquad \forall w \in \mathcal{D}_\chi,$$

where Lemma 6.4.15 was used in the first and last equality. As $\mathcal{D}_\chi$ is dense in V_σ, it follows for every $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_\chi$ that $T\chi(\xi)v = \chi(\xi)Tv$. Thus $T \in \mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$. Hence $\mathcal{N} = \mathcal{B}(\mathcal{H}_\rho)^{H,\chi}$. ☐

Combined with Lemma 6.4.34, Lemma 6.4.35 below completes the proof of Theorem 6.4.29.

Lemma 6.4.35. *Assume that $q_V \in \rho(G)''$. Then the map*

$$r : \mathcal{B}(\mathcal{H}_\rho)^G \to \mathcal{B}(V_\sigma)^{H,\chi}, \quad r(T) := T|_{V_\sigma}$$

defines an isomorphism of von Neumann algebras.

Proof. We know using Lemma 6.4.34 that $\mathcal{B}(V_\sigma)^{H,\chi}$ is a von Neumann algebra. Notice that the assumption $q_V \in \rho(G)''$ is equivalent with $TV_\sigma \subseteq V_\sigma$ for every $T \in \mathcal{B}(\mathcal{H}_\rho)^G$. Let $T \in \mathcal{B}(\mathcal{H}_\rho)^G$. Then $T\mathcal{H}_\rho^\omega \subseteq \mathcal{H}_\rho^\omega$ and $TV_\sigma \subseteq V_\sigma$. Recalling that $\mathcal{D}_\chi = V_\sigma \cap \mathcal{H}_\rho^\omega$, it follows that $T\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$. Since both T and T^* are in $\mathcal{B}(\mathcal{H}_\rho)^G$, it follows that $r(T) \in \mathcal{B}(V_\sigma)^{H,\chi}$, where we recall that $\rho(h)|_{V_\sigma} = \sigma(h)$ and

$d\rho(\xi)|_{\mathcal{D}_\chi} = \chi(\xi)$ for $h \in H$ and $\xi \in \mathfrak{b}_-$. It is clear that r is a weakly-continuous unital $*$-preserving homomorphism. It is also injective, because $r(T) = 0$ implies $T\rho(G)V_\sigma = \rho(G)TV_\sigma = \{0\}$, which in turn implies that $T = 0$ because V_σ is cyclic for the G-representation $\mathcal{H}_\rho$. As r is weakly continuous, its image is weakly closed [Mur90, Thm. 4.3.4]. Thus, to see that r is surjective, it suffices to show that its image contains all orthogonal projections in $\mathcal{B}(V_\sigma)^{H,\chi}$. Let $p_1 \in \mathcal{B}(V_\sigma)^{H,\chi}$ be an orthogonal projection and let $p_2 := 1 - p_1$. For $k \in \{1,2\}$, define $V_k, \mathcal{D}_k, \sigma_k$ and χ_k as in Remark 6.4.28, so that $(\sigma, \chi) \cong (\sigma_1, \chi_1) \oplus (\sigma_2, \chi_2)$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the closed G-invariant subspaces of $\mathcal{H}_\rho$ generated by V_1 and V_2, respectively. It suffices to show that $\mathcal{H}_1 \perp \mathcal{H}_2$. Let $v_1 \in V_1$ and $v_2 \in V_2$. As $p_1 \in \mathcal{B}(V_\sigma)^{H,\chi}$, it follows from equation (6.4.8) that

$$\langle v_1, \rho(g)v_2\rangle = \langle v_1, p_1\rho(g)v_2\rangle = \langle v_1, \rho(g)p_1v_2\rangle = 0, \qquad \forall g \in G.$$

It follows that $V_1 \perp \rho(G)V_2$, which in turn implies $\mathcal{H}_1 \perp \mathcal{H}_2$. □

Finally, it remains to prove Theorem 6.4.30:

Proof of Theorem 6.4.30: It remains only to prove the first item. The second will follow using Theorem 6.4.29. It is clear that $\mathcal{B}(V_\sigma)^{H,\chi} \subseteq \mathcal{B}(V_\sigma)^H$. Conversely, take $T \in \mathcal{B}(V_\sigma)^H$. Let $v, w \in \mathcal{D}_\chi$. Then in particular the orbit maps $G \to \mathcal{H}_\rho, g \mapsto \rho(g)v$ and $g \mapsto \rho(g)w$ are real-analytic. Notice that $T\mathcal{D}_\chi \subseteq V_\sigma^\infty$ and similarly $T^*\mathcal{D}_\chi \subseteq V_\sigma^\infty$. Using Lemma 6.4.23, we obtain for all ξ in a small-enough 0-neighborhood that

$$\begin{aligned}
\langle w, T\rho(e^\xi)v\rangle &= \sum_{n=0}^\infty \frac{1}{n!}\langle w, Td\rho(\xi^n)v\rangle = \sum_{n=0}^\infty \frac{1}{n!}\langle w, Td\sigma(E_0(\xi^n))v\rangle \\
&= \sum_{n=0}^\infty \frac{1}{n!}\langle w, d\sigma(E_0(\xi^n))Tv\rangle = \sum_{n=0}^\infty \frac{(-1^n)}{n!}\langle d\rho(\xi^n)w, Tv\rangle \\
&= \langle \rho(e^{-\xi})w, Tv\rangle = \langle w, \rho(e^\xi)Tv\rangle.
\end{aligned}$$

Using Lemma 6.4.33, it follows that $T\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$. Hence $\mathcal{B}(V_\sigma)^H\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$ and in particular $T^*\mathcal{D}_\chi \subseteq \mathcal{D}_\chi$. Suppose that $v \in \mathcal{D}_\chi, \xi_0 \in \mathfrak{h}_\mathbb{C}$ and $\xi_- \in \mathfrak{n}_-$. Then

$$T\chi(\xi_0 + \xi_-)v = Td\sigma(\xi_0)v = d\sigma(\xi_0)Tv = \chi(\xi_0 + \xi_-)Tv.$$

Hence $T\chi(\xi)v = \chi(\xi)Tv$ for all $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_\chi$. We conclude that $T \in \mathcal{B}(V_\sigma)^{H,\chi}$. Thus $\mathcal{B}(V_\sigma)^{H,\chi} = \mathcal{B}(V_\sigma)^H$. □

6.4.5 Holomorphic induction in stages

Let us next consider holomorphic induction in stages. We specialize to the context of trivial extensions. Recall from Section 6.4.5 that $\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_+$ and that $H \subseteq G$ is a connected Lie subgroup with $\mathbf{L}(H) = \mathfrak{h}$. Assume similarly that $\mathfrak{h}_\mathbb{C} = \mathfrak{a}_- \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_+$, where $\mathfrak{a}_\pm$ and $\mathfrak{t}_\mathbb{C}$ are closed subalgebras with $\theta(\mathfrak{t}_\mathbb{C}) \subseteq \mathfrak{t}_\mathbb{C}$, $\theta(\mathfrak{a}_\pm) \subseteq \mathfrak{a}_\mp$ and $[\mathfrak{t}_\mathbb{C}, \mathfrak{a}_\pm] \subseteq \mathfrak{a}_\pm$. Let T be a connected and closed locally exponential Lie subgroup of H integrating $\mathfrak{t} \subseteq \mathfrak{h}$. Using the notation of Definition 6.4.25:

Proposition 6.4.36 (Induction In Stages). *Let $(\rho, \mathcal{H}_\rho)$, $(\sigma, \mathcal{H}_\sigma)$ and $(\nu, \mathcal{H}_\nu)$ be analytic unitary representations of G, H and T, respectively. Then*

1. $\rho = \mathrm{HolInd}_T^G(\nu)$ *and* $\sigma = \mathrm{HolInd}_T^H(\nu) \implies \rho = \mathrm{HolInd}_H^G(\sigma)$.
2. *Suppose that* $\sigma = \mathrm{HolInd}_T^H(\nu)$ *and* $\rho = \mathrm{HolInd}_H^G(\sigma)$. *Assume w.l.o.g. that* $\mathcal{H}_\nu \subseteq \mathcal{H}_\sigma \subseteq \mathcal{H}_\rho$ *using Theorem 6.4.13, the inclusions being* T*- and* H*-equivariant, respectively. If* $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ *is dense in* $\mathcal{H}_\nu$*, then* $\rho = \mathrm{HolInd}_T^G(\nu)$.

Proof. These observations follow from a repeated application of Theorem 6.4.13.

1. In view of Theorem 6.4.13, we may assume that $\mathcal{H}_\nu \subseteq \mathcal{H}_\rho$ as T-representations and that $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ is dense in $\mathcal{H}_\nu$ and killed by $d\rho(\mathfrak{n}_- \oplus \mathfrak{a}_-)$. Let $(\pi, \mathcal{H}_\pi)$ denote the unitary H-representation in $\mathcal{H}_\rho$ generated by $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega \subseteq \mathcal{H}_\rho$. Using Theorem 6.4.13 it follows that $\pi = \mathrm{HolInd}_T^H(\nu)$. By Theorem 6.4.21, it follows that $\pi \cong \sigma$ as unitary H-representations. Thus we may assume $\mathcal{H}_\sigma = \mathcal{H}_\pi \subseteq \mathcal{H}_\rho$, the last inclusion being H-equivariant. The H-orbit of $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ under $\rho|_H$ in $\mathcal{H}_\sigma$ is contained in $\mathcal{H}_\sigma \cap \mathcal{H}_\rho^\omega$ and is trivially total for $\mathcal{H}_\sigma$. Thus $\mathcal{H}_\sigma \cap \mathcal{H}_\rho^\omega$ is dense in $\mathcal{H}_\sigma$. As $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ is already cyclic for $(\rho, \mathcal{H}_\rho)$, so is the larger space $\mathcal{H}_\sigma \cap \mathcal{H}_\rho^\omega$. To see that $\rho = \mathrm{HolInd}_H^G(\sigma)$, it just remains to show that $\mathcal{H}_\sigma \cap \mathcal{H}_\rho^\omega$ is killed by $d\rho(\mathfrak{n}_-)$. As $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ is killed by $d\rho(\mathfrak{n}_-)$ and $\mathrm{Ad}_H(\mathfrak{n}_-) \subseteq \mathfrak{n}_-$, it follows that $d\rho(\mathfrak{n}_-)\rho(H)\psi \subseteq \rho(H)d\rho(\mathfrak{n}_-)\psi = \{0\}$ for any $\psi \in \mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$. Thus $d\rho(\mathfrak{n}_-)$ kills $\rho(H)(\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega)$. As $\rho(H)(\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega)$ is total in $\mathcal{H}_\sigma$, it follows that $d\rho(\mathfrak{n}_-)$ kills $\mathcal{H}_\sigma \cap \mathcal{H}_\rho^\omega$. Having shown all conditions of Theorem 6.4.13, we conclude that $\rho = \mathrm{HolInd}_H^G(\sigma)$.
2. As $\sigma = \mathrm{HolInd}_T^H(\nu)$ we may assume that $\mathcal{H}_\nu \subseteq \mathcal{H}_\sigma$ as T-representations and that $\mathcal{H}_\sigma^\omega \cap \mathcal{H}_\nu$ is dense in $\mathcal{H}_\nu$, cyclic for the H-representation $\mathcal{H}_\sigma$, and killed by $d\sigma(\mathfrak{a}_-)$. Similarly, as $\rho = \mathrm{HolInd}_H^G(\sigma)$ we may assume that $\mathcal{H}_\sigma \subseteq \mathcal{H}_\rho$ as H representations and moreover that $\mathcal{H}_\rho^\omega \cap \mathcal{H}_\sigma$ is dense in $\mathcal{H}_\sigma$, cyclic for the G-representation $\mathcal{H}_\rho$ and killed by $d\rho(\mathfrak{n}_-)$. Then $\mathcal{H}_\nu \subseteq \mathcal{H}_\sigma \subseteq \mathcal{H}_\rho$ the inclusions being T- and H equivariant, respectively. By assumption $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ is dense in $\mathcal{H}_\nu$. Since $\mathcal{H}_\nu$ is cyclic for $(\sigma, \mathcal{H}_\sigma)$ and $\mathcal{H}_\sigma$ for $(\rho, \mathcal{H}_\rho)$, it follows that $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ is cyclic for $(\rho, \mathcal{H}_\rho)$. For any $\psi \in \mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega \subseteq \mathcal{H}_\sigma \cap \mathcal{H}_\rho^\omega$ we have $d\rho(\mathfrak{a}_- \oplus \mathfrak{n}_-)\psi \subseteq d\sigma(\mathfrak{a}_-)\psi + d\rho(\mathfrak{n}_-)\psi = \{0\}$. Thus $\mathcal{H}_\nu \cap \mathcal{H}_\rho^\omega$ is killed by $d\rho(\mathfrak{a}_- \oplus \mathfrak{n}_-)$. By Theorem 6.4.13 it follows that $\rho = \mathrm{HolInd}_T^G(\nu)$. □

6.5 A complex-geometric approach

In this section, a definition of holomorphically induced representations is presented which ensures that $\mathcal{H}_\rho^\infty$ embeds in a space of holomorphic mappings. Contrary to Section 6.4, this approach involves complex geometry. It is not as generally applicable, and in particular requires access to a dense set of b-strongly-entire vectors in the representation being induced, a condition that is well-understood for finite-dimensional Lie groups but barely studied for infinite-dimensional ones.

We begin in Section 6.5.1 and Section 6.5.2 with a detailed consideration of analytic manifold structures on the homogeneous space G/H. The results in these two sections are not original, as they are mentioned in [Nee14a, Appendix C]. Rather, it is the purpose of these sections to provide detailed proofs. In Section 6.5.3, we consequently equip the homogeneous vector bundle $G \times_H V_\sigma^{\mathcal{O}_b}$ with a suitable complex-analytic bundle structure. Here, we closely follow the construction of [Nee13, Thm. 2.6]. We then proceed in Section 6.5.4 to define geometric holomorphic induction and compare the notion with the one studied in Section 6.4.

6.5.1 Real-analytic manifold structure on G/H

Let G be a BCH Fréchet-Lie group with Lie algebra $\mathfrak{g}$. Let $H \subseteq G$ be a closed locally exponential Lie subgroup of G, in the sense of [Nee06, Def. IV.3.2]. In this case, H is necessarily BCH and an analytic embedded Lie subgroup of G (cf. Lemma 6.4.1). Let $\mathfrak{h} := \mathbf{L}(H)$ be the Lie algebra of H, which we identify as Lie subalgebra of $\mathfrak{g}$ using the pushforward of the inclusion $H \hookrightarrow G$.

Theorem 6.5.1 below is the main result of this section. Its proof follows that of the Quotient Manifold Theorem in the finite-dimensional context, replacing the use of the Inverse Function Theorem by (A1).

Theorem 6.5.1. *Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a closed complement of $\mathfrak{h}$ in $\mathfrak{g}$. Assume that $U_\mathfrak{p} \subseteq \mathfrak{p}$ and $U_\mathfrak{h} \subseteq \mathfrak{h}$ are open 0-neighborhoods for which*

$$U_\mathfrak{p} \times U_\mathfrak{h} \to \mathfrak{g}, \quad (x, y) \mapsto x * y, \tag{A1}$$

is an analytic diffeomorphism onto an open subset $U_\mathfrak{g} \subseteq \mathfrak{g}$, where $x * y$ is defined by the BCH series. Then $M = G/H$ carries a unique real-analytic manifold structure satisfying the following properties:

1. *The left G-action $G \times G/H \to G/H$ is real-analytic.*
2. *$q : G \to G/H$ is a real-analytic principal H-bundle.*

Moreover, if N is a real-analytic manifold, then a map $f : G/H \to N$ is real-analytic if and only if its lift $\widetilde{f} : G \to N$ is so.

Notice that the uniqueness of the analytic structure follows immediately from the second property in Theorem 6.5.1. Indeed, for $k \in \{1, 2\}$, let M_k denote the space G/H equipped with a real-analytic manifold structure for which the quotient map $q_k : G \to M_k$ is a real-analytic principal H-bundle. The identity on G/H defines a diffeomorphism $I : M_1 \to M_2$. If $\sigma_1 : U \to G$ is a real-analytic local section of $q_1 : G \to M_1$, defined on some open set $U \subseteq M_1$, then $I|_U = q_2 \circ \sigma_1$ is real-analytic. It follows that I is an analytic diffeomorphism, so $M_1 \cong M_2$.

Before diving into analytic manifold structures on the homogeneous space G/H, let us verify that it is indeed a Hausdorff space.

Lemma 6.5.2. *The quotient space G/H is Hausdorff.*

Proof. The map

$$f : G \times H \hookrightarrow G \times G, \quad (g, h) \mapsto (g, gh)$$

is closed, because it is the composition of the inclusion $G \times H \hookrightarrow G \times G$, which is a closed map because $H \subseteq G$ is closed, and the map

$$G \times G \to G \times G, \quad (x, y) \mapsto (x, xy),$$

which is closed because it is a diffeomorphism with inverse $(x, y) \mapsto (x, x^{-1}y)$. It follows that the equivalence relation $R := f(G, H) \subseteq G \times G$ is closed in $G \times G$. Since $q : G \to G/H$ is an open map, and the image of the open set $(G \times G) \setminus R$ under $q \times q$ is the complement of $\Delta(G/H)$ in $G/H \times G/H$, it follows that $\Delta(G/H)$ is closed in $G/H \times G/H$. We conclude that G/H is Hausdorff. □

We proceed with the proof of existence in Theorem 6.5.1, which requires some preparatory lemmas. Throughout the following, we denote the multiplication of G by $\mu : G \times G \to G$.

We first show that for any point $g \in G$, there exists an analytically embedded submanifold $S \subseteq G$ that contains g, is analytically diffeomorphic to an open subset of $\mathfrak{p}$, and for which $\mu|_{S\times H} : S \times H \to G$ is an analytic diffeomorphism onto some open subset of G. In this case, we call S a *transversal* (through g).

Lemma 6.5.3. *Let $S \subseteq G$ be a transversal. Then $q(S) \subseteq G/H$ is open and $q|_S$ is a homeomorphism onto $q(S) \subseteq G/H$.*

Proof. The map $q|_S : S \to q(S)$ is injective because $\mu|_{S\times H} : S \times H \to G$ is injective, so if $g_1, g_2 \in S$ and $g_2 = g_1 h$ for some $h \in H$, then $g_1 = g_2$ and $h = e$ is the identity. It also an open map, because $q^{-1}(q(W)) = WH = \mu(W, H)$ is open in G for any open set $W \subseteq S$. Hence $q|_S$ is a homeomorphism. □

Transversals thus define charts for the quotient space G/H. For these to define a manifold structure on G/H, there should exist a transversal through any $g \in G$.

Lemma 6.5.4. *If there exists a transversal through $e \in G$, then there exists a transversal through any $g \in G$.*

Proof. If S_e is a transversal through e and $g \in G$, then $S_g := gS$ is a transversal through g, because

$$\mu|_{S_g\times H} = l_g \circ \mu|_{S_e\times H} \circ (l_g^{-1} \times \mathrm{id}_H)$$

is an analytic diffeomorphism onto its image, where $l_g(x) = gx$ for $x \in G$. □

We show next that there exists a transversal through the identity $e \in G$. Assume that $U_{\mathfrak{p}} \subseteq \mathfrak{p}$ and $U_{\mathfrak{h}} \subseteq \mathfrak{h}$ are open 0-neighborhoods for which (A1) is an analytic diffeomorphism onto an open subset $U_{\mathfrak{g}}$ of $\mathfrak{g}$. Shrinking these open sets, we may furthermore assume that $\exp_G|_{U_{\mathfrak{g}}}$ is an analytic diffeomorphism onto an open 1-neighborhood U_G of G. Define also the open sets $U_P := \exp_G(U_{\mathfrak{p}}) \subseteq G$ and

$U_H := \exp_G(U_{\mathfrak{h}}) \subseteq H.$

Define the map

$$\phi : U_{\mathfrak{p}} \times U_{\mathfrak{h}} \to U_G, \qquad \phi(x, y) := \exp_G(x) \exp_G(y) = \exp_G(x * y).$$

Notice that the assumptions on $U_{\mathfrak{p}}$ and $U_{\mathfrak{h}}$ imply that both $\mu|_{U_P \times U_H}$ and ϕ are analytic diffeomorphisms. Notice also that ϕ is a foliated chart for the foliation on G defined by the right H-action. The analytic submanifolds $\phi(\{\eta\} \times U_{\mathfrak{h}})$ of U_G are plaques of the foliated chart defined by ϕ, where $\eta \in U_{\mathfrak{p}}$.

Lemma 6.5.5. *There exist open* 0*-neighborhoods* $V_{\mathfrak{p}} \subseteq U_{\mathfrak{p}}$ *and* $V_{\mathfrak{h}} \subseteq U_{\mathfrak{h}}$ *such that for any right* H*-orbit* $\mathcal{O}$ *in* G*, there exists at most one* $\eta \in V_{\mathfrak{p}}$ *satisfying* $\phi(\{\eta\} \times V_{\mathfrak{h}}) \cap \mathcal{O} \neq \emptyset$.

Proof. Assume that there exist no open subsets $V_{\mathfrak{p}} \subseteq U_{\mathfrak{p}}$ and $V_{\mathfrak{h}} \subseteq U_{\mathfrak{h}}$ as in the assertion. Since $\mathfrak{g}$ is metrizable, we can find a countable and decreasing neighborhood basis $\{V_i\}_{i \in \mathbb{N}}$ of the identity $e \in G$, where the open 0-neighborhoods V_i are of the form $V_i = \phi(V_{i,\mathfrak{p}} \times V_{i,\mathfrak{h}})$ for some open 0-neighborhoods $V_{i,\mathfrak{p}} \subseteq U_{\mathfrak{p}}$ and $V_{i,\mathfrak{h}} \subseteq U_{\mathfrak{h}}$ and distinct points $g_i, g_i' \in V_{i,P} := \phi(V_{i,\mathfrak{p}} \times \{0\})$ lying in the same H-orbit, for every $i \in \mathbb{N}$. Then $g_i' = g_i h_i$ for some $h_i \in H$. The construction of $\{V_i\}_{i \in \mathbb{N}}$ ensures that $g_i \to e$ and $g_i' \to e$ in G. We then also have $h_i = g_i^{-1} g_i' \to e$. Notice that $\mu(g_i', e) = \mu(g_i, h_i)$ for all $i \in \mathbb{N}$. Since $g_i \neq g_i'$ for $i \in \mathbb{N}$ and $h_i \in U_H$ for large-enough $i \in \mathbb{N}$, this contradicts the injectivity of $\mu|_{U_P \times U_H}$. ☐

By Lemma 6.5.5, we may and do assume that for any right H-orbit $\mathcal{O}$ in G, there exists at most one $\eta \in U_{\mathfrak{p}}$ satisfying $\phi(\{\eta\} \times U_{\mathfrak{h}}) \cap \mathcal{O} \neq \emptyset$, by shrinking $U_{\mathfrak{p}}$ and $U_{\mathfrak{h}}$ if necessary.

Lemma 6.5.6. U_P *is a transversal through* $e \in G$.

Proof. We must show that $\mu|_{U_P \times H}$ is an analytic diffeomorphism onto its image. To see that $\mu|_{U_P \times H}$ is injective, let $x, y \in U_{\mathfrak{p}}$ and assume that $\exp_G(y) = \exp_G(x)h$ for some $h \in H$. Since $\exp_G(x) = \phi(x, 0)$ and $\exp_G(y) = \phi(y, 0)$ lie on the same right H-orbit in G, and any right H-orbit intersects U_G in at most one plaque of the foliated chart defined by ϕ, we must have $x = y$. Hence $\mu|_{U_P \times H}$ is injective. We already know that $\mu|_{U_P \times U_H}$ is an analytic diffeomorphism onto U_G. Since $\mu|_{U_P \times (U_H \cdot h)} = r_h \circ \mu|_{U_P \times U_H}$ for any $h \in H$, where $r_h(g) = gh$ for $g \in G$, it follows that $\mu|_{U_P \times H}$ is an analytic local diffeomorphism. Since it is also injective, we are done. ☐

Proof of Theorem 6.5.1:
The quotient space is Hausdorff by Lemma 6.5.2. Combining Lemma 6.5.4 with Lemma 6.5.6, it follows that there exists a transversal through any point in G. In particular, the set $\{\, q(S) : S \text{ is a transversal} \,\}$ is an open cover of G/H. Suppose that $S_1, S_2 \subseteq G$ are two transversals and assume that $U_{G/H} := q(S_1) \cap q(S_1) \neq \emptyset$. Let $U_G := q^{-1}(U_{G/H}) \subseteq G$. For $k \in \{1, 2\}$, let $\psi_k : V_k \to S_k$ be an analytic diffeomorphism for some open neighborhood $V_k \subseteq \mathfrak{p}$. Consider the charts

$$\sigma_k := (q|_{S_k} \circ \psi_k)^{-1} : q(S_k) \to V_k \tag{6.5.1}$$

of G/H, let $W_k := \sigma_k(U_{G/H}) \subseteq V_k$ be the domain of the transition function, and let $Z_k := \psi_k(W_k) \subseteq S_k$. The maps $\mu|_{Z_1 \times H}$ and $\mu|_{Z_2 \times H}$ are both real-analytic diffeomorphisms onto U_G, because $U_G = Z_1 H = Z_2 H$. We thus have commutative diagrams

$$\begin{array}{ccccc} W_k \times H & \xrightarrow{\psi_k \times \mathrm{id}_H} & Z_k \times H & \xrightarrow{\mu} & U_G \\ \big\downarrow{\scriptstyle \pi_{W_k}} & & \big\downarrow & & \big\downarrow{\scriptstyle q} \\ W_k & \xrightarrow{\psi_k} & Z_k & \xrightarrow{q|_{Z_k}} & U_{G/H} \end{array} \qquad \text{for } k \in \{1,2\}. \tag{6.5.2}$$

Notice that the upper horizontal arrows in (6.5.2) are analytic H-equivariant diffeomorphisms. Define

$$\Psi_k := (\mu|_{Z_k \times H} \circ (\psi_k \times \mathrm{id}_H))^{-1} : U_G \to W_k \times H, \qquad k \in \{1,2\}$$

and $\Psi_{12} := \Psi_1 \circ \Psi_2^{-1}$. It follows from the commutativity of (6.5.2) that the transition function $\sigma_{12} := \sigma_1 \circ \sigma_2^{-1}$ is given by $\sigma_{12}(x) = \pi_{W_1}(\Psi_{12}(x, e))$ for $x \in W_2$, which is real-analytic because both π_{W_1} and Ψ_{12} are so. It follows that charts of the form (6.5.1) equip G/H with the structure of a real-analytic manifold. Moreover, taking $S_1 = S_2$ in the preceding argument, the diagram (6.5.2) shows that $q : G \to G/H$ is an analytic principal H-bundle. This in turn implies for a real-analytic manifold N that a map $G/H \to N$ is real-analytic if and only if its lift to G is so. Similarly, the action map $G \times G/H \to G/H, (g, q(x)) \mapsto q(gx)$ is real-analytic because $G \times G \xrightarrow{\mathrm{id}_G \times q} G \times G/H$ is a principal H-bundle and the group multiplication $G \times G \to G$ is real-analytic. □

6.5.2 Complex-analytic manifold structure on G/H

As in Section 6.4, we let $\theta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ denote the conjugation $\theta(\xi + i\eta) = \xi - i\eta$ for $\xi, \eta \in \mathfrak{g}$. Throughout this section, we continue in the setting of Section 6.5.1 and assume in addition that we have a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_- \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_+, \tag{6.5.3}$$

where $\mathfrak{n}_\pm$ and $\mathfrak{h}_{\mathbb{C}}$ are closed Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$ satisfying $\theta(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\mp$, $\theta(\mathfrak{h}_{\mathbb{C}}) \subseteq \mathfrak{h}_{\mathbb{C}}$ and $[\mathfrak{h}_{\mathbb{C}}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$. We assume further that $G_{\mathbb{C}}$ is a complex-analytic BCH Fréchet-Lie group with Lie algebra $\mathbf{L}(G_{\mathbb{C}}) = \mathfrak{g}_{\mathbb{C}}$. We also assume the existence of a real-analytic embedding $G \hookrightarrow G_{\mathbb{C}}$ whose pushforward is the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$. Finally, we assume that $\mathrm{Ad}_H(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\pm$. Let $\mathfrak{b}_\pm := \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_\pm$ and define the closed complement $\mathfrak{p} := (\mathfrak{n}_- \oplus \mathfrak{n}_+) \cap \mathfrak{g}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Notice that (6.5.3) induces an isomorphism $\mathfrak{g}_{\mathbb{C}}/\mathfrak{b}_- \cong \mathfrak{n}_+$, and that the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$ descends to an isomorphism $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{b}_-$ of real topological vector spaces. As in Section 6.5.1, we let $q : G \to G/H$ denote the quotient map.

The main result of this section is the following complex analogue of Theorem 6.5.1:

Theorem 6.5.7. *Assume that there exist open* 0*-neighborhoods* $U_{\mathfrak{h}} \subseteq \mathfrak{h}$*,* $U_{\mathfrak{p}} \subseteq \mathfrak{p}$*,* $U_{\mathfrak{n}_+} \subseteq \mathfrak{n}_+$ *and* $U_{\mathfrak{b}_-} \subseteq \mathfrak{b}_-$ *such that the maps*

$$U_{\mathfrak{p}} \times U_{\mathfrak{h}} \to \mathfrak{g}, \quad (x, y) \mapsto x * y, \tag{A1}$$

$$U_{\mathfrak{p}} \times U_{\mathfrak{b}_-} \to \mathfrak{g}_{\mathbb{C}}, \quad (x, y) \mapsto x * y, \tag{A2}$$

$$U_{\mathfrak{n}_+} \times U_{\mathfrak{b}_-} \to \mathfrak{g}_{\mathbb{C}}, \quad (x, y) \mapsto x * y, \tag{A3}$$

are analytic diffeomorphisms onto an open subset in their codomain, where $x * y$ *is defined by the BCH series. Then* G/H *carries a unique complex-analytic manifold structure compatible with the real-analytic structure obtained from Theorem 6.5.1, and which satisfies the following properties:*

1. *The left* G*-action* $l_g : G/H \to G/H$ *is complex-analytic for any* $g \in G$*.*
2. *The* $\mathbb{C}$*-linear extension of the map*

$$\mathfrak{g} \to T_{eH}(G/H), \quad \xi \mapsto \left.\frac{d}{dt}\right|_{t=0} q(e^{t\xi})$$

   descends to a $\mathbb{C}$*-linear isomorphism* $\mathfrak{g}_{\mathbb{C}}/\mathfrak{b}_- \cong T_e(G/H)$ *of topological complex vector spaces.*

Moreover, if Q *is a complex-analytic manifold, then a smooth map* $f : G/H \to Q$ *is holomorphic if and only if its lift* $\widetilde{f} : G \to Q$ *satisfies* $T_g^{\mathbb{C}}(\widetilde{f})(\boldsymbol{v}_g(\xi)) = 0$ *for all* $g \in G$ *and* $\xi \in \mathfrak{b}_-$*, where* $T_g^{\mathbb{C}}(\widetilde{f}) : T_g^{\mathbb{C}}(G) \to T_{\widetilde{f}(g)}Q$ *is the* $\mathbb{C}$*-linear extension of* $T_g(\widetilde{f})$ *and where* $\boldsymbol{v}(\xi)$ *is the left-invariant* $T^{\mathbb{C}}G$*-valued vector field on* G *satisfying* $\boldsymbol{v}_e(\xi) = \xi$*.*

Remark 6.5.8. In [Nee14a, Example C.4], sufficient conditions are discussed that guarantee that (A1), (A2) and (A3) are satisfied. Using the Inverse Function Theorem, this is in particular the case if G is a simply connected Banach-Lie group and G/H is a Banach homogeneous space. In [Nee14a, Sec. 5.2], these conditions are also shown to be satisfied in the context where G is (a central $\mathbb{T}$-extensions of) a (twisted) loop group, and where $H \subseteq G$ is the subgroup of fixed points under a particular $\mathbb{R}$-action on G.

We proceed with the proof of Theorem 6.5.7. Its proof will parallel that of Theorem 6.5.1. Throughout, we let $\mu : G_{\mathbb{C}} \times G_{\mathbb{C}}$ denote the multiplication of $G_{\mathbb{C}}$. As (A1) is satisfied, we equip G/H with the corresponding real-analytic manifold structure obtained from Theorem 6.5.1. So $q|_S : S \to q(S) \subseteq G/H$ is a real-analytic diffeomorphism for any transversal $S \subseteq G$.

Let us first prove the uniqueness assertion. Suppose that M_1 and M_2 are two complex manifolds that are equal to G/H as real-analytic manifold, and which both satisfy the two conditions in Theorem 6.5.7. Then the identity map on G/H defines a real-analytic diffeomorphism $I : M_1 \to M_2$. The two conditions in Theorem 6.5.7 moreover ensure that its tangent map $T(I)$ is fiber-wise $\mathbb{C}$-linear. It follows using

Proposition 6.2.11 that I is holomorphic, and consequently that $M_1 \cong M_2$ as complex manifolds.

We next introduce some terminology. If A and B are topological spaces, we let π_A and π_B denote the canonical projections of $A \times B$ onto A and B, respectively. We also write $c_g(x) := gxg^{-1}$ for $x, g \in G_{\mathbb{C}}$.

Definition 6.5.9. A *good triple* (S, N, U_B) consists of:

1. A transversal $S \subseteq G$.
2. An embedded complex submanifold $N \subseteq G_{\mathbb{C}}$ that is biholomorphic to an open 0-neighborhood of $\mathfrak{n}_+$.
3. An embedded complex submanifold $U_B \subseteq G_{\mathbb{C}}$ of the form $U_B = \exp_{G_{\mathbb{C}}}(U_{\mathfrak{b}_-})$, where $U_{\mathfrak{b}_-} \subseteq \mathfrak{b}_-$ is a symmetric 0-neighborhood such that $\exp_{G_{\mathbb{C}}}|_{U_{\mathfrak{b}_-}}$ is biholomorphic onto U_B, and such that the BCH series of $\mathfrak{g}_{\mathbb{C}}$ defines a holomorphic map $U_{\mathfrak{b}_-} \times U_{\mathfrak{b}_-} \to \mathfrak{b}_-$,

such that:

— $\mu|_{S \times U_B}$ is a real-analytic diffeomorphism onto an open set in $G_{\mathbb{C}}$.

— $\mu|_{N \times U_B}$ is biholomorphic onto an open set in $G_{\mathbb{C}}$.

— $S \cap NU_B \neq \emptyset$.

— The implication

$$n_1 b_1 = n_2 b_2 c_h(b_3) \implies n_1 = n_2 \tag{6.5.4}$$

holds true for every $n_1, n_2 \in N$ and $b_1, b_2, b_3 \in U_B$ and $h \in H$.

We call the (non-empty) open subset $S_0 := S \cap NU_B$ of S the *real domain* of the good triple, and we say that the open subset $N_0 := N \cap SU_B$ of N is its *complex domain*. We say that (S, N, U_B) contains g if $g \in S_0$.

Lemma 6.5.10. *Assume that (S, N, U_B) is a good triple with real and complex domains S_0 and N_0, respectively. Let ζ be the inverse of $\mu|_{N \times U_B} : N \times U_B \to NU_B$. Then $\pi_N \circ \zeta|_{S_0}$ is a real-analytic diffeomorphism $S_0 \to N_0$.*

Proof. Let χ be the inverse of $\mu|_{S \times U_B} : S \times U_B \to SU_B$. Write $\nu(p) := \pi_N(\zeta(p))$ for $p \in S_0$. It is clear that $\nu : S_0 \to N$ is real-analytic, being the composition of $\zeta|_S$ and π_N. Notice for $p \in S$, $n \in N$ and $b \in U_B$ that $p = nb \iff n = pb^{-1}$. Since U_B is symmetric, these equivalent conditions imply that $p \in S_0$ and $n \in N_0$. It moreover follows that the image of ν is precisely N_0. With p, n and b as above, it also follows that $p = \pi_S(\chi(n))$, so that ν is bijective onto N_0, with inverse given by the real-analytic map $\nu^{-1}(n) = \pi_S(\chi(n))$ for $n \in N_0$. ∎

In view of Lemma 6.5.10, notice in the notation of Definition 6.5.9 that $q(S_0) \cong S_0 \cong N_0$ are diffeomorphic as real-analytic manifolds. Since N_0 is a complex manifold, we can use this isomorphism to turn $q(S_0)$ into a complex manifold, yielding candidates for complex charts of the homogeneous space G/H. We first have to verify that there are sufficiently many good triples, which is our next aim.

Lemma 6.5.11. *Let (S, N, U_B) be a good triple containing $e \in G$, and let $g \in G$. Then (gS, gN, U_B) is a good triple containing g.*

Proof. Notice that gS is a transversal by the proof of Lemma 6.5.4. Moreover, $\mu|_{gN\times U_B}$ is biholomorphic onto its image because

$$\mu|_{gN\times U_B} = l_g \circ \mu|_{N\times U_B} \circ (l_g^{-1} \times \mathrm{id}_{U_B}),$$

where $l_g(x) := gx$ for $x \in G_{\mathbb{C}}$. It similarly follows that $\mu|_{gS\times U_B}$ is a real-analytic diffeomorphism onto its image. Assume next that $n_1, n_2 \in U_N$, $b_1, b_2, b_3 \in U_B$ and $h \in H$ are s.t. $gn_1b_1 = gn_2b_2c_h(b_3)$. Then $n_1b_1 = n_2b_2c_h(b_3)$, so $n_1 = n_2$ by (6.5.4). Finally, from $e \in S \cap NU_B$ we find $g \in gS \cap gNU_B$. Hence (gS, gN, U_B) is a good triple containing g. □

We show next that there exists a good triple containing the identity $e \in G$. Let $U_{\mathfrak{p}} \subseteq \mathfrak{p}$, $U_{\mathfrak{n}_+} \subseteq \mathfrak{n}_+$, $U_{\mathfrak{b}_-} \subseteq \mathfrak{b}_-$ and $U_{\mathfrak{h}} \subseteq \mathfrak{h}$ be open 0-neighborhoods s.t. (A1), (A2) and (A3) are analytic diffeomorphisms onto an open 0-neighborhood in their codomain. In view of Lemma 6.5.6, we may assume that $U_P := \exp_G(U_{\mathfrak{p}})$ is a transversal in G, by shrinking $U_{\mathfrak{p}}$ if necessary. Define $U_{\mathfrak{g}_{\mathbb{C}}} := (U_{\mathfrak{n}_+} * U_{\mathfrak{b}_-}) \cup (U_{\mathfrak{p}} * U_{\mathfrak{b}_-})$, which is open in $\mathfrak{g}_{\mathbb{C}}$. Shrinking all these sets further, we may additionally assume that $\exp_{G_{\mathbb{C}}}|_{U_{\mathfrak{g}_{\mathbb{C}}}}$ is biholomorphic onto some open subset of $G_{\mathbb{C}}$, that the BCH series defines a holomorphic map $U_{\mathfrak{g}_{\mathbb{C}}} \times U_{\mathfrak{g}_{\mathbb{C}}} \to \mathfrak{g}_{\mathbb{C}}, (x, y) \mapsto x * y$, and that $U_{\mathfrak{b}_-}$ is symmetric.

Recall that $U_P := \exp_G(U_{\mathfrak{p}})$. Define also $U_N := \exp_{G_{\mathbb{C}}}(U_{\mathfrak{n}_+})$ and $U_B := \exp_{G_{\mathbb{C}}}(U_{\mathfrak{b}_-})$. By construction, all desired properties for (U_P, U_N, U_B) to be a good triple are satisfied, except for (6.5.4).

Lemma 6.5.12. *There exist open 1-neighborhoods $V_N \subseteq U_N$ and $V_B \subseteq U_B$ s.t.*

$$n_1b_1 = n_2b_2c_h(b_3) \quad \Longrightarrow \quad n_1 = n_2 \tag{6.5.5}$$

for all $n_1, n_2 \in V_N$, $b_1, b_2, b_3 \in V_B$ and $h \in H$.

Proof. Assume that there exist no open 1-neighborhoods $V_N \subseteq U_N$ and $V_B \subseteq U_B$ as in the assertion. Since $\mathfrak{g}_{\mathbb{C}}$ is metrizable, we can find a countable and decreasing neighborhood basis $\{V_i\}_{i\in\mathbb{N}}$ of the identity $e \in G_{\mathbb{C}}$, where each $V_i \subseteq G_{\mathbb{C}}$ is of the form $V_i = V_{i,N}V_{i,B}$ for some open 1-neighborhoods $V_{i,N} \subseteq U_N$ and $V_{i,B} \subseteq U_B$, elements $b_1^{(i)}, b_2^{(i)}, b_3^{(i)} \in V_{i,B}$, $h_i \in H$ and distinct $n_1^{(i)}, n_2^{(i)} \in V_{i,N}$, such that $n_1^{(i)}b_1^{(i)} = n_2^{(i)}b_2^{(i)}c_{h_i}(b_3^{(i)})$ for every $i \in \mathbb{N}$. The construction of $\{V_i\}_{i\in\mathbb{N}}$ ensures that the sequences $(n_1^{(i)})$, $(n_2^{(i)})$, $(b_1^{(i)})$, $(b_2^{(i)})$ and $(b_3^{(i)})$ all converge to $e \in G_{\mathbb{C}}$. Hence $c_{h_i}(b_3^{(i)}) \to e$ in $G_{\mathbb{C}}$.

The map $H \times G_{\mathbb{C}} \to G_{\mathbb{C}} \times G_{\mathbb{C}}, (h, g) \mapsto (g, c_h(g))$ is proper, because it equals the composition of the inclusion $H \times G_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}} \times G_{\mathbb{C}}$ and the diffeomorphism $G_{\mathbb{C}} \times G_{\mathbb{C}} \to G_{\mathbb{C}} \times G_{\mathbb{C}}, (x, y) \mapsto (y, c_x(y))$. Since $c_{h_i}(b_3^{(i)}) \to e$ and $b_3^{(i)} \to e$, we may therefore assume, after passing to a subsequence, that (h_i) is convergent in H.

Notice that there exists an open neighborhood $\mathcal{O}_B \subseteq U_B$ of $e \in U_B$ such that $\mathcal{O}_B \mathcal{O}_B \subseteq U_B$, because the BCH product $*$ on $U_{\mathfrak{g}_{\mathbb{C}}}$ is continuous, $\mathfrak{b}_-$ is a closed subalgebra in $\mathfrak{g}_{\mathbb{C}}$ and $\exp_{G_{\mathbb{C}}}|_{U_{\mathfrak{b}}}$ is a homeomorphism onto U_B. Notice similarly that $c_{h_i}(b_3^{(i)}) \in \mathcal{O}_B$ for large-enough $i \in \mathbb{N}$, because $\mathrm{Ad}_H(\mathfrak{b}_-) \subseteq \mathfrak{b}_-$ and the adjoint action of H on $\mathfrak{b}$ is jointly continuous. Thus $b_2^{(i)} c_{h_i}(b_3^{(i)}) \in U_B$ for large-enough $i \in \mathbb{N}$. Since $n_1^{(i)} \neq n_2^{(i)}$ and $\mu(n_1^{(i)}, b_1^{(i)}) = \mu(n_2^{(i)}, b_2^{(i)} c_{h_i}(b_3^{(i)}))$ for all $i \in \mathbb{N}$, this contradicts the injectivity of $\mu|_{U_N \times U_B}$. □

We may thus assume that (6.5.5) holds true for $V_N := U_N$ and $V_B := U_B$, by shrinking U_N and U_B further if necessary. Then (U_P, U_N, U_B) is a good triple containing $e \in G$. We therefore have shown:

Lemma 6.5.13. *There exists a good triple (S, N, U_B) containing $e \in G$.*

Proof of Theorem 6.5.7:
By Lemma 6.5.13, there exists a good triple $(S^{(e)}, N^{(e)}, U_B^{(e)})$ containing the identity $e \in G$. By Lemma 6.5.4, we then find that for any $g \in G$, there exists a good triple of the form $(S, N, U_B^{(e)})$ containing g. Consequently, the set $\left\{ q(S) \,:\, (S, N, U_B^{(e)}) \text{ is a good triple} \right\}$ is an open cover of G/H. Henceforth, we write $U_B := U_B^{(e)}$. Let $U_{\mathfrak{b}_-} \subseteq \mathfrak{b}_-$ be a symmetric 0-neighborhood such that $\exp_{G_{\mathbb{C}}}|_{U_{\mathfrak{b}_-}}$ is biholomorphic onto U_B, and such that the BCH series defines a holomorphic map $* : U_{\mathfrak{b}_-} \times U_{\mathfrak{b}_-} \to \mathfrak{b}_-$.

For $k \in \{1, 2\}$, let $(S^{(k)}, N^{(k)}, U_B)$ be a good triple with real and complex domains $S_0^{(k)}$ and $N_0^{(k)}$, respectively. Denote the inverse of $\mu|_{N^{(k)} \times U_B} : N^{(k)} \times U_B \to N^{(k)} U_B$ by ζ_k. Let $\nu_k := \pi_{N^{(k)}} \circ \zeta_k|_{S_0^{(k)}}$ be the associated real-analytic diffeomorphism $S_0^{(k)} \to N_0^{(k)}$ (cf. Lemma 6.5.10). Concretely, for $g_k \in S_0^{(k)}$ there exists unique $(n_k, b_k) \in N_0^{(k)} \times U_B$ s.t. $g_k = n_k b_k$, and then $\nu_k(g_k) = n_k$. Denote the associated chart of G/H by $\sigma_k := \nu_k \circ q|_{S_0^{(k)}}^{-1} : q(S_0^{(k)}) \to N_0^{(k)}$. (We avoid choosing concrete charts of the complex manifold $N_0^{(k)}$, to avoid unnecessary notation.) The situation is depicted in the diagram below:

$$\begin{array}{ccc} S_0^{(k)} & \xrightarrow{\ \zeta_k|_{S_0^{(k)}}\ } & N^{(k)} \times U_B \\ {\scriptstyle q|_{S_0^{(k)}}} \big\downarrow & \searrow^{\nu_k} & \big\downarrow {\scriptstyle \pi_{N^{(k)}}} \\ q(S_0^{(k)}) & \xrightarrow{\ \sigma_k\ } & N_0^{(k)} \end{array} .$$

Suppose that $V_{G/H} := q(S_0^{(1)}) \cap q(S_0^{(2)}) \neq \emptyset$. For $k \in \{1, 2\}$, define the open set

$Z^{(k)} := q|_{S_0^{(k)}}^{-1}(V_{G/H}) \subseteq S_0^{(k)}$ and let $\mathcal{D}^{(k)} := \sigma_k(V_{G/H}) \subseteq N_0^{(k)}$ be the domain of the transition function. Notice concretely that

$$n \in \mathcal{D}^{(k)} \iff \exists (g,b) \in Z^{(k)} \times U_B \,:\, g = nb.$$

Notice that the transition function $\sigma_{12} := \sigma_1 \circ \sigma_2^{-1} : \mathcal{D}^{(2)} \to \mathcal{D}^{(1)}$ is real-analytic. We must show that it is also holomorphic. To determine σ_{12} explicitly, observe that $\sigma_{12} = \nu_1 \circ q|_{S^{(1)}}^{-1} \circ q|_{S^{(2)}} \circ \nu_2^{-1}$. Take $n_2 \in \mathcal{D}^{(2)}$ and let $g_2 := \nu_2^{-1}(n_2) \in Z^{(2)}$ be the corresponding element in the transversal, so that $g_2 = n_2 b_2$ for a unique $b_2 \in U_B$. There exists unique $h \in H$ such that

$$(\,q|_{S^{(1)}}^{-1} \circ q|_{S^{(2)}})(g_2) = g_2 h \in Z^{(1)}.$$

Letting $(n_1, b_1) \in \mathcal{D}^{(1)} \times U_B$ be s.t. $g_2 h = n_1 b_1$, we have $\nu_1(g_2 h) = n_1 = \pi_{N^{(1)}}(\zeta_1(g_2 h))$, and $\sigma_{12}(n_2)$ is explicitly given by

$$\begin{aligned}\sigma_{12}(n_2) &= (\nu_1 \circ q|_{S^{(1)}}^{-1} \circ q|_{S^{(2)}})(g_2) = \nu_1(g_2 h) \\ &= n_1 = \pi_{N^{(1)}}(\zeta_1(n_2 b_2 h)).\end{aligned}$$

Let us henceforth write $g_2(n_2), n_1(n_2), b_1(n_2), b_2(n_2)$ and $h_2(n_2)$ instead of g_2, n_1, b_1, b_2 and h_2, to emphasize their (real-analytic) dependence on n_2. We thus have

$$\sigma_{12}(n_2) = n_1(n_2) = \pi_{N^{(1)}}(\zeta_1(g_2(n_2)h(n_2))) = \pi_{N^{(1)}}(\zeta_1(n_2 b_2(n_2) h(n_2))). \quad (6.5.6)$$

We must show that σ_{12} is holomorphic near any point. Let $n_2^\bullet \in \mathcal{D}^{(2)}$. Define $b_1^\bullet := b_1(n_2^\bullet)$, $b_2^\bullet := b_2(n_2^\bullet)$, $h^\bullet := h(n_2^\bullet)$ and $h_\Delta(n_2) := (h^\bullet)^{-1}h(n_2)$. Using the continuity of the BCH product $* : U_{\mathfrak{b}_-} \times U_{\mathfrak{b}_-} \to \mathfrak{b}_-$, observe that we can find open neighborhoods $\mathcal{O}_B \subseteq U_B$ and $\mathcal{O}_H \subseteq H$ of $b_1^\bullet \in U_B$ and $e \in H$, respectively, such that $\mathcal{O}_B \mathcal{O}_H \subseteq U_B$. Now, from

$$n_1(n_2) b_1(n_2) = g(n_2) h(n_2) = g(n_2) h^\bullet h_\Delta(n_2),$$

we obtain that $g(n_2)h^\bullet = n_1(n_2) b_1(n_2) h_\Delta(n_2)^{-1}$. Since $n_1(n_2), b_1(n_2)$ and $h_\Delta(n_2)^{-1}$ depend continuously on n_2, we have $b_1(n_2) \in \mathcal{O}_B$ and $h_\Delta(n_2)^{-1} \in \mathcal{O}_H$ for all n_2 in some open neighborhood $\mathcal{O}_N^{(2)} \subseteq \mathcal{D}^{(2)}$ of $n_2^\bullet$, so that $b_1(n_2) h_\Delta(n_2)^{-1} \in U_B$. We then have $g(n_2)h^\bullet \in S_0^{(1)}$ and

$$\zeta_1(g(n_2)h^\bullet) = (n_1(n_2),\, b_1(n_2) h_\Delta(n_2)^{-1}) \in \mathcal{D}^{(1)} \times U_B, \qquad \forall n_2 \in \mathcal{O}_N^{(2)}. \quad (6.5.7)$$

We thus see for any $n_2 \in \mathcal{O}_N^{(2)}$ that $g(n_2)h(n_2)$ and $g(n_2)h^\bullet$ have the same $N^{(1)}$-component, namely $n_1(n_2)$. Using (6.5.6) we therefore find that

$$\sigma_{12}(n_2) = \pi_{N^{(1)}}(\zeta_1(n_2 b_2(n_2) h^\bullet)), \qquad \forall n_2 \in \mathcal{O}_N^{(2)}.$$

Shrinking $\mathcal{O}_N^{(2)}$ if necessary, we may further assume that $b_2(n_2)^{-1} b_2^\bullet \in U_B$ (using the continuity of the BCH product $* : U_{\mathfrak{b}_-} \times U_{\mathfrak{b}_-} \to \mathfrak{b}_-$), and that $n_2 b_2^\bullet h^\bullet \in \mathcal{D}^{(1)} U_B$ (because $n_2^\bullet b_2^\bullet h^\bullet \in \mathcal{D}^{(1)} U_B$ and $\mathcal{D}^{(1)} U_B$ is open in $N^{(1)} U_B \subseteq G_{\mathbb{C}}$), for any $n_2 \in \mathcal{O}_N^{(2)}$. We show that we then have

$$\sigma_{12}(n_2) = \pi_{N^{(1)}}(\zeta_1(n_2 b_2^\bullet h^\bullet)), \qquad \forall n_2 \in \mathcal{O}_N^{(2)}. \quad (6.5.8)$$

Take $n_2 \in \mathcal{O}_N^{(2)}$. Suppose that $(n_1, \beta_1) = \zeta_1(n_2 b_2(n_2) h^\bullet)$ and $(n_1', \beta_1') = \zeta_1(n_2 b_2^\bullet h^\bullet)$ for some $n_1, n_1' \in \mathcal{D}^{(1)}$ and $\beta_1, \beta_1' \in U_B$, so $n_1\beta_1 = n_2 b_2(n_2) h^\bullet$ and $n_1'\beta_1' = n_2 b_2^\bullet h^\bullet$. (In fact, notice that $\beta_1 = b_1(n_2) h_\Delta(n_2)^{-1}$ by (6.5.7).) Then

$$n_1'\beta_1' = n_1\beta_1 c_{h^\bullet}^{-1}(b_2(n_2)^{-1} b_2^\bullet).$$

Using that $b_2(n_2)^{-1} b_2^\bullet \in U_B$, this implies by (6.5.4) that $n_1 = n_1'$. Thus (6.5.8) holds true. Since $\pi_{N^{(1)}}$, $r_{h^\bullet b_2^\bullet}$ and ζ_1 are all holomorphic, it follows from (6.5.8) that $\sigma_{12}|_{\mathcal{O}_N^{(2)}}$ is so, too. As the point $n_2^\bullet \in \mathcal{D}^{(2)}$ was arbitrary, we conclude that σ_{12} is holomorphic.

The charts defined by good triples (S, N, U_B) therefore turn the real-analytic manifold G/H into a complex manifold modeled on $\mathfrak{n}_+$, that has the same underlying real-analytic structure.

Let us show next that the $\mathbb{C}$-linear extension of the map

$$\mathfrak{g} \to T_e(G/H), \quad \xi \mapsto \left.\frac{d}{dt}\right|_{t=0} q(\exp_G(t\xi)) \tag{6.5.9}$$

descends to a $\mathbb{C}$-linear isomorphism $\mathfrak{g}_\mathbb{C}/\mathfrak{b}_- \cong T_e(G/H)$ of topological complex vector spaces. Consider the good triple (U_P, U_N, U_B) defined in the proof of Lemma 6.5.13, with associated real and complex domains $S_0 \subseteq U_P$ and $N_0 \subseteq U_N$, respectively. Let $\sigma : q(S_0) \to N_0$ be the associated chart of G/H. Let ζ be the inverse of $\mu|_{U_N \times U_B} : U_N \times U_B \to U_N U_B$, and let $\nu := \pi_{U_N} \circ \zeta|_{S_0}$ be the corresponding map $S_0 \to N_0$ (cf. Lemma 6.5.10). Finally, let $U_H \subseteq U_B \cap H$ be an open 1-neighborhood of H.

Let $\xi \in \mathfrak{g}$. Using (A1), there exist smooth paths $p(t)$ and $h(t)$ in S_0 and U_H respectively, both defined for all t in some interval $I \subseteq \mathbb{R}$ containing zero, such that $\exp_G(t\xi) = p(t)h(t)$ for all $t \in I$. Then $q|_{S_0}^{-1}(q(\exp_G(t\xi))) = p(t)$ for $t \in I$. Letting $(n(t), b(t)) = \zeta(p(t))$, so that $p(t) = n(t)b(t)$ for $t \in I$, we have that $\nu(p(t)) = n(t)$. The smooth path $q(\exp_G(t\xi))$ in G/H is in the local coordinates defined by σ therefore represented by the path $n(t)$ in $N_0 \subseteq U_N$. Since

$$\exp_G(t\xi) = p(t)h(t) = n(t)b(t)h(t)$$

and $b(t)h(t) \in U_B$ for all $t \in I$ with $|t|$ small-enough, we also have

$$n(t) = \pi_{U_N}(\zeta(\exp_G(t\xi)))$$

for all t in some interval $I_0 \subseteq I$ containing zero. We thus find that

$$\left.\frac{d}{dt}\right|_{t=0} n(t) = \left.\frac{d}{dt}\right|_{t=0} \pi_{U_N}\zeta(\exp_G(t\xi)) = \pi_{U_{\mathfrak{n}_+}}\big(T_e(\zeta)(\xi)\big). \tag{6.5.10}$$

Since both $\pi_{U_{\mathfrak{n}_+}}$ and $T_e(\zeta)$ are $\mathbb{C}$-linear, we see using (6.5.10) that the $\mathbb{C}$-linear extension of the map in (6.5.9) is in these local coordinates of G/H represented by the map

$$\mathfrak{g}_\mathbb{C} \to \mathfrak{n}_+, \qquad \xi \mapsto \pi_{U_{\mathfrak{n}_+}}\big(T_e(\zeta)(\xi)\big). \tag{6.5.11}$$

Notice that $T_e(\zeta) : \mathfrak{g}_\mathbb{C} \to \mathfrak{n}_+ \oplus \mathfrak{b}_-$ is simply the inverse of the linear isomorphism

$$T_{(e,e)}(\mu|_{U_N \times U_B}) : \mathfrak{n}_+ \oplus \mathfrak{b}_- \to \mathfrak{g}_\mathbb{C}, \quad (\xi_+, \xi_-) \mapsto \xi_+ + \xi_-.$$

It is therefore clear that the kernel of the map in (6.5.11) is precisely $\mathfrak{b}_-$. Consequently, (6.5.11) induces a $\mathbb{C}$-linear isomorphism $\mathfrak{g}_\mathbb{C}/\mathfrak{b}_- \cong T_e(G/H) \cong \mathfrak{n}_+$ of topological complex vector spaces.

We verify next that $l_g : G/H \to G/H$ is complex-analytic for any $g \in G$. In the charts defined by the good triples (S, N, U_B) and (gS, gN, U_B), the map l_g is simply represented by $N_0 \to gN_0$, $n \mapsto gn$, where N_0 is the complex domain of (S, N, U_B). This map is holomorphic because $G_\mathbb{C}$ is a complex Lie group.

Finally, suppose that Q is a complex manifold and that $f : G/H \to Q$ is a smooth map with lift $\widetilde{f} : G \to Q$. By Proposition 6.2.11, f is holomorphic if and only if $T(f)$ is fiberwise $\mathbb{C}$-linear. Identifying $T_g(G/H) \cong T_e(G/H) \cong \mathfrak{g}_\mathbb{C}/\mathfrak{b}_-$ for any $g \in G$ using the left G-action, this is the case if and only if $T_g^\mathbb{C}(\widetilde{f})(\boldsymbol{v}_g(\xi)) = 0$ for all $g \in G$ and $\xi \in \mathfrak{b}_-$. □

6.5.3 Complex bundle structures on $\mathbb{E}_\sigma = G \times_H V_\sigma^{\mathcal{O}_b}$

Throughout the following, we continue in the setting of Section 6.5.2 and assume in addition that the assumptions of Theorem 6.5.7 are satisfied. We write M for the homogeneous space G/H, endowed with the unique G-invariant complex manifold structure satisfying the conditions in Theorem 6.5.7.

Fix a unitary H-representation (σ, V_σ). Recall from Definition 6.2.19 that $V_\sigma^{\mathcal{O}_b}$ denotes the space of b-strongly-entire vectors for the H-representation σ on V_σ. Assume that the H-action σ on $V_\sigma^{\mathcal{O}_b}$ is real-analytic $H \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}$. Since $G \to G/H$ is a real-analytic principal H-bundle, by Theorem 6.5.1, we have in this case that the G-homogeneous vector bundle $\mathbb{E}_\sigma := G \times_H V_\sigma^{\mathcal{O}_b}$ over G/H with typical fiber is $V_\sigma^{\mathcal{O}_b}$ carries a natural real-analytic bundle structure.

Remark 6.5.14. If H is actually a Banach-Lie group, so that $\mathfrak{h}$ is a Banach space w.r.t. the subspace topology inherited from $\mathfrak{g}$, then the H-action on $V_\sigma^{\mathcal{O}_b}$ is always real-analytic $H \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}$, cf. Remark 6.3.15.

In the following, we adapt the proof of [Nee13, Thm. 2.6] to endow $\mathbb{E}_\sigma$ with a complex-analytic bundle structure, using the notion of entire extensions $\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ of $d\sigma$ to $\mathfrak{b}_-$, see Definition 6.5.17 below. We write $L_g : \mathbb{E}_\sigma \to \mathbb{E}_\sigma$ for the left G-action on $\mathbb{E}_\sigma$.

Definition 6.5.15. Let W be a complete Hausdorff complex (resp. real) locally convex vector space and let $F : W \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ be a function. We say that F is complex-analytic (resp. real-analytic, smooth) if the corresponding map

$$F^\vee : W \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}$$

is complex-analytic (resp. real-analytic, smooth).

Lemma 6.5.16. *Consider the setting of Definition 6.5.15. If F is smooth then F is complex-analytic if and only if the map $T_x(F) : W \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ is $\mathbb{C}$-linear for every $x \in W$, where $T_x(F)(w)v := \frac{d}{dt}\big|_{t=0} F(x+tw)v$*

Proof. By Proposition 6.2.11, the map $F^\vee : W \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}$ is complex-analytic if and only if it is smooth and $T(F^\vee)$ is fiber-wise $\mathbb{C}$-linear. $F^\vee$ is smooth by assumption and $v \mapsto T_x(F^\vee)(0,v)$ is trivially $\mathbb{C}$-linear. Thus F is complex-analytic if and only if $w \mapsto T_x(F^\vee)(w,0)$ is $\mathbb{C}$-linear for any $x, w \in W$, which is the statement. $\square$

Definition 6.5.17. An *entire extension* χ of $d\sigma : \mathfrak{h}_\mathbb{C} \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ to $\mathfrak{b}_-$ is a homomorphism $\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ of Lie algebras such that:

1. $\chi|_{\mathfrak{h}_\mathbb{C}} = d\sigma$.
2. $\chi(\mathrm{Ad}_h(\xi)) = \sigma(h)\chi(\xi)\sigma(h)^{-1}$ for all $h \in H$ and $\xi \in \mathfrak{b}_-$.
3. The series $\sum_{n=0}^\infty \frac{1}{n!}\chi(\xi)^n v$ converges in $V_\sigma^{\mathcal{O}_b}$ for all $\xi \in \mathfrak{b}_-$ and $v \in V_\sigma^{\mathcal{O}_b}$, and defines an entire map

$$\mathfrak{b}_- \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}, \qquad (\xi, v) \mapsto \sum_{n=0}^\infty \frac{1}{n!}\chi(\xi)^n v.$$

In this case, we write $e^{\chi(\xi)}\psi := \sum_{n=0}^\infty \frac{1}{n!}\chi(\xi)^n\psi$.

Notice that an entire extension is in particular an extension in the sense of Definition 6.4.4.

The following example and the discussion in Remark 6.3.15 reflect the main reason for working with the space $V_\sigma^{\mathcal{O}_b}$ of b-strongly-entire vectors, instead of $V_\sigma^{\mathcal{O}}$:

Example 6.5.18. Assume that H is a Banach-Lie group. In view of Remark 6.3.15, we then know that the following map is entire:

$$\mathfrak{h}_\mathbb{C} \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}, \qquad (\eta, v) \mapsto \sum_{n=0}^\infty \frac{1}{n!} d\sigma(\eta^n)v \tag{6.5.12}$$

Consequently, the trivial extension $\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ of $d\sigma$ to $\mathfrak{b}_-$ with domain $V_\sigma^{\mathcal{O}_b}$ is an entire extension.

Using the notion of entire extensions, [Nee13, Thm. 2.6] adapts straightforwardly to the present setting:

Theorem 6.5.19. *Let $\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ be an entire extension of $d\sigma$ to $\mathfrak{b}_-$. Then $\mathbb{E}_\sigma = G \times_H V_\sigma^{\mathcal{O}_b}$ carries a unique complex-analytic bundle structure satisfying the following properties:*

1. *The left G-action L_g is complex-analytic for any $g \in G$.*
2. *The quotient map $G \times V_\sigma \to \mathbb{E}_\sigma$ is real-analytic.*
3. *Let $U \subseteq G$ be a neighborhood of $g \in G$. A smooth function $f \in C^\infty(UH, V_\sigma^{\mathcal{O}_b})^H$ corresponds to a local holomorphic section of $\mathbb{E}_\sigma$ if and only if*

$$\mathcal{L}_{\boldsymbol{v}(\xi)} f = -\chi(\xi) f, \qquad \forall \xi \in \mathfrak{n}_-.$$

If the two entire extensions χ_1 and χ_2 of $d\sigma$ to $\mathfrak{b}_-$ define the same complex-bundle structure, then $\chi_1 = \chi_2$.

Definition 6.5.20. Let $\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ be an entire extension of $d\sigma$ to $\mathfrak{b}_-$. We denote by $\mathbb{E}_{(\sigma,\chi)} \to M$ the vector bundle $\mathbb{E}_\sigma \to M$ equipped with the unique complex-analytic bundle structure satisfying the conditions in Theorem 6.5.19.

Proof of Theorem 6.5.19: This proof essentially follows from trivial adaptations of [Nee13, Thm. 2.6]. Let us indicate the required changes and recall the construction of the local charts, for later use.

Let $q_M : G \to G/H$ denote the quotient map. Let $U_\mathfrak{g} \subseteq \mathfrak{g}$ and $U_G \subseteq G$ be neighborhoods of $0 \in \mathfrak{g}$ and $e \in G$, respectively, s.t. $\exp_G|_{U_\mathfrak{g}} : U_\mathfrak{g} \to U_G$ is an analytic diffeomorphism. Shrinking $U_\mathfrak{g}$ if necessary, there exists by (A1) some 0-neighborhoods $U_\mathfrak{p} \subseteq \mathfrak{p}$ and $U_\mathfrak{h} \subseteq \mathfrak{h}$ s.t. the BCH series defines an analytic diffeomorphism $U_\mathfrak{p} \times U_\mathfrak{h} \to U_\mathfrak{g}$. Define $U_P := \exp_G(U_\mathfrak{p})$ and $U_H := \exp_G(U_\mathfrak{h})$, so $U_G = U_P U_H$. (Comparing with the proof of [Nee13, Thm. 2.6], U_P takes the role of U_Z.) Define for any $x \in G$ the open subsets

$$U_x := x q_M(U_P) \subseteq M \qquad \text{and} \qquad \widetilde{U}_x := x U_P H \subseteq G. \tag{6.5.13}$$

Using (A3), and replacing $\beta : \mathfrak{b}_- \to \mathcal{B}(V_\sigma)$ in steps $2-4$ of the proof of [Nee13, Thm. 2.6] by the entire extension $\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$, we obtain after shrinking $U_\mathfrak{g}$ if necessary for each $x \in G$ a smooth function $F_x : \widetilde{U}_x \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})^\times$ satisfying the following properties:

1. $F_x(gh) = \sigma(h)^{-1} F_x(g)$ for all $g \in \widetilde{U}_x$ and $h \in H$.
2. $\mathcal{L}_{\boldsymbol{v}(\xi)} F_x = -\chi(\xi) F_x$ for all $\xi \in \mathfrak{b}_-$.
3. $F_x(x) = \mathrm{id}_{V_\sigma^{\mathcal{O}_b}}$.

Moreover, using (A1), (A3), that $e^\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ is complex-analytic and that the action $H \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}$ is real-analytic, in view of Remark 6.3.15, observe from its construction that both F_x and F_x^{-1} are actually real-analytic. These functions moreover satisfy $F_{yx}(yg) = F_x(g)$ for any $x, y \in G$ and $g \in \widetilde{U}_x$, as is immediate from their construction. As in [Nee13, Thm. 2.6], we now define for each $x \in G$ the trivialization

$$\phi_x : U_x \times V_\sigma^{\mathcal{O}_b} \to \mathbb{E}_\sigma|_{U_x}, \qquad (gH, v) \mapsto [g, F_x(g)v], \tag{6.5.14}$$

so that the transition function $\phi_x^{-1} \circ \phi_y : U_x \cap U_y \times V_\sigma^{\mathcal{O}_b} \to U_x \cap U_y \times V_\sigma^{\mathcal{O}_b}$ is given by $(gH, v) \mapsto (gH, \phi_{xy}(gH)v)$, where

$$\phi_{xy} : U_x \cap U_y \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})^\times, \quad \phi_{xy}(gH) = F_x(g)^{-1}F_y(g).$$

Let us check as in [Nee13, Thm. 2.6] that these transition functions are complex-analytic. It suffices by Lemma 6.5.16 to show that the lift $\widetilde{\phi}_{xy} : \widetilde{U}_x \cap \widetilde{U}_y \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ of ϕ_{xy} satisfies $\mathcal{L}_{\boldsymbol{v}(\xi)}\widetilde{\phi}_{xy} = 0$ for all $\xi \in \mathfrak{b}_-$. This follows from the three properties of the functions F_x mentioned above:

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{v}(\xi)}\widetilde{\phi}_{xy} &= \big(\mathcal{L}_{\boldsymbol{v}(\xi)}F_x^{-1}\big)F_y + F_x^{-1}\big(\mathcal{L}_{\boldsymbol{v}(\xi)}F_y\big)\\
&= -F_x^{-1}\big(\mathcal{L}_{\boldsymbol{v}(\xi)}F_x\big)F_x^{-1}F_y + F_x^{-1}\big(\mathcal{L}_{\boldsymbol{v}(\xi)}F_y\big)\\
&= F_x^{-1}\chi(\xi)F_y - F_x^{-1}\chi(\xi)F_y\\
&= 0.
\end{aligned}$$

Thus the trivializations $\{\phi_x\}_{x\in G}$ define a complex-analytic bundle structure on $\mathbb{E}_\sigma$. Let $\mathbb{E}_{(\sigma,\chi)}$ denote the corresponding complex-analytic bundle. We show that the properties $1-3$ in Theorem 6.5.19 are satisfied:

1. Let $x, g \in G$. In the local charts defined by ϕ_x and ϕ_{gx}, L_g is represented by $l_g \times \mathrm{id}_{V_\sigma^{\mathcal{O}_b}}$, which is complex-analytic from the corresponding property of $l_g : M \to M$.

2. Let $x \in G$. Consider the local coordinates of $\mathbb{E}_{(\sigma,\chi)}$ defined by ϕ_x. In these local coordinates, the quotient map $G \times V_\sigma^{\mathcal{O}_b} \to \mathbb{E}_{(\sigma,\chi)}$ is represented by the real-analytic function
$$\widetilde{U}_x \times V_\sigma^{\mathcal{O}_b} \to U_x \times V_\sigma^{\mathcal{O}_b}, \qquad (g, v) \mapsto (gH, F_x(g)^{-1}v).$$

3. Take $f \in C^\infty(UH, V_\sigma^{\mathcal{O}_b})^H$. The corresponding local section of $\mathbb{E}_{(\sigma,\chi)}$ is obtained by descending the function $\widetilde{f} : UH \to \mathbb{E}_\sigma, \widetilde{f}(g) := [g, f(g)]$ to the quotient $q_M(U)$. Let $x \in U$ and define $W_x := U_x \cap U$ and $\widetilde{W}_x := \widetilde{U}_x \cap UH$. Using the local chart ϕ_x, the map $\widetilde{f}\big|_{\widetilde{W}_x}$ is represented by the smooth function
$$\overline{f} : \widetilde{W}_x \to U_x \times V_\sigma^{\mathcal{O}_b}, \qquad \overline{f}(g) = (gH, F_x(g)^{-1}f(g)),$$
which is complex-analytic if and only if $\mathcal{L}_{\boldsymbol{v}(\xi)}h = 0$ for any $\xi \in \mathfrak{b}_-$, where h is given by
$$h : \widetilde{W}_x \to V_\sigma^{\mathcal{O}_b}, \qquad h(g) := F_x(g)^{-1}f(g).$$

We compute that

$$\begin{aligned}\mathcal{L}_{\boldsymbol{v}(\xi)}h &= \big(\mathcal{L}_{\boldsymbol{v}(\xi)}F_x^{-1}\big)f + F_x^{-1}\big(\mathcal{L}_{\boldsymbol{v}(\xi)}f\big)\\ &= -F_x^{-1}\big(\mathcal{L}_{\boldsymbol{v}(\xi)}F_x\big)F_x^{-1}f + F_x^{-1}\big(\mathcal{L}_{\boldsymbol{v}(\xi)}f\big)\\ &= F_x^{-1}\chi(\xi)f + F_x^{-1}\big(\mathcal{L}_{\boldsymbol{v}(\xi)}f\big).\end{aligned}$$

Thus $\mathcal{L}_{\boldsymbol{v}(\xi)}h = 0$ if and only if $\mathcal{L}_{\boldsymbol{v}(\xi)}f = -\chi(\xi)f$ for any $\xi \in \mathfrak{b}_-$. Consequently f corresponds to a holomorphic local section of $\mathbb{E}_\sigma \to M$ if and only if $\mathcal{L}_{\boldsymbol{v}(\xi)}f = -\chi(\xi)f$ for every $\xi \in \mathfrak{b}_-$. The equation is automatically satisfied for any $\xi \in \mathfrak{h}_\mathbb{C}$ by the H-equivariance of f. The conclusion follows.

Step 5 in [Nee13, Thm. 2.6] shows that if the two entire extensions χ_1 and χ_2 define the same complex bundle structure, then $\chi_1 = \chi_2$. To see that the complex-bundle structure is unique, we simply remark that if $\mathbb{E}_\sigma^1$ and $\mathbb{E}_\sigma^2$ denote the vector bundle $\mathbb{E}_\sigma$ equipped à priori with possibly different complex-analytic bundle structures satisfying the properties $1-3$ in Theorem 6.5.19, then by the third property they have the same holomorphic local sections. This implies $\mathbb{E}_\sigma^1 = \mathbb{E}_\sigma^2$ as complex-analytic vector bundles over M. □

6.5.4 Geometric holomorphic induction

Having the complex-analytic G-homogeneous vector bundles $\mathbb{E}_{(\sigma,\chi)}$ at hand, we are now in a position to define a stronger notion of holomorphic induction, which guarantees that $\mathcal{H}_\rho^\infty$ actually embeds into a space of holomorphic mappings. We continue under the assumptions of Section 6.5.3. In particular, σ is a unitary representation of H on V_σ for which the H-action on $V_\sigma^{\mathcal{O}_b}$ is real-analytic $H \times V_\sigma^{\mathcal{O}_b} \to V_\sigma^{\mathcal{O}_b}$. Let $\chi : \mathfrak{b}_- \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ be an entire extension of $d\sigma : \mathfrak{h}_\mathbb{C} \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ to $\mathfrak{b}_-$, and let $(\rho, \mathcal{H}_\rho)$ be a unitary G-representation.

Definition 6.5.21. We say that $(\rho, \mathcal{H}_\rho)$ is *geometrically holomorphically induced* from (σ, χ) if σ is b-strongly-entire and there exists a G-equivariant injective linear map $\Phi : \mathcal{H}_\rho \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ satisfying:

1. The point evaluation $\mathcal{E}_x : \mathcal{H}_\rho \to V_\sigma,\ \mathcal{E}_x(\psi) := \Phi_\psi(x)$ is continuous for every $x \in G$.
2. $\mathcal{E}_x\mathcal{E}_x^* = \mathrm{id}_{V_\sigma}$ for every $x \in G$.
3. For every $w \in V_\sigma^{\mathcal{O}_b}$, the following function is holomorphic:

$$f_w : \mathbb{E}_{(\sigma,\chi)} \to \mathbb{C}, \qquad f_w([g,v]) := \langle \mathcal{E}_e^* w, \rho(g)\mathcal{E}_e^* v\rangle.$$

Remark 6.5.22. The first condition in Definition 6.5.21 entails that $(\rho, \mathcal{H}_\rho)$ is unitarily equivalent to the natural G-representation on the reproducing kernel Hilbert space $\mathcal{H}_Q$, where $Q \in C(G \times G, \mathcal{B}(V_\sigma))^{H\times H}$ is the positive definite and G-invariant kernel defined by $Q(x,y) := \mathcal{E}_x\mathcal{E}_y^*$. Combined with the second property, we additionally have that $\mathcal{E}_e^*$ is an H-equivariant isometry and that the subspace $\mathcal{E}_e^* V_\sigma \subseteq H_\rho$ is cyclic for G, cf. Theorem 6.9.3 and Proposition 6.9.5 below.

We start with a lemma:

Lemma 6.5.23. *Assume that $V_\sigma \subseteq \mathcal{H}_\rho$ as unitary H-representations and that V_σ is cyclic for G in $\mathcal{H}_\rho$. Assume further that σ is b-strongly-entire. Then the following assertions are equivalent:*

1. *$V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\infty$ and $d\rho(\xi)v = \chi(\xi)v$ for all $\xi \in \mathfrak{b}_-$ and $v \in V_\sigma^{\mathcal{O}_b}$.*

2. *$f_w \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$ for every $w \in V_\sigma^{\mathcal{O}_b}$, where $f_w([g,v]) := \langle w, \rho(g)v\rangle$.*

If these assertions are satisfied, then we even have $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\omega$. Moreover, we have $f_\psi \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$ for any $\psi \in \mathcal{H}_\rho^\infty$, where $f_\psi([g,v]) := \langle \psi, \rho(g)v\rangle$.

Proof. Let $\psi \in \mathcal{H}_\rho^\infty$ and consider the function

$$f_\psi : \mathbb{E}_{(\sigma,\chi)} \to \mathbb{C}, \qquad f_\psi([g,v]) = \langle \psi, \rho(g)v\rangle.$$

Consider its lift to $G \times V_\sigma^{\mathcal{O}_b}$, defined by $\widetilde{f}_\psi : G \times V_\sigma^{\mathcal{O}_b} \to \mathbb{C}$, $\widetilde{f}_\psi(g,v) = f_\psi([g,v])$. Let $x \in G$. Define the open sets $\widetilde{U}_x \subseteq G$ and $U_x \subseteq M$ as in (6.5.13), so U_x is an open neighborhood of $xH \in M$. Let $F_x : \widetilde{U}_x \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})^\times$ be defined as in the proof of Theorem 6.5.19. In particular, F_x satisfies $\mathcal{L}_{\boldsymbol{v}(\xi)}F_x = -\chi(\xi)F_x$ for any $\xi \in \mathfrak{b}_-$. Let $\phi_x : U_x \times V_\sigma^{\mathcal{O}_b} \to \mathbb{E}_{(\sigma,\chi)}\big|_{U_x}$ be the corresponding chart of the holomorphic vector bundle $\mathbb{E}_{(\sigma,\chi)}$, defined in (6.5.14). In these local coordinates, f_ψ and $\widetilde{f}_\psi$ are represented by $h_{\psi,x}$ and $\widetilde{h}_{\psi,x}$, respectively, where

$$\begin{aligned} h_{\psi,x} &: U_x \times V_\sigma^{\mathcal{O}_b} \to \mathbb{C}, & h_{\psi,x}(gH,v) &= \langle \rho(g)^{-1}\psi, F_x(g)v\rangle, \\ \widetilde{h}_{\psi,x} &: \widetilde{U}_x \times V_\sigma^{\mathcal{O}_b} \to \mathbb{C}, & \widetilde{h}_{\psi,x}(g,v) &= \langle \rho(g)^{-1}\psi, F_x(g)v\rangle. \end{aligned}$$

As F_x is smooth, and because $\psi \in \mathcal{H}_\rho^\infty$, this shows in particular that f_ψ is smooth for the underlying real manifold structure. Then $h_{\psi,x}$ is complex-analytic if and only if $\mathcal{L}_{\boldsymbol{v}(\xi)}\widetilde{h}_{\psi,x} = 0$ for any $\xi \in \mathfrak{b}_-$. Let $\xi \in \mathfrak{b}_-$. Using $\mathcal{L}_{\boldsymbol{v}(\xi)}F_x = -\chi(\xi)F_x$, we compute for any $(g,v) \in \widetilde{U}_x \times V_\sigma^{\mathcal{O}_b}$ that

$$(\mathcal{L}_{\boldsymbol{v}(\xi)}\widetilde{h}_{\psi,x})(g,v) = \langle d\rho(\xi^*)\rho(g)^{-1}\psi, F_x(g)v\rangle - \langle \rho(g)^{-1}\psi, \chi(\xi)F_x(g)v\rangle. \quad (6.5.15)$$

Thus if (1) holds true, then (6.5.15) shows that $\mathcal{L}_{\boldsymbol{v}(\xi)}\widetilde{h}_{\psi,x} = 0$ for any $\xi \in \mathfrak{b}_-$, so that $h_{\psi,x}$ is complex-analytic for any $x \in G$. We then conclude that $f_\psi \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$ for any $\psi \in \mathcal{H}_\rho^\infty$. Since $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\infty$ by assumption, we in particular notice that (2) holds true.

Assume conversely that $f_w \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$ for any $w \in V_\sigma^{\mathcal{O}_b}$. Let $v \in V_\sigma^{\mathcal{O}_b}$. Then the function $g \mapsto \langle v, \rho(g)v\rangle = f_v([g,v])$ is real-analytic $G \to \mathbb{C}$, where we have used that the quotient map $G \times V_\sigma^{\mathcal{O}_b} \to \mathbb{E}_{(\sigma,\chi)}$ is real-analytic. As G is a BCH Fréchet-Lie group, this implies by [Nee11, Thm. 5.2] that $v \in \mathcal{H}_\rho^\omega$. Hence $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\omega$. We know from Corollary 6.2.24 that $\mathcal{H}_\rho^\omega \subseteq \mathcal{H}_\rho^\infty$, so we also obtain that $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\infty$. To see that (1) holds true, it remains to show that $d\rho(\xi)v = \chi(\xi)v$ for all $\xi \in \mathfrak{b}_-$ and $v \in V_\sigma^{\mathcal{O}_b}$. Consider the set $\mathcal{D} := \{\, \psi \in \mathcal{H}_\rho^\infty \,:\, f_\psi \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)}) \,\}$. The preceding shows that $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{D}$. The set $\mathcal{D}$ is moreover G-invariant. Indeed, if $\psi \in \mathcal{D}$ and $g \in G$, then $f_{\rho(g)\psi} = f_\psi \circ L_{g^{-1}}$ defines a holomorphic map on $\mathbb{E}_{(\sigma,\chi)}$, because $L_{g^{-1}} : \mathbb{E}_{(\sigma,\chi)} \to \mathbb{E}_{(\sigma,\chi)}$ is holomorphic. As $V_\sigma^{\mathcal{O}_b}$ is dense in V_σ and V_σ is cyclic for G, it follows that $\mathcal{D}$ is dense in $\mathcal{H}_\rho$. Let $\xi \in \mathfrak{b}_-$, $\psi \in \mathcal{D}$ and $v \in V_\sigma^{\mathcal{O}_b}$. Recall that $F_e(e) = \mathrm{id}_{V_\sigma^{\mathcal{O}_b}}$. As $f_\psi \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$, we know that $(\mathcal{L}_{\boldsymbol{v}(\xi)}\widetilde{h}_{\psi,e})(e) = 0$. Using that $v \in \mathcal{H}_\rho^\infty$, it follows by evaluating (6.5.15) at $(e,v) \in \widetilde{U}_e \times V_\sigma^{\mathcal{O}_b}$ that

$$\langle \psi, d\rho(\xi)v\rangle = \langle d\rho(\xi^*)\psi, v\rangle = \langle \psi, \chi(\xi)v\rangle.$$

As $\mathcal{D}$ is dense, it follows that $d\rho(\xi)v = \chi(\xi)v$ for all $\xi \in \mathfrak{b}_-$ and $v \in V_\sigma^{\mathcal{O}_b}$, so that (1) holds true. We have also shown that if these equivalent conditions are satisfied, then $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\omega$ and $f_\psi \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$ for any $\psi \in \mathcal{H}_\rho^\infty$. ☐

The following entails that $\mathcal{H}_\rho^\infty$ can be seen as a space of of holomorphic functions on the complex-analytic bundle $\overline{\mathbb{E}}_{(\sigma,\chi)} \to \overline{M}$ conjugate to $\mathbb{E}_{(\sigma,\chi)} \to M$:

Proposition 6.5.24. *Assume that ρ is geometrically holomorphically induced from (σ,χ). Then there is an injective G-equivariant $\mathbb{C}$-linear map $\mathcal{H}_\rho^\infty \hookrightarrow \mathcal{O}(\overline{\mathbb{E}}_{(\sigma,\chi)})$ for which all point evaluations are continuous.*

Proof. Assume that ρ is geometrically holomorphically induced from (σ,χ). In particular, this implies that σ is b-strongly-entire. Let $\Phi : \mathcal{H}_\rho \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ satisfy the conditions in Definition 6.5.21. We may consider V_σ as a subspace of $\mathcal{H}_\rho$ using the H-equivariant isometry $\mathcal{E}_e^*$. We know by Theorem 6.9.3 that $V_\sigma \subseteq \mathcal{H}_\rho$ is cyclic. From Lemma 6.5.23 we obtain that $f_\psi \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$ for any $\psi \in \mathcal{H}_\rho^\infty$. The map $\psi \mapsto f_\psi$ defines a G-equivariant $\mathbb{C}$-linear map $\mathcal{H}_\rho^\infty \to \overline{\mathcal{O}(\mathbb{E}_{(\sigma,\chi)})}$ that has continuous point evaluations, where $\overline{\mathcal{O}(\mathbb{E}_{(\sigma,\chi)})}$ denotes the vector space complex conjugate to $\mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$, which may be identified with $\mathcal{O}(\overline{\mathbb{E}}_{(\sigma,\chi)})$. This map is injective because $f_\psi = 0$ implies that $\psi \perp \rho(G)V_\sigma^{\mathcal{O}_b}$, which in turn implies $\psi = 0$ because $V_\sigma^{\mathcal{O}_b}$ is cyclic for $\mathcal{H}_\rho$. ☐

Let us next compare the notion of geometric holomorphic induction with Definition 6.4.11. Recall that χ is an entire extension of $d\sigma : \mathfrak{h}_\mathbb{C} \to \mathcal{B}(V_\sigma^{\mathcal{O}_b})$ to $\mathfrak{b}_-$. We assume in the following that G and H are both connected.

Theorem 6.5.25. *Assume that σ is b-strongly-entire. The following assertions are equivalent:*

1. *$(\rho, \mathcal{H}_\rho)$ is geometrically holomorphically induced from (σ,χ).*
2. *There is a subspace $\mathcal{D}_{\widetilde{\chi}} \subseteq V_\sigma^\omega$ containing $V_\sigma^{\mathcal{O}_b}$ and an extension $\widetilde{\chi} : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_{\widetilde{\chi}})$ of $d\sigma$ to $\mathfrak{b}_-$ such that $\chi(\xi) = \widetilde{\chi}(\xi)|_{V_\sigma^{\mathcal{O}_b}}$ for every $\xi \in \mathfrak{b}_-$, and such that $\rho = \mathrm{HolInd}_H^G(\sigma, \widetilde{\chi})$.*

Suppose that χ is the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $V_\sigma^{\mathcal{O}_b}$. Then these assertions are equivalent to:

3. *$\rho = \mathrm{HolInd}_H^G(\sigma)$ and $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\infty$, where we considered V_σ as a subspace of $\mathcal{H}_\rho$ using Theorem 6.4.13.*

Proof. Assume that $(\rho, \mathcal{H}_\rho)$ is geometrically holomorphically induced from (σ, χ). Let $\Phi : \mathcal{H}_\rho \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ satisfy the conditions in Definition 6.5.21. Identify V_σ with a cyclic subspace of $\mathcal{H}_\rho$ using $\mathcal{E}_e^*$. Define $\mathcal{D}_{\widetilde{\chi}} := V_\sigma \cap \mathcal{H}_\rho^\omega$. From Lemma 6.5.23 we obtain that $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{D}_{\widetilde{\chi}}$ and that $d\rho(\xi)v = \chi(\xi)v$ for all $\xi \in \mathfrak{b}_-$ and $v \in V_\sigma^{\mathcal{O}_b}$. As $V_\sigma^{\mathcal{O}_b}$ is dense in V_σ, the latter in particular implies that $d\rho(\mathfrak{b}_-)\mathcal{D}_{\widetilde{\chi}} \subseteq V_\sigma$, which in turn implies $d\rho(\mathfrak{b}_-)\mathcal{D}_{\widetilde{\chi}} \subseteq \mathcal{D}_{\widetilde{\chi}}$. From Theorem 6.4.13 it follows that $\rho = \mathrm{HolInd}_H^G(\sigma, \widetilde{\chi})$, where $\widetilde{\chi} : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_{\widetilde{\chi}})$ is the extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $\mathcal{D}_{\widetilde{\chi}}$, defined by $\widetilde{\chi}(\xi)v = d\rho(\xi)v$. This extension satisfies $\widetilde{\chi}(\xi)|_{V_\sigma^{\mathcal{O}_b}} = \chi(\xi)$ for any $\xi \in \mathfrak{b}_-$, as required.

Conversely, let $\widetilde{\chi} : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_{\widetilde{\chi}})$ satisfy the conditions in (2), so in particular $\rho = \mathrm{HolInd}_H^G(\sigma, \widetilde{\chi})$. By Theorem 6.4.13 we may assume that $V_\sigma \subseteq \mathcal{H}_\rho$ as unitary H-representations, that $\mathcal{D}_{\widetilde{\chi}} = V_\sigma \cap \mathcal{H}_\rho^\omega$ and that $\widetilde{\chi}(\xi)v = d\rho(\xi)v$ for all $\xi \in \mathfrak{b}_-$ and $v \in \mathcal{D}_{\widetilde{\chi}}$. As $\mathcal{D}_{\widetilde{\chi}}$ contains $V_\sigma^{\mathcal{O}_b}$ by assumption, it follows in particular that $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\omega$. From Lemma 6.5.23, we obtain that $f_w \in \mathcal{O}(\mathbb{E}_{(\sigma,\chi)})$ for any $w \in V_\sigma^{\mathcal{O}_b}$, where $f_w([g, v]) = \langle w, \rho(g)v\rangle$. So the map

$$\Phi : \mathcal{H}_\rho \to \mathrm{Map}(G, V_\sigma)^H, \quad \Phi_\psi(g) := p_V \rho(g)^{-1}\psi$$

satisfies the conditions in Definition 6.5.21, where $p_V : \mathcal{H}_\rho \to V_\sigma$ is the orthogonal projection.

Assume that χ is the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $V_\sigma^{\mathcal{O}_b}$. Assume that (2) holds true. Let the subspace $\mathcal{D}_{\widetilde{\chi}} \subseteq V_\sigma$ and the extension $\widetilde{\chi} : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_{\widetilde{\chi}})$ satisfy the conditions in (2). We may consider V_σ as a closed H-invariant linear subspace of $\mathcal{H}_\rho$ satisfying the conditions in Theorem 6.4.13. In particular, we have $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{D}_{\widetilde{\chi}} = V_\sigma \cap \mathcal{H}_\rho^\omega$, so certainly $V_\sigma^{\mathcal{O}_b} \subseteq V_\sigma^\infty$. As χ is the trivial extension on $V_\sigma^{\mathcal{O}_b}$ and $\chi(\xi) = \widetilde{\chi}(\xi)|_{V_\sigma^{\mathcal{O}_b}}$ for every $\xi \in \mathfrak{b}_-$, we also have $d\rho(\mathfrak{n}_-)V_\sigma^{\mathcal{O}_b} = \{0\}$. As $V_\sigma^{\mathcal{O}_b}$ is dense in V_σ, this further implies that $d\rho(\mathfrak{n}_-)\mathcal{D}_{\widetilde{\chi}} = \{0\}$, so $\widetilde{\chi}$ is the trivial extension on $\mathcal{D}_{\widetilde{\chi}}$. Hence (3) holds true. Assume conversely that (3) is valid. Let $\widetilde{\chi}$ denote the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ on the domain $\mathcal{D}_{\widetilde{\chi}} := V_\sigma \cap \mathcal{H}_\rho^\omega$. By assumption $\rho = \mathrm{HolInd}_H^G(\sigma, \widetilde{\chi})$ and $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\infty$. As $\mathcal{D}_{\widetilde{\chi}}$ is killed by $d\rho(\mathfrak{n}_-)$ and dense in V_σ, it follows that $d\rho(\mathfrak{n}_-)V_\sigma^{\mathcal{O}_b} = \{0\}$. Thus (1) in Lemma 6.5.23 is satisfied, from which we obtain that $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{H}_\rho^\omega$. This means that $V_\sigma^{\mathcal{O}_b} \subseteq \mathcal{D}_{\widetilde{\chi}}$. So (2) is satisfied using the trivial extension $\widetilde{\chi}$ on the subspace $\mathcal{D}_{\widetilde{\chi}} \subseteq V_\sigma$. ☐

6.6 Arveson spectral theory

In Section 6.7 below, we shall have need for a suitably general notion of Arveson spectral subspaces. As such, we extend the already existing notion to a more general setting. Let V be a complete locally convex vector space over $\mathbb{C}$ that is Hausdorff. We define Arveson spectral subspaces of V associated to a strongly continuous $\mathbb{R}$-representation α on V that satisfies a suitable condition, using the convolution algebra $\mathcal{S}(\mathbb{R})$ of $\mathbb{C}$-valued Schwartz functions on $\mathbb{R}$. The results are adaptations of those in [Arv74, Sec. 2], [NSZ15, Sec. A.3] and [Nee13, Sec. A.2].

6.6.1 Certain classes of $\mathbb{R}$-representations

Throughout the section, let $\alpha : \mathbb{R} \to \mathcal{B}(V)^\times$ be a strongly continuous representation of $\mathbb{R}$ on V. In [NSZ15, Sec. A.3], the $\mathbb{R}$-action α is required to be polynomially bounded (see Definition 6.6.1 below). It will however be convenient to define both a stronger and a weaker notion, that in turn are both still weaker than equicontinuity, which is used in [Nee13, Sec. A.2].

Definition 6.6.1. Let $\alpha : \mathbb{R} \to \mathcal{B}(V)^\times$ be a strongly continuous representation of $\mathbb{R}$ on V.

— α is said to be *equicontinuous* if there is a basis of absolutely convex α-invariant 0-neighborhoods in V. Equivalently, if the topology of V is defined by a family of α-invariant continuous seminorms.

— α is said to have *polynomial growth* if there is a basis $\mathcal{B}$ of absolutely convex 0-neighborhoods in V such that for every $U \in \mathcal{B}$ there is a monic polynomial $r \in \mathbb{R}[t]$ such that $\alpha_t(U) \subseteq r(|t|)U$ for all $t \in \mathbb{R}$. Equivalently, if there is a family $\mathcal{P}$ of defining seminorms on V such that for every $p \in \mathcal{P}$ there exists a monic polynomial $r \in \mathbb{R}[t]$ such that $p(\alpha_t(v)) \leq r(|t|)p(v)$ for all $t \in \mathbb{R}$ and $v \in V$.

— $\alpha : \mathbb{R} \to \mathcal{B}(V)^\times$ is called *polynomially bounded* if for every continuous seminorm p on V, there is a 0-neighborhood $U \subseteq V$ and some $N \in \mathbb{N}$ such that

$$\sup_{v \in U} \sup_{t \in \mathbb{R}} \frac{p(\alpha_t(v))}{1 + |t|^N} < \infty.$$

— $\alpha : \mathbb{R} \to \mathcal{B}(V)^\times$ is said to be *pointwise polynomially bounded* if for every $v \in V$ and continuous seminorm p on V, there exists $N \in \mathbb{N}$ such that

$$\sup_{t \in \mathbb{R}} \frac{p(\alpha_t(v))}{1 + |t|^N} < \infty.$$

Remark 6.6.2. Notice that we have the following implications:

$$\alpha \text{ is equicontinuous} \implies \alpha \text{ has polynomial growth} \implies \alpha \text{ is polynomially bounded.}$$

If V is a Banach space, then α has polynomial growth if and only if it is polynomially bounded.

Example 6.6.3.

1. The $\mathbb{R}$-representations on both $L^2(\mathbb{R})$ and $C^\infty(\mathbb{T})$ by translation are equicontinuous.

2. The $\mathbb{R}$-action α on $\mathcal{S}(\mathbb{R})$ by translation is not equicontinuous but does have polynomial growth. Indeed, one checks that the open set

$$U := \left\{ f \in \mathcal{S}(\mathbb{R}) \,:\, \sup_{x \in \mathbb{R}} |xf(x)| < 1 \right\} \subseteq \mathcal{S}(\mathbb{R})$$

satisfies $\bigcap_{t\in\mathbb{R}} \alpha_t(U) = \{0\}$. By [Nee13, Prop. A.1], this implies that α is not equicontinuous. It does have polynomial growth, because the topology on $\mathcal{S}(\mathbb{R})$ is generated by the seminorms

$$p_{n,m}(f) := \sup_{x\in\mathbb{R}} (1+|x|)^n |(\partial^m f)(x)|, \qquad \text{for } n,m \in \mathbb{N}_{\geq 0},$$

which satisfy $p_{n,m}(\alpha_t f) \leq \left[\sum_{k=0}^n \binom{n}{k} |t|^{n-k}\right] p_{n,m}(f)$ for $t\in\mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$.

3. The action of $\mathbb{R}$ on $C^\infty(\mathbb{R})$ by translations is not pointwise polynomially bounded. For example, the smooth function $f(x) = e^x$ satisfies

$$\|\alpha_t(f)\|_{C([0,1])} = \|f\|_{C[t,t+t]} \geq e^t, \qquad \forall t \in \mathbb{R}.$$

Let $\mathcal{P}$ denote the set of continuous seminorms on V. For $p\in\mathcal{P}$, let

$$\mathcal{N}_p := \{\, v \in V \;:\; p(v) = 0 \,\}$$

denote its kernel. Let $V_p := \overline{V/\mathcal{N}_p}$ be the corresponding Banach space. If $p,q\in\mathcal{P}$ and $p\leq q$, then $\mathcal{N}_q \subseteq \mathcal{N}_p$ and hence there is a canonical contraction $\eta_{p,q} : V_q \to V_p$.

Lemma 6.6.4. *Assume that α is strongly continuous and has polynomial growth. Then α descends for each $p\in\mathcal{P}$ to a representation of $\mathbb{R}$ on V_p with polynomial growth. Moreover $V = \varprojlim V_p$ as $\mathbb{R}$-representations.*

Proof. Let $p \in \mathcal{P}$. Since α has polynomial growth, we have $\alpha_t(\mathcal{N}_p) \subseteq \mathcal{N}_p$ for every $t\in\mathbb{R}$. Consequently, α descends to a strongly continuous $\mathbb{R}$-representation $\alpha^{(p)}$ on V_p that again has polynomial growth. If $p,q\in\mathcal{P}$ and $t\in\mathbb{R}$, then $\eta_{p,q}\circ\alpha_t^{(q)} = \alpha_t^{(p)}$. We thus obtain an $\mathbb{R}$-action on the projective limit $\varprojlim V_p$ for which the canonical isomorphism $V \cong \varprojlim V_p$ is $\mathbb{R}$-equivariant. □

Proposition 6.6.5. *Assume that α is strongly continuous and has polynomial growth. Then the action $\alpha : \mathbb{R}\times V \to V$ is continuous.*

Proof. By Lemma 6.6.4 it follows that $V = \varprojlim V_p$ as $\mathbb{R}$-representation on locally convex space. If $p\in\mathcal{P}$, then since V_p is a Banach space and the $\mathbb{R}$-representation on V_p is strongly continuous, it follows from [Nee10a, Prop. 5.1] that the action map $\mathbb{R}\times V_p \to V_p$ is jointly continuous. Using that $V \cong \varprojlim V_p$ as topological representations of $\mathbb{R}$, it follows that the action $\alpha : \mathbb{R}\times V \to V$ is jointly continuous. □

6.6.2 Arveson spectral subspaces

Let V be a complete locally convex vector space over $\mathbb{C}$ that is Hausdorff. Let $\alpha : \mathbb{R} \to \mathcal{B}(V)^\times$ be a strongly continuous representation of $\mathbb{R}$ on V. Assume that α is pointwise polynomially bounded. In the following, we define the Arveson spectral subspaces of V associated to subsets E of $\mathbb{R}$. We extend the results in [NSZ15, A.3] to the case where α is only required to be pointwise polynomial bounded. We will use the convention that the Fourier transform $f\mapsto \widehat{f}$ on $\mathcal{S}(\mathbb{R})$ is given by

$$\widehat{f}(p) := \int_{\mathbb{R}} f(t) e^{itp} dt. \qquad (6.6.1)$$

Definition 6.6.6.

— If $I \subseteq \mathcal{S}(\mathbb{R})$ is an ideal, define its *hull* $h(I) \subseteq \mathbb{R}$ by

$$h(I) := \left\{ p \in \mathbb{R} \,:\, \widehat{f}(p) = 0 \text{ for all } f \in I \right\}.$$

— If $E \subseteq \mathbb{R}$ is a closed subset, define the ideal $I_0(E)$ of $\mathcal{S}(\mathbb{R})$ by

$$I_0(E) := \left\{ f \in \mathcal{S}(\mathbb{R}) \,:\, \operatorname{supp}(\widehat{f}) \cap E = \emptyset \right\}.$$

Lemma 6.6.7 ([NSZ15, Prop. A.8])**.**

1. *If $E \subseteq \mathbb{R}$ is a closed subset, then $h(I_0(E)) = E$.*
2. *If $I \subseteq \mathcal{S}(\mathbb{R})$ is a closed ideal, then $I_0(h(I)) \subseteq I$.*

Corollary 6.6.8. *Let $I \subseteq \mathcal{S}(\mathbb{R})$ be a closed ideal with $h(I) = \emptyset$. Then $I = \mathcal{S}(\mathbb{R})$.*

Proof. Since $I_0(\emptyset) = \mathcal{S}(\mathbb{R})$ it follows using Lemma 6.6.7 that

$$\mathcal{S}(\mathbb{R}) = I_0(\emptyset) = I_0(h(I)) \subseteq I. \quad \square$$

We proceed by defining a representation of the convolution algebra $(\mathcal{S}(\mathbb{R}), *)$ on V.

Lemma 6.6.9. *Let $f \in \mathcal{S}(\mathbb{R})$ and $v \in V$. Then the weak integral $\int_{\mathbb{R}} f(t)\alpha_t(v)dt$ exists in V.*

Proof. For any $a > 0$, the weak integral $\int_{-a}^{a} f(t)\alpha_t(v)dt$ exists in V because $\mathbb{R} \to V$, $t \mapsto f(t)\alpha_t(v)$ is continuous and V is complete (cf. [Mil84, p. 1021] or [GN, Prop. 1.1.15]). As α is pointwise polynomially bounded and $f \in \mathcal{S}(\mathbb{R})$ is a Schwartz function, the limit $v_* := \lim_{a\to\infty} \int_{-a}^{a} f(t)\alpha_t(v)dt$ exists in V, and $v_* = \int_{\mathbb{R}} f(t)\alpha_t(v)dt \in V$. $\square$

Definition 6.6.10. For any Schwartz function $f \in \mathcal{S}(\mathbb{R})$, define the linear operator $\alpha_f \in \mathcal{L}(V)$ by

$$\alpha_f(v) := \int_{\mathbb{R}} f(t)\alpha_t(v)dt.$$

Then $f \mapsto \alpha_f$ defines a strongly continuous representation of the convolution algebra $(\mathcal{S}(\mathbb{R}), *)$ on V.

Remark 6.6.11. If α is polynomially bounded, then $\alpha_f \in \mathcal{B}(V)$ is a continuous operator for every $f \in \mathcal{S}(\mathbb{R})$.

Definition 6.6.12.

— Let $\operatorname{Spec}_\alpha(V) := h(\ker \alpha) \subseteq \mathbb{R}$ be the hull of the closed ideal $\ker \alpha$ in $\mathcal{S}(\mathbb{R})$.

— For $v \in V$, let $\mathcal{S}(\mathbb{R})_v := \{ f \in \mathcal{S}(\mathbb{R}) \,:\, \alpha_f(v) = 0 \}$ denote the annihilator of v in $\mathcal{S}(\mathbb{R})$, which is a closed ideal in $\mathcal{S}(\mathbb{R})$, and let $\operatorname{Spec}_\alpha(v) := h(\mathcal{S}(\mathbb{R})_v) \subseteq \mathbb{R}$ be its hull.

— If $E \subseteq \mathbb{R}$ is a subset, define

$$V_\alpha(E)_0 := \left\{ v \in V \,:\, \mathrm{Spec}_\alpha(v) \subseteq \overline{E} \right\}$$

and let $V_\alpha(E) := \overline{V_\alpha(E)_0}$ be its closure in V. Define moreover

$$V_\alpha^+(E) := \bigcap_N V_\alpha(E+N),$$

where N runs over all 0-neighborhoods in $\mathbb{R}$.

If the action α is clear from the context, we drop α from the notation and simply write $V(E)_0, V(E)$ and $V^+(E)$ instead of $V_\alpha(E)_0, V_\alpha(E)$ and $V_\alpha^+(E)$.

Example 6.6.13. Let $U : \mathbb{R} \to \mathrm{U}(\mathcal{H})$ be a strongly continuous unitary representation of $\mathbb{R}$. Then $U_t = e^{tH}$ for some self-adjoint operator H on $\mathcal{H}$. Suppose that $U_t = \int_{\mathbb{R}} e^{itp} dP(p)$ is the corresponding spectral decomposition of U, for some projection-valued measure P on $\mathbb{R}$. With the convention (6.6.1) we have $U_f := \int_{\mathbb{R}} f(t) U_t dt = \int_{\mathbb{R}} \widehat{f}(p) dP(p)$ for $f \in \mathcal{S}(\mathbb{R})$. The Arveson spectrum $\mathrm{Spec}_U(\mathcal{H})$ coincides with $\mathrm{Spec}(H)$, the spectrum of the self-adjoint operator H. Moreover, for a closed subset $E \subseteq \mathbb{R}$, the corresponding spectral subspace is given by $\mathcal{H}_U(E) = P(E)\mathcal{H}$.

Remark 6.6.14. Let $\{E_i\}_{i \in \mathcal{I}}$ be a family of closed subsets of $\mathbb{R}$. Observe that $\bigcap_{i \in \mathcal{I}} V(E_i)_0 = V\big(\bigcap_{i \in \mathcal{I}} E_i\big)_0$.

Remark 6.6.15. Notice for any $v \in V$ that $\ker \alpha \subseteq \mathcal{S}(\mathbb{R})_v$, so $\mathrm{Spec}_\alpha(v) \subseteq \mathrm{Spec}_\alpha(V)$. Thus

$$V = V(\mathrm{Spec}_\alpha(V))_0 = V\big(\mathrm{Spec}_\alpha(V)\big) = V^+\big(\mathrm{Spec}_\alpha(V)\big).$$

Combining this with Remark 6.6.14, we obtain for any closed subset $E \subseteq \mathbb{R}$ that $V(E)_0 = V(E \cap \mathrm{Spec}_\alpha(V))_0$.

Lemma 6.6.16. *Let $f \in \mathcal{S}(\mathbb{R})$ and $v \in V$. Then* $\mathrm{Spec}_\alpha(\alpha_f(v)) \subseteq \mathrm{supp}(\widehat{f})$.

Proof. Let $p \in \mathbb{R} \setminus \mathrm{supp}(\widehat{f})$ and choose $g \in \mathcal{S}(\mathbb{R})$ s.t. $\widehat{g}(p) \neq 0$ and $\widehat{g}|_{\mathrm{supp}(\widehat{f})} = 0$. Then $g * f = 0$, because $\widehat{g}\widehat{f} = 0$. It follows that $\alpha_g \alpha_f v = \alpha_{f*g} v = 0$. Since we also have $\widehat{g}(p) \neq 0$ it follows that $p \notin \mathrm{Spec}_\alpha(\alpha_f v)$. ☐

Proposition 6.6.17. *Let $v \in V$. Then $\mathcal{S}(\mathbb{R})_v = \mathcal{S}(\mathbb{R})$ implies $v = 0$. Moreover $v \neq 0$ implies* $\mathrm{Spec}_\alpha(v) \neq \emptyset$.

Proof. Assume that $\mathcal{S}(\mathbb{R})_v = \mathcal{S}(\mathbb{R})$. If $\lambda \in V'$ is a continuous functional, it follows that $\int_{\mathbb{R}} f(t) \lambda(\alpha_t v) dt = 0$ for any $f \in \mathcal{S}(\mathbb{R})$. As $t \mapsto \lambda(\alpha_t v)$ is continuous, this implies that $\lambda(\alpha_t v) = 0$ for all $t \in \mathbb{R}$. In particular $\lambda(v) = 0$. As V' separates the points of V by the Hahn-Banach Theorem [Rud91, Thm. I.3.4], it follow that $v = 0$. Finally, if $\mathrm{Spec}_\alpha(v) = \emptyset$ then by Corollary 6.6.8 it follows that $\mathcal{S}(\mathbb{R})_v = \mathcal{S}(\mathbb{R})$ and hence $v = 0$. ☐

Corollary 6.6.18. *If $E_1, E_2 \subseteq \mathbb{R}$ are two disjoint closed subsets, then*

$$V(E_1)_0 \cap V(E_2)_0 = \{0\}.$$

Proof. We have $V(E_1)_0 \cap V(E_2)_0 = V(E_1 \cap E_2) = V(\emptyset) = \{0\}$ by Remark 6.6.14 and Proposition 6.6.17. □

If $E \subseteq \mathbb{R}$ is a subset, recall from Definition 6.6.6 that $I_0(\overline{E}) \subseteq \mathcal{S}(\mathbb{R})$ denotes the ideal of functions $f \in \mathcal{S}(\mathbb{R})$ whose Fourier transform $\widehat{f}$ vanishes on a neighborhood of $\overline{E} \subseteq \mathbb{R}$. Proposition 6.6.19 below provides a convenient characterization of $V(E)_0$ in terms of $I_0(\overline{E})$, which will be used repeatedly.

Proposition 6.6.19 ([NSZ15, Prop. A.8])**.**
For any subset $E \subseteq \mathbb{R}$ we have

$$\begin{aligned} V(E)_0 &= \left\{ v \in V \ : \ I_0(\overline{E}) \subseteq \mathcal{S}(\mathbb{R})_v \right\} \\ &= \left\{ v \in V \ : \ \forall f \in \mathcal{S}(\mathbb{R}) \ : \ \mathrm{supp}(\widehat{f}) \cap \overline{E} = \emptyset \implies \alpha_f(v) = 0 \right\}. \end{aligned}$$

In particular $V(E)_0, V(E)$ and $V^+(E)$ are linear subspaces of V.

Proof. The proof of [NSZ15, Prop. A.8] continues to hold when α is only pointwise polynomially bounded. □

Corollary 6.6.20. *Assume that α is polynomially bounded. Then for any $E \subseteq \mathbb{R}$ we have*

$$V(E)_0 = V(E) = V^+(E).$$

Proof. Let $E \subseteq \mathbb{R}$ be a subset. By Remark 6.6.11 we know that α_f is a continuous linear operator for every $f \in \mathcal{S}(\mathbb{R})$. It then follows from Proposition 6.6.19 that $V(E)_0$ is closed, so $V(E)_0 = V(E)$. Using Remark 6.6.14, we further obtain that

$$V^+(E) = \bigcap_N V(E+N) = \bigcap_N V(E+N)_0 = V\left(\bigcap_N E+N\right)_0 = V(\overline{E})_0 = V(E)_0. \quad \square$$

The following will also be used frequently:

Corollary 6.6.21. *Let $E \subseteq \mathbb{R}$ be a subset. The following assertions are equivalent:*

1. $\mathrm{Spec}_\alpha(V) \subseteq \overline{E}$.
2. $V \subseteq V(E)_0$.
3. $I_0(\overline{E}) \subseteq \ker \alpha$.

Proof. Assume that $\mathrm{Spec}_\alpha(V) \subseteq \overline{E}$. Then for any $v \in V$ we have

$$\mathrm{Spec}_\alpha(v) \subseteq \mathrm{Spec}_\alpha(V) \subseteq \overline{E},$$

by Remark 6.6.15. This means that $V \subseteq V(E)_0$. Assume next that $V \subseteq V(E)_0$. By Proposition 6.6.19, this means that $I_0(\overline{E}) \subseteq \mathcal{S}(\mathbb{R})_v$ for all $v \in V$. So elements of $I_0(\overline{E})$ annihilate every $v \in V$. Thus $I_0(\overline{E}) \subseteq \ker \alpha$. If $I_0(\overline{E}) \subseteq \ker \alpha$, then $\mathrm{Spec}_\alpha(V) = h(\ker \alpha) \subseteq h(I_0(\overline{E})) = \overline{E}$, where the last equality uses Lemma 6.6.7. □

Corollary 6.6.22. $\mathrm{Spec}_\alpha(V) = \overline{\bigcup_{v\in V} \mathrm{Spec}_\alpha(v)}$.

Proof. Define $E := \bigcup_{v\in V} \mathrm{Spec}_\alpha(v)$. By Remark 6.6.15 we have $\mathrm{Spec}_\alpha(v) \subseteq \mathrm{Spec}_\alpha(V)$ for any $v \in V$. As $\mathrm{Spec}_\alpha(V)$ is closed, it follows that $\overline{E} \subseteq \mathrm{Spec}_\alpha(V)$. Conversely, recall that $V(E)_0 = \{ v \in V : \mathrm{Spec}_\alpha(v) \in \overline{E} \}$. So from our definition of E, we trivially have $V \subseteq V(E)_0$. Then $\mathrm{Spec}_\alpha(V) \subseteq \overline{E}$ follows by Corollary 6.6.21. □

Let us next record the behavior of spectral subspaces under continuous linear and multi-linear maps:

Proposition 6.6.23. *For $j \in \{1, 2\}$, let $\alpha_j : \mathbb{R} \to \mathcal{B}(V_j)^\times$ be a strongly continuous representation of $\mathbb{R}$ on the complete and Hausdorff complex locally convex vector space V_j. Assume that α_j is pointwise polynomially bounded. Let $T : V_1 \to V_2$ be a continuous $\mathbb{R}$-equivariant linear map. Then for every subset $E \subseteq \mathbb{R}$ we have $T(V_1(E)) \subseteq V_2(E)$. If T is injective, then $\mathrm{Spec}_{\alpha_1}(V_1) \subseteq \mathrm{Spec}_{\alpha_2}(V_2)$.*

Proof. Let $v \in V$. As T is equivariant, we have that $\mathcal{S}(\mathbb{R})_v \subseteq \mathcal{S}(\mathbb{R})_{Tv}$. Hence $h(\mathcal{S}(\mathbb{R})_{Tv}) \subseteq h(\mathcal{S}(\mathbb{R})_v)$, which is to say that $\mathrm{Spec}_{\alpha_2}(Tv) \subseteq \mathrm{Spec}_{\alpha_1}(v)$. Thus if $E \subseteq \mathbb{R}$ is a subset then $TV_1(E)_0 \subseteq V_2(E)_0$. As T is continuous, it also follows that $TV_1(E) \subseteq V_2(E)$. If T is injective, we have $\mathcal{S}(\mathbb{R})_v = \mathcal{S}(\mathbb{R})_{Tv}$ for any $v \in V_1$. Consequently, $\mathrm{Spec}_{\alpha_1}(v) = \mathrm{Spec}_{\alpha_2}(Tv) \subseteq \mathrm{Spec}_{\alpha_2}(V_2)$. As $\mathrm{Spec}_{\alpha_2}(V_2)$ is closed, it follows using Corollary 6.6.22 that $\mathrm{Spec}_{\alpha_1}(V_1) = \overline{\bigcup_{v\in V_1} \mathrm{Spec}_{\alpha_1}(v)} \subseteq \mathrm{Spec}_{\alpha_2}(V_2)$. □

In the multi-linear context, we have the following analogue of [NSZ15, A.10]:

Proposition 6.6.24. *For $j \in \{1, 2, 3\}$, let $\alpha_j : \mathbb{R} \to \mathcal{B}(V_j)^\times$ be a strongly continuous representation of $\mathbb{R}$ on the complete and Hausdorff complex locally convex vector space V_j. Assume that α_j is pointwise polynomially bounded. Let $\beta : V_1 \times V_2 \to V_3$ be a continuous $\mathbb{R}$-equivariant bilinear map. Let $E_1, E_2 \subseteq \mathbb{R}$ be closed subsets. Then*

$$\beta(V_1(E) \times V_2(E)) \subseteq V_3^+(E_1 + E_2).$$

In particular, if α_3 is polynomially bounded then $\beta(V_1(E) \times V_2(E)) \subseteq V_3(E_1 + E_2)$.

Before proceeding to to proof of Proposition 6.6.24, let us mention the following immediate consequence:

Corollary 6.6.25. *Consider the setting of Proposition 6.6.24. Assume additionally that β has dense span and that α_3 is polynomially bounded. Then*

$$\mathrm{Spec}_{\alpha_3}(V_3) \subseteq \overline{\mathrm{Spec}_{\alpha_1}(V_1) + \mathrm{Spec}_{\alpha_2}(V_2)}.$$

Proof. We know by Proposition 6.6.24 that $\beta(V_1, V_2) \subseteq V_3^+\big(\mathrm{Spec}_{\alpha_1}(V_1) + \mathrm{Spec}_{\alpha_2}(V_2)\big)$. In view of Proposition 6.6.19 and Corollary 6.6.20, we further know that

$$V_3^+\big(\mathrm{Spec}_{\alpha_1}(V_1) + \mathrm{Spec}_{\alpha_2}(V_2)\big) = V_3\big(\mathrm{Spec}_{\alpha_1}(V_1) + \mathrm{Spec}_{\alpha_2}(V_2)\big)_0,$$

and this is a closed linear subspace of V_3. As $\beta(V_1, V_2)$ has dense linear span in V_3, it follows that

$$V_3 \subseteq V_3\big(\operatorname{Spec}_{\alpha_1}(V_1) + \operatorname{Spec}_{\alpha_2}(V_2)\big)_0.$$

According to Corollary 6.6.21, this equivalent with

$$\operatorname{Spec}_{\alpha_3}(V_3) \subseteq \overline{\operatorname{Spec}_{\alpha_1}(V_1) + \operatorname{Spec}_{\alpha_2}(V_2)}. \quad \square$$

The proof of Proposition 6.6.24 requires some preparation. It closely follows that of [Arv74, Prop. 2.2] and [Nee13, Prop. A.14]. We first introduce some additional notation:

Definition 6.6.26. For a subset $E \subseteq \mathbb{R}$, define the ideal $J(E) \subseteq \mathcal{S}(\mathbb{R})$ and the subspace $R_\alpha(E)_0 \subseteq V$ by

$$J(E) := \left\{ f \in \mathcal{S}(\mathbb{R}) \,:\, \widehat{f} \in C_c^\infty(\mathbb{R}) \text{ and } \operatorname{supp} \widehat{f} \subseteq E \right\},$$
$$R_\alpha(E)_0 := \{ \alpha_f v \,:\, f \in J(E),\ v \in V \} \subseteq V$$

Let $R_\alpha(E) := \overline{R_\alpha(E)_0}$ be its closure. If α is clear from the context, we write simply $R(E)_0$ and $R(E)$ instead of $R_\alpha(E)_0$ and $R_\alpha(E)$, respectively.

If $E \subseteq \mathbb{R}$ is a subset, recall from Definition 6.6.6 that $I_0(\overline{E})$ consists of all Schwartz functions f whose Fourier transform $\widehat{f}$ vanishes on a neighborhood of $\overline{E} \subseteq \mathbb{R}$. On the other hand, $J(E)$ is the ideal in $\mathcal{S}(\mathbb{R})$ generated by those $f \in \mathcal{S}(\mathbb{R})$ for which $\widehat{f}$ has compact support contained in E.

Lemma 6.6.27. *Let $E \subseteq \mathbb{R}$ be a closed subset and let $N \subseteq \mathbb{R}$ be a 0-neighborhood. Then*

$$\overline{I_0(E) + J(E + N)} = \mathcal{S}(\mathbb{R}).$$

Proof. Let $J_2 := \overline{J(E + N)_0 + I_0(E)}$ be the closed ideal of $\mathcal{S}(\mathbb{R})$ generated by $J(E + N)$ and $I_0(E)$. Observe that $h(J(E + N)) \subseteq \mathbb{R} \setminus E$. On the other hand, $h(I_0(E)) \subseteq E$ by Lemma 6.6.7. We thus find that

$$h(J_2) \subseteq h(I_0(E)) \cap h(J(E + N)) \subseteq \emptyset$$

and hence $h(J_2) = \emptyset$. It follows using Corollary 6.6.8 that $J_2 = \mathcal{S}(\mathbb{R})$. $\square$

Lemma 6.6.28. *Let $v \in V$ and $N \subseteq \mathbb{R}$ be a 0-neighborhood. Assume that $J(\operatorname{Spec}_\alpha(V) + N) \subseteq \mathcal{S}(\mathbb{R})_v$. Then $v = 0$.*

Proof. Let $E := \operatorname{Spec}_\alpha(V)$. Assume that $J(E + N) \subseteq \mathcal{S}(\mathbb{R})_v$. Recall from Remark 6.6.15 that $V = V(E)_0$. By Proposition 6.6.19, this means that $I_0(E) \subseteq \mathcal{S}(\mathbb{R})_v$. On the other hand, $J(E + N) \subseteq \mathcal{S}(\mathbb{R})_v$, by assumption. Since $\mathcal{S}(\mathbb{R})_v$ is closed we obtain using Lemma 6.6.27 that $\mathcal{S}(\mathbb{R}) = \overline{I_0(E) + J(E + N)} \subseteq \mathcal{S}(\mathbb{R})_v$. By Proposition 6.6.17, this implies that $v = 0$. $\square$

Lemma 6.6.29. *Let $E \subseteq \mathbb{R}$ be closed. Then $V(E) \subseteq \bigcap_N R(E + N) \subseteq V^+(E)$, where N runs over all open 0-neighborhoods in $\mathbb{R}$.*

Proof. This proof follows that of [Arv74, Prop. 2.2]. Lemma 6.6.16 entails that $\mathrm{Spec}_\alpha(\alpha_f v) \subseteq \mathrm{supp}(\widehat{f})$ for any $f \in \mathcal{S}(\mathbb{R})$ and $v \in V$. If $N \subseteq \mathbb{R}$ is a 0-neighborhood and $f \in J(E+N)$, then by definition $\mathrm{supp}\,\widehat{f} \subseteq E+N$ and hence $\mathrm{Spec}_\alpha(\alpha_f v) \subseteq E+N$ for any $v \in V$. Recalling that $R(E+N)_0$ is the subspace of V generated by $J(E+N)$, we obtain that $R(E+N)_0 \subseteq V(E+N)_0$. Consequently $\bigcap_N R(E+N) \subseteq \bigcap_N V(E+N) = V^+(E)$. Next, take $v \in V(E)_0$. We show that $v \in \bigcap_N R(E+N)$. Let N be a 0-neighborhood in $\mathbb{R}$. Let $\lambda \in V'$ be a continuous functional with $\lambda(R(E+N)) = \{0\}$. Trivially, $\alpha_f(v) \in R(E+N)_0$ for any $f \in J(E+N)$, and hence $\lambda(\alpha_f v) = 0$. We further have $I_0(E) \subseteq \mathcal{S}(\mathbb{R})_v$, by Proposition 6.6.19, and consequently $\lambda(\alpha_g v) = 0$ for any $g \in I_0(E)$. Thus $\lambda(\alpha_f v) = 0$ for any f in the closed ideal $J_2 := \overline{I_0(E) + J(E+N)}$ of $\mathcal{S}(\mathbb{R})$ spanned by $I_0(E)$ and $J(E+N)$. By Lemma 6.6.27 this ideal equals $\mathcal{S}(\mathbb{R})$, so $\int_\mathbb{R} f(t)\lambda(\alpha_t v)dt = \lambda(\alpha_f v) = 0$ for any $f \in \mathcal{S}(\mathbb{R})$. As $t \mapsto \lambda(\alpha_t v)$ is continuous, it follows that $\lambda(\alpha_t v) = 0$ for all $t \in \mathbb{R}$. In particular $\lambda(v) = 0$. Using the Hahn-Banach Theorem [Rud91, Thm. I.3.5], it follows that $v \in \bigcap_N R(E+N)$. Thus $V(E)_0 \subseteq \bigcap_N R(E+N)$ and consequently also $V(E) \subseteq \bigcap_N R(E+N)$. ☐

Proof of Proposition 6.6.24: Having Lemma 6.6.29 at hand, we proceed as in [Nee13, Prop. A.14]. Let $N \subseteq \mathbb{R}$ be an open 0-neighborhood. Let $N_1, N_2 \subseteq \mathbb{R}$ be open 0-neighborhoods s.t. $N_1 + N_2 \subseteq N$. We show that

$$\beta\Big(R_{\alpha_1}(E_1+N_1)_0 \times R_{\alpha_2}(E_2+N_2)_0\Big) \subseteq V\big(E_1+E_2+N\big)_0. \tag{6.6.2}$$

As such, for $k \in \{1,2\}$, take $v_k \in V$ and $f_k \in J(E_k+N_k)$, meaning that $\mathrm{supp}(\widehat{f_k}) \subseteq E_k + N_k$. We show that $\beta(\alpha_1(f_1)v_1, \alpha_2(f_2)v_2) \in V\big(E_1+E_2+N\big)_0$. In view of Proposition 6.6.19, we must show that it is annihilated by $I_0(\overline{E_1+E_2+N})$. Let $f_3 \in I_0(\overline{E_1+E_2+N})$, so $\mathrm{supp}(\widehat{f_3}) \cap \overline{E_1+E_2+N} = \emptyset$. Then

$$\begin{aligned}\alpha_{f_3}\beta(\alpha_{f_1}(v_1), \alpha_{f_2}(v_2)) &= \int_\mathbb{R}\int_\mathbb{R} f_1(t_1)f_2(t_2)f_3(t_3)\beta(\alpha_1(t_1+t_3)v_1, \alpha_2(t_2+t_3)v_2)dt_1dt_2dt_3,\\ &= \int_\mathbb{R}\int_\mathbb{R} F(t_1,t_2)\beta(\alpha_1(t_1)v_1, \alpha_2(t_2)v_2)dt_1dt_2,\end{aligned} \tag{6.6.3}$$

where $F \in \mathcal{S}(\mathbb{R}^2)$ is defined by

$$F(t_1,t_2) := \int_\mathbb{R} f_3(t_3)f_1(t_1-t_3)f_2(t_2-t_3)dt_3.$$

The Fourier transform $\widehat{F} \in \mathcal{S}(\mathbb{R}^2)$ of F is given by

$$\widehat{F}(p_1,p_2) = \widehat{f_1}(p_1)\widehat{f_2}(p_2)\widehat{f_3}(p_1+p_2).$$

Observe that $\mathrm{supp}(\widehat{f_1}) + \mathrm{supp}(\widehat{f_2}) \subseteq (E_1+N_1)+(E_2+N_2) \subseteq E_1+E_2+N$. Since $\widehat{f_3}$ vanishes on E_1+E_2+N, we find that $\widehat{F} = 0$. Hence $F = 0$. From Equation (6.6.3) we obtain that $\alpha_3(f_3)\beta(\alpha_1(f_1)v_1, \alpha_2(f_2)v_2) = 0$. By Proposition 6.6.19 we conclude that $\beta(\alpha_1(f_1)v_1, \alpha_2(f_2)v_2) \in V\big(E_1 + E_2 + N\big)_0$. Thus (6.6.2) is valid. As β is continuous, it follows that

$$\beta(V_1(E) \times V_2(E)) \subseteq \beta\Big(R_{\alpha_1}(E_1 + N_1) \times R_{\alpha_2}(E_2 + N_2)\Big) \subseteq V\big(E_1 + E_2 + N\big),$$

where the first inclusion uses Lemma 6.6.29. Thus

$$\beta(V_1(E) \times V_2(E)) \subseteq \bigcap_N V\big(E_1 + E_2 + N\big) = V^+(E_1 + E_2).$$

Assume next that α_3 is polynomially bounded. Then α_f is continuous for every $f \in \mathcal{S}(\mathbb{R})$, by Remark 6.6.11. By Corollary 6.6.20 it follows that

$$V^+(E_1 + E_2) = V(E_1 + E_2). \quad \square$$

Let us next consider the behavior of spectra under tensor products and spaces of continuous linear maps:

Proposition 6.6.30. *Let α and σ be $\mathbb{R}$-representation on the complete and Hausdorff locally convex vector spaces V and W over $\mathbb{C}$, respectively. Assume that α and σ are strongly continuous and have polynomial growth. Let $n \in \mathbb{N}$.*

1. *The $\mathbb{R}$-representation $\alpha\widehat{\otimes}\sigma$ on the completed projective tensor product $V\widehat{\otimes}W$ has a continuous action $\mathbb{R} \times V\widehat{\otimes}W \to V\widehat{\otimes}W$, polynomial growth and satisfies*

$$\mathrm{Spec}_{\alpha\widehat{\otimes}\sigma}(V\widehat{\otimes}W) \subseteq \overline{\mathrm{Spec}_\alpha(V) + \mathrm{Spec}_\alpha(W)}. \tag{6.6.4}$$

2. *Equip $\mathcal{B}(V, W)$ either with the strong topology or that of uniform convergence on compact sets. The $\mathbb{R}$-representation γ on $\mathcal{B}(V, W)$ defined by*

$$\gamma_t T = \sigma_t \circ T \circ \alpha_{-t}$$

is strongly continuous, pointwise polynomially bounded and satisfies

$$\mathrm{Spec}_\gamma(\mathcal{B}(V, W)) \subseteq \overline{\mathrm{Spec}_\sigma(W) - \mathrm{Spec}_\alpha(V)}. \tag{6.6.5}$$

Proof. Notice by Proposition 6.6.5 that the actions $\alpha : \mathbb{R}\times V \to V$ and $\sigma : \mathbb{R}\times W \to W$ are continuous.

1. We write $\gamma_t := \alpha_t\widehat{\otimes}\sigma_t$ for $t \in \mathbb{R}$. We first show that the $\mathbb{R}$-representation γ on $V\widehat{\otimes}W$ has polynomial growth. Let p and q be continuous seminorms on V and W respectively. Assume that $p(\alpha_t v) \leq r_\alpha(|t|)p(v)$ and $q(\alpha_t w) \leq r_\sigma(|t|)q(w)$ for all $t \in \mathbb{R}$, $v \in V$ and $w \in W$, where $r_\alpha, r_\sigma \in \mathbb{R}[t]$ are monic polynomials. Using this inequality, it follows from the definition of the seminorm $p \otimes q$ on $V\widehat{\otimes}W$ (see Equation (6.2.1)) that $(p \otimes q)(\gamma_t\psi) \leq r_\alpha(|t|)r_\sigma(|t|)(p \otimes q)(\psi)$ for all $t \in \mathbb{R}$ and $\psi \in V\widehat{\otimes}W$. Thus $\alpha\widehat{\otimes}\sigma$ has polynomial growth.

   To see that $\alpha\widehat{\otimes}\sigma$ has a continuous action, it suffices by Proposition 6.6.5 to show it is strongly continuous. Let $\psi \in V\widehat{\otimes}W$. It suffices to show that

$t \mapsto \gamma_t\psi$ is continuous at $t = 0$. Assume first that $\psi \in V \otimes W$, so that $\psi = \sum_{k=1}^n v_k \otimes w_k$ for some $v_k \in V$ and $w_k \in W$. Let p and q be continuous seminorms on V and W, respectively. Let $r_\alpha, r_\sigma \in \mathbb{R}[t]$ be as above. Let $\epsilon > 0$. As α and σ are strongly continuous, we can find $\delta > 0$ s.t. $p(\alpha_t v_k - v_k)q(\sigma_t w_k) < \epsilon$ and $p(v_k)q(\sigma_t w_k - w_k) < \epsilon$ for all $t \in (-\delta, \delta)$ and $k \in \{1, \ldots, n\}$. Writing $\alpha_t v_k \otimes \sigma_t w_k - v_k \otimes w_k = (\alpha_t v_k - v_k) \otimes \sigma_t w_k + v_k \otimes (\sigma_t w_k - w_k)$, we obtain for any $t \in (-\delta, \delta)$ that

$$(p \otimes q)(\gamma_t\psi - \psi) \leq \sum_{k=1}^n p(\alpha_t v_k - v_k)q(\sigma_t w_k) + p(v_k)q(\sigma_t w_k - w_k) < 2k\epsilon.$$

This proves that $\gamma_t\psi \to \psi$ as $t \to 0$, for any ψ in the dense subspace $V \otimes W$. Let us next consider general $\psi \in V\widehat{\otimes}W$. Let $\eta \in V \otimes W$ be such that $(p \otimes q)(\psi - \eta) < \epsilon$. For small enough $\delta > 0$ we have $r_\alpha(|t|)r_\sigma(|t|) \leq 2$ and $(p \otimes q)(\gamma_t\eta - \eta) < \epsilon$ for all $t \in (-\delta, \delta)$. Using

$$(p \otimes q)(\gamma_t(\psi - \eta)) \leq r_\alpha(|t|)r_\sigma(|t|)(p \otimes q)(\psi - \eta) < 2\epsilon,$$

we find for all $t \in (-\delta, \delta)$ that

$$\begin{aligned}(p \otimes q)(\gamma_t\psi - \psi) &\leq (p \otimes q)(\gamma_t(\psi - \eta)) + (p \otimes q)(\psi - \eta) + (p \otimes q)(\gamma_t\eta - \eta) \\ &< 4\epsilon\end{aligned}$$

Thus $\mathbb{R} \to V\widehat{\otimes}W$, $t \mapsto \gamma_t\psi$ is continuous.

As the canonical bilinear map $\widehat{\otimes} : V \times W \to V\widehat{\otimes}W$ is continuous, $\mathbb{R}$-equivariant and has dense span in $V\widehat{\otimes}W$, the remaining assertion is immediate from Corollary 6.6.25.

2. It suffices to consider only the topology of uniform convergence on compact sets. Let $T \in \mathcal{B}(V, W)$. Let q be a continuous seminorm on W and let $K \subseteq V$ be compact. Consider the continuous seminorm on $\mathcal{B}(V, W)$ defined by $q_K(T) := \sup_{v \in K} q(Tv)$. As T is bounded, there is a continuous seminorm p on V s.t. $q(Tv) \leq p(v)$ for all $v \in v$. Let $r_\sigma, r_\alpha \in \mathbb{R}[t]$ be monic polynomials s.t. $q(\sigma_t w) \leq r_\sigma(|t|)q(w)$ and $p(\alpha_t v) \leq r_\alpha(|t|)p(v)$ for all $t \in \mathbb{R}$, $v \in V$ and $w \in W$. Then

$$q_K(\gamma_t(T)) = \sup_{v \in K} q(\sigma_t T \alpha_{-t} v) \leq r_\sigma(|t|)r_\alpha(|t|) \sup p(K).$$

This implies that γ is pointwise polynomially bounded.

We next show that γ is strongly continuous. Let $T \in \mathcal{B}(V, W)$, $\epsilon > 0$ and $\mathcal{O} := q^{-1}([0, \epsilon)) \subseteq W$. The map

$$\Phi : \mathbb{R} \times V \to W, \quad \Phi(t, v) := \gamma_t(T)v - Tv = \sigma_t T \alpha_{-t} v - Tv$$

is continuous, because the map $T : V \to W$ and the actions $\alpha : \mathbb{R} \times V \to V$ and $\sigma : \mathbb{R} \times W \to W$ are all continuous. Since $\{0\} \times K \subseteq \Phi^{-1}(\mathcal{O})$ and K

is compact, it follows from the Tube Lemma (cf. [Mun00, Lem. 26.8]) that there is an interval $I \subseteq \mathbb{R}$ containing 0 s.t. $\Phi(I \times K) \subseteq \mathcal{O}$. This means that $q_K(\gamma_t(T) - T) < \epsilon$ for all $t \in I$, so γ is strongly continuous.

It remains to show (6.6.5). Define $E_V := \mathrm{Spec}_\alpha(V)$ and $E_W := \mathrm{Spec}_\alpha(W)$. Let $N \subseteq \mathbb{R}$ be a 0-neighborhood. Let $T \in \mathcal{B}(V, W)$ and $f_3 \in I_0(\overline{E_W - E_V + N})$, so $f_3 \in \mathcal{S}(\mathbb{R})$ satisfies $\mathrm{supp}(\widehat{f_3}) \cap \overline{E_W - E_V + N} = \emptyset$. We show $\gamma_{f_3}(T) = 0$. Let $N_1, N_2 \subseteq \mathbb{R}$ be symmetric 0-neighborhoods such that $N_1 + N_2 \subseteq N$. Let $v \in V$, $f_1 \in J(E_V + N_1)$ and $f_2 \in J(E_W + N_2)$. So $\widehat{f_1}$ and $\widehat{f_2}$ have compact support contained in $E_V + N_1$ and $E_W + N_2$, respectively. One verifies that

$$\begin{aligned}\sigma_{f_2}\gamma_{f_3}(T)\alpha_{f_1}v &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f_1(t_1)f_2(t_2)f_3(t_3)\sigma_{t_2+t_3}T\alpha_{t_1-t_3}v\, dt_1dt_2dt_3 \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} F(t_1,t_2)\sigma_{t_2}T\alpha_{t_1}dt_1dt_2,\end{aligned} \tag{6.6.6}$$

where $F \in \mathcal{S}(\mathbb{R}^2)$ is given by

$$F(t_1, t_2) = \int_{\mathbb{R}} f_1(t_1 + t_3)f_2(t_2 - t_3)f_3(t_3)dt_3.$$

The Fourier transform $\widehat{F} \in \mathcal{S}(\mathbb{R}^2)$ of F is given by

$$\widehat{F}(p_1, p_2) = \widehat{f_1}(p_1)\widehat{f_2}(p_2)\widehat{f_3}(p_2 - p_1).$$

Recalling that N_1 is symmetric, notice that

$$\begin{aligned}\mathrm{supp}(\widehat{f_2}) - \mathrm{supp}(\widehat{f_1}) &\subseteq (E_W + N_2) - (E_V + N_V) \subseteq E_W - E_V + (N_1 + N_2) \\ &\subseteq E_W - E_V + N.\end{aligned}$$

As $\widehat{f_3}$ vanishes on $E_W - E_V + N$, it follows that $\widehat{F} = 0$ and hence $F = 0$. From (6.6.6) we conclude that $\sigma_{f_2}\gamma_{f_3}(T)\alpha_{f_1}v = 0$ for all $f_2 \in J(E_W + N_2)$. This implies $\gamma_{f_3}(T)\alpha_{f_1}v = 0$, by Lemma 6.6.28. Consequently, if $\lambda \in W'$ is any continuous functional, then $\int_{\mathbb{R}} f_1(t_1)\langle\lambda, \gamma_{f_3}(T)\alpha_{t_1}v\rangle dt = 0$. As the map $t \mapsto \langle\lambda, \gamma_{f_3}(T)\alpha_{t_1}v\rangle$ is continuous it follows that $\langle\lambda, \gamma_{f_3}(T)\alpha_{t_1}v\rangle = 0$ for all $t \in \mathbb{R}$. In particular $\langle\lambda, \gamma_{f_3}(T)v\rangle = 0$. As W' separates the points of W by the Hahn-Banach Theorem [Rud91, Thm. I.3.4], it follows that $\gamma_{f_3}(T)v = 0$. As $v \in V$ was arbitrary we find that $\gamma_{f_3}(T) = 0$. We have thus shown that $I_0(\overline{E_W - E_V + N}) \subseteq \ker\gamma$. By Corollary 6.6.21, this is equivalent to $\mathrm{Spec}_\gamma(\mathcal{B}(V, W)) \subseteq \overline{E_W - E_V + N}$. Hence

$$\mathrm{Spec}_\gamma(\mathcal{B}(V, W)) \subseteq \bigcap_N \overline{E_W - E_V + N} = \overline{E_W - E_V}.$$

□

Recall from Section 6.2.1 that $P(V,W) = \prod_{k=0}^{\infty} P^k(V,W)$ is equipped with the product topology, where each $P^k(E,F)$ carries the topology of uniform convergence on compact sets. We will have need for the following result in Section 6.7 below:

Corollary 6.6.31. *Consider the setting of Proposition 6.6.30. Assume that V is Fréchet. Define the representation γ of $\mathbb{R}$ on $P(V,W)$ by $\gamma_t(f)(v) := \sigma_t(f(\alpha_{-t}(v)))$. Then γ is strongly continuous and pointwise polynomially bounded. Moreover, if $\mathrm{Spec}_\alpha(V) \subseteq (-\infty, 0]$, then*

$$\inf \mathrm{Spec}_\gamma(P(V,W)) = \inf \mathrm{Spec}_\sigma(W) \in \{-\infty\} \cup \mathbb{R}.$$

Proof. Let $n \in \mathbb{N}_{\geq 0}$. Notice that γ leaves the homogeneous component $P^n(V,W) \subseteq P(V,W)$ invariant. Recall from Proposition 6.2.4 and Proposition 6.2.3 that

$$P^n(V,W) \cong \mathrm{Sym}^n(V,W) \subseteq \mathrm{Mult}(V^n,W) \cong \mathcal{B}(V^{\widehat{\otimes} n},W)$$

as locally convex vector spaces. The thus-obtained continuous linear embedding $\Phi_n : P^n(V,W) \hookrightarrow \mathcal{B}(V^{\widehat{\otimes} n},W)$ is $\mathbb{R}$-equivariant when $\mathcal{B}(V^{\widehat{\otimes} n},W)$ is equipped with the $\mathbb{R}$-action defined by $\widetilde{\gamma}_t(T) := \sigma_t T \alpha_{-t}$. By Proposition 6.6.30, this action is strongly continuous and pointwise polynomially bounded. Consequently, also γ is strongly continuous and pointwise polynomially bounded on $P^n(V,W)$. As $P(V,W)$ carries the product topology, the same holds for the $\mathbb{R}$-action γ on $P(V,W)$.

For the final statement, notice that $W = P^0(V,W) \subseteq P(V,W)$. By Proposition 6.6.23 it follows that $\mathrm{Spec}_\sigma(W) \subseteq \mathrm{Spec}_\gamma(P(V,W))$, thereby showing

$$\inf \mathrm{Spec}_\gamma(P(V,W)) \leq \inf \mathrm{Spec}_\sigma(W)$$

Conversely, let $n \in \mathbb{N}$. As Φ_n is continuous, injective and $\mathbb{R}$-equivariant, we know that $\mathrm{Spec}_\gamma(P^n(V,W)) \subseteq \mathrm{Spec}_{\widetilde{\gamma}}\left(\mathcal{B}(V^{\widehat{\otimes} n},W)\right)$, by Proposition 6.6.23. Furthermore, using Proposition 6.6.30 we notice that $\mathrm{Spec}_{\alpha^{\otimes n}}(V^{\widehat{\otimes} n}) \subseteq (-\infty, 0]$ and therefore also that $\mathrm{Spec}_{\widetilde{\gamma}}\left(\mathcal{B}(V^{\widehat{\otimes} n},W)\right) \subseteq \mathrm{Spec}_\sigma(W) + [0,\infty) =: E$. Thus $\mathrm{Spec}_\gamma(P^n(V,W)) \subseteq E$ for any $n \in \mathbb{N}_{\geq 0}$. By Corollary 6.6.21 this means that $\gamma_f \psi_n = 0$ for any $f \in I_0(E)$, $\psi_n \in P^n(V,W)$ and $n \in \mathbb{N}$. Consequently, $\gamma_f \psi = 0$ for any $f \in I_0(E)$ and $\psi \in P(V,W)$. So $I_0(E) \subseteq \ker \gamma$. By Corollary 6.6.21, this is equivalent with $\mathrm{Spec}_\gamma(P(V,W)) \subseteq E$. Hence $\inf \mathrm{Spec}_\sigma(W) = \inf E \leq \inf \mathrm{Spec}_\gamma(P(V,W))$. ☐

Finally, we record some useful facts regarding the space of smooth vectors of a unitary G-representation:

Proposition 6.6.32. *Let G be a regular locally convex Fréchet-Lie group. Let $\boldsymbol{d} \in \mathfrak{g}$ and assume that the $\mathbb{R}$-action $\dot{\alpha} : \mathbb{R} \to Aut(\mathfrak{g})$ defined by $\dot{\alpha}_t := Ad(\exp(t\boldsymbol{d}))$ is polynomially bounded. Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of G. Let $E \subseteq \mathbb{R}$ be a closed subset. Then the following assertions are valid:*

1. *The $\mathbb{R}$-representation $t \mapsto \rho(\exp(t\boldsymbol{d}))|_{\mathcal{H}_\rho^\infty}$ on $\mathcal{H}_\rho^\infty$ is strongly continuous and pointwise polynomially bounded, where $\mathcal{H}_\rho^\infty$ is equipped with the strong topology.*

2. *The operator $\pi(f) := \int_{\mathbb{R}} f(t)\rho(\exp(t\boldsymbol{d}))dt$ on $\mathcal{H}_\rho$ leaves $\mathcal{H}_\rho^\infty$ invariant for any $f \in \mathcal{S}(\mathbb{R})$.*

3. $\mathcal{H}_\rho^\infty(E) = \mathcal{H}_\rho(E) \cap \mathcal{H}_\rho^\infty$

4. *For any open subset $U \subseteq \mathbb{R}$, $\mathcal{H}_\rho^\infty(U)$ is dense in $\mathcal{H}_\rho(U)$.*

5. *If $E_1, E_2 \subseteq \mathbb{R}$ are closed subsets then $d\rho(\mathfrak{g}_{\mathbb{C}}(E_1))\mathcal{H}^\infty(E_2) \subseteq \mathcal{H}_\rho^\infty(E_1 + E_2)$.*

Proof. The second item follows from [NSZ15, Thm. 2.3], the fourth from [NSZ15, Prop. 3.2] and the fifth from [NSZ15, Thm. 3.1]. We provide an alternative proof of the second assertion and prove the first and third.

By [JN19, Prop. 3.19], the locally convex space $\mathcal{H}_\rho^\infty$ is complete. Let $n \in \mathbb{N}$ and $B \subseteq \mathfrak{g}^n$ be a bounded subset. Consider continuous seminorm

$$p_B(\psi) := \sup_{\xi \in B} \|d\rho(\xi_1 \cdots \xi_n)\psi\|$$

on $\mathcal{H}_\rho^\infty$. Let $\psi \in \mathcal{H}_\rho^\infty$. By [JN19, Lem. 3.24], the orbit map $\rho^\phi : G \to \mathcal{H}_\rho^\infty, g \mapsto \rho(g)\psi$ is smooth. It follows in particular that the $\mathbb{R}$-representation $t \mapsto \rho(\exp(t\boldsymbol{d}))$ on $\mathcal{H}_\rho^\infty$ is strongly continuous. It follows moreover that the multi-linear map $\mathfrak{g}^n \to \mathcal{H}_\rho^\infty, (\xi_1, \ldots, \xi_n) \mapsto d\rho(\xi_1 \cdots \xi_n)\psi$ is continuous. Using Proposition 6.2.3, we find that there exists a continuous seminorm p on $\mathfrak{g}$ such that $\|d\rho(\xi_1 \cdots \xi_n)\psi\| \leq \prod_{k=1}^n p(\xi_k)$ for every $\xi \in \mathfrak{g}^n$. Let $N \in \mathbb{N}$ and the 0-neighborhood $U \subseteq \mathfrak{g}$ be s.t. $C := \sup_{\xi \in U} \sup_{t \in \mathbb{R}} \frac{1}{1+|t|^N} p(\dot{\alpha}_t(\xi)) < \infty$. As $B \subseteq \mathfrak{g}^n$ is bounded, so is its projection $B_k \subseteq \mathfrak{g}$ onto the k^{th} factor for every $k \in \{1, \ldots, n\}$. Thus there exists $s > 0$ such that $B_k \subseteq sU$ for all $1 \leq k \leq n$. We obtain that

$$\sup_{\xi_k \in B_k} \sup_{t \in \mathbb{R}} \frac{1}{1+|t|^N} p(\dot{\alpha}_t(\xi_k)) \leq sC$$

for every $1 \leq k \leq n$. Using that ρ is unitary we find for all $t \in \mathbb{R}$ that

$$p_B(\rho(e^{-t\boldsymbol{d}})\psi) = \sup_{\xi \in B} \|d\rho(\dot{\alpha}_t(\xi_1) \cdots \dot{\alpha}_t(\xi_n))\psi\| \leq \sup_{\xi \in B} \prod_{k=1}^n p(\dot{\alpha}_t(\xi_k)) \leq C^n s^n (1+|t|^N)^n.$$

This implies that the $\mathbb{R}$-action $t \mapsto \rho(\exp(t\boldsymbol{d}))$ on $\mathcal{H}_\rho^\infty$ is pointwise polynomially bounded. As in Definition 6.6.10, we conclude that $\pi^\infty(f)\psi := \int_{\mathbb{R}} f(t)\rho(\exp(t\boldsymbol{d}))\psi dt$ defines a representation $\pi^\infty : \mathcal{S}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}_\rho^\infty)$ of $\mathcal{S}(\mathbb{R})$ on $\mathcal{H}_\rho^\infty$ by linear operators. It is clear that $\pi^\infty(f) := \pi(f)|_{\mathcal{H}_\rho^\infty}$, so this proves that $\pi(f)$ leaves $\mathcal{H}_\rho^\infty$ invariant for every $f \in \mathcal{S}(\mathbb{R})$. It is further immediate from Definition 6.6.12 that $\mathcal{H}_\rho^\infty(E) = \mathcal{H}_\rho(E) \cap \mathcal{H}_\rho^\infty$. □

6.7 Positive energy representations and holomorphic induction

In this section we explore the connection between positive energy representations and holomorphic induction. It is shown in Theorem 6.7.6 and Theorem 6.7.17 that these two are intimately related, as is to be expected from similar known results in more restrictive settings, such as [PS86, Thm. 11.1.1], [Nee13, Sec. 4] and [Nee14a, Thm. 6.1]. This is used to transfer various results from holomorphic induction to the context of positive energy representations, under suitable assumptions. Before proceeding to the main results, let us clarify the setting and make some preliminary observations.

Notation and preliminary observations

Let G be a connected regular BCH Fréchet-Lie group with Lie algebra $\mathfrak{g}$. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(G)$ be a homomorphism having a smooth action $\mathbb{R} \times G \to G$ and let $\dot{\alpha}$ be the corresponding $\mathbb{R}$-action on $\mathfrak{g}_\mathbb{C}$, defined by $\dot{\alpha}_s(\xi) := \mathbf{L}(\alpha_s)\xi := \frac{d}{dt}\big|_{t=0} \alpha_s(e^{t\xi})$ for $s \in \mathbb{R}$. Let $D\xi := \frac{d}{ds}\big|_{s=0} \dot{\alpha}_s(\xi)$ be the corresponding derivation on $\mathfrak{g}_\mathbb{C}$. Assume that $\dot{\alpha}$ has polynomial growth, in the sense of Definition 6.6.1. Define the Lie group $G^\sharp := G \rtimes_\alpha \mathbb{R}$, which has Lie algebra $\mathfrak{g}^\sharp := \mathfrak{g} \rtimes_D \mathbb{R}\boldsymbol{d}$, where we have written $\boldsymbol{d} := 1 \in \mathbb{R} \subseteq \mathfrak{g}^\sharp$ for the standard basis element. Then $G^\sharp$ is again a connected regular Fréchet-Lie group, using [Nee06, Thm. V.I.8], but not necessarily BCH.

As $\dot{\alpha}$ is assumed to have polynomial growth, we can define the Arveson spectral subspaces of $\mathfrak{g}_\mathbb{C}$ as in Definition 6.6.12. If $E \subseteq \mathbb{R}$ is any subset, we write $\mathfrak{g}_\mathbb{C}(E)$ for the spectral subspace of $\mathfrak{g}_\mathbb{C}$ associated to E. Define $\mathfrak{h}_\mathbb{C} := \ker D \subseteq \mathfrak{g}_\mathbb{C}(\{0\})$, $\mathfrak{h} := \mathfrak{h}_\mathbb{C} \cap \mathfrak{g}$ and

$$\mathfrak{n}_- := \overline{\bigcup_{\delta>0} \mathfrak{g}_\mathbb{C}((-\infty, -\delta])}, \qquad \mathfrak{n}_+ := \overline{\bigcup_{\delta>0} \mathfrak{g}_\mathbb{C}([\delta, \infty))}.$$

We assume that $(\mathfrak{g}_\mathbb{C}, \alpha)$ satisfies the so-called *splitting condition*, meaning that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_+.$$

Define $\mathfrak{b}_\pm := \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_\pm \subseteq \mathfrak{g}_\mathbb{C}$. Let $H := (G^\alpha)_0 \subseteq G$ be the connected subgroup of α-fixed points in G. Let us first establish that the assumptions on H, $\mathfrak{n}_\pm$ and $\mathfrak{h}_\mathbb{C}$ made in Section 6.4.2 are presently satisfied.

Lemma 6.7.1. *H is a locally exponential Lie subgroup of G with Lie algebra $\mathfrak{h}$.*

Proof. Since G is locally exponential, we can find a 0-neighborhood $U_\mathfrak{g} \subseteq \mathfrak{g}$ s.t. $\exp_G$ restricts to a diffeomorphism on $U_\mathfrak{g}$. Let $\xi \in U_\mathfrak{g}$ arbitrary. Using the fact that $\alpha_t(\exp_G(\xi)) = \exp_G(\dot{\alpha}_t(\xi))$ for all $t \in \mathbb{R}$, observe that

$$\xi \in \ker D \iff \exp_G(\xi) \in G^\alpha.$$

This implies that $\exp_G(U_\mathfrak{g} \cap \mathfrak{h}) = \exp_G(U_\mathfrak{g}) \cap H$. We also obtain that

$$\mathfrak{h} = \{ \xi \in \mathfrak{g} \, : \, \exp_G(\mathbb{R}\xi) \subseteq H \}.$$

It follows that H is a locally exponential Lie subgroup with Lie algebra $\mathfrak{h}$, by [Nee06, Thm. IV.3.3]. $\square$

Lemma 6.7.2. *The subspaces $\mathfrak{n}_\pm$, $\mathfrak{h}_\mathbb{C}$ and $\mathfrak{b}_\pm$ are Lie subalgebras of $\mathfrak{g}_\mathbb{C}$ and we have $[\mathfrak{h}_\mathbb{C}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$. Moreover, $Ad_H(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\pm$. Finally, $\theta(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\mp$ and $\theta(\mathfrak{h}_\mathbb{C}) \subseteq \mathfrak{h}_\mathbb{C}$.*

Proof. The Lie bracket $[-,-] : \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ is bilinear, continuous and $\mathbb{R}$-equivariant, meaning that $\dot{\alpha}_s([\xi, \eta]) = [\dot{\alpha}_s(\xi), \dot{\alpha}_s(\eta)]$ for all $s \in \mathbb{R}$ and $\xi, \eta \in \mathfrak{g}_\mathbb{C}$. Using Proposition 6.6.24 we obtain for any two closed subsets $E_1, E_2 \subseteq \mathbb{R}$ that $[\mathfrak{g}_\mathbb{C}(E_1), \mathfrak{g}_\mathbb{C}(E_2)] \subseteq \mathfrak{g}_\mathbb{C}(E_1 + E_2)$. This implies that $\mathfrak{n}_\pm$, $\mathfrak{h}_\mathbb{C}$ and $\mathfrak{b}_\pm$ are Lie subalgebras of $\mathfrak{g}_\mathbb{C}$ and that $[\mathfrak{h}_\mathbb{C}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$. We next show that $\mathrm{Ad}_H(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\pm$. Let $h \in H$. Then $\dot{\alpha}_s$ and Ad_h commute for any $s \in \mathbb{R}$, so $\mathrm{Ad}_h : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ is a continuous equivariant linear map. It follows using Proposition 6.6.23 that $\mathrm{Ad}_h(\mathfrak{g}_\mathbb{C}(E)) \subseteq \mathfrak{g}_\mathbb{C}(E)$ for any closed subset $E \subseteq \mathbb{R}$. Hence $\mathrm{Ad}_H(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\pm$. Let us next consider the conjugation θ. Using that $\dot{\alpha}_t$ commutes with θ for any $t \in \mathbb{R}$, observe that $\theta\dot{\alpha}_f\theta = \dot{\alpha}_{\overline{f}}$ for any $f \in \mathcal{S}(\mathbb{R})$. Consequently, $\mathcal{S}(\mathbb{R})_{\theta(\xi)} = \{ \overline{f} : f \in \mathcal{S}(\mathbb{R})_\xi \}$. Using that $\mathcal{F}(\overline{f})(p) = \overline{\mathcal{F}f}(-p)$ for $p \in \mathbb{R}$, we obtain for any $\xi \in \mathfrak{g}_\mathbb{C}$ that

$$\mathrm{Spec}_{\dot{\alpha}}(\theta(\xi)) = h(\mathcal{S}(\mathbb{R})_{\theta(\xi)}) = -h(\mathcal{S}(\mathbb{R})_\xi) = -\mathrm{Spec}_{\dot{\alpha}}(\xi).$$

So we have $\theta(\mathfrak{g}_\mathbb{C}(E)) = \mathfrak{g}_\mathbb{C}(-E)$ for any closed $E \subseteq \mathbb{R}$. This implies that $\theta(\mathfrak{n}_\pm) \subseteq \mathfrak{n}_\mp$. Since $\mathfrak{h}_\mathbb{C} = \ker D$ and θ commutes with D, we also have $\theta(\mathfrak{h}_\mathbb{C}) \subseteq \mathfrak{h}_\mathbb{C}$. $\square$

As the Lie group $G^\sharp = G \rtimes_\alpha \mathbb{R}$ need not be analytic, we only have access to the analytic structure of G:

Definition 6.7.3. If $(\rho, \mathcal{H}_\rho)$ is a unitary representation of $G^\sharp$, we write $\mathcal{H}_\rho^{\omega_G}$ for the space of G-analytic vectors in $\mathcal{H}_\rho$. We further define

$$\mathcal{H}_\rho^{\infty, \mathfrak{n}_-} := \{ \psi \in \mathcal{H}_\rho^\infty : d\rho(\mathfrak{n}_-)\psi = \{0\} \}$$

and we write $V(\rho) := \overline{\mathcal{H}_\rho^{\infty, \mathfrak{n}_-}}$ for its closure.

Let us first clarify that $V(\rho)$ can equivalently be defined as the closure of the set of G-smooth vectors in $\mathcal{H}_\rho$ that are killed by $\mathfrak{n}_-$, as opposed to the $G^\sharp$-smooth ones:

Lemma 6.7.4. *Let ρ be a unitary $G^\sharp$-representation. Let $W(\rho) \subseteq \mathcal{H}_\rho$ be the closed linear subspace generated by the set of G-smooth vectors in $\mathcal{H}_\rho$ that are killed by $d\rho(\mathfrak{n}_-)$. Then $W(\rho) = V(\rho)$.*

Proof. It is trivial that $V(\rho) \subseteq W(\rho)$. Let $\psi \in \mathcal{H}_\rho$ be a G-smooth vector s.t. $d\rho(\mathfrak{n}_-)\psi = \{0\}$. Let $f \in C_c^\infty(\mathbb{R})$ and define $\pi_f\psi := \int_\mathbb{R} f(t)\rho(t)\psi dt \in \mathcal{H}_\rho$. Then $\pi_f\psi$ is a smooth vector for $G^\sharp$, e.g. using [NSZ15, Lem. A.4]. Let $\xi \in \mathfrak{n}_-$. Then $\dot{\alpha}_{-t}(\xi) \in \mathfrak{n}_-$ and hence $d\rho(\dot{\alpha}_{-t}(\xi))\psi = 0$ for every $t \in \mathbb{R}$. Using [NSZ15, Lem. A.4] to differentiate under the integral, we obtain:

$$d\rho(\xi)\pi_f\psi = \int_\mathbb{R} f(t) d\rho(\xi)\rho(t)\psi dt = \int_\mathbb{R} f(t)\rho(t) d\rho(\dot{\alpha}_{-t}(\xi))\psi dt = 0.$$

So $\pi_f\psi \in \mathcal{H}_\rho^{\infty, \mathfrak{n}_-}$ for any $f \in C_c^\infty(\mathbb{R})$. Approximating ψ by vectors of the form $\pi_f\psi$, we conclude that $\psi \in V(\rho)$. So $V(\rho) = W(\rho)$. $\square$

To keep a uniform notation for G- and $G^\sharp$-representations, we complement Definition 6.7.3 with:

Definition 6.7.5. If $(\rho, \mathcal{H}_\rho)$ is a smooth unitary representation of G, we write $\mathcal{H}_\rho^{\omega_G} := \mathcal{H}_\rho^\omega$ for the space of G-analytic vectors in $\mathcal{H}_\rho$. Define

$$\mathcal{H}_\rho^{\infty,\mathfrak{n}_-} := \{ \psi \in \mathcal{H}_\rho^\infty \,:\, d\rho(\mathfrak{n}_-)\psi = \{0\} \}$$

and let $V(\rho) := \overline{\mathcal{H}_\rho^{\infty,\mathfrak{n}_-}}$ denote its closure.

Let us proceed with the task of relating the positive energy condition with holomorphic induction. Notice that $V(\rho) \subseteq \mathcal{H}_\rho$ is $H \times \mathbb{R}$-invariant for any smooth unitary G-representation ρ, because $\mathfrak{n}_-$ is invariant under $\dot{\alpha}_t$ and Ad_h for any $t \in \mathbb{R}$ and $h \in H$. The following makes use of the notation specified in Definition 6.4.25:

Theorem 6.7.6. *Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of $G^\sharp$ and let σ be the unitary representation of $H \times \mathbb{R}$ on $V_\sigma := V(\rho)$ defined by $\sigma(h,t) := \rho(h,t)|_{V(\rho)}$. The following assertions are equivalent:*

1. *ρ is of positive energy at $\boldsymbol{d}$, $V(\rho)$ is cyclic for ρ and $V(\rho) \cap \mathcal{H}_\rho^{\omega_G}$ is dense in $V(\rho)$.*
2. *σ is of positive energy at $\boldsymbol{d}$ and $\rho|_G = \mathrm{HolInd}_H^G(\sigma|_H)$.*

If these conditions are satisfied, then $\inf \mathrm{Spec}(-i\overline{d\rho(\boldsymbol{d})}) = \inf \mathrm{Spec}(-i\overline{d\sigma(\boldsymbol{d})}) \geq 0$.

We start the proof of Theorem 6.7.6 with two lemmas:

Lemma 6.7.7. *Let $W \subseteq V(\rho)$ be a H-invariant closed linear subspace that is cyclic for G and contains a dense set of G-analytic vectors. Then $W = V(\rho)$.*

Proof. Let $W^\perp$ be the orthogonal complement of W in $V(\rho)$, so $V(\rho) = W \oplus W^\perp$ as unitary H-representations. It suffices to show that $W^\perp \perp \rho(G)W$. Define $W^{\omega_G} := W \cap \mathcal{H}_\rho^{\omega_G}$. Let $w \in W^{\omega_G}$ and $v \in W^\perp \subseteq V(\rho)$. Consider the analytic function $f : G \to \mathbb{C},\ f(g) := \langle v, \rho(g)w\rangle$. Let $E_0 : \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathcal{U}(\mathfrak{h}_\mathbb{C})$ be defined as in Definition 6.4.22. As $d\rho(\mathfrak{n}_-)$ kills both $\mathcal{H}_\rho^{\infty,\mathfrak{n}_-}$ and W^{ω_G}, observe that $\langle v, d\rho(x)w\rangle = \langle v, d\sigma(E_0(x))w\rangle = 0$ for any $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. It follows that $j_e^\infty(f) = 0$. As G is connected and f is analytic, we conclude using Proposition 6.2.14 that $f = 0$. Because W^{ω_G} is dense in W, it follows that $W^\perp \perp \rho(G)W$. □

Lemma 6.7.8. *Let $\mathcal{D} \subseteq \mathcal{H}_\rho^\omega$ be a linear subspace. Then $\overline{d\rho(\mathcal{U}(\mathfrak{g}_\mathbb{C}))\mathcal{D}}$ is the closed G-invariant linear subspace of $\mathcal{H}_\rho$ generated by $\mathcal{D}$.*

Proof. Define $\mathcal{F} := \overline{d\rho(\mathcal{U}(\mathfrak{g}_\mathbb{C}))\mathcal{D}}$ and let $\mathcal{F}'$ denote the closed G-invariant linear subspace generated by $\mathcal{D}$. The inclusion $\mathcal{F} \subseteq \mathcal{F}'$ is clear. Let $v \in \mathcal{F}^\perp \subseteq \mathcal{H}_\rho$ and take $\psi \in \mathcal{D}$. Consider the analytic function $f : G \to \mathbb{C},\ f(g) := \langle v, \rho(g)\psi\rangle$. Notice for $x \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ that $\langle v, d\rho(x)\psi\rangle = 0$, because $d\rho(x)\psi \in \mathcal{F}$. It follows that $j_e^\infty(f) = 0$. As G is connected and f is analytic, we conclude using Proposition 6.2.14 that $f = 0$. We therefore find that $v \perp \rho(G)\mathcal{D}$, so in fact $v \perp \mathcal{F}'$. Hence $\mathcal{F}^\perp \subseteq (\mathcal{F}')^\perp$, which is equivalent to $\mathcal{F}' \subseteq \mathcal{F}$. □

Proof of Theorem 6.7.6: Define $\mathcal{D}_\chi := V(\rho) \cap \mathcal{H}_\rho^{\omega_G}$. Assume that (1) holds true. Then in particular, σ is of positive energy at $\boldsymbol{d}$. Let $\chi : \mathfrak{b}_- \to \mathcal{L}(\mathcal{D}_\chi)$ be the trivial extension of $d\sigma$ to $\mathfrak{b}_-$ with domain $\mathcal{D}_\chi$. By definition of $V(\rho)$, $\mathcal{D}_\chi$ is killed by $d\rho(\mathfrak{n}_-)$. The conditions for V_σ in Theorem 6.4.13 are satisfied for the $(H, \mathfrak{b}_-)$-extension pair $(\sigma|_H, \chi)$, so (2) follows from Theorem 6.4.13.

Conversely, assume that $\rho|_G = \mathrm{HolInd}_H^G(\sigma|_H)$ and that σ is of p.e. at $\boldsymbol{d}$. It follows from Theorem 6.4.13 that there is a H-invariant closed linear subspace $W \subseteq \mathcal{H}_\rho$ s.t. W is cyclic for ρ and $W \cap \mathcal{H}_\rho^{\omega_G}$ is both dense in W and killed by $d\rho(\mathfrak{n}_-)$. The last condition implies using Lemma 6.7.4 that $W \subseteq V(\rho)$. By Lemma 6.7.7 we obtain that $W = V(\rho)$. To see that (1) holds true, it only remains to show that ρ is of positive energy at $\boldsymbol{d}$. Define

$$\Phi : \mathcal{H}_\rho^\infty \to C^\infty(G, V_\sigma)^H, \qquad \Phi_\psi(g) := p_V \rho(g)^{-1}\psi$$

for $\psi \in \mathcal{H}_\rho^\infty$, where $p_V : \mathcal{H}_\rho \to V(\rho)$ is the orthogonal projection. Using the exponential map as a local chart, identify $J_e^\infty C^\infty(G, V_\sigma) \cong P(\mathfrak{g}_\mathbb{C}, V_\sigma)$ G-equivariantly. Let A denote the composition

$$A : \mathcal{H}_\rho^\infty \xrightarrow{\Phi} C^\infty(G, V_\sigma)^H \xrightarrow{j_e^\infty} P(\mathfrak{g}_\mathbb{C}, V_\sigma) \xrightarrow{restr} P(\mathfrak{n}_-, V_\sigma).$$

Observe that

$$\begin{aligned}\Phi_{\rho(t)\psi}(g) &= p_V\rho(g)^{-1}\rho(t)\psi = p_V\rho(t)\rho(\alpha_{-t}(g))^{-1}\psi = \sigma(t)p_V\rho(\alpha_{-t}(g))^{-1}\psi\\ &= \sigma(t)\Phi_\psi(\alpha_{-t}(g)).\end{aligned}$$

Consequently, A is $\mathbb{R}$-equivariant if we equip $P(\mathfrak{n}_-, V_\sigma)$ with the $\mathbb{R}$-action defined by $(\nu_t f)(\xi) := \sigma(t)f(\dot{\alpha}_{-t}(\xi))$ for $t \in \mathbb{R}$ and $f \in P^n(\mathfrak{n}_-, V_\sigma)$. Equip $\mathcal{H}_\rho^\infty$ with the strong topology (cf. Definition 6.2.17), with respect to which it is complete because G is a regular Fréchet-Lie group [JN19, Prop. 3.19]. Recall that $P(\mathfrak{n}_-, V_\sigma) = \prod_{n=0}^\infty P^n(\mathfrak{n}_-, V_\sigma)$ carries the product topology and each $P^n(\mathfrak{n}_-, V_\sigma)$ carries the topology of uniform convergence on compact sets. We show that A is continuous with respect to these topologies. For any $\psi \in \mathcal{H}_\rho^\infty$, let $f_\psi \in C^\infty(G, \mathcal{H}_\rho)$, $f_\psi(g) := \rho(g)\psi$ denote the orbit map. Using that ρ is unitary, observe that the linear map $\mathcal{H}_\rho^\infty \to C^\infty(G, \mathcal{H}_\rho), \psi \mapsto f_\psi$ is continuous w.r.t. the smooth compact-open topology on $C^\infty(G, \mathcal{H}_\rho)$. This implies that Φ is continuous. As j_e^∞ is continuous by Proposition 6.2.15, the continuity of A follows. We remark further that the $\mathbb{R}$-representation $t \mapsto \rho(t)$ on $\mathcal{H}_\rho^\infty$ is strongly continuous and pointwise polynomially bounded by Proposition 6.6.32, so that its Arveson spectrum can be defined according to Definition 6.6.12. Similarly, because the $\mathbb{R}$-actions on $\mathfrak{n}_-$ and V_σ both have polynomially growth and are strongly continuous, it follows from Corollary 6.6.31 that the $\mathbb{R}$-action ν on $P(\mathfrak{n}_-, V_\sigma)$ is strongly continuous and pointwise polynomially bounded. Since $\mathfrak{n}_-$ and V_σ have non-positive and non-negative spectrum, respectively (relative to the $\mathbb{R}$-actions $\dot{\alpha}_t$ and $\sigma(t)$, respectively), we further obtain from Corollary 6.6.31 and Example 6.6.13 that

$$\inf \mathrm{Spec}_\nu\left(P(\mathfrak{n}_-, V_\sigma)\right) = \inf \mathrm{Spec}(V_\sigma) = \inf \mathrm{Spec}(-i\overline{d\sigma(\boldsymbol{d})}) \geq 0$$

We show next that A is injective. Let $\psi \in \mathcal{H}_\rho^\infty$ and suppose that $A(\psi) = 0$. Then $p_V d\rho(\mathcal{U}(\mathfrak{n}_-))\psi = \{0\}$, which implies $\psi \perp \overline{d\rho(\mathcal{U}(\mathfrak{n}_+))\mathcal{D}_\chi}$. Since $\mathcal{D}_\chi$ is $d\rho(\mathfrak{b}_-)$-invariant, notice that $\overline{d\rho(\mathcal{U}(\mathfrak{n}_+))\mathcal{D}_\chi} = \overline{d\rho(\mathcal{U}(\mathfrak{g}_\mathbb{C}))\mathcal{D}_\chi}$ by the PBW Theorem. By Lemma 6.7.8, this is the closed G-invariant subspace of $\mathcal{H}_\rho$ generated by $\mathcal{D}_\chi$, which equals all of $\mathcal{H}_\rho$ because $\mathcal{D}_\chi$ is dense in $V(\rho)$ and $V(\rho)$ is cyclic for ρ. Thus $\psi \perp \mathcal{H}_\rho$ and so $\psi = 0$. Hence A is injective, continuous and $\mathbb{R}$-equivariant. It follows by Proposition 6.6.23 that $\mathrm{Spec}(\mathcal{H}_\rho^\infty) \subseteq \mathrm{Spec}_\nu\big(P(\mathfrak{n}_-, V_\sigma)\big)$, where we consider the $\mathbb{R}$-action $t \mapsto \rho(t)$ on $\mathcal{H}_\rho^\infty$. Thus

$$\inf \mathrm{Spec}(\mathcal{H}_\rho^\infty) \geq \inf\ \mathrm{Spec}_\nu\big(P(\mathfrak{n}_-, V_\sigma)\big) = \inf \mathrm{Spec}(-i\overline{d\sigma(\boldsymbol{d})}),$$

Notice that $\mathcal{H}_\rho$ and $\mathcal{H}_\rho^\infty$ have the same spectrum, because $\mathcal{H}_\rho^\infty$ is dense in $\mathcal{H}_\rho$. So

$$\inf\ \mathrm{Spec}(-i\overline{d\rho(\boldsymbol{d})}) = \inf\ \mathrm{Spec}(\mathcal{H}_\rho) = \inf\ \mathrm{Spec}(\mathcal{H}_\rho^\infty) \geq \inf\ \mathrm{Spec}(-i\overline{d\sigma(\boldsymbol{d})}) \geq 0.$$

Thus, ρ is of positive energy at $\boldsymbol{d}$. So (1) holds true. Finally, the inclusion $V(\rho) \subseteq \mathcal{H}_\rho$ is $\mathbb{R}$-equivariant, so by Proposition 6.6.23 we also have the reverse inequality $\inf\ \mathrm{Spec}(-i\overline{d\rho(\boldsymbol{d})}) \leq \inf\ \mathrm{Spec}(-i\overline{d\sigma(\boldsymbol{d})})$.

□

Let us state some important immediate consequences of Theorem 6.7.6.

Lemma 6.7.9. *Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary G-representation. Let $q_V \in \mathcal{B}(\mathcal{H}_\rho)$ denote the orthogonal projection onto $V(\rho)$. Then $q_V \in \rho(G)''$.*

Proof. Let $T \in \rho(G)' = \mathcal{B}(\mathcal{H}_\rho)^G$. Then $T\mathcal{H}_\rho^\infty \subseteq \mathcal{H}_\rho^\infty$ and

$$d\rho(\mathfrak{n}_-)T\mathcal{H}_\rho^{\infty,\mathfrak{n}_-} = Td\rho(\mathfrak{n}_-)\mathcal{H}_\rho^{\infty,\mathfrak{n}_-} \subseteq \{0\}.$$

Thus $T\mathcal{H}_\rho^{\infty,\mathfrak{n}_-} \subseteq \mathcal{H}_\rho^{\infty,\mathfrak{n}_-}$. It follows that $TV(\rho) \subseteq V(\rho)$. As $\mathcal{B}(\mathcal{H}_\rho)^G$ is $*$-closed, we have also shown that $T^*V(\rho) \subseteq V(\rho)$. Thus $q_V T = Tq_V$, and so $q_V \in \rho(G)''$. □

Corollary 6.7.10. *Suppose that the unitary $G^\sharp$-representation ρ satisfies the equivalent conditions of Theorem 6.7.6. Then $T \mapsto T|_{V(\rho)}$ defines isomorphisms of von Neumann algebras*

$$\mathcal{B}(\mathcal{H}_\rho)^G \cong \mathcal{B}(V(\rho))^H \qquad \textit{and} \qquad \mathcal{B}(\mathcal{H}_\rho)^{G^\sharp} \cong \mathcal{B}(V(\rho))^{H\times\mathbb{R}}.$$

Proof. That $T \mapsto T|_{V(\rho)}$ defines an isomorphism $\mathcal{B}(\mathcal{H}_\rho)^G \to \mathcal{B}(V(\rho))^H$ is immediate from Lemma 6.7.9 and Theorem 6.4.30. Consequently, it suffices to show that any $T \in \mathcal{B}(\mathcal{H}_\rho)^G$ with $T|_{V(\rho)} \in \mathcal{B}(V(\rho))^{H\times\mathbb{R}}$ automatically commutes with the $\mathbb{R}$-action $t \mapsto \rho(t)$ on $\mathcal{H}_\rho$. Consider such T and let $t \in \mathbb{R}$. Then

$$\begin{aligned}\rho(t)T\rho(g)v &= \rho(t)\rho(g)Tv = \rho(\alpha_t(g))\rho(t)Tv = \rho(\alpha_t(g))T\rho(t)v \\ &= T\rho(\alpha_t(g))\rho(t)v = T\rho(t)\rho(g)v\end{aligned} \tag{6.7.1}$$

for any $g \in G$ and $v \in V(\rho)$. As $V(\rho)$ is cyclic for G, it follows that $T\rho(t) = \rho(t)T$ for all $t \in \mathbb{R}$. □

Corollary 6.7.11. *Suppose that the unitary $G^\sharp$-representations ρ_1 and ρ_2 satisfy the equivalent conditions of Theorem 6.7.6. Then the following assertions are valid:*

1. *If $V(\rho_1) \cong V(\rho_2)$ as unitary H-representations, then $\rho_1|_G \cong \rho_2|_G$.*
2. *If $V(\rho_1) \cong V(\rho_2)$ as unitary $H \times \mathbb{R}$-representations, then $\rho_1 \cong \rho_2$.*

Proof. The first assertion is immediate from Theorem 6.4.21. Assume that the unitary $u : V(\rho_1) \to V(\rho_2)$ intertwines the $H \times \mathbb{R}$-actions. Consider the unitary $G^\sharp$-representation $\rho = \rho_1 \oplus \rho_2$ on $\mathcal{H}_{\rho_1} \oplus \mathcal{H}_{\rho_2}$. Notice that

$$V(\rho_1 \oplus \rho_2) = V(\rho_1) \oplus V(\rho_2) =: W$$

and that ρ satisfies the conditions in Theorem 6.7.6. Define $S \in \mathcal{B}(W)^{H\times\mathbb{R}}$ by $S(v_1, v_2) := (0, uv_1)$. By Corollary 6.7.10, there is some $T \in \mathcal{B}(\mathcal{H}_{\rho_1} \oplus \mathcal{H}_{\rho_2})^{G^\sharp}$ s.t. $T|_W = S$. As $V(\rho_1)$ and $V(\rho_2)$ are cyclic for G in $\mathcal{H}_{\rho_1}$ and $\mathcal{H}_{\rho_2}$, respectively, T is of the form $T(\psi_1, \psi_2) = (0, U\psi_1)$ for some $U : \mathcal{H}_{\rho_1} \to \mathcal{H}_{\rho_2}$ intertwining the $G^\sharp$-actions. Notice that S^*S and SS^* are the orthogonal projections onto $V(\rho_1)$ and $V(\rho_2)$, respectively. By Corollary 6.7.10 it follows that T^*T and TT^* are the orthogonal projections onto $\mathcal{H}_{\rho_1}$ and $\mathcal{H}_{\rho_2}$, respectively. This implies that U is unitary. ☐

The spectral gap condition

We will next assume that the so-called spectral gap condition is satisfied, a stronger variant of the splitting condition. We show that in this case, $V(\rho)$ is always cyclic for positive energy representations.

Definition 6.7.12. We say that *the spectral gap* (SG) condition is satisfied if there is some $\delta > 0$ such that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g}_\mathbb{C}((-\infty, -\delta]) \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{g}_\mathbb{C}([\delta, \infty)). \tag{6.7.2}$$

If ρ is a smooth unitary representation of $G^\sharp$ and $E \subseteq \mathbb{R}$ is a subset, we write $\mathcal{H}_\rho(E)$ and $\mathcal{H}_\rho^\infty(E)$ for the closed spectral subspaces associated to the $\mathbb{R}$-representation $t \mapsto \rho(t)$ on $\mathcal{H}_\rho$ and $\mathcal{H}_\rho^\infty$, respectively, where we recall that the $\mathbb{R}$-action on $\mathcal{H}_\rho^\infty$ is pointwise polynomially bounded by Proposition 6.6.32. Recall also from Proposition 6.6.32 that $H_\rho^\infty(E) = \mathcal{H}_\rho(E) \cap \mathcal{H}_\rho^\infty$.

Lemma 6.7.13. *Assume that (SG) is satisfied. Let ρ be a smooth unitary representation of $G^\sharp$ which is of p.e. at $\boldsymbol{d} \in \mathfrak{g}^\sharp$. If $\mathcal{H}_\rho \neq \{0\}$, then $V(\rho) \neq \{0\}$.*

Proof. Let $\delta > 0$ be such that (6.7.2) is satisfied. Set $E_0 := -i \inf \operatorname{Spec}(d\rho(\boldsymbol{d}))$. Let $0 < \epsilon < \delta$ and define $U := [E_0, E_0 + \epsilon)$. By definition of E_0, the spectral subspace $\mathcal{H}_\rho(U)$ is nonzero. By Proposition 6.6.32(4), $\mathcal{H}_\rho^\infty(U)$ is dense in $\mathcal{H}_\rho(U) = \mathcal{H}_\rho((E_0 - \epsilon, E_0 + \epsilon))$. Since $\mathcal{H}_\rho(U)$ is nonzero, so is $\mathcal{H}_\rho^\infty(U)$. By the last point in Proposition 6.6.32, we obtain that $d\rho(\mathfrak{n}_-)\mathcal{H}^\infty(U) \subseteq \mathcal{H}_\rho^\infty((-\infty, E_0 + \epsilon - \delta]) = \{0\}$. Hence $\mathcal{H}^\infty(U) \subseteq \mathcal{H}_\rho^{\infty,\mathfrak{n}_-} \subseteq V(\rho)$. It follows that $V(\rho) \neq \{0\}$. ☐

Proposition 6.7.14. *Assume that (SG) is satisfied. Let ρ be a smooth unitary representation of $G^\sharp$ which is of p.e. at $\boldsymbol{d} \in \mathfrak{g}^\sharp$. Then $V(\rho)$ is cyclic for ρ.*

Proof. Let W be the closed $G^\sharp$-invariant subspace of $\mathcal{H}_\rho$ generated by $V(\rho)$. Then $W^\perp$ carries a smooth representation of $G^\sharp$ that is of positive energy at $\boldsymbol{d} \in \mathfrak{g}^\sharp$. From $(W^\perp)^{\infty,\mathfrak{n}_-} \subseteq \mathcal{H}_\rho^{\infty,\mathfrak{n}_-} \subseteq V(\rho)$, we obtain that $(W^\perp)^{\infty,\mathfrak{n}_-} \subseteq W^\perp \cap V(\rho) = \{0\}$. Using Lemma 6.7.13 we conclude that $W^\perp = \{0\}$, so $W = \mathcal{H}_\rho$. □

Ground state representations

We now shift our attention to ground state representations, where Theorem 6.7.6 simplifies somewhat. If ρ is a smooth unitary representation of $G^\sharp$ on $\mathcal{H}_\rho$, we define $\mathcal{H}_\rho(0) = \ker \overline{d\rho(\boldsymbol{d})}$, $\mathcal{H}_\rho^\infty(0) := \mathcal{H}_\rho(0) \cap \mathcal{H}_\rho^\infty$ and $\mathcal{H}_\rho^{\omega_G}(0) := \mathcal{H}_\rho(0) \cap \mathcal{H}_\rho^{\omega_G}$. It will be convenient to make the following definition:

Definition 6.7.15. Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of $G^\sharp$ that is ground state at $\boldsymbol{d} \in \mathfrak{g}^\sharp$. We say that ρ is *analytically ground state* at $\boldsymbol{d} \in \mathfrak{g}^\sharp$ if $\mathcal{H}_\rho^{\omega_G}(0)$ is dense in $\mathcal{H}_\rho(0)$.

Lemma 6.7.16. *Let $(\rho, \mathcal{H}_\rho)$ be a smooth unitary representation of $G^\sharp$ that is of positive energy at $\boldsymbol{d} \in \mathfrak{g}^\sharp$. Then $\mathcal{H}_\rho^\infty(0) \subseteq \mathcal{H}_\rho^{\infty,\mathfrak{n}_-}$. If ρ is analytically ground state at $\boldsymbol{d}$, then $V(\rho) = \mathcal{H}_\rho(0)$.*

Proof. Using Proposition 6.6.24, we obtain that

$$d\rho(\mathfrak{g}_\mathbb{C}((-\infty, -\delta]))\mathcal{H}_\rho^\infty(0) \subseteq \mathcal{H}_\rho^\infty((-\infty, -\delta]) = \{0\}, \qquad \forall \delta > 0.$$

Hence $\mathcal{H}_\rho^\infty(0) \subseteq V(\rho)$. If ρ is analytically ground state at $\boldsymbol{d}$, the preceding implies $\mathcal{H}_\rho(0) \subseteq V(\rho)$. Using Lemma 6.7.7 we conclude that $\mathcal{H}_\rho(0) = V(\rho)$. □

The following clarifies the tight relation between unitary representations of $G \rtimes_\alpha \mathbb{R}$ that are analytically ground state at $\boldsymbol{d} \in \mathfrak{g}^\sharp$ and holomorphic induction:

Theorem 6.7.17. *Consider the setting of Theorem 6.7.6. The following assertions are equivalent:*

1. *ρ is analytically ground state at $\boldsymbol{d} \in \mathfrak{g}^\sharp$.*
2. *$\rho|_G = \mathrm{HolInd}_H^G(\sigma|_H)$ and $V(\rho) = \mathcal{H}_\rho(0)$.*

Proof. Assume that (1) is valid. Then $V(\rho) = \mathcal{H}_\rho(0)$, by Lemma 6.7.16, so (2) follows from Theorem 6.7.6. Suppose conversely that (2) holds true. Theorem 6.7.6 then yields that ρ is of positive energy at $\boldsymbol{d}$, that $\mathcal{H}_\rho(0)$ is cyclic for G and that $\mathcal{H}_\rho^{\omega_G}(0)$ is dense in $\mathcal{H}_\rho(0)$. Thus (1) is valid. □

Let us complement Theorem 6.7.17 with the following observation:

Proposition 6.7.18. *Let ρ be a smooth unitary p.e. representation of G. Let ρ_0 denote its minimal positive extension to $G^\sharp$. Assume that ρ_0 satisfies the equivalent conditions of Theorem 6.7.6. If ρ is irreducible, then ρ_0 is analytically ground state and $V(\rho) = \mathcal{H}_{\rho_0}(0)$.*

Proof. Define $V_\sigma := V(\rho)$, $\sigma_0(h,t) := \rho_0(h,t)|_{V_\sigma}$ and $\sigma(h) := \rho(h)|_{V_\sigma}$. Let $\mathcal{M} := \rho(G)''$ be the von Neumann algebra generated by $\rho(G)$. By Corollary 6.7.10 we have $\mathcal{B}(V_\sigma)^H = \mathbb{C}\,\mathrm{id}_{V_\sigma}$. Thus $\sigma_0(t) = e^{itp}\,\mathrm{id}_{V_\sigma}$ for some $p \in \mathbb{R}$. As σ_0 is of p.e., we have $p \geq 0$. By Theorem 6.7.6 we know that $\inf \mathrm{Spec}(-\overline{id\rho_0(\boldsymbol{d})}) = p$, so $\rho_1(t) := \rho_0(t)e^{-itp}$ defines a positive inner implementation of $\mathbb{R} \to \mathrm{Aut}(\mathcal{M})$, $t \mapsto \mathrm{Ad}(\rho_0(t))$. As $\rho_0(t)$ is minimal, it follows that $p \leq 0$. Hence $p = 0$, and so $V(\rho) \subseteq \mathcal{H}_{\rho_0}(0)$. On the other hand, we know from Lemma 6.7.16 that $\mathcal{H}_{\rho_0}^\infty(0) \subseteq V(\rho)$. Since $\mathcal{H}_{\rho_0}^\infty(0)$ contains all vectors of the form $\int_\mathbb{R} f(t)\rho_0(t)v$ for $f \in C_c^\infty(\mathbb{R})$ and $v \in V(\rho) \cap \mathcal{H}_\rho^{\omega_G}$, $\overline{\mathcal{H}_{\rho_0}^\infty(0)}$ contains $V(\rho) \cap \mathcal{H}_\rho^{\omega_G}$, which is dense in $V(\rho)$. So $\overline{\mathcal{H}_{\rho_0}^\infty(0)} = V(\rho) = \mathcal{H}_{\rho_0}(0)$. This implies that ρ_0 is analytically ground state. □

Strongly-entire ground state representations for $\mathbb{T}$-actions

The preceding results become particularly applicable for representations ρ which are both strongly-entire and ground state w.r.t. a $\mathbb{T}$-action. In this case, we can always guarantee that they are analytically ground state:

Lemma 6.7.19. *Suppose that α descends to a $\mathbb{T}$-action. Let ρ be a unitary p.e. representation of $G \rtimes_\alpha \mathbb{T}$. We write $\mathcal{H}_\rho^{\mathcal{O}_G}$ for the vectors in $\mathcal{H}_\rho$ that are strongly-entire for the G-action. Let $P : \mathcal{H}_\rho \to \mathcal{H}_\rho(0)$ denote the orthogonal projection. Then $P\mathcal{H}_\rho^{\mathcal{O}_G} \subseteq \mathcal{H}_\rho^{\mathcal{O}_G}$. In particular, if $\rho|_G$ is strongly-entire then $\mathcal{H}_\rho(0) \cap \mathcal{H}_\rho^{\mathcal{O}_G}$ is dense in $\mathcal{H}_\rho(0)$.*

Proof. For a compact subset $B \subseteq \mathfrak{g}_\mathbb{C}$ and $\psi \in \mathcal{H}_\rho^\infty$, we write

$$p_B^n(\psi) := \sup_{\xi_j \in B} \|d\rho(\xi_1 \cdots \xi_n)\psi\|$$

for $n \in \mathbb{N}_{\geq 0}$ and set $q_B(\psi) := \sum_{n=0}^\infty \frac{1}{n!} p_B^n(\psi)$. Let $\psi \in \mathcal{H}_\rho^{\mathcal{O}_G}$ and let $B \subseteq \mathfrak{g}_\mathbb{C}$ be compact. Then $B' := \alpha(\mathbb{T} \times B) \subseteq \mathfrak{g}_\mathbb{C}$ is compact, $\mathbb{T}$-invariant and satisfies $B \subseteq B'$. Observe that

$$p_B^n(\rho(t)\psi) \leq p_{B'}^n(\rho(t)\psi) = p_{\alpha_{-t}(B')}^n(\psi) = p_{B'}^n(\psi), \qquad \forall t \in \mathbb{T}.$$

Identifying $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$, recall that $P = \frac{1}{2\pi}\int_0^{2\pi} \rho(t)dt$. Notice using e.g. [NSZ15, Lem. A.4] that $P\mathcal{H}_\rho^\infty \subseteq \mathcal{H}_\rho^\infty$, and moreover that

$$p_B^n(P\psi) \leq \frac{1}{2\pi}\int_0^{2\pi} p_B^n(\rho(t)\psi)dt \leq p_{B'}^n(\psi), \qquad \forall \psi \in \mathcal{H}_\rho^\infty,\ n \in \mathbb{N}_{\geq 0}.$$

We thus find that $q_B(P\psi) \leq q_{B'}(\psi)$. So $P\mathcal{H}_\rho^{\mathcal{O}_G} \subseteq \mathcal{H}_\rho^{\mathcal{O}_G}$. □

Combining Theorem 6.7.17 and Lemma 6.7.19 we obtain the following:

Theorem 6.7.20. *Assume that α is a $\mathbb{T}$-action. Let $(\rho, \mathcal{H}_\rho)$ be a unitary representation of $G \rtimes_\alpha \mathbb{T}$. Assume that $\rho|_G$ is strongly-entire. Let σ be the unitary representation of $H \times \mathbb{T}$ on $V(\rho)$. The following are equivalent:*

1. *ρ is ground state at $\boldsymbol{d} \in \mathfrak{g}^\sharp$.*
2. *$\rho|_G = \mathrm{HolInd}_H^G(\sigma|_H)$ and $V(\rho) = \mathcal{H}_\rho(0)$.*

In this case, also σ is strongly-entire.

By Proposition 6.2.30(3), we know that any smooth unitary representation ρ of G which is of p.e. w.r.t. a smooth $\mathbb{T}$-action α is automatically ground state, and also that the minimal positive extension ρ_0 of ρ to $G^\sharp$ descends to $G \rtimes_\alpha \mathbb{T}$. Combining Theorem 6.7.20, Corollary 6.7.10 and Corollary 6.7.11, we obtain:

Corollary 6.7.21. *Assume that α is a smooth $\mathbb{T}$-action and that every irreducible unitary representation of G that is of positive energy w.r.t. α is strongly-entire. Then there is an injective map $\widehat{G}_{pos(\alpha)} \hookrightarrow \widehat{H}$, obtained by sending $\rho \in \widehat{G}_{pos(\alpha)}$ to the irreducible unitary H-representation on $V(\rho)$.*

Remark 6.7.22. Recall from Theorem 6.3.6 that if G is a finite-dimensional Lie group of type R, then every continuous unitary G-representation is in fact strongly-entire.

It would be beneficial to obtain sufficient conditions for $V(\rho) \cap \mathcal{H}_\rho^{\omega_G}$ to be dense in $V(\rho)$. We state the following related open problem:

Problem 6.7.23. Assume there are 0-neighborhoods $U \subseteq \mathfrak{g}_\mathbb{C}$, $U_- \subseteq \mathfrak{n}_-$, $U_0 \subseteq \mathfrak{h}_\mathbb{C}$ and $U_+ \subseteq \mathfrak{n}_+$ for which the map

$$U_+ \times U_0 \times U_- \to U, \qquad (\xi_+, \xi_0, \xi_-) \mapsto \xi_+ * \xi_0 * \xi_-$$

is biholomorphic, where $*$ is defined by the BCH series. We write $\xi \mapsto (\xi_+, \xi_0, \xi_-)$ for its inverse. Let ρ be a unitary representation of G that is of positive energy. Set $V_\sigma := V(\rho)$, considered as a unitary H-representation. Assume that V_σ is cyclic for ρ. Is it true that $V_\sigma^\omega \subseteq \mathcal{H}_\rho^{\omega_G}$? Taking $v \in V_\sigma^\omega$, the assumptions imply that the map $U \to \mathbb{C},\ \xi \mapsto \langle v, \sigma(e^{\xi_0})v\rangle$ is analytic on some 0-neighborhood. If it can be shown to locally extend the map $\mathfrak{g} \to \mathbb{C},\ \xi \mapsto \langle v, \rho(e^{\xi})v\rangle$ on some 0-neighborhood in $\mathfrak{g}$, then it would follow from [Nee11, Thm. 5.2] that $v \in \mathcal{H}_\rho^{\omega_G}$.

6.8 Examples

Example 6.8.1 (Finite-dimensional Lie groups of type R)**.**
Let G be a connected finite-dimensional Lie group of type R and let α be a $\mathbb{T}$-action on G. Let $H := (G^\alpha)_0$ be the connected subgroup of α-fixed points. In view of Theorem 6.3.6 and Theorem 6.7.20, any continuous ground state representation ρ of G is holomorphically induced from $V(\rho)$. According to Corollary 6.7.21, this defines an injection $\widehat{G}_{\mathrm{pos}(\alpha)} \hookrightarrow \widehat{H}$.

Example 6.8.2 (Holomorphically induced, but not geometrically).

1. Consider $G = \mathrm{SL}(2,\mathbb{R})$ and let ρ be any non-trivial continuous unitary representation. Trivially, we have $\rho = \mathrm{HolInd}_G^G(\rho)$. However, as ρ admits no non-trivial strongly-entire vectors by Theorem 6.3.6, it is *not* geometrically holomorphically induced from itself.

2. For a slightly less trivial example, consider the group $G = K \times \mathrm{SL}(2,\mathbb{R})$, where K is a connected compact simple Lie group. Let $T \subseteq K$ be a maximal torus and set $\mathfrak{t} := \mathbf{L}(T)$. Pick a regular element $H \in \mathfrak{t}_{reg}$ and let $\Delta_+ := \{\, \alpha \in \Delta \,:\, -i\alpha(H) > 0 \,\}$ be the corresponding system of positive roots, where $\Delta \subseteq i\mathfrak{t}^*$ denotes the set of all roots of $\mathfrak{k}$. Consider the $\mathbb{T}$-action on G defined by $\alpha_t(k,x) = (e^{tH}ke^{-tH}, x)$. Let $(\rho, \mathcal{H}_\rho)$ be a continuous irreducible unitary representation of G. Then ρ decomposes as $\mathcal{H}_\rho = \mathcal{H}_\nu \otimes \mathcal{H}_\sigma$ for some irreducible unitary K- and $\mathrm{SL}(2,\mathbb{R})$-representations $(\nu, \mathcal{H}_\nu)$ and $(\sigma, \mathcal{H}_\sigma)$, respectively. Then ρ is of positive energy w.r.t. α and $V(\rho) = \mathbb{C}_\lambda \otimes \mathcal{H}_\sigma$, where $\mathbb{C}_\lambda \subseteq \mathcal{H}_\nu$ is a lowest-weight subspace. Since $\mathcal{H}_\rho^\omega = \mathcal{H}_\nu \otimes \mathcal{H}_\sigma^\omega$, Theorem 6.4.13 implies that ρ is holomorphically induced from the $T\times\mathrm{SL}(2,\mathbb{R})$-representation on $\mathbb{C}_\lambda \otimes \mathcal{H}_\sigma$. The latter admits no strongly-entire vectors by Theorem 6.3.6, so ρ is not geometrically holomorphically induced from the $T \times \mathrm{SL}(2,\mathbb{R})$-representation on $\mathbb{C}_\lambda \otimes \mathcal{H}_\sigma$.

Example 6.8.3 (Positive energy representations of Heisenberg groups).
Let V be a real Fréchet space equipped with a non-degenerate continuous skew-symmetric bilinear form ω. Let $\beta : \mathbb{T} \to \mathrm{Sp}(V,\omega)$ be a homomorphism with smooth action $\mathbb{T} \times V \to V$. Then β is equicontinuous by [Nee13, Lem. A.3]. Define $Dv := \frac{d}{dt}\big|_{t=0} \beta_t v$ and consider the closed subspaces

$$\begin{aligned} V_0 &:= \ker D = \{\, v \in V \,:\, \beta_t v = v \quad \forall t \in \mathbb{R} \,\}, \\ V_{\mathrm{eff}} &:= \overline{\mathrm{Span}}\,\{\, \beta_t v - v \,:\, t \in \mathbb{R}, v \in V \,\}. \end{aligned} \tag{6.8.1}$$

As $\beta_t^*\omega = \omega$ for all $t \in \mathbb{R}$, notice that V_0 and V_{eff} are symplectic complements, so $(V,\omega) \cong (V_0,\omega_0) \oplus (V_{\mathrm{eff}},\omega_1)$, where ω_0 and ω_1 are the restrictions of ω to V_0 and V_{eff}, respectively. Assume that $(V_{\mathrm{eff}})_\mathbb{C}$ decomposes as $(V_{\mathrm{eff}})_\mathbb{C} \cong L_+ \oplus L_-$ into the positive (L_+) and negative (L_-) Fourier modes of the $\mathbb{T}$-action β. Let $\mathfrak{heis}(V,\omega) \rtimes_D \mathbb{R}\boldsymbol{d}$ be the Lie algebra of $\mathrm{Heis}(V,\omega) \rtimes_\beta \mathbb{T}$. By Theorem 6.7.17, we know for any unitary representation ρ of $\mathrm{Heis}(V,\omega) \rtimes_\beta \mathbb{T}$ which is analytically ground state at $\boldsymbol{d}$ that $\rho|_{\mathrm{Heis}(V,\omega)}$ is holomorphically induced by some analytic unitary representation of $\mathrm{Heis}(V_0,\omega_0)$.

Let us consider a concrete example. Assume that $\omega_1(v, Dv) > 0$ for every nonzero $v \in V_{\mathrm{eff}}$. Let $\mathcal{J}_1$ be the complex structure on V defined by $\mathcal{J}_1(v+w) := iv - iw$ for $v \in L_+$ and $w \in L_-$. Then $\mathcal{J}_1^*\omega_1 = \omega_1$ and $\omega_1(v, \mathcal{J}_1 v) > 0$ for every $v \in V_{\mathrm{eff}}$, so $\mathcal{J}_1$ defines a compatible positive polarization on V_{eff}. If $\mathcal{J}_0$ is a compatible positive polarization on V_0, then $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1$ defines one on V. As in Example 6.3.9, we equip the (now complex) vector space V with the inner product $\langle v, w\rangle_\mathcal{J} := \omega(v, \mathcal{J}w) + i\omega(v,w)$, making V into a complex pre-Hilbert space, on which β_t acts unitarily for any $t \in \mathbb{T}$. Let $\mathcal{H}$ be its Hilbert space completion, and let $\mathcal{H}_0$ and $\mathcal{H}_1$ be the closures in $\mathcal{H}$ of V_0 and V_{eff}, respectively. Consider the unitary representation $u : \mathbb{T} \to \mathrm{U}(\mathcal{H})$ of $\mathbb{T}$ on $\mathcal{H}$ defined by $t \mapsto u_t$, where u_t is the unitary operator on $\mathcal{H}$ extending β_t. Notice that as unitary $\mathbb{T}$-representations, we have

$$(V_{\text{eff}}, \langle -, - \rangle_{\mathcal{J}_1}) \cong (L_+, \langle -, - \rangle_{L_+}),$$

where $\langle v, w \rangle_{L_+} := 2i\omega(\overline{v}, w)$ for $v, w \in L_+$. The unitary $\mathbb{T}$-representation u on $\mathcal{H}$ is therefore of positive energy. Let $\mathcal{F}(\mathcal{H})$ be the Hilbert space completion of the symmetric algebra $\mathrm{S}^\bullet(\mathcal{H})$ w.r.t. the inner product (6.3.1), and let ρ be the b-strongly-entire unitary representation of $\mathrm{Heis}(V, \omega)$ on $\mathcal{F}(\mathcal{H})$ constructed in Example 6.3.9. Similarly, we write ρ_0 and ρ_1 for the representations of $\mathrm{Heis}(V_0, \omega_0)$ and $\mathrm{Heis}(V_{\text{eff}}, \omega_1)$ on $\mathcal{F}(\mathcal{H}_0)$ and $\mathcal{F}(\mathcal{H}_1)$, respectively. Letting $\mathbb{T}$ act on $\mathcal{F}(\mathcal{H})$ according to the second quantization $\mathcal{F}(u)$ of u, we obtain an extension of ρ to a smooth representation of $\mathrm{Heis}(V, \omega) \rtimes_\beta \mathbb{T}$ on $\mathcal{F}(\mathcal{H})$, which we denote again by ρ. Explicitly, we have $\rho(g, t) = \rho(g)\mathcal{F}(u_t)$ for $(g, t) \in \mathrm{Heis}(V, \omega) \rtimes_\beta \mathbb{T}$. This extension is ground state w.r.t. β. We have $\mathcal{F}(\mathcal{H}) \cong \mathcal{F}(\mathcal{H}_0) \otimes \mathcal{F}(\mathcal{H}_1)$ and

$$V(\rho) = \mathcal{F}(\mathcal{H}_0) \otimes \Omega_1 \subseteq \mathcal{F}(\mathcal{H}),$$

where $\Omega_1 \in \mathcal{F}(\mathcal{H}_1)$ is the vacuum vector. Theorem 6.7.20 implies that $\rho|_{\mathrm{Heis}(V,\omega)}$ is holomorphically induced from the representation ρ_0 of $\mathrm{Heis}(V_0, \omega_0)$ on $\mathcal{F}(\mathcal{H}_0)$. Moreover, we have $\mathcal{F}(\mathcal{H}_0)^\infty \otimes \Omega_1 \subseteq \mathcal{H}_\rho^\infty$. Indeed, the vacuum vector Ω_1 is smooth for $\mathrm{Heis}(V_{\text{eff}}, \omega_1)$, so if $\psi \in \mathcal{F}(\mathcal{H}_0)^\infty$ is a smooth vector for $\mathrm{Heis}(V_0, \omega_0)$ then

$$(z, v) \mapsto \rho(z, v)\psi = z\rho_0(v_0)\psi_0 \otimes \rho_1(v_1)\Omega_1$$

is a smooth map $\mathrm{Heis}(V, \omega) \to \mathcal{F}(\mathcal{H})$. So provided that V_0 is a Banach space, it follows using Theorem 6.5.25 and Example 6.5.18 that $\rho|_{\mathrm{Heis}(V,\omega)}$ is also geometrically holomorphically induced from ρ_0.

Example 6.8.4 (Metaplectic representation)**.**
We continue in the notation and setting of Example 6.8.3. Let $\mathcal{H}_\mathbb{R}$ be the real vector space underlying $\mathcal{H}$. The symplectic form ω on V extends to $\mathcal{H}_\mathbb{R}$ by setting $\omega(v, w) := \mathrm{Im}\langle v, w \rangle_\mathcal{J}$ for $v, w \in \mathcal{H}_\mathbb{R}$. Define

$$\mathcal{B}_{\text{res}}(\mathcal{H}_\mathbb{R}) := \{ A \in \mathcal{B}(\mathcal{H}_\mathbb{R}) : [\mathcal{J}, A] \in \mathcal{B}_2(\mathcal{H}) \},$$

whose elements are 'close' to being $\mathbb{C}$-linear. It is a real Banach algebra with norm $\|A\|_{\text{res}} := \|A\| + \|[\mathcal{J}, A]\|_2$, where $\mathcal{B}_2(\mathcal{H})$ denotes the space of Hilbert-Schmidt operators on $\mathcal{H}$. The restricted symplectic group is defined by

$$\mathrm{Sp}_{\text{res}}(\mathcal{H}_\mathbb{R}, \omega) := \mathrm{Sp}(\mathcal{H}_\mathbb{R}, \omega) \cap \mathcal{B}_{\text{res}}(\mathcal{H}_\mathbb{R}),$$

equipped with the subspace topology. Being an algebraic subgroup of $\mathcal{B}_{\text{res}}(\mathcal{H}_\mathbb{R})^\times$, we obtain using [Nee04, Prop. IV.14] that $\mathrm{Sp}_{\text{res}}(\mathcal{H}_\mathbb{R}, \omega)$ is a Banach-Lie group modeled on the Banach-Lie algebra

$$\mathfrak{sp}_{\text{res}}(\mathcal{H}_\mathbb{R}, \omega) := \mathfrak{sp}(\mathcal{H}_\mathbb{R}, \omega) \cap \mathcal{B}_{\text{res}}(\mathcal{H}_\mathbb{R}).$$

By Example 6.8.3, there is an irreducible projective unitary representation $\overline{\rho}$ of the Abelian Banach-Lie group $(\mathcal{H}_\mathbb{R}, +)$ on the symmetric Fock space $\mathcal{F}(\mathcal{H})$, which is well-known to extend to the semi-direct product $\mathrm{HSp}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega) := \mathcal{H}_\mathbb{R} \rtimes \mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ [Nee10b, Rem. 9.12]. We denote this extension again by $\overline{\rho}$. Let $\widetilde{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ and $\widetilde{\mathrm{HSp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ be the central $\mathbb{T}$-extensions of $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ and $\mathrm{HSp}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$, respectively, obtained by pulling back the central $\mathbb{T}$-extension $\mathrm{U}(\mathcal{F}(\mathcal{H})) \to \mathrm{PU}(\mathcal{F}(\mathcal{H}))$ along $\overline{\rho}$. Let

$$\rho : \widetilde{\mathrm{HSp}}_{\mathrm{res}}(H_\mathbb{R}, \omega) \to \mathrm{U}(\mathcal{F}(\mathcal{H}))$$

be the corresponding lift of $\overline{\rho}$. It is proven in [Nee10b, Thm. 9.3, Rem. 9.12] that both $\widetilde{\mathrm{HSp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ and $\widetilde{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ are again Banach-Lie groups, and that the (cyclic) vacuum vector $\Omega \in \mathcal{F}(\mathcal{H})$ is smooth for ρ, which implies that ρ is a smooth representation of $\widetilde{\mathrm{HSp}}_{\mathrm{res}}(H_\mathbb{R}, \omega)$. Identify $\mathcal{F}(\mathcal{H}_0)$ as a subspace of $\mathcal{F}(\mathcal{H})$ via $\mathcal{F}(\mathcal{H}_0) \hookrightarrow \mathcal{F}(\mathcal{H})$, $\psi_0 \mapsto \psi_0 \otimes \Omega_1$. We observe first that ρ is in fact analytic, and that $\mathcal{F}(\mathcal{H}_0)$ contains a dense set of $\widetilde{\mathrm{HSp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$-analytic vectors. To see this, notice using equation (35) and the subsequent remarks in the proof of [Nee10b, Thm. 9.3] that Ω is not just a smooth-, but even an analytic vector for the action of $\widetilde{\mathrm{Sp}}_{\mathrm{res}}(H_\mathbb{R}, \omega)$ on $\mathcal{F}(\mathcal{H})$. We furthermore know from Example 6.8.3 that Ω is analytic for $\mathrm{Heis}(\mathcal{H}_\mathbb{R}, \omega)$ (and even b-strongly-entire), so the function

$$\begin{aligned}
&\mathrm{Heis}(\mathcal{H}_\mathbb{R}, \omega) \times \widetilde{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega) \to \mathbb{C}, \\
&(v, A) \mapsto \langle \Omega, \rho(v)\rho(A)\Omega \rangle = \langle \rho(v)^{-1}\Omega, \rho(A)\Omega \rangle
\end{aligned}$$

is real-analytic. This implies using [Nee11, Thm. 5.2] that Ω is an analytic vector for the representation ρ of $\widetilde{\mathrm{HSp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ on $\mathcal{F}(\mathcal{H})$. As Ω is cyclic for the action ρ_0 of $\mathrm{Heis}((\mathcal{H}_0)_\mathbb{R}, \omega_0)$ on $\mathcal{F}(\mathcal{H}_0)$, it follows that the set of vectors in $\mathcal{F}(\mathcal{H}_0) \subseteq \mathcal{F}(\mathcal{H})$ that are analytic for the action of $\widetilde{\mathrm{HSp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ is dense in $\mathcal{F}(\mathcal{H}_0)$. It also follows that ρ is analytic, because Ω is cyclic for ρ.

Now, suppose that G is a connected regular BCH Fréchet-Lie group with Lie algebra $\mathfrak{g}$, and that the homomorphism $\alpha : \mathbb{T} \to \mathrm{Aut}(G)$ defines a smooth $\mathbb{T}$-action on G. Assume that α satisfies the assumptions made in Section 6.7. Let $H := (G^\alpha)_0$ be the connected subgroup of α-fixed points in G. Consider the $\mathbb{T}$-actions on $\mathrm{HSp}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$ and $\mathrm{U}(\mathcal{F}(\mathcal{H}))$ defined by $t \cdot (v, x) := (u_t v, u_t x u_t^{-1})$ and $t \cdot U := \mathcal{F}(u_t) U \mathcal{F}(u_t)^{-1}$ respectively, where $(v, x) \in \mathrm{HSp}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$, $U \in \mathrm{U}(\mathcal{F}(\mathcal{H}))$ and $t \in \mathbb{T}$. This also equips the $\mathbb{T}$-invariant subgroup

$$\widetilde{\mathrm{HSp}}_{\mathrm{res}}(H_\mathbb{R}, \omega) \subseteq \mathrm{HSp}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega) \times \mathrm{U}(\mathcal{F}(\mathcal{H}))$$

with a $\mathbb{T}$-action. Assume that

$$\eta : G \to \widetilde{\mathrm{HSp}}_{\mathrm{res}}(\mathcal{H}_\mathbb{R}, \omega)$$

is a continuous and $\mathbb{T}$-equivariant homomorphism. Then η is automatically analytic by [Nee06, Thm. IV.1.18]. Letting $\mathbb{T}$ act on $\mathcal{F}(\mathcal{H})$ by $t \mapsto \mathcal{F}(u_t)$, notice that $\rho \circ \eta$ extends to a smooth unitary representation of $G \rtimes_\alpha \mathbb{T}$ which is of positive energy at $(0, 1) \in \mathfrak{g} \rtimes \mathbb{R}$. Moreover, it follows from the preceding that $\mathcal{F}(\mathcal{H}_0)$ contains a dense set of vectors that are analytic for the G-action $\rho \circ \eta$ on $\mathcal{F}(\mathcal{H})$. Letting $\mathcal{K} \subseteq \mathcal{F}(\mathcal{H})$ be the closed $(G \rtimes_\alpha \mathbb{T})$-invariant linear subspace generated by $\mathcal{F}(\mathcal{H}_0)$, it follows that the unitary representation of $G \rtimes_\alpha \mathbb{T}$ on $\mathcal{K}$ is analytically ground-state at $(0, 1) \in \mathfrak{g} \rtimes \mathbb{R}$. According to Theorem 6.7.17, it further follows that the G-representation $\rho \circ \eta$ on $\mathcal{K}$ is holomorphically induced from the H-representation on $\mathcal{F}(\mathcal{H}_0)$.

Example 6.8.5 (Groups of jets).
Let K be a 1-connected compact simple Lie group with Lie algebra $\mathfrak{k}$. Let V be a finite-dimensional real vector space. We consider the Lie group $J^n_0(V, K)$ of n-jets at $0 \in V$ of smooth maps $V \to K$. Let $\gamma : \mathbb{R} \to \mathrm{GL}(V)$ be a continuous representation of $\mathbb{R}$ on V and let $\phi \in \mathfrak{k}$. Assume that the $\mathbb{R}$-action $\widetilde{\alpha}_t(f)(x) := e^{t\phi} f(\gamma_{-t}(x))e^{-t\phi}$ on $C^\infty_c(V, K)$ factors through $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. As γ fixes the origin, $\widetilde{\alpha}$ descends to a smooth $\mathbb{T}$-action on $J^n_0(V, K)$, denoted α. Let $D := \frac{d}{dt}\big|_{t=0} \dot{\alpha}_t$ be the corresponding derivation on the Lie algebra $J^n_0(V, \mathfrak{k})$. Let G be a central $\mathbb{T}$-extension of $J^n_0(V, K) \rtimes_\alpha \mathbb{T}$, and let $\boldsymbol{d} \in \mathfrak{g} := \mathrm{Lie}(G)$ cover $(0, 1) \in J^n_0(V, \mathfrak{k}) \rtimes_D \mathbb{R}$. As usual, we write $H := (G^\alpha)_0 \subseteq G$ for the connected Lie subgroup of α-fixed points in G, whose Lie algebra is $\mathfrak{h} = \ker D$. As $G \cong N \rtimes K$ for some nilpotent Lie group N, it follows from Proposition 6.3.5 that G is of type R. By Example 6.8.1, we thus obtain that any continuous unitary G-representation which is of positive energy w.r.t. α is holomorphically by some unitary H-representation. A classification of $\widehat{G}_{\mathrm{pos}(\alpha)}$ amounts to determining the holomorphically inducible elements in $\widehat{H}$. Unitary positive energy representations of groups of jets are studied in more detail in Chapter 4.

To make the preceding concrete, suppose that $V = \mathbb{R}^2$, $n = 2k$ for some $k \in \mathbb{N}$, that γ is the action of $\mathbb{T}$ on $\mathbb{R}^2$ by rotations and that $\phi = 0$. Then

$$\mathfrak{h} \cong \mathbb{R} \oplus_\omega (\mathbb{R}_k[x^2 + y^2] \otimes \mathfrak{k}),$$

where $\mathbb{R}_k[c]$ denotes the polynomial ring in c truncated at the k^{th} degree, and where ω is a 2-cocycle on the Lie algebra $\mathbb{R}_k[x^2+y^2]\otimes\mathfrak{k}$ (which in this case actually must be a coboundary). Every continuous unitary representation ρ of G that is of positive energy w.r.t. α is holomorphically induced from the H-representation on $V(\rho)$.

Example 6.8.6 (Gauge groups).
Let M be a compact manifold and let $P \to M$ be a principal bundle with structure group K, a simple compact Lie group with Lie algebra $\mathfrak{k}$. Consider the group of gauge transformations $\mathrm{Gau}(P) = \Gamma(M, \mathrm{Ad}(P))$, where $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} K$ is the adjoint bundle. This group is a regular BCH Fréchet-Lie group with Lie algebra $\mathfrak{gau}(P) = \Gamma(M, P \times_{\mathrm{Ad}} \mathfrak{k})$ [Nee06, Thm. IV.1.12]. Suppose that $\gamma : \mathbb{T} \to \mathrm{Aut}(P)$ is a smooth $\mathbb{T}$-action on P by automorphisms of P. Let $\eta : \mathbb{T} \to \mathrm{Aut}(\mathrm{Ad}(P))$ and $\overline{\gamma} : \mathbb{T} \to \mathrm{Diff}(M)$ denote the induced $\mathbb{T}$-actions on $\mathrm{Ad}(P)$ and M, respectively. Explicitly, η is given by $\eta_t([p, k]) := [\gamma_t(p), k]$ for $p \in P$, $k \in K$ and $t \in \mathbb{T}$. Then $\mathbb{T}$ acts smoothly on $\mathrm{Gau}(P)$ by $\alpha_t(s) := \eta_t \circ s \circ \overline{\gamma}_{-t}$ for $s \in \mathrm{Gau}(P)$ and $t \in \mathbb{T}$. The paper [JN21] studies projective unitary representations of $\mathrm{Gau}(P)$ which are smooth in the sense of admitting a dense set of smooth rays. According to [JN19, Cor. 4.5, Thm. 7.3], these correspond to smooth unitary representations of a central $\mathbb{T}$-extension of $\mathrm{Gau}(P)$. One of the main results of [JN19] is the full classification of

smooth projective unitary of the identity component $\mathrm{Gau}(P)_0$ which are of positive energy w.r.t α, provided that M has no $\mathbb{T}$-fixed points for $\overline{\gamma}$ [JN21, Thm. 8.10]. Let us consider a central $\mathbb{T}$-extension G of the connected component $\mathrm{Gau}(P)_0$ of the identity. Suppose that α lifts to a smooth $\mathbb{T}$-action $\widetilde{\alpha}$ on G. In view of [JN21, Prop. 8.6], a consequence of the classification [JN21, Thm. 8.10] is that every smooth unitary representation ρ of G which is of positive energy w.r.t. $\widetilde{\alpha}$ is holomorphically induced from the corresponding representation of $H := (G^{\widetilde{\alpha}})_0$ on $V(\rho)$. The more general case where M is allowed to have $\mathbb{T}$-fixed points is not yet fully understood. One approach would be to determine, in specific cases, the irreducible unitary factor representations of H that are holomorphically inducible to G, as an intermediate step towards the classification of the possibly larger class of all p.e. factor representations of G.

Example 6.8.7 (Unitary groups of CIA's)**.**
An interesting class of examples to which the theory of Section 6.7 applies can be obtained using so-called continuous inverse algebras (CIAs). Suppose that $\mathcal{A}$ is a unital complex Fréchet algebra that is a CIA, meaning that its group of units $\mathcal{A}^\times$ is open in $\mathcal{A}$ and that the inversion $a \mapsto a^{-1}$ is continuous $\mathcal{A} \to \mathcal{A}$. Let us suppose further that $\mathcal{A}$ carries a continuous conjugate-linear algebra involution $\mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$. In this setting, $\mathcal{A}^\times$ is a complex BCH Fréchet-Lie group modeled on $\mathcal{A}$ [Glö02a, Thm. 5.6]. Assume that the Lie group $\mathcal{A}^\times$ is moreover regular. A sufficient condition for this is provided in [GN12]. The unitary subgroup

$$\mathrm{U}(\mathcal{A}) := \left\{ a \in \mathcal{A}^\times \,:\, a^* = a^{-1} \right\}$$

is a real Lie subgroup of $\mathcal{A}^\times$, so that it is an embedded submanifold. It is modeled on the Lie algebra

$$\mathfrak{u}(\mathcal{A}) := \{ a \in \mathcal{A} \,:\, a^* = -a \},$$

equipped with the commutator bracket. To see this, let $U \subseteq \mathcal{A}$ be a 0-neighborhood s.t. $\exp_{\mathcal{A}}$ maps U diffeomorphically onto its image in $\mathcal{A}^\times$. We may assume that $U = -U$ and that $U^* = U$, by shrinking U if necessary. By [Glö02a, Cor. 4.11] we know that $\exp_{\mathcal{A}}(a) = \sum_{n=0}^\infty \frac{1}{n!}a^n$ for all $a \in \mathcal{A}$. Using that both $a \mapsto a^{-1}$ and $a \mapsto a^*$ are continuous, it follows that $\exp_{\mathcal{A}}(a)^* = \exp(a^*)$ and $\exp_{\mathcal{A}}(a)^{-1} = \exp_{\mathcal{A}}(-a)$ for all $a \in U$. This implies that $\exp_{\mathcal{A}}(U \cap \mathfrak{u}(\mathcal{A})) = \exp_{\mathcal{A}}(U) \cap \mathrm{U}(\mathcal{A})$. As $\mathrm{U}(\mathcal{A})$ is a closed subgroup of the locally exponential Lie group $\mathcal{A}^\times$, it follows from [Nee06, Thm. IV.3.3] that $\mathrm{U}(\mathcal{A}) \subseteq \mathcal{A}^\times$ is a locally exponential Lie subgroup. It is therefore a regular BCH Fréchet-Lie group. Notice further that $\mathfrak{u}(\mathcal{A})_\mathbb{C} = (\mathcal{A}, [-,-])$ as complex Lie algebras.

Suppose that $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ is a homomorphism that has a smooth action $\mathbb{R} \times \mathcal{A} \to \mathcal{A}$ and that has polynomial growth. Assume further that the splitting condition

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+$$

is satisfied. Setting $G := \mathrm{U}(\mathcal{A})_0$ and $H := \mathrm{U}(\mathcal{A}_0)_0 = (G^\alpha)_0$, all assumptions of both Section 6.4.2 and Section 6.7 are satisfied.

Typically, such triples $(\mathcal{A}, \mathbb{R}, \alpha)$ can be obtained as the set of smooth points of a C^*-dynamical system $(\mathcal{B}, G, \gamma)$, where $\mathcal{B}$ is a unital C^*-algebra, G is a Banach-Lie group and $\gamma : G \to \mathrm{Aut}(\mathcal{B})$ is a strongly continuous G-action on $\mathcal{B}$ by automorphisms. By [Nee10a, Def. 4.1, Thm. 6.2], we know in this setting that the set of smooth points $\mathcal{A} := \mathcal{B}^\infty$ is a G-invariant and $*$-closed subalgebra which naturally carries a Fréchet topology. Moreover, $\mathcal{A}$ is a CIA and the G-action $\gamma : G \times \mathcal{A} \to \mathcal{A}$ is smooth w.r.t. this topology. If G is finite-dimensional, then this topology coincides with the one obtained from the embedding $\mathcal{A} \hookrightarrow C^\infty(G, \mathcal{B})$, where $C^\infty(G, \mathcal{B})$ carries the smooth compact-open topology [Nee10a, Prop. 4.6]. If $\iota : \mathbb{R} \hookrightarrow G$ is a one-parameter subgroup of G for which the corresponding $\mathbb{R}$-action $\alpha := \gamma \circ \iota$ on $\mathcal{A}$ has polynomial growth and satisfies the splitting condition $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+$, then the triple $(\mathcal{A}, \mathbb{R}, \alpha)$ satisfies all the above assumptions.

As a concrete example, let $\mathcal{A}_\theta := C^\infty_\theta(\mathbb{T}^2)$ be the smooth non-commutative 2-torus with parameter $\theta \in [0, \frac{1}{2}]$:

$$\mathcal{A}_\theta := \left\{ \sum_{n,m \in \mathbb{Z}} a_{n,m} u^n v^m \ : \ \sum_{n,m \in \mathbb{Z}} (1 + |n| + |m|)^k |a_{n,m}| < \infty \text{ for all } k \in \mathbb{N} \right\},$$

where u and v are unitary operators satisfying $uv = e^{i2\pi\theta} vu$, and where $\mathcal{A}_\theta$ is equipped with the seminorms $p_k(a) := \sum_{n,m \in \mathbb{Z}} (1 + |n| + |m|)^k |a_{n,m}|$ for $k \in \mathbb{N}_{\geq 0}$. This is a unital Fréchet CIA carrying a continuous involution, obtained as the smooth points of the natural $\mathbb{T}^2$-action on the 'continuous' non-commutative 2-torus $C_\theta(\mathbb{T}^2)$ with parameter θ. It is moreover shown in [GN] that the Lie group $\mathcal{A}_\theta^\times$ is regular. Consider the smooth and equicontinuous $\mathbb{T}$-action α on $C^\infty_\theta(\mathbb{T}^2)$ satisfying $\alpha_z(u^n v^m) := z^m u^n v^m$ for all $n, m \in \mathbb{Z}$ and $z \in \mathbb{T}$. Define $G := \mathrm{U}(\mathcal{A}_\theta)_0$. Then for any unitary representation ρ of $G \rtimes_\alpha \mathbb{T}$ that is analytically ground state w.r.t. α, we obtain from Theorem 6.7.17 that $\rho|_G$ is holomorphically induced from the corresponding unitary representation of the connected Abelian group $H := (\mathrm{U}(\mathcal{A}_\theta)^\alpha)_0 \cong C^\infty(\mathbb{T}, \mathbb{T})_0$ on $\mathcal{H}_\rho(0)$. In particular, if $\rho(G)''$ is a factor, then as H is Abelian, we obtain with Corollary 6.7.10 that $\rho|_G$ is holomorphically induced from a character of H. By Corollary 6.7.10 this implies that $\rho|_G$ is irreducible.

6.9 Appendix

6.9.1 Representations on reproducing kernel Hilbert spaces

In the following we summarize relevant properties concerning reproducing kernel Hilbert spaces in the context of unitary group representations. Let $\mathcal{H}$ and V be Hilbert spaces and let G be a group. We write V^G or $\mathrm{Map}(G, V)$ for the space of functions $G \to V$ and $V^{(G)}$ for the space of finitely-supported functions $G \to V$.

Definition 6.9.1. Suppose that $\mathcal{H} \subseteq V^G$. Then $\mathcal{H}$ is said to *have continuous evaluation maps* if for every $x \in G$ the linear map $\mathcal{E}_x : \mathcal{H} \to V, \psi \mapsto \psi(x)$ is bounded.

Definition 6.9.2. A function $Q : G \times G \to \mathcal{B}(V)$ is said to be *positive definite* if

$$\|v\|_Q^2 := \sum_{x,y \in \mathrm{supp}(v)} \langle v_x, Q(x,y)v_y \rangle_V \geq 0, \qquad \forall v \in V^{(G)}.$$

Theorem 6.9.3 ([Nee00, Thm. I.1.4])**.**
Let $Q : G \times G \to \mathcal{B}(V)$ be a function. The following assertions are equivalent:

1. *Q is positive definite*
2. *There exists a Hilbert space $\mathcal{H}_Q \subseteq V^G$ with continuous point-evaluations $\mathcal{E}_x : \mathcal{H}_Q \to V$ s.t. $Q(x,y) = \mathcal{E}_x \mathcal{E}_y^*$ for all $(x,y) \in G \times G$.*

In this case $\mathcal{H}_Q$ is unique up to unitary equivalence and $\{ \mathcal{E}_x^* v : x \in G, v \in V \}$ is total in $\mathcal{H}_Q$.

Definition 6.9.4. A function $Q : G \times G \to \mathcal{B}(V)$ is said to be a *reproducing kernel* for the Hilbert space $\mathcal{H}$ if Q is positive definite and $\mathcal{H} \cong \mathcal{H}_Q$.

Proposition 6.9.5. *Let G be a topological group and $H \subseteq G$ be a closed subgroup. Let (σ, V_σ) be a strongly continuous unitary H-representation. Let $Q \in C(G \times G, \mathcal{B}(V_\sigma))^{H \times H}$, so $Q(xh_1, yh_2) = \sigma(h_1)^{-1}Q(x,y)\sigma(h_2)$ for all $x_1, x_2 \in G$ and $h_1, h_2 \in H$. Assume that Q is positive definite.*

1. *The left-regular action of G on $V_\sigma^{(G)}$ induces a unitary G-action π on $\mathcal{H}_Q$ if and only if Q is G-invariant. In this case, there is a function $F : G \to \mathcal{B}(V_\sigma)$ such that $Q(x,y) = F(x^{-1}y)$.*
2. *Assume that Q is G-invariant. There exists a G-equivariant linear map $\mathcal{H}_Q \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ with continuous point-evaluations $\mathcal{E}_x$ for $x \in G$. These satisfy the equivariance condition $\mathcal{E}_x \pi(g) = \mathcal{E}_{g^{-1}x}$ for all $x, y \in G$.*
3. *Assume that $Q : G \times G \to \mathcal{B}(V_\sigma)$ is G-invariant and strongly continuous. Then the unitary G-representation $\mathcal{H}_Q$ is strongly continuous.*
4. *Suppose that $(\rho, \mathcal{H}_\rho)$ is a unitary G-representation and that there is a G-equivariant injective linear map $\Phi : \mathcal{H}_\rho \hookrightarrow \mathrm{Map}(G, V_\sigma)^H$ having continuous point evaluations $\mathcal{E}_x := \mathrm{ev}_x \circ \Phi$ for $x \in G$. Then the corresponding kernel Q is G-invariant, and $\mathcal{H}_\rho \cong \mathcal{H}_Q$ as unitary G-representations.*

Proof. Let l_g denote the left G-action on itself by left-multiplication. Recall that $\mathcal{H}_Q = \overline{V_\sigma^{(G)}/\mathcal{N}_Q}^{\langle -,- \rangle_Q}$, where $\mathcal{N}_Q := \left\{ f \in V_\sigma^{(G)} : \|f\|_Q = 0 \right\}$. For any $x \in G$ we have a map $\delta_x : V_\sigma \hookrightarrow V_\sigma^{(G)}$ defined by considering elements of V_σ as functions on G with support $\{x\}$. Let $q_x : V_\sigma \to \mathcal{H}_Q, v \mapsto [\delta_x(v)]$ be its composition with the quotient map $V_\sigma^{(G)} \to \mathcal{H}_Q$. We then have $\mathcal{E}_x = q_x^*$ (cf. [Nee00, Thm. I.1.4] for more details). The embedding $\mathcal{H}_Q \hookrightarrow V_\sigma^G$ is defined by $f \mapsto f_\psi$, where $f_\psi(x) = \mathcal{E}_x(\psi)$.

1. For $g \in G$ and $f \in V_\sigma^{(G)}$, we write $g.f := f \circ l_g^{-1}$ for the left-regular action of G on $V_\sigma^{(G)}$. Let $x, y \in G$. Take $v, w \in V_\sigma$. Then $g.\delta_x(v) = \delta_{gx}(v)$

and $g.\delta_y(w) = \delta_{gy}(w)$ have support on $\{gx\}$ and $\{gy\}$, respectively. Thus $\langle g.\delta_x(v), g.\delta_y(w)\rangle_Q = \langle v, Q(gx, gy)w\rangle$ whereas $\langle q_x(v), q_y(w)\rangle_Q = \langle v, Q(x, y)w\rangle$. The first assertion follows. If Q is G-invariant, then $F(x) := Q(e, x)$ satisfies $F(x^{-1}y) = Q(x, y)$.

2. Let $x \in G$ and $h \in H$. From $Q(xh, y) = \sigma(h)^{-1}Q(x, y)$ it follows that $\mathcal{E}_{xh}\mathcal{E}_y^*v = \sigma(h)^{-1}\mathcal{E}_x\mathcal{E}_y^*v$ for any $y \in G$ and $v \in V_\sigma$. As $\{\mathcal{E}_y^*v : y \in G, v \in V_\sigma\}$ is total in $\mathcal{H}_Q$ by Theorem 6.9.3, it follows that $\mathcal{E}_{xh} = \sigma(h)^{-1}\mathcal{E}_x$. Thus $f_\psi \in \mathrm{Map}(G, V_\sigma)^H$ for any $\psi \in \mathcal{H}_Q$. We show that $\psi \mapsto f_\psi$ is G-equivariant. We have $\pi(g)\mathcal{E}_x^*v = \pi(g)q_x(v) = q_{gx}(v) = \mathcal{E}_{gx}^*(v)$ for every $x, g \in G$ and $v \in V_\sigma$. Hence $\pi(g)\mathcal{E}_x^* = \mathcal{E}_{gx}^*$ and $\mathcal{E}_x\pi(g) = \mathcal{E}_{g^{-1}x}$ for every $x, g \in G$. Thus for $\psi \in \mathcal{H}_Q$ we obtain $f_\psi(g^{-1}x) = \mathcal{E}_{g^{-1}x}\psi = \mathcal{E}_x\pi(g)\psi = f_{\pi(g)\psi}(x)$, so $\psi \mapsto f_\psi$ is G-equivariant.

3. As G acts unitarily on $\mathcal{H}_Q$, it suffices to show that $G \to \mathbb{C}\ g \mapsto \langle\psi, \pi(g)\psi\rangle_Q$ is continuous for any ψ in some total subspace. Consider $\psi = \mathcal{E}_x^*v$ for arbitrary $x \in G$ and $v \in V_\sigma$. Such vectors form a total set in $\mathcal{H}_Q$ by Theorem 6.9.3. For $g \in G$, we have

$$\langle\psi, \pi(g)\psi\rangle_Q = \langle v, \mathcal{E}_x\pi(g)\mathcal{E}_x^*v\rangle_V = \langle v, \mathcal{E}_x\mathcal{E}_{gx}^*v\rangle_V = \langle v, Q(x, gx)v\rangle_V. \quad (6.9.1)$$

As $Q : G \times G \to \mathcal{B}(V_\sigma)$ is continuous w.r.t. the strong topology, the map $g \mapsto \langle\psi, \pi(g)\psi\rangle_Q$ is continuous.

4. As Φ is G-equivariant, we have $\mathcal{E}_x\rho(g) = \mathcal{E}_{g^{-1}x}$ for every $x, g \in G$. As ρ is unitary this implies that the corresponding kernel $Q(x, y) := \mathcal{E}_x\mathcal{E}_y^*$ is G-invariant. This kernel is also positive definite by Theorem 6.9.3, so $\mathcal{H}_Q$ is a unitary G-representation by the first item. We already know from Theorem 6.9.3 that $\mathcal{H}_Q \cong \mathcal{H}_\rho$ as Hilbert spaces. The unitary isomorphism $U : \mathcal{H}_Q \to \mathcal{H}_\rho$ is on the dense subspace $V_\sigma^{(G)}/\mathcal{N}_Q$ given by $Uq(f) := \sum_{x\in\mathrm{supp}(f)} \mathcal{E}_x^*f(x)$, where $q : V_\sigma^{(G)} \to \mathcal{H}_Q$ denotes the quotient map. Write π for the unitary G-action on $\mathcal{H}_Q$. Using $q_x = \mathcal{E}_x^*$, $q_{gx} = \pi(g)q_x$ and $\rho(g)\mathcal{E}_x^* = \mathcal{E}_{gx}^*$, we obtain that

$$U\pi(g)q_x(v) = Uq_{gx}(v) = \mathcal{E}_{gx}^*v = \rho(g)\mathcal{E}_x^*v = \rho(g)Uq_x(v), \qquad \forall v \in V_\sigma. \quad \square$$

Curriculum Vitae

Milan Niestijl was born on May 26, 1995, in Delft, the Netherlands. He completed his high school education at Libanon Lyceum in Rotterdam in 2013 and earned a bachelor's degree in applied physics from the TU Delft in 2016. He earned a master's degree in applied mathematics at the TU Delft in 2019, with a thesis titled "*Positive energy representations of gauge groups*" under the supervision of Dr. Bas Janssens. He began his PhD research in October 2019 at the TU Delft with the same supervisor, and with promotor Prof. dr. J.M.A.M. van Neerven.

List of publications

1. **M. Niestijl**, *Generalized positive energy representations of groups of jets.* Doc. Math, 28(3):709–763, 2023.

2. **M. Niestijl**, *Holomorphic induction beyond the norm-continuous setting, with applications to positive energy representations.* In preparation, available at arXiv:2301.05129 (2023).

3. **B. Janssens and M. Niestijl**, *Central extensions and generalized positive energy representations of the group of compactly supported diffeomorphisms.* In preparation.

Bibliography

[AAL04] B. Agrebaoui, F. Ammar, and P. Lecomte. On the cohomology of the spaces of differential operators acting on skewsymmetric tensor fields or on forms, as modules of the Lie algebra of vector fields. *Differ. Geom. Appl.*, 20(2):241–249, 2004.

[ADGV16] S. Albeverio, B. Driver, M. Gordina, and A. Vershik. Equivalence of the brownian and energy representations. *J. of Math. Sc.*, 219:612–630, 12 2016.

[AE18] F. Alessio and G. Esposito. On the structure and applications of the Bondi-Metzner-Sachs group. *Int. J. Geom. Methods Mod. Phys.*, 15(2):1830002, 61, 2018.

[AHK78] S. Albeverio and R. Høegh-Krohn. The energy representation of Sobolev-Lie groups. *Comp. Math.*, 36(1):37–51, 1978.

[AHKM+93] S. Albeverio, R. Høegh-Krohn, J. Marion, D. Testard, and B. Torrésani. *Noncommutative Distributions*, volume 175. Marcel Dekker, Inc., New York, 1993.

[AHKT81] S. Albeverio, R. Høegh-Krohn, and D. Testard. Irreducibility and reducibility for the energy representation of the group of mappings of a Riemannian manifold into a compact semisimple Lie group. *J. Funct. Anal.*, 41(3):378–396, 1981.

[AK98] V. Arnold and B. Khesin. *Topological Methods in Hydrodynamics*, volume 125. Springer-Verlag, New York, 1998.

[AM66] L. Auslander and C.C Moore. *Unitary Representations of Solvable Lie Groups*. Mem. Amer. Math. Soc., 1966.

[AM22] A. Alekseev and E. Meinrenken. On the coadjoint Virasoro action, 2022. Available at arXiv:2211.06216v1.

[Ara74] H. Araki. On the equivalence of the KMS condition and the variational principle for quantum lattice systems. *Commun. Math. Phys.*, 38:1—-10, 1974.

[Arn88] V. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*, volume 250. Springer-Verlag, New York, second edition, 1988.

[Arv74] W. Arveson. On groups of automorphisms of operator algebras. *J. Func. Anal.*, 15:217–243, 1974.

[AS77a] H. Araki and G. Sewell. KMS conditions and local thermodynamical stability of quantum lattice systems. *Comm. Math. Phys.*, 52(2):103–109, 1977.

[AS77b] H. Araki and G. Sewell. KMS conditions and local thermodynamical stability of quantum lattice systems. II. *Comm. Math. Phys.*, 55(1):53–61, 1977.

[AS81] A. Ashtekar and M. Streubel. Symplectic geometry of radiative modes and conserved quantities at null infinity. *Proc. Roy. Soc. London Ser. A*, 376(1767):585–607, 1981.

[Ash15] A. Ashtekar. Geometry and physics of null infinity. In *Surveys in Differential Geometry*, volume 20 of *Surv. Differ. Geom.*, pages 99–122. Int. Press, Boston, MA, 2015.

[AT94] S. Albeverio and B. Torrésani. Some remarks on representations of jet groups and gauge groups. *J. Math. Phys.*, 35(9):4897–4908, 1994.

[Ban97] A. Banyaga. *The Structure of Classical Diffeomorphism Groups*, volume 400 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.

[Bas64] A. Bastiani. Applications différentiables et variétés différentiables de dimension infinie. *J. Analyse Math.*, 13:1–114, 1964.

[BDFS00] D. Buchholz, O. Dreyer, M. Florig, and S. Summers. Geometric modular action and spacetime symmetry groups. *Rev. Math. Phys.*, 12(4):475–560, 2000.

[BEK80] O. Bratteli, G.A. Elliott, and A. Kishimoto. The temperature state space of a C^*-dynamical system. I. *Yokohama Math. J.*, 28(1-2):125–167, 1980.

[BEK86] O. Bratteli, G.A. Elliott, and A. Kishimoto. The temperature state space of a C^*-dynamical system. II. *Ann. of Math. (2)*, 123(2):205–263, 1986.

[Bel05] D. Beltiţă. Integrability of analytic almost complex structures on Banach manifolds. *Ann. Global Anal. Geom.*, 28(1):59–73, 2005.

[BGL93] R. Brunetti, D. Guido, and R. Longo. Modular structure and duality in conformal quantum field theory. *Comm. Math. Phys.*, 156(1):201–219, 1993.

[BGL02] R. Brunetti, D. Guido, and R. Longo. Modular localization and Wigner particles. *Rev. Math. Phys.*, 14(7-8):759–785, 2002.

[BGN20] D. Beltita, H. Grundling, and K.-H. Neeb. Covariant representations for possibly singular actions on C^*-algebras. *Diss. Math.*, 549:1–94, 2020.

[BMT88] D. Buchholz, G. Mack, and I. Todorov. The current algebra on the circle as a germ of local field theories. *Nuclear Phys. B Proc. Suppl.*, 5B:20–56, 1988.

[BN08a] D. Beltiţă and K.-H. Neeb. A nonsmooth continuous unitary representation of a Banach-Lie group. *J. Lie Theory*, 18(4):933–936, 2008.

[BN08b] Y. Billig and K.-H. Neeb. On the cohomology of vector fields on parallelizable manifolds. *Ann. Inst. Fourier*, 58(6):1937–1982, 2008.

[Boc45] S. Bochner. Compact groups of differentiable transformations. *Ann. of Math.*, 46:372–381, 1945.

[Bor66] H.-J. Borchers. Energy and momentum as observables in quantum field theory. *Comm. Math. Phys.*, 2:49–54, 1966.

[Bor87] H.-J. Borchers. On the interplay between spectrum condition and locality in quantum field theory. In *Operator Algebras and Mathematical Physics*, volume 62, pages 143–152. Amer. Math. Soc., 1987.

[Bor92] H.-J. Borchers. The CPT-theorem in two-dimensional theories of local observables. *Comm. Math. Phys.*, 143(2):315–332, 1992.

[Bor00] H. J. Borchers. On revolutionizing quantum field theory with Tomita's modular theory. *J. Math. Phys.*, 41(6):3604–3673, 2000.

[Bot73] R. Bott. Notes on Gelfand-Fuks cohomology and characteristic classes. In *Proceedings of the Eleventh Annual Holliday Symposium*, 1973. Notes by M. Mostow and J. Perchik.

[Boy80] R. P. Boyer. Representation theory of the Hilbert-Lie group $\mathrm{U}(H)_2$. *Duke Math. J.*, 47(2):325–344, 1980.

[Boy88] R. P. Boyer. Representation theory of $\mathrm{U}_1(H)$ in the symmetric tensors. *J. Funct. Anal.*, 78(1):13–23, 1988.

[Boy93] R. P. Boyer. Representation theory of infinite-dimensional unitary groups. In *Representation Theory of Groups and Algebras*, volume 145 of *Contemp. Math.*, pages 381–391. Amer. Math. Soc., Providence, RI, 1993.

[Boy03] R. P. Boyer. Factor representations of diffeomorphism groups. *Stud. Math.*, 156(2):105–120, 2003.

[BR87] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1.* Springer-Verlag, New York, second edition, 1987. C^*- and W^*-algebras, symmetry groups, decomposition of states.

[BR97] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2.* Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.

[Bre97] G. E. Bredon. *Sheaf Theory*, volume 170. Springer-Verlag, New York, second edition, 1997.

[BS71a] J. Bochnak and J. Siciak. Analytic functions in topological vector spaces. *Studia Math.*, 39:77–112, 1971.

[BS71b] J. Bochnak and J. Siciak. Polynomials and multilinear mappings in topological vector spaces. *Studia Math.*, 39:59–76, 1971.

[BS77] R. Bott and G. Segal. The cohomology of the vector fields on a manifold. *Topology*, 16(4):285–298, 1977.

[BvdBM62] H. Bondi, M. van der Burg, and A. Metzner. Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems. *Proc. Roy. Soc. London Ser. A*, 269:21–52, 1962.

[BW48] V. Bargmann and E.P. Wigner. Group theoretical discussion of relativistic wave equations. *Proc. Nat. Acad. Sci. U.S.A.*, 34:211–223, 1948.

[BY99] H. Borchers and J. Yngvason. Modular groups of quantum fields in thermal states. *J. Math. Phys.*, 40(2):601–624, 1999.

[Car84] A. L. Carey. Projective representations of the Hilbert Lie group $\mathcal{U}(H)_2$ via quasifree states on the CAR algebra. *J. Funct. Anal.*, 55(3):277–296, 1984.

[CDVIT21] S. Carpi, S. Del Vecchio, S. Iovieno, and Y. Tanimoto. Positive energy representations of Sobolev diffeomorphism groups of the circle. *Anal. Math. Phys.*, 11(1):36, 2021.

[CG90] L.J. Corwin and F.P. Greenleaf. *Representations of Nilpotent Lie Groups and Their Applications. Part I*, volume 18. Cambridge University Press, Cambridge, 1990.

[CG97] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry.* Birkhäuser Boston, Inc., Boston, MA, 1997.

[CH87] A. L. Carey and K. C. Hannabuss. Temperature states on loop groups, theta functions and the Luttinger model. *J. Funct. Anal.*, 75(1):128–160, 1987.

[CH92] A. L. Carey and K. C. Hannabuss. Temperature states on gauge groups. *Ann. Inst. H. Poincaré Phys. Théor.*, 57(3):219–257, 1992.

[Con73] A. Connes. Une classification des facteurs de type III. *Ann. Sci. École Norm. Sup. (4)*, 6:133–252, 1973.

[CR87] A. L. Carey and S. Ruijsenaars. On fermion gauge groups, current algebras and Kac-Moody algebras. *Acta Appl. Math.*, 10(1):1–86, 1987.

[Dix77] J. Dixmier. *C^*-Algebras.* North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[DJNV21] T. Diez, B. Janssens, K.-H. Neeb, and C. Vizman. Central extensions of Lie groups preserving a differential form. *Int. Math. Res. Not.*, 2021(5):3794–3821, 2021.

[DK00] J.J. Duistermaat and J.A.C. Kolk. *Lie Groups.* Springer-Verlag, Berlin, 2000.

[DP99] I. Dimitrov and I. Penkov. Weight modules of direct limit Lie algebras. *Int. Math. Res. Not.*, 1999(5):223–249, 1999.

[dR84] G. de Rham. *Differentiable Manifolds*, volume 266 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1984. Forms, Currents, Harmonic Forms.

[Dul04] H. Dulac. Recherches sur les points singuliers des équations différentielles. *Journ. École Polyt. Série 2*, 9:1–25, 1904.

[FH91] W. Fulton and J. Harris. *Representation Theory*, volume 129. Springer-Verlag, New York, 1991.

[FM04] R.L. Fernandes and P. Monnier. Linearization of Poisson brackets. *Lett. Math. Phys.*, 69:89–114, 2004.

[FNÓ23] J. Frahm, K.-H. Neeb, and G. Ólafsson. Nets of standard subspaces on non-compactly causal symmetric spaces, 2023. Available at arXiv:2303.10065.

[Fuk86] D. B. Fuks. *Cohomology of infinite-dimensional Lie algebras.* Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. Translated from the Russian by A. B. Sosinskiĭ.

[FV77] M. Fannes and A. Verbeure. Correlation inequalities and equilibrium states II. *Comm. Math. Phys.*, 57(2):165–171, 1977.

[Går47] L. Gårding. Note on continuous representations of Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 33:331–332, 1947.

[GF68] I. M. Gelfand and V.B. Fuks. The cohomologies of the Lie algebra of the vector fields in a circle. *Funct Anal Its Appl*, 2:342–343, 1968.

[GF69] I. M. Gelfand and V.B. Fuks. Cohomologies of Lie algebra of tangential vector fields of a smooth manifold. *Funct Anal Its Appl*, 3:194–210, 1969.

[GF70a] I. M. Gel'fand and D. B. Fuks. Cohomologies of Lie algebra of vector fields with nontrivial coefficients. *Funct. Anal. Appl.*, 4:181–192, 1970.

[GF70b] I. M. Gelfand and V.B. Fuks. Cohomologies of Lie algebra of tangential vector fields of a smooth manifold II. *Funct Anal Its Appl*, 4:110–116, 1970.

[GF70c] I. M. Gelfand and V.B. Fuks. Cohomology of the Lie algebra of formal vector fields. *Izv. Akad. Nauk SSSR Ser. Mat*, 4(2):327–342, 1970.

[GG68] I.M. Gelfand and M.I. Graev. Representations of quaternion groups over locally compact fields and function fields. *Funkcional. Anal. i Priložen*, 2(1):20–35, 1968.

[GGV77] I.M. Gelfand, M.I. Graev, and A.M. Veršik. Representations of the group of smooth mappings of a manifold X into a compact Lie group. *Comp. Math.*, 35(3):299–334, 1977.

[GL95] D. Guido and R. Longo. An algebraic spin and statistics theorem. *Comm. Math. Phys.*, 172(3):517–533, 1995.

[Glö02a] H. Glöckner. Algebras whose groups of units are Lie groups. *Studia Math.*, 153(2):147–177, 2002.

[Glö02b] H. Glöckner. Infinite-dimensional Lie groups without completeness restrictions. In *Geometry and Analysis on Finite- and Infinite-dimensional Lie Groups*, volume 55, pages 43–59. Polish Acad. Sci. Inst. Math., 2002.

[Glö05] H. Glöckner. Fundamentals of direct limit Lie theory. *Compos. Math.*, 141(6):1551–1577, 2005.

[GMS80] G. Goldin, R. Menikoff, and D. Sharp. Particle statistics from induced representations of a local current group. *J. Math. Phys.*, 21(4):650–664, 1980.

[GMS81] G. Goldin, R. Menikoff, and D. Sharp. Representations of a local current algebra in nonsimply connected space and the Aharonov-Bohm effect. *J. Math. Phys.*, 22(8):1664–1668, 1981.

[GMS83] G. Goldin, R. Menikoff, and D. Sharp. Diffeomorphism groups, gauge groups, and quantum theory. *Phys. Rev. Lett.*, 51(25):2246–2249, 1983.

[GN] H. Glöckner and K.-H. Neeb. Infinite Dimensional Lie Groups. Book in preparation.

[GN12] H. Glöckner and K.-H. Neeb. When unit groups of continuous inverse algebras are regular Lie groups. *Stud. Math.*, 211(2):95–109, 2012.

[Gol04] G. Goldin. Lectures on diffeomorphism groups in quantum physics. In *Contemporary Problems in Mathematical Physics*, pages 3–93. World Sci. Publ., Hackensack, NJ, 2004.

[Goo69] R.W. Goodman. Analytic and entire vectors for representations of Lie groups. *Trans. Amer. Math. Soc.*, 143:55–76, 1969.

[GS68] V.W. Guillemin and S. Sternberg. Remarks on a paper of Hermann. *Trans. Amer. Math. Soc.*, 130:110–116, 1968.

[Gui73] V. W. Guillemin. Cohomology of vector fields on a manifold. *Advances in Math.*, 10:192–220, 1973.

[GW84] R. Goodman and N.R. Wallach. Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle. *J. Reine Angew. Math.*, 347:69–133, 1984.

[GW85] R. Goodman and N. R. Wallach. Projective unitary positive-energy representations of $Diff(S^1)$. *J. Funct. Anal.*, 63:299–321, 1985.

[Haa75] U. Haagerup. The standard form of von Neumann algebras. *Math. Scand.*, 37(2):271–283, 1975.

[Haa92] R. Haag. *Local Quantum Physics*. Springer-Verlag, Berlin, 1992.

[Hae76] A. Haefliger. Sur la cohomologie de l'algèbre de Lie des champs de vecteurs. *Ann. Sci. Éc. Norm. Supér.*, 9:503–532, 1976.

[Hat02] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[Her68] R. Hermann. The formal linearization of a semisimple Lie algebra of vector fields about a singular point. *Trans. Amer. Math. Soc.*, 130:105–109, 1968.

[HHW67] R. Haag, N. Hugenholtz, and M. Winnink. On the equilibrium states in quantum statistical mechanics. *Comm. Math. Phys.*, 5:215–236, 1967.

[HK64] R. Haag and D. Kastler. An algebraic approach to quantum field theory. *J. Mathematical Phys.*, 5:848–861, 1964.

[HKTP74] R. Haag, D. Kastler, and B. Trych-Pohlmeyer. Stability and equilibrium states. *Comm. Math. Phys.*, 38:173–193, 1974.

[HM07] K.H. Hofmann and S.A. Morris. *The Lie Theory of Connected Pro-Lie Groups*, volume 2. European Mathematical Society (EMS), Zürich, 2007.

[HN09] K.H. Hofmann and K.-H. Neeb. Pro-Lie groups which are infinite-dimensional Lie groups. *Math. Proc. Cambridge Philos. Soc.*, 146(2):351–378, 2009.

[Hör03] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis.

[Ism76] R.S. Ismagilov. Unitary representations of the group $C_0^\infty(X, G)$, $G = \mathrm{SU}_2$. *Mat. Sb.*, 100(142)(1):117–131, 166, 1976.

[Ism96] R.S. Ismagilov. *Representations of Infinite-Dimensional Groups*, volume 152 of *Translations of mathematical monographs.* American Mathematical Society, 1996.

[Jac79] N. Jacobson. *Lie Algebras.* Dover Publications, Inc., New York, 1979.

[Jan16] B. Janssens. Infinitesimally natural principal bundles. *J. Geom. Mech.*, 8(2):199–220, 2016.

[Jen73] J.W. Jenkins. Growth of connected locally compact groups. *J. Func. Anal.*, 12:113–127, 1973.

[JN] B. Janssens and K.-H. Neeb. Positive energy representations of gauge groups II: Conformal completion of space-time. To appear.

[JN17] B. Janssens and K.-H. Neeb. Covariant central extensions of gauge Lie algebras. In *Stochastic Geometric Mechanics*, volume 202, pages 101–114. Springer, Cham, 2017.

[JN19] B. Janssens and K.-H. Neeb. Projective unitary representations of infinite-dimensional Lie groups. *Kyoto J. Math.*, 59(2):293–341, 2019.

[JN21] B. Janssens and K.-H. Neeb. Positive energy representations of gauge groups I: Localization, 2021. In preparation. Available at arXiv:2108.03501.

[JN23] B. Janssens and M. Niestijl. Central extensions and generalized positive energy representations of the group of compactly supported diffeomoprhisms, 2023. In preparation.

[JP04] D. Jialing and D. Pickrell. Coadjoint orbits for the central extension of $\mathrm{Diff}^+(S^1)$ and their representatives. *Acta Math. Sci., Ser. B, Engl. Ed.*, 24(2):185–205, 2004.

[JRV23] B. Janssens, L. Ryvkin, and C. Vizman. Universal central extension of the Lie algebra of exact divergence-free vector fields, 2023. In preparation.

[JV16] B. Janssens and C. Vizman. Universal central extension of the Lie algebra of Hamiltonian vector fields. *Int. Math. Res. Not.*, 2016(16):4996–5047, 2016.

[JV19] B. Janssens and C. Vizman. Integrability of central extensions of the Poisson Lie algebra via prequantization. *J. Symplectic Geom.*, 16(5):1351–1375, 2019.

[Kac90] V.G. Kac. *Infinite-dimensional Lie Algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[Kir73] A. A. Kirillov. Representations of the infinite-dimensional unitary group. *Dokl. Akad. Nauk. SSSR*, 212:288–290, 1973.

[Kir76] A.A. Kirillov. *Elements of the Theory of Representations*. Springer-Verlag, Berlin-New York, 1976.

[Kir81] A. A. Kirillov. Orbits of the group of diffeomorphisms of a circle and local Lie superalgebras. *Funct. Anal. Appl.*, 15:135–137, 1981.

[Kir04] A. A. Kirillov. *Lectures on the orbit method*, volume 64 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2004.

[KM97] A. Kriegl and P.W. Michor. *The Convenient Setting of Global Analysis*, volume 53. AMS, Providence, 1997.

[Kna96] A.W. Knapp. *Lie Groups Beyond an Introduction*, volume 140. Birkhäuser Boston, Inc., Boston, MA, 1996.

[Kna01] A.W. Knapp. *Representation Theory of Semisimple Groups*. Princeton University Press, Princeton, NJ, 2001.

[KR87] V.G. Kac and A.K. Raina. *Bombay Lectures on Highest Weight Representations of Infinite-dimensional Lie Algebras*, volume 2. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987.

[Kub57] R. Kubo. Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems. *J. Phys. Soc. Japan*, 12:570–586, 1957.

[KW01] R. Kähler and H. Wiesbrock. Modular theory and the reconstruction of four-dimensional quantum field theories. *J. Math. Phys.*, 42(1):74–86, 2001.

[KW09] B. Khesin and R. Wendt. *The Geometry of Infinite-Dimensional Groups*, volume 51. Springer-Verlag, Berlin, 2009.

[KYN57] R. Kubo, M. Yokota, and S. Nakajima. Statistical-mechanical theory of irreversible processes. II. Response to thermal disturbance. *J. Phys. Soc. Japan*, 12:1203–1211, 1957.

[Lee13] J. M. Lee. *Introduction to Smooth Manifolds*, volume 218. Springer, New York, second edition, 2013.

[Lis95] W. Lisiecki. Coherent state representations. A survey. *Rep. Math. Phys.*, 35(2-3):327–358, 1995.

[LL15] G. Lechner and R. Longo. Localization in nets of standard spaces. *Comm. Math. Phys.*, 336(1):27–61, 2015.

[LM75] M. Lüscher and G. Mack. Global conformal invariance in quantum field theory. *Comm. Math. Phys.*, 41:203–234, 1975.

[LO00] P. B. A. Lecomte and V. Yu. Ovsienko. Cohomology of the vector fields Lie algebra and modules of differential operators on a smooth manifold. *Compos. Math.*, 124(1):95–110, 2000.

[Lon08] R. Longo. Lectures on conformal nets, November 2008.

[Los98] M. V. Losik. Diagonal cohomology of the Lie algebra of vector fields. *Differential Geom. Appl.*, 8(1):71–86, 1998.

[LP76] V. Lazutkin and T. Pankratova. Normal forms and versal deformations for Hill's equation. *Funct. Anal. Appl.*, 9:306–311, 1976.

[Mai02] P. Maier. Central extensions of topological current algebras. In *Geometry and Analysis on Finite- and Infinite-dimensional Lie Groups*, volume 55, pages 61–76. Polish Acad. Sci. Inst. Math., Warsaw, 2002.

[Mal67] B. Malgrange. *Ideals of Differentiable Functions*, volume 3. Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1967.

[Man04] M. Manetti. Lectures on deformations of complex manifolds. *Rend. Mat. Appl.*, 24(1):1–183, 2004.

[MC73] P. J. McCarthy and M. Crampin. Representations of the Bondi-Metzner-Sachs group. III. Poincaré spin multiplicities and irreducibility. *Proc. Roy. Soc. London Ser. A*, 335:301–311, 1973.

[McC72] P. J. McCarthy. Representations of the Bondi-Metzner-Sachs group. I. Determination of the representations. *Proc. Roy. Soc. London Ser. A*, 330:517–535, 1972.

[McC73] P. J. McCarthy. Representations of the Bondi-Metzner-Sachs group. II. Properties and classification of the representations. *Proc. Roy. Soc. London Ser. A*, 333:317–336, 1973.

[McC78] P. J. McCarthy. Lifting of projective representations of the Bondi-Metzner-Sachs group. *Proc. Roy. Soc. London Ser. A*, 358(1693):141–171, 1978.

[Mia22] L. Miaskiwskyi. A local-to-global analysis of gelfand-fuks cohomology. *Pre-print*, 2022. Available at arXiv:2202.09666.

[Mil84] J. Milnor. Remarks on infinite-dimensional Lie groups. In *Relativity, Groups and Topology*, pages 1007–1057. North-Holland, Amsterdam, 1984.

[MN16] T. Marquis and K.-H. Neeb. Positive energy representations for locally finite split Lie algebras. *Int. Math. Res. Not.*, 2016(21):6689–6712, 2016.

[MN21] V. Morinelli and K.-H. Neeb. Covariant homogeneous nets of standard subspaces. *Comm. Math. Phys.*, 386(1):305–358, 2021.

[MNO23] V. Morinelli, K.-H. Neeb, and G. Ólafsson. From Euler elements and 3-gradings to non-compactly causal symmetric spaces. *J. Lie Theory*, 33(1):377–432, 2023.

[MNS10] C. Müller, K.-H. Neeb, and H. Seppänen. Borel-Weil theory for root graded Banach-Lie groups. *Int. Math. Res. Not. IMRN*, 2010(5):783–823, 2010.

[Mor18] V. Morinelli. The Bisognano-Wichmann property on nets of standard subspaces, some sufficient conditions. *Ann. Henri Poincaré*, 19(3):937–958, 2018.

[MS59] P. Martin and J. Schwinger. Theory of many-particle systems. I. *Phys. Rev. (2)*, 115:1342–1373, 1959.

[Mun00] J.R. Munkres. *Topology.* Prentice Hall, Inc., 2000.

[Mun01] J. Mund. The Bisognano-Wichmann theorem for massive theories. *Ann. Henri Poincaré*, 2(5):907–926, 2001.

[Mur90] G.J. Murphy. *C^*-algebras and Operator Theory.* Academic Press, Inc., Boston, MA, 1990.

[Nat94] L. Natarajan. Unitary highest weight-modules of inductive limit Lie algebras and groups. *J. Algebra*, 167(1):9–28, 1994.

[Nee98a] K.-H. Neeb. Holomorphic highest weight representations of infinite-dimensional complex classical groups. *J. Reine Angew. Math.*, 497:171–222, 1998.

[Nee98b] K.-H. Neeb. Some open problems in representation theory related to complex geometry. In *Positivity in Lie Theory: Open Problems*, volume 26, pages 195–220. de Gruyter, Berlin, 1998.

[Nee00] K.-H. Neeb. *Holomorphy and Convexity in Lie Theory*, volume 28. Walter de Gruyter & Co., Berlin, 2000.

[Nee01a] K.-H. Neeb. Borel-Weil theory for loop groups. In *Infinite Dimensional Kähler Manifolds*, volume 31, pages 179–229. Birkhäuser Verlag, 2001.

[Nee01b] K.-H. Neeb. Borel-Weil theory for loop groups. In *Infinite Dimensional Kähler Manifolds*, volume 31, pages 179–229. Birkhäuser, Basel, 2001.

[Nee02] K.-H. Neeb. Central extensions of infinite-dimensional Lie groups. *Ann. Inst. Fourier*, 52(5):1365–1442, 2002.

[Nee04] K.-H. Neeb. Infinite-dimensional groups and their representations. In *Lie Theory*, volume 228, pages 213–328. Birkhäuser Boston, 2004.

[Nee06] K.-H. Neeb. Towards a Lie theory of locally convex groups. *Jpn. J. Math.*, 1(2):291–468, 2006.

[Nee10a] K.-H. Neeb. On differentiable vectors for representations of infinite dimensional Lie groups. *J. Funct. Anal.*, 259(11):2814–2855, 2010.

[Nee10b] K.-H. Neeb. Semibounded representations and invariant cones in infinite dimensional Lie algebras. *Confl. Math.*, 2(1):37–134, 2010.

[Nee10c] K.-H. Neeb. Unitary highest weight modules of locally affine Lie algebras. In *Quantum Affine Algebras, Extended Affine Lie Algebras, and Their Applications*, volume 506 of *Contemp. Math.*, pages 227–262. Amer. Math. Soc., Providence, RI, 2010.

[Nee11] K.-H. Neeb. On analytic vectors for unitary representations of infinite dimensional Lie groups. *Ann. Inst. Fourier*, 61(5):1839–1874, 2011.

[Nee12] K.-H. Neeb. Semibounded representations of Hermitian Lie groups. *Travaux mathématiques.*, 21(5):29–109, 2012.

[Nee13] K.-H. Neeb. Holomorphic realization of unitary representations of Banach-Lie groups. In *Lie Groups: Structure, Actions, and Representations*, volume 306, pages 185–223. Birkhäuser/Springer, New York, 2013.

[Nee14a] K.-H. Neeb. Semibounded unitary representations of double extensions of Hilbert-loop groups. *Ann. Inst. Fourier*, 64(5):1823–1892, 2014.

[Nee14b] K.-H. Neeb. Unitary representations of unitary groups. In *Developments and Retrospectives in Lie Theory*, volume 37, pages 197–243. Springer, Cham, 2014.

[Nee17] K.-H. Neeb. Bounded and semibounded representations of infinite dimensional Lie groups. In *Representation Theory—Current Trends and Perspectives*, pages 541–563. Eur. Math. Soc., 2017.

[Nie23a] M. Niestijl. Generalized positive energy representations of groups of jets. *Doc. Math*, 28(3):709–763, 2023.

[Nie23b] M. Niestijl. Holomorphic induction beyond the norm-continuous setting, with applications to positive energy representations, 2023. In preparation. Available at arXiv:2301.05129.

[NØ98] K.-H. Neeb and B. Ørsted. Unitary highest weight representations in Hilbert spaces of holomorphic functions on infinite-dimensional domains. *J. Funct. Anal.*, 156(1):263–300, 1998.

[NO17] K.-H. Neeb and G. Ólafsson. Antiunitary representations and modular theory. In *50th Seminar "Sophus Lie"*, volume 113, pages 291–362. Polish Acad. Sci. Inst. Math., Warsaw, 2017.

[NO21] K.-H. Neeb and G. Ólafsson. Nets of standard subspaces on Lie groups. *Adv. Math.*, 384:Paper No. 107715, 69, 2021.

[NÓ22] K.-H. Neeb and G. Ólafsson. Algebraic quantum field theory and causal symmetric spaces, 2022. Available at arXiv:2210.01299.

[NR22] K.-H. Neeb and F.G. Russo. Ground state representations of topological groups. *Math. Ann.*, pages 1–60, 2022.

[NRCW01] L. Natarajan, E. Rodríguez-Carrington, and J. Wolf. The Bott-Borel-Weil theorem for direct limit groups. *Trans. Amer. Math. Soc.*, 353(11):4583–4622, 2001.

[NS11] K.-H. Neeb and H. Seppänen. Borel-Weil theory for groups over commutative Banach algebras. *J. Reine Angew. Math.*, 655:165–187, 2011.

[NS15] K.-H. Neeb and H. Salmasian. Classification of positive energy representations of the Virasoro group. *Int. Math. Res. Not.*, Volume 2015(18):8620–8656, 2015.

[NSZ15] K.-H. Neeb, H. Salmasian, and C. Zellner. On an invariance property of the space of smooth vectors. *Kyoto J. Math.*, 55(3):501–515, 2015.

[NU62] E. Newman and T. Unti. Behavior of asymptotically flat empty spaces. *J. Math. Phys.*, 3:891–901, 1962.

[NV03] K.-H. Neeb and C. Vizman. Flux homomorphisms and principal bundles over infinite dimensional manifolds. *Monatsh. Math.*, 139(4):309–333, 2003.

[Ol'78] G. I. Ol'shanskiĭ. Unitary representations of the infinite-dimensional classical groups $\mathrm{U}(p, \infty)$, $\mathrm{SO}_0(p, \infty)$, $\mathrm{Sp}(p, \infty)$, and of the corresponding motion groups. *Funktsional. Anal. i Prilozhen.*, 12(3):32–44, 96, 1978.

[Ol'81] G. I. Ol'shanskiĭ. Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series. *Funktsional. Anal. i Prilozhen.*, 15(4):53–66, 1981.

[Ol'84] G. I. Ol'shanskiĭ. Infinite-dimensional classical groups of finite R-rank: description of representations and asymptotic theory. *Funktsional. Anal. i Prilozhen.*, 18(1):28–42, 1984.

[Ott95] J. T. Ottesen. *Infinite-Dimensional Groups and Algebras in Quantum Physics*, volume 27. Springer-Verlag, Berlin, 1995.

[Pan81] S.M. Paneitz. Invariant convex cones and causality in semisimple Lie algebras and groups. *J. Func. Anal.*, 43(3):313–359, 1981.

[Pee60] J. Peetre. Réctification à l'article "Une caractérisation abstraite des opérateurs différentiels". *Math. Scand.*, 8:116–120, 1960.

[Pen64] R. Penrose. Conformal treatment of infinity. In *Relativité, Groupes et Topologie*, pages 565–584. Gordon and Breach, New York, 1964.

[Pen74] R. Penney. Entire vectors and holomorphic extension of representations. *Trans. Amer. Math. Soc.*, 198:107–121, 1974.

[Per86] A. Perelomov. *Generalized Coherent States and Their Applications.* Springer-Verlag, Berlin, 1986.

[Pia77a] A. Piard. Representations of the Bondi-Metzner-Sachs group with the Hilbert topology. *Rep. Mathematical Phys.*, 11(2):279–283, 1977.

[Pia77b] A. Piard. Unitary representations of semi-direct product groups with infinite dimensional abelian normal subgroup. *Rep. Mathematical Phys.*, 11(2):259–278, 1977.

[Pic88] D. Pickrell. The separable representations of $\mathrm{U}(H)$. *Proc. Amer. Math. Soc.*, 102(2):416–420, 1988.

[Pic90] D. Pickrell. Separable representations for automorphism groups of infinite symmetric spaces. *J. Funct. Anal.*, 90(1):1–26, 1990.

[Poi79] H. Poincaré. *Sur les Propriétés des Fonctions Définies par les Équations aux Différences Partielles.* Gauthier-Villars, 1879.

[PR94] R. J. Plymen and P. L. Robinson. *Spinors in Hilbert space*, volume 114 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1994.

[PS76] K.R. Parthasarathy and K. Schmidt. A new method for constructing factorisable representations for current groups and current algebras. *Comm. Math. Phys.*, 50(2):167–175, 1976.

[PS86] A. Pressley and G. Segal. *Loop Groups.* Oxford University Press, 1986.

[PS22] D. Prinz and A. Schmeding. Lie theory for asymptotic symmetries in general relativity: the BMS group. *Class. Quantum Grav.*, 39(6):065004, 2022.

[PW78] W. Pusz and S. L. Woronowicz. Passive states and KMS states for general quantum systems. *Comm. Math. Phys.*, 58(3):273–290, 1978.

[Rud91] W. Rudin. *Functional Analysis.* McGraw-Hill, Inc., New York, second edition, 1991.

[Ruz20] R. Ruzziconi. *On the Various Extensions of the BMS Group.* PhD thesis, Université Libre de Bruxelles and International Solvay Institutes, 2020.

[Sac62] R. K. Sachs. Gravitational waves in general relativity. VIII. Waves in asymptotically flat space-time. *Proc. Roy. Soc. London Ser. A*, 270:103–126, 1962.

[Sch04] R. Schmid. Infinite dimensional Lie groups with applications to mathematical physics. *J. Geom. Symmetry Phys.*, 1:54–120, 2004.

[Seg57] I. E. Segal. The structure of a class of representations of the unitary group on a Hilbert space. *Proc. Amer. Math. Soc.*, 8:197–203, 1957.

[Seg81] G. Segal. Unitary representations of some infinite-dimensional groups. *Comm. Math. Phys.*, 80(3):301–342, 1981.

[Shn76] S. Shnider. Continuous cohomology for compactly supported vectorfields on R^n. *Trans. Amer. Math. Soc.*, 217:364–377, 1976.

[Sim23] T. Simon. Factorial type I KMS states of Lie groups. *Pre-print*, 2023. Available at arXiv:2301.03444v1.

[SP54] M. E. Shanks and L. E. Pursell. The Lie algebra of a smooth manifold. *Proc. Amer. Math. Soc.*, 5:468–472, 1954.

[SS65] D. Shale and W. F. Stinespring. Spinor representations of infinite orthogonal groups. *J. Math. Mech.*, 14:315–322, 1965.

[SV75] S. Strătilă and D. Voiculescu. *Representations of AF-Algebras and of the Group $U(\infty)$*. Lecture Notes in Mathematics, Vol. 486. Springer-Verlag, Berlin-New York, 1975.

[SV78] S. Strătilă and D. Voiculescu. On a class of KMS states for the unitary group $\mathrm{U}(\infty)$. *Math. Ann.*, 235(1):87–110, 1978.

[SW64] R.F. Streater and A.S. Wightman. *PCT, Spin and Statistics, and all That.* Benjamin, New York, 1964.

[Tak02] M. Takesaki. *Theory of Operator Algebras I*, volume 124. Springer-Verlag, Berlin, 2002.

[Tak03a] M. Takesaki. *Theory of Operator Algebras II*, volume 125. Springer-Verlag, Berlin, 2003.

[Tak03b] M. Takesaki. *Theory of Operator Algebras III*, volume 127. Springer-Verlag, Berlin, 2003.

[Tan11] Y. Tanimoto. Ground state representations of loop algebras. *Ann. Henri Poincaré*, 12(4):805–827, 2011.

[TL99] V. Toledano Laredo. Positive energy representations of the loop groups of non-simply connected Lie groups. *Comm. Math. Phys.*, 207(2):307–339, 1999.

[Tre67] F. Treves. *Topological Vector Spaces, Distributions and Kernels.* Academic Press, New York-London, 1967.

[Tsu81] T. Tsujishita. *Continuous Cohomology of the Lie Algebra of Vector Fields*, volume 253 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society, 1981.

[TW71] J.A. Tirao and J.A. Wolf. Homogeneous holomorphic vector bundles. *Indiana Univ. Math. J.*, 20:15–31, 1970/71.

[vEK64] W. T. van Est and Th. J. Korthagen. Non-enlargible Lie algebras. *Indag. Math.*, 26:15–31, 1964.

[Viz09] C. Vizman. Central extensions of coverings of symplectomorphism groups. *J. Lie Theory*, 19(2):237–249, 2009.

[Wal84] R. Wald. *General Relativity.* University of Chicago Press, Chicago, IL, 1984.

[Wal87] N.R. Wallach. On the irreducibility and inequivalence of unitary representations of gauge groups. *Comp. Math.*, 64(1):3–29, 1987.

[War72] G. Warner. *Harmonic Analysis on Semi-simple Lie Groups I.* Springer-Verlag, 1972.

[Was98] A. Wassermann. Operator algebras and conformal field theory. III. Fusion of positive energy representations of LSU(N) using bounded operators. *Invent. Math.*, 133(3):467–538, 1998.

[Wel80] R. O. Wells. *Differential Analysis on Complex Manifolds*, volume 65. Springer-Verlag, New York-Berlin, second edition, 1980.

[Wig39] E. Wigner. On unitary representations of the inhomogeneous Lorentz group. *Ann. of Math.*, 40(1):149–204, 1939.

[Wil07] D. Williams. *Crossed products of C^*-algebras*, volume 134. American Mathematical Society, Providence, RI, 2007.

[Wit88] E. Witten. Coadjoint orbits of the Virasoro group. *Commun. Math. Phys.*, 114(1):1–53, 1988.

[Wol05] J. A. Wolf. Principal series representations of direct limit groups. *Compos. Math.*, 141(6):1504–1530, 2005.

[Wol13] J. A. Wolf. Principal series representations of infinite dimensional Lie groups, II: construction of induced representations. In *Geometric Analysis and Integral Geometry*, volume 598 of *Contemp. Math.*, pages 257–280. Amer. Math. Soc., Providence, RI, 2013.

[Wol14] J. A. Wolf. Principal series representations of infinite-dimensional Lie groups, I: Minimal parabolic subgroups. In *Symmetry: Representation Theory and its Applications*, volume 257 of *Progr. Math.*, pages 519–538. Birkhäuser/Springer, New York, 2014.