STRESSES DUE TO GRAVITY IN AN ELASTIC HALF-PLANE WITH NOTCHES OR MOUNDS

PROEFSCHRIFT TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETEN-SCHAPPEN AAN DE TECHNISCHE HOGESCHOOL TE DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS DR. IR. C.J.D.M. VERHAGEN, HOOGLERAAR IN DE AFDELING DER TECH-NISCHE NATUURKUNDE, VOOR EEN COM-MISSIE UIT DE SENAAT TE VERDEDIGEN OP DINSDAG 1 JULI 1969 TE 14 UUR

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ARNOLD VERRUIJT

civiel-ingenieur

geboren te Alphen aan den Rijn

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Chapter 1

INTRODUCTION

The study presented in this thesis is concerned with the calculation of the stresses in a homogeneous, isotropic, linear elastic material, initially occupying a lower half space, and in equilibrium under its own weight. The investigations refer to the state of stress in the material when the originally plane upper boundary is modified. The modification may consist of taking out some parts of the material (notch problem), or it may consist of adding some more of the same material (mound problem). Due to the addition or removal of this material the stresses will change, and it is this change of stress that is to be calculated. Restriction will be made to regions of which the boundary remains free of external stresses. In the case of addition of material the resulting stresses may depend upon the mode of construction. The mathematical problem to be considered in this thesis corresponds to the hypothetical case of a non-stressed half space with the mound already present, in which stresses are generated by gradually letting the body force of gravity increase from zero to its real value.

Restriction will be made to modifications of the boundary which have the same form in all planes parallel to some given vertical plane, for instance a long straight notch or a long straight mound. The problem will then be of the plane strain type, and it is sufficient to consider the stresses in a plane. In general the problem is that of a heavy elastic material occupying the part of the entre palne below a certain line. This line must have the property that near infinity the region approximates a lower half plane.

A general method for the solution of certain problems of this type has been described by MUSKHELISHVILI (1953). This method is based upon the conformal transformation of the region occupied by the body onto a half plane by means of a rational function (i.e. a quotient of two polynomials). This method has been used to solve several problems, such as problems for regions having a parabolic boundary (MUSKHELISHVILI, 1953; PARIA, 1957; NEUBER, 1962; VERMA, 1966), and problems for a half plane with a particular type of smooth notch (WARREN & MICHELL, 1965; KUNERT, 1966). The solution of these problems requires a considerable amount of analytical and numerical work to be done. The amount of work strongly depends upon the character of the mapping function. Moreover, when for instance the general solution for mapping functions having as their denominator a second degree polynomial has been found, this is of little value value for the solution of the problem involving a mapping function having a third degree polynomial as its denominator. Not only the numerical work to be performed is different, but also a substantial part of the analytical work. Finally, in case that the conformal transformation mapping the region under consideration onto half plane is known, but is not a rational function, the approximation of the mapping function by a rational function is not a simple matter, for which no general procedure seems to exist. In this respect it might be mentioned here that the mapping function cannot be a rational function in case that the contour possesses corner points, and this is a case of considerable interest. It is probably for the resaon just mentioned that Muskhelishvili himself has called for more effective methods for problems involving corner points (MUSKHELISHVILI, 1965, p. 75).

In this thesis a method of solution will be presented in which the confomrla transformation onto the interior of a unit circle is used. In his treatise on complex variable methods MUSKHELISHVILI (1953) used such a transformation for problems involving a finite region, or an infinite region with a single hole. By making use of the analytic character of the mapping function in the interior of the unit circle it is possible to obtain its Taylor series expansion around the center of the circle, even in case of complicated mapping functions, such as arising when the boundary of the region considered is a contour with several contour points. By taking into account only a finite number of terms of the series, an approximation is obtained. This approximation results in the corners of the contour to be rounded oof. The elasticity problem for the region mapped onto the unit circle by the approximate mapping function can then be solved with the aid of complex variable techniques.

When a semi-infinite region is mapped onto the interior of a unit circle the mapping function will possess a pole on the unit circle. It will be shown in this thesis that this complication can be incorporated into the existing complex variable methods by writing the mapping function as the sum of a singular and a regular part. The analytical work to be performed is somewhat more than for the cases of a finite region or an infinite region with a single hole. However, this analytical work needs to be performed only once for all problems of the class considered. Only the numerical work is different for different problems.

It may be mentioned that the class of problems considered here includes the case of a single notch in a half plane. For several types of notches some other rather effective methods have been developed (LING, 1947; SEIKA, 1960; MITCHELL, 1965; BOWIE, 1966). These solutions all refer to a notched half plane under tension, but they could be adapted to the case of stresses due to gravity. In fact, a method similar to the one used by LING (1947), in which use is made of Fourier integrals, will be used in section 7.3 of this thesis as a verification of the results obtained by the complex variable method.

The class of problems treated in this thesis occurs in applied soil mechanics, and the question arises whether the solutions obtained in this thesis area applicable to soils. Although these solutions might indeed be regarded as giving, in first approximation, an impression of the change of stress in a soil body when making a long straight excavation, or when constructing a long straight embankment, it is to be noted that the mechanical properties of natural soils are much more complicated than those of a homogeneous, isotropic, linear elastic material. Usually soils are inhomogeneous, sometimes also anisotropic, and the relationship between stresses and strains is, at least partly, non-linear and inelastic. Moreover, the behaviour of soils under the influence of external loadings is sometimes further complicated by the presence of water in the pores. This pore water retards volumetric deformations of the soil. These complications prohibit an anlytical treatment of the general problem. Certain aspects, such as anisotropy or the retardation due to the presence of pore water, might admit a theoretical investigation, but such a course will not be pursued in this thesis.

Thus, in conclusion, since the solutions presented here refer to a mechanically much simpler material than soils, these solutions should be handled with great caution and under great reservation when applied in soil mechanics practice.

Chapter 2

DESCRIPTION OF THE PROBLEM

Let there be given a soil body which at a certain instant of time occupies a lower half space. Cartesian coordinates x, y, z are introduced such that the upper surface of the soil body is represented by the plane y = 0. The y-axis is directed upwards. The soil mass is assumed to be in equilibrium under the influence of its own weight. Therefore the components of stress should satisfy the following equations of equilibrium (see e.g. TIMOSHENKO & GOODIER, 1951),

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0, \qquad \tau_{xy} = \tau_{yx},
\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} - \rho g = 0, \qquad \tau_{yz} = \tau_{zy},
\frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0, \qquad \tau_{zx} = \tau_{xz},$$
(2.1)

where ρ is the density of the material (which is assumed to be a constant) and g is the acceleration of gravity. The state of stress in the soil body is also required to be such that the surface y = 0 is free of stress, i.e.

$$y = 0$$
 : $\tau_{yx} = \tau_{yy} = \tau_{yz} = 0.$ (2.2)

The problem defined by the equations of equilibrium (2.1) and the boundary condition (2.2) does not possess a unique solution. The state of stress depends upon the mechanical properties of the soil material, and also upon the geological history, i.e. upon the way in which the soil body has been formed. It is especially the influence of the, usually unknown, history upon the state of stress that prohibits a calculation of the stresses in such a soil body. The determination of these stresses is more s problem of experimental measurement than theoretical solution.

A possible stress state is

$$\begin{aligned}
\tau_{xx} &= K_0 \rho g y, & \tau_{xy} = \tau_{yx} = 0, \\
\tau_{yy} &= \rho g y, & \tau_{yz} = \tau_{zy} = 0, \\
\tau_{zz} &= K_0 \rho g y, & \tau_{zx} = \tau_{xz} = 0,
\end{aligned} \tag{2.3}$$

where K_0 is a constant. The state of stress defined by (2.3) has the property that the vertical direction is everywhere a principal direction and furthermore this stress state is invariant for translations and rotations in the horizontal plane. This state of stress can be expected to be acting in the hypothetical case of a soil that has been deposited uniformly over a large horizontal area. The coefficient K_0 in (2.3) is called the coefficient of neutral earth pressure.



Figure 2.1: Half plane with modified boundary.

In this thesis restriction will be made to soil bodies in which the initial state of stress is given by (2.3). The coefficient of neutral earth pressure K_0 will be considered a given constant.

The stress state (2.3) will change as a result of a modification of the upper boundary of the soil mass, as for instance occurs when part of the soil is taken away by excavation or erosion, or when an embankment is built upon the soil, thereby using the same material. The incremental stresses will be caluclated by means of the theory of linear elasticity. Therefore the initial state, with stresses (2.3), is considered as a reference state, and it is assumed that the relationship between incremental stresses and icremental strains can be described with sufficient accuracy by Hooke's law. The material is furthermore assumed to be isotropic as regards its mechanical properties, and the two elastic coefficients which describe the response of an isotropic linear elastic material are assumed to be constant throughout the body. This latter assumption expresses homogeneity of the soil with regard to incremental deformations. Finally, it will be assumed that the incremental deformations are small enough to ensure the applicability of the first order (infinitesimal strain) theory.

Restriction will be made to such excavations, embankments, etc., of which the form is independent of one of the horizontal directions, e.g. the z-coordinate. Then the deformation will be independent of z, and it is sufficient to consider the deformation in an arbitrary plane perpendicular to the z-axis. The twodimensional problem in this x, y-plane is of the plane strain type.

Figure 2.1 shows the boundary of the region after its modification. AS the figure suggests the modification of the boundary occurs only in the finite part of the soil body. At infinity the soil suface is not affected, and this ensures that the total area of the excavation or embankment is finite.

Chapter 3

MATHEMATICAL FORMULATION OF THE PROBLEM

In this chapter the problem of elastic equilibrium in a semi-infinite plane, deforming under plane strain conditions, will be formulated. The region occupied by the elastic body in the plane z = x + iy is denoted by R, and its boundary by C. The boundary C is assumed to be an open line, extending towards infinity in both directions (fig. 2.1). A positive direction is defined on C such that the region R lies to the left of C when a particle moves along C in the positive direction. The considerations in this thesis will be restricted to regions that approximate a half plane at infinity. Therefore the line C is assumed to have the property that its two ends approach the real axis asymptotically. As a boundary condition it is considered that line C is free from external stresses. The problem is to determine the stresses in R due to the action of gravity.

Stresses

0

In the case of plane strain the relevant equations of equilibrium are

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0,
\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} - \rho g = 0,
\tau_{xy} = \tau_{yx},$$
(3.1)

in which ρg is the constant specific weight of the material. The components of total stress $(\tau_{xx}, \tau_{yy}, \tau_{xy}, \tau_{yx})$ are now decomposed into "initial stresses", to be denoted by $\tau_{xx}^0, \tau_{yy}^0, \tau_{xy}^0, \tau_{yx}^0$, and "incremental stresses", to be denoted by $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx}$. The initial stresses are assumed to be

$$\begin{aligned}
\tau^0_{xx} &= K_0 \rho g y, \\
\tau^0_{yy} &= \rho g y, \\
\tau^0_{xy} &= \tau^o_{yx} = 0,
\end{aligned} \tag{3.2}$$

in which K_0 is considered as a given constant (the coefficient of neutral earth pressure). The state of stress (3.2) has the property that it satisfies the equations of equilibrium (3.1). Hence if one now writes

$$\tau_{xx} = \tau_{xx}^{0} + \sigma_{xx},$$

$$\tau_{yy} = \tau_{yy}^{0} + \sigma_{yy},$$

$$\tau_{xy} = \tau_{xy}^{0} + \sigma_{xy},$$

$$\tau_{yx} = \tau_{yx}^{0} + \sigma_{yx},$$

(3.3)

then the components of incremental stress $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx})$ satisfy the homogeneous equations obtained from (3.1) by taking $\rho g = 0$, i.e.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0,
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0,
\sigma_{xy} = \sigma_{yx},$$
(3.4)

In this thesis restriction is made to problems in which the stresses at infinity do not change as a result of the excavation or the construction of the embankment. This means that at infinity the incremental stresses vanish. It can be expected that the assumption that the modification of the originally straihjt and horizontal boundary in the line C does not affect the stresses at infinity, is justified only when the total area between the line C and the original boundary (the horizontal axis) is finite. The mathematical formulation of the conditions to be imposed on the problem in order that the incremental stresses indeed vanish at infinity will be given in chapter 4, eqs. (4.18). In the present considerations the vanishing of these incremental stresses at infinity is merely postulated, as an assumption.

Under the conditions expressed above it is not yet certain that there exists a unique solution of the problem. A uniqueness theorem for half space problems satisfying the condition of vanishing stresses at infinity (TURTELTAUB & STERNBERG, 1967) provides some support for the probable uniqueness of the solution of the present problem. Actually the procedures to be used in the sequel (chapters 4 and 5) lead to a single solution, thus proving the existence of a unique solution. The necessary conditions will be presented as the solution proceeds.

Complex potentials

In terms of Muskhelishvili's complex potentials the mathematical problem is to determine two functions, $\Phi_1(z)$ and $\Psi_1(z)$, holomorphic (i.e. single-valued and analytic) in the region R, continuous in R+C, and satisfying certain conditions, to be expressed below, along C and at infinity. The stresses can be derived from these functions by means of the formulas of KOLOSOV-MUSKHELISHVILI,

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi_1(z) + \overline{\Phi_1(z)}],$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\overline{z}\Phi_1'(z) + \Psi_1(z)],$$
(3.5)

where the bar denotes the complex conjugate, and the accent denotes differentiation with respect to the argument. For a derivation of (3.5), and the proof that $\Phi_1(z)$ and $\Psi_1(z)$ are holomorphic in R, the reader is referred to MUSKHELISHVILI (1953, chapter 5).

Condition at infinity

Since the incremental stresses are assumed to vanish at infinity it follows from (3.5) that both $\Phi_1(z)$ and $\Psi_1(z)$ must vanish at infinity,

$$\lim_{z \to \infty} \Phi_1(z) = 0,$$

$$\lim_{z \to \infty} \Psi_1(z) = 0.$$
(3.6)

Condition along C

The condition along C is that this boundary must be free of external stresses. The mathematical formulation of this condition is obtained by equating to zero the components of total stress acting upon an element of C. Using the familiar rules for the transformation of stress components this gives

$$(\tau_{xx} + \tau_{yy}) + (\tau_{yy} - \tau_{xx} + 2i\tau_{xy})\exp(2i\alpha) = 0, \ z \in C,$$
(3.7)

where α denoted the angle over which the real axis is to be rotated, in counterclockwise direction, to coincide with the tangent to C. The total stresses consist of initial stresses and incremental stresses, see (3.3). With the expressions (3.2) for the initial stresses and the formulas (3.3) for the incremental stresse, the boundary condition (3.7) can be expressed in terms of $\Phi_1(z)$ and $\Psi_1(z)$ as follows

$$\Phi_1(z) + \overline{\Phi_1(z)} + [\overline{z}\Phi_1'(z) + \Psi_1(z)] \exp(2i\alpha) = -\frac{1}{2}(1+K_0)\rho gy - \frac{1}{2}(1-K_0)\rho gy \exp(2i\alpha), \ z \in C.$$
(3.8)

Conformal transformation

Let the region R in the z-plane be mapped conformally onto the interior of the unit circle γ ($|\zeta| = 1$) in the ζ -plane by means of the function



Figure 3.1: Unit circle in ζ -plane.

$$z = \omega(\zeta), \ |\zeta| \le 1. \tag{3.9}$$

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Because of the semi-infinite character of the region R the function $\omega(\zeta)$ will certainly have a first order pole on γ . Furthermore $\omega(\zeta)$ may have branch points on γ , which correspond to corner points in the boundary C in the z-plane. The properties of the mapping function $\omega(\zeta)$ will be discussed in detail in chapter 4. For the considerations of this chapter it is sufficient to assume that $\omega(\zeta)$ is a given function, defined at least for all ζ inside and on γ , holomorphic inside γ and continuous inside and on γ , with the exception of a single point.

In the sequel the interior of γ will be denoted by S^+ and its exterior by S^- (fig. 3.1). Points on γ will be denoted by σ , and the counterclockwise direction on γ is considered as positive. The point σ may also be written as

$$\sigma = \exp(i\theta),\tag{3.10}$$

where θ is the so-called argument of the complex number σ (fig. 3.1). It is to be noted that σ has the property that its complex conjugate, $\overline{\sigma}$, is also its inverse, $1/\sigma$,

$$\overline{\sigma} = 1/\sigma = \exp(-i\theta). \tag{3.11}$$

Through the conformal transformation (3.9) the functions $\Phi_1(z)$ and $\Psi_1(z)$ can be transformed into functions of ζ , to be denoted by $\Phi(\zeta)$ and $\Psi(\zeta)$, respectively. Since $\Phi_1(z)$ and $\Psi_1(z)$ are holomorphic in R and continuous in R + C(including the point at infinity, see (3.6)), both $\Phi(\zeta)$ and $\Psi(\zeta)$ are holomorphic in S^+ and continuous in $S^+ + \gamma$. The derivative of $\Phi_1(z)$ becomes

$$\Phi_1'(z) = \frac{d\Phi_1(z)}{dz} = \frac{d\Phi(\zeta)}{d\zeta} \frac{d\zeta}{dz} = \frac{\Phi'(\zeta)}{\omega'(\zeta)}.$$
(3.12)

For future reference the following formulas, valid for points on γ , are needed,

$$y = -\frac{1}{2}i[\omega(\sigma) - \overline{\omega(\sigma)}], \qquad (3.13)$$

$$\exp(2i\alpha) = -\sigma^2 [\omega'(\sigma)/\overline{\omega'(\sigma)}].$$
(3.14)

Eq. (3.13) expresses that y is the imaginary part of z. Eq. (3.14) can be established by starting from the well-known property of conformal transformations that the rotation of an infinitesimal line element equals the argument of the derivative of the transformation function. In the present case the angle of the line element $d\sigma$ with the real axis in the ζ -plane is $\pi/2 + \theta$ (fig. 3.2), and the angle of its image along C with the real axis in the z-plane is $\pi + \alpha$ (fig. 3.2). Hence

$$\pi + \alpha - (\pi/2 + \theta) = \arg[\omega'(\sigma)].$$

Now one may write, in general

$$\omega'(\sigma) = |\omega'(\sigma)| \exp\{i \arg[\omega'(\zeta)]\},\$$



Figure 3.2: Boundary elements in planes of ζ and z.

and from this it follows that

$$\frac{\omega'(\sigma)}{|\omega'(\sigma)|} = \exp\{i \arg[\omega'(\sigma)]\} = \exp[i(\pi/2 + \alpha - \theta)] = \frac{i}{\sigma} \exp(i\alpha).$$

Here use has been made of the relationship $\sigma = \exp(i\theta)$, see (3.10). It now follows that

$$\exp(i\alpha) = -i\sigma \,\omega'(\sigma)/|\omega'(\sigma)|.$$

Division of this equation by its complex conjugate yields (3.14).

Boundary conditions

The conditions (3.6) and (3.7) for the functions $\Phi_1(z)$ and $\Psi_1(z)$ are transformed into conditions for $\Phi(\zeta)$ and $\Psi(\zeta)$ along γ , when the variable z is replaced by ζ through the confomal transformation $z = \omega(\zeta)$.

Let the point on the unit circle γ which corresponds to $z = \infty$ be denoted by σ_0 , i.e.

$$\omega(\sigma_0) = \infty. \tag{3.15}$$

Then the vanishing of the functions $\Phi_1(z)$ and $\Psi_1(z)$ at infinity (in the z-plane) implies that $\Phi(\zeta)$ and $\Psi(\zeta)$ are to vanish at σ_0 ,

$$\lim_{\zeta \to \sigma_0} \Phi(\zeta) = 0,$$
$$\lim_{\zeta \to \sigma_0} \Psi(\zeta) = 0.$$

More precisely, it will be assumed that near σ_0 the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ are of order $O(\zeta - \sigma_0)$,

$$\zeta \to \sigma_0 : \Phi(\zeta) = O(\zeta - \sigma_0),$$

$$\zeta \to \sigma_0 : \Psi(\zeta) = O(\zeta - \sigma_0).$$
(3.16)

It will appear later (see chapter 5) that this assumption is related to a restriction in the class of mapping functions $\omega(\zeta)$ for which the solution given in this section applies.

With (3.12), (3.13) and (3.14) the boundary condition (3.8) becomes, after some elaboration,

$$\omega'(\sigma)\Phi(\sigma) + \omega'(\sigma)\overline{\Phi(\sigma)} - \sigma^{-2}[\omega(\sigma)\overline{\Phi'(\sigma)} + \overline{\omega'(\sigma)}\,\overline{\Psi(\sigma)}] = F(\sigma), \quad (3.17)$$

where

$$F(\sigma) = \frac{1}{4}i\rho g[\omega(\sigma) - \overline{\omega(\sigma)}] \{ [\omega'(\sigma) - \frac{\overline{\omega'(\sigma)}}{\sigma^2}] + K_0[\omega'(\sigma) + \frac{\overline{\omega'(\sigma)}}{\sigma^2}] \}.$$
(3.18)

Conclusion

The mathematical problem that is to be solved is now to determine two functions, $\Phi(\zeta)$ and $\Psi(\zeta)$, holomorphic inside γ and continuous up to γ , satisfying the condition (3.17) on γ , and conditions (3.16) near σ_0 .

In chapter 5 the solution of the problem for mapping functions of the special form

$$\omega(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k, \qquad (3.19)$$

will be established. Before presenting this solution, however, it will first be shown in chapter 4 how any mapping function $\omega(\zeta)$, transforming the unit circle into a region approximating a half plane at infinity, can be brought, at least approximately, in this special form.

Chapter 4

THE CONFORMAL TRANSFORMATION

The function $\omega(\zeta)$, of which the boundary value $\omega(\sigma)$ appears in eqs. (3.17) and (3.18), must be of a special form in order that the region R will approximate a half plane near infinity. To ensure this property, the function $\omega(\zeta)$ must have a first order pole at some point of γ , denoted by σ_0 . The mapping function must approximate the mapping function for a half plane near σ_0 , hence one may write, if ζ approaches σ_0 from the interior of γ ,

$$\omega(\zeta) = \frac{p}{\zeta - \sigma_0} + O(1), \ \zeta \to \sigma_0, \ \zeta \in S^+,$$

where p is a constant. In all other points of γ , other than σ_0 , $\omega(\zeta)$ must be bounded and continuous from the interior of γ . Therefore, if a function $\omega_0(\zeta)$ is defined by the relationship

$$\omega(\zeta) = \frac{p}{\zeta - \sigma_0} + \omega_0(\zeta), \ \zeta \in S^+ + \gamma, \tag{4.1}$$

then this function is holomorphic in S^+ and it is continuous (and thus bounded) in $S^+ + \gamma$. The function $\omega_0(\zeta)$ can therfore be expanded in a Taylor series, which will surely be convergent inside the unit circle γ (TITCHMARSH, 1939, p. 8). Hence one may write

$$\omega_0(\zeta) = \sum_{k=0}^{\infty} c_k \,\zeta^k, \ \zeta \in S^+, \tag{4.2}$$

where the coefficients c_k are given by one of the following equivalent expressions,

$$c_k = \frac{1}{2\pi i} \int_L \omega_0(\zeta) \,\zeta^{-k-1} \,d\zeta = \frac{\omega_0^{(k)}(0)}{k!}, \ k = 0, 1, 2, \dots$$
(4.3)

In (4.3) L is an arbitrary closed contour lying entirely in S^+ and encircling the origin once, and $\omega_0^{(k)}(0)$ denotes the value of the k-th derivative of $\omega_0(\zeta)$, in the origin $\zeta = 0$.

Convergence of Taylor series

It will next be shown that under certain, physically wide, conditions the series expansion (4.2) converges not only in S^+ , but also on its boundary γ .

By means of partial integration eq. (4.3) can be re-expressed as follows,

$$kc_k = \frac{1}{2\pi i} \int_L \omega'_0(\zeta) \,\zeta^{-k} \, d\zeta = \frac{\omega_0^{(k)}(0)}{(k-1)!}, \ k = 1, 2, \dots$$
(4.4)

where use has been made of the singlevaluedness of $\omega_0(\zeta) \zeta^{-k}$ on L. It is now assumed that $\omega_0(\zeta)$ has only a finite number of isolated branch points on γ , or, in other words : that the boundary C of the region R in the z-plane has only a finite number of corner points. Denoting a typical branch point by σ_m one may write, if ζ approaches σ_m from the interior of γ ,

$$\omega_0'(\zeta) = A(\zeta - \sigma_m)^{\alpha_m/\pi} [1 + o(1)], \ \zeta \to \sigma_m, \ \zeta \in S^+,$$

$$(4.5)$$

where A is some constant, the symbol o(1) denotes a quantity tending to zero when $\zeta \to \sigma_m$, and α_m denotes the value of the abrupt change of direction along C in the corner point corresponding to σ_m . For a re-entrant angle the value of α_m is limited to $0 < \alpha_m \le \pi$, and for a salient angle to $-\pi \le \alpha_m < 0$. The value $\alpha_m = -\pi$ will now be excluded (which means physically that the boundary C of the region R is supposed to have no cusps). Then

$$-\pi < \alpha_m < \pi. \tag{4.6}$$

The integrand of eq. (4.4) is holomorphic in S^+ , except at the origin. The contour L may thus be transformed into a contour γ' , where γ' coincides with γ , except near the branch points, where γ' consists of small semi-circles around σ_m (inside γ). When the radius of these semi-circles tends to zero the contribution to the total values of the integrals will vanish because of (4.5) and (4.6). Thus one may write

$$kc_k = \frac{1}{2\pi i} \int_{\gamma} \omega'_0(\zeta) \,\zeta^{-k} \,d\zeta, \ k = 1, 2, \dots,$$
(4.7)

and these integrals will exist in the sense of improper integrals. For points on γ one may write $\zeta = \sigma = \exp(i\theta)$, and thus by writing $\omega'_0(\sigma) = \omega'_*(\theta)$ eq. (4.7) becomes

$$kc_k = \frac{1}{2\pi} \int_0^{2\pi} \omega'_*(\theta) \, \exp[-(k-1)i\theta] \, d\theta, \ k = 1, 2, \dots$$
(4.8)

It follows from (4.5) and (4.6) that $\omega'_*(\theta)$ is an integrable function over the interval $0 < \theta < 2\pi$. Therefore by the Riemann-Lebesque theorem (see e.g. TITCHMARSH, 1939, p. 403)

$$\lim_{k \to \infty} kc_k = 0. \tag{4.9}$$

Now returning to the expression (4.2) it is recalled that the left hand member, $\omega_0(\zeta)$, is continuous from the interior of γ , and bounded on γ . Thus $\omega_0(\zeta)$ will tend to a definite limit if ζ approaches a point σ on γ from the interior. Together with (4.9) this means that the series expansion (4.2) satisfies the conditions under which Tauber's theorem (see e.g. TITCHMARSH, 1939, p. 230; THRON, 1953, p. 131) is valid. This theorem expresses that under these conditions the series expansion (4.2) converges also on the unit circle γ .

General formula for c_k

For the calculation of the coefficients c_k in a particular case either of the expressions (4.3) may be used. The second formula may be preferred when the derivatives of $\omega_0(\zeta)$ are easy to determine. When these derivatives are not so easily determined (as occurs for instance when there are branch points on γ) then the first formula can be used more profitably. In a similar way as done in the transition from (4.4) to (4.8) this formula can be transformed into

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \omega_*(\theta) \, \exp(-ki\theta) \, d\theta, \ k = 0, 1, 2, \dots,$$
(4.10)

where $\omega_*(\theta) = \omega_0(\sigma) = \omega_0(\exp[i\theta])$. The values of the function $\omega_0(\zeta)$ for $\zeta = \sigma = \exp(i\theta)$ are usually easy to calculate.

Filon's method of integration

For large values of k the numerical calculation of the coefficients c_k by means of forumla (4.10) deserves some special care, since in that case the integrand is a rapidly oscillating function. Profitable use can then be made of a method devised by FILON (1928), which consists of dividing the interval $(0, 2\pi)$ into an even number of equal parts, and then in each subinterval approximating the function $\omega_*(\theta)$ by a second order polynomial coninciding with the function in the end points and the mid point of the subinterval. For k = 0 the formulas obtained by Filon reduce to the familiar Simpson's rule. In the appendix an extension of Filon's method, based on an approximation by a fourth order polynomial in each subinterval, is presented.

Approximate conformal transformation

By taking only a finite number of terms of the series expansion (4.2) into account an approximation to the conformal transformation is obtained. The function $\omega_n(\zeta)$ defined by

$$\omega_n(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k, \tag{4.11}$$

will represent an approximation to $\omega(\zeta)$. According to the considerations given above the difference between $\omega(\zeta)$ and $\omega_n(\zeta)$ in any arbitrary point inside or on the unit circle γ can be made as small as one pleases by taking *n* sufficiently large. The function $\omega_n(\zeta)$ maps not the region *R* onto the interior of the unit circle, but rather a region R_n that will closely resemble *R* when *n* is chosen large enough. In chapter 5 it will be shown that for mapping functions of the form (4.11) the boundary value problem can be solved in a general way.

Elaboration of $F(\sigma)$

For the purpose of future considerations it is most convenient to present first some consequences of the adopted character (4.11) of the conformal transformation, especially with regard to the function $F(\sigma)$, which appears in the boundary condition, see eqs. (3.10) and (3.11). Therefore the following alternative form of eq. (4.11) will appear to be useful,

$$\omega_n(\zeta) = \frac{p}{\zeta - \sigma_0} + d_0 + d_1(\zeta - \sigma_0) + (\zeta - \sigma_0)^2 \sum_{k=2}^n d_k \zeta^{k-2}.$$
 (4.12)

The coefficients d_k , k = 0, 1, 2, ..., n can easily be calculated from the coefficients c_k by means of the following recurrent relations

$$d_{k} = c_{k}, \ k = n,$$

$$d_{k} = c_{k} + 2d_{k+1}\sigma_{0}, \ k = n - 1,$$

$$d_{k} = c_{k} + 2d_{k+1}\sigma_{0} - d_{k+2}\sigma_{0}^{2}, \ k = n - 2, \dots, 1,$$

$$d_{k} = c_{k} + d_{k+1}\sigma_{0} - d_{k+2}\sigma_{0}^{2}, \ k = 0.$$
(4.13)

The general solution of the system of equations (4.13) is

$$d_{0} = \sum_{j=0}^{n} c_{j} \sigma_{0}^{j},$$

$$d_{k} = \sum_{j=k}^{n} (j-k+1)c_{j} \sigma_{0}^{j-k}, \quad k = 1, \dots, n,$$
(4.14)

as can be verified without difficulty by substitution into (4.13).

The three factors appearing in the expression (3.11) for $F(\sigma)$, which involve the conformal transformation $\omega(\zeta)$ in different ways, can be expressed in terms of the coefficients c_k or d_k as follows.

$$\omega_n(\sigma) - \overline{\omega_n(\sigma)} = (p + \overline{p}\sigma_0^2)/(\sigma - \sigma_0) + \overline{p}\sigma_0 + \sum_{k=0}^n (c_k\sigma^k - \overline{c_k}\sigma^{-k}) = (p + \overline{p}\sigma_0^2)/(\sigma - \sigma_0) + (\overline{p}\sigma_0 + d_0 - \overline{d_0}) + (d_1 + \overline{d_1}/\sigma_0^2)(\sigma - \sigma_0) + (\sigma - \sigma_0)^2 \{\sum_{k=2}^n d_k\sigma^{k-2} - \sigma_0^{-2}\sum_{k=1}^n \overline{d_k}\sigma^{-k}\}, \quad (4.15)$$

$$\omega_n'(\sigma) - \overline{\omega_n'(\sigma)} / \sigma^2 = -(p - \overline{p}\sigma_0^2) / (\sigma - \sigma_0)^2 + \sum_{k=1}^n (kc_k \sigma^{k-1} - k\overline{c_k} \sigma^{-k-1}), \quad (4.16)$$

$$\omega_n'(\sigma) + \overline{\omega_n'(\sigma)} / \sigma^2 = -(p + \overline{p}\sigma_0^2) / (\sigma - \sigma_0)^2 + \sum_{k=1}^n (kc_k \sigma^{k-1} + k\overline{c_k} \sigma^{-k-1}), \quad (4.17)$$

In deriving these expressions use has been made of the property that $\overline{\sigma} = \sigma^{-1}$.

It is now postulated that $F(\sigma)$ is to be bounded near σ_0 , and it will be investigated what implications this has for the approximate mapping function. By inspection of eq. (3.11), together with the expressions (5.15), (4.16) and (4.17), it is observed that, if no restrictive conditions are imposed on the conformal transformation, then $F(\sigma)$ will have a third order pole at σ_0 . In order that $F(\sigma)$ be bounded near σ_0 the coefficients of the terms with $(\sigma - \sigma_0)^{-3}$, $(\sigma - \sigma_0)^{-2}$ and $(\sigma - \sigma_0)^{-1}$ must vanish. This leads to the following conditions

$$p + \overline{p}\sigma_0^2 = 0,$$

$$d_0 - \overline{d_0} + \overline{p}\sigma_0 = 0,$$

$$d_1 + \overline{d_1}/\sigma_0^2 = 0.$$
(4.18)

The physical meaning of the first two conditions is best understood by inspecting their implications for the quantity $\omega_n(\sigma) - \overline{\omega_n(\sigma)}$, which represents the value of the *y*-coordinate (multiplied with 2*i*) along the boundary of the region *R* in the *z*-plane. It is immediately seen from the second expression in eq. (4.15) that the first two conditions of (4.18) ensure that the *y*-coordinate of a point on the boundary vanishes near the end points of this boundary at infinity (the point at infinity in the *z*-plane corresponds to $\sigma = \sigma_0$).

The physical meaning of the third condition is best understood by considering the implications for the *y*-coordinate as well as the *x*-coordinate of a boundary point near infinity. From (4.15) it follows that near infinity (hence for σ near σ_0)

$$2iy = (d_1 + \overline{d_1}/\sigma_0^2)(\sigma - \sigma_0) + O(\sigma - \sigma_0)^2, \qquad (4.19)$$

where the first two of the conditions (4.18) have already been used, but not yet the third. On the other hand one may obtain from (4.12) that the x-coordinate of a boundary point near infinity is given by

$$2x = 2p/(\sigma - \sigma_0) + 2d_0 + O(\sigma - \sigma_0), \qquad (4.20)$$

where again use has been made of the first two of the conditions (4.18). It follows from (4.19) and (4.20) that near infinity

$$2iy = \frac{p(d_1 + \overline{d_1}/\sigma_0^2)}{x - d_0} + \dots$$
(4.21)

Thus the vanishing of the factor $d_1 + \overline{d_1}/\sigma_0^2$, as required by the third of the conditions (4.18), implies that the *y*-coordinate of a boundary pooint should got zero for $x \to \infty$ more rapidly than according to the hyperbolic formula

(4.21). In fact, the area between a hyperbola and its asymptote is known to be unbounded and a boundary curve behaving as a hyperbola near its ends would correspond to an edge notch of infinite total area. For such an edge notch the assumption of vanishing incremental stresses near infinity would clearly not be applicable. Hence for this assumption to be applicable the coefficient of $1/(x-d_0)$ in eq. (4.18) should be equal to zero, and this is just the third of the conditions (4.18). It is to be noted that the considerations just given constitute the mathematical formulation of the properties of the line C mentioned at the end of chapter 2 and in the beginning of chapter 3.

The first of conditions (4.18) expresses that p/σ_0 is to be an imaginary quantity. With eqs. (4.14) for k = 0, respectively k = 1, the last two conditions can be expressed in terms of the coefficients c_k . This leads to

$$\sum_{k=0}^{n} (c_k \sigma_0^k - \overline{c_k} \sigma_0^{-k}) + \overline{p} \sigma_0 = 0,$$

$$\sum_{k=1}^{n} (k c_k \sigma_0^k + k \overline{c_k} \sigma_0^{-k}) = 0.$$
(4.22)

These two equations are purely imaginary, respectively purely real. The can be satisfied by giving the last coefficient (c_n) the following, in general complex, value

$$c_n = \frac{1}{2}p\sigma_0^{-n-1} - \frac{1}{2}\sigma_0^{-n}\sum_{k=0}^{n-1} [(1+k/n)c_k\sigma_0^k - (1-k/n)\overline{c_k}\sigma_0^{-k}].$$
 (4.23)

It should be noted that in general the above conditions are automatically satisfied by the conformal transformation if the region mapped by it onto the unit circle is indeed a half plane with edge notches having a finite total area. For the purpose of carrying out numerical calculations, however, it is convenient to have a mathematical formulation of this property. This is especially useful when the original conformal transformation is approximated by a formula of the form (4.11), with the coefficients c_k calculated by some numerical procedure. By choosing the last coefficient, c_n , in accordance with (4.23) it is then ensured that the approximate region R_n is itself in the class of regions considered. In the sequel it will be assumed that p/σ_0 is imaginary, and that c_n has been given the value following from (4.23). Then the conditions (4.18) are satisfied, and hence the function $F(\sigma)$ is bounded near σ_0 .

To facilitate future considerations it is convenient to present here first the consequences of the conditions (4.18) for the appearance of the function $F(\sigma)$. Substitution of (4.15), (4.16) and (4.17) into (3.11) gives, using (4.18),

$$\frac{F(\sigma)}{\rho g} = -\frac{1}{2} i p \left\{ \sum_{k=2}^{n} d_{k} \sigma^{k-2} - \sigma_{0}^{-2} \sum_{k=1}^{n} \overline{d_{k}} \sigma^{-k} \right\} + \frac{1}{4} i \left\{ \overline{p} \sigma_{0} + \sum_{k=0}^{n} (c_{k} \sigma^{k} - \overline{c_{k}} \sigma^{-k}) \right\} \times \left\{ (1+K_{0}) \sum_{k=1}^{n} k c_{k} \sigma^{k-1} - (1-K_{0}) \sum_{k=1}^{n} k \overline{c_{k}} \sigma^{-k-1} \right\}. \quad (4.24)$$

This form can be elaborated by making use of the rules for the multiplication of polynomials. This gives

$$\frac{F(\sigma)}{\rho g} = -\frac{1}{2}ip\left\{\sum_{k=2}^{n} d_{k}\sigma^{k-2} - \sigma_{0}^{-2}\sum_{k=1}^{n} \overline{d_{k}}\sigma^{-k}\right\} + \frac{1}{4}i\overline{p}\,\sigma_{0} \times \left\{\left(1+K_{0}\right)\sum_{k=0}^{n-1}(k+1)c_{k+1}\sigma^{k} - \left(1-K_{0}\right)\sum_{k=2}^{n+1}(k-1)\overline{c_{k-1}}\sigma^{-k}\right\} + \frac{1}{4}i(1+K_{0})\left\{\sum_{k=0}^{2n-1}e_{k}\sigma^{k} - \sum_{k=0}^{n-1}\overline{f_{n-k-1}}\sigma^{k} - \sum_{k=1}^{n}\overline{f_{n+k-1}}\sigma^{-k}\right\} + \frac{1}{4}i(1-K_{0})\left\{\sum_{k=2}^{2n+1}\overline{e_{k-2}}\sigma^{-k} - \sum_{k=0}^{n-2}f_{n+k+1}\sigma^{k} - \sum_{k=1}^{n+1}f_{n-k+1}\sigma^{-k}\right\}, \quad (4.25)$$

where

$$e_k = \sum_{\max(0,k-n)}^{\min(k,n-1)} (j+1)c_{j+1}c_{k-j}, \ k = 0, 1, \dots, 2n-1,$$
(4.26)

and

$$f_k = \sum_{\max(0,k-n)}^{\min(k,n-1)} (n-j)\overline{c_{n-j}}c_{k-j}, \ k = 0, 1, \dots, 2n+1,$$
(4.27)

It appears that the function $F(\sigma)$ is now uniquely composed of positive and negative powers of σ , and that the coefficients of these terms can directly be calculated from the coefficients c_k , which describe the conformal transformation.

Chapter 5

SOLUTION OF THE BOUNDARY VALUE PROBLEM

In this chapter the general boundary value problem will be solved. This will be done in three stages. In the first stage (section 5.1) the general character of the solution is investigated. This will lead to an expression containing some constants, which are determined in section 5.2. Finally, in section 5.3 the solution is elaborated until expressions for the stress components are obtained.

General character of the solution

The mathematical problem, as formulated in chapter 3 is to determine two functions $\Phi(\zeta)$ and $\Psi(\zeta)$, holomorphic inside the unit circle γ and continuous inside and on γ , satisfying the condition (3.17) on γ , i.e.

$$\omega_n'(\sigma)\Phi_+(\sigma) + \omega_n'(\sigma)\overline{\Phi_+(\sigma)} - \sigma^{-2}[\omega_n(\sigma)\overline{\Phi_+'(\sigma)} + \overline{\omega_n'(\sigma)}\,\overline{\Psi_+(\sigma)}] = F(\sigma),$$
(5.1)

where $F(\sigma)$ is now given by (4.25), and where

$$\omega_n(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k.$$
(5.2)

In eq. (5.1) the subscript + indicates that the limiting value of a function for ζ tending to a point σ on γ from the interior (S⁺) is intended.

The boundary value problem will be solved by means of the method proposed by MUSKHELISHVILI (1953), chapters 18-21. In this method the problem is reduced to a Hilbert problem (the problem of linear relationship) from the theory of functions. In order to perform the reduction to a Hilbert problem it is necessary to extend the regions of definition of the functions $\omega_n(\zeta)$ and $\Phi(\zeta)$. Originally the region of definition of these functions is, for physical reasons, restricted to the interior S^+ of the unit circle γ and the unit circle γ itself. For points outside γ the functions $\omega_n(\zeta)$ and $\Phi(\zeta)$ have not been defined. Since points outside γ do not appear in the analysis one may, if one wishes, attribute any value to the functions $\omega_n(\zeta)$ and $\Phi(\zeta)$ for any ζ in S^- . This is completely irrelevant for the problem as formulated above. The essence of Muskhelishvili's method is, however, to choose very particular values for the functions $\omega_n(\zeta)$ and $\Phi(\zeta)$ for ζ outside γ , namely in such a way that the mathematical problem reduces to a Hilbert problem, which can be solved.

The function $\omega_n(\zeta)$

In the first place it is stated that the definition (5.2) for $\omega_n(\zeta)$ from now on applies to all values of ζ in the entire plane. For ζ in S^+ or for ζ on γ this function represents, as before, the values of the complex variable ζ which describes the region occupied by the elastic body and its boundary. For values of ζ in S^- (the exterior of the unit circle γ) this function has no physical meaning. Mathematically spekaing the mapping function $\omega_n(\zeta)$ has now been *continued* analytically in S^- (TITCHMARSH, 1939, chapter 4; THRON, 1953, chapter 23). Such an analytic continuation into parts of the plane outside the original region of definition of the function has the property that it is continuous across the boundary of this region. It is immediately observed form the definition (5.2)that the function $\omega_n(\zeta)$ is now holomorphic in the entire plane with the exception of the singular point $\zeta = \sigma_0$, where the function possesses a first order pole, and with the further exception of the point at infinity. It is to be noted that the mapping function $\omega_n(\zeta)$, which is an approximation of the original mapping function $\omega(\zeta)$, does not possess branch points on γ , in contrast with the original function $\omega(\zeta)$. In fact, such branch points prohibit an analytic continuation of $\omega(\zeta)$ into S⁻ which is everywhere continuous across γ , and it is this circumstance that makes it impossible to apply Muskhelishvili's method to the case with the mapping function $\omega(\zeta)$. Muskhelishvili's method will be shown to be applicable, however, to the approximate mapping function $\omega_n(\zeta)$, which tends uniformly to $\omega(\zeta)$ for $n \to \infty$.

The function $\Phi(\zeta)$

Next the region of definition of the stress function $\Phi(\zeta)$ will be extended, but in this case the extension will not be taken as an analytic continuation. For points ζ in the exterior S^- of γ the value of $\Phi(\zeta)$ will be taken in accordance with the following expression

$$\omega_n'(\zeta)\Phi(\zeta) = -\omega_n'(\zeta)\overline{\Phi}(1/\zeta) + \zeta^{-2}\omega_n(\zeta)\overline{\Phi'}(1/\zeta) + \zeta^{-2}\overline{\omega_n'}(1/\zeta)\overline{\Psi}(1/\zeta), \ \zeta \in S^-.$$
(5.3)

In this definition use has been made of the notation (see MUSKHELISHVILI, 1953, p. 288)

$$\overline{f}(\zeta) = \overline{f(\overline{\zeta})},$$

from which it follows that

$$\overline{f}(1/\zeta) = \overline{f(1/\overline{\zeta})}.$$
(5.4)

It will appear later, see eq. (5.7), that the definition (5.3) of $\Phi(\zeta)$ for $\zeta \in S^$ implies that the function $\Phi(\zeta)$ is not continuous across γ , but that the limiting values of $\Phi(\zeta)$ for ζ tending towards a point σ on γ from the exterior (S^-) or the interior (S^+) differ by a given amount. Before presenting this, however, it will first be shown that eq. (5.3) indeed represents a definition of $\Phi(\zeta)$ for $\zeta \in S^-$, and that this function is holomorphic in S^- .

First it is noted (fig. 5.1) that if ζ is some point in S^- then the points $1/\zeta$ and $1/\overline{\zeta}$ are points in S^+ . Thus if $f(\zeta)$ is a given function for all $\zeta \in S^+$, then the value of $f(1/\overline{\zeta})$ for $\zeta \in S^-$ can be calculated since $1/\overline{\zeta}$ is then a point in



Figure 5.1: Points in ζ -plane.

 S^+ . Taking the complex conjugate of the value of $f(1/\zeta)$ yields $\overline{f}(1/\zeta)$. From this it follows that, if the solution of the problem, as described by $\Phi(\zeta)$ and $\Psi(\zeta)$ for $\zeta \in S^+$ where known, then it would be possible to calculate the values of $\overline{\Phi}(1/\zeta)$, $\overline{\Phi'}(1/\zeta)$ and $\overline{\Psi}(1/\zeta)$ for all $\zeta \in S^-$. Moreover $\omega_n(\zeta)$ is a function known for all ζ , and the function $\overline{\omega'_n}(1/\zeta)$ is found to be

$$\overline{\omega'_n}(1/\zeta) = -\frac{\overline{p}\,\zeta^2\sigma_0^2}{(\zeta-\sigma_0)^2} + \sum_{k=1}^n k\,\overline{c_k}\,\zeta^{-k+1},$$

which shows that this function is holomorphic in the entire plane with the exception of the points $\zeta = \sigma_0$ and $\zeta = 0$, where the function possesses singularities in the form of poles. It now follows that the function $\omega'_n(\zeta)\Phi(\zeta)$, as defined by (5.3) for $\zeta \in S^-$, is expressed in terms of the given mapping function $\omega_n(\zeta)$ and the basic unknown stress functions $\Phi(\zeta)$ and $\Psi(\zeta)$ as defined for $\zeta \in S^+$. Since $\omega'_n(\zeta)$ is known for $\zeta \in S_-$ it now follows that (5.3) indeed defines a value for $\Phi(\zeta)$ for $\zeta \in S^-$ in terms of functions defined before.

In the second place it will be shown that the function $\Phi(\zeta)$ as defined for $\zeta \in S^-$ by (5.3) is holomorphic in the region S^- . In order to prove this, use is made of the definition (5.4). Let it be given that $f(\zeta)$ is holomorphic in S^+ . Then this function can be expanded into a Taylor series around $\zeta = 0$,

$$f(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \dots,$$

which will be convergent for all $\zeta \in S^+$. It now follows immediately from (5.4) that

$$\overline{f}(1/\zeta) = \overline{a_0} + \overline{a_1}/\zeta + \overline{a_2}/\zeta^2 + \dots$$

and this will be a convergent series for all values of ζ such that $1/\zeta \in S^+$, i.e. for all $\zeta \in S^-$. This means that $\overline{f}(1/\zeta)$ is holomorphic in S^- when $f(\zeta)$ is holomorphic in S^+ . Application of this result to the functions $\Phi(1/\zeta)$, $\Phi'(1/\zeta)$ and $\Psi(1/zeta)$, which appear in (5.3), now shows that the function $\omega'_n(\zeta)\Phi(\zeta)$ is holomorphic in S^- , with the possible exception of the point at infinity. The point at infinity may be a singular point of $\omega'_n(\zeta)\Phi(\zeta)$ because of the appearance of the functions $\omega_n(\zeta)$ and $\omega'_n(\zeta)$ in the right hand member of (5.3). It now follows that $\Phi(\zeta)$ as defined for $\zeta \in S^-$ by (5.3) is a holomorphic function in S^- , with the possible exception of the point at infinity.

In a later stage of the considerations (section 5.2) it will be necessary to investigate in detail the behaviour of $\omega'_n(\zeta)\Phi(\zeta)$ near infinity. For the present considerations it is sufficient to observe that by taking ζ very large in (5.3) one obtains, using the definition of $\omega_n(\zeta)$ and the functions $\omega'_n(\zeta)$ and $\overline{\omega'_n(1/\zeta)}$ derived from it,

$$nc_n\Phi(\zeta)[\zeta^{n-1} + O(\zeta^{n-2})] = -nc_n\zeta^{n-1}\overline{\Phi(0)} + O(\zeta^{n-2}), \ \zeta \to \infty.$$

From this it follows that $\Phi(\zeta)$ tends to a definite limit for $\zeta \to \infty$, namely

$$\Phi(\infty) = -\overline{\Phi(0)}.$$

This means that the function $\omega'_n(\zeta)\Phi(\zeta)$ will possess a pole of order n-1 at infinity, i.e.

$$\omega'_n(\zeta)\Phi(\zeta) = O(\zeta^{n-1}), \ \zeta \to \infty.$$
(5.5)

Reduction to Hilbert problem

Now the boundary value problem (5.1) will be reduced to a Hilbert problem. Therefore it is noted that if in eq. (5.3) ζ is replaced by $1/\overline{\zeta}$ then an equation valid for $\zeta \in S^+$ is obtained, and if in this equation ζ is made to approach a point σ on γ one obtains

$$-\sigma^{-2}\overline{\omega_n'(\sigma)}\overline{\Psi_+(\sigma)} = -\omega_n'(\sigma)\Phi_-(\sigma) - \omega_n'(\sigma)\overline{\Phi_+(\sigma)} + \sigma^{-2}\omega_n(\sigma)\overline{\Phi_+'(\sigma)}.$$
 (5.6)

Here use has been made of the fact (fig. 5.1) that when $\zeta \to \sigma$ from the interior of γ , then $1/\overline{\zeta} \to \sigma$ from the exterior of γ . In (5.6) $\Phi_{-}(\sigma)$ denotes the limit of $\Phi(\zeta)$ when $\zeta \to \sigma$ from the exterior S^{-} of γ . Substitution of (5.6) into (5.1) gives

$$\omega'_n(\sigma)\Phi_+(\sigma) - \omega'_n(\sigma)\Phi_-(\sigma) = F(\sigma), \tag{5.7}$$

which represents the discontinuity condition for the so-called Hilbert problem from the theory of functions (by some authors designated as the Riemannproblem, the Riemann-Hilbert problem or the problem of linear relationship). The problem is to determine a function $\omega'_n(\zeta)\Phi(\zeta)$, holomorphic in S^- and S^+ , except possibly at infinity (where the function in the present case may have a pole of order n-1, see (5.5)), and satisfying the condition (5.7) on γ . In the present case the function $\omega'_n(\zeta)\Phi(\zeta)$ is to tend to a definite limit when ζ approaches any point σ on γ from the positive or the negative side, with the single exception of the point σ_0 . In fact, the function $\omega'_n(\zeta)\Phi(\zeta)$ may have a first order pole in σ_0 , since $\omega'_n(\zeta)$ has a second order pole there and $\Phi(\zeta)$ has been assumed to be of order $O(\zeta - \sigma)$ near σ_0 .

Solution of the Hilbert problem

Problems of the type (5.7) have been discussed extensively by MUSKHELISHVILI (1953). They can be solved with the aid of Cauchy integrals. Therefore the solution is decomposed into three parts,

$$\omega'_n(\zeta)\Phi(\zeta) = \chi(\zeta) + \chi^*(\zeta) + \chi^{**}(\zeta).$$
(5.8)

The function $\chi(\zeta)$ will be taken such that the discontinuity condition (5.7) is satisfied, $\chi^*(\zeta)$ will account for the pole at $\zeta = -\sigma_0$, and $\chi^{**}(\zeta)$ will account for the behaviour at infinity.

The essential part of the solution is the part that takes care of the discontinuity along γ . This part of the solution is taken in the form of the Cauchy integral

$$\chi(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\sigma) \, d\sigma}{\sigma - \zeta}, \quad \zeta \in S^- + S^+.$$
(5.9)

The function $F(\sigma)$, as given by (4.25), is bounded, and even continuous, on γ , and therefore the integral (5.9) exists in the entire plane, with the provible exception of the line γ . The function $\chi(\zeta)$ is holomorphic in S^- and S^+ ; it is said to be a *sectionally holomorphic* function, with the unit circle γ as line of discontinuity (MUSKHELISHVILI, 1953, p. 427). Since $F(\sigma)$ is bounded on γ , and since the length of γ is finite, it follows from (5.8) that $\chi(\zeta)$ tends to zero when ζ tends to infinity.

The limiting values of $\chi(\zeta)$, when ζ approaches a point σ on γ from the exterior or the interior of γ exist, and they are related by the so-called Sokhotski-Plemelj formula (MUSKHELISHVILI, 1953, p. 262)

$$\chi_+(\sigma) - \chi_-(\sigma) = F(\sigma),$$

which shows that indeed $\chi(\zeta)$ possesses the discontinuity that $\omega'_n(\zeta)\Phi(\zeta)$ should have, see eq. (5.7).

In the second place let

$$\chi^*(\zeta) = \frac{A}{\zeta - \sigma_0},\tag{5.10}$$

where A is the residue of the function $\omega'_n(\zeta)\Phi(\zeta)$ in its first order pole $\zeta = \sigma_0$ (this residue is as yet unknown).

It now follows that the remaining part $\chi^{**}(\zeta)$ of the solution (5.8) must have the following properties:

- 1. It must be holomorphic in S^- and S^+ with the exception of the point at infinity;
- 2. It must be of order $O(\zeta^{n-1})$ near infinity, because of (5.5) and since both $\chi(\zeta)$ and $\chi^*(\zeta)$ vanish at infinity;

3. It must be continuous across γ , since the discontinuity of $\omega'_n(\zeta)\Phi(\zeta)$ has been accounted for by $\chi(\zeta)$.

The requirements 1 and 3 imply that the holomorphic functions $\chi^{**}(\zeta)$ for $\zeta \in S^+$ and $\chi^{**}(\zeta)$ for $\zeta \in S^-$ are each others analytic continuation across the unit circle γ . Hence the function $\chi^{**}(\zeta)$ must be holomorphic in the entire plane, except at infinity, and it now follows from the generalized Liouville theorem (TITCHMARSH, 1939, p. 85) that $\chi^{**}(\zeta)$ is a polynomial of order n-1, i.e.

$$\chi^{**}(\zeta) = \sum_{k=0}^{n-1} A_k \zeta^k, \text{ all } \zeta.$$
 (5.11)

The coefficients $A_k(k = 0, 1, ..., n - 1)$ are as yet unknown.

Substitution of (5.9), (5.10) and (5.11) into (5.8) shows that the solution of the problem is

$$\omega_n'(\zeta)\Phi(\zeta) = \chi(\zeta) + \frac{A}{\zeta - \sigma_0} + \sum_{k=0}^{n-1} A_k \zeta^k, \quad \zeta \in S^- + S^+.$$
(5.12)

The coefficients A and $A_k(k = 0, 1, 2, ..., n - 1)$ will be determined in section 5.2. Before proceeding to the evaluation of these coefficients, however, it is convenient to first elaborate the function $\chi(\zeta)$.

Evaluation of $\chi(\zeta)$

In order to calculate the Cauchy integral (5.9), i.e.

$$\chi(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\sigma) \, d\sigma}{\sigma - \zeta}, \quad \zeta \in S^- + S^+,$$

it is recalled from the previous chapter, see eq. (4.25), that $F(\sigma)$ consists of positive and neagtive powers of σ . The Cauchy integral can therefore easily be evaluated by making use of the following elementary formulas, valid for integer values of k,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\sigma^k \, d\sigma}{\sigma - \zeta} = \begin{cases} \zeta^k, & \zeta \in S^+, \\ 0, & \zeta \in S^-, \end{cases} \quad (k \ge 0),$$
(5.13)

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\sigma^{-k} \, d\sigma}{\sigma - \zeta} = \begin{cases} 0, & \zeta \in S^+, \\ \zeta^{-k}, & \zeta \in S^-, \end{cases} \quad (k > 0).$$
(5.14)

Thus the function $\chi(\zeta)$ appears to be

$$\chi(\zeta) = \sum_{k=0}^{2n-1} B_k \,\zeta^k, \ \zeta \in S^+,$$

$$\chi(\zeta) = \sum_{k=1}^{2n+1} C_k \,\zeta^{-k}, \ \zeta \in S^-.$$
(5.15)

In these expressions the coefficients B_k and C_k represent the following quantities

$$B_{k}/(\frac{1}{4}i\rho g) = (1+K_{0})e_{k} - 2pd_{k+2}E_{n-k-2} + (1+K_{0})\overline{p}\sigma_{0}(k+1)c_{k+1}E_{n-k-1} - (1+K_{0})\overline{f_{n-k-1}}E_{n-k-1} - (1-K_{0})f_{n+k+1}E_{n-k-2}, \qquad k = 0, 1, \dots, 2n-1, \quad (5.16)$$

$$C_{k}/(\frac{1}{4}i\rho g) = -2p\sigma_{0}^{-2}\overline{d_{k}}E_{n-k} + (1-K_{0})\overline{p}\sigma_{0}(k-1)\overline{c_{k-1}}E_{n-k-1} + (1+K_{0})\overline{f_{n+k-1}}E_{n-k} + (1-K_{0})f_{n-k+1}E_{n-k+1} - (1-K_{0})\overline{e_{k-2}}E_{k-2}, \qquad k = 1, \dots, 2n+1. \quad (5.17)$$

In the above expressions the symbol E_j has been used to denote

$$E_j = \begin{cases} 1, & j \ge 0, \\ 0, & j < 0. \end{cases}$$
(5.18)

Since the coefficients d_k , e_k and f_k can all be easily be calculated from the elementary coefficients c_k (see chapter 4), the coefficients B_k and C_k can also be calculated without difficulty by simple arithmetic operations.

It now remains to determine the coefficients A and A_k which appear in the solution (5.12). These constants will be determined in the next section.

General considerations

Replacement of ζ in eq. (5.3) by $1/\zeta$ leads to an equation valid for $\zeta \in S^+$, and when subsequently the complex conjugate of this equation is taken, the result is, after some reaarangement,

$$\zeta^{2}\omega_{n}'(\zeta)\Psi(\zeta) = \overline{\omega_{n}'(1/\zeta)}\overline{\Phi}(1/\zeta) + \overline{\omega_{n}'(1/\zeta)}\Phi(\zeta) - \zeta^{2}\overline{\omega_{n}}(1/\zeta)\Phi'(\zeta), \quad \zeta \in S^{+}.$$
(5.19)

This equation will now be investigated in detail, especially for values of ζ close to the origin. Since $\zeta = 0$ corresponds to $1/\zeta = \infty$ this will enable to relate the behaviour of $\omega'_n(\zeta)\Phi(\zeta)$ near infinity to the behaviour of $\omega_n(\zeta)$ and $\Phi(\zeta)$ near $\zeta = 0$.

The left hand member of (5.19) is holomorphic inside γ , and therefore so must be the right hand member. Multiplication of (5.19) with ζ^{k-1} , with $k = -1, 0, 1, \ldots, n-1$ and subsequent integration over a contour L, lying entirely inside γ and encircling the origin, leads to zero since $\zeta^{k+1}\omega'_n(\zeta)\Psi(\zeta)$ is holomorphic inside L for all integer values of k not less than -1. Elaboration of the right hand members will be shown to lead to equations from which Aand A_k , for $k = 0, 1, \ldots, n-1$ can be calculated.

In performing the integration of the right hand member of (5.19) around L it will be useful to have an expression for the first term of this right hand member. With (5.12) and (5.15) this first term can be expressed as

$$\overline{\omega_n'}(1/\zeta)\overline{\Phi}(1/\zeta) = \sum_{j=1}^{2n+1} \overline{C_j}\zeta^j + \sum_{j=0}^{n-1} \overline{A_j}\zeta^{-j} - \frac{\overline{A}\,\zeta\,\sigma_0}{\zeta - \sigma_0}, \quad \zeta \in S^+.$$
(5.20)

The two remaining terms of eq. (5.19) can not so easily be expressed in a simple form.

Determination of A

The case k = -1 will first be investigated separately. Therefore eq. (5.19) is multiplied by ζ^{-2} and the resulting expression is integrated over L. This gives, with (5.20),

$$2\pi i(\overline{C_1} + \overline{A}) + \int_L \zeta^{-2} \left\{ \overline{\omega'_n}(1/\zeta) \Phi(\zeta) - \zeta^2 \overline{\omega_n}(1/\zeta) \Phi'(\zeta) \right\} d\zeta = 0,$$

where use has been made of the fact that the integral $\int_L \zeta^j d\zeta$ is equal to $2\pi i$ for j = -1 and yields zero for all other integral values of j. The above result can also be written as

$$2\pi i(\overline{C_1} + \overline{A}) - \int_L \frac{d}{d\zeta} \left\{ \overline{\omega_n}(1/\zeta) \Phi(\zeta) \right\} d\zeta = 0,$$

and since the function $\overline{\omega_n}(1/\zeta)\Phi(\zeta)$ is holomorphic inside and on L the integral equals zero, hence it now follows that

$$A = -C_1. \tag{5.21}$$

Thus one of the coefficients has now been found.

Determination of A_k

Next the case of multiplication of eq. (5.19) by ζ^{k-1} , (k = 0, 1, ..., n-1) and subsequent integration over L will be investigated. This requires elaboration of the following equation

$$2\pi i\overline{A_k} - \int_L \zeta^{k+1} \frac{d}{d\zeta} \left\{ \overline{\omega_n}(1/\zeta) \Phi(\zeta) \right\} d\zeta = 0, \quad k = 0, 1, \dots, n-1.$$

Using partial integration this can be transformed into

$$2\pi i \overline{A_k} + (k+1) \int_L \zeta^k \,\overline{\omega_n}(1/\zeta) \Phi(\zeta) \, d\zeta = 0, \quad k = 0, 1, \dots, n-1.$$
 (5.22)

This system of equations will be further investigated below.

Since $\Phi(\zeta)$ is holomorphic in S^+ one may write

$$\Phi(\zeta) = \sum_{k=0}^{n-1} q_k \zeta^k + \zeta^n G(\zeta), \quad \zeta \in S^+,$$
(5.23)

where $G(\zeta)$ is holomorphic in S^+ , and the constants q_k (k = 0, 1, ..., n - 1) respresent the first *n* coefficients in the Taylor series expansion of $\Phi(\zeta)$ around the origin. It will be shown that the coefficients A_k can easily be expressed

into the coefficients q_k and that the equations (5.22) can be transformed into a system of equations for the determination of q_k .

The conformal transformation $\omega_n(\zeta)$ can be written as

$$\omega_n(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k = \sum_{k=0}^n (c_k - p\sigma_0^{-k-1})\zeta^k + \zeta^{n+1}g(\zeta), \quad (5.24)$$

where

$$g(\zeta) = \frac{p\sigma_0^{-n-1}}{\zeta - \sigma_0}.$$
 (5.25)

Furthermore, the first derivative of $\omega_n(\zeta)$ can be written as

$$\omega_n'(\zeta) = \sum_{k=0}^{n-1} (k+1)(c_{k+1} - p\sigma_0^{-k-2})\zeta^k + \zeta^n h(\zeta),$$
(5.26)

where

$$h(\zeta) = (n+1)g(\zeta) + \zeta g'(\zeta) = \frac{p\sigma_0^{-n-1}[n\zeta - (n+1)\sigma_0]}{(\zeta - \sigma_0)^2}.$$
 (5.27)

Both $g(\zeta)$ and $h(\zeta)$ are clearly holomorphic inside γ . With (5.23) it now follows that near $\zeta = 0$

$$\omega_n'(\zeta)\Phi(\zeta) = \sum_{k=1}^{n-1} \left\{ \sum_{j=0}^k (k-j+1)q_j(c_{k-j+1} - p\sigma_0^{-k+j-2}) \right\} \zeta^k + O(\zeta^n).$$
(5.28)

On the other hand it follows from (5.12) and (5.15) that near $\zeta = 0$

$$\omega_n'(\zeta)\Phi(\zeta) = \sum_{k=0}^{n-1} \{A_k + B_k - A\sigma_0^{-k-1}\}\zeta^k + O(\zeta^n).$$
(5.29)

Hence, by equating coefficients of like powers of ζ in eqs. (5.28) and (5.29) one obtains

$$A_{k} = -B_{k} + A\sigma_{0}^{-k-1} + \sum_{j=0}^{k} (k-j+1)(c_{k-j+1} - p\sigma_{o}^{-k+j-2})q_{j},$$

$$k = 0, 1, \dots, n-1, \quad (5.30)$$

which shows that indeed the coefficients A_k can be calculated if the coefficients q_k are known.

Now returning to eqs. (5.22) one observes that the first term, $2\pi i \overline{A_k}$, can be expressed linearly into q_k , through (5.30). It remains to express the second term of (5.22) into these coefficients. Therfore the conformal transformation $\omega_n(\zeta)$ is expanded in a series valid for large ζ . This gives, with (5.2),

$$\omega_n(\zeta) = \sum_{j=0}^n c_j \zeta^j + O(\zeta^{-1}), \quad \zeta \to \infty.$$

Hence near $\zeta = 0$,

$$\overline{\omega_n(1/\zeta)}\Phi(\zeta) = \overline{\omega_n(1/\overline{\zeta})} = \sum_{j=0}^n \overline{c_j}\zeta^{-j} + O(\zeta), \quad \zeta \to 0.$$
(5.31)

From (5.23) and (5.31) it follows that near $\zeta = 0$,

$$\overline{\omega_n}(1/\zeta)\Phi(\zeta) = \sum_{k=1}^n \left\{ \sum_{j=0}^{n-k} \overline{c_{j+k}} q_j \right\} \zeta^{-k} + O(1), \quad \zeta \to 0.$$
(5.32)

Substitution of (5.32) into the integral appearing in (5.22) gives, when the length of the contour L is taken to be extremely small,

$$\int_{L} \zeta^{k} \overline{\omega_{n}}(1/\zeta) \Phi(\zeta) \, d\zeta) = 2\pi i \sum_{j=0}^{n-k-1} \overline{c_{j+k+1}} q_{j}, \quad k = 0, 1, \dots, n-1.$$
(5.33)

Substitution of this result and of (5.30) into (5.22) now finally yields, after taking complex conjugates, the following system of equations

$$-B_{k} + A\sigma_{0}^{-k-1} + \sum_{j=0}^{k} (k-j+1)(c_{k-j+1} - p\sigma_{o}^{-k+j-2})q_{j} + (k+1)\sum_{j=0}^{n-k-1} c_{j+k+1}\overline{q_{j}} = 0, \quad k = 0, 1, \dots, n-1.$$
(5.34)

This is a system of n linear complex equations with n complex unknowns $(q_0, q_1, \ldots, q_{n-1})$. From this system the coefficients q_k can be determined. Once these coefficients are known, the coefficients A_k can be calculated using (5.30). Then all the coefficients in the solution, eq. (5.12), i.e.

$$\omega'_{n}(\zeta)\Phi(\zeta) = \chi(\zeta) + \frac{A}{\zeta - \sigma_{0}} + \sum_{k=0}^{n-1} A_{k}\zeta^{k}, \quad \zeta \in S^{-} + S^{+}.$$
(5.35)

are known. The problem is therefore, in principle, completely solved.

Elaboration

In the preceding sections of this chapter the mathematical problem has been solved in terms of the function $\Phi(\zeta)$, which for $\zeta \in S^+$ represents a stress function, and which for $\zeta \in S^-$ has been introduced as an auxiliary mathematical quantoty without direct physical meaning. In this section the solution will be elaborated with the final aim of obtaing formulas for the incremental stresses.

The components of incremental stress can be calculated using the following formulas, which correspond to eqs. (3.5),

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}], \quad \zeta \in S^+,$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\overline{\omega_n(\zeta)}\Phi'(\zeta)/\omega'_n(\zeta) + \Psi(\zeta)], \quad \zeta \in S^+.$$
(5.36)

The solution of the problem has been given in terms of the function $\omega'_n(\zeta)\Phi(\zeta)$, see (5.35). The value of $\Phi(\zeta)$ in a certain point ζ can be found by division of the value of $\omega'_n(\zeta)\Phi(\zeta)$ by $\omega'_n(\zeta)$. The value of $\Phi'(\zeta)$ is also not difficult to obtain, thereby starting from the derivative of $\omega'_n(\zeta)\Phi(\zeta)$, which can easily be obtained from (5.35). A complication arises, however in the calculation of $\Psi(\zeta)$. This complication, and a method to circumvent it, will be presented in this section.

The function $\Psi(\zeta)$, which has been defined only for $\zeta \in S^+ + \gamma$, and represents a stress function in the region $S^+ + \gamma$, can be expressed in terms of the function $\Phi(\zeta)$, of which the region of definition has been extended into S^- . The functional relationship between $\Phi(\zeta)$ and $\Psi(\zeta)$ is expressed by (5.19), i.e.

$$\omega_n'(\zeta)\Psi(\zeta) = \zeta^{-2}\overline{\omega_n'}(1/\zeta)\overline{\Phi}(1/\zeta) + \zeta^{-2}\overline{\omega_n'}(1/\zeta)\Phi(\zeta) - \overline{\omega_n}(1/\zeta)\Phi'(\zeta), \quad \zeta \in S^+.$$
(5.37)

All quantities in the right hand member of (5.37) can be calculated in a relatively simple way, thereby starting from the solution (5.35) and expressions for the conformal transformation $\omega_n(\zeta)$ and related functions. Therefore the value of $\Psi(\zeta)$ can, in principle, be determined from (5.37). This expression is, however, very inconvenient for numerical calculations, since implicitly it contains terms of the form $\varepsilon_k \zeta^{-k}$, where the coefficients ε_k are extremely small, but not exactly zero. In fact, all ε_k have to be zero, since the left hand member of (5.37) is holomorphic in S^+ , and therefore terms with ζ^{-k} cannot appear in the right hand member.

In the previous section, in the determination of the constants A and A_k , use was made of the fact that the function $\omega'_n(\zeta)\Psi(\zeta)$ is holomorphic in S^+ , see the considerations following eq. (5.19). Hence if A and A_k are given the values found in the previous sections, the quantities ε_k vanish identically. The numerical calculation of the constants A abd A_k , however, which involves the numerical solution of a system of equations, is never completely exact. As a consequence the coefficients ε_k are not made equal to zero, but equal to some small quantity, say $\varepsilon_k \approx 10^{-10}$. Since k may be as large as 50, this means that in the calculation of (5.37) an error is made of magnitude $10^{-10}|\zeta|^{-50}$. Only values of ζ with $|\zeta|/le1$ are of physical relevance, and one observes that if for instance $\zeta = 0.3$, then the error is magnitude of 10^{+16} , which is by no means small. Thus the error may greatly transcend the correct value, and this means that the calculations will be very inaccurate, except for values of ζ close to unity. Hence, only the stresses close to the boundary are calculated accurately.

The inaccuracies mentioned above can be removed by elaborating eq. (5.37) in such a way that all terms giving rise to negative powers of ζ are separated from the remaining terms. In other words, it is to be attempted to write $\omega'_n(\zeta)\Psi(\zeta)$ in the following form

$$\omega_n'(\zeta)\Psi(\zeta) = \sum_{k=1}^{n+1} \varepsilon_k \zeta^{-k} + f(\zeta), \qquad (5.38)$$

where $f(\zeta)$ is holomorphic in S^+ . If indeed the considerations of the previous sections are correct, the coefficients ε_k must vanish identically. That this is the case will be shown the sequel. In the numerical calculations the function $\omega'_n(\zeta)\Psi(\zeta)$ can then be replaced by $f(\zeta)$. Thus the problem now is to show that indeed all ε_k 's vanish, and to find an analytical expression for $f(\zeta)$.

It might be mentioned here that the considerations to be given below in a way duplicate the considerations of section 5.2. In fact, the equations for the determination of A and A - k as obtained in section 5.2, can also be found by requiring that the coefficients ε_k in (5.38) vanish. This corresponds to the procedure generally used by Muskhelishvili in order to determine the unknown parameters figuring in the solution of a Hilbert problem (see MUSKHELISHVILI, 1953, chapter 21). In this thesis the equations for the determination of A and A_k have been derived in section 5.2 in a different way, which in this case is much simpler. The elaborations of this section are necessary only to improve the accuracy of the numerical solution.

Rather than retain the function $\Psi(\zeta)$ as an intermediate between $\Phi(\zeta)$ and the stresses, it will be more convenient to eliminate $\Psi(\zeta)$ from the second of eqs. (5.36) and eq. (5.37). This leads to

$$\frac{1}{2}\omega_n'(\zeta)[\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}] = [\overline{\omega_n(\zeta)} - \overline{\omega_n}(1/\zeta)]\Phi'(\zeta) + \zeta^{-2}\overline{\omega_n'}(1/\zeta)\overline{\Phi}(1/\zeta) + \zeta^{-2}\overline{\omega_n'}(1/\zeta)\Phi(\zeta), \quad \zeta \in S^+.$$
(5.39)

It is this formula that will be elaborated now, until a stage is reached in which it is evident that the right hand member remains bounded for $\zeta \to 0$. For this purpose the factors that give rise to negative powers of ζ (and which should cancel when taken all together) will be separated from the remaining regular factors. The three terms of the right hand member of (5.39) will be investigated separately.

First term

It follows from (5.2) that

$$\overline{\omega_n(\zeta)} = \frac{\overline{p}}{\overline{\zeta - \sigma_0}} + \sum_{k=0}^n \overline{c_k} \overline{\zeta}^k,$$

and

$$\overline{\omega_n}(1/\zeta) = -\frac{\overline{p}\sigma_0\zeta}{\zeta - \sigma_0} + \sum_{k=0}^n \overline{c_k}\zeta^{-k}$$

Hence the first term of the right hand member of (5.39) becomes

$$[\overline{\omega_n(\zeta)} - \overline{\omega_n}(1/\zeta)] \Phi'(\zeta) = -\frac{\overline{p}\sigma_0(1-\zeta\overline{\zeta})}{|\zeta - \sigma_0|^2} \Phi'(\zeta) + \Phi'(\zeta) \sum_{k=0}^n \overline{c_k} \overline{\zeta}^k - \Phi'(\zeta) \sum_{k=0}^n \overline{c_k} \zeta^{-k}, \quad \zeta \in S^+.$$
(5.40)

In this expression only the last term is unbounded for $\zeta \to 0.$

Second term

From (5.12) and (5.15) one obtains for the second term of (5.39)

$$\zeta^{-2}\overline{\omega'_n}(1/\zeta)\overline{\Phi}(1/\zeta) = \sum_{k=1}^{2n+1} \overline{C_k}\zeta^{k-2} + \sum_{k=0}^{n-1} \overline{A_k}\zeta^{-k-2} - \overline{A_k}\overline{\zeta(\zeta-\sigma_0)}, \quad \zeta \in S^+.$$

Since $A = -C_1$, see (5.21), this can also be written as

$$\zeta^{-2}\overline{\omega'_n}(1/\zeta)\overline{\Phi}(1/\zeta) = \frac{\overline{C_1}}{\zeta - \sigma_0} + \sum_{k=0}^{2n-1} \overline{C_{k+2}}\zeta^k + \sum_{k=0}^{n-1} \overline{A_k}\zeta^{-k-2}, \quad \zeta \in S^+.$$
(5.41)

In this expression it is also only the last term that is unbounded for $\zeta \to 0$.

Third term

From (5.2) it follows that

$$\overline{\omega'_n}(1/\zeta) = -\frac{\overline{p}\sigma_0^2\zeta^2}{(\zeta - \sigma_0)^2} + \sum_{k=1}^n k\overline{c_k}\zeta^{-k+1},$$

and hence the third term of (5.39) becomes

$$\zeta^{-2}\overline{\omega_n'}(1/\zeta)\Phi(\zeta) = -\frac{\overline{p}\sigma_0^2\Phi(\zeta)}{(\zeta-\sigma_0)^2} + \Phi(\zeta)\sum_{k=1}^n k\overline{c_k}\zeta^{-k-1}, \quad \zeta \in S^+.$$
(5.42)

Factors unbounded for $\zeta \to 0$ appear only in the last term.

Addition of terms

Substitution of the three expressions (5.40), (5.41) and (5.42) into (5.39) yields

$$\frac{1}{2}\omega'_n(\zeta)[\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}] = H(\zeta) + K(\zeta), \quad \zeta \in S^+,$$
(5.43)

where $H(\zeta)$ contains all terms that are directly seen to be bounded for $\zeta \to 0$, i.e.

$$H(\zeta) = -\frac{\overline{p}\sigma_0(1-\zeta|\zeta|)\Phi'(\zeta)}{|\zeta-\sigma_0|^2} + \Phi'(\zeta)\sum_{k=0}^n \overline{c_k}\zeta^k + \frac{\overline{C_1}}{\zeta-\sigma_0} + \sum_{k=0}^{2n-1} \overline{C_{k+2}}\zeta^k - \frac{\overline{p}\sigma_0^2\Phi(\zeta)}{(\zeta-\sigma_0)^2}, \quad \zeta \in S^+.$$
(5.44)

The remaining terms are assembled in the function $K(\zeta)$,

$$K(\zeta) = -\Phi'(\zeta) \sum_{k=0}^{n} \overline{c_k} \zeta^{-k} + \sum_{k=0}^{n-1} \overline{A_k} \zeta^{-k-2} + \Phi(\zeta) \sum_{k=1}^{n} k \overline{c_k} \zeta^{-k-1}, \quad \zeta \in S^+,$$

which can also be written as

$$K(\zeta) = -\frac{d}{d\zeta} \left\{ \Phi(\zeta) \sum_{k=0}^{n} \overline{c_k} \zeta^{-k} \right\} + \sum_{k=0}^{n-1} \overline{A_k} \zeta^{-k-2}, \quad \zeta \in S^+.$$
(5.45)

Although the right hand member of (5.45) contains several factors which are not bounded for $\zeta \to 0$, the function $K(\zeta)$ itself is bounded for $\zeta \to 0$. This will be proved by elaborating (5.45).

Elaboration of $K(\zeta)$

As already indicated in section 5.2, see eq. (5.23), the function $\Phi(\zeta)$, which is holomorphic in S^+ , can be written as

$$\Phi(\zeta) = \sum_{k=0}^{n} q_k \zeta^k + \zeta^n G(\zeta), \quad \zeta \in S^+,$$
(5.46)

where $G(\zeta)$ is iself a holomorphic function in S^+ . This function, which is an auxiliary function for the calculation of $K(\zeta)$, will first be determined. Therfore the expression (5.26) for $\omega'_n(\zeta)$ will also be needed, i.e.

$$\omega_n'(\zeta) = \sum_{k=0}^{n-1} (k+1)(c_{k+1} - p\sigma_o^{-k-2})\zeta^k + \zeta^n h(\zeta), \qquad (5.47)$$

where

$$h(\zeta) = \frac{p\sigma_o^{-n-1}[n\zeta - (n+1)\sigma_0]}{(\zeta - \sigma_0)^2}.$$
(5.48)

Multiplication of (5.46) and (5.47), and some subsequent elaboration, leads to the following expression for $\omega'_n(\zeta)\Phi(\zeta)$,

$$\omega_n'(\zeta)\Phi(\zeta) = \omega_n'(\zeta)\zeta^n G(\zeta) + \zeta^n h(\zeta) \sum_{k=0}^{n-1} q_k \zeta^k + \sum_{k=0}^{n-1} \left\{ \sum_{j=0}^k (k-j+1)(c_{k-j+1} - p\sigma_0^{-k+j-2})q_j \right\} \zeta^k + \sum_{k=n}^{2n-2} \left\{ \sum_{j=k-n+1}^{n-1} (k-j+1)(c_{k-j+1} - p\sigma_0^{-k+j-2})q_j \right\} \zeta^k, \quad \zeta \in S^+.$$
(5.49)

On the other hand an expression for $\omega'_n(\zeta)\Phi(\zeta)$, valid for $\zeta \in S^+$, is given by the solution (5.12). With (5.13) this gives

$$\omega'_{n}(\zeta)\Phi(\zeta) = \sum_{k=0}^{n-1} A_{k}\zeta^{k} + \sum_{k=0}^{2n-1} b_{k}\zeta^{k} + \frac{A}{\zeta - \sigma_{0}},$$

which can also be written as

$$\omega'_{n}(\zeta)\Phi(\zeta) = \sum_{k=0}^{n-1} (A_{k} + B_{k} - A\sigma_{0}^{k-1})\zeta^{k} + \zeta^{n}R(\zeta), \quad \zeta \in S^{+},$$
(5.50)

where

$$R(\zeta)) = \sum_{k=n}^{2n-1} B_k \zeta^{k-n} + \frac{A\sigma_0^{-n}}{\zeta - \sigma_0}.$$
 (5.51)

The two expressions (5.49) and (5.50) for $\omega'_n(\zeta)\Phi(\zeta)$ must of course be equal, and this will enable to find an expression for $G(\zeta)$. Therefore it is first recalled from (5.30) that

$$A_k + B_k - A\sigma_0^{-k-1} = \sum_{j=0}^k (k-j+1)(c_{k-j+1} - p\sigma_o^{-k+j-2})q_j,$$

$$k = 0, 1, \dots, n-1, \quad (5.52)$$

and thus the terms with ζ^k , for k = 0, 1, ..., n - 1 in (5.49) and (5.50) are equal. Hence equating $\omega'_n(\zeta)\Phi(\zeta)$ from (5.49) and (5.50) gives, after cancelling these terms,

$$\omega_n'(\zeta)\zeta^n G(\zeta) + \zeta^n h(\zeta) \sum_{k=0}^{n-1} q_k \zeta^k + \sum_{k=n}^{2n-2} \left\{ \sum_{j=k-n+1}^{n-1} (k-j+1)(c_{k-j+1} - p\sigma_0^{-k+j-2})q_j \right\} \zeta^k = \zeta^n R(\zeta).$$

All terms in this equation contain a factor ζ^n . After division by this common factor the result is

$$\omega_n'(\zeta)G(\zeta) = R(\zeta) - h(\zeta) \sum_{k=0}^{n-1} q_k \zeta^k - \sum_{k=0}^{n-2} \left\{ \sum_{j=k+2}^n j(c_j - p\sigma_0^{-j-1})q_{n+k-j+1} \right\} \zeta^k, \quad \zeta \in S^+.$$
(5.53)

From this equation the function $G(\zeta)$ can be calculated.

Now returning to the elaboration of $K(\zeta)$, see eq. (5.45), it is observed that the expression between parentheses in (5.45) can be expressed in terms of $G(\zeta)$, using (5.46). This gives

$$\begin{split} \Phi(\zeta) \sum_{k=0}^{n} \overline{c_k} \zeta^{-k} &= G(\zeta) \sum_{k=0}^{n} \overline{c_{n-k}} \zeta^k + \\ & \sum_{k=1}^{n} \Bigl\{ \sum_{j=0}^{n-k} q_j \overline{c_{j+k}} \Bigr\} \zeta^{-k} + \sum_{k=0}^{n-1} \Bigl\{ \sum_{j=k}^{n-1} q_j \overline{c_{j-k}} \Bigr\} \zeta^k, \quad \zeta \in S^+. \end{split}$$

Hence, after substitution into (5.45),

$$K(\zeta) = \sum_{k=0}^{n-1} \overline{A_k} \zeta^{-k-2} - G(\zeta) \sum_{k=1}^n k \overline{c_{n-k}} \zeta^{k-1} - G'(\zeta) \sum_{k=0}^n \overline{c_{n-k}} \zeta^k + \sum_{k=0}^{n-1} (k+1) \Big\{ \sum_{j=0}^{n-k-1} q_j \overline{c_{j+k+1}} \Big\} \zeta^{-k-2} - \sum_{k=1}^{n-1} k \Big\{ \sum_{j=k}^{n-1} q_j \overline{c_{j-k}} \Big\} \zeta^{k-1}, \quad \zeta \in S^+.$$
(5.54)

It follows from (5.30) and (5.34) that

$$A_k = -(k+1) \sum_{j=0}^{n-k-1} c_{j+k+1} \overline{q_j}, \quad k = 0, 1, \dots, n-1,$$

and thus, by taking complex conjugates

$$\overline{A_k} = -(k+1) \sum_{j=0}^{n-k-1} \overline{c_{j+k+1}} q_j, \quad k = 0, 1, \dots, n-1.$$

This means that the terms with ζ^{-k-2} for $k = 0, 1, \ldots, n-1$ in eq. (5.54) cancel, and therefore (5.54) reduces to

$$K(\zeta) = -G(\zeta) \sum_{k=0}^{n-1} (k+1)\overline{c_{n-k-1}}\zeta^k - G'(\zeta) \sum_{k=0}^n \overline{c_{n-k}}\zeta^k - \sum_{k=1}^{n-1} k \Big\{ \sum_{j=k}^{n-1} q_j \overline{c_{j-k}} \Big\} \zeta^{k-1}, \quad \zeta \in S^+.$$
(5.55)

This shows that the function $K(\zeta)$ is indeed bounded for $\zeta \to 0$, which was to be proved.
Conclusion

The functions $H(\zeta)$ and $K(\zeta)$, which together constitute the quantity

$$\frac{1}{2}\omega_n'(\zeta)[\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}] = H(\zeta) + K(\zeta), \quad \zeta \in S^+, \tag{5.56}$$

see eq. (5.43), can be calculated without essential difficulties from (5.44) and (5.55). The stress components σ_{xx} , σ_{yy} and σ_{xy} can then easily be determined from the quantities $\sigma_{xx} + \sigma_{yy}$ and $\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}$. Since the formulas are rather complicated, and the calculations involve a large number of intermediate quantities, the calculations to be performed are assembled in the next chapter.

Chapter 6

RECAPITULATION OF FORMULAS

In this chapter the formulas derived in the preceding chapters are assembled. Some minor changes of notation will be introduced for the convenience of computation. The order in which the formulas are given is the same as that in which numerical calculations have to be performed. No reference to the preceding chapters, from which the formulas are taken, will be given.

Let the conformal transformation

$$z = \omega(\zeta),\tag{6.1}$$

map the interior of the unit circle γ in the ζ -plane onto a region R in the zplane, where R is the lower half plane $\Im(z) < 0$, provided with edge notches and mounds of a finite total area. Let σ_0 be the point on γ corresponding to the point at infinity in the z-plane. The conformal transformation (6.1) is approximated by the series expansion

$$\omega(\zeta) \approx \omega_n(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k < \quad n \ge 2,$$
(6.2)

where p is the residue of $\omega(\zeta)$ at its simple pole $\zeta = \sigma_0$,

$$p = \lim_{\zeta \to \sigma_0} \{ (\zeta - \sigma_0) \omega(\zeta) \}, \tag{6.3}$$

and where the coefficients c_k (k = 0, 1, 2, ..., n) are to be determined from the following equations

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \{\omega(\exp(i\theta)) - \frac{p}{\exp(i\theta) - \sigma_0}\} \exp(-ki\theta) d\theta,$$

$$k = 0, 1, \dots, n-1, \quad (6.4)$$

$$c_n = \frac{1}{2}p\sigma_0^{-n-1} - \frac{1}{2}\sigma_0^{-n}\sum_{k=0}^{n-1} \{(1+k/n)c_k\sigma_0^k - (1-k/n)\overline{c_k}\sigma_0^{-k}\}.$$
 (6.5)

From the coefficients c_k the following sets of other coefficients are to be calculated successively,

$$d_{0} = \sum_{j=0}^{n} c_{j} \sigma_{0}^{j},$$

$$d_{k} = \sum_{j=k}^{n} (j-k+1)c_{j} \sigma_{0}^{j-k}, \quad k = 1, \dots, n,$$
(6.6)

$$e_k = \sum_{\max(0,k-n)}^{\min(k,n-1)} (j+1)c_{j+1}c_{k-j}, \ k = 0, 1, \dots, 2n-1,$$
(6.7)

$$f_k = \sum_{\max(0,k-n)}^{\min(k,n-1)} (n-j)\overline{c_{n-j}}c_{k-j}, \ k = 0, 1, \dots, 2n+1,$$
(6.8)

$$B_{k} = (1+K_{0})e_{k} - 2pd_{k+2}E_{n-k-2} + (1+K_{0})\overline{p}\sigma_{0}(k+1)c_{k+1}E_{n-k-1} - (1+K_{0})\overline{f_{n-k-1}}E_{n-k-1} - (1-K_{0})f_{n+k+1}E_{n-k-2}, \qquad k = 0, 1, \dots, 2n-1,$$
(6.9)

$$C_{k} = -2p\sigma_{0}^{-2}\overline{d_{k}}E_{n-k} + (1-K_{0})\overline{p}\sigma_{0}(k-1)\overline{c_{k-1}}E_{n-k-1} + (1+K_{0})\overline{f_{n+k-1}}E_{n-k} + (1-K_{0})f_{n-k+1}E_{n-k+1} - (1-K_{0})\overline{c_{k-2}}E_{k-2}, \qquad k = 1, \dots, 2n+1,$$
(6.10)

where K_0 is the (given) coefficient of neutral earth pressure at infinity, and the symbol E_j denotes, for integer values of j,

$$E_j = \begin{cases} 1, & j \ge 0, \\ 0, & j < 0. \end{cases}$$
(6.11)

Next the coefficients q_k (k = 0, 1, ..., n - 1) are to be calculated from the following system of n linear equations

$$-B_{k} - C_{1}\sigma_{0}^{-k-1} + \sum_{j=0}^{k} (k-j+1)(c_{k-j+1} - p\sigma_{0}^{-k+j-2})q_{j} + (k+1)\sum_{j=0}^{n-k-1} c_{j+k+1}\overline{q_{j}} = 0, \quad k = 0, 1, \dots, n-1.$$
(6.12)

When these coefficients q_k have been calculated the following coefficients can be calculated

$$A = -C_1, \tag{6.13}$$

$$A_{k} = -B_{k} - C_{1}\sigma_{0}^{-k-1} + \sum_{j=0}^{k} (k-j+1)(c_{k-j+1} - p\sigma_{0}^{-k+j-2})q_{j},$$

$$k = 0, 1, \dots, n-1.$$
(6.14)

When this stage is reached, all coefficients necessary for the calculation of the stresses are, in principle, available. To facilitate the actual computation of the stresses the following sets of constants are introduced,

$$r_k = \sum_{j=k+1}^{n-1} (j+1)(c_{j+1} - p\sigma_0^{-j-2})q_{n+k-j}, \quad k = 0, 1, \dots, n-2,$$
(6.15)

$$s_k = \sum_{j=k}^{n-1} q_j \overline{c_{j-k}}, \quad k = 1, 2, \dots, n-1,$$
(6.16)

The formulas in the sequel of this chapter refer to the actual evaluation of the stress components.

The choice of a point ζ , such that $|\zeta| \leq 1$ and $\zeta \neq 0$, defines a point in the physical z-plane,

$$z = \omega_n(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k.$$
(6.17)

The stresses in this point will be calculated.

First the following auxiliary functions (the first and second derivative of the conformal transformation function $\omega_n(\zeta)$) are needed,

$$\omega_n'(\zeta) = -\frac{p}{(\zeta - \sigma_0)^2} + \sum_{k=1}^n k c_k \zeta^{k-1},$$
(6.18)

$$\omega_n''(\zeta) = \frac{2p}{(\zeta - \sigma_0)^3} + \sum_{k=2}^n k(k-1)c_k \zeta^{k-2}.$$
(6.19)

The stress function $\Phi(\zeta)$ and its first derivative $\Phi'(\zeta)$ can be calculated from the following two formulas

$$\Phi(\zeta) = \frac{1}{4}i \left\{ \sum_{k=0}^{2n-1} B_k \zeta^k + \sum_{k=0}^{n-1} A_k \zeta^k + \frac{A}{\zeta - \sigma_0} \right\} / \omega'_n(\zeta),$$
(6.20)

$$\Phi'(\zeta) = \frac{1}{4}i \left\{ \sum_{k=1}^{2n-1} kB_k \zeta^{k-1} + \sum_{k=1}^{n-1} kA_k \zeta^{k-1} - \frac{A}{(\zeta - \sigma_0)^2} - \omega_n''(\zeta) \Phi(\zeta) \right\} / \omega_n'(\zeta), \qquad (6.21)$$

Next the following auxiliary functions must be calculated consecutively

$$h(\zeta) = p\sigma_0^{-n-1} [n\zeta - (n+1)\sigma_0] / (\zeta - \sigma_0)^2,$$
(6.22)

$$h'(\zeta) = -p\sigma_0^{-n-1}[n\zeta - (n+2)\sigma_0]/(\zeta - \sigma_0)^3,$$
(6.23)

$$R(\zeta) = \sum_{k=0}^{n-1} B_{n+k} \zeta^k + A \sigma_0^{-n} / (\zeta - \sigma_0), \qquad (6.24)$$

$$R'(\zeta) = \sum_{k=1}^{n-1} k B_{n+k} \zeta^{k-1} - A\sigma_0^{-n} / (\zeta - \sigma_0)^2, \qquad (6.25)$$

$$F(\zeta) = R(\zeta) - h(\zeta) \sum_{k=0}^{n-1} q_k \zeta^k - \sum_{k=0}^{n-2} r_k \zeta^k,$$
(6.26)

$$F'(\zeta) = R'(\zeta) - h'(\zeta) \sum_{k=0}^{n-1} q_k \zeta^k - h(\zeta) \sum_{k=1}^{n-1} k q_k \zeta^{k-1} - \sum_{k=1}^{n-2} k r_k \zeta^{k-1}, \quad (6.27)$$

$$G(\zeta) = F(\zeta)/\omega'_n(\zeta), \tag{6.28}$$

$$G'(\zeta) = \left\{ F'(\zeta) - G(\zeta)\omega_n''(\zeta) \right\} / \omega_n'(\zeta), \tag{6.29}$$

$$H(\zeta) = -\frac{\overline{p}\sigma_0(1-\zeta|\zeta|)\Phi'(\zeta)}{|\zeta-\sigma_0|^2} + \Phi'(\zeta)\sum_{k=0}^n \overline{c_k}\,\overline{\zeta}^k + \frac{\overline{C_1}}{\zeta-\sigma_0} + \sum_{k=0}^{2n-1} \overline{C_{k+2}}\zeta^k - \frac{\overline{p}\sigma_0^2\Phi(\zeta)}{(\zeta-\sigma_0)^2}$$
(6.30)

$$K(\zeta) = -G(\zeta) \sum_{k=0}^{n-1} (k+1)\overline{c_{n-k-1}}\zeta^k - G'(\zeta) \sum_{k=0}^n \overline{c_{n-k}}\zeta^k - \sum_{k=1}^{n-1} ks_k \zeta^{k-1}, \qquad (6.31)$$

$$L(\zeta) = \frac{1}{2}i[H(\zeta) + K(\zeta)]/\omega'_{n}(\zeta),$$
(6.32)

$$M(\zeta) = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}].$$
(6.33)

The components of incremental stress are now obtained as

$$\sigma_{xx}/\rho g = \frac{1}{2} [M(\zeta) - \Re\{L(\zeta)\}], \tag{6.34}$$

$$\sigma_{yy}/\rho g = \frac{1}{2} [M(\zeta) + \Re\{L(\zeta)\}], \tag{6.35}$$

$$\sigma_{xy}/\rho g = \frac{1}{2} \Im\{L(\zeta)\}. \tag{6.36}$$

Finally, when the total stresses (i.e. initial stresses plus incremental stresses) are needed, these can be calculated from

$$\tau_{xx}/\rho g = \sigma_{xx}/\rho g + K_0 \Im\{\omega_n(\zeta)\},\tag{6.37}$$

$$\tau_{yy}/\rho g = \sigma_{yy}/\rho g + \Im\{\omega_n(\zeta)\},\tag{6.38}$$

$$\tau_{xy}/\rho g = \sigma_{xy}/\rho g. \tag{6.39}$$

It may be mentioned that, when ζ is considered as being dimensionless, then, since z has the dimension of a length, p and c_k will also have the dimension of a

length. The dimensions of all other quantities can then easily be investigated, and it turns out that $L(\zeta)$ and $M(\zeta)$ both have the dimension of a length. This implies that the quantities σ_{xx} , σ_{yy} and σ_{xy} indeed have the dimension of a stress, as they should.

The above formulas are somewhat simplified when the region R in the zplane is symmetric with respect to the y-axis. In that case when $\sigma_0 = -i$ (so that in the ζ -plane the axis of symmetry is likewise the imaginary axis), the constant p is real and the coefficients c_k are alternatively purely real and purely imaginary, i.e.

$$\Re\{c_k i^k\} = 0, \quad k = 0, 1, \dots, n.$$

This then enables to write the equations (6.6) - (6.16) in such a way that they involve only real quantities. No particular difficulties are encountered and no particularly interesting results are obtained, therefore this will not be elaborated here.

Chapter 7

CIRCULAR ARC NOTCH

In this chapter the case of a single circular arc notch in a half plane (fig. 7.1) is considered. In order to apply the results of chapters 5 and 6 it is necessary to find the conformal transformation of the region into the interior of the unit circle $||\zeta| = 1$, and to bring this transformation into the form



Figure 7.1: Half plane with circular arc notch.

Once that the parameters p, σ_0 and c_k have been found the calculation of the stresses along the lines of chapter 6 requires only simple algebraic operations. These operations have been assembled in a computer program and need not be reconsidered here. Therefore the considerations of this chapter can be restricted to the indication of a procedure for the calculation of p, σ_0 and c_k . The same is true for any specific case in the class of problems considered in this thesis for which the stresses are to be calculated.

In section 7.1 such a procedure for the determination of the parameters p, σ_0 and c_k will be presented. In section 7.2 some results for two special cases will be given. Finally, in section 7.3 an alternative method of solution will be presented, using Fourier integrals. This will enable a comparison of results obtained by the two methods.

7.1 The conformal transformation

In the complex plane let the region R be the region below the open line C, which consists of the parts $(-\infty, -1)$ and $(1, +\infty)$ of the real axis, and, between the points z = -1 and z = +1, of a circular arc. The interior angle at $z = \pm 1$ is denoted by α (fig. 7.1). When $\alpha = \pi$ the region R is a half plane, when $0 < \alpha < \pi$ the region R is a half plane with a circular arc noth, and when $\pi < \alpha < 2\pi$ the region R is a half plane with a circular arc mound.

The interior of the unit circle γ in the ζ -plane can be mapped onto the region R by the conformal transformation

$$z = \omega(\zeta) = \frac{a(\zeta+1)^{\alpha/\pi} + (\zeta-1)^{\alpha/\pi}/a}{a(\zeta+1)^{\alpha/\pi} - (\zeta-1)^{\alpha/\pi}/a}, \quad |\zeta| \le 1,$$
(7.1)

where

$$a = \exp(3i\alpha/4). \tag{7.2}$$

In the formula (7.1) the arguments of the quantities $\zeta + 1$ and $\zeta - 1$ are to be taken as follows

$$\begin{aligned} |\zeta| &\le 1 : \quad -\pi/2 \le \arg(\zeta + 1) \le \pi/2, \\ |\zeta| &\le 1 : \quad \pi/2 \le \arg(\zeta - 1) \le 3\pi/2. \end{aligned}$$
(7.3)

These conditions ensure that the function $\omega(\zeta)$ as defined by (7.1) is single valued.

That equation (7.1) indeed represents the conformal transformation of S^+ (the interior of γ) onto R can easily be verified. In the first place for a point σ on the upper half of γ one has (fig. 7.2),



Figure 7.2: Point on upper half of unit circle.

$$\arg(\zeta + 1) = \theta/2, \\ \arg(\zeta - 1) = (\pi + \theta)/2, \\ |\zeta + 1| = 2\cos(\theta/2), \\ |\zeta - 1| = 2\sin(\theta/2),$$

where θ is the argument of ζ , i.e. $\zeta = \sigma = \exp(i\theta)$, with $0 \le \theta \le \pi$. Then after some elaboration and simplification equation (7.1) reduces to

$$0 \le \theta \le \pi : z = \omega(\zeta) = \frac{c^{2\alpha/\pi} - s^{2\alpha/\pi} - 2i(cs)^{\alpha/\pi}\sin(\alpha)}{c^{2\alpha/\pi} + s^{2\alpha/\pi} - 2(cs)^{\alpha/\pi}\cos(\alpha)},$$
(7.4)

where

$$c = \cos(\theta/2), \quad s = \sin(\theta/2). \tag{7.5}$$

Taking the real and imaginary parts of the expression (7.4) enables to verify that

$$x^{2} + [y + \cot(\alpha)]^{2} = 1/\sin^{2}(\alpha), \tag{7.6}$$

which is the equation of the circle of which the circular arc forms a part.

for a point σ on the lower half of the unit circle γ one has (fig. 7.3), in accordance with eqs. (7.3),

$$\arg(\zeta + 1) = (\theta - 2\pi)/2, \\ \arg(\zeta - 1) = (\pi + \theta)/2, \\ |\zeta + 1| = 2\sin[(\theta - \pi)/2], \\ |\zeta - 1| = 2\cos[(\theta - \pi)/2].$$



Figure 7.3: Point on lower half of unit circle.

This then leads to the following reduced form of (7.1),

$$\pi \le \theta \le 2\pi : \ z = \omega(\sigma) = \frac{s_*^{\alpha/\pi} + c_*^{\alpha/\pi}}{s_*^{\alpha/\pi} - c_*^{\alpha/\pi}},\tag{7.7}$$

where

$$c_* = \cos[(\theta - \pi)/2], \quad s_* = \sin[(\theta - \pi)/2].$$
 (7.8)

Since (7.7) is real it follows that to the lower half of the unit circle γ there correspond points on the real axis in the z-plane, as required. The points $\theta = \pi$ and $\theta = 2\pi$ correspond to z = -1, respectively z = +1. For $\theta = 3\pi/2$ the denominator in (7.7) is zero, indicating that then $x = \pm \infty$. The behaviour near this point, which is a singularity of the conformal transformation (7.1) deserves some special attention.

By writing $\zeta = -i + \varepsilon$ in eq. (7.1) and then elaborating each factor, one obtaines a series expansion of $\omega(\zeta)$ increasing powers of ε . If in this series ε is replaced by $\zeta + i$ the result is the following Laurent series expansion of $\omega(\zeta)$ around $\zeta = -i$:

$$|\zeta + i| \ll 1 : \ \omega(\zeta) = \frac{2\pi/\alpha}{\zeta + i} + \frac{\pi i}{\alpha} - \frac{1}{6} \left(\frac{\pi}{\alpha} - \frac{\alpha}{\pi}\right) (\zeta + i) + \dots$$
(7.9)

This shows that the singularity $\zeta = -i$ is a first order pole, and that the region in the z-plane indeed approximates a half plane at infinity. For a point on the unit circle γ close to $\zeta = -i$ one may write

$$\zeta = \sigma = \exp(i\theta) = \exp[i(3\pi/2 + \delta)].$$

Expanding this for small values of δ shows that

 $\zeta = -i + \delta + i\delta^2/2 - \delta^3/6,$

hence

$$\zeta + i = \delta + i\delta^2/2 + \delta^3/6.$$

Using this result equation (7.9) can be rewritten in terms of increasing powers of δ . When δ is then replaced by $\theta - 3\pi/2$ one obtains

$$|\theta - 3\pi/2| \ll 1 : \omega(\sigma) = \frac{2\pi/\alpha}{\theta - 3\pi/2} - \frac{1}{6} \left(\frac{2\pi}{\alpha} - \frac{\alpha}{\pi}\right) (\theta - 3\pi/2) + \dots (7.10)$$

In chapter 4 the general conformal transformation was written as follows (see eq. (4.1)),

$$\omega(\zeta) = \frac{p}{\zeta - \sigma_0} + \omega_0(\zeta), \quad \zeta \in S^+ + \gamma.$$
(7.11)

where the function $\omega_0(\zeta)$ is holomorphic in S^+ and continuous (and thus bounded) in $S^+ + \gamma$. It follows from (7.9) that for the class of problems considered in this chapter

$$\sigma_0 = -i, \quad p = 2\pi/\alpha. \tag{7.12}$$

It may be observed from (7.12) that p/σ_0 is an imaginary quantity, which was obtained in chapter 4 as a condition to be imposed on the conformal transformation, see (4.18).

The regular part of the conformal transformation, the function $\omega_0(\zeta)$, is now given by

$$\omega_0(\zeta) = \omega(\zeta) - \frac{2\pi/\alpha}{\zeta + i},\tag{7.13}$$

and, in accordance with the considerations of chapter 4, this function is expanded in a Taylor series around $\zeta = 0$,

$$\omega_0(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k.$$
(7.14)

An approximation to the conformal transformation is obtained by taking into account the first n terms only,

$$\omega(\zeta) \approx \omega_n(\zeta) = \frac{2\pi/\alpha}{\zeta+i} + \sum_{k=0}^n c_k \zeta^k.$$
(7.15)

The coefficients c_k may be calculated from the equations (4.10), i.e.

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \omega_*(\theta) \, \exp(-ki\theta) \, d\theta, \ k = 0, 1, 2, \dots,$$
(7.16)

where

$$\omega_*(\theta) = \omega_0(\sigma) = \omega_0(\exp[i\theta]). \tag{7.17}$$

In principle the problem of determining the approximate conformal transformation is now solved, since σ_0 and p are known, and the coefficients c_k can be calculated from (7.16).

In general, the integrals (7.16) may be calculated numerically using Filon's methods (see appendix A). The values of the function $\omega_*(\theta)$, needed for the application of this method, can be determined by first calculating the value of $\omega(\sigma)$ in the point $\sigma = \exp(i\theta)$, using either (7.4) or (7.7), and then subtracting the value of $2\pi/[\alpha(\sigma + i)]$. This procedure works well, except near $\sigma = -i$, where the value of $\omega_*(\theta)$ is best obtained by combining (7.9) and (7.13). This gives

$$|\theta - 3\pi/2| \ll 1$$
 : $\omega_*(\theta) = \frac{\pi i}{6} - \frac{1}{6} \left(\frac{\pi}{\alpha} - \frac{\alpha}{\pi}\right) (\theta - 3\pi/2) + \dots,$

where the error is of magnitude $O(\theta - 3\pi/2)^3$.

For the special case $\alpha = \pi/2$ the integrals (7.16) happen to admit an exact evaluation. The details of this integration will not be given here, but only the result, which is

$$\alpha = \pi/2$$
 : $c_0 = (3 - \sqrt{2})i$, $c_k = 2i^{k+1} - i\sqrt{2}\sum_{j=0}^{k-j} i^k b_j$, $k = 1, 2, \dots$,

where

$$b_0 = 1, \ b_1 = 0, \ b_k = [(k-3)/k]b_{k-2}, \ k = 2, 3, \dots$$

For this special case ($\alpha = \pi/2$), which is the case of a semi-circular arc notch, some numerical results are given in table 7.1. Since the coefficients are alternatively imaginary and real, only the values of $\Im(c_k i^k)$ are given, the values of $\Re(c_k i^k)$ being zero for all values of k. In the second column of the table the exact values of the coefficients are given. The third column (marked Filon2) shows the values obtained using Filon's method. In this method (FILON, 1928) the function $\omega_*(\theta)$, see eq. (7.16), is approximated by a second order polynomial, using a subdivision of the integration interval $(0, 2\pi)$ into a large number of equal parts. The fourth column (marked Filon4) shows the results obtained using an extension of Filon's method (see appendix A), in which a fourth order polynomial approximation is used. The interval $(0, 2\pi)$ was subdivided into 40 equal parts. Computer time for both calculations was about the same (approximately 150 seconds for the calculation of 50 coefficients, on the Telefunken

	$\Im(c_k i^k)$		
k	exact	Filon2	Filon4
0	+1.58579	+1.58586	+1.58582
1	-0.58579	-0.58586	-0.58582
2	-0.12132	-0.12139	-0.12135
3	+0.12132	+0.12139	+0.12135
4	+0.05546	+0.05553	+0.05549
5	-0.05546	-0.05553	-0.05549
6	-0.03293	-0.03301	-0.03296
7	+0.03293	+0.03301	+0.03293
8	+0.02231	+0.02238	+0.02234
9	-0.02231	-0.02238	-0.02233
10	-0.01636	-0.01643	-0.01640
20	+0.00606	+0.00613	+0.00609
30	-0.00334	-0.00343	-0.00338
40	+0.00219	+0.00227	+0.00222
50	-0.00157	-0.00163	-0.00160

Table 7.1: Coefficients c_k for $\alpha = \pi/2$.

TR4 of the Delft University of Technology) and it is found that the extended Filon rule yields somewhat more accurate results. It has been observed that by using the extended Filon rule with a subdivision into 20 equal parts about the same accuracy is achieved as with the normal Filon rule with a subdivision into 200 equal parts. The gain in computer time then is about 50 %.

Examples

The region corresponding to the interior of the circle $|\zeta| = 1$ by the approximate conformal transformation is shown in figure 7.4. The left half of the figure has been determined using 50 terms in the Taylor series expansion. In the right half of the figure the results are shown of an approximation using 100 terms. Both parts of the figure have been drawn using the exact values of the Taylor series coefficients, but use of the coefficients calculated by Filon's method would not lead to discernible differences.

As a further illustration some more results are shown in figures 7.5 and 7.6. These figures show the approximate regions obtained by using 50 terms of the Taylor series for $\alpha = 5\pi/6$ and $\alpha = 3\pi/2$.

The most striking difference between the figures 7.4 and 7.6 is that in the former figure the approximation close to the corner point is rather bad, whereas in the latter figure the approximation is extremely good. The accuracy for this case is further illustrated in figure 7.7, which shows a detail of figure 7.6, close to the corner point. The drawn line is the original contour, whereas the dots respresent points on the approximate contour. The very good approximation



Figure 7.4: Approximation of half plane with semi-circular notch.



Figure 7.5: Approximation of half plane with circular arc notch.

near a re-entrant angle is in accordance with results for a plate with a square or triangular hole (see MUSKHELISHVILI, 1963). It appears that the approximation near a salient angle is much worse. Fortunately, such a corner point is, in the absence of external loadings, a dead corner with respect to the stresses, i.e. the stresses tend to zero near such a point. A closer investigation of this property is presented below.



Figure 7.6: Approximation of half plane with semi-circular mound.



Figure 7.7: Detail of corner point.

Generalization of the approximation procedure

The approximation of a complex function $f(\theta)$, continuous for $0 \le \theta \le 2\pi$, and satisfying the condition $f(0) = f(2\pi)$, by the first *n* terms of its Fourier expansion, represents a least squares approximation with a uniform weight function. This means that the quantity

$$E = \int_0^{2\pi} \left\{ f(\theta) - \sum_{k=0}^n c_k \exp(ki\theta) \right\} \left\{ \overline{f(\theta)} - \sum_{k=0}^n \overline{c_k} \exp(-ki\theta) \right\} d\theta,$$

attains the least possible value when the coefficients c_k are chosen as

$$c_k = \int_0^{2\pi} f(\theta) \exp(-ki\theta) d\theta$$

Thus the deviations from the original function $f(\theta)$ are minimized with a weight $A \Delta \theta$ per elementary arc of length $\Delta \theta$, where A is a constant. Since in general for a conformal transformation $z = \omega(\zeta)$ one has

$$|\Delta z| = |\Delta \zeta| |\omega'(\zeta)|,$$

and since for points on the unit circle $|\Delta\zeta| = \Delta\theta$, it now follows that the error on an arc of length $|\Delta z|$ in the z-plane is counted with a weight $A|\Delta z|/|\omega'(\zeta)|$, and this is certainly not the same for all elementary arcs of the same length $|\Delta z|$. In fact, it has been seen in chapter 4, see eq. (4.5), that near a point σ_m corresponding to a corner point of the contour in the z-plane,

$$\omega'(\zeta) = B(\zeta - \sigma_m)^{\alpha/\pi} [1 + o(1)],$$

where $0 < \alpha < \pi$ for a re-entrant angle and $-\pi < \alpha < 0$ for a salient angle. Hence in a re-entrant angle $\omega'(\zeta)$ vanishes, and in a salient angle $\omega'(\zeta)$ is unbounded. From this it follows that the weight $A|\Delta z|/|\omega'(\zeta)|$ becomes zero near a salient angle, and this weight is infinite near a re-entrant angle. This explains why the Taylor series approximation is so bad for a salient angle, and leads to such extremely good results near a re-entrant angle.

The considerations just given suggest to investigate the possibility of improving the approximation near salient angles by the introduction of a variable weight function in the least squares approximation. This then leads to different values for the coefficients c_k , which then have to be calculated from a system of linear equations. Indeed, it has been found that, by using weight functions which concentrate the weight in points corresponding to salient angles, a considerable improvement for the immediate neighbourhood of salient angles can be achieved. However, the result is also that then in the remaining parts of the contour the approximation becomes much worse than it was before, and this completely spoils the total results. It may thus be possible to find a better approximation of the region, but then larger errors are made in parts of the region where the stresses are not small.

7.2 Numerical results

In this section some numerical results, which may serve as an illustration of the considerations of chapter 5, will be presented. This will also give the opportunity to compare certain results with an exact solution for a special case. Some of the results have already been presented elsewhere (VERRUIJT, 1969). This section also contains some general considerations on the accuracy of the method.

In the first place attention will be paid to the case $\alpha = \pi/2$, i.e. the case of a semi-circular notch. For this case an exact solution, valid only for $K_0 = 1$ exists¹, namely

$$\tau_{xx}/\rho g R = -(R/r)\sin\psi\cos^2\psi + (r/R)\sin\psi,$$

$$\tau_{yy}/\rho g R = -(R/r)\sin^3\psi + (r/R)\sin\psi,$$

$$\tau_{xy}/\rho g R = -(R/r)\sin^2\psi\cos\psi,$$

(7.18)

where r and ψ are polar coordinates and R is the radius of the semi-circle. That this solution satisfies the conditions of a stress free boundary can be verified without difficulty. The incremental stresses (due to the excavation only) are represented by the first terms in the right hand members of eqs. (7.18). Actually, these stresses also represent the solution of the problem of a concentrated load of magnitude $\pi \rho g R^2$ (the weight of the material in the excavation), acting on the lower half plane $y \leq 0$ (the Boussinesq-Flamant problem, see e.g. scshape Timoshenko & Goodier, 1951, p. 87).

It can be shown that in this case the contours of constant $(\sigma_{xx} + \sigma_{yy})/2$, the isotropic part of the incremental stress tensor, and those of constant maximum shear stress,

$$\tau = \sqrt{(\sigma_{xx} - \sigma_{yy})^2 / 4 + \sigma_{xy}^2},$$

are circles passing through the origin. In figure 7.8 the contours of constant $(\sigma_{xx} + \sigma_{yy})/2$ for the exact solution are represented by the drawn lines. In the same figure the results of calculations by the approximate method presented in this thesis are represented by the dots. The results were obtained by approximating the conformal transformation by a series containing the first 50 terms of the Taylor series expansion of the regular part. The stresses were calulated in 240 points, namely those for which $|\zeta| = 0, 0.1, \ldots, 1$ and $\arg(\zeta) = 0, \pi/12, \ldots, 2\pi$. The points in which $(\sigma_{xx} + \sigma_{yy})/2$ has a certain value were obtained by linear interpolation (in the ζ -plane) between the 240 points for which the stresses were calculated.

In figure 7.9 the contours of maximum incremental shear stress are shown. Again the fully drawn lines repesent the circles corresponding to the incremental part of the exact solution (7.18), and the dots are the results of the approximate complex variable method. In both figures 7.8 and 7.9 the correspondence between the exact and approximate resluts is fairly good, except

¹This solution was brought to the author's attention by H.L. Koning



Figure 7.8: Contours of $(\sigma_{xx} + \sigma_{yy})/2$ for semi-circular notch, $K_0 = 1$.

close to the boundary, where considerable errors (of about 10 %) occur. A general consideration on errors, which explains why these errors occur, is presented below.

In general the conformal transformation is approximated by a formula of the form

$$\omega_n(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k.$$

It can be expected that the deviation (ε) of the approximate boundary from the exact one is of the order of magnitude of the last term,

$$\varepsilon \approx |c_n|. \tag{7.19}$$



Figure 7.9: Contours of $(\tau/2 \text{ for semi-circular notch}, K_0 = 1.$

The last term of the series, $c_n \zeta^n$, results in a wave of constant amplitude and variable period in the boundary in the z-plane (fig. 7.10). The wave length (2ℓ) can be calculated by noting that an increase of $\arg(\zeta)$ by $2\pi/n$ results in the term $c_n \zeta^n$ to return to its original value. In general the relation between the lengths of small elementary arcs in the planes of ζ and z is

$$|\Delta z|/|\Delta \zeta| = |\omega'_n(\zeta)|.$$

Now, since in this case $|\Delta \zeta| = 2\pi/n$ corresponds to $|\Delta z = 2\ell$, it follows that

$$\ell \approx \pi |\omega_n'(\zeta)|/n. \tag{7.20}$$

In order to investigate the influence of this wave on the radius of curvature of the arc it is noted that in general the height f of an arc of chord length ℓ and



Figure 7.10: Wave on boundary.

radius R (fig. 7.11) is

 $f = \ell^2 / 8R,$



Figure 7.11: Stresses on boundary element.

provided that $\ell^2 \ll R^2$. When in f an error ε is made, the radius of curvature will be different, say R^* . Then

$$f \pm \varepsilon = \ell^2 / 8R^*$$

It now follows that

$$\frac{1}{R} - \frac{1}{R^*} \pm \frac{8\varepsilon}{\ell^2},$$

or, with (7.19) and (7.20),

$$\frac{1}{R^*} \approx \frac{1}{R} \pm \frac{8n^2 |c_n|}{\pi^2 |\omega_n'(\zeta)|^2}.$$
(7.21)

The influence of this error in the radius of curvature of the boundary on the stresses can be realized by investigating its effect on the equation of equilibrium in the direction normal to the boundary (fig. 7.11),

$$\frac{\partial \tau_{nn}}{\partial n} + \frac{\tau_{nn} - \tau_{tt}}{R} + \frac{\partial \tau_{nt}}{\partial t} + \rho g \sin \beta = 0,$$

where β is the angle of the normal to the boundary with the vertical direction. In order to obtain an over-estimation of the error in the stresses it will now be investigated what error $(\Delta \tau_{tt})$ must occur in the stress component τ_{tt} to balance the error in the radius of curvature. This means that in the equation of equilibrium all stresses are assumed to remain unchanged, except τ_{tt} . This equation then becomes, with (7.21),

$$\frac{\partial \tau_{nn}}{\partial n} + \frac{\tau_{nn} - \tau_{tt} - \Delta \tau_{tt}}{R} \pm (\tau_{nn} - \tau_{tt} - \Delta \tau_{tt}) \frac{8n^2 |c_n|}{\pi^2 |\omega'_n(\zeta)|^2} + \frac{\partial \tau_{nt}}{\partial t} + \rho g \sin \beta = 0$$

With the aid of the original equation of equilibrium this gives

$$-\frac{\Delta\tau_{tt}}{R} \pm (\tau_{nn} - \tau_{tt} - \Delta\tau_{tt}) \frac{8n^2 |c_n|}{\pi^2 |\omega'_n(\zeta)|^2} \approx 0.$$

Hence, since along the boundary $\tau_{nn} = 0$, one obtains, taking the largest of the two possible values,

$$\left|\frac{\Delta\tau_{tt}}{\tau_{tt}}\right| \le \frac{1}{|1 - \pi^2|\omega_n'(\zeta)|^2/(8n^2|c_n|R)|}.$$
(7.22)

It should be noted that eq. (7.22) provides merely a probable upper boud for the error in the stresses.

In the case of a semi-circular arc notch of radius R = 1, the values n = 49and $c_n = 0.00167$ correspond to each other. The smallest value of $|\omega'_n(\zeta)|$ along the boundary occurs for $\zeta = i$, which corresponds to z = -i, the deepest point of the notch. For $\zeta = i$ the value of $|\omega'(\zeta)|$ is obtained from (7.1) as 0.5. With these numerical values it is found that

$$\pi^2 |\omega'(\zeta)|^2 / (8n^2 |c_n| R = 0.00769$$

and hence with (7.22) this gives

$$\left|\frac{\Delta \tau_{tt}}{\tau_{tt}}\right| \le 1.08$$

It can now be concluded that the approximate calculations will lead to values for the stresses in points of the boundary which may differ from the exact values by errors at the most of the same order of magnitude as the stresses themselves. It is to be remembered that the considerations just given can merely be interpreted as providing some insight in the order of accuracy. In reality the situation is more complicated because of the fact that the shape of the boundary in the z-plane is the result of a large number of waves, of which only the one with the highest frequency has been taken into account here.

The conclusion to be drawn from the above considerations is that the approximate method of this thesis is not well suited for the calculation of the stresses along the boundary. This general conclusion is confirmed by the results shown in figures 7.8 and 7.9, in which along the boundary errors of about 10 % occur. In section 7.3 (fig. 7.19) results of another example will be presented in which even larger errors occur. There it will also appear that these errors occur in the form of a wave of short period, with a wave length corresponding to the predicted one.

The expression (7.22) for the order of magnitude of the relative errors in the stresses along the boundary also shows that taking more terms into account results in an improvement only when the series converges more rapidly than $1/n^2$. In the case of a semi-circular arc notch this is not the case (then $nc_n \to 0$, but n^2c_n is unbounded for $n \to \infty$) and therefore the errors may increase when more terms are taken into account. In general it may stated that the approximate complex variable method of this thesis yields accurate results for the stresses along the boundary if

$$8n^2 |c_n| R/(\pi^2 |\omega'(\zeta)|^2) \ll 1.$$
(7.23)

The effect mentioned above is somewhat disturbing and is certainly a disadvantage of the approximate method considered here. On the other hand, however, it is to be expected that this effect occurs only in the immediate vicinty of the boundary. More specifically, it can be expected, on the basis of De Siant-Venant's principle, that at a distance large compared to ℓ and ε the edge disturbance is no longer of importance. In the case of a semi-circular arc notch this is certainly confirmed by the results shown in figures 7.8 and 7.9, away from the boundary. Also, many other specific examples that have been elaborated (and of which some results will be presented in other sections of this thesis) have reconfirmed the validity of De Saint-Venant's principle for the type of problems considered.



Figure 7.12: Vertical stress τ_{yy} on plane y = -R.

As a further illustration of the numerical results obtained for the case of a semi-circular arc notch, figure 7.12 shows the vertical normal stress τ_{yy} along the horizontal line y = -R (a horizontal line passing through the deepest point of the notch) for two particular values of the coefficient of neutral earth pressure (indicated by θ in the figure), namely $K_0 = 0$ and $K_0 = 1$. This figure shows

that the coefficient of neutral earth pressure has a considerable influence also on the vertical stresses.

As a final illustration the stresses along the base of a semi-circular mound $(\alpha = 3\pi/2)$ are shown in figure 7.13, for $K_0 = 0$ and n = 19. It appears from this figure that the stresses (in particular the normal stress in a direction tangent to the boundary) have a singularity in the corner point z = R. It is to be noted that the results of figure 7.13 can be expected to be rather accurate, since the approximation of the boundary has been found to be very good in this case, see figures 7.6 and 7.7. It may also be mentioned that in this case formula (7.22) leads to an estimate of the relative error in the stresses along the boundary of abou 0.08. Moreover, by taking more terms into account the



Figure 7.13: Stresses on the base of a semi-circular mound, $K_0 = 0$.

accuracy is improved in this case, since now it happens that the series converges more rapidly, in fact,

$$\lim_{n \to \infty} n^2 c_n = 0.$$

Apart from the singularity in the corner point the vertical components of stress, τ_{yy} , along the base of the mound appear to be fairly constant.

7.3 Comparison with Fourier integral method

This section presents an alternative method of solution of the problem of a circular arc notch in an elastic half plane. The solution is exact, and therefore it provides a suitable test for the verification of the approximate complex variable solution of the preceding chapters. The method was suggested by KOITER (1968), and is analogous to a method for the calculation of stress concentrations in a stretched half plane with a circular arch notch ((LING, 1947; see also GREEN & ZERNA, 1954, p. 317). Since the method of this section is completely different from the methods generally used in this thesis, it requires a somewhat different terminology, and therefore the problem is setup and elaborated independently.



Figure 7.14: Half plane with circular arc notch.

Statement of the problem

The problem of an elastic half plane $-\infty < x < +\infty$, $y \leq 0$ with a circular arc notch is considered (fig. 7.14)/ The stresses due to gravity are to be calculated. Following KOITER (1968) the stresses are decomposed as follows,

$$\tau_{xx} = K_0 \rho g y + s_{xx} + t_{xx},$$

$$\tau_{yy} = \rho g y + s_{yy} + t_{yy},$$

$$\tau_{xy} = s_{xy} + t_{xy},$$

(7.24)

where K_0 is the coefficient of neutral earth pressure (which is considered as a given constant). In eq. (7.24) the stresses s_{xx} , s_{yy} , s_{xy} are the stresses corresponding to the solution for a given concentrated force P, equal in magnitude to the weight of the excavated soil, acting in the origin of a lower half plane (fig. 7.15). The value of P is

$$P = \rho g R^2 (\beta - \sin \beta \cos \beta), \tag{7.25}$$

where R and β define the circular arc notch (fig. 7.14). The solution of the problem for a concentrated force P on a lower half plane (the Boussinesq-



Figure 7.15: Half plane with concentrated force.

Flamant problem) is, see TIMOSHENKO & GOODIER (1951, p. 87),

$$s_{xx} = -\frac{2P}{\pi r} \sin \psi \cos^2 \psi,$$

$$s_{yy} = -\frac{2P}{\pi r} \sin^3 \psi,$$

$$s_{xy} = -\frac{2P}{\pi r} \sin^2 \psi \cos \psi,$$

(7.26)

where r and ψ are polar coordinates in the x, y-plane.

The stresses t_{xx} , t_{yy} , t_{xy} have to satisfy the equations of equilibrium in the absence of body forces. The boundary conditions are to be determined from the requirement that the boundary EABCD (fig. 7.14) is to be free of external stress.

Stress function

The stresses t_{xx} , t_{yy} , t_{xy} can be derived from a biharmonic function (Airy's stress function) $U_1(x, y)$ according to the following formulas

$$t_{xx} = -\frac{\partial^2 U_1}{\partial y^2}, \quad t_{yy} = -\frac{\partial^2 U_1}{\partial x^2}, \quad t_{xy} = -\frac{\partial^2 U_1}{\partial x \partial y}.$$
(7.27)

The boundary conditions for the total stresses are :

EA :
$$\tau_{yy} = \tau_{xy} = 0,$$

CD : $\tau_{yy} = \tau_{xy} = 0,$
ABC : $\tau_{xx}\nu_x + \tau_{xy}\nu_y = \tau_{xy}\nu_x + \tau_{yy}\nu_y = 0,$
(7.28)

where ν_x and ν_y are the components in x and y direction of the outwardly directed unit vector $\boldsymbol{\nu}$, normal to the boundary. With (7.24) and (7.26) the first two conditions are easily transformed into conditions for t_{xx} , t_{yy} , t_{xy} ,

EA :
$$t_{yy} = t_{xy} = 0,$$

CD : $t_{yy} = t_{xy} = 0.$
(7.29)

The boundary condition along ABC requires some more investigation. Therefore, a coordinate φ , defining points on the arc ABC is introduced, see figure 7.16. If the cartesian coordinates of a point on ABC are denoted by x_0 , y_0 ,



Figure 7.16: Point on arc ABC of boundary.

the relation with φ is

$$x_0 = R\sin\varphi, \quad y_0 = R(\cos\beta - \cos\varphi). \tag{7.30}$$

From this it follows that

$$dx_0/d\varphi = R\cos\varphi, \quad dy_0/d\varphi = R\sin\varphi.$$
 (7.31)

Furthermore, along ABC the values of ν_x and ν_y are (fig. 7.16),

$$\nu_x = -\sin\varphi = -\frac{1}{R}\frac{dy_0}{d\varphi}, \quad \nu_y = \cos\varphi = \frac{1}{R}\frac{dx_0}{d\varphi}.$$
(7.32)

Together with (7.27) these expressions enable to write

$$t_{xx}\nu_x + t_{xy}\nu_y = -\frac{1}{R} \left\{ \frac{\partial^2 U_1}{\partial y^2} \frac{dy_0}{d\varphi} + \frac{\partial^2 U_1}{\partial x \partial y} \frac{dx_0}{d\varphi} \right\} = -\frac{1}{R} \frac{d}{d\varphi} \left(\frac{\partial U_1}{\partial y} \right)_0,$$

$$(7.33)$$

$$t_{xy}\nu_x + t_{yy}\nu_y = \frac{1}{R} \left\{ \frac{\partial^2 U_1}{\partial x \partial y} \frac{dy_0}{d\varphi} + \frac{\partial^2 U_1}{\partial x^2} \frac{dx_0}{d\varphi} \right\} = \frac{1}{R} \frac{d}{d\varphi} \left(\frac{\partial U_1}{\partial x} \right)_0.$$

In these equations the subscript $_0$ indicates that the values are to be taken in points of the arc ABC.

In order to relate the stresses t_{xx} , t_{yy} , t_{xy} to the stresses τ_{xx} , τ_{yy} , τ_{xy} along ABC the expressions (7.24) and (7.26) must be used. The polar coordinates r_0 and ψ_0 of a point on ABC are related to φ as follows

$$r_{0} = \sqrt{x_{0}^{2} + y_{0}^{2}} = R\sqrt{1 - 2\cos\beta\cos\varphi + \cos^{2}\beta},$$

$$\cos\psi_{0} = x_{0}/r_{0} = \sin\varphi/\sqrt{1 - 2\cos\beta\cos\varphi + \cos^{2}\beta},$$

$$\sin\psi_{0} = y_{0}/r_{0} = (\cos\beta - \cos\varphi)/\sqrt{1 - 2\cos\beta\cos\varphi + \cos^{2}\beta}.$$
(7.34)

It now follows that, for points on ABC,

$$t_{xx} = \tau_{xx} - K_0 \rho g R(\cos\beta - \cos\varphi) + \frac{2P}{\pi R} \frac{(\cos\beta - \cos\varphi)\sin^2\varphi}{(1 - 2\cos\beta\cos\varphi + \cos^2\beta)^2},$$
$$t_{yy} = \tau_{yy} - \rho g R(\cos\beta - \cos\varphi) + \frac{2P}{\pi R} \frac{(\cos\beta - \cos\varphi)^3}{(1 - 2\cos\beta\cos\varphi + \cos^2\beta)^2},$$
$$t_{xy} = \tau_{xy} + \frac{2P}{\pi R} \frac{(\cos\beta - \cos\varphi)^2\sin\varphi}{(1 - 2\cos\beta\cos\varphi + \cos^2\beta)^2},$$

Hence, with (7.32) and the third of eqs. (7.28),

ABC :
$$t_{xx}\nu_x + t_{xy}\nu_y = X(\varphi),$$
 (7.35)

ABC :
$$t_{xy}\nu_x + t_{yy}\nu_y = Y(\varphi),$$
 (7.36)

where

$$X(\varphi) = K_0 \rho g R(\cos\beta - \cos\varphi) \sin\varphi - \frac{2P}{\pi R} \frac{\sin\varphi(\cos\beta - \cos\varphi)(1 - \cos\beta\cos\varphi)}{(1 - 2\cos\beta\cos\varphi + \cos^2\beta)^2},$$
(7.37)

$$Y(\varphi) = -\rho g R(\cos\beta - \cos\varphi)\cos\varphi - \frac{2P}{\pi R} \frac{\sin\varphi(\cos\beta - \cos\varphi)^2 (1 - \cos\beta\cos\varphi)}{(1 - 2\cos\beta\cos\varphi + \cos^2\beta)^2},$$
(7.38)

Equations (7.35) and (7.36) are the boundary conditions for the stresses t_{xx} , t_{yy} , t_{xy} along the circular arc ABC. With (7.33) these conditions can be expressed in terms of the stress function U_1 . This gives

ABC :
$$-\frac{1}{R} \frac{d}{d\varphi} \left(\frac{\partial U_1}{\partial y}\right)_0 = X(\varphi),$$
 (7.39)

ABC :
$$\frac{1}{R} \frac{d}{d\varphi} \left(\frac{\partial U_1}{\partial x} \right)_0 = Y(\varphi).$$
 (7.40)

The boundary conditions along EA and CD, see (7.29), lead to

EA, CD :
$$\frac{\partial^2 U_1}{\partial x^2} = \frac{\partial^2 U_1}{\partial x \partial y} = 0.$$

Since along EA and CD the value of y is constant (y = 0) these equations can be integrated in the x direction,

EA, CD :
$$\frac{\partial U_1}{\partial x} = \text{constant}, \quad \frac{\partial U_1}{\partial y} = \text{constant}.$$

Along one part of the boundarys the constants can be chosen arbitrarily. Choosing the constants zero along EA gives

EA :
$$\frac{\partial U_1}{\partial x} = 0$$
, $\frac{\partial U_1}{\partial y} = 0$, (7.41)

which are the boundary conditions for U_1 along EA. With (7.39) and (7.40) the boundary conditions along ABC can be written as

ABC :
$$\frac{\partial U_1}{\partial x} = R \int_{-\beta}^{\varphi} Y(\varphi) \, d\varphi,$$

ABC : $\frac{\partial U_1}{\partial y} = -R \int_{-\beta}^{\varphi} X(\varphi) \, d\varphi.$
(7.42)

By taking the lower limit of integration of the integrals (7.42) as $\varphi = -\beta$ it has been ensured that in the point A the quantities $\partial U_1/\partial x$ and $\partial U_1/\partial y$ are continuous. The values of $\partial U_1/\partial x$ and $\partial U_1/\partial y$ in the point C are given by

C :
$$\frac{\partial U_1}{\partial x} = R \int_{-\beta}^{\beta} Y(\varphi) \, d\varphi, \quad \frac{\partial U_1}{\partial y} = -R \int_{-\beta}^{\beta} X(\varphi) \, d\varphi.$$
 (7.43)

These integrals can be evaluated using (7.37) and (7.38). In both cases the result is exactly zero, which is, physically speaking, a consequence of the fact that the loading system for the stresses t_{xx} , t_{yy} , t_{xy} along the arc ABC is itself in equilibrium. In its turn this is a consequence of the separation (7.24) of the total stresses τ in a part accounting for the body forces, in a part s which accounts for the resulting force P of the loading, and a remaining part t. Beacuse of the vanishing of the integrals (7.43) the constants in the boundarys condition along CD are equal to those on EA. Hence these conditions are

$$CD : \frac{\partial U_1}{\partial x} = 0, \quad \frac{\partial U_1}{\partial y} = 0,$$
 (7.44)

The boundary condition $\partial U_1/\partial x = 0$, valid along EA and CD expresses that U_1 is constant along these parts of the boundary. In order that U_1 be continuous at infinity these constant values must be equal along EA and CD. Without loss

of generality these constants may be taken as zero. The mathematical problem is now

$$\nabla^2 \nabla^2 U_1 = 0, \tag{7.45}$$

with the boundary conditions

EA, CD :
$$U_1 = 0$$
, $\partial U_1 / \partial y = 0$, (7.46)

ABC :
$$\frac{\partial U_1}{\partial x} = R \int_{-\beta}^{\varphi} Y(\varphi) \, d\varphi,$$
 (7.47)

ABC :
$$\frac{\partial U_1}{\partial y} = -R \int_{-\beta}^{\varphi} X(\varphi) \, d\varphi,$$
 (7.48)

where $X(\varphi)$ and $Y(\varphi)$ are given by (7.37) and (7.38).

Introduction of potentials

The problem will be solved using potentials. Therefore it is used that every solution of the biharmonic equation (7.45), which is to hold in the region occupied by the body, can be written as

$$U_1(x,y) = -2ay\Phi_1(x,y) + (x^2 + y^2 - a^2)\Psi_1(x,y),$$
(7.49)

where Φ_1 and Ψ_1 are harmonic functions,

$$\nabla^2 \Phi_1 = 0, \quad \nabla^2 \Psi_1 = 0, \tag{7.50}$$

and where

$$a = R\sin\beta. \tag{7.51}$$

The decomposition (7.49) of the biharmonic function U_1 into two harmonic functions deserves some further clarification. The general solution of Goursat of the biharmonic equation (MUSKHELISHVILI, 1953, p. 110) is

$$2U_1 = \overline{z}f_1(z) + z\overline{f_1(z)} + f_2(z) + \overline{f_2(z)},$$

where $f_1(z)$ and $f_2(z)$ are holomorphic functions of z in the region occupied by the body. Let this region be denoted by S. It is now assumed that there exists a real number a such that z = a and z = -a are points outside a. Then the functions $g_1(z)$ and $g_2(z)$ defined by

$$g_1(z) = \frac{f_2(z) + zf_1(z)}{z^2 - a^2},$$
$$g_2(z) = \frac{a^2 f_1(z) + zf_2(z)}{a(z^2 - a^2)}$$

are also holomorphic in S. The functions $f_1(z)$ and $f_2(z)$ can be expressed into $g_1(z)$ and $g_2(z)$ by

$$f_1(z) = zg_1(z) - ag_2(z),$$

$$f_2(z) = azg_2(z) - a^2g_1(z),$$

and the biharmonic function U_1 can therefore be written as

$$2U_1 = (z\overline{z} - a^2)[g_1(z) + \overline{g_1(z)}] + a(z - \overline{z})[g_2(z) - \overline{g_2(z)}].$$

By writing

$$\Re\{g_1(z)\} = \Psi_1,$$

$$\Im\{g_2(z)\} = \Phi_1,$$

it follows that

$$U_1(x,y) = -2ay\Phi_1(x,y) + (x^2 + y^2 - a^2)\Psi_1(x,y)$$

which is precisely equation (7.49). The representation of the solution of the biharmonic equation in terms of the two complex functions $f_1(z)$ and $f_2(z)$ is complete, i.e. for any solution of the biharmonic equation there exists at least one pair of functions f_1 and $f_2(z)$. Furthermore, from any pair of functions $f_1(z)$, $f_2(z)$ there can be derived a pair of functions $g_1(z)$ and $g_2(z)$. This means that for any solution of the biharmonic equation there exists at least one pair of functions $\Phi_1(=\Im g_2)$ and $\Psi_1(=\Re g_2)$. Or, in other words, the representation (7.49) is complete, provided that $x \pm a$, y = 0 are points outside the region occupied by the body.

Conformal transformation

In a similar way as done by LING (1947), see also GREEN & ZERNA (1954, p. 317), the variables x and y are repolaced by variables ξ and η through the conformal transformation

$$z = x + iy = a \coth(\frac{1}{2}\zeta) = a \coth(\frac{1}{2}\xi + \frac{1}{2}i\eta).$$
(7.52)

Separation into real and imaginary parts gives

$$x = \frac{a \sinh \xi}{\cosh \xi - \cos \eta}, \quad y = -\frac{a \sin \eta}{\cosh \xi - \cos \eta}.$$
 (7.53)

The conformal transformation (7.52) maps the half plane $y \leq 0$ with a circular arc notch onto the interior of an infinite strip of width $\pi - \beta$, see figure 7.17.

Two quantities useful for future reference are

$$-2ay = \frac{2a^2 \sin \eta}{\cosh \xi - \cos \eta},\tag{7.54}$$



Figure 7.17: Conformal transformation.

$$x^{2} + y^{2} - a^{2} = \frac{2a^{2}\cos\eta}{\cosh\xi - \cos\eta}.$$
(7.55)

Furthermore, it follows from (7.53) that

$$\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta} = a \frac{1 - \cosh \xi \cos \eta}{(\cosh \xi - \cos \eta)^2},\tag{7.56}$$

$$\frac{\partial x}{\partial \eta} = -\frac{\partial y}{\partial \xi} = -a \frac{\sinh \xi \sin \eta}{(\cosh \xi - \cos \eta)^2}.$$
(7.57)

Through the relations (7.53) the original independent variables x and y are relaced by ξ and η . The functions $U_1(x, y)$, $\Phi(x, y)$ and $\Psi_1(x, y)$ then become functions of ξ and η . These functions will be denoted by $U(\xi, \eta)$, $\Phi(\xi, \eta)$ and

 $\Psi(\xi, \eta)$. Since Laplace's equation is invariant for conformal transformations the functions Φ and Ψ are harmonic,

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0, \quad \frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} = 0.$$
(7.58)

It should be noted that the function $U(\xi, \eta)$ is not biharmonic, because the biharmonic equation is not invariant for conformal transformations. The functional relationship between U, Φ and Ψ can be obtained by expressing the coefficients in eq. (7.49) in terms of ξ and η with the aid of (7.54) and (7.55). This gives

$$U = \frac{2a^2(\Phi\sin\eta + \Psi\cos\eta)}{\cosh\xi - \cos\eta}.$$
(7.59)

Operational solution

Because of the symmetry of the problem and the conformal transformation with respect to the y-axis, it can be expected that U will be an even function of ξ ,

$$U(-\xi,\eta) = U(\xi,\eta),\tag{7.60}$$

and hence it is also expected that Φ and Ψ are even functions of ξ . This suggests to express the solutions of the differential equations (7.58) by means of Fourier cosine integrals (TITCHMARSH, 1948; SNEDDON, 1951),

$$\Phi(\xi,\eta) = \int_0^\infty [F_1(\lambda)\cosh(\lambda\eta) + F_2(\lambda)\sinh(\lambda\eta)]\cos(\lambda\xi)\,d\lambda,\tag{7.61}$$

$$\Psi(\xi,\eta) = \int_0^\infty [G_1(\lambda)\cosh(\lambda\eta) + G_2(\lambda)\sinh(\lambda\eta)]\cos(\lambda\xi)\,d\lambda.$$
(7.62)

It may be mentioned here that the introduction of these Fourier integrals can be considered as the result of solving eqs. (7.58) by means of Fourier transforms. The success of the Fourier method for the type of problem considered here is a consequence of the fact that in the ζ -plane the geometry of the region is that of an infinite strip. It will be shown that the four functions F_1 , F_2 , G_1 , G_2 can be determined from the boundary conditions of the problem.

First the boundary conditions along EA and CD (fig. 7.14) will be considered. In the ζ -plane the parts EA and CD of the boundary are mapped on the real axis, $\eta = 0$. Hence conditions (7.46) can be written as

$$\eta = 0 : U_1 = 0, \quad \partial U_1 / \partial y = 0.$$
 (7.63)

In general one has

$$\frac{\partial U}{\partial \eta} = \frac{\partial U_1}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial U_1}{\partial y} \frac{\partial y}{\partial \eta},$$

and because along EA and CD both $\partial U_1/\partial x$ and $\partial U_1/\partial y$ vanish, the boundary conditions (7.63) become, in terms of ξ and η ,

$$\eta = 0 : U = 0, \tag{7.64}$$

$$\eta = 0 : \partial U / \partial \eta = 0. \tag{7.65}$$

In order to transform these conditions into terms of Φ and Ψ use is to be made of (7.59), and of its partial derivative with respect to η ,

$$\frac{\partial U}{\partial \eta}(\cosh\xi - \cos\eta) + U\sin\eta = 2a^2 \Big(\frac{\partial\Phi}{\partial\eta}\sin\eta + \Phi\cos\eta + \frac{\partial\Psi}{\partial\eta}\cos\eta - \Psi\sin\eta\Big).$$
(7.66)

It now follows from (7.59) and (7.60) that the conditions (7.64) and (7.65) can only be satisfied if and only if

$$\eta = 0 \ : \Psi = 0, \tag{7.67}$$

$$\eta = 0 : \Phi + \partial \Psi / \partial \eta = 0. \tag{7.68}$$

Substitution of these conditions into the general expressions (7.61) and (7.62) gives

$$\int_0^\infty G_1(\lambda)\cos(\lambda\xi) \, d\lambda = 0,$$
$$\int_0^\infty F_1(\lambda)\cos(\lambda\xi) \, d\lambda + \int_0^\infty \lambda G_2(\lambda)\cos(\lambda\xi) \, d\lambda = 0.$$

These equations should be satisfied for all ξ . Hence

$$G_1(\lambda) = 0, \tag{7.69}$$

$$F_1(\lambda) + \lambda G_2(\lambda) = 0. \tag{7.70}$$

Using (7.69) and (7.70) the general solutions (7.61) and (7.62) reduce to

$$\Phi(\xi,\eta) = \int_0^\infty \left[-\lambda G_2(\lambda) \cosh(\lambda\eta) + F_2(\lambda) \sinh(\lambda\eta)\right] \cos(\lambda\xi) \, d\lambda, \qquad (7.71)$$

$$\Psi(\xi,\eta) = \int_0^\infty G_2(\lambda) \sinh(\lambda\eta) \cos(\lambda\xi) \, d\lambda.$$
(7.72)

The two remaining unknown functions F_2 and G_2 must be determined from the boundary conditions along ABC.

The boundary ABC corresponds to the straight line $\eta = \pi - \beta$ in the ζ -plnae. The boundary conditions (7.47) and (7.48) express that along ABC the quantities $\partial U_1/\partial x$ and $\partial U_1/\partial y$ are prescribed. From these quantities the

values of U_1 and its derivative normal to the boundary can be determined (at least in principle), thereby starting from the value $U_1 = 0$ in A or C. Expressed in terms of ξ and η this means that U and $\partial U/\partial \eta$ can be considered as given along the line $\eta = \pi - \beta$. We now write

$$\eta = \pi - \beta : U(\cosh \xi - \cos \eta) = \int_0^\infty A(\lambda) \cos(\lambda \xi) \, d\lambda, \tag{7.73}$$

$$\eta = \pi - \beta : \frac{\partial U}{\partial \eta} (\cosh \xi - \cos \eta) + U \sin \lambda = \int_0^\infty B(\lambda) \cos(\lambda \xi) \, d\lambda. \quad (7.74)$$

These equations can be considered as the definitions of $A(\lambda)$ and $B(\lambda)$. It is assumed that $A(\lambda)$ and $B(\lambda)$ can be determined from the boundary conditions. From (7.59) and (7.73) it now follows, with (7.71) and (7.72), that

$$2a^{2}\sin\beta\{-\lambda G_{2}\cosh[\lambda(\pi-\beta)] + F_{2}\sinh[\lambda(\pi-\beta)]\} - 2a^{2}\cos\beta G_{2}\sinh[\lambda(\pi-\beta)] = A. \quad (7.75)$$

Furthermore, it follows from (7.66) and (7.74), with (7.71) and (7.72), that

$$2a^{2} \sin \beta \{-\lambda^{2} G_{2} \sinh[\lambda(\pi-\beta)] + F_{2} \cosh[\lambda(\pi-\beta)]\} - 2a^{2} \cos \beta \{-\lambda G_{2} \cosh[\lambda(\pi-\beta)] + F_{2} \sinh[\lambda(\pi-\beta)]\} - 2a^{2} \cos \beta \lambda G_{2} \cosh[\lambda(\pi-\beta)] - 2a^{2} \sin \beta G_{2} \sinh[\lambda(\pi-\beta)] = B. \quad (7.76)$$

From the system of equations (7.75) and (7.76) it is possible to determine F_2 and G_2 . This gives

$$2a^{2}F_{2}(\lambda) = (1+\lambda^{2})A\sin\beta\sinh[\lambda(\pi-\beta)]/N(\lambda) - B\{\lambda\sin\beta\cosh[\lambda(\pi-\beta)] + \cos\beta\sinh[\lambda(\pi-\beta)]\}/N(\lambda), \quad (7.77)$$

$$2a^{2}G_{2}(\lambda) = A\{\lambda \sin\beta \cosh[\lambda(\pi-\beta)] - \cos\beta \sinh[\lambda(\pi-\beta)]\}/N(\lambda) - B\sin\beta \sinh[\lambda(\pi-\beta)]/N(\lambda), \quad (7.78)$$

in which

$$N(\lambda) = \sinh^2[\lambda(\pi - \beta)] - \lambda^2 \sin^2 \beta.$$
(7.79)

In principle, the problem has now been solved. Elaboration of the solution for a specific case requires the following operations

- 1. Evaluation of U and $\partial U/\partial \eta$ along ABC from (7.47) and (7.48),
- 2. Representation of the left hand members of equations (7.73) and (7.74) as Fourier integrals, in order to obtain expressions for A and B,
- 3. Calculation of integrals of the type (7.71) and (7.79).

It will be shown below that for the special case of a semi-circular arc notch $(\beta = \pi/2)$ the first two operations can be performed analytically, as predicted by KOITER (1968). This then leads to expressions for the stresses in the form of definite integrals, which may be evaluated numerically. In the general case of $\beta \neq \pi/2$ the final fomulas will be multiple integrals, which can be calculated numerically, at the cost of considerable computation effort.

Elaboration for $\beta = \pi/2$

When $\beta = \pi/2$ the weight P of the excavation is, with (7.25),

 $P = \frac{1}{2}\pi\rho g R^2.$

The functions $X(\varphi)$ and $Y(\varphi)$, as defined by (7.37) and (7.38), then are

$$X(\varphi) = (1 - K_0)\rho gR\sin\varphi\cos\varphi, \qquad (7.80)$$

$$Y(\varphi) = 0. \tag{7.81}$$

Equations (7.47) and (7.48) now become

ABC :
$$\partial U_1 / \partial x = 0,$$
 (7.82)

ABC :
$$\partial U_1 / \partial y = \frac{1}{2} (1 - K_0) \rho g R^2 \cos^2 \varphi.$$
 (7.83)

It may be noted that when the coefficient of neutral earth pressure $K_0 = 1$, the values of $\partial U_1/\partial x$ and $\partial U_1/\partial y$ both vanish identically along ABC. In that special case the problem is particularly simple, since then the part of the solution described by the stresses t_{xx} , t_{yy} , t_{xy} vanishes. This reconfirms the statement in section 7.2 that for $K_0 = 1$ the problem of a semi-circular arc notch has as its solution the Boussinesq-Flamant solution.

Along ABC one has

$$\frac{\partial U_1}{\partial \varphi} = \frac{\partial U_1}{\partial x} \frac{dx_0}{d\varphi} + \frac{\partial U_1}{\partial y} \frac{dy_0}{d\varphi}$$

Hence, with (7.31), (7.82) and (7.83),

ABC :
$$\frac{dU_1}{d\varphi} = \frac{1}{2}(1-K_0)\rho g R^3 \cos^3 \varphi \sin \varphi.$$

Integration gives

ABC :
$$U_1 = -\frac{1}{6}(1 - K_0)\rho g R^3 \cos^3 \varphi,$$
 (7.84)

in which the integration constant has been omitted in order to obtain that $U_1 = 0$ for $\varphi = -\beta = -\pi/2$.

Next the equations (7.82) – (7.84) will be transformed into terms a and ξ rather than R and φ . In general $a = R \sin \beta$, so that with $\beta = \pi/2$ it follows that

$$a = R. \tag{7.85}$$

Furthermore, with $\beta = \pi/2$ the second of equations (7.30) gives

 $\cos\varphi = -y_0/R,$

and from the second of equations (7.53) it is found that for $\eta + \pi - \beta = \pi/2$,

 $y_0/a = y_0/R = -1/\cosh\xi.$

Hence, along ABC :

$$\cos\varphi = 1/\cosh\xi. \tag{7.86}$$

With (7.85) and (7.86) equations (7.82) - (7.84) can be written as

ABC :
$$U_1 = -\frac{1}{6}(1 - K_0)\rho g a^3 / \cosh^3 \xi,$$
 (7.87)

$$ABC : \partial U_1 / \partial x = 0, \tag{7.88}$$

ABC :
$$\partial U_1 / \partial y = \frac{1}{2} (1 - K_0) \rho g a^2 / \cosh^2 \xi.$$
 (7.89)

In equations (7.73) and (7.74) the following quantities are needed

$$\overline{A}(\xi) = \int_0^\infty A(\lambda) \cos(\lambda\xi) \, d\lambda = (\cosh\xi + \cos\beta) \, U(\xi, \pi - \beta), \tag{7.90}$$

$$\overline{B}(\xi) = \int_0^\infty B(\lambda) \cos(\lambda\xi) \, d\lambda = (\cosh\xi + \cos\beta) \Big(\frac{\partial U}{\partial\eta}\Big)_{\eta=\pi-\beta} + \sin\beta \, U(\xi, \pi-\beta),$$
(7.91)

or, with $\beta = \pi/2$,

$$\overline{A}(\xi) = \cosh \xi U(\xi, \pi/2), \tag{7.92}$$

$$\overline{B}(\xi) = \cosh \xi \left(\frac{\partial U}{\partial \eta}\right)_{\eta = \pi/2} + U(\xi, \pi/2).$$
(7.93)

Beacuse the boundary ABC corresponds to the line $\eta = \pi/2$ in the ζ -plane it follows directly from (7.87) and (7.92) that

$$\overline{A}(\xi) = -\frac{1}{6}(1 - K_0)\rho g a^3 / \cosh^2 \xi.$$
(7.94)

In order to determine $\overline{B}(\xi)$ it is noted that laong ABC

$$\frac{\partial U}{\partial \eta} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial \eta} = \frac{1}{2}(1 - K_0)\rho g a^3 / \cosh^4 \xi,$$

where use has been made of (7.56), (7.57), (7.57), (7.88) and (7.89). Thus equation (7.93) now gives

$$\overline{B}(\xi) = \frac{1}{3}(1 - K_0)\rho g a^3 / \cosh^3 \xi.$$
(7.95)

With the aid of the integral representations (B.10) and (B.11), see appendix B, equations (7.94) and (7.95) can be written as

$$\overline{A}(\xi) = -\int_0^\infty \frac{(1 - K_0)\rho g a^3 \lambda/6}{\sinh(\pi \lambda/2)} \cos(\xi \lambda) \, d\lambda, \tag{7.96}$$

$$\overline{B}(\xi) = \int_0^\infty \frac{(1 - K_0)\rho g a^3 (1 + \lambda^2)/6}{\cosh(\pi\lambda/2)} \,\cos(\xi\lambda) \,d\lambda.$$
(7.97)

This means, see equations (7.90) and (7.91), that $A(\lambda)$ and $B(\lambda)$ in this case are

$$A(\lambda) = -\frac{1}{6}(1 - K_0)\rho g a^3 \lambda / \sinh(\pi \lambda/2), \qquad (7.98)$$

$$B(\lambda) = \frac{1}{6}(1 - K_0)\rho g a^3 (1 + \lambda^2) / \cosh(\pi \lambda/2).$$
(7.99)

Using these results the functions $F_2(\lambda)$ and $G(\lambda)$ are obtained from (7.77) and (7.78) as follows

$$F_2(\lambda) = \frac{1}{6} (1 - K_0) \rho g a \, \frac{\lambda (1 + \lambda^2)}{\sinh^2(\pi \lambda/2) - \lambda^2},\tag{7.100}$$

$$G_2(\lambda) = -\frac{1}{6}(1 - K_0)\rho ga \frac{\sinh^2(\pi\lambda/2) + \lambda^2 \cosh(\pi\lambda)}{\sinh(\pi\lambda)[\sinh^2(\pi\lambda/2) - \lambda^2]}.$$
(7.101)

Now that $F_2(\lambda)$ and $G_2(\lambda)$ are known, the integral expressions (7.71) and (7.72) for the stress functions Φ and Ψ are completely defined. Analytical expressions of these integrals is, unfortunately, impossible. However, for a given value of ξ and η (that is : for given x and y) the values of Φ and Ψ as well as their partial derivatives can be calculated numerically without essential difficulties. Since the Fourier integral method in the context of this thesis has merely the purpose of a check on the complex variable method, only one quantity, namely the sum of the principal stresses, will be elaborated here.

The sum of the principal stresses

It follows from (7.27) that the sum of the principal stresses, which is equal to $t_{xx} + t_{yy}$, is given by

$$t_{xx} + t_{yy} = \frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2}$$

With (7.49) this gives, because Φ and Ψ are harmonic,

$$\frac{1}{4}(t_{xx} + t_{yy}) = -a\frac{\partial\Phi_1}{\partial y} + \Psi_1 + x\frac{\partial\Psi_1}{\partial x} + y\frac{\partial\Psi_1}{\partial y}.$$

With equations (7.53) - (7.57) this can be expressed in terms of ξ and η instead of x and y. The result is

$$\frac{1}{4}(t_{xx} + t_{yy}) = -\sinh\xi\sin\eta\frac{\partial\Phi}{\partial\xi} - (1-\cosh\xi\cos\eta)\frac{\partial\Phi}{\partial\eta} + \Psi - \sinh\xi\cos\eta\frac{\partial\Psi}{\partial\xi} - \cosh\xi\sin\eta\frac{\partial\Psi}{\partial\eta}.$$
 (7.102)
Expressions for the partial derivatives of Φ and Ψ can be obtained from (7.71) and (7.72). This gives

$$\begin{aligned} \frac{\partial \Phi}{\partial \xi} &= \int_0^\infty [\lambda^2 G_2 \cosh(\lambda \eta) - \lambda F_2 \sinh(\lambda \eta)] \sin(\lambda \xi) d\lambda, \\ \frac{\partial \Phi}{\partial \eta} &= \int_0^\infty [-\lambda^2 G_2 \sinh(\lambda \eta) + \lambda F_2 \cosh(\lambda \eta)] \cos(\lambda \xi) d\lambda, \\ \frac{\partial \Psi}{\partial \xi} &= \int_0^\infty [-\lambda G_2 \sinh(\lambda \eta)] \sin(\lambda \xi) d\lambda, \\ \frac{\partial \Psi}{\partial \eta} &= \int_0^\infty [\lambda G_2 \cosh(\lambda \eta)] \cos(\lambda \xi) d\lambda. \end{aligned}$$

Because F_2 and G_2 can be calculated for any value of λ from (7.100) and (7.101), equation (7.102) can now be evaluated numerically. The integrals appear to be rapidly convergent, because for $\lambda \to \infty$ the integrands tend towards zer as $\exp(-b\lambda)$, where b is always greater than $\pi/2$. For the numerical calculations the upper limit of integration has been taken equal to 10 instead of ∞ . Some care is needed to evaluate the integrands for values of λ close to zero, since for $\lambda = 0$ the denominators vanish. It may be verified, however, that all terms in the integrands are bounded for $\lambda \to 0$. In the computer program used for the numerical calculation, the quantities λF_2 and λG_2 are calculated first, and then these are used to calculate the values of the integrands. The integrals are then calculated by means of a standard program based on Simpson's rule.

Once that $t_{xx} + t_{yy}$ is known, the value of $\tau_{xx} + \tau_{yy}$ can be obtained with (7.24) and (7.26). In the expressions (7.26) the quantities $\sin \psi$, $\cos \psi$ and r may be replaced by

$$\sin \psi = \frac{x}{\sqrt{x^2 + y^2}}, \quad \cos \psi = \frac{y}{\sqrt{x^2 + y^2}}, \quad r = \sqrt{x^2 + y^2},$$

and the value of P in this case is, because $\beta = \pi/2$,

$$P = \frac{1}{2}\pi\rho g R^2.$$

The final expression for $\tau_{xx} + \tau_{yy}$ is

$$\frac{\tau_{xx} + \tau_{yy}}{\rho gR} = (1 + K_0)\frac{y}{R} - \frac{yR}{x^2 + y^2} + \frac{t_{xx} + t_{yy}}{\rho gR}.$$
(7.103)

The results obtained for points on the y-axis (i.e. x = 0, or $\xi = 0$), for $k_0 = 0$, are shown in figure 7.18. In the same figure the values obtained by the approximate complex variable method are indicated by the heavy dots, taking 39 terms into account, see section 7.2. It appears from the figure that the correspondence in general is extremely good (differences in numerical values of at the most 0.003, which can not be visualized on the scale of the figure), except in the upper most point. In this point, which is the deepest point of



Figure 7.18: Sum of principal stresses along x = 0.

the notch, the exact value (as calculated by the Fourier integral method) is 1.298, whereas the complex variable method leads to a value of 1.650. This confirms the conclusion, already given in section 7.2, that along the boundary the approximate complex variable method is not accurate, but in the interior of the body it is very accurate. In fact, by taking a somewhat different number of terms in the approximate conformal transformation, the results for the interior of the body are hardly modified, but the values along the boundary may be completely different. Thus, for instance, instead of the 1.650, values up to 2.100 have been obtained by taking a somewhat smaller number of terms into account. For other values of K_0 similar results as those shown in figure 7.18 are obtained: extremely good agreement in interior points, and considerable errors in boundary points.

K_0	$\tau_{\xi\xi}/\rho gR$
0.00	+1.298
0.25	+0.724
0.50	+0.149
0.75	-0.425
1.00	-1.000

Table 7.2: Influence of K_0 .

Since the quantity $\tau_{xx} + \tau_{yy}$ is invariant for rotations of the coordinate system, and since along the boundary the stress component normal to it is zero, this quantity equals the normal stress acting on a plane normal to the

boundary. This stress may be denoted by $\tau_{\xi\xi}$. For various values of K_0 the stress in the deepest point of the notch ($\xi = 0$, $\eta = \pi/2$, or x = 0, y = -R) is recorded in table 7.2.

As a final illustration the values of $\tau_{xi\xi}$ along the boundary of the notch are represented in figure 7.19, for the case $K_0 = 0$, togethre with some values (represented by the dots) obtained by the approximate complex variable method (with n = 49). In this case the agreement is very bad. Only the general tendency is the same, but locally large deviations occur. The reason for these differences is that locally the shape of the boundary is irregular, thus resulting in large errors in the stresses in points located on the boundary. In fact, figure



Figure 7.19: Stresses along boundary of notch.

7.19 very clearly confirms the considerations of section 7.2. There it was argued that errors of about 100 % can be expected, and it was also shown that these errors will occur in the form of a wave. Even the wave length predicted in section 7.2 corrsponds to the one observed in figure 7.19. It is observed from figure 7.19 that, for small values of φ , the wave length 2ℓ is equal to about 4°,

or $2\pi/90$ radians. Hence

 $\ell \approx \pi/90.$

On the other hand, near the deepest point of the notch, the value of $|\omega'(\zeta)|$ is 0.5, and therefore with the formula (7.20) the predicted value of ℓ is

 $\ell \approx \pi/98$

and this corresponds very well with the observe value of about $\pi/90$.

In conclusion it may be said that the comparison with the Fourier integral method shows that the approximate complex variable method leads to accurate results in interior points of the region. However, the values of stress in points in the immediate vicinity of the boundary (say at a distance from it of the order of magnitude of the irregularities in the approximation of the boundary), are inaccurate. The conclusion must also be that if one is particularly interested in the stresses along the boundary (which is not the case in applications in Soil Engineering), then Fourier methods are to be preferred to an approximate complex variable method.

Chapter 8

DIKE PROBLEM

As a second class of problems which can be solved with the aid of the procedures described in chapters 5 and 6, the case of a homogeneous symmetric dike will be considered. The dike has been built upon a half space of the same elastic material. This problem has been discussed earlier by PERLOFF, BALADI & HARR (1967), but their method of solution is based upon the (implicit) assumption that the biharmonic equation is invariant for conformal transformations, which is incorrect.

Just as in the preceding chapter restriction can be made to an approximation of the conformal transformation of the standard form

$$\omega(\zeta) \approx \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^n c_k \zeta^k.$$

Procedures for the determination of p, σ_0 and c_k will be given in section 8.1. Once these parameters are known, the stresses can be calculated in a straightforward way, along the lines of chapter 6. Some results of numerical calculations will be presented in section 8.2.

8.1 The conformal transformation



Figure 8.1: Half plane with symmetrical dike.

Let there be given a symmetric dike on a half plane (fig. 8.1), defined by the four corner points z_1 , z_2 , z_3 , z_4 , with coordinates

$$z_{1} = L, z_{2} = L - H \cot \alpha + iH, z_{3} = -L + H \cot \alpha + iH, z_{4} = -L.$$
(8.1)

Here 2L represents the width of the dike (measured at the toe), H its height, and α is the inclination of the slopes. The ratio of H and L will be called the relative height h,

$$h = H/L. \tag{8.2}$$

The region occupied by the dike and the sub-soil is mapped onto the interior of the unit circle $|\zeta| = 1$, see fig. 8.2. The points $\zeta = \sigma_1$ and $\zeta = \sigma_4$, which are to correspond with $z = z_1$ and $z = z_4$, are taken as $\sigma_1 = +1$, $\sigma_4 = -1$. The point $\zeta = \sigma_0 = -i$ corresponds to $z = z_0 = \infty$. The location of these three points has been taken as convenient as possible, in accordance with the general property of the conformal transformation that the images of three points can be located arbitrarily on the unit circle. The points $\zeta = \sigma_2$ and $\zeta = \sigma_3$, which correspond to $z = z_2$ and $z = z_4$, can be assumed to lie symmetrically with respect to the imaginary axis in the ζ -plane. Hence



Figure 8.2: Unit circle in ζ -plane.

Here the value of θ_0 is as yet unknown. It can be expected that θ_0 depends upon the ratio of height and width of the dike. This will be elaborated in a later stage.

General character of the conformal transformation

The conformal transformation from the interior of the unit circle $|\zeta| = 1$ onto the region in the z-plane occupied by the dike and the sub-soil is denoted by

$$z = \omega(\zeta), \quad |\zeta| \le 1. \tag{8.4}$$

In accordance with the general formula for the conformal transformation of a polygon onto a unit circle (a variant of the Schwartz-Christoffel transformation, see e.g. NEHARI, 1952), the first derivative of the mapping function will be of the following form

$$\omega'(\zeta) = -A(\zeta - 1)^{\alpha/\pi} [\zeta - \exp(i\theta_0)]^{-\alpha/\pi} \times [\zeta + \exp(-i\theta_0)]^{-\alpha/\pi} (\zeta + 1)^{\alpha/\pi} (\zeta + i)^{-2}$$



Figure 8.3: Arguments of elementary functions of ζ .

In order that the rational powers are uniquely defined for all values of ζ inside or on the unit circle, the arguments of $\zeta - 1$, $\zeta - \exp(i\theta_0)$, etc., have to be assigned definite intervals. For a certain point ζ the arguments of these functions are indicated in figure 8.3. These arguments correspond to the following choices for their intervals,

$$\pi/2 < \arg(\zeta - 1) < 3\pi/2, \pi/2 + \theta_0 < \arg[\zeta - \exp(i\theta_0)] < 3\pi/2 + \theta_0, -\pi/2 - \theta_0 < \arg[\zeta + \exp(-i\theta_0)] < \pi/2 - \theta_0, -\pi/2 < \arg(\zeta + 1) < \pi/2.$$
(8.5)

The expression for $\omega'(\zeta)$ as given above may alternatively be written as

$$\omega'(\zeta) = -A \left(\frac{\zeta^2 - 1}{\zeta^2 - 2i\zeta \sin \theta_0 - 1}\right)^{\alpha/\pi} (\zeta + i)^{-2}, \quad |\zeta| \le 1.$$
(8.6)

Here the value of A as well as the value of θ_0 is as yet undetermined. The function $\omega(\zeta)$ itself can be found by integration. This involves another arbitrary constant B, according to

$$\omega(\zeta) = \int_{i}^{\zeta} \omega'(\zeta) d\zeta + B.$$
(8.7)

The constants A, B, and θ_0 , which will be called the transformation parameters, will be determined below.

Determination of transformation parameters

The constants A, B and θ_0 are to be taken such that the corner points in the z-plane are located in the correct positions. Varying of the constant B results in a translation of the region in the z-plane, and thus B can be determined by fixing a certain point. The constant A is merely a multiplication factor, see equation (8.6). Thus, by changing |A| only the scale of the figure is affected, and a modification of $\arg(A)$ merely results in a rotation of the entire region. This means that only the parameter θ_0 depends upon the shape of the region. This shape is determined by the relative height h and by the slope α . The slope α has already been taken into account, and thus for a certain value of α ,

$$\theta_0 = \theta_0(h). \tag{8.8}$$

Choosing $\zeta = i$ as the lower limit of integration in equation (8.7) means that z = B corresponds to $\zeta = i$. The point in the z-plane corresponding to $\zeta = i$ is z = iH. Hence

$$B = iH. ag{8.9}$$

Thus one the transformation parameters has been found. Next a procedure for the calculation of θ_0 and A will be presented.

For the determination of θ_0 and A from a given set of values of H and L a semi-inverse procedure will be followed. Therefore let there be chosen a certain value for θ_0 . As stated above this determines the shape of the dike, and the constant A must now be chosen such that for instance the location of the point corresponding to $\zeta = 1$ is z = L. Hence the problem is to determine A and h = H/L when L and θ_0 are considered as given.

With (8.7) and (8.9) the locations of the points z_2 and z_1 , as given in (8.1), are found as

$$z_{2} = L - H \cot \alpha + iH = \int_{i}^{\exp(i\theta_{0})} \omega'(\zeta) d\zeta + iH$$
$$z_{1} = L = \int_{i}^{1} \omega'(\zeta) d\zeta + iH,$$

where use has been made of $\sigma_2 = \exp(i\theta_0)$ and $\sigma_1 = 1$. The first of these equations may be written as

$$L - H \cot \alpha = A \int_{i}^{\exp(i\theta_0)} [\omega'(\zeta)/A] d\zeta, \qquad (8.10)$$

and in the second equation it is convenient to eliminate L with the aid of (8.10). This leads to

$$H(\cot \alpha - i) = A \int_{\exp(i\theta_0)}^{1} [\omega'(\zeta)/A] d\zeta.$$
(8.11)

Since θ_0 is considered as given, the integrand $\omega'(\zeta)/A$ of (8.10) and (8.11) is completely determined, see (8.6). Thus the integrals appearing in (8.10) and (8.11) can be calculated. For convenience, the path of integration in both integrals will be taken along the unit circle. Then

$$\zeta = \sigma = \exp(i\theta), \qquad d\zeta = i \exp(i\theta) \, d\theta. \tag{8.12}$$

The arguments of the quantities $\sigma - \sigma_1$, $\sigma - \sigma_2$, $\sigma - \sigma_3$, $\sigma - \sigma_4$, must be taken in accordance with (8.5). After some elaboration of the integrand $\omega'(\zeta)/A$ the integral in equation (8.10) then appears to be real,

$$\int_{i}^{\exp(i\theta_0)} [\omega'(\zeta)/A] d\zeta = a, \qquad (8.13)$$

and its value is

$$a = \frac{1}{2} \int_{\theta_0}^{\pi/2} \left(\frac{\sin\theta}{\sin\theta - \sin\theta_0}\right)^{\alpha/\pi} \frac{d\theta}{1 + \sin\theta}.$$
(8.14)

On the other hand, elaboration of the integral in equation (8.11) with (8.6) shows that this integral contains a factor ($\cot \alpha - i$). Hence one may write

$$\int_{\exp(i\theta_0)}^{1} [\omega'(\zeta)/A] d\zeta = b(\cot\alpha - i), \qquad (8.15)$$

in which b is a real quantity, namely

$$b = \frac{1}{2}\sin\alpha \int_0^{\theta_0} \left(\frac{\sin\theta}{\sin\theta_0 - \sin\theta}\right)^{\alpha/\pi} \frac{d\theta}{1 + \sin\theta}.$$
(8.16)

In a later part of this section the numerical calculation of the quantities a and b will be considered. For the present considerations it is sufficient to note that they are completely determined.

Substitution of (8.13) and (8.15) into (8.10) and (8.11) gives

$$L - H \cot \alpha = Aa,$$

H = Ab,

from which A and H/L can be solved, in terms of L, α , a and b. The solution can be written as

$$h = H/L = b/(a + b \cot \alpha), \tag{8.17}$$

$$A = L/(a + b\cot\alpha). \tag{8.18}$$

The formulas (8.17) and (8.18) enable to calculate h and A when θ_0 and α are given.

The results of numerical computations have been used to construct the graphs presented in figures 8.4 and 8.5. These figures not only serve as an illustration, but can also be used to determine the values of θ_0 and A corresponding to certain given values of α , H and L.



Figure 8.4: Relationship between h = H/L and θ_0 and α .



Figure 8.5: A/L as a function of θ_0 and α .

Numerical calculation of a and b

The numerical calculation of the real parameters a and b, as defined by (8.14) and (8.16), is somewhat complicated by the circumstance that the integrals are improper. For $\theta = \theta_0$ the integrand becomes infinitely large, but since α/π is always smaller than 1, this does not prevent the existence of the integrals. In order to calculate the integrals the interval is split into two parts, one of length ε (ε being small) near the end $\theta - \theta_0$, and the other one being the remaining part. For the small part of length ε the integral is evaluated by expanding the integrand around θ_0 . This leads to

$$a = a_0 + \frac{1}{2} \int_{\theta_0 + \varepsilon}^{\pi/2} \left(\frac{\sin \theta}{\sin \theta - \sin \theta_0} \right)^{\alpha/\pi} \frac{d\theta}{1 + \sin \theta},$$
(8.19)

$$b = b_0 + \frac{1}{2}\sin\alpha \int_0^{\theta_0 - \varepsilon} \left(\frac{\sin\theta}{\sin\theta_0 - \sin\theta}\right)^{\alpha/\pi} \frac{d\theta}{1 + \sin\theta},\tag{8.20}$$

where

0

$$a_0 = \frac{\varepsilon^{1-\alpha/\pi} \tan \theta_0}{2(1-\alpha/\pi)(1+\sin \theta_0)} \Big\{ 1 + \frac{\pi-\alpha}{2\pi-\alpha} \Big[\frac{\alpha}{\pi} (\cot \theta_0 + \frac{1}{2} \tan \theta_0) - \frac{\cos \theta_0}{1+\sin \theta_0} \Big] \varepsilon + \ldots \Big\},$$
(8.21)

$$b_{0} = \frac{\varepsilon^{1-\alpha/\pi} \sin \alpha \tan \theta_{0}}{2(1-\alpha/\pi)(1+\sin \theta_{0})} \Big\{ 1 - \frac{\pi-\alpha}{2\pi-\alpha} \Big[\frac{\alpha}{\pi} (\cot \theta_{0} + \frac{1}{2} \tan \theta_{0}) - \frac{\cos \theta_{0}}{1+\sin \theta_{0}} \Big] \varepsilon + \dots \Big\}.$$
(8.22)

In (8.21) and (8.22) the truncation errors in the expressions between parentheses are of order $O(\varepsilon^2)$. The integrals in (8.19) and (8.20) can be calculated by any standard numerical integration procedure. The result should of course be independent of the choice of ε . In order to check this some integrals were calculated for various (small) values of ε . A change of a factor 10 in ε ($\varepsilon = 10^4$ instead of $\varepsilon = 10^{-5}$) appeared to result in a relative modification of the values of h and A by at the most 0.0004. This has been considered as sufficient evidence for the accuracy of the calculation of h and A, as represented graphically in figures 8.4 and 8.5

Approximation of the conformal transformation

The considerations presented above have shown that, when the shape of the dike is given by the slope α and the relative height h, then the values of A/L and θ_0 can be determined from the figures 8.4 and 8.5. Hence the derivative of the conformal transformation, $\omega'(\zeta)$, is completely known, see (8.6), i.e.

$$\frac{\omega'(\zeta)}{L} = -\frac{A}{L} \left(\frac{\zeta^2 - 1}{\zeta^2 - 2i\zeta \sin \theta_0 - 1} \right)^{\alpha/\pi} (\zeta + i)^{-2}, \quad |\zeta| \le 1.$$
(8.23)

Throughout this section the length L will be considered as a reference length. In order to apply the stress calculation porcedures of chapters 5 and 6, the conformal transformation $\omega(\zeta)$ must be brought into the standard form

$$\omega(\zeta) = \frac{p}{\zeta - \sigma_0} + \sum_{k=0}^{n} c_k \zeta^k, \qquad (8.24)$$

and the values p, σ_0, n and c_k (k = 0, 1, ..., n) ought to be given as these are the basic parameters for the stress calculation. In chapter 4 it was suggested that these parameters can be calculated by starting from an expression for $\omega(\zeta)$. In the present case this procedure fails, since an expression for $\omega(\zeta)$ can not be found by analytical integration of (8.23). Rather than use a numerical integration procedure to find $\omega(\zeta)$ and retain the original procedure for the calculation of the coefficients, it is simpler to calculate the coefficients c_k from equations (4.7), i.e.

$$kc_k = \frac{1}{2\pi i} \int_{\gamma} \omega'_0(\zeta) \zeta^{-k} d\zeta, \quad k = 1, 2, \dots,$$
 (8.25)

where $\omega'_0(\zeta)$ is the first derivative of the function $\omega_0(\zeta)$, which is defined as

$$\omega_0(\zeta) = \omega(\zeta) - \frac{p}{\zeta - \sigma_0}$$

hence

$$\omega_0'(\zeta) = \omega'(\zeta) + \frac{p}{(\zeta - \sigma_0)^2}.$$
(8.26)

In the present case, see (8.23), the second order pole of $\omega'(\zeta)$ is located at $\zeta = -i$, hence

$$\sigma_0 = -i. \tag{8.27}$$

The coefficient p can be found by determining the limit

$$\lim_{\zeta \to -i} [(\zeta + i)^2 \omega'(\zeta)].$$

Determining this limit from (8.23) as well as (8.24) leads to

$$p/L = (A/L)(1 + \sin\theta_0)^{-\alpha/\pi}.$$
(8.28)

Now that p and σ_0 have been found, the function $\omega'_0(\zeta)$, as defined by (8.26), is completely known. Hence now the coefficients c_1, c_2, \ldots, c_n can be calculated from (8.23). For points on γ

$$\zeta = \sigma = \exp(i\theta),\tag{8.29}$$

and if one now writes

$$\omega_0'(\sigma) = \omega_0'(\exp(i\theta)) = \omega_*'(\theta), \tag{8.30}$$

then equation (8.25) can be transformed into

$$c_k/L = \frac{1}{2\pi k} \int_0^{2\pi} [\omega'_*(\theta)/L] \exp[-(k-1)i\theta] \, d\theta, \quad k = 1, 2, \dots$$
(8.31)

A detailed evaluation of these integrals will be presented later. For the moment it suffices to note that the integrals (8.31) in general are improper, since near a corner point the function $\omega'_*(\zeta)$ may be infinite, but the integrals in general do exist as improper integrals, because the function $\omega'_*(\theta)$ possesses singularities near which

$$\omega'_*(\theta) = O(\theta - \theta_m)^{\alpha_m/\pi},$$

with $\alpha_m > -\pi$, see section 4, equations (4.5) and (4.6).

All necessary coefficients c_k can be calculated from (8.31), except c_0 . Since this coefficient only affects the position of the region R in the z-plane, and not the shape of R, see equation (8.24), its determination should not be essential or difficult. Two possibilities have been examined, both in connection with the formulat (4.23) for the last coefficient, i.e.

$$c_n = \frac{1}{2}p\sigma_0^{-n-1} - \frac{1}{2}\sigma_0^{-n}\sum_{k=0}^{n-1} [(1+k/n)c_k\sigma_0^k - (1-k/n)\overline{c_k}\sigma_0^{-k}].$$
 (8.32)

Since the regions in the ζ -plane and the z-plane are symmetric with respect to the imaginary axis, one may write

$$c_k = \underline{c}_k i^{k+1}, \tag{8.33}$$

where now the coefficients \underline{c}_k are real. Substitution of (8.33) into (8.32) gives, using $\sigma_0 = -i$,

$$\underline{c}_n = \frac{1}{2}p - \sum_{k=0}^{n-1} \underline{c}_k.$$
(8.34)

The condition (8.34) must always be satisfied for the solution of chapter 5 to be applicable. When it is required that the approximate conformal transformation passes exactly through the point z = iH, which corresponds to $\zeta = i$ (see figures 8.1 and 8.2), a second requirement is obtained from (8.24) with $\zeta = i$,

$$iH = \frac{p}{2i} + \sum_{k=0}^{n} \underline{c}_k i^{k+1} i^k$$

hence

$$H = -\frac{1}{2}p + \sum_{k=0}^{n} \underline{c}_{k}(-1)^{k}.$$
(8.35)

From the two equations (8.34) and (8.35) both c_0 and c_n can be determined. If n is chosen uneven, one obtains after elemination of \underline{c}_n from (8.34) and (8.35),

$$\underline{c}_0 = \frac{1}{2}(H+p) - \sum_{k=2,4,\dots}^{n-1} \underline{c}_k.$$
(8.36)

The coefficient \underline{c}_n can then be calculated from (8.34).

Experience with this scheme of calculating c_0 and c_n has shown, however, that the condition that the boundary of the region should pass excatly through the point z = iH may lead to a fairly large value for the coefficient c_n (in fact, all errors made are then balanced by the value of c_n), and this means that the series expansion (8.24) contains a last term with a fairly large last coefficient. This results in a "high frequency" wave of considerable amplitude in the boundary of the region R.

In order to prevent this irregularity it has been found more convenient to use a different scheme of calculating c_0 and c_n . In this alternative method the condition that the boundary should pass exactly through the point z - iH is dropped. Instead, the value of c_n is taken as zero, $c_n = 0$, and equation (8.34) is considered as an equation for the determination of c_0 , i.e.

$$\underline{c}_0 = \frac{1}{2}p - \sum_{k=1}^{n-1} \underline{c}_k.$$
(8.37)

Calculation of the improper integrals

Now returning to equation (8.31) it will finally be elaborated how these integrals can be calculated. According to (8.30) and (8.26) one has, with (8.23)

$$\omega_*'(\theta) = -\frac{A}{(\sigma+i)^2} \Big\{ \Big(\frac{\sigma^2 - 1}{\sigma^2 - 2i\sigma\sin\theta_0 - 1}\Big)^{\alpha/\pi} - \Big(\frac{1}{1 + \sin\theta_0}\Big)^{\alpha/\pi} \Big\}, \quad (8.38)$$

where $\sigma = \exp(i\theta)$, $0 \le \theta \le 2\pi$. In general the factors involving σ in (8.38) can be written as

$$\frac{\sigma^2 - 1}{\sigma^2 - 2i\sigma\sin\theta_0 - 1} = \frac{\sin\theta}{\sin\theta - \sin\theta_0},$$
$$\frac{1}{(\sigma + i)^2} = -\frac{\sin\theta + i\cos\theta}{2(1 + \sin\theta)},$$

as can be verified by using $\sigma = \exp(i\theta)$. If $\sin\theta/(\sin\theta - \sin\theta_0)$ is non-negative, i.e. when either $\theta_0 \le \theta \le \pi - \theta_0$ or $\pi \le \theta \le 2\pi$, equation (8.36) gives

$$\theta_0 \le \theta \le \pi - \theta_0 \text{ or } \pi \le \theta \le 2\pi :$$

$$\omega'_*(\theta) = \frac{A(\sin\theta + i\cos\theta)}{2(1+\cos\theta)} \Big\{ \Big(\frac{\sin\theta}{\sin\theta - \sin\theta_0}\Big)^{\alpha/\pi} - \Big(\frac{1}{1+\sin\theta_0}\Big)^{\alpha/\pi} \Big\}.$$
(8.39)

A minor difficulty arises near $\theta = 3\pi/2$, where both the term between parentheses as the denominator vanish. A series expansion near $\theta = 3\pi/2$ shows that near this point equation (8.39) tends towards the limit

$$\theta - 3\pi/2 \ll 1 :$$

$$\omega'_{*}(\theta) = -\frac{A(\sin\theta + i\cos\theta)}{(1 + \sin\theta_{0})^{\alpha/\pi}} \frac{\alpha}{2\pi} \frac{\sin\theta_{0}}{1 + \sin\theta_{0}} + O(\theta - 3\pi/2)^{2}. \quad (8.40)$$

When $0 \leq \theta \leq \theta_0$ the expression $\sin \theta / (\sin \theta - \sin \theta_0)$ is negative, and before raising it to the power α / π it must be investigated whether the minus sign stands for $\exp(i\pi)$ or $\exp(-i\pi)$. This can best be done by letting a point σ on the unit circle γ pass through one of the points $\theta = 0$ or $\theta = \theta_0$. For instance, when σ passes through the point $\exp(i\theta_0)$, see figure 8.6, with θ decreasing, the argument of $\sigma - \exp(i\theta_0)$ increases by π , in accordance with (8.5). That



Figure 8.6: Detail of unit circle near σ_2 .

this is necessary can be understood by noting that points on γ are the limiting states of points inside γ , or, in other words, the branch cuts necessary to make the functions single valued should not reach into the interior of γ . It now follows that for the path shown in figure 8.6 the argument of the expression $(\sigma^2 - 1)/(\sigma^2 - 2i\sigma\sin\theta_0 - 1)$ decreases by π , which means that in this case -1 stands for $\exp(-i\pi)$, and thus the first term between brackets in (8.38) will be $\exp(-i\alpha)$. Hence, after some elaboration,

$$0 \leq \theta \leq \theta_{0} :$$

$$\omega_{*}'(\theta) = \frac{A[\sin(\theta + \alpha) + i\cos(\theta + \alpha)]}{2(1 + \sin\theta)} \left(\frac{\sin\theta}{\sin\theta_{0} - \sin\theta}\right)^{\alpha/\pi} - \frac{A(\sin\theta + i\cos\theta)}{2(1 + \sin\theta)} \left(\frac{1}{1 + \sin\theta_{0}}\right)^{\alpha/\pi}.$$
(8.41)

In a similar way it is possible to show that in the interval $\pi - \theta_0 \leq \theta \leq \pi$ the function $\omega'_*(\theta)$ should be calculated from the formula

$$\pi - \theta_0 \le \theta \le \pi :$$

$$\omega'_*(\theta) = \frac{A[\sin(\theta - \alpha) + i\cos(\theta - \alpha)]}{2(1 + \sin\theta)} \left(\frac{\sin\theta}{\sin\theta_0 - \sin\theta}\right)^{\alpha/\pi} - \frac{A(\sin\theta + i\cos\theta)}{2(1 + \sin\theta)} \left(\frac{1}{1 + \sin\theta_0}\right)^{\alpha/\pi}.$$
(8.42)

The equations (8.39) - (8.42) enable to calculate $\omega'_*(\theta)$ for all values of θ in the interval from 0 to 2π .

It should be noted that for $\theta = \theta_0$ and $\theta = \pi - theta_0$ the function $\omega'_*(\theta)$ has a singularity. This singularity does not prohibit the integrability, but it presents a small difficulty for the numerical calculation of the integrals (8.31). This difficulty can be removed by splitting up the interval of integration into three parts, i.e. by writing (8.31) as

$$2\pi kc_k = \int_0^{\theta_0 - \varepsilon} \omega'_*(\theta) \exp[-(k-1)i\theta] d\theta + \int_{\theta_0 + \varepsilon}^{\pi - \theta_0 - \varepsilon} \omega'_*(\theta) \exp[-(k-1)i\theta] d\theta + \int_{\pi - \theta_0 + \varepsilon}^{2\pi} \omega'_*(\theta) \exp[-(k-1)i\theta] d\theta + J,$$
(8.43)

where

$$J = \int_{\theta_0 - \varepsilon}^{\theta_0 + \varepsilon} \omega'_*(\theta) \, \exp[-(k-1)i\theta] \, d\theta + \int_{\pi - \theta_0 - \varepsilon}^{\pi - \theta_0 + \varepsilon} \omega'_*(\theta) \, \exp[-(k-1)i\theta] \, d\theta.$$
(8.44)

The two integrals in (8.44) can be calculated approximately by expanding $\omega'_*(\theta)$ around $\theta = thea_0$, respectively $\theta = \pi - \theta_0$. After some elabortation this leads to the following first approximation for J,

$$J = \frac{iA\varepsilon \exp(-i\pi k/2)}{1+\sin\theta_0} \Big\{ -2\Big(\frac{1}{1+\sin\theta_0}\Big)^{\alpha/\pi} \cos[k(\pi/2-\theta_0)] + \frac{(\tan\theta_0/\varepsilon)^{\alpha/\pi}}{1-\alpha/\pi} \Big(\cos[k(\pi/2-\theta_0)] + \cos[k(\pi/2-\theta_0-\alpha]]\Big) \Big\}.$$
(8.45)

In equation (8.45) the error is of order $O(\varepsilon^{2-\alpha/\pi})$, which is at least of order $O(\varepsilon)$, since $\alpha < \pi$. The three remaining integrals in equation (8.45) have bounded integrands, and can therefore directly be calculated using Filon's method (see appendix A).

Conclusion

The above considerations have resulted in expressions for the parameters p, σ_0 and c_k , (k = 0, 1, ..., n), which together define the approximate conformal transformation. On the basis of these coefficients the stresses can be calculated with the aid of the procedures developed in chapter 5 and assembled in chapter 6. In the following section some results of numerical calculations will be presented.

8.2 Numerical results

The procedures described in the previous section have been used to compose a computer program. Some numerical results will be presented here.

The conformal transformation

First some attention is paid to the approximation of the conformal transformation. In this part of the numerical calculations the basic parameters are the relative height h = H/L of the dike, and the slope angle α (see figure 8.1). From figures 8.4 and 8.5 the Scwarz-Christoffel transformation parameters A/Land θ_0 can then be obtained. On the basis of these four parameters the computer program evaluates the coefficients p and c_k of the approximate conformal transformation

$$\omega n(\zeta) = \frac{p}{\zeta + i} + \sum_{k=0}^{n} c_k \zeta^k$$

using the procedures developed in section 8.1.

In figure 8.7 the results are presented for the case $\alpha = \pi/4$ and h = 0.5, for which $\theta_0/\pi = 0.3165$ and A/L = 1.802. The number of coefficients used was



Figure 8.7: Approximation of half plane with dike, n = 39.

n = 39, the value of ε in equation (8.43) was taken as 0.02, and the integrals were evaluated by the extension of Filon's method, described in appendix A. The integration intervals were subdivided into 20 equal parts (m = 10 in equation (A.3)). It appears from the figure that the approximation is rather good. As before, see section 7.1, the approximation near the re-entrant angle at the toe of the dike is very good, but near the salient angle at the upper end of the slope the deviations are somewhat larger. very good, but near the salient angle at the upper end of the slope the deviations are somewhat larger. Fortunately, this is not a very serious defect of the method, since it can be expected that the stresses vanish at such a point. A slight improvement can be obtained by



Figure 8.8: Approximation of corner point with n = 39 (scale as in figure 8.7).

taking more terms into account, as is illustrated in figure 8.8, which has been obtained by taking n = 99.

$The \ stresses$

Using the approximation of the conformal transformation with 39 terms the stresses in the interior of the dike have been calculated along the lines of chapter



Figure 8.9: Vertical stress τ_{yy} along base of dike, $K_0 = 3/7$.

6. Some results, for the case $K_0 = 3/7$, are presented in the figures 8.9, 8.10 and 8.11. Figure 8.9 shows the vertical stresses at the base of the dike (i.e. the original surface y = 0 of the half plane), and figure 8.10 shows the distribution



Figure 8.10: Shear stress τ_{xy} along base of dike, $K_0 = 3/7$.

of shear stresses along this surface. In these two figures the results of PERLOFF,

BALADI & HARR (1967) are represented by dashed lines. As mentioned before, these results were obtained by an incorrect method.

In applied soil mechanics it is often assumed, for reasons of simplicity, that the distribution of vertical stress along y = 0 is conformal to the shape of the dike, and that the shear stresses vanish along y = 0. The stresses in the subsoil are then calculated by solving the problem of a half plane with a vertical load in the shape of the dike. It appears from figure 8.9 that in the correct solution the vertical stresses at the base of the dike are more homogeneous than in this traditional approach.



Figure 8.11: Horizontal stress τ_{xx} along x = 0, $K_0 = 3/7$.

Figure 8.11 shows the distribution of horizontal stress along the vertical axis (x = 0). Again the dashed line represents the results obtained by PERLOFF, BALADI & HARR (1967), who recorded values only for y < 0, that is: in the original half plane. It appears that there is a considerable difference between the two curves.

Conclusion

As a final illustration the approximate boundary for the case $\alpha = \pi/2$, i.e. the case of a dike with vertical faces, is shown in figure 8.12. The left half



Figure 8.12: Approximation of dike with vertical faces on half plane.

of the figure was obtained with 39 terms, and the right half with 99 terms. In this case the approximation is much worse than in the case of figure 8.7. Fortunately, however, most dikes in practice are not built with vertical faces, but rather with slopes having an inclination of $\pi/6$ or less. For such cases the method used here gives sufficiently accurate results.

Chapter 9

POLYGONAL EDGE NOTCH

In this chapter the general case of a polygonal edge notch is investigated (see figure 9.1), with the aim to obtain a general procedure for the determination of the conformal transformation onto the unit circle $|\zeta| \leq 1$, starting from the location of the corner points on the contour. A method for the determination of the Schwarz-Christoffel parameters, which define such a transformation, has been presented by KANTOROVICH & KRYLOV (1964). This is in fact a gener-



Figure 9.1: Half plane with polygonal edge notch.

alization to n dimensions of the Newton-Raphson method for the solution of non-linear equations. The method presented here, though less rigorous, seems somewhat simpler to operate, and it will appear that it leads to sufficiently accurate results.

Once that the conformal transformation is known, the stresses in the half plane with a polygonal edge notch can be calculated by the method described in chapters 4 and 5.

Schwarz-Christoffel transformation

Let there be given an open line C, composed of non-intersecting straight line segments connecting the points

$$z_0 = +\infty, \ z_1, \ z_2, \ \dots, \ z_n = -\infty,$$
 (9.1)

in the complex z-plane, see figure 9.1. Here $+\infty$ and $-\infty$ denote the point at infinity when approached along the real axis to its right and left ends, respectively. The points z_1 and z_{n-1} are also located on the real axis,

$$\Im(z_1) = \Im(z_{n-1}) = 0$$

The positive direction on C is defined by the order of subscripts in the sequence of points (9.1). The line C divides the entire z-plane into two parts. The region to the left of C is denoted by R.

The properties of the line C specified above ensure that the region R is the lower half plane $\Im(z) < 0$ with an edge notch in the shape of an open polygon. The conformal transformation of R onto the interior of a unit circle will be determined, but first the conformal transformation onto the lower half plane $\Im(w) < 0$ will be considered. Subsequent transformation onto a circle is then a simple matter.

The conformal transformation of the lower half plane $\Im(w) < 0$ onto R is denoted by

$$z = f(w). \tag{9.2}$$

The function f(w) will be of the following general form (the Schwarz-Christoffel transformation, see NEHARI, 1952),

$$f(w) = \alpha^* \int_0^w (\lambda - u_1)^{-k_1} (\lambda - u_2)^{-k_2} \dots (\lambda - u_n)^{-k_n} d\lambda + \beta, \qquad (9.3)$$

where α^* and β are complex constants, and u_1, u_2, \ldots, u_n are the points on the real axis $\Im(w) = 0$ corresponding to the corner points of the boundary Cof R. The value of $k_j\pi$ is the abrupt change in direction (in counter-clockwise direction) at the corner point corresponding to u_j . An alternative form of (9.3) is

$$\frac{df}{dw} = \alpha^* (w - u_1)^{-k_1} (w - u_2)^{-k_2} \dots (w - u_n)^{-k_n}.$$
(9.4)

The point at infinity in the z-plane is chosen to correspond to the point at infinity in the w-plane. Thus u_n is taken at infinity. Letting u_n become very large in (9.4) and at the same time letting α^* tend to zero or infinity in such a way that $\alpha^*(w-u_n)^{-k_n}$ tends to a definite constant, say α , leads to

$$\frac{df}{dw} = \alpha (w - u_1)^{-k_1} (w - u_2)^{-k_2} \dots (w - u_{n-1})^{-k_{n-1}}.$$
(9.5)

In the present case of a lower half plane with a polygonal edge notch, the directions of the line segments for ∞ to z_1 and from z_{n-1} to $-\infty$ must be the same, hence

$$\sum_{i=1}^{n-1} k_i = 0. (9.6)$$

This condition is necessary, but not sufficient for the region corresponding to the lower half plane $\Im(w) < 0$ according to the transformation z = f(w) to approximate a half plane at infinity. Sufficient conditions are obtained by requiring that near infinity the function f(w) is of the form

$$w \to \infty : f(w) = aw + b + cw^{-1} + \dots,$$
 (9.7)

or, by requiring that df/dw is the form

$$w \to \infty : df/dw = a - cw^{-2} + \dots, \tag{9.8}$$

The essential feature of condition (9.8) is that it does not contain a term of order w_{-1} . Such a term would correspond to a logarithmic term in (9.7). Elaboration of (9.5) shows that in general for $w \to \infty$

$$\frac{df}{dw} = \alpha \Big[1 + \frac{k_1 u_1}{w} + \frac{k_2 u_2}{w} + \ldots + \frac{k_{n-1} u_{n-1}}{w} + O(\frac{1}{w^2}) \Big], \tag{9.9}$$

where use has been made of (9.6). Thus, in order that the coefficient of w^{-1} vanishes,

$$\sum_{i=1}^{n-1} k_i u_i = 0. (9.10)$$

Determination of parameters

In general it is impossible to integrate (9.5) analyticially. Numerical integration presents the difficulty that the parameters $u_1, u_2, \ldots, u_{n-1}$ must be given in order to perform the numerical integration process. These parameters are usually not given beforehand, however, but rather the points $z_1, z_2, \ldots, z_{n-1}$ in the z-plane, corresponding to $u_1, u_2, \ldots, u_{n-1}$. For the mathematical problem of determining the parameters u_i such that their images z_i are located in the appropriate points of the z-plane an approximate method will be presented.

As is well known from the theory of conformal transformations (NEHARI, 1952) three parameters can be taken arbitrarily without loss of generality. Since u_n has been taken at infinity, this property leaves two more parameters to be chosen arbitrarily. As such the parameters u_1 and u_{n-1} will be taken, assuming them to be located at

$$\begin{array}{c} u_1 = 1, \\ \\ u_{n-1} = -1. \end{array} \right\}$$
 (9.11)

Equation (9.5) then still contains the following independent and unknown parameters

$$u_2, \ldots, u_{n-2}, |\alpha|, \arg(\alpha).$$
 (9.12)

The parameter $\arg(\alpha)$ produces merely a rotation of the region in the z-plane, and does not influence the shape of the region. In fact, it follows from (9.9) that $\arg(\alpha)$ is the rotation of the line segments near infinity. Since both the regions in the w-plane and the z-plane approximate a half plane at infinity, and their orientation is the same, it follows that

$$\arg(\alpha) = 0,\tag{9.13}$$

and thus there remain n-2 parameters,

$$u_2, \ldots, u_{n-2}, |\alpha|.$$
 (9.14)

In the z-plane the shape of the region R is determined by the following quantities, which can be considered as given,

$$\begin{cases} l_1 = |z_2 - z_1|, \\ l_2 = |z_3 - z_2|, \\ \dots \\ l_{n-2} = |z_{n-1} - z_{n-2}|, \end{cases}$$

$$(9.15)$$

and, of course, by the values of the coefficients k_j (j = 1, 2, ..., n - 1). The quantity l_j represents the length of the line segment from z_j towards z_{j+1} . The sum of all l_j 's will be denoted by l_j .

$$l = \sum_{j=1}^{n-2} l_j. \tag{9.16}$$

The problem now is to determine the n-2 parameters $u_2, \ldots, u_{n-2}, |\alpha|$, such that the n-2 lengths have the preassigned values, given by (9.15).

The following quantities are now introduced,

$$\begin{array}{c} v_1 = u_1 - u_2, \\ v_2 = u_2 - u_3, \\ \dots \\ v_{n-2} = u_{n-2} - u_{n-1}. \end{array} \right\}$$

$$(9.17)$$

Then

$$\sum_{j=1}^{n-2} v_j = u_1 - u_{n-1} = 2.$$
(9.18)

The newly introduced parameters v_j represent the length of the line segment between the two succesive points u_j and u_{j+1} on the axis $\Im(w) = 0$. These parameters v_j are related to the lengths l_j in the z-plane. The parameter $|\alpha|$ (or α , because $\arg(\alpha) = 0$ anyway) represents a constant multiplication factor. Thus α cal always be chosen such that the total length of the notch in the z-plane is l. The parameters v_j should be chosen in such a ratio to each other that the total length l is correctly subdivided into parts l_j . Although each parameter v_j wil be dependent upon all the lengths l_j , it is to be expected that v_j depends most strongly upon that length l_j which has the same subscript. This suggests the following iterative procedure for the determination of the parameters.

Iterative calculation of parameters

As an initial estimate the parameters v_j are chosen as

$$v_j = v_j^1 = 2l_j/l, \qquad j = 1, 2, \dots, n-2.$$
 (9.19)

Because of (9.16) the condition (9.18) is now identically satisfied.

With (9.17) and (9.19) the initial values of $u_1, u_2, \ldots, u_{n-2}$ can now be determined as

$$u_1 = 1,$$

$$u_j = u_j^1 = 1 - 2\sum_{i=1}^{j-1} l_i/l, \qquad j = 2, \dots, n-1.$$
(9.20)

It follows from (9.5) that in general the lengths l_j can be calculated from

$$l_j = |\alpha| \int_{u_{j+1}}^{u_j} |w - u_1|^{-k_1} |w - u_2|^{-k_2} \dots |w - u_{n-1}|^{-k_{n-1}} dw.$$
(9.21)

Thus the lengths l_j^1 , corresponding to the values u_j as given by (9.20), can be calculated. This gives

$$l_{j}^{1} = |\alpha| \int_{u_{j+1}^{1}}^{u_{j}^{1}} |w - u_{1}^{1}|^{-k_{1}} |w - u_{2}^{1}|^{-k_{2}} \dots |w - u_{n-1}^{1}|^{-k_{n-1}} dw, \qquad (9.22)$$

where $|\alpha|$ should be such that the total length is l, hence

$$|\alpha| = l / \int_{u_1}^{u_{n-1}} |w - u_1^1|^{-k_1} |w - u_2^1|^{-k_2} \dots |w - u_{n-1}^1|^{-k_{n-1}} dw.$$
(9.23)

The values of l_j^1/l will in general not be equal to the prescribed values l_j/l , since the initial estimate (9.19) may not be accurate enough. When for a certain value of j the value of l_j^1/l is greater than l_j/l it can be expected that v_j has been chosen too large. As a second series of estimates one may take

$$v_j = v_j^2 = v_j^1 + c(l_j - l_j^1)/l, \qquad j = 1, 2, \dots, n-2.$$
 (9.24)

where c is some constant, which can be determined experimentally by requiring that the approximation procedure converges rapidly.

As may be clear from the considerations given above, the iterative method is based upon two assumptions: 1) the distance v_j depends most strongly upon that value of l_i for which i = j, and much less upon the other values of l_i ; 2) v_j is a monotonic increasing function of l_j .

Transformation onto unit circle

Once that the transformation from a lower half plane onto R is known, it remains to derive from this the transformation from the unit circle $|\zeta| < 1$. This can be done by the function

$$w = (i\zeta + 1)/(\zeta + i), \tag{9.25}$$

which maps the interior of the unit circle $|\zeta| = 1$ onto the lower half plane $\Im(w) < 0$. The points $\zeta = +1$, $\zeta = i$, $\zeta = -1$ and $\zeta = -i$ correspond to w = 1, w = 0, w = -1 and $w = \infty$, respectively, see figure 9.2. Denoting the



Figure 9.2: Conformal transformation $w = (i\zeta + 1)/(\zeta + i)$.

conformal transformation from the interior of the unit circle $|\zeta| = 1$ onto the region R in the z-plane as before by $z = \omega(\zeta)$ it follows with (9.2) that

$$\omega'(\zeta) = \frac{df}{dw} \frac{dw}{d\zeta},$$

or, with (9.5) and (9.25),

$$\omega'(\zeta) = \alpha \left(\frac{i\zeta+1}{\zeta+i} - u_1\right)^{-k_1} \dots \left(\frac{i\zeta+1}{\zeta+i} - u_{n-1}\right)^{-k_{n-1}} \frac{i-1}{(\zeta+i)^2}.$$
 (9.26)

With (9.6) this can be written in the following form

$$\omega'(\zeta) = A(\zeta - b_1)^{-k_1} \dots (\zeta - b_{n-1})^{-k_{n-1}} (\zeta + i)^{-2}, \qquad (9.27)$$

where

$$A = 2\alpha (1 + iu_1)^{-k_1} \dots (1 + iu_{n-1})^{-k_{n-1}},$$
(9.28)

$$b_j = i(1 - iu_j)/(1 + iu_j), \qquad j = 1, \dots, n-1.$$
 (9.29)

The coefficients A and b_j can be calculated from (9.28) and (9.29) when α and u_j are known, and thus now a possible procedure for the determination of the derivative $\omega'(\zeta)$ of the mapping function has been found. The coefficients of the apporximate mapping function

$$\omega_n(\zeta) + \frac{p}{\zeta - \sigma_0} + \sum_{k=1}^n c_k \zeta^k, \tag{9.30}$$

can next be determined in the same way as done in chapter 8 for the case of a dike. The calculation of the stresses due to gravity acting in the region R can then be executed along the lines of chapter 5 or 6.

For the procedure outline above a computer program has been written. To test the program and the iteration procedure the case of a dike has been



Figure 9.3: Dike on a half plane

investigated once more, see figure 9.3. It turned out that after 10 iteratiuons (with c = 1) the relative accuracy of the locations of the corner points is about 10^{-3} , whereas after 15 iterations the error is less than 10^{-4} . Each iteration too about 20 seconds computer time on the TR4 of the Delft University of Technology. For the case $\alpha = \pi/4$, h = H/L = 0.5, the following values for A, b_1, b_2, b_3, b_4 were obtained:

A/L = 1.803, $b_1 = 1,$ $b_2 = \exp(0.317i\pi),$ $b_3 = -\exp(0.683i\pi),$ $b_4 = -1.$

These results may be compared with the results of the direct computations of chapter 8. There the value of A/L in the same case was found as 1.802, and the argument of b_2 was found to be 0.316π . The correspondence of these critical values is close enough to justify the conclusion that the method exposed here leads to sufficiently accurate results.

Remarks

Two remarks may be made to conclude this section. The first is that the calculation of the lengths l_j^1 by the formulas (9.22) presents the difficulty that the points u_1^1 , u_2^1 , etc. are singular points of the integrand. When k_j is positive (as it is for a re-entrant corner) the integrand is even unbounded at $w = u_j$, although the integrals always exist as improper integrals (provide that $k_j < 1$, which is a restriction of no significance). A method for the calculation of such integrals has been describe by KANTOROVICH & KRYLOV (1964). This method is based, as usual, upon the decomposition of the integrals into a singular part which is evaluated by an exact formula, and a regular part (having a bounded integrand) which is evaluated by a numerical technique. A similar technique has been described in section 8.1, where the parameters for the case of a dike were calculated. The computer program written for the approximate calculations described in this section, uses an even simpler (and less accurate) method, in which the improper integral is simply approximated by the proper integral obtained by letting the interval of integration start (or end) at a very small

distance ε from the singular point. Of course in this way an error is introduced into the calculations, of order $O(\varepsilon^{1-k_j})$. That this is not intolerable follows from the fact that the Schwarz-Christoffel parameters are not the final objectives of the calculations. Once they are known, the conformal transformation must be approximated anyway to bring it in the form suitable for the application of the stress-calculation procedures of chapters 5 and 6. It is clearly of little importance to calculate the Schwarz-Christoffel parameters with an accuracy that greatly surpasses the accuracy of the approximation procedure used in the second stage of the computations.

A second final remark is that the procedure of this section is a direct method, in contrast with the method used in section 8.1 for the dike problem. The latter is an inverse method, leading to the graphs in figures 8.4 and 8.5 for the determination of the Schwarz-Christoffel parameters. The advantage of the present direct technique is that it can be used in a chain of computer programs, which then enables to obtain directly the approximate shape of the dike, and the stresses in its interior, using as input for the computer program only the corner points of the dike, and some parameters related to the accuracy that is required.

Appendix A

AN EXTENSION OF FILON'S INTEGRATION METHOD

The usual numerical integration procedures, such as those associated with the names of Simpson, Newton-Cotes, Gauss, etc., are not very suited for the approximate evaluation of integrals of the form

$$\int_{a}^{b} f(x) \exp(-kix) \, dx, \qquad k = 0, 1, 2, \dots,$$
(A.1)

for large values of the parameter k. This is caused by the rapidly oscillating character of the integrand of equation (A.1), which in its turn is due to the factor $\exp(-kix)$. A special method for the evaluation of this type of integral was devised by FILON (1928). In this method the interval of integration is subdivided into a number of relatively small subintervals, and in each subinterval the function f(x) is approximated by a second order polynomial coinciding with the function in the end points and the midpoint of the subinterval. After replacing f(x) in the integral by its apporximation the integration can be performed exactly, and thus an approximation of the integral is obtained (see also ABRAMOWITZ & STEGUN, 1965). In this appendix Filon's method is extended by approximating the function f(x) in each subinterval by a fourth order polynomial. A measure for the error will also be derived.

Consider the integrals

$$s_k = \frac{1}{b-a} \int_a^b f(x) \exp(-kix) \, dx, \qquad k = 0, 1, 2, \dots,$$
(A.2)

where f(x) is a given (complex) function, bounded in the interval $a \le x \le b$. This interval is now divided into a number of equal subintervals. For reasons of convenience, the number of subintervals is chosen even, 2m. Then equation (A.2) can be rewritten as

$$s_k = \frac{1}{b-a} \sum_{p=0}^{2m-1} \exp(-kix_p) s_{kp}, \qquad k = 0, 1, 2, \dots,$$
(A.3)

where

$$s_{kp} = \int_{x_p}^{x_{p+1}} f(x) \exp[-ki(x - x_p)] \, dx, \tag{A.4}$$

and where

$$x_p = a + (b - a)p/(2m).$$
 (A.5)

By writing $y = x - x_p$ in equation (A.4), and introducing a quantity h defined as

$$h = (b - a)/(8m),$$
 (A.6)

one obtains

$$s_{kp} = \int_0^{4h} f(x_p + y) \exp(-kiy) \, dy, \qquad k = 0, 1, 2, \dots$$
 (A.7)

For this elementary contribution to the original integral (A.2) an approximation will be developed below.

In the small interval of length 4h the function $f(x_p + y)$ is approximated by a Lagrangian formula (see e.g. HILDEBRAND, 1956), or, stated more explicitly, by a fourth order polynomial which coincides with the function $f(x_p + y)$ in the points for which y = 0, y = h, y = 2h, y = 3h and y = 4h. It is easily verified that this is accomplished by writing

$$24h^{4}f(x_{p} + y) = (y - h)(y - 2h)(y - 3h)(y - 4h)f(x_{p}) - 4y(y - 2h)(y - 3h)(y - 4h)f(x_{p} + h) + 6y(y - h)(y - 3h)(y - 4h)f(x_{p} + 2h) - 4y(y - h)(y - 2h)(y - 4h)f(x_{p} + 3h) + y(y - h)(y - 2h)(y - 3h)f(x_{p} + 4h) + 24h^{4}E_{p}(y).$$
(A.8)

In this expression the term $E_p(y)$ represents the error. When the function f(x) possesses at least 5 continuous derivatives in the interval $x_p < x < x_p + 4h$ the error term is given by the following formula (see e.g. HILDEBRAND, 1956, p. 63),

$$E_p(y) = \frac{1}{5!} y(y-h)(y-2h)(y-3h)(y-4h)f^5(x_p+4\xi h),$$
(A.9)

where the superscript 5 indicates the fifth derivative and ξ is some number in the interval $0 < \xi < 1$.

Substitution from (A.8) into (A.7) gives

$$24h^{4}s_{kp} = f(x_{p} \int_{0}^{4h} (y-h)(y-2h)(y-3h)(y-4h) \exp(-kiy) dy - 4f(x_{p}+h) \int_{0}^{4h} y(y-2h)(y-3h)(y-4h) \exp(-kiy) dy + 6f(x_{p}+2h) \int_{0}^{4h} y(y-h)(y-3h)(y-4h) \exp(-kiy) dy - 4f(x_{p}+3h) \int_{0}^{4h} y(y-h)(y-2h)(y-4h) \exp(-kiy) dy + f(x_{p}+4h) \int_{0}^{4h} y(y-h)(y-2h)(y-3h) \exp(-kiy) dy + 24h^{4} \int_{0}^{4h} E_{p}(y) \exp(-kiy) dy.$$
(A.10)

The first five integrals appearing in equation (A.10) can be evaluated using the general integration formula

$$\int_{0}^{4h} y^{n} \exp(-kiy) \, dy = \frac{(-i)^{n+1} n!}{k^{n+1}} \Big\{ 1 - \exp(-4ikh) \sum_{j=0}^{n} \frac{(4ikh)^{j}}{j!} \Big\}.$$
(A.11)

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After some elaboration the following result is obtained

where t = kh, and where the coefficients are given by

$$a_{1} = -24i - 60t + 70it^{2} + 50t^{3} - 24it^{4},$$

$$b_{1} = 24i - 36t - 22it^{2} + 6t^{3},$$

$$a_{2} = 4(24i + 54t - 52it^{2} - 24t^{3}),$$

$$b_{2} = 4(-24i + 42t + 28it^{2} - 8t^{3}),$$

$$a_{3} = 6(-24i - 48t + 38it^{2} + 12t^{3}),$$

$$b_{3} = 6(24i - 48t - 38it^{2} + 12t^{3}),$$

$$a_{4} = 4(24i + 42t - 28it^{2} - 8t^{3}),$$

$$b_{4} = 4(-24i + 54t + 52it^{2} - 24t^{3}),$$

$$a_{5} = -24i - 36t + 22it^{2} + 6t^{3},$$

$$b_{5} = 24i - 60t - 70it^{2} + 50t^{3} + 24it^{4},$$
(A.13)

Equation (A.12) enables to calculate an approximation to the true value of the contribution s_{kp} to the original integral. By summing the contributions of all subintervals, according to (A.3), an approximation to the integral itself is obtained.

For small values of the parameter t = kh the expression (A.12) is inaccurate. By using the Taylor series expansion of $\exp(-4it)$ around t = 0, and retaining the first ten terms only, the following formula can be derived

$$s_{kp} = \frac{2h}{45} \left\{ \left(7 + \frac{20}{7}t^2 - \frac{80}{21}it^3 - \frac{32}{9}t^4 + \dots\right)f(x_p) + \left(32 - 32it - \frac{192}{7}t^2 + \frac{512}{21}it^3 + \frac{1280}{64}t^4 + \dots\right)f(x_p + h) + \left(12 - 24it - \frac{48}{7}t^2 - \frac{128}{7}it^3 - \frac{640}{21}t^4 + \dots\right)f(x_p + 2h) + \left(32 - 96it - \frac{1088}{7}t^2 + \frac{512}{3}it^3 + \frac{2801}{9}t^4 + \dots\right)f(x_p + 3h) + \left(7 - 28it - \frac{372}{7}t^2 + \frac{1408}{21}it^3 + \frac{4000}{63}t^4 + \dots\right)f(x_p + 2h) \right\} + \varepsilon_{kn}.$$
(A.14)

In the limit $t \to 0$ equation (A.14) reduces to one of the well known Newton-Cotes integration formulas (HILDEBRAND, 1956, p. 73).

An estimation of the truncation error can be found in the following way. Substitution from (A.9) into (A.10) gives

$$\varepsilon_{kp} = \frac{1}{5!} \int_0^{4h} y(y-h)(y-2h)(y-3h)(y-4h) \exp(-kiy) \times f^5(x_p+4\xi h) \, dy.$$

An estimation of the error is obtained by replacing $f^5(x_p + 4\xi h)$ (in which ξ depends upon an unknown way on y) by some average vale $\overline{f^5}$. Taking an upper bound for the remaining integral leads to

$$|\varepsilon_{kp}| \le \frac{\overline{f^5}}{5!} \int_0^{4h} |y||y-h||y-2h||y-3h||y-4h|dy = \frac{19}{360} h^6 \overline{f^5}.$$

The average value of $\overline{f^5}$ of the fifth derivative of f(x) in the neighbourhood of x_p can be estimated by some finite difference approximation. In order to obtain a simple symmetric expression it is most convenient to write

$$4h^4 f^5 \approx f^4(x_p + 4h) - f^4(x_p),$$

and to use forward difference expressions for the fourth derivatives. This leads to

$$4h^4 f^5 \approx f(x_p + 8h) - 4f(x_p + 7h) + 6f(x_p + 6h) - 4f(x_p + 5h) + 4h^4 f(x_p + 3h) - 6f(x_p + 2h) + 4f(x_p + h) - f(x_p).$$
(A.15)

This approximation of the fifth derivative can be applied in the intervals (x_p, x_{p+1}) and (x_{p+1}, x_{p+2}) . It involves only points in which the functional value has to be calculated anyway for the approximate evaluation of the integral. In order to obtain the contribution to the error in the original integral ε_{kp} is to be divided by (b-a), see equation (A.3). Thus one obtains for the total error

$$E = 2\sum_{p=0,2,4,\dots}^{2m-2} E_p, \tag{A.16}$$

where

$$E_p = \frac{|\varepsilon_{kp}|}{b-a} = \frac{1}{8mh} \frac{19}{360} h^6 f^5 = \frac{19}{11520 m} (4h^4 f^5),$$

or, with (A.15),

$$E_p = \frac{19}{11520 m} \Big\{ f(x_p + 8h) - 4f(x_p + 7h) + 6f(x_p + 6h) - 4f(x_p + 5h) + 4f(x_p + 3h) - 6f(x_p + 2h) + 4f(x_p + h) - 6f(x_p) \Big\}.$$
 (A.17)

Example

In several chapters of this thesis it was found necessary to evaluate integrals of the type considered in this appendix. These were all calculated by means of a standard ALGOL-procedure (procedure FOURIER), which determines the integrals (A.2) for k = 0, 1, ..., n when the lower and upper limits of integration (a and b), the number of subintervals (2m), a function procedure $(f(x) = \Re(f(x)) + i \Im(f(x)))$ and n are given. The following example may serve as an illustration and as a check on the formulas and their representation in the computer program. This example concerns the first 10 coefficients of the Fourier expansion of the function $\exp(x)$,

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \exp(x) \exp(-kix) \, dx, \qquad k = 0, 1, \dots, 10.$$
 (A.18)

The coefficients c_3, c_4, \ldots, c_{10} were calculated with m = 10 and with the formula (A.12). For the coefficients c_0, c_1 and c_2 the corresponding values of hkwould have been rather small (less than 0.2), and therefore these were calculated using (A.14), and by taking m 5 times as large as before (i.e. m = 50) in order to make hk small enough for the expansions in (A.14) to be convergent. The results are shown in table A.1, together with the exact results, which happen to be easily calculated in this case,

	Exact		Filon's method	
k	$\Re(c)$	$\Im(c)$	$\Re(c)$	$\Im(c)$
0	85.066989	0.000000	85.066989	0.000000
1	42.533494	42.533494	42.533494	42.533495
2	17.013398	34.026795	17.013396	34.026796
3	8.506699	25.520096	8.506699	25.520097
4	5.003941	20.015762	5.003941	20.015762
5	3.271807	16.359036	3.271807	16.359036
6	2.299108	13.794647	2.299108	13.794647
7	1.701340	11.909378	1.701340	11.909378
8	1.308723	10.469783	1.308723	10.469783
9	1.037403	9.336621	1.037403	9.336621
10	0.842247	8.422474	0.842247	8.422474

 $c_k = \frac{1+ik}{2\pi} \frac{\exp(2\pi) - 1}{k^2 - 1}, \qquad k = 0, 1, \dots$ (A.19)

Table A.1: Example.

It can be seen from the table that the approximate and exact results agree to at least 6 significant figures. The truncation error estimated by the numerical procedure itself, according to (A.16) was found to be 3×10^{-6} , whereas the maximum total error (which includes accumulated errors due to rounding off) appears to be about 2×10^{-6} .

Appendix B

SOME FOURIER TRANSFORMS

In this appendix some definite integrals of the Fourier transform type will be evaluated. Some of these integrals are used in section (7.3).

The basic integral to be considered is

$$J_1 = \int_{-\infty}^{+\infty} \frac{\sinh(ax)}{\sinh(\pi x)} \exp(i\xi x) \, dx, \quad -\pi < \alpha < +\pi, \tag{B.1}$$

where π/α is not an integer, and where ξ is some real number. In the complex *x*-plane the integrand has simple poles at the zeroes of $\sinh(\pi x)$, i.e. for

 $x = ki, \quad k = \pm 1, \pm 2, \pm 3, \dots$

By extending the line of integration with a large semi-circle in the upper half plane the integral is replaced by a contour integral (see figure B.1). It can be



Figure B.1: Contour in complex x-plane

shown that the contribution of the integration along the semi-circle tends to zero, provided that $\xi > 0$ and $|\alpha| < \pi$. The integral (B.1) then is equal to the contour integral, which, by the residue theorem, equals $2\pi i$ times the sum of the residues in the poles encircled by the contour. This leads to

$$J_1 = -2\Im\{\sum_{k=1}^{\infty} (-1)^k \exp[-k(\xi - i\alpha)]\}$$

The expression between parentheses is the sum of a geometrical series, hence, if $\xi < 0$,

$$J_1 = 2\Im \frac{\exp[-(\xi - i\alpha)]}{1 + \exp[-(\xi - i\alpha)]} = \frac{\sin \alpha}{\cosh \xi + \cos \alpha}$$

Hence

$$\int_{-\infty}^{+\infty} \frac{\sinh(\alpha x)}{\sinh(\pi x)} \exp(i\xi x) \, dx = \frac{\sin\alpha}{\cosh\xi + \cos\alpha}, \quad |\alpha| < \pi. \tag{B.2}$$

This formula (see also SNEDDON, 1951, p. 523) is also valid for negative values of ξ . In that case the derivation requires extension of the line of integration with a semi-circle in the lower half plane.

An alternative form of (B.2) is

$$\int_0^\infty \frac{\sinh(\alpha x)}{\sinh(\pi x)} \cos(\xi x) \, dx = \frac{\sin \alpha}{2(\cosh \xi + \cos \alpha)}. \quad |\alpha| < \pi. \tag{B.3}$$

In the sequel a number of integrals will be derived from (B.3). This will be done by differentiation or integration of expressions such as (B.3) with respect to the parameters ξ and α . This is permitted since (B.3) and all similar integrals to be presented below are uniformly convergent for all ξ , and for all α such that $|\alpha| < \pi - \delta$, where δ is an arbitrarily small positive fixed number.

Differentiation of (B.3) with respect to α gives

$$\int_0^\infty \frac{x \cosh(\alpha x)}{\sinh(\pi x)} \cos(\xi x) \, dx = \frac{1 + \cosh \xi \cos \alpha}{2(\cosh \xi + \cos \alpha)^2}.$$
 (B.4)

Differentiating this again with respect to α gives

$$\int_{0}^{\infty} \frac{x^{2} \sinh(\alpha x)}{\sinh(\pi x)} \cos(\xi x) dx = \frac{2 \sin \alpha + \sin \alpha \cos \alpha \cosh \xi - \sin \alpha \cosh^{2} \xi}{2(\cosh \xi + \cos \alpha)^{3}}.$$
 (B.5)

By taking various combinations of (B.3) – (B.5) the following three integral representations can be obtained, all valid for $|\alpha| < \pi$,

$$\frac{1}{\cosh\xi + \cos\alpha} = \int_0^\infty \frac{2\sinh(\alpha x)}{\sin\alpha\sinh(\pi x)} \cos(\xi x) \, dx,\tag{B.6}$$

$$\frac{1}{(\cosh\xi + \cos\alpha)^2} = \int_0^\infty \frac{2x\sin\alpha\cosh(\alpha x) - 2\cos\alpha\sinh(\alpha x)}{\sin^3\alpha\sinh(\pi x)} \cos(\xi x) \, dx, \qquad (B.7)$$

$$\frac{1}{(\cosh\xi + \cos\alpha)^3} = \int_0^\infty \frac{[(1+x^2)\sin^2\alpha + 3\cos^2\alpha]\sinh(\alpha x) - 3x\sin\alpha\cos\alpha\cosh(\alpha x)}{\sin^5\alpha\sinh(\pi x)} \times \cos(\xi x) \, dx, \quad (B.8)$$

For $\alpha = \pi/2$ equations (B.6) – (B.8) reduce to

$$\frac{1}{\cosh\xi} = \int_0^\infty \frac{1}{\cosh(\pi x/2)} \,\cos(\xi x) \,dx,\tag{B.9}$$

$$\frac{1}{\cosh^2 \xi} = \int_0^\infty \frac{x}{\sinh(\pi x/2)} \, \cos(\xi x) \, dx,\tag{B.10}$$

$$\frac{1}{\cosh^3 \xi} = \int_0^\infty \frac{1+x^2}{2\cosh(\pi x/2)} \,\cos(\xi x) \,dx. \tag{B.11}$$

The limiting values of (B.6) – (B.8) for $\alpha \rightarrow 0$ can be determined with the aid of l'Hopital's rule. This leads to

$$\frac{1}{1+\cosh\xi} = \int_0^\infty \frac{2x}{\sinh(\pi x)} \cos(\xi x) \, dx,\tag{B.12}$$

$$\frac{1}{(1+\cosh\xi)^2} = \int_0^\infty \frac{2x(1+x^2)}{3\sinh(\pi x)} \cos(\xi x) \, dx,\tag{B.13}$$

$$\frac{1}{(1+\cosh\xi)^3} = \int_0^\infty \frac{x(1+x^2)(4+x^2)}{15\sinh(\pi x)} \cos(\xi x) \, dx. \tag{B.14}$$

Another set of integrals can be obtained by returning to (B.4) and integrating with respect to ξ . This gives

$$\int_0^\infty \frac{\cosh(\alpha x)}{\sinh(\pi x)} \sin(\xi x) \, dx = \frac{\sinh \xi}{2(\cosh \xi + \cos \alpha)},\tag{B.15}$$

as can easily be verified by differentiation of (B.15) with respect to ξ . No integration constant needs to be added to (B.15) since for $\xi = 0$ both members are zero. The integral (B.15) can also be found in BATEMAN (1954), p. 88.

Differentiation of (B.15) with respect to α gives

$$\int_0^\infty \frac{x \sinh(\alpha x)}{\sinh(\pi x)} \sin(\xi x) \, dx = \frac{\sin \alpha \sinh \xi}{2(\cosh \xi + \cos \alpha)^2},\tag{B.16}$$

Differentiating this again with respect to α gives

$$\int_{0}^{\infty} \frac{x^{2} \cosh(\alpha x)}{\sinh(\pi x)} \sin(\xi x) dx = \frac{\sinh \xi [\cos \alpha (\cosh \xi + \cos \alpha) + 2 \sin^{2} \alpha]}{2(\cosh \xi + \cos \alpha)^{3}}.$$
 (B.17)

For $\alpha = \pi/2$ the formulas (B.15) – (B.17) reduce to

$$\frac{\sinh\xi}{\cosh\xi} = \int_0^\infty \frac{1}{\sinh(\pi x/2)} \,\sin(\xi x) \,dx,\tag{B.18}$$
$$\frac{\sinh\xi}{\cosh^2\xi} = \int_0^\infty \frac{x}{\cosh(\pi x/2)} \sin(\xi x) \, dx,\tag{B.19}$$

$$\frac{\sinh\xi}{\cosh^3\xi} = \int_0^\infty \frac{x^2}{2\sinh(\pi x/2)} \sin(\xi x) \, dx. \tag{B.20}$$

The last two integral representations could also have been obtained from (B.9) and (B.10) by differentiating these with respect to ξ .

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SAMENVATTING

In dit proefschrift wordt een algemene oplossingsmethode voor een bepaalde klasse van problemen uit de twee-dimensionale elasticiteitstheorie aangegeven. Het betreft hier problemen die als gemeenschappelijk kenmerk hebben dat het elastische lichaam de vorm heeft van een beneden-halfvlak met een iets gewijzigde bovenrand, waarop als belasting uitsluitend het eigen gewicht van het materiaal werkt. Deze klasse van problemen omvat onder meer het probleem van een dijk, bestaande uit elastisch materiaal, gebouwd op een elastisch halfvlak, en ook de problemen van een ontgraving of een ontgronding in een lichaam dat aanvankelijk een halfvlak innam.

Na een beschrijving van het algemene probleem in hoofdstuk 2, en de wiskundige formulering ervan in hoofdstuk 3. wordt in de hoofdstukken 4 en 5 de eigenlijke oplossingsmethode behandeld. Deze valt uiteen in twee gedeelten. In de eerste plaats wordt in hoofdstuk 4 beschreven hoe de conforme afbeelding van het gebied ingenomen door het elastische lichaam, op het inwendige van de eenheidscirkel in standaardvorm kan worden gebracht. Deze standaardvorm bestaat uit een singulier gedeelte, dat er voor zorgt dat op het oneindige het gebied een halfvlak benadert, en een regulier gedeelte, dat geschreven kan worden in de vorm van een reeks van Taylor. Bewezen wordt dat deze Taylor-reeks in het algemeen convergeert, niet alleen binnen, maar ook op de eenheidscirkel, mits keerpunten in de contour worden uitgesloten. Ook wordt aangegeven hoe de coëfficiënten van de termen uit de Taylor-reeks kunnen worden berekend. Daarbij wordt gebruik gemaakt van een methode van Filon voor de numerieke berekening van trigonometrische integralen. Van de Taylor-reeks worden vervolgens alleen de eerste n termen in aanmerking genomen. Dit betekent dat niet het oorspronkelijke probleem wordt opgelost, maar een probleem met een iets afwijkende vorm van de rand. Door voldoende termen van de Taylorreeks in aanmerking te nemen kan de afwijking kleinder gemaakt worden dan een willekeurig kleine maat. Afgezien van deze benadering van de rand is de oplossingsmethode exact.

De oplossing van het randvoorwaarde-probleem wordt, in een algemene vorm, gegeven in hoofdstuk 5. Deze oplossing wijkt alleen af van de bekende technieken (zoals ontwikkeld door Muskhelishvili) doordat de afbeeldingsfunctie een singulariteit, namelijk een pool van de eerste orde, heeft op de rand van de eenheidscirkel. Dit leidt tot enige complicaties, maar verhindert niet dat een algemene oplossing kan worden gegeven. Uitwerking van de oplossing voor een specifiek geval vereist alleen algebraïsche bewerkingen. Daarbij is de meest gecompliceerde bewerking het oplossen van een stelsel van 2n lineaire vergelijkingen met 2n onbekenden (n is het aantal termen van de afgebroken reeks van Taylor). In hoofdstuk 6 zijn de uit te voeren bewerkingen verzameld.

Bij wijze van voorbeeld wordt vervolgens in hoofdstuk 7 beschouwd het geval van een halfvlak met een uitsnijding in de vorm van een cirkelboog. Enige numerieke resultaten worden gegeven. Voor een speciaal geval blijkt een exacte oplossing te betsaan, en dit maakt een toetsing van de benaderingsmethode mogelijk. Een goede overeenstemming met de exacte oplossing toont aan dat de methode tot vrij nauwkeurige resultaten leidt, behalve voor punten juist op (of zeer dicht bij) de rand gelegen. In paragraaf 7.2 wordt ook verklaard waarom de methode onnauwkeurig is voor randpunten. In paragraaf 7.3 worden de resulaten van de benaderde complexe berekeningsmethode nog vergeleken met enige resultaten verkregen met een (exacte) methode die gebruik maakt van Fourier-integralen. Ook daar blijkt de overeenstemming uitstekend te zijn in punten in het inwendige van het gebied.

Als een tweede type van voorbeelden wordt in hoofdstuk 8 het probleem van een dijk op een halfvlak beschouwd. Ook voor dit geval worden enige numerieke resultaten gegeven. In hoofdstuk 9 tenslotte wordt een generalisatie hiervan, namelijk het geval van een halfvlak met een kerf in de vorm van een willekeurige veelhoek, behandeld. Het blijkt mogelijk te zijn een keten van procedures samen te stellen, waarbij, uitgaande van de coöordinaten van de hoekpunten van de rand, als resultaat de spanningen in het inwendige worden verkregen.

STELLINGEN

Ι

De door Biot aangegeven potentiaalfuncties voor het gekoppelde thermoelastische probleem vormen, onder zekere regulariteitscondities, een volledig stelsel.

M.A. Biot, J. Appl. Phys. 27, 240-253 (1956).

A. Verruijt, Quart. Appl. Math. 26, 485-490 (1969).

Π

De verplaatsingsfuncties van McNamee-Gibson voor het consolidatieprobleem kunnen zodanig worden uitgebreid dat de samendrukbaarheid van het water in rekening kan worden gebracht. In deze gewijzigde vorm zijn de verplaatsingsfuncties ook geschikt voor de oplossing van twee-dimensionale en axiaalsymmetrische problemen uit de theorie der thermo-elasticiteit.

J. McNamee and R.E. Gibson, Quart. J. Mech. Appl. Math. 13, 98-111 (1960)

III

Een mogelijke verklaring van het Noordbergum effekt (dat is: een plaatselijke verhoging van de waterspanningen in de grond bij een pompproef) kan worden gevonden met behulp van de algemene consolidatie-theorie. Dit effekt is dan een direkt gevolg van de koppeling van de grondwaterstroming met de deformatie van de grond.

V

De grafische methode van Schmidt voor het benaderend oplossen van de één-dimensionale warmtediffusie-vergelijking kan op eenvoudige wijze worden uitgebreid tot een oplossingsmethode voor quasi-lineaire en niet-lineaire problemen.

E. Schmidt, Beiträge zur technische Mechanik und technischen Physik,

Springer, Berlin, 1924.

A. Verruijt, Proc. Eur. Conf. Soil Mech. & Found. Eng. Wiesbaden, 305-312 1963.

Voor realistische berekeningen van het gedrag van een grondmassief onder invloed van uitwenige belastingen zou men de mechanische eigenschappen statistisch moeten invoeren. De elementenmethode biedt deze mogelijkheid.

VII

Voor de oplossing van sommige vraagstukken verband houdende met het transport van warmte door stromend grondwater is het zinvol de potentiaal en de stroomfunctie, die de grondwaterstroming beschrijven, als de onafhankelijk veranderlijken voor het warmteprobleem te beschouwen.

VIII

De oplossing van Szelagowski voor het probleem van de getrokken halfoneindige schijf met een half-cirkelvormige kerf (een oplossing die leidt tot de waarde 3 voor de spanningsconcentratie-faktor in het diepste punt van de kerf) is gebaseerd op een ongeoorloofde toepassing van de stelling van Cauchy, en is daarom onjuist.

F. Szelagowski, Bull. Acad. Pol. Sci., Sér. Sci. Tech. 12, no. 7, 353-360 (1964).

IX

De door Gakhov voorgestelde oplossingsmethoden voor de varianten III en IV van het Riemann-probleem (in dit proefschrift aangeduid als het Hilbertprobleem) bevatten een fundamentele fout.

F.D. Gakhov, Boundary valkue problems, Pergamon Press, Oxford, 1966, p. 136-137.

Х

De controle op overeenstemming van naam en nummer van de begunstigde op een girokaart kan worden geautomatiseerd door het rekeningnummer op te bouwen uit twee gedeelten.

\mathbf{XI}

Van de mogelijkheid verenigingen en commissies op te heffen wordt in het algemeen te weinig gebruik gemaakt.