THE ULTIMATE IMAGE SINGULARITIES FOR EXTERNAL SPHEROIDAL AND ELLIPSOIDAL HARMONICS

by

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Abstract

The image system of singularities of an arbitrary exterior potential field within a tri-axial ellipsoid is derived. It is found that the image system consists of a source and doublet distribution over the fundamental ellipsoid. The present contribution is a generalization of previous theories on the image system of an exterior potential field within a sphere and spheroid. A proof of Havelock's spheroid theorem which apparently is not available in the literature is also given.

The knowledge of the image system is required, for example, when hydrodynamical forces and moments acting on an ellipsoid immersed in a potential flow are computed by the Lagally theorem.

The two examples given consider the image system of singularities of an ellipsoid in a uniform translatory motion and in pure rotation.

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1. Introduction

The Lagally theorem [1] together with its recent generalizations [2, 3] yields exact expressions for the forces and moments on Rankine bodies immersed in arbitrary inviscid, potential flows. The main difficulty with the application of the Lagally theorem is that it is necessary to know the image system of singularities associated with the analytic continuation of the external potential flow into the body.

The sphere is the only three-dimensional shape for which there exists a "sphere theorem" [4,5,6] yielding the disturbance potential due to the presence of the sphere in terms of the undisturbed potential. Also a theorem due to Hobson [7] (p. 134) provides an expression for any spherical harmonic in terms of singularities (sources, doublets or multipoles) at the center of the sphere,

$$\frac{P_n^s(\mu)e^{is\phi}}{R^{n+1}} = \frac{(-1)^n}{(n-s)!}\frac{\partial^{n-s}}{\partial x^{n-s}}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)^s\frac{1}{R} \qquad (1)$$

where (R, θ, ϕ) are spherical coordinates, $\mu = \cos\theta$, $P_n^s(\mu)$ is the Legendre function of the first kind defined by (5), and n and s are positive integers.

An inversion theorem similar to the sphere theorem, is not available for a spheroid. However, a most useful relation was given without proof by Havelock [8]. This relation expresses an exterior spheroidal harmonic in terms of singularities distributed on the major axis of the spheroid between the two foci. If the foci are chosen to be at (±1,0,0) in Cartesian notation, the Havelock formula may be written as

$$P_n^s(\mu)Q_n^s(\zeta)e^{is\phi} = \frac{1}{2}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)^s \int_{-1}^{1} \frac{(1-\xi^2)^{\frac{s}{2}} \quad P_n^s(\xi)}{\sqrt{(x-\xi)^2+y^2+z^2}} \, d\xi \qquad (2)$$

where (μ,ζ,ϕ) are spheroidal coordinates defined by (4), and $Q_n^s(\zeta)$ is the Legendre function of the second kind given by (7).

A proof of the Havelock formula (2) is not available in the literature. However proofs of (2) are given in an unpublished thesis [9] and in class notes of Professor L. Landweber. These proofs are based on a suggestion of Havelock to expand the function 1/R in terms of an infinite series of spheroidal harmonics. A briefer proof of the Havelock formula will be presented in section 2. This proof is based on analytic continuation applied to a generalization of the Neumann formula for the Legendre function of the second kind $Q_n^s(\zeta)$.

Equations (1) and (2) provide exact expressions for the image system within a sphere or spheroid of an arbitrary undisturbed external potential flow. Once the image system of singularities is known within the body, the disturbance potential may be written immediately in terms of these singularities.

According to Morse and Feshbach [10] the most general coordinates for the separability of the Laplace equation are the ellipsoidal or the focal coordinates. The ellipsoidal harmonics are also the most general harmonics which are a solution of the three-dimensional Laplace equation. In fact the tri-axial ellipsoid is a truly three-dimensional form, while both the sphere and the spheroid are axisymmetrical forms.

An ellipsoid theorem, similar to the sphere (1) and the spheroid (2) theorems, would be most important in ship hydrodynamics since ship forms can be better approximated by a tri-axial ellipsoid than by a spheroid. Such a theorem, which yields the singularity system within the ellipsoid of an arbitrary external potential function, will be developed in section 3.

2. The Image System for a Spheroid

A spheroidal harmonic of degree n and order s, which vanishes at infinity, is of the form

$$H_n^s(\mu, \zeta, \phi) = P_n^s(\mu)\, Q_n^s(\zeta)\, e^{is\phi} \tag{3}$$

where (μ, ζ, ϕ) are spheroidal coordinates defined by

$$x = \mu\zeta; \qquad y+iz = \sqrt{(1-\mu^2)(\zeta^2-1)}\; e^{i\phi} \tag{4}$$

and the two foci of the spheroid are at $(\pm 1, 0, 0)$ in Cartesian representation.

The Legendre functions of the first kind, P_n^s, and of the second kind, Q_n^s, are defined by [11] (p. 142).

$$P_n^s(\mu) = \frac{1}{2^n n!}(1-\mu^2)^{\frac{s}{2}} \frac{d^{n+s}}{d\mu^{n+s}}(\mu^2-1)^n, \quad \mu\leq 1 \tag{5}$$

$$P_n^s(\zeta) = \frac{1}{2^n n!}(\zeta^2-1)^{\frac{s}{2}} \frac{d^{n+s}}{d\zeta^{n+s}}(\zeta^2-1)^n, \quad \zeta\geq 1 \tag{6}$$

$$Q_n^s(\zeta) = (\zeta^2-1)^{\frac{s}{2}} \frac{d^s}{d\zeta^s}\left\{ P_n(\zeta) \int_\zeta^\infty \frac{d\zeta}{[P_n(\zeta)]^2\,(\zeta^2-1)} \right\} \tag{7}$$

The following relation is given by Hobson (p. 97):

$$\frac{(\mu^2-1)^{\frac{s}{2}}}{(n+s)!} \frac{d^{n+s}}{d\mu^{n+s}}(\mu^2-1)^n = \frac{(\mu^2-1)^{-\frac{s}{2}}}{(n-s)!} \frac{d^{n-s}}{d\mu^{n-s}}(\mu^2-1)^n \tag{8}$$

The above relation together with the Rodriguez formula (5) yield the following expression for $P_n(\mu)$:

$$P_n(\mu) = (-1)^s \frac{(n-s)!}{(n+s)!} \frac{d^s}{d\mu^s} [(1-\mu^2)^{\frac{s}{2}} \; P_n{}^s(\mu)] \tag{9}$$

In addition, the well-known Neumann formula for $Q_n(x)$,

$$Q_n(x) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(\xi)}{x-\xi} \, d\xi \tag{10}$$

is available.

When equation (9) is substituted into (10), and when the latter is integrated by parts s times, equation (10) becomes

$$Q_n(x) = \frac{s!(n-s)!}{2(n+s)!} \int_{-1}^{1} \frac{(1-\xi^2)^{\frac{s}{2}} P_n{}^s(\xi) \, d\xi}{(x-\xi)^{s+1}} \tag{11}$$

The above relation may be considered as a generalization of the Neumann formula (10), which corresponds to the case $s = 0$.

Equations (4), (5), and (6) imply that the exterior spheroidal harmonic (3) may also be written in the form

$$P_n{}^s(\mu) \, Q_n{}^s(\zeta) e^{is\phi} = \frac{d^s P_n(\mu)}{d\mu^s} \frac{d^s Q_n(\zeta)}{d\zeta^s} (y+iz)^s \tag{12}$$

On the part of the x-axis where $|x|>1$, $\mu = 1$ and

$$\frac{d^s P_n(1)}{d\mu^s} = \frac{(n+s)!}{2^s . s!(n-s)!} \tag{13}$$

Substituting equations (11) and (13) into (12) yields

$$\lim_{\mu\to 1} \frac{P_n{}^s(\mu) Q_n{}^s(\zeta) e^{is\phi}}{(y+iz)^s} = \frac{(-1)^s . (2s)!}{2^{s+1} . s!} \int_{-1}^{1} \frac{(1-\xi^2)^{\frac{s}{2}} \; P_n{}^s(\xi)}{(x-\xi)^{2s+1}} \, d\xi \tag{14}$$

Analytic continuation arguments applied to (14) imply that for points off the x axis

$$P_n^s(\mu)\, Q_n^s(\zeta) e^{is\phi} = \frac{(-1)^s.(2s)!}{2^{s+1}.s!} \int_{-1}^{1} \frac{(1-\xi^2)^{\frac{s}{2}} P_n^s(\xi)(y+iz)^s \, d\xi}{[(x-\xi)^2+y^2+z^2]^{s+\frac{1}{2}}} \tag{15}$$

The following relation is easily verified by mathematical induction;

$$\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)^s [(x-\xi)^2 + y^2+z^2]^{-\frac{1}{2}} = \frac{(-1)^s.(2s)!}{2^s.s!} \frac{(y+iz)^s}{[(x-\xi)^2+y^2+z^2]^{s+\frac{1}{2}}} \tag{16}$$

When the above relation is substituted into (15), the latter yields the Havelock formula (2).

3. The Image System for an Ellipsoid

Let the equation of the ellipsoid be given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \; ; \; a>b>c \tag{17}$$

The ellipsoidal coordinates (ρ, μ, ν) are defined by the solution of the cubic equation in λ,

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2-h^2} + \frac{z^2}{\lambda^2-k^2} = 1 \tag{18}$$

for fixed values of (x, y, z) where

$$k^2 = a^2-c^2 \; ; \quad h^2 = a^2-b^2 \tag{19}$$

The three roots of (18) are chosen so that

$$\infty>\rho^2\geq k^2 \; ; \quad k^2\geq\mu^2\geq h^2 \; ; \quad h^2\geq\nu^2\geq 0 \tag{20}$$

The three surfaces, ρ = const (ellipsoids), μ = const (hyperboloids of one sheet) and ν = const (hyperboloids of two sheets) then form a triply-orthogonal coordinate system in space.

The transformation between the Cartesian and the ellipsoidal coordinates is given by (Hobson p. 455)

$$x^2 = \frac{\rho^2\mu^2\nu^2}{h^2k^2} \tag{21}$$

$$y^2 = \frac{(\rho^2-h^2)(\mu^2-h^2)(h^2-\nu^2)}{h^2(k^2-h^2)} \tag{22}$$

and

$$z^2 = \frac{(\rho^2-k^2)(k^2-\mu^2)(k^2-\nu^2)}{k^2(k^2-h^2)} \tag{23}$$

An ellipsoidal harmonic which is regular at infinity is defined as

$$H_n^m(\rho,\mu,\nu) = F_n^m(\rho)\ E_n^m(\mu)\ E_n^m(\nu) \tag{24}$$

where n and m are positive integers such that $m \leq 2n + 1$. Here E_n^m denotes the Lamé function of the first kind which is regular at the origin, and F_n^m is the Lamé function of the second kind which is regular at infinity. The Lamé function of the second kind $F_n^m(\rho)$ is defined in terms of $E_n^m(\rho)$ (Hobson p. 472),

$$F_n^m(\rho) = (2n+1)\ E_n^m(\rho) \int_\rho^\infty \frac{d\rho}{[E_n^m(\rho)]^2\sqrt{(\rho^2-h^2)(\rho^2-k^2)}} \tag{25}$$

Following Hobson (p. 460), there exist four different classes of Lamé functions given by $P(\rho)$, $\sqrt{\rho^2-h^2}\ P(\rho)$, $\sqrt{\rho^2-k^2}\ P(\rho)$ and $\sqrt{(\rho^2-k^2)(\rho^2-h^2)}\ P(\rho)$, where $P(\rho)$ denotes a polynomial in ρ. Using Hobson's notation, the four classes of the Lamé functions will be denoted by $K(\rho)$, $L(\rho)$, $M(\rho)$ and $N(\rho)$ respectively. The normal functions $K(\rho)\ K(\mu)\ K(\nu)$ and $L(\rho)\ L(\mu)\ L(\nu)$ then yield interior ellipsoidal harmonics which are even with respect to z, while

Proof of Theorem 1.

Let us assume that the image system of an even exterior ellipsoidal harmonic consists of multipoles of the order $\frac{\partial^{p+q+2r}}{\partial x^p \partial y^q \partial z^{2r}}$ distributed over the fundamental ellipsoid where p, q and r are positive integers. The above differential operator operates on 1/R, where R denotes the distance between the point $(\xi, \eta, 0)$ on the fundamental ellipsoid and a field point (x, y, z). Since 1/R is an harmonic function, we have

$$\frac{\partial^{p+q+2r}}{\partial x^p \partial y^q \partial z^{2r}} \left(\frac{1}{R}\right) = (-1)^r \frac{\partial^{p+q}}{\partial x^p \partial y^q} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^r \left(\frac{1}{R}\right)$$

$$= (-1)^{p+q+r} \frac{\partial^{p+q}}{\partial \xi^p \partial \eta^q} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^r \left(\frac{1}{R}\right) \qquad (31)$$

The integration of (27) is carried out over both ξ and η hence the multipoles given by (31) may be reduced by an integration by parts to simple source-sink distribution over the fundamental ellipsoid and a line distribution of multipoles over the contour of the ellipse given by (26).

The potential of a line multipole distribution is singular at points of the distribution . On the other hand, the Stieltjes theorem implies that (25) is a convergent integral for $\rho=k$ for all points on the fundamental ellipsoid. Hence we exclude the possibility of a contour distribution of multipoles on the ellipse (26).

In order to determine the source strength, use will be made of the Gauss flux theorem which, since we are dealing with a plane distribution, yields the following relation for the source distribution:

$$s(\mu', \nu') = \frac{1}{2\pi} \lim_{z\to 0}\left[\frac{\partial}{\partial z} H_n^m(x,y,z)\right] = \frac{1}{2\pi} \lim_{\rho\to k}\left[\frac{1}{h_\rho} \frac{\partial}{\partial \rho} H_n^m(\rho,\mu,\nu)\right] \qquad (32)$$

where h_ρ is the linearizing factor in the ρ direction given by

$$h_\rho^2 = \frac{(\rho^2-\mu^2)(\rho^2-\nu^2)}{(\rho^2-h^2)(\rho^2-k^2)} \tag{33}$$

Substituting equations (24), (25) and (33) into (32) yields

$$s(\mu',\nu') = -\frac{(2n+1)\ E_n^m(\mu')\ E_n^m(\nu')}{2\pi\ E_n^m(k)\ \sqrt{(k^2-\mu'^2)(k^2-\nu'^2)}} \tag{34}$$

For the case where $\rho=k$ equations (21) and (22) may be solved explicitly for $\mu'(x,y)$ and $\nu'(x,y)$. The resulting expressions are given by (29) and (30) respectively. Similarly, the denominator of (34) is given in Cartesian representation by

$$(k^2-\mu'^2)(k^2-\nu'^2) = k^2(k^2-h^2)(1 - \frac{x^2}{k^2} - \frac{y^2}{k^2-h^2}) \tag{35}$$

Substituting (35) into (34) yields (28), and the proof of Theorem 1 is completed.

Theorem 2

An odd (in z) exterior ellipsoidal harmonic may be generated by a normal doublet distribution in the z-direction over the fundamental ellipsoid,

$$F_n^m(\rho)\ E_n^m(\mu)\ E_n^m(\nu) = -\frac{\partial}{\partial z}\int_{S_o} \frac{d(\xi,\eta)d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}} \tag{36}$$

where

$$d(x,y) = \frac{(2n+1)\ E_n^m(\mu')\ E_n^m(\nu')}{2\pi k\ \overline{E}_n^m(k)\ \sqrt{k^2-h^2}} \tag{37}$$

and

$$\overline{E}_n^m(k) = \lim_{\rho\to k} \frac{E_n^m(\rho)}{\sqrt{\rho^2-k^2}} \tag{38}$$

Proof of Theorem 2.

By the same arguments as were used to prove Theorem 1, one can show that an odd exterior ellipsoidal harmonic may be generated by a distribution of doublets oriented in the z-direction over the fundamental ellipsoid .

An odd exterior ellipsoidal harmonic may be written as

$$H_n^m(\rho,\mu,\nu) = (2n+1)\,\bar{E}_n^m(\rho)\,E_n^m(\mu)\,E_n^m(\nu)$$

$$\left\{\sqrt{\rho^2-k^2}\int_\rho^\infty \frac{d\rho}{[\bar{E}_n^m(\rho)]^2\,(\rho^2-k^2)^{3/2}\,(\rho^2-h^2)^{1/2}}\right\} \tag{39}$$

The discontinuity in the potential across a normal doublet distribution then implies that the normal doublet distribution over the fundamental ellipsoid, which is the image system of an odd exterior ellipsoidal harmonic, is given by

$$d(\mu',\nu') = \frac{1}{2\pi}\lim_{\rho\to k} H_n^m(\rho,\mu,\nu) \tag{40}$$

By the Stieltjes theorem, the only singularity of the integrand of (39) as ρ approaches k is $(\rho^2-k^2)^{-3/2}$. Integration by parts of (39) yields

$$H_n^m(\rho,\mu,\nu) = (2n+1)\,\bar{E}_n^m(\rho)\,E_n^m(\mu)\,E_n^m(\nu)$$

$$\left\{\left\{\frac{1}{\rho[\bar{E}_n^m(\rho)]^2\,\sqrt{\rho^2-h^2}} + \sqrt{\rho^2-k^2}\int_\rho^\infty \frac{d}{d\rho}\left\{\frac{1}{\rho[E_n^m(\rho)]^2\,\sqrt{\rho^2-h^2}}\right\}\frac{d\rho}{\sqrt{\rho^2-k^2}}\right\}\right\} \tag{41}$$

Applying the limit $\rho\to k$ to $H_n^m(\rho,\mu,\nu)$, equations (40) and (41) yield the doublet distribution given by (37), and the proof of Theorem 2 is completed.

4. Examples

Let $\phi_o(\rho,\mu,\nu)$ be a given potential function free of singularities in the region $\rho \leq a$. Introducing an ellipsoid disturbs the flow and the velocity potential in the region $\rho \geq a$ is then given by

$$\Phi_E(\rho,\mu,\nu) = \phi_o(\rho,\mu,\nu) + \phi_e(\rho,\mu,\nu) \tag{42}$$

where $\phi_e(\rho,\mu,\nu)$ is the velocity potential due to the image system within the ellipsoid.

Since both ϕ_o and ϕ_e are harmonic functions, they may be expanded in terms of ellipsoidal harmonics in the form

$$\phi_o(\rho,\mu,\nu) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} A_n^m E_n^m(\rho)\, E_n^m(\mu)\, E_n^m(\nu) \qquad \rho \leq a \tag{43}$$

and

$$\phi_e(\rho,\mu,\nu) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} B_n^m\, F_n^m(\rho)\, E_n^m(\mu)\, E_n^m(\nu) \qquad \rho \geq a \tag{44}$$

where A_n^m and B_n^m are constants to be determined. For the Neumann problem, the normal derivative of Φ_E must vanish on the ellipsoid $\rho = a$. Since the normal derivative on the ellipsoid is that with respect to ρ, we then obtain from (42), (43) and (44) the following expression in the region exterior to the ellipsoid $\rho = a$:

$$\phi_e(\rho,\mu,\nu) = -\sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{A_n^m}{C_n^m}\, F_n^m(\rho)\, E_n^m(\mu)\, E_n^m(\nu) \qquad \rho \geq a \tag{45}$$

where

$$C_n^m = \frac{\dot{F}_n^m(a)}{\dot{E}_n^m(a)} \tag{46}$$

Here the dot denotes differentiation with respect to the argument.

Equation (45) is an expression for the disturbance velocity potential when an ellipsoid is introduced in a potential flow field given by (43).

Example 1: Pure Translation.

Let $\phi_0(x,y,z)$ be given by

$$\phi_0(x,y,z) = Ux + Vy + Wz \tag{47}$$

which represents a uniform stream with velocities U, V and W in the x,y and z directions respectively.

As already mentioned there exists three Lamé functions of the first order (n=1). These three functions are of the class K, L and M defined earlier and will be denoted herein as K_1, L_1 and M_1. In terms of these functions, equation (47), written in ellipsoidal coordinates, is of the form

$$\phi_0(\rho,\mu,\nu) = \frac{UK_1(\rho)\ K_1(\mu)\ K_1(\nu)}{hk} + \frac{VL_1(\rho)\ L_1(\mu)\ L_1(\nu)}{h\sqrt{k^2-h^2}} + \frac{WM_1(\rho)\ M_1(\mu)\ M_1(\nu)}{k\sqrt{k^2-h^2}} \tag{48}$$

Consider now the case where a solid ellipsoid is introduced into the stream. The disturbance velocity potential at points in the exterior region may be considered to be given by a certain singularity distribution over the fundamental ellipsoid.

Equations (34), (37) and (45) then imply that the image system consists of a source distribution of strength

$$s(\mu',\nu') = \frac{3}{2\pi h\sqrt{(k^2-\mu'^2)(k^2-\nu'^2)}}\left[\frac{U\mu'\nu'}{k^2C_1(K)} + \frac{V\sqrt{(\mu'^2-h^2)(h^2-\nu'^2)}}{(k^2-h^2)\ C_1(L)}\right] \tag{49}$$

and a doublet distribution in the z-direction of strength

$$d(\mu',\nu') = \frac{3W}{2\pi k^2(k^2-h^2)}\ \frac{\sqrt{(k^2-\mu'^2)(k^2-\nu'^2)}}{C_1(M)} \tag{50}$$

Here $C_1(K)$, $C_1(L)$ and $C_1(M)$ denote the three values of C_1^m defined in (46) corresponding to the three possible forms of E_1^m, i.e., K_1, L_1 and M_1.

It is more convenient to use Cartesian representation. The source distribution (49) is then given by

$$s(x,y) = \frac{3}{2\pi k\sqrt{k^2-h^2}}\left(1-\frac{x^2}{k^2}-\frac{y^2}{k^2-h^2}\right)^{-\frac{1}{2}}\left[\frac{Ux}{k^2\, C_1(K)}+\frac{Vy}{(k^2-h^2)\, C_1(L)}\right] \tag{51}$$

and the doublet distribution (50) is

$$d(x,y) = \frac{3W}{2\pi k\, C_1(M)\sqrt{k^2-h^2}}\left(1-\frac{x^2}{k^2}-\frac{y^2}{k^2-h^2}\right)^{\frac{1}{2}} \tag{52}$$

The above expressions are a reduced form of Havelock's [13] results for the image system of an ellipsoid in a uniform stream.

The coefficients $C_1(K)$, $C_1(L)$ and $C_1(M)$ are given in the Appendix in terms of tabulated elliptic integrals.

Example 2: Pure Rotation

Let us assume that the ellipsoid is rotating about its principal axes in an infinite inviscid fluid otherwise at rest with angular velocity,

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z) \tag{53}$$

The boundary condition to be satisfied on the ellipsoid is

$$\frac{\partial \phi_o}{\partial n} = (\vec{\omega} \times \vec{r}) \cdot \hat{n} \tag{54}$$

where $\vec{r}$ is a unit vector from the origin to a point (x,y,z) on the ellipsoid and $\hat{n}$ denotes a unit normal vector to the surface of the ellipsoid. Following Lamb (p. 147) the interior velocity potential is

$$\phi_o(x,y,z) = \frac{b^2-c^2}{b^2+c^2}\,\omega_x yz + \frac{c^2-a^2}{c^2+a^2}\,\omega_y zx + \frac{a^2-b^2}{a^2+b^2}\,\omega_z xy \tag{55}$$

When n=2 there exist five Lamé functions, two of class K, and one function each of classes L, M and N (Hobson p. 465). These five functions will be denoted as K_2^1, K_2^2, L_2, M_2 and N_2 respectively. The interior velocity potential expressed in terms of ellipsoidal coordinates is then given by

$$\phi_o(\rho,\mu,\nu) = \frac{\omega_x(b^2-c^2)\, N_2(\rho)\, N_2(\mu)\, N_2(\nu)}{hk(b^2+c^2)\,(k^2-h^2)} + \frac{\omega_y(c^2-a^2)\, M_2(\rho)\, M_2(\mu)\, M_2(\nu)}{hk^2(c^2+a^2)\,\sqrt{k^2-h^2}} + \frac{\omega_z(a^2-b^2)\, L_2(\rho)\, L_2(\mu)\, L_2(\nu)}{h^2k(a^2+b^2)\,\sqrt{k^2-h^2}} \qquad (56)$$

Again equations (45) and (46) yield the expansion in ellipsoidal harmonics of the exterior velocity potential in terms of the ellipsoidal harmonic expansion of the interior potential. The image system of the exterior potential is given by Theorems 1 and 2. This system consists of source distribution

$$s(\mu',\nu') = -\frac{5\omega_z(a^2-b^2)\mu'\nu'\sqrt{(\mu'^2-h^2)(h^2-\nu'^2)}}{2\pi k^2h^2(a^2+b^2)(k^2-h^2)\; C_2(L)\; \sqrt{(k^2-\mu'^2)(k^2-\nu'^2)}} \qquad (57)$$

and normal doublet distribution

$$d(\mu',\nu') = -\frac{5\sqrt{(k^2-\mu'^2)(k^2-\nu'^2)}}{2\pi hk^2(k^2-h^2)} \left[\frac{\omega_y(c^2-a^2)\mu'\nu'}{k^2(c^2+a^2)\; C_2(M)} + \frac{\omega_x(b^2-c^2)\;\sqrt{(\mu'^2-h^2)(h^2-\nu'^2)}}{(b^2+c^2)(k^2-h^2)\; C_2(N)} \right] \qquad (58)$$

Here the coefficients $C_2(L)$, $C_2(M)$ and $C_2(N)$ are the three values of $C_2^{\,m}$ defined in (46), which correspond to the replacement of $E_2^{\,m}$ by L_2, M_2 and N_2 respectively.

The equivalent expressions in Cartesian representation are

$$s(x,y) = -\frac{5\omega_z(a^2-b^2)xy}{2\pi k^3(a^2+b^2)(k^2-h^2)^{3/2}\; C_2(L)} \left(1 - \frac{x^2}{k^2} - \frac{y^2}{k^2-h^2}\right)^{-\frac{1}{2}} \qquad (59)$$

and

$$d(x,y) = -\frac{5}{2\pi k\sqrt{k^2-h^2}}\left(1-\frac{x^2}{k^2}-\frac{y^2}{k^2-h^2}\right)^{\frac{1}{2}} \tag{60}$$

$$\left[\frac{\omega_y(c^2-a^2)x}{k^2(c^2+a^2)\ C_2(M)}+\frac{\omega_x(b^2-c^2)y}{(k^2-h^2)(b^2+c^2)\ C_2(N)}\right]$$

Expressions for $C_2(L)$, $C_2(M)$ and $C_2(N)$ in terms of elliptic integrals are given in the Appendix.

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Appendix

Expressions for the coefficients C_n^m in terms of elliptic integrals.

Since $K_1(\rho) = \rho$, equation (25) yields

$$F_1(\rho) = 3\rho \int_\rho^\infty \frac{d\rho}{\rho^2\sqrt{(\rho^2-k^2)(\rho^2-h^2)}} \tag{1}$$

Hence (46) implies that

$$C_1(K) = 3\int_a^\infty \frac{d\rho}{\rho^2\sqrt{(\rho^2-k^2)(\rho^2-h^2)}} - \frac{3}{abc} \tag{2}$$

The above equation may be expressed in terms of tabulated elliptic integrals [14] as

$$C_1(K) = \frac{3}{kh^2}\left[\mathbb{F}(\phi,t) - \mathbb{E}(\phi,t)\right] - \frac{3}{abc} \tag{3}$$

where $\mathbb{F}$ and $\mathbb{E}$ denote the Legendre incomplete elliptic integrals of the first and second kind respectively, and

$$t = \frac{h}{k}\ ; \quad \phi = \sin^{-1}\frac{k}{a}\ . \tag{4}$$

Similarly, the rest of the coefficients may be expressed in terms of the tabulated incomplete elliptic integrals [14]. Here we will give only the final results:

$$C_1(L) = \frac{3k}{h^2(k^2-h^2)}\left[\mathbb{F}(\phi,t) - \left(1-\frac{h^2}{k^2}\right)\mathbb{E}(\phi,t) - \frac{h^2}{k^2}\frac{\sin\phi\cos\phi}{\sqrt{1-\frac{h^2}{k^2}\sin^2\phi}}\right] - \frac{3}{abc} \tag{5}$$

$$C_1(M) = \frac{3}{k(k^2-h^2)}\left[\mathrm{tg}\phi\sqrt{1-\frac{h^2}{k^2}\sin^2\phi} - \mathbb{E}(\phi,t)\right] - \frac{3}{abc} \tag{6}$$

Appendix (continued)

$$C_2(L) = \frac{5}{kh^2(k^2-h^2)} [\mathbb{F}(\phi,t) - \frac{\sin\phi \cos\phi}{\sqrt{1 - \frac{h^2}{k^2}\sin^2\phi}}] - \frac{5}{abc(a^2+b^2)} \qquad (7)$$

$$C_2(M) = \frac{5}{k^3(k^2-h^2)} [tg\phi \sqrt{1 - \frac{h^2}{k^2}\sin^2\phi}$$

$$- (\frac{k^2}{h^2} - 1)\ \mathbb{F}(\phi,t) + (\frac{k^2}{h^2} - 2)\ \mathbb{E}(\phi,t)] - \frac{5}{abc(a^2+c^2)} \qquad (8)$$

$$C_2(N) = \frac{5}{k(k^2-h^2)^2} [\ \frac{tg\phi(1 - \frac{h^2}{k^2}\sin^2\phi) + \sin\phi \cos\phi}{\sqrt{1 - \frac{h^2}{k^2}\sin^2\phi}}$$

$$+ (\frac{k^2}{h^2} - 2)\ \mathbb{E}(\phi,t) - \frac{k^2}{h^2}\mathbb{F}(\phi,t)] - \frac{5}{abc(b^2+c^2)} \qquad (9)$$