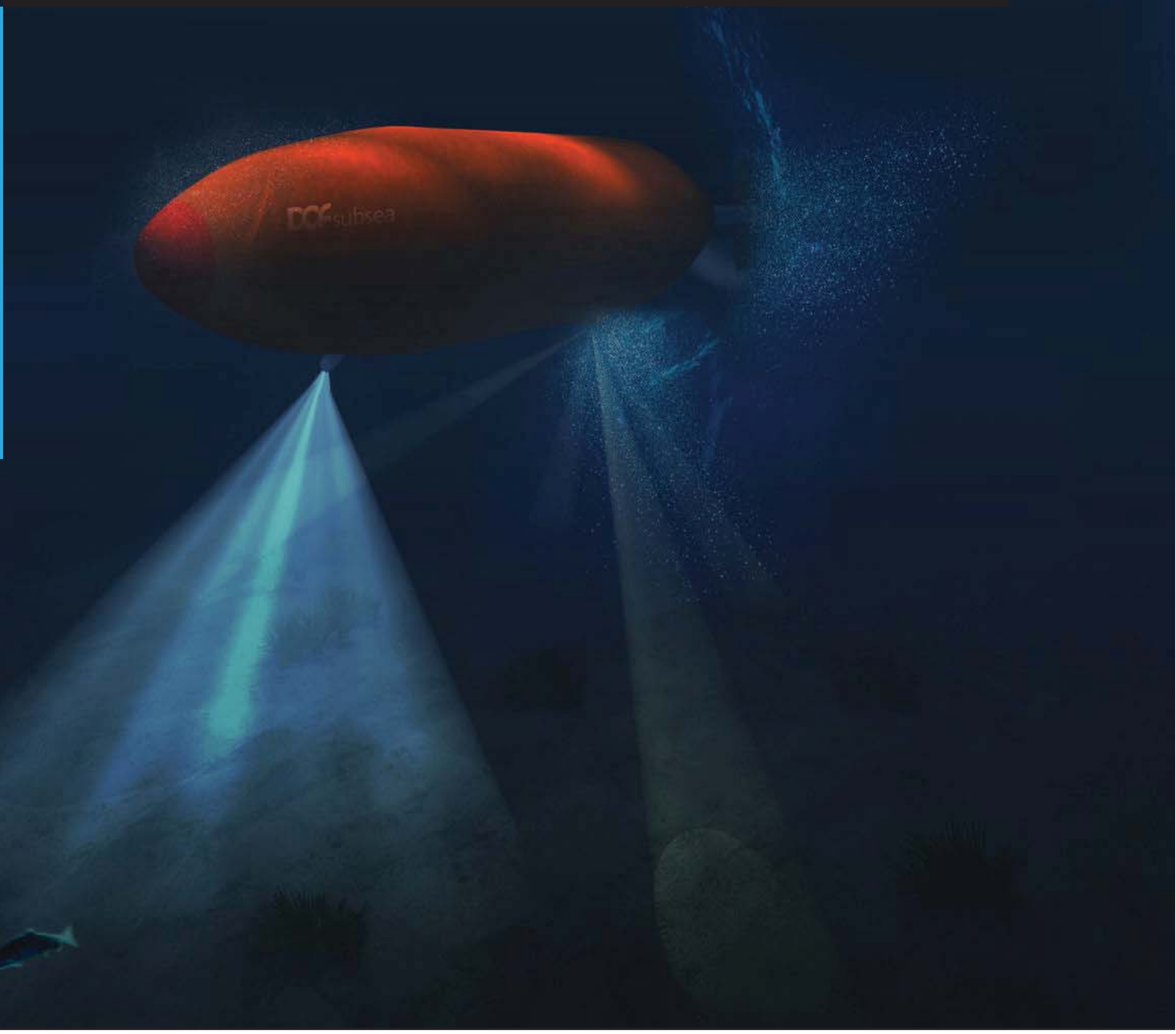
LQG Coordination Control

Optimal Control Theory for Coordinated Linear Systems with Application to Autonomous Underwater Vehicles

Nicola Pambakian

Master of Science Thesis



Delft University of Technology

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For the degree of Master of Science in Systems and Control at Delft University of Technology

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Nicola Pambakian

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Abstract

The topic of this thesis is the solution of the 'LQG coordination control' problem, which is the Linear-Quadratic-Gaussian (LQG) control problem for the class of Coordinated Linear Systems (CLSs). Decentralized and Distributed Control have attracted the interest of many researchers in the last decades. During the past years, numerous methodologies and approaches have been introduced. This thesis focuses on Linear Coordination Control, a theoretical framework developed to control CLSs. A CLS is a hierarchical system composed of a number of subsystems and a 'coordinator'. The subsystems are independent of each other, but they depend on the coordinator, while the coordinator is independant of the subsystems. Thus, a 'nested information pattern' is available to the control law. The following issues are investigated: (1) the applicability of the separation principle; (2) the synthesis of the optimal state-feedback gain and (3) the synthesis of the optimal observer gain (or Kalman gain). The solution to these three issues, and therefore to the LQG coordination control problem, are discussed for different categories of problem formulations: general problems, decomposable problems and virtual coordination problems. No results were found for general problems. For decomposable problems, we show that the separation property holds if certain parameters are fixed, and we present a control synthesis procedure that involves numerical optimization of cost functions; for these cost functions, the convexity in the optimization parameters is conjectured. For virtual coordination problems, we derive optimal analytical results.

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Chapter 1

Introduction

The goal of the present work is to provide a solution to the LQG (Linear Quadratic Gaussian) optimal control problem for a particular class of linear, time-invariant systems: the coordinated linear systems (CLSs). This class of systems is used to model distributed stochastic systems which are independent of each other, with a coordinator subsystem having influence on all of them. The influence of the coordinator on the subsystems describes a particular set of data that is available to the control system, or *information pattern*.

Classical LQG control theory assumes a *classical pattern*, which roughly consists of a non-delayed sharing for all the present and past inputs and outputs within the whole system. In this master thesis we consider a non-classical, *nested information pattern*. Limiting our focus to this kind of patterns introduces an intrinsic difficulty for the control synthesis, as the classical LQG optimal control theory does not provide for a control synthesis for every information pattern, but only for the classical one. By the nested information structure, the coordinator of the CLS shares its information with the other subsystems subsystems, which are independent of each other, without having access to their iformation (its present and past inputs and outputs). In other words, the information of each subsystem is only locally available, while that of the coordinator is shared with every other subsystem.

In order to introduce the reader to our problem, in Section 1-1 we motivate this work by showing the possible benefits it would provide, for example, to the problems of vehicles coordination. A generic problem description is given in Section 1-2, while the most crucial objectives to be reached are introduced in Section 1-3. The general approach adopted to solve the LQG coordination control problem is summarized in Section 1-3-2. At last, an outline of the thesis is found at the end of the chapter, in Section 1-4.

1-1 Motivation

An improvement in the theory of linear coordination control would be of benefit for problems requiring dynamical coordination of agents. An important field of application of this class of problems is given by autonomous vehicles coordination problems. To these kind of problems, much attention has been given in the last decades.

In practice, coordinated groups of autonomous vehicles can provide significant benefits to many applications. These include environmental sampling, mapping, surveillance, fire-detection, mine-sweeping, military use and sensor networks (see, for instance, [13, 17, 19, 31, 32] and the references therein). In these years, different approaches have been formulated to tackle vehicles coordination problems. Some of these make use of advanced control techniques such as model predictive control [29] and non-linear control [2]. Others use different approaches, such as auction bidding systems [52] or other optimization methods. A glance of the state-of-the-art of coordination of groups of vehicles is given in [37], while a more generic overview on coordination control and cooperative control can be found in [30].

Some issues still appear to be poorly treated in the literature framework.

1. The necessity of communication from the vehicles, hidden by the frequently assumed full or partial knowledge of their state by the rest of the system (see for example [3, 19]) often reflects into the need of a complex communication apparatus, and represents both an energetic and an economical cost.

2. Stochastic systems have been considered relatively little in the coordination control literature. Stochastic disturbances (e.g. air and water turbulence, measurement noises, or pavement irregularities) may play a role in the performance of the control laws, as it could happen, for instance, in the leader-following problems.

3. Computationally efficient approaches might be required for some applications. More and more the control systems tend to be decentralized and embedded in vehicles. The computational power of these systems cannot always afford complex operations, and limited time is usually available for computations.

We aim to offer a solution for these issues by extending the boundaries of the LQG optimal control theory to the class of coordinated linear systems (CLSs). A CLS is in fact a useful abstraction to model coordinated vehicles systems. It does not only give the possibility to represent leader-followers dynamical systems, but also allows the representation of virtually coordinated agents (i.e. coordinated by virtually-built operators, like computer programs).

An important aspect of CLSs is that of communication. There, subsystems do not communicate any data to the coordinator nor to their peers. As communication can represent an expensive cost to the vehicles, this translates to an extremely important advantage. Avoidance of communication is particularly welcome for Autonomous Underwater Vehicles (AUVs) coordination problems, where the communication, carried out by sonars, induces a huge energetic cost.

1-2 Problem Description

The classical version of the LQG optimal control theory provides for an instrument to synthesize optimal control laws for linear systems with a classical information pattern[22]. As

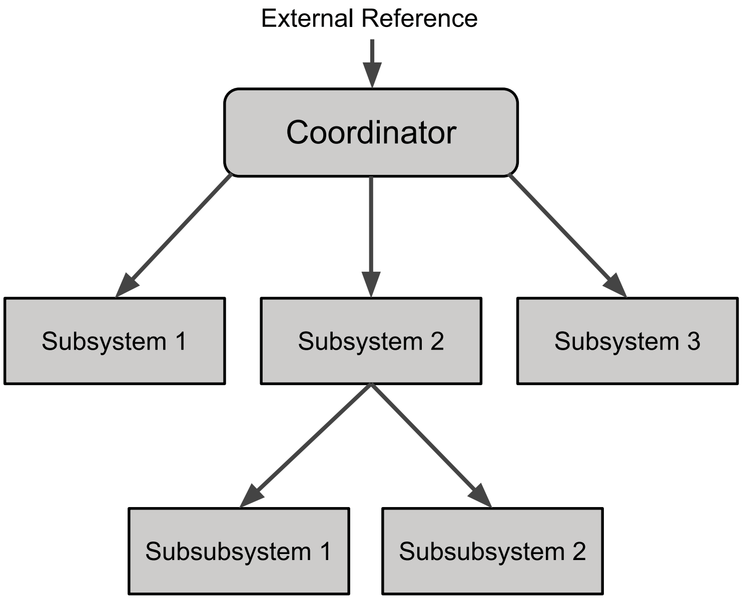

Figure 1-1: Example of coordination control structure.

mentioned before, in this thesis we consider a nested information pattern instead. The nested information structure of the system is illustrated in Figure 1-1 by means of an example.

In the figure, a coordinator influences three subsystems which are independent of each other. The coordinator can be seen as the leading part of the system. It does not receive or measure the states of the subsystems, but it influences them. Any of these subsystems can be a coordinated system itself, allowing in this way a nested structure. Since no feedback is applied from the subsystems to the coordinator, there is no communication from the subsystems. In general, the absence of communication from the subsystems inevitably introduces a loss in performance for the global control system but might represent a good approach for problems where avoidance of communication is strongly appreciated.

It is our goal to build a theoretical framework which fully reproduces the advantages of the classical LQG optimal control theory for this particular class of problems. We want, therefore, to offer a procedure to find the optimal state-feedback gain separately from the optimal observer gain by applying the *separation principle*.

1-3 Ends and Means

Final goals of LQG coordination control and the approach taken to solve the problem are explained here.

1-3-1 Objectives

In the process of extending the LQG optimal control theory to the class of coordinated linear systems, the following objectives will be pursued.

1. Establish that the separation property holds. If it holds then the problem of control with partial observations separates into two problems: one for the optimal state estimator and one for the state-feedback control law based on complete observations [49].

2. Develop a control synthesis procedure. Derive the procedure by which the optimal control law and the state estimator can be determined.

3. Show the validity of the developed procedures through an implementation in a case study. For this purpose, the problem of coordination control of AUVs will be taken into account, in Chapter 4.

These three objectives will be formulated as problems in Chapter 2, and solved in Chapter 3.

1-3-2 Approach

We choose to represent our systems in the discrete-time domain. This decision lies on the relative ease of implementation of discrete-time controllers in relation to analog controllers. We assume the reader is familiar with the concepts of *stable* and *asymptotically stable matrices*, and the concepts of *controllable* and *stabilizabl*e *pairs*, of which a definition can be found, e.g. in [48].

We will divide the problem formulations of LQG coordination control in three categories: the *general problems*, the *decomposable problems* and the *virtual coordination problems*. For each of these categories, the following approach is taken.

1. At first, we determine the main passages required for solving the classical LQG control problem.

2. We check whether, and under which conditions, these passages are valid for the optimal control synthesis for CLSs. If they cannot be applied to our problem, we look for a decomposition of the problem itself that allows for it.

3. An analytical solution is then sought. If this is not found, a numerical optimization is indicated instead.

As we will see, this approach produced good results for decomposable problems and for virtual coordination problems.

1-4 Outline of the Thesis

The rest of the thesis is organized as follows. In Chapter 2, the problem of LQG coordination control is formulated after defining the main actors of the thesis, and after overviewing some existing approaches in the literature. In Chapter 3 we treat the LQG coordination control problem by dividing it into the three categories introduced above, and the respective achieved results are presented. After that, in Chapter 4 the developed theory is implemented to a case study: coordination control of a group of autonomous underwater vehicles (AUVs). Chapter 5 collects the conclusions to the work, the main contributions given by the thesis, its strengths and weaknesses, and some suggestions for future work.

Chapter 2

Problem Formulation

This chapter is dedicated to the formulation of the thesis problem in a mathematical way. In order to introduce the problem of LQG coordination control, in Section 2-1 we define the class of *coordinated linear systems* (CLSs), and show the properties of two classes of matrices which will play an important role in in the coming pages: the *swallow* and the *arrow* matrices. To introduce the results already available about the argument, in Section 2-2 we briefly explain the classical LQG optimal control theory, then, in Section 2-3, we make an overview of correlated works in the literature. At last, we formulate the problem of *LQG coordination control* in Section 2-4. General conclusions about the chapter are found in Section 2-5.

2-1 Introduction to Coordinated Linear Systems

A *coordinated linear system* (CLS) is a special class of hierarchical systems in which every subsystem can be influenced by one coordinator. A CLS has a nested structure which is reflected by its information pattern. In this section, we define CLSs and explain some of their important properties.

2-1-1 Definition

Let us indicate with N_s the total number of subsystems (excluding the coordinator) of a CLS. To simplify notation, we will always represent only two subsystems in formulas, even if $N_s > 2$. The class of CLSs, on which our focus is going to be set for the whole thesis, is defined as follows.

Definition 2-1.1. A *Coordinated Linear System (CLS)* with N_s subsystems is a linear state-space system of the form

$$\left\{\begin{array}{l} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_c(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix} + \\ \qquad + \begin{bmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{bmatrix} \begin{bmatrix} v_{x,1}(t) \\ v_{x,2}(t) \\ v_{x,c}(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t+1) \\ y_2(t+1) \\ y_c(t+1) \end{bmatrix} = \begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} N_{11} & 0 & N_{1c} \\ 0 & N_{22} & N_{2c} \\ 0 & 0 & N_{cc} \end{bmatrix} \begin{bmatrix} v_{y,1}(t) \\ v_{y,2}(t) \\ v_{y,c}(t) \end{bmatrix}; \end{array}\right. \tag{2-1}$$

$$x_1(t_0) = x_{1,0};\; x_2(t_0) = x_{2,0};\; x_c(t_0) = x_{c,0};$$

where $v_{x,i}, v_{y,i}$, for $i = 1, \ldots, N_s, c$ are Gaussian white noises, and A, B, C, M, N are matrices of appropriate sizes.

General references about CLSs and their properties are found in [26, 27, 25, 40]. For ease of notation, we will often indicate states, inputs and outputs as

$$\begin{aligned} x(t) &= \left[x_1^T(t), x_2^T(t), x_c^T(t)\right]^T; \\ u(t) &= \left[u_1^T(t), u_2^T(t), u_c^T(t)\right]^T; \\ y(t) &= \left[y_1^T(t), y_2^T(t), y_c^T(t)\right]^T, \end{aligned}$$

recalling that wherever we will display a subsystem of two subsystems only, we intend its general version with N_s subsystems instead.

2-1-2 Swallow and Arrow Matrices

In CLSs, it is very common to find matrices of, as we define it, the *swallow* form. A swallow matrix (named after the shape it describes by its non-zero elements, that recalls a flying bird) reflects the information pattern of CLSs. We hereby define this class of matrices and study its properties.

Definition 2-1.2. A matrix is said to be *swallow* if it is structured as

$$\begin{bmatrix} S_{11} & 0 & \cdots & 0 & S_{1c} \\ 0 & S_{22} & \cdots & 0 & S_{2c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S_{N_sN_s} & S_{N_sc} \\ 0 & 0 & \cdots & 0 & S_{cc} \end{bmatrix},$$

where $S_{11} \in \mathbb{R}^{l_1 \times c_1}, \ldots, S_{N_sN_s} \in \mathbb{R}^{l_{N_s} \times c_{N_s}}$, $S_{cc} \in \mathbb{R}^{l_c \times c_c}$, and $S_{1c} \in \mathbb{R}^{l_1 \times c_c}, \ldots, S_{N_sc} \in \mathbb{R}^{l_{N_s} \times c_c}$.

Some conservation properties hold for swallow matrices.

1. If A and B are two swallow matrices with compatible dimensions, then $A + B$ is also swallow.
2. If A and B are two swallow matrices with compatible dimensions, then $A \cdot B$ is also swallow.
3. If an invertible matrix A is swallow, then its inverse $A^{-1}$is also swallow.

Following from these properties, important conclusions can be inferred regarding CLSs. For example, applying a control action $u(t) = Fx(t)$, F being swallow with compatible dimensions, the shape of the closed-loop state-space matrices $(A + BF)$ remains swallow.

Another form of matrix we will often mention in the thesis is the *arrow form.* This is defined below.

Definition 2-1.3. A matrix is said to be in the *arrow form* if it is structured as

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 & A_{1c} \\ 0 & A_{22} & \cdots & 0 & A_{2c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{N_sN_s} & A_{N_sc} \\ A_{c1} & A_{c2} & \cdots & A_{cN_s} & A_{cc} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{l_1 \times c_1}$,..., $A_{N_sN_s} \in \mathbb{R}^{l_{N_s} \times c_{N_s}}$, $A_{cc} \in \mathbb{R}^{l_c \times c_c}$, and $A_{1c} \in \mathbb{R}^{l_1 \times c_c}$,..., $A_{N_sc} \in \mathbb{R}^{l_{N_s} \times c_c}$,$A_{c1} \in \mathbb{R}^{l_c \times c_1}$,..., $A_{N_sc} \in \mathbb{R}^{l_c \times c_{N_s}}$.

Another conservation property involving the product of arrow and swallow matrices is given by the following lemma.

Lemma 2-1.4. *(Arrow-Swallow Multiplication) Let S be a swallow matrix and A be a matrix in the arrow form. The products AS, $S^T A$ and $S^T AS$ result into a matrix of the arrow form.*

Proof. This follows from term-by-term multiplication. □

2-2 LQG Optimal Control Theory

The most popular, and better explored area of optimal stochastic control theory is that of LQG (Linear-Quadratic-Gaussian) control. This is an optimal control synthesis technique applicable to linear systems affected by Gaussian white noise of known variance. A general reference on the topic can be found in [22], but thousands of pages have been written about LQG control. For interested readers, a huge collection of references divided by argument can be found in the bibliography [33].

In the classical LQG control theory, the control synthesis procedure is divided in two parts. An *optimal state-feedback gain F* is computed in order to minimize the variance of the closed-loop system's state, having the possibility to emphasize the importance of states and control inputs by selecting proper weighing matrices Q, R and S. Secondly, solving a dual problem, the optimal gain of the observer is computed. This gain, denoted by K, is often referred to, in literature, as the *Kalman gain*. The introduction of an observer in a system modifies its closed-loop dynamics. In general, this fact couples the computation of the two gains. However, if a control problem has the *separation property*, the optimal feedback gain F and the Kalman gain K can be computed separately, in a fully decoupled way [51]. This idea follows what it is called the *separation principle*.

In this section we summarize the separation principle and the procedures to obtain the two gains in the classical version of the LQG control theory. The following generic linear system is considered,

$$\begin{cases} x\left(t+1\right) = Ax\left(t\right) + Bu\left(t\right) + Mv_x\left(t\right) \\ y\left(t\right) = Cx\left(t\right) + Nv_y\left(t\right), \end{cases}$$

where $v_x(t)$ and $v_y(t)$ are jointly Gaussian noises with zero average and covariance matrix

$$V_{tot} = V_{tot}^T = \begin{bmatrix} V_x & V_{xy} \\ V_{xy}^T & V_y \end{bmatrix} \geq 0$$

with $V_y > 0$.

2-2-1 Separation Principle

Since the gain of the observer modifies the dynamics of the closed-loop system, the optimal values of F and K are, in general, coupled. In LQG optimal control, the *separation property* allows the syntheses of the optimal gains F and K to be performed separately.

The validity of this principle is strongly connected with the information pattern of the controller and the observer, with the presence of delays and with the structure of the controller [51]. The separation property holds for LTI systems with *classical information pattern* [11], as it is the case of LQG optimal control. Thus, for the classical LQG control, the synthesis of the two gains can be separated.

2-2-2 LQG Optimal State-Feedback Gain Synthesis

To obtain the optimal feedback gain F, an appropriate cost function J to be minimized is defined. In this thesis, we focus on the type of *infinite-horizon cost function*, as we are interested in a static control action minimizing the steady-state behavior of the system. Let

$$J = \lim_{t\to\infty} \frac{1}{t} E\left[\sum_{s=0}^{t-1} \begin{pmatrix} x(s) \\ u(s) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(s) \\ u(s) \end{pmatrix}\right], \tag{2-2}$$

where

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$$

and $R > 0$. Let us assume a *classical information pattern* to be available for the controller. This means that the controller has access to the value of every present and past state of the whole system. Now, choosing $u(t) = Fx(t)$, J can be rewritten as follows [5],

$$J = trace\left[W\left(Q + SF + F^T S^T + F^T RF\right)\right], \tag{2-3}$$

where W is the steady-state variance of the closed-loop system, and it is given by the solution of the Lyapunov equation

$$W = (A + BF)\, W\, (A + BF)^T + MV_x M^T.$$

The gain F which minimizes this cost function is then given by

$$F = -\left(B^T PB + R\right)^{-1}\left(B^T PA + S^T\right),$$

where P is the solution of the Discrete-time Algebraic Riccati Equation (DARE) [1],

$$P = A^T PA + Q - \left(A^T PB + S\right)\left(B^T PB + R\right)^{-1}\left(B^T PA + S^T\right).$$

This equation is proved to have only one, positive-definite solution if and only if two conditions hold: (A, B) is a stabilizable pair and (A_F, L_F) is an observable pair, where $A_F = A - BR^{-1}S^T$ and $L_F L_F^T = Q - SR^{-1}S^T$ [49]. We denote the solution of this problem by the short-hand notation

$$F = LQG\,(A, B, Q, R, S)\,.$$

2-2-3 LQG Optimal Observer Gain Synthesis

LQG optimal control provides a procedure to synthesize an optimal gain to minimize the variance of the state-observation error. Denoting by $\hat{x}\,(t)$ the observed state, and by $e(t) = x(t) - \hat{x}(t)$ the observation error, we have

$$e(t + 1) = (A + KC)\, e(t) + Mv_x(t) + KNv_y(t). \tag{2-4}$$

As we wish to minimize the steady-state variance of this process, a cost function H is defined as

$$H = \lim_{t\to\infty} \frac{1}{t} E\left[\sum_{s=0}^{t-1} e\,(t)^T\, e\,(t)\right].$$

This can be rewritten as follows [5],

$$H = trace\,(W_e)\,, \tag{2-5}$$

where W_e is the solution of the Lyapunov equation

$$W_e = \quad (A + KC)\, W_e\, (A + KC)^T + \begin{bmatrix} M & KN \end{bmatrix} \begin{bmatrix} V_x & V_{xy} \\ V_{xy}^T & V_y \end{bmatrix} \begin{bmatrix} M^T \\ N^T K^T \end{bmatrix}.$$

As for the feedback gain, the information pattern is assumed to be classical. In this case, the theory provides the following optimal solution,

$$K = -\left(AXC^T + MV_{xy}N^T\right)\left(CXC^T + NV_yN^T\right)^{-1},$$

where X is the solution of the DARE [1],

$$X = AXA^T + MV_xM^T - \left(AXC^T + MV_{xy}N^T\right)\left(CXC^T + NV_yN^T\right)^{-1}\left(CXA^T + NV_{xy}^TM^T\right).$$

This Lyapunov equation has a unique, positive-definite solution, if and only if the following conditions hold: (A, C) is a detectable pair and (A_K, L_K) is an observable pair, being $A_K = A - MV_{xy}N^T\left(NV_yN^T\right)^{-1}C$ and $L_KL_K^T = MV_xM^T - MV_{xy}N^T\left(NV_yN^T\right)^{-1}NV_{xy}^TM^T$ [49]. Consistently with the notation introduced before, the solution to this problem can be written as

$$K = \left[LQG\left(A^T, C^T, MV_xM^T, NV_yN^T, MV_{xy}N^T\right)\right]^T.$$

Note that this problem is dual to that of the synthesis of the state-feedback gain F. A duality table is shown in 2-1.

Optimal Gain F	Optimal Gain K^T
A	A^T
B	C^T
Q	MV_xM^T
R	NV_yN^T
S	$MV_{xy}N^T$

Table 2-1: Duality table for the problems of synthesis of the optimal gains F and K in classical LQG control theory.

2-3 Literature Overview

We present here a brief literature overview related to the topic of this thesis.

2-3-1 Literature on Linear Coordination Control

Linear coordination control is a recently developed branch of the linear control theory. The approach was originally introduced by Ran and van Schuppen in [40], with the aim of building a framework to control CLSs. The theory was then extended by the two authors and Kempker: in [25] it is shown how to construct a coordinated linear system by a state transformation starting from a generic linear system, while in[26], controllability and observability properties of CLSs are considered. There, interesting patterns and totally new concepts of controllable and reachable states are introduced to describe the complexity of states and inputs interactions.

The theory of LQ-optimal control, very close in both ideal and practical development to that of LQG control, is extended by Kempker to the class of CLSs, in [28]. Results and approaches there described forms the basis of the work of this thesis. The results of Kempker's extension of LQ-optimal control for CLSs were used to implement a formation-flying control system for AUVs, in [27].

2-3-2 Alternative Approaches

Theoretical frameworks to build stabilizing and even optimal controllers for CLSs exist already. We depict a selection of these approaches.

One of the most generic frameworks that could be implemented is that of *Model Predictive Control* (MPC). Model predictive control is one of the most important existing approaches to coordination control. Its intrinsic power to guarantee stabilizing (in some cases optimal) control laws, already allowed for successful studies in the field of coordination control of autonomous vehicles [7, 16, 29]. When considering the problem of coordination control, the choice of MPC has several advantages. Among others, it can handle constraints, and it can be applied to non-linear stochastic systems. The main restriction of the technique lies in its computational power requirements. In fact, at every time-step, each agent and the coordinator are required to run an optimization problem whose difficulty varies depending on the type of problem considered. Another possible disadvantage of the approach could depend on the information needed by each controller, and we recall here that information may imply either data communication or sensing. To explore possible available solutions to the communication issue, a recent survey on architectures for distributed and hierarchical MPC can be found in [46].

LQG problems with a non-classical information patterns have already been studied by Gupta et al. [23]. There, an approach is developed to compute the LQG optimal feedback gain F lying in particular vector spaces, as it is in the case for CLSs. However, the approach requires the state-space matrix A to be block-diagonal, reducing in this way the generality of the results. Furthermore, for the optimal controller to be found, nT coupled matrix equation are to be solved at every time instant, where n is the number of independent subsystems and T is the horizon of the quadratic cost function taken into account. A simplified version of the algorithm is also offered in [23] to overcome this problem, but a theoretical loss of the technique's optimality occurs. Moreover, it requires the covariance matrices of disturbances, states and inputs to be computed in every equation, resulting then in a computationally cumbersome solution.

Another important approach to coordination control is provided by Rotkowitz and Lall. In [44], they provide a condition to determine the optimal control laws to be applied to a class of problems in which the information constraints given to the controller are *quadratically invariant* (see [43] for more information). This class of problems includes LQG coordination control. In particular, the approach was analyzed and explored for *partially ordered* sets by Shah and Parrilo [47]. These papers give an important alternative to the approach studied in this thesis. However, they do not provide an equipotent extension to the LQG classical theory. In fact, no results are shown about how to compute optimal observer and state-feedback gains separately, but only a less transparent, yet optimal, transfer function from output to input.

To conclude, a wide class of distributed and decentralized control approaches are available. Because of the vastness of this literature, we will not cover it in this section, and we refer the reader to a famous general survey as, for instance, [45].

2-4 LQG Coordination Control Problem Formulation

Application of the classical LQG control synthesis procedure to CLSs can be performed to compute the optimal gains F and K for CLSs. However, we recall that every subsystem only has access to its own states and those of the coordinator. It could be necessary, or at least preferable, to restrict the information pattern of the controller to the same data.

In this case, we look for swallow state-feedback and observer gain matrices. In fact, the swallow matrix reflects the nested information pattern.

2-4-1 Problems Formulation

Three problems are to be formulated. The first problem to be solved is to show the validity of the separation property.

Problem 2-4.1. (separation property) Prove that the separation property holds for the control problem.

Consequently, if this problem is solved, we can proceed with the synthesis of the two gains, F and K, in the swallow form.

Problem 2-4.2. (swallow state-feedback gain synthesis) Consider the state-equation of a generic CLS, as displayed in (2-1). Consider a nested past-state information pattern, and the set of linear control laws $u(t) = Fx(t)$ in which F is restricted to

$$F = \begin{bmatrix} F_{11} & 0 & \cdots & 0 & F_{1c} \\ 0 & F_{22} & \cdots & 0 & F_{2c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & F_{N_sN_s} & F_{N_sc} \\ 0 & 0 & \cdots & 0 & F_{cc} \end{bmatrix}.$$

Given the infinite-horizon average cost function

$$J = \lim_{t\to\infty} \frac{1}{t} E \left\{ \sum_{s=1}^{t} \left[x^T(s)\, Qx(s) + u^T(F,s)\, Ru(F,s) \right] \right\},$$

solve

$$\inf_{F_{11}, F_{1c}, \ldots, F_{N_sN_s} F_{N_sc}, F_{cc}} J\left(F_{11}, F_{1c}, \ldots, F_{N_sN_s} F_{N_sc}, F_{cc}\right).$$

such that the closed-loop system is stable.

In a dual way, the problem of finding the optimal observer gain follows.

Problem 2-4.3. (swallow observer gain synthesis) Consider the state-equation of a generic CLS, as displayed in (2-1). Consider a nested past-input and past-output information pattern for the observer system

$$\begin{cases} \hat{x}(t+1) = A\hat{x}(t) + Bu(t) + K\left[\hat{y}(t) - y(t)\right] \\ \hat{y}(t) = C\hat{x}(t) + Du(t). \end{cases}$$

with $\hat{x}(t_0) = \hat{x}_0$ in which K is restricted to the swallow form,

$$K = \begin{bmatrix} K_{11} & 0 & \cdots & 0 & K_{1c} \\ 0 & K_{22} & \cdots & 0 & K_{2c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & K_{N_sN_s} & K_{N_sc} \\ 0 & 0 & \cdots & 0 & K_{cc} \end{bmatrix}.$$

Given the infinite-horizon average cost function

$$H = \lim_{t\to\infty} \frac{1}{t} E\left[\sum_{s=0}^{t-1} e\left(K,s\right)^T e\left(K,s\right)\right],$$

where $e(t)$ is the error signal defined in eq. (2-4), solve

$$\inf_{K_{11},K_{1c},\ldots,K_{N_sN_s}K_{N_sc},K_{cc}} H\left(K_{11},K_{1c},\ldots,K_{N_sN_s}K_{N_sc},K_{cc}\right).$$

The solution to these three problems corresponds to the solution to the LQG coordination control problem.

Problem 2-4.4. (LQG Coordination Control Problem) Solve Problem 2-4.1, Problem 2-4.2 and Problem 2-4.3 for a given CLS, and for given weighing matrices and given disturbance covariance matrices V_x, V_{xy} and V_y.

2-4-2 Control Objectives

Goal of LQG coordination control is to solve Problem 2-4.4. A direct consequence of its solution is the achievement of the following closed-loop properties:

- stability,
- optimality with respect to the performance criterion (minimization of the cost functions).

Ancillary results are sought in the solution to the problem:

- computational efficiency,
- elasticity to changes.

Comments regarding these properties will be discussed in Chapter 5, where general strengths and weaknesses of the results will be considered.

2-5 Conclusions

In this chapter, the problem of LQG coordination control was formulated. LQG coordination control is required to emulate the characteristics of the classical LQG optimal control theory. For CLSs, the gain matrices F and K are required to be swallow matrices, which were introduced in Section 2-1. The shape of this matrices reflects the information constraints of the control laws, and important invariance properties hold for them.

As overviewed in Section 2-3, a few results are already available in the literature to control CLSs. LQG coordination control could be a suitable alternative to these approaches where requirements impose a low computational burden and imply the separation of observer and state-feedback gains syntheses.

Chapter 3

LQG Coordination Control

In this chapter we present all the results obtained for LQG coordination control. An introduction of the methodology applied to the problem is given in Section 3-1. Then, different results are explained for three categories of problems. In Section 3-2, the *general problems* formulation is treated. For *decomposable problems*, a class of problems with a slightly stricter problem formulation, results are shown in Section 3-3. Going down along the chain, a very specific class of problems, defined as *virtual coordination problems* is considered in Section 3-4. At last, conclusions are drawn in Section 3-6.

3-1 Introduction

The results obtained in this thesis are classified through a problem categorization. The main bottleneck preventing general results to be achieved is the validity of the separation property. Due to this, the main theorems we provide only cover particular subclasses of the general problem formulation. A graphical representation of the problem categories considered is given by Figure (3-1). We distinguish the following:

1. *General problems*: no restrictions are set for weighing and covariance matrices. The LQG coordination control theory cannot always be applied to this class of problems because of the invalidity of the separation property.

2. *Decomposable problems*: weighing and covariance matrices are restricted to a form which allow a problem decomposition. A control synthesis involving a numerical optimization is available for this class of problems.

3. *Virtual coordination problems*: the decomposable problem's limitations are applied with the ulterior assumption that no noise affects the coordinator. For this class of problems, the local LQG control solution is proved to produce globally optimal control laws.

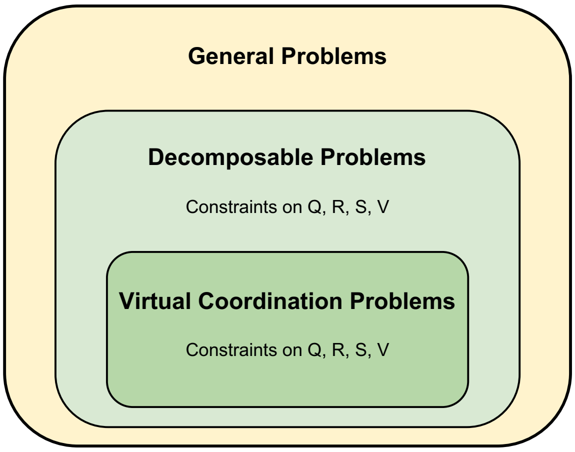

Figure 3-1: The three categories of problems considered. Constraints are set for the weighing matrices Q and R, and for the covariance matrix V .

To reduce the notational load on the following pages, in formulas we display every CLS as if it only had $N_s = 2$ subsystems. No loss of generality is introduced by this decision, as all the expressions can easily be extended. Although the formulas contain only two subsystems, we in fact provide results covering any number N_s of subsystems.

3-2 General Problems

We here define the LQG coordination control problem in its most generic formulation.

Problem 3-2.1. (General Problem Formulation) Solve Problem 2-4.4 given the weighing matrices

$$Q_{tot} = Q_{tot}^T = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0; \qquad Q = Q^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{1c} \\ Q_{12}^T & Q_{22} & Q_{2c} \\ Q_{1c}^T & Q_{2c}^T & Q_{cc} \end{bmatrix};$$
$$S = \begin{bmatrix} S_{11} & S_{12} & S_{1c} \\ S_{12}^T & S_{22} & S_{2c} \\ S_{1c}^T & S_{2c}^T & S_{cc} \end{bmatrix}, \quad R = R^T = \begin{bmatrix} R_{11} & R_{12} & R_{1c} \\ R_{12}^T & R_{22} & R_{2c} \\ R_{1c}^T & R_{2c}^T & R_{cc} \end{bmatrix} > 0.$$

and a disturbance vector $v_{tot}(t) = \left[v_x^T(t), v_y^T(t)\right]^T$ being a Gaussian zero-average noise with covariance matrices

$$V_{tot} = V_{tot}^T \begin{bmatrix} V_x & V_{xy} \\ V_{xy}^T & V_y \end{bmatrix} \geq 0; \qquad V_x = V_x^T = \begin{bmatrix} V_{x,11} & V_{x,12} & V_{x,1c} \\ V_{x,12}^T & V_{x,22} & V_{x,2c} \\ V_{x,1c}^T & V_{x,2c}^T & V_{x,cc} \end{bmatrix};$$
$$V_{xy} = \begin{bmatrix} V_{xy,11} & V_{xy,12} & V_{xy,1c} \\ V_{xy,12}^T & V_{xy,22} & V_{xy,2c} \\ V_{xy,1c}^T & V_{xy,2c}^T & V_{xy,cc} \end{bmatrix}; \quad V_y = V_y^T = \begin{bmatrix} V_{y,11} & V_{y,12} & V_{y,1c} \\ V_{y,12}^T & V_{y,22} & V_{y,2c} \\ V_{y,1c}^T & V_{y,2c}^T & V_{y,cc} \end{bmatrix} > 0.$$

For this class of problems, the author is not able to present a solution to the LQG coordination control problem. Therefore, after discussing the issues which prevent results from being produced, the reader is referred to the only external result given in the literature that could be of interest for the topic.

3-2-1 Issues

The first issue about general problems is related to the separation principle. The information pattern of our controller is non-classical, which means the controller and the observer do not have access to the information of the whole system, but only to a selection of it. The separation property does not hold, in general, for problems with an information pattern different from the classical one [42, 51]. Unfortunately, no proof has been found for the validity of the separation principle for the general problem formulation.

A second important issue is related to the coupling of subsystems' information introduced by the cost functions. In fact, if the weighing matrices Q, R and S can be any symmetric positive semi-definite, and definite matrices respectively, the closed-loop covariance between the states of the subsystems is weighted in the cost function. Results are hard to be achieved even under the assumption that no observation of the states needed, and the present and past states of the local subsystem and the coordinator are known at every time instant.

3-2-2 Available Results

An important theoretical result is available for this class of problems. The solution to the LQG optimal control problem is a linear control law. In fact, Ho and Chu [24] proved that for problems with partially nested information structure, the optimal control law that solves the LQG problem is linear. The rough meaning of nested information structure is that if a subsystem is influenced by another one, it also has access to all its information. This, of course, is the case of CLSs. The result is therefore applicable to the General Problem formulation, and therefore to all its subclasses.

3-3 Decomposable Problems

Decomposable problems are a particular class of problems for which a problem decomposition can be performed. The definition of the LQG coordination control problem to them related follows.

Problem 3-3.1. (Decomposable Problems) Solve Problem 2-4.4 given the weighing matrices

$$Q_{tot} = Q_{tot}^T = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0; \qquad Q = Q^T = \begin{bmatrix} Q_{11} & 0 & Q_{1c} \\ 0 & Q_{22} & Q_{2c} \\ Q_{1c}^T & Q_{2c}^T & Q_{cc} \end{bmatrix};$$
$$S = \begin{bmatrix} S_{11} & 0 & S_{1c} \\ 0 & S_{22} & S_{2c} \\ 0 & 0 & S_{cc} \end{bmatrix}, \quad R = R^T = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{cc} \end{bmatrix} > 0.$$

and a disturbance vector $v_{tot}(t) = \left[v_x^T(t), v_y^T(t)\right]^T$ being a Gaussian zero-average noise with covariance matrices

$$V_{tot} = V_{tot}^T \quad \begin{bmatrix} V_x & V_{xy} \\ V_{xy}^T & V_y \end{bmatrix} \geq 0; \qquad V_x = V_x^T = \begin{bmatrix} V_{x,11} & 0 & V_{x,1c} \\ 0 & V_{x,22} & V_{x,2c} \\ V_{x,1c}^T & V_{x,2c}^T & V_{x,cc} \end{bmatrix};$$

$$V_{xy} = \begin{bmatrix} V_{xy,11} & 0 & V_{xy,1c} \\ 0 & V_{xy,22} & V_{xy,2c} \\ V_{xy,1c}^T & V_{xy,2c}^T & V_{xy,cc} \end{bmatrix}; \quad V_y = V_y^T = \begin{bmatrix} V_{y,11} & 0 & V_{y,1c} \\ 0 & V_{y,22} & V_{y,2c} \\ V_{y,1c}^T & V_{y,2c}^T & V_{y,cc} \end{bmatrix} > 0.$$

The approach adopted to solve these problems is that of fixing part of the swallow gains F and K to be determined (precisely, F_{cc} and K_{ii}, $i = 1, \ldots, N_s$) in order to decompose the global optimization problem into decoupled subproblems for which the separation principle holds. Then, we solve each of the subproblems optimally by use of the classical LQG control theory.

3-3-1 Applicability of the Separation Principle

Topic of this section is the solution of Problem 2-4.1 for the category of decomposable problems, a crucial step to solve the LQG coordination control problem. We show hereby that if the parameters F_{cc} and K_{ii}, for $i = 1, \ldots, N_s$, are fixed, the separation principle can be applied to the class of decomposable problems. The procedure adopted to prove it is explained by the next passages.

1. We show that every CLS (including its observer system) can be decomposed into N_s independent systems.

2. We prove that the separation property holds for each of these N_s systems if the gains F_{cc} and K_{ii} are fixed.

3. We show that the cost functions of the global problem can be decomposed additively, and that proper subcost functions can be defined to be associated to the N_s independent systems representing the CLS.

4. A theorem combines all the above results to show that, for decomposable problems, the control synthesis problem is equivalent to the control synthesis of N_s independent subproblems, for which the separation principle holds.

We prove in this way both the possibility of decomposing the problem into N_s subproblems and the validity of the separation principle for given values of F_{cc} and K_{ii}, for $i = 1, \ldots, N_s$.

Step 1: Decomposition of Coordinated Linear Systems

Every CLS, together with their state-observers, can be decomposed in N_s independent subsystems. We show how by means of two lemmas.

Lemma 3-3.2. *(CLS System Decomposition) Let $F_{cc} \in \mathbb{R}^{m_c \times n_c}$ be given such that $(A_{cc} + B_{cc}F_{cc})$ is asymptotically stable, m_c and n_c being the number of inputs and states of the coordinator respectively. Then, a CLS can be decomposed into N_s subsystems of the form*

$$\begin{cases} \begin{bmatrix} x_i(t+1) \\ x_c(t+1) \end{bmatrix} = \begin{bmatrix} A_{ii} & A_{ic} + B_{ic}F_{cc} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_{ii} \\ 0 \end{bmatrix} u_i(t) + \begin{bmatrix} M_{ii} & M_{ic} \\ 0 & M_{cc} \end{bmatrix} \begin{bmatrix} v_{x,i}(t) \\ v_{x,c}(t) \end{bmatrix} \\ \begin{bmatrix} y_i(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} C_{ii} & C_{ic} \\ 0 & C_{cc} \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} N_{ii} & N_{ic} \\ 0 & N_{cc} \end{bmatrix} \begin{bmatrix} v_{y,i}(t) \\ v_{y,c}(t) \end{bmatrix} \end{cases}. \tag{3-1}$$

Proof. Trivially, after setting $u_c(t) = F_{cc}x_c(t)$, all the remaining inputs act in a decoupled way on the system, allowing the given decomposition. □

Notice that the system obtained after fixing F_{cc} is stabilizable, but not controllable. This decomposition approach was at first discovered by Kempker for the continuous-time case, studying an extension of the LQ-optimal control theory for CLSs [28]. The same idea is applied in the following lemma also to the state-observer system,

$$\begin{cases} \hat{x}(t+1) = \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \hat{x}(t) + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} u(t) + \begin{bmatrix} K_{11} & 0 & K_{1c} \\ 0 & K_{22} & K_{2c} \\ 0 & 0 & K_{cc} \end{bmatrix} (\hat{y}(t) - y(t)) \\ \hat{y}(t) = \begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix} \hat{x}(t); \\ \hat{x}(t_0) = \hat{x}_0. \end{cases} \tag{3-2}$$

Lemma 3-3.3. *(CLS Observer System Decomposition) Let $F_{cc} \in \mathbb{R}^{m_c \times n_c}$ and $K_{ii} \in \mathbb{R}^{n_i \times o_i}$, for $i = 1, \ldots, N_s$, be given such that $(A_{ii} + K_{ii}C_{ii})$, for $i = 1, \ldots, N_s$, and $(A_{cc} + B_{cc}F_{cc})$ are asymptotically stable, m_c and n_c being the number of inputs and states of the coordinator respectively; n_i and o_i the number of states and outputs of the i^{th} subsystem, for $i = 1, \ldots, N_s$. Then, the state-observer of a CLS can be decomposed into N_s observer systems of the form*

$$\begin{cases} \begin{bmatrix} \hat{x}_i(t+1) \\ \hat{x}_c(t+1) \end{bmatrix} = \begin{bmatrix} A_{ii} + K_{ii}C_{ii} & A_{ic} + B_{ic}F_{cc} + K_{ii}C_{ic} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} \begin{bmatrix} \hat{x}_i(t) \\ \hat{x}_c(t) \end{bmatrix} + \\ \qquad + \begin{bmatrix} B_{ii} \\ 0 \end{bmatrix} u_i(t) + \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix} (\hat{y}_c(t) - y_c(t)) \\ \begin{bmatrix} \hat{y}_i(t) \\ \hat{y}_c(t) \end{bmatrix} = \begin{bmatrix} C_{ii} & C_{ic} \\ 0 & C_{cc} \end{bmatrix} \begin{bmatrix} \hat{x}_i(t) \\ \hat{x}_c(t) \end{bmatrix}. \end{cases} \tag{3-3}$$

Proof. The decomposition follows from fixing F_{cc} and K_{ii}, for $i = 1, \ldots, N_s$ in equation (3-2). □

Again, we notice that the system obtained after fixing the two gains is detectable, but not observable. As we will see, the system decomposition here retrieved plays an important role in the control synthesis problem decomposition.

Step 2: Investigation on Information Patterns and Separation Property

By the following proposition we show that the information pattern of each of the N_s systems equivalently representing a CLS is classical. This is the most important result needed to apply the separation principle to Problem 3-3.1.

Proposition 3-3.4. *(Classical Information Pattern) Let* $F_{cc} \in \mathbb{R}^{m_c \times n_c}$ *and* $K_{ii} \in \mathbb{R}^{n_i \times o_i}$, *for* $i = 1, \ldots, N_s$, *be given such that* $(A_{ii} + K_{ii}C_{ii})$, *for* $i = 1, \ldots, N_s$, *and* $(A_{cc} + B_{cc}F_{cc})$ *are asymptotically stable,* m_c *and* n_c *being the number of inputs and states of the coordinator respectively;* n_i *and* o_i *the number of states and outputs of the* i^{th} *subsystem, for* $i = 1, \ldots, N_s$. *Let us consider the decomposed CLS system (3-1) and its observer (3-3), for any* $i \in [1, N_s]$. *Let the control law be determined by the two gains*

$$F_i = \begin{bmatrix} F_{ii} & F_{ic} \end{bmatrix}; \; K_i = \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix}.$$

Then, the control law is associated to a classical information pattern, for $i = 1, \ldots, N_s$.

Proof. The gains $F_i = [F_{ii}, F_{ic}]$ and $K_i = [K_{ic}; K_{cc}]$, for $i = 1, \ldots, N_s$, have no structural constraints (i.e., they are, in general, filled by non-zero entries). As a direct consequence, the information pattern related to each of the considered control problems is classical. □

The fact that each of the N_s systems composing the CLS has a classical information pattern, this does not directly imply that the control synthesis problem can be approached in a separated way. This is a necessary condition, yet not sufficient. In fact, not only the system, but also the cost functions have to be decomposed in the same way as we did for CLS. In this way, the whole problem can be separated into N_s equivalent subproblems. We treat the cost function decomposition in the following paragraph.

Step 3: Decomposition of the Cost Functions

By means of two propositions, we show that the two cost functions J and H as in eq. (2-3) and (2-5) can be decomposed additively. This decomposition will be used to associate to each of the N_s systems previously obtained an appropriate subcost function.

Lemma 3-3.5. *(Decomposition of the Cost Function* J*) The cost function* J *of eq. (2-3) decomposes additively as*

$$\begin{aligned} J = \;& trace\left[W\left(Q + SF + F^T S^T + F^T RF\right)\right] = \\ = \;& J_{11} + J_{1c} + \cdots + J_{N_s N_s} + J_{N_s c} + J_{cc} = \\ = \;& \textstyle\sum_{i=1}^{N_s} \left[J_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ic}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ci}\left(F_{ic}, F_{cc}\right)\right] + J_{cc}\left(F_{cc}\right), \end{aligned}$$

with

$$\begin{aligned} J_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right) = \;& trace\left\{W_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right)\left[Q_{ii} + S_{ii}F_{ii} + F_{ii}^T S_{ii}^T + F_{ii}^T R_{ii} F_{ii}\right]\right\}; \\ J_{ic}\left(F_{ii}, F_{ic}, F_{cc}\right) = \;& 2 \times trace\left[W_{ic}\left(Q_{ic}^T + F_{ic}^T S_{ii}^T + F_{cc}^T S_{ic}^T + F_{ic}^T R_{ii} F_{ii}\right)\right]; \\ J_{ci}\left(F_{ic}, F_{cc}\right) = \;& trace\left\{W_{cc} F_{ic}^T R_{ii} F_{ic}\right\}, \end{aligned}$$

for $i = 1, \ldots, N_s$, *and*

$$J_{cc}(F_{cc}) = trace\left[W_{cc}\left(Q_{cc} + S_{cc}F_{cc} + F_{cc}^T S_{cc}^T + F_{cc}^T R_{cc} F_{cc}\right)\right].$$

Proof. The steady-state variance of the closed-loop system's state is given by the solution of the Lyapunov equation

$$W = (A + BF)\, W\, (A + BF)^T + M V_x M^T,$$

which can be separated into the following,

$$\begin{array}{rl} W_{ij} = & A_{ii,cl} W_{ij} A_{jj,cl}^T + A_{ic,cl} W_{cj} A_{jj,cl}^T + A_{ii,cl} W_{ic} A_{jc,cl}^T + A_{ic,cl} W_{cc} A_{jc,cl}^T + \\ & + M_{ii} V_{x,ij} M_{jj}^T + M_{ic} V_{x,cj} M_{jj}^T + M_{ii} V_{x,ic} M_{jc}^T + M_{ic} V_{x,cc} M_{jc}^T; \\ W_{ic} = & A_{ii,cl} W_{ic} A_{cc,cl}^T + A_{ic,cl} W_{cc} A_{cc,cl}^T + M_{ii} V_{x,ic} M_{cc}^T + M_{ic} V_{x,cc} M_{cc}^T; \\ W_{cc} = & A_{cc,cl} W_{cc} A_{cc,cl}^T + M_{cc} V_{x,cc} M_{cc}^T. \end{array} \tag{3-4}$$

for $i, j = 1, \ldots, N_s$, where $A_{ii,cl} = A_{ii} + B_{ii}F_{ii}$; $A_{ic,cl} = A_{ic} + B_{ii}F_{ic} + B_{ic}F_{cc}$, for $i = 1, \ldots, N_s$, and $A_{cc,cl} = A_{cc} + B_{cc}F_{cc}$. The term W is weighted by

$$\overline{Q} = \left(Q + SF + F^T S^T + F^T R F\right).$$

Because of Proposition 2-1.4,$\overline{Q}$ is in the arrow form,

$$\overline{Q} = \begin{bmatrix} \overline{Q}_{11} & 0 & \overline{Q}_{1c} \\ 0 & \overline{Q}_{22} & \overline{Q}_{2c} \\ \overline{Q}_{1c}^T & \overline{Q}_{2c}^T & \overline{Q}_{cc} \end{bmatrix},$$

where

$$\begin{array}{rl} \overline{Q}_{ii} = & Q_{ii} + S_{ii}F_{ii} + F_{ii}^T S_{ii}^T + F_{ii}^T R_{ii} F_{ii}; \\ \overline{Q}_{ic} = & Q_{ic} + S_{ii}F_{ic} + S_{ic}F_{cc} + F_{ii}^T R_{ii} F_{ic}; \\ \overline{Q}_{cc} = & Q_{cc} + S_{cc}F_{cc} + F_{cc}^T S_{cc}^T + \sum_{k=1,2,c} F_{kc}^T R_{cc} F_{kc}. \end{array}$$

The cost function is therefore

$$\begin{array}{rl} J = & trace\left[\begin{pmatrix} W_{11}(F_{11}, F_{1c}, F_{cc}) & W_{12} & W_{1c}(F_{1c}, F_{cc}) \\ W_{12}^T & W_{22}(F_{22}, F_{2c}, F_{cc}) & W_{2c}(F_{2c}, F_{cc}) \\ W_{1c}^T(F_{ic}, F_{cc}) & W_{1c}^T(F_{ic}, F_{cc}) & W_{cc}(F_{cc}) \end{pmatrix}\begin{pmatrix} \overline{Q}_{11} & 0 & \overline{Q}_{1c} \\ 0 & \overline{Q}_{22} & \overline{Q}_{2c} \\ \overline{Q}_{1c}^T & \overline{Q}_{2c}^T & \overline{Q}_{cc} \end{pmatrix}\right] = \\ = & \sum_{i=1}^{N_s}\left(trace\left\{W_{ii}(F_{ii}, F_{ic}, F_{cc})\left[Q_{ii} + S_{ii}F_{ii} + F_{ii}^T S_{ii}^T + F_{ii}^T R_{ii} F_{ii}\right]\right\}\right) + \\ & + \sum_{i=1}^{N_s}\left(2 \times trace\left\{W_{ic}(F_{ic}, F_{cc})\left[Q_{ic} + S_{ii}F_{ic} + S_{ic}F_{cc} + F_{ii}^T R_{ii} F_{ic}\right]\right\}\right) + \\ & + \sum_{i=1}^{N_s}\left(trace\left\{W_{cc}(F_{cc}) F_{ic}^T R_{ii} F_{ic}\right\}\right) + \\ & + trace\left(W_{cc}(F_{cc})\left[Q_{ic}^T + F_{ic}^T S_{ii}^T + F_{cc}^T S_{ic}^T + F_{ic}^T R_{ii} F_{ii}\right]\right) = \\ = & \sum_{i=1}^{N_s}\left[J_{ii}(F_{ii}, F_{ic}, F_{cc}) + J_{ic}(F_{ii}, F_{ic}, F_{cc}) + J_{ci}(F_{ic}, F_{cc})\right] + J_{cc}(F_{cc}), \end{array}$$

as it was to be proved. □

The corrisponding result for the cost function H follows.

Lemma 3-3.6. *(Decomposition of the Cost Function H) The cost function H of eq. (2-5) decomposes additively,*

$$\begin{array}{rl} H = & trace(W_e) = \\ = & H_{11} + \cdots + H_{N_sN_s} + H_{cc} = \\ = & \sum_{i=1}^{N_s}\left[H_{ii}(K_{ii}, K_{ic}, K_{cc})\right] + H_{cc}(K_{cc}), \end{array}$$

with

$$H_{ii}\left(K_{ii}, K_{ic}, K_{cc}\right) = \quad trace\left[W_{e,ii}\left(K_{ii}, K_{ic}, K_{cc}\right)\right],$$

for $i = 1, \ldots, N_s,$ *and*

$$H_{cc}\left(K_{cc}, F_{cc}\right) = \quad trace\left[W_{e,cc}\left(K_{cc}, F_{cc}\right)\right].$$

Proof. The covariance matrix W_e of the steady-state observation error is given by the solution of the Lyapunov equation

$$W_e = (A + KC)\, W_e\, (A + KC)^T + \begin{bmatrix} M & KN \end{bmatrix} \begin{bmatrix} V_x & V_{xy} \\ V_{xy}^T & V_y \end{bmatrix} \begin{bmatrix} M^T \\ N^T K^T \end{bmatrix}.$$

Expanding the terms, we observe

$$\begin{aligned}
W_{e,ii}\left(K_{ii}, K_{ic}, K_{cc}, F_{cc}\right) = \quad & \left[A_{ii} + K_{ii}C_{ii}\right] W_{e,ii}\left(K_{ii}, K_{ic}, K_{cc}\right) \left[A_{ii} + K_{ii}C_{ii}\right]^T + \\
+ \quad & \left[A_{ic} + B_{ic}F_{cc} + C_{ii}K_{ic} + C_{ic}K_{cc}\right] W_{e,ic}^T\left(K_{ii}, K_{ic}, K_{cc}\right) \left[A_{ii} + K_{ii}C_{ii}\right]^T + \\
+ \quad & \left[A_{ii} + K_{ii}C_{ii}\right] W_{e,ic}\left(K_{ii}, K_{ic}, K_{cc}\right) \left[A_{cc} + B_{cc}F_{cc} + K_{cc}C_{cc}\right]^T + \\
+ \quad & \left[A_{ic} + B_{ic}F_{cc} + C_{ii}K_{ic} + C_{ic}K_{cc}\right] W_{e,cc} \left[A_{ic} + B_{ic}F_{cc} + C_{ii}K_{ic} + C_{ic}K_{cc}\right]^T + \\
+ \quad & M_{ii}V_{x,ii}M_{ii}^T + M_{ii}V_{x,ic}^T M_{ic}^T + M_{ic}V_{x,cc}M_{ic}^T + \\
+ \quad & M_{ii}V_{xy,ii}N_{ii}^T K_{ii}^T + M_{ic}V_{xy,ic}^T N_{ii}^T K_{ii}^T + M_{ii}V_{xy,ic}N_{ic}^T K_{ii}^T + M_{ic}V_{xy,cc}N_{ic}^T K_{ii}^T + \\
+ \quad & M_{ii}V_{xy,ic}N_{cc}^T K_{ic}^T + M_{ic}V_{xy,cc}N_{cc}^T K_{ic}^T + K_{ii}N_{ii}V_{xy,ii}M_{ii}^T + K_{ii}N_{ic}V_{xy,ic}^T M_{ii}^T + \\
+ \quad & K_{ic}N_{cc}V_{xy,ic}M_{ii}^T + K_{ii}N_{ii}V_{xy,ic}M_{ic}^T + K_{ii}N_{ic}V_{xy,cc}M_{ic}^T + K_{ic}N_{cc}V_{xy,cc}M_{ic}^T + \\
+ \quad & K_{ii}N_{ii}V_{y,ii}N_{ii}^T K_{ii}^T + K_{ii}N_{ic}V_{y,ic}^T N_{ii}^T + K_{ic}N_{cc}V_{y,ic}^T N_{ii}^T + \\
+ \quad & K_{ii}N_{ii}V_{y,ic}N_{ic}^T K_{ii}^T + K_{ii}N_{ii}V_{y,ic}N_{cc}^T K_{ic}^T + K_{ii}N_{ic}V_{y,cc}N_{ic}^T K_{ii}^T + \\
+ \quad & K_{ic}N_{cc}V_{y,cc}N_{ic}^T K_{ii}^T + K_{ii}N_{ic}V_{y,cc}N_{cc}^T K_{ic}^T + K_{ic}N_{cc}V_{y,cc}N_{cc}^T K_{ic}^T; \\[1em]
W_{e,ic}\left(K_{ii}, K_{ic}, K_{cc}, F_{cc}\right) = \quad & \left[A_{ii} + K_{ii}C_{ii}\right] W_{e,ic}\left(K_{ii}, K_{ic}, K_{cc}, F_{cc}\right) \left[A_{cc} + B_{cc}F_{cc} + K_{cc}C_{cc}\right]^T \\
+ \quad & M_{ii}V_{x,ic}M_{cc}^T + M_{ic}V_{x,cc}M_{cc}^T + M_{ii}V_{xy,ic}N_{cc}^T K_{cc}^T + M_{ic}V_{xy,cc}N_{cc}^T K_{cc}^T + \\
+ \quad & K_{ii}N_{ii}V_{xy,ic}M_{cc}^T + K_{ii}N_{ic}V_{xy,cc}M_{cc}^T + K_{ic}N_{cc}V_{xy,cc}M_{cc}^T \\
+ \quad & K_{ii}N_{ii}V_{y,ic}N_{cc}K_{cc}^T + K_{ii}N_{ic}V_{y,cc}N_{cc}^T K_{cc}^T + K_{ic}N_{cc}V_{y,cc}N_{cc}^T K_{cc}^T; \\[1em]
W_{e,cc}\left(K_{cc}, F_{cc}\right) = \quad & \left[A_{cc} + B_{cc}F_{cc} + K_{cc}C_{cc}\right] W_{e,cc}\left(K_{cc}, F_{cc}\right) \left[A_{cc} + B_{cc}F_{cc} + K_{cc}C_{cc}\right]^T + \\
+ \quad & M_{cc}V_{x,cc}M_{cc}^T + M_{cc}V_{xy,cc}N_{cc}^T K_{cc}^T + K_{cc}N_{cc}V_{xy,cc}M_{cc}^T + K_{cc}N_{cc}V_{y,cc}N_{cc}^T K_{cc}^T.
\end{aligned} \tag{3-5}$$

Since $(A_{ii} + K_{ii}C_{ii})$, for $i = 1, \ldots, N_s$ and $(A_{cc} + B_{cc}F_{cc} + K_{cc}C_{cc})$ are asymptotically stable, the Lyapunov equation has a unique, symmetric, positive-definite solution (see [49], appendix E). By the definition of the *trace* operator, it follows that

$$H = \sum_{i=1}^{N_s} \left[H_{ii}\left(K_{ii}, K_{ic}, K_{cc}, F_{cc}\right)\right] + H_{cc}\left(K_{cc}, F_{cc}\right),$$

as it was to be proved. □

Step 4: Separation Principle for Decomposable Problems

The results obtained so far allow to decompose the system into N_s equivalent systems for which the separation principle holds. We now show that, as the cost functions decompose in the same way of the system, the whole problem can be decomposed into N_s subproblems.

Theorem 3-3.7. *(Separation Principle for Decomposable Problems) Consider a decomposable problem. Let the gains $F_{cc} \in \mathbb{R}^{m_c \times n_c}$ and $K_{ii} \in \mathbb{R}^{n_i \times o_i}$, for $i = 1, \ldots, N_s$, be fixed such that $(A_{ii} + K_{ii}C_{ii})$, for $i = 1, \ldots, N_s$, and $(A_{cc} + B_{cc}F_{cc})$ are asymptotically stable, m_c and n_c being the number of inputs and states of the coordinator respectively; n_i and o_i the number of states and outputs of the i^{th} subsystem, $i = 1, \ldots, N_s$. Then,*

a) *Problem 2-4.2 can be decomposed into N_s subproblems of the form*

$$\inf_{F_{ii}, F_{ic}} \left\{ \sum_{i=1}^{N_s} \left[J_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ic}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ci}\left(F_{ic}, F_{cc}\right) \right] \right\},$$

for $i = 1, \ldots, N_s$;

b) *Problem 2-4.3 can be decomposed in N_s subproblems of the form*

$$\inf_{K_{ic}} \left\{ \sum_{i=1}^{N_s} \left[H_{ii}\left(K_{ii}, K_{ic}, K_{cc}, F_{cc}\right) \right] \right\},$$

for $i = 1, \ldots, N_s$;

c) *the separation property holds for each of these subproblems.*

Proof. Each claim is proved in order of appearance.

a) Problem 2-4.2 requires to find

$$\inf_{F_{11}, F_{1c}, \ldots, F_{N_sN_s} F_{N_sc}, F_{cc}} J\left(F_{11}, F_{1c}, \ldots, F_{N_sN_s} F_{N_sc}, F_{cc}\right).$$

As it was proved in Lemma 3-3.5, the cost function J can be decomposed additively. Therefore, we have

$$\inf_{F_{11}, F_{1c}, \ldots, F_{N_sN_s} F_{N_sc}, F_{cc}} \left\{ \sum_{i=1}^{N_s} \left[J_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ic}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ci}\left(F_{ic}, F_{cc}\right) \right] + J_{cc}\left(F_{cc}\right) \right\}.$$

As F_{cc} is fixed, it has to be excluded from the minimization domain. For this reason, the term $J_{cc}\left(F_{cc}\right)$ can be separated from the cost function. The problem results into the following.

$$\inf_{F_{11}, F_{1c}, \ldots, F_{N_sN_s} F_{N_sc}} \left\{ \sum_{i=1}^{N_s} \left[J_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ic}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ci}\left(F_{ic}, F_{cc}\right) \right] \right\} =$$

$$= \sum_{i=1}^{N_s} \inf_{F_{ii}, F_{ic}} \left\{ \left[J_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ic}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ci}\left(F_{ic}, F_{cc}\right) \right] \right\},$$

which is the same as minimizing the N_s subproblems

$$\inf_{F_{ii}, F_{ic}} \left\{ J_{ii}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ic}\left(F_{ii}, F_{ic}, F_{cc}\right) + J_{ci}\left(F_{ic}, F_{cc}\right) \right\}$$

for $i = 1, \ldots, N_s$ separately. The initial minimization problem is therefore decomposed into N_s subproblems.

b) Similarly, Problem 2-4.3 requires to find

$$\inf_{K_{11},K_{1c},\ldots,K_{N_sN_s}K_{N_sc},K_{cc}} H\left(K_{11},K_{1c},\ldots,K_{N_sN_s},K_{N_sc},K_{cc},F_{cc}\right).$$

As it was proved in Lemma 3-3.6, the cost function H can be decomposed additively. Therefore, we have

$$\inf_{K_{11},K_{1c},\ldots,K_{N_sN_s}K_{N_sc},K_{cc}} \left\{\sum_{i=1}^{N_s}\left[H_{ii}\left(K_{ii},K_{ic},K_{cc},F_{cc}\right)\right] + H_{cc}\left(K_{cc},F_{cc}\right)\right\}.$$

As F_{cc} and K_{ii}, for $i = 1,\ldots,N_s$, are fixed, the problem becomes

$$\inf_{K_{1c},\ldots,K_{N_sc},K_{cc}} \sum_{i=1}^{N_s}\left[H_{ii}\left(K_{ii},K_{ic},K_{cc},F_{cc}\right)\right].$$

To furtherly decompose the cost function, an external result is necessary. This result is proved in a future section, in Theorem 3-3.15. There, we show that the optimal value of K_{cc}, once the parameters K_{ii}, for $i = 1,\ldots,N_s$ and F_{cc} are fixed, is computed independently as

$$K_{cc} = \left[LQG\left(A_{cc} + B_{cc}F_{cc}, C_{cc}^T, M_{cc}V_{x,cc}M_{cc}^T, N_{cc}V_{y,cc}N_{cc}^T, M_{cc}V_{xy,cc}N_{cc}^T\right)\right]^T.$$

This fact allows the problem to be decomposed into the N_s subproblems of the form

$$\inf_{K_{ic}}\left\{H_{ii}\left(K_{ii},K_{ic},K_{cc},F_{cc}\right)\right\},$$

for $i = 1,\ldots,N_s$.

c) The subcost functions J_{ii}, J_{ic}, J_{ci} and H_{ii} are associated to the i^{th} of the N_s subsystems, for $i = 1,\ldots,N_s$. Since the LQG coordination control problem can be decomposed into N_s subproblems, and since we have proved in Proposition 3-3.4 that the separation property holds for the N_s systems, then this property holds for the whole system, if and only if the parameters F_{cc} and K_{ii}, for $i = 1,\ldots,N_s$ are fixed.

The proof is therefore complete. □

The decomposition of the original control synthesis problem into N_s independent subproblems, for which the separation principle is applicable, is an extremely important result. We will show, in Section 3-3-2 and Section 3-3-3, that this result is the key to obtain a procedure to synthesize a swallow state-feedback gain F and a swallow observer gain K. A schematic representation of this procedure is illustrated in Figure 3-2.

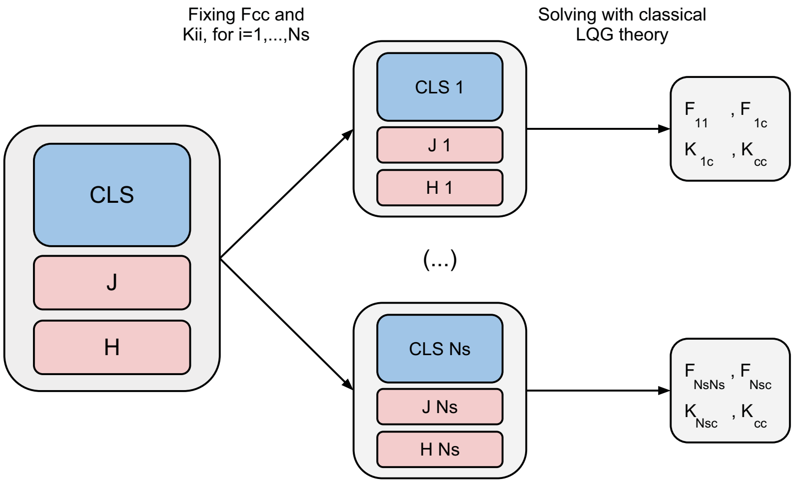

Figure 3-2: Control synthesis problem decomposition for decomposable problems. Fixing the parameters F_{cc}, K_{ii} for $i = 1, \ldots, N_s$ the problem is decomposed into N_s subproblems. Each of these problems can be solved independently by use of the classical LQG control theory.

3-3-2 State-Feedback Gain Synthesis

In this section we discuss the solution to Problem 2-4.2 for decomposable problems. We look for the optimal state-feedback gain F such that the control law is defined as

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix} = \begin{bmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \\ 0 & 0 & F_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix}.$$

The synthesis of this gain is done as explained below.

1. We make use of the decomposability of the problem to define a procedure to obtain $F(F_{cc})$ in an optimal way, given an arbitrary F_{cc}.
2. We define a procedure to minimize $J(F_{cc})$, also offering a way to derive symbolic expressions for the gradient and the Hessian of J with respect to the optimization parameter, F_{cc}.
3. The state-feedback gain F is then obtained by numerical optimization of $J(F_{cc})$ on the parameter F_{cc}. We conjecture J to be convex in F_{cc}.

Decomposition of the State-Feedback Gain Synthesis Problem

Following from the results of Theorem 3-3.7, Problem 2-4.2 can be decomposed in N_s subproblems if we provide a value for F_{cc}. This result is an adaptation and revision of the results obtained by Kempker, Ran and van Schuppen in the field of LQ-optimal control for continuous-time, deterministic CLSs [28]. Detailed results regarding the synthesis of a swallow gain F follow.

Theorem 3-3.8. *(Problem Decomposition for the Feedback Gain F) Given a decomposable problem, let a CLS with N_s subsystems be given, where*

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_c(t) \end{bmatrix} = \begin{bmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \\ 0 & 0 & F_{cc} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_c(t) \end{bmatrix}.$$

Let F_{cc} be fixed. For $i = 1, \ldots, N_s$, define

$$\begin{aligned} A_i &= \begin{bmatrix} A_{ii} & A_{ic} + B_{ic}F_{cc} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix}; & B_i &= \begin{bmatrix} B_{ii} \\ 0 \end{bmatrix}; \\ Q_i &= \begin{bmatrix} Q_{ii} & Q_{ic} \\ Q_{ic}^T & Q_{cc} \end{bmatrix}; & M_i &= \begin{bmatrix} M_{ii} & M_{ic} \\ 0 & M_{cc} \end{bmatrix}; \\ S_i &= \begin{bmatrix} S_{ii} \\ 0 \end{bmatrix}; & F_i &= \begin{bmatrix} F_{ii} & F_{ic} \end{bmatrix}; \\ L_{F,i}L_{F,i}^T &= Q_i - S_iR_i^{-1}S_i^T; & A_{F,i} &= A_i - B_iR_i^{-1}S_i^T; \\ x_{i,pair}(t) &= \begin{bmatrix} x_i(t) \\ x_c(t) \end{bmatrix}; & v_{i,pair}(t) &= \begin{bmatrix} v_{x,i}(t) \\ v_{x,c}(t) \end{bmatrix}. \end{aligned}$$

If

1) $(A_{cc} + B_{cc}F_{cc})$ is asymptotically stable;

2) (A_{ii}, B_{ii}) is stabilizable, for $i = 1, ..., N_s$;

3) $(A_{F,i}, L_{F,i})$ is detectable, for $i = 1, ..., N_s$;

then

a) the optimal value of $F_i = \begin{bmatrix} F_{ii} & F_{ic} \end{bmatrix}$ can be retrieved from

$$F_i = LQG\left(A_i, B_i, Q_i, R_{ii}, S_i\right),$$

b) and, in particular, F_{ii} is independently computed as

$$F_{ii} = LQG\left(A_{ii}, B_{ii}, Q_{ii}, R_{ii}, S_{ii}\right).$$

Proof. The two claims are hereby proved.

a) Recall that, by Theorem 3-3.7, by imposing $u_c(t) = F_{cc}x_c(t)$, our state-space system can be rewritten as follows,

$$x(t+1) = \begin{bmatrix} A_{11} & 0 & A_{1c} + B_{1c}F_{cc} \\ 0 & A_{22} & A_{2c} + B_{2c}F_{cc} \\ 0 & 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} x(t) + \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{bmatrix} v(t),$$

and the control synthesis problem separates into N_s independent subproblems, for which the separation property holds. For each subproblem, we are required to find the optimal state-feedback gain for a system of the form,

$$x_{i,pair}(t+1) = A_i x_{i,pair}(t) + B_i u_i(t) + M_i v_{i,pair}(t),$$

for $i = 1, ..., N_s$. The state-feedback gain of each system can be synthesized by classical LQG control theory, since by assumption (A_i, B_i) is stabilizable, $(A_{F,i}, L_{F,i})$ is detectable, $Q_i \geq 0$ and $R_{ii} > 0$, for $i = 1, \ldots, N_s$. These hypotheses imply that there is a unique solution to the DARE (see [49], Chapter 11)

$$P_i = A_i^T P_i A_i + Q_i - \left[A_i^T P_i B_i + S_i\right] \left[B_i^T P_i B_i + R_{ii}\right]^{-1} \left[A_i^T P_i B_i + S_i\right]^T, \tag{3-6}$$

for $i = 1, ..., N_s$, and that the optimal feedback gain F_i is given by

$$F_i = - \left[B_i^T P_i B_i + R_{ii}\right]^{-1} \left[A_i^T P_i B_i + S_i\right]^T, \tag{3-7}$$

for $i = 1, ..., N_s$.

b) By mere expansion of (3-6) and (3-7), it follows that

$$\begin{aligned} P_{ii} &= A_{ii}^T P_{ii} A_{ii} + Q_{ii} - \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right] \left[B_{ii} P_{ii} B_{ii} + R_{ii}\right]^{-1} \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right]^T \\ F_{ii} &= -\left[B_{ii} P_{ii} B_{ii} + R_{ii}\right]^{-1} \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right]^T, \end{aligned}$$

for $i = 1, ..., N_s$. Following from our notation,

$$F_{ii} = LQG\left(A_{ii}, B_{ii}, Q_{ii}, R_{ii}, S_{ii}\right).$$

This completes the proof. □

We have therefore indicated how to compute the optimal swallow gain F, given an arbitrary value of F_{cc}, by solving N_s subproblems. We will now formally indicate a procedure for the synthesis of the swallow F gain.

Feedback Gain Synthesis Procedure

As we saw, by fixing a value of F_{cc}, the problem decomposes into N_s subproblems. Each of these can be solved by the classical LQG control theory, and the remaining values of the F matrix are filled by optimal entries which depend on F_{cc}, as it was shown in Theorem 3-3.8. The cost function J can, in other words, be defined as a mere function of the gain F_{cc}, which, up to now, is chosen arbitrarily.

Within the stability region of the system, the cost function $J(F_{cc})$ is continuous in the elements of the matrix F_{cc} [15, 38]. As a consequence, we can minimize its value numerically, using the elements of F_{cc} as optimization domain. A schematic procedure for the synthesis of a (locally) optimal state-feedback gain F is given below.

Algorithm 3-3.9. *(Feedback Gain Synthesis Procedure)*

1) Fix an initial value for $F_{cc} \in \mathbb{R}^{m_c \times n_c}$ *such that* $(A_{cc} + B_{cc}F_{cc})$ *is asymptotically stable, being* m_c *the number of inputs of the coordinator and* n_c *the number of its states.*

2) Use a numerical optimization method to solve

$$F_{cc,opt} = \arg\inf_{F_{cc}} \; J(F_{cc})$$

3) Determine F_{ii}*,* F_{ic}*, for* $i = 1, \ldots, N_s$ *by use of Theorem 3-3.8, fixing* $F_{cc} = F_{cc,opt}$*.*

4) Construct the swallow state-feedback gain F*.*

Remark 3-3.10. Notice that, given any F_{cc} that stabilizes the coordinator, Theorem 3-3.8 provides the corresponding optimal stabilizing F. This is not claimed to be the globally optimal result.

The convexity of the numerical problem to be run in Step 2 of the algorithm is conjectured. If this convexity could be proven, the optimal solution to the problem would converge to the global minimum. We do not suggest a particular numerical method to solve this problem. Instead, we provide the reader with useful tools for choosing the algorithm that best fits the situation, by indicating a way to compute the symbolic expressions of the gradient and the Hessian of the cost function.

Gradient and Hessian of the Cost Function J

Finding the expression of the gradient of $J(F_{cc})$ is not a trivial issue. The cost function is given by

$$J(F_{cc}) = trace\left[W(F_{cc})\left(Q + SF(F_{cc}) + F^T(F_{cc})S^T + F^T(F_{cc})RF(F_{cc})\right)\right],$$

where

$$W(F_{cc}) = \begin{bmatrix} W_{11}(F_{cc}) & W_{12}(F_{cc}) & W_{1c}(F_{cc}) \\ W_{12}^T(F_{cc}) & W_{22}(F_{cc}) & W_{2c}(F_{cc}) \\ W_{1c}^T(F_{cc}) & W_{2c}^T(F_{cc}) & W_{cc}(F_{cc}) \end{bmatrix}; \quad F(F_{cc}) = \begin{bmatrix} F_{11} & 0 & F_{1c}(F_{cc}) \\ 0 & F_{22} & F_{2c}(F_{cc}) \\ 0 & 0 & F_{cc} \end{bmatrix}.$$

In fact, the terms W_{ii}, W_{ic}, W_{cc}, F_{ii} and F_{ic}, for $i = 1, \ldots, N_s$, all depend in a direct and/or indirect way on the parameter F_{cc}. As we have seen, these terms are computed through solutions of Riccati and Lyapunov equations. There is no known procedure for the analytical solutions to Riccati equations in their generic form, and numerical methods are usually applied to determine their solution. However, all the equations that are to be solved in order to find a symbolic expression of the cost function $J(F_{cc})$ can be reduced to the class of Sylvester's equations. This is due to the fact that each matrix equation is linear in the variable to be determined.

Definition 3-3.11. (Sylvester's Equations) A matrix equation is called *Sylvester's equation* if it is of the form

$$X = LXR + M,$$

with $X \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{m \times m}$, $R \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{m \times n}$.

The solution of such equations is treated in the following lemma.

Lemma 3-3.12. *(Solution Sylvester's Equations) The solution to the Sylvester's equation is given by*

$$X = -vec^{-1}\left([A - I]^{\dagger} B\right),$$

where $A = L \otimes R^T - I$; $B = vec(M)$. *Here,* $\otimes$ *indicates the Kronecker's product;* $\dagger$ *indicates the pseudo-inverse of the matrix; the vec operator builds a vector from the columns of its argument;* $vec^{-1}(V)$ *builds a matrix of appropriate dimensions by juxtaposition of columns taken by the vector* V.

Proof. The Sylvester's equation is a set of $m \times n$ linear equations in $m \times n$ variables. The solution to the equations comes straightforward from the explicitation of the expression $X - LXR + M = 0$. □

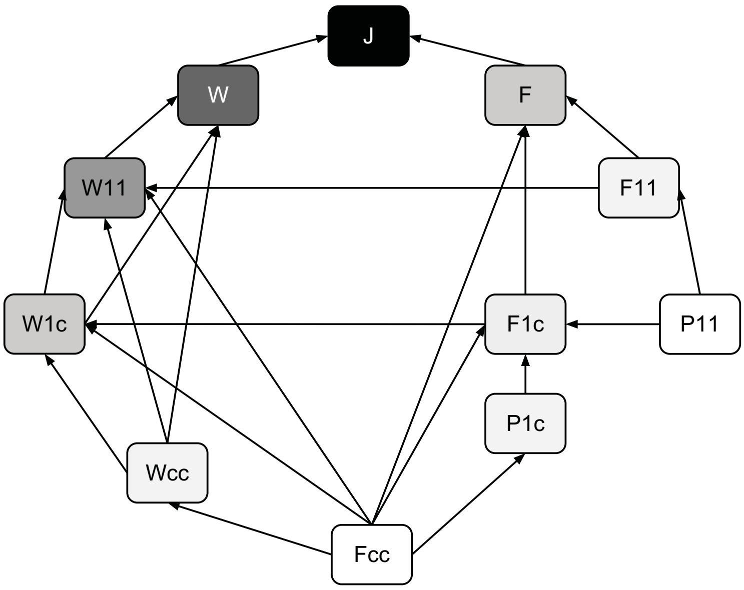

Figure 3-3: Variables interconnections and dependence on the parameter F_{cc}. The dark tonality of an element indicates the amount of previous (parallel) operations needed in order to compute its analytic expression.

In Figure 3-3, we sketch the structure of direct dependencies of all the variables playing a role in determining the cost function J. These variables appear in a nested structure. Starting from the computation of the bottom elements (in white), it is possible to arrive to an expression for the upper ones; the darker a block is, the more steps are required to get its analytic expression.

By the following procedure, it is possible to express the cost function $J(F_{cc})$, its gradient and its Hessian explicitly in terms of the parameter F_{cc}.

Algorithm 3-3.13. *(Analytic Expression of J, its Gradient and its Hessian in Terms of F_{cc})* *Let $F_{cc} \in \mathbb{R}^{m_c \times n_c}$ be given, where m_c is the number of inputs of the coordinator and n_c is the number of its states. Define*

$$\begin{aligned} A_{cc,cl}(F_{cc}) &= A_{cc} + B_{cc}F_{cc}; \\ A_{ic,cl}(F_{cc}) &= A_{ic} + B_{ii}F_{ic} + B_{ic}F_{cc}; \\ A_{ii,cl} &= A_{ii} + B_{ii}F_{ii}, \end{aligned}$$

for $i = 1, \ldots, N_s$.

1) *Solve the following Sylvester's equation for $W_{cc}(F_{cc})$ analytically,*

$$W_{cc}(F_{cc}) = \quad A_{cc,cl}(F_{cc}) W_{cc} A_{cc,cl}^T (F_{cc}) + M_{cc} V_{x,cc} M_{cc}^T,$$

being $W_{cc}(F_{cc})$ a rational function of the elements of F_{cc}.

2) *For $i = 1, \ldots, N_s$, do*

   2a) *compute the numerical solution to the Riccati equation*

$$P_{ii} = \quad A_{ii}^T P_{ii} A_{ii} + Q_{ii} - \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right] \left[B_{ii}^T P_{ii} B_{ii} + R_{ii}\right]^{-1} \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right]^T;$$

   note that P_{ii} does not depend on F_{cc};

   2b) *compute*

$$F_{ii} = \quad -\left[B_{ii}^T P_{ii} B_{ii} + R_{ii}\right]^{-1} \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right]^T \quad ;$$

   note that F_{ii} does not depend on F_{cc};

   2c) *solve the following Sylvester's equation for $P_{ic}(F_{cc})$ analytically,*

$$\begin{aligned} P_{ic}(F_{cc}) = \quad & \left\{A_{ii}^T - \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right] \left[B_{ii}^T P_{ii} B_{ii} + R_{ii}\right]^{-1} B_{ii}^T\right\} P_{ic} \left[A_{cc} + B_{cc}F_{cc}\right] + \\ & + Q_{ic} + \left\{A_{ii}^T P_{ii} \left[A_{ic} + B_{ic}F_{cc}\right]\right\} + \\ & - \left[A_{ii}^T P_{ii} B_{ii} + S_{ii}\right] \left[B_{ii}^T P_{ii} B_{ii} + R_{ii}\right]^{-1} \left\{B_{ii}^T P_{ii} \left[A_{ic} + B_{ic}F_{cc}\right]\right\}; \end{aligned}$$

   $P_{ic}(F_{cc})$is a rational function of the elements of F_{cc} because of the way it occurs in the equation;

   2d) *define the symbolic expression*

$$F_{ic}(F_{cc}) = \quad -\left[B_{ii} P_{ii} B_{ii} + R_{ii}\right]^{-1} \left\{B_{ii}^T P_{ii} \left[A_{ic} + B_{ic}F_{cc}\right] + B_{ii}^T P_{ic}(F_{cc}) \left[A_{cc} + B_{cc}F_{cc}\right]\right\};$$

   $F_{ic}(F_{cc})$ is a rational function of the elements of F_{cc} via the term F_{cc} itself and via the rational dependence on the term $P_{ic}(F_{cc})$;

2e) solve the following Sylvester's equation for $W_{ic}(F_{cc})$ analytically,

$$\begin{aligned} W_{ic}(F_{cc}) = \; & A_{ii,cl} W_{ic}(F_{cc}) A^T_{cc,cl}(F_{cc}) + A_{ic,cl}(F_{cc}) W_{cc}(F_{cc}) A^T_{cc,cl}(F_{cc}) + \\ & + M_{ii} V_{x,ic} M^T_{cc} + M_{ic} V_{x,cc} M^T_{cc}; \end{aligned}$$

$W_{ic}(F_{cc})$ is a rational function of the elements of F_{cc} because of its dependence on the terms $A_{ic,cl}(F_{cc})$ and $A_{cc,cl}(F_{cc})$, which are linear in F_{cc};

2f) solve the following Sylvester's equation for $W_{ii}(F_{cc})$ analytically,

$$\begin{aligned} W_{ii}(F_{cc}) = \; & A_{ii,cl} W_{ii}(F_{cc}) A^T_{ii,cl} + A_{ic,cl}(F_{cc}) W^T_{ic}(F_{cc}) A^T_{ii,cl} + \\ & + A_{ii,cl} W_{ic}(F_{cc}) A^T_{ic,cl}(F_{cc}) + A_{ic,cl}(F_{cc}) W_{cc}(F_{cc}) A^T_{ic,cl}(F_{cc}) + \\ & + M_{ii} V_{x,ii} M^T_{ii} + M_{ic} V^T_{x,ic} M^T_{ii} + M_{ii} V_{x,ic} M^T_{ic} + M_{ic} V_{x,cc} M^T_{ic}; \end{aligned}$$

$W_{ii}(F_{cc})$ is a rational function of the elements of F_{cc} because the equation depends on $A_{ic,cl}(F_{cc})$, $W_{ic}(F_{cc})$ and *$W_{cc}(F_{cc})$.*

3) Define

$$F(F_{cc}) = \begin{bmatrix} F_{11} & 0 & \cdots & 0 & F_{1c}(F_{cc}) \\ 0 & F_{22} & \cdots & 0 & F_{2c}(F_{cc}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & F_{N_sN_s} & F_{N_sc}(F_{cc}) \\ 0 & 0 & \cdots & 0 & F_{cc} \end{bmatrix};$$

$$W(F_{cc}) = \begin{bmatrix} W_{11}(F_{cc}) & 0 & \cdots & 0 & W_{1c}(F_{cc}) \\ 0 & W_{22}(F_{cc}) & \cdots & 0 & W_{2c}(F_{cc}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & W_{N_sN_s}(F_{cc}) & W_{N_sc}(F_{cc}) \\ W^T_{1c}(F_{cc}) & W^T_{2c}(F_{cc}) & \cdots & W^T_{N_sc}(F_{cc}) & W_{cc}(F_{cc}) \end{bmatrix};$$

the two matrices $F(F_{cc})$ and $W(F_{cc})$ are rational functions of the elements of F_{cc}.

4) Find the expression of the cost function,

$$J(F_{cc}) = trace\left\{W\left[Q + SF(F_{cc}) + F^T(F_{cc})S^T + F^T(F_{cc})RF(F_{cc})\right]\right\},$$

which is then a rational function of the elements of F_{cc}.

5) A symbolic expression of the gradient of $J(F_{cc})$ is obtained as

$$Grad(J(F_{cc})) = \frac{\partial J}{\partial F_{cc}},$$

where each element $\left(\frac{\partial J}{\partial F_{cc}}\right)_{i,j} = \frac{\partial J}{\partial F_{cc,ij}}$ for $i = 1, \ldots, N_s$. The gradient of $J(F_{cc})$ is a rational function of the elements of F_{cc}.

6) A symbolic expression of the Hessian of $J(F_{cc})$ is obtained as

$$Hess(J(F_{cc})) = \frac{\partial^2 J}{\partial F^2_{cc}}$$

where each element $\left(\frac{\partial J}{\partial F^2_{cc}}\right)_{i,j} = \frac{\partial J}{\partial F^2_{cc,ij}}$ for $i = 1, \ldots, N_s$. The Hessian of $J(F_{cc})$ is a rational function of the elements of F_{cc}.

Remark 3-3.14. The passages indicated are equivalent to those of the proof of Theorem 3-3.8. Note that the concatenation of interdependent equations introduces a symbolic complexity that translates into huge expressions for the gradient and the Hessian, even for the simplest cases.

Notice that no claim of convexity of $J(F_{cc})$ has been made. In fact, rational functions are not convex in general. However, this property will be conjectured. We now extend the results obtained for the state-feedback gain F to the synthesis of the observer gain K.

3-3-3 Observer Gain Synthesis

A control system based on state-feedback requires information regarding the states. When these states cannot be directly measured, a state-observer system is necessary to construct an estimate $\hat{x}(t)$ of the state by using the information coming from the system's inputs and outputs. The observer system requires a gain which multiplies the output error $(y(t) - \hat{y}(t))$, being $\hat{y}(t)$ the observer's expected output. This chapter treats the synthesis of this gain, K. Deeper insights into state-observers can be found, for instance, in [48].

In this section we build a solution of Problem 2-4.3 for decomposable problems. After having shown the validity of the separation property for CLSs once the parameters F_{cc} and K_{ii},for $i, \ldots, N_s$ are fixed, and after having indicated an expression for the synthesis of the swallow state-feedback gain F, we here define a similar procedure to obtain the swallow optimal observer gain K.

Although steps and approaches are basically the same as those made for F, proofs and algorithms needed to generate an adequate framework for the synthesis of K are intrinsically more complex, and the results are less straightforward. This is due to the fact that the observer influences the state-space dynamics and couples the disturbances acting on the outputs to those acting on the inputs.

The following steps will be taken.

1. We begin by showing how to compute the whole gain K exploiting the problem's decomposition once the parameters F_{cc} and K_{ii} are fixed.
2. The cost function $H(K_{11}, \ldots, K_{N_sN_s}, K_{cc}, F_{cc})$ is separated into N_s subcost functions. We define a procedure to minimize the subcost functions $H_i(F_{cc}, K_{ii})$, for $i = 1, \ldots, N_s$, also offering a way to obtain symbolic expressions for gradient and Hessian with respect to the parameter K_{ii}, for a fixed value of F_{cc}.
3. The swallow observer gain K is obtained by numerical optimization of the functions $H_i(F_{cc}, K_{ii})$, for $i = 1, \ldots, N_s$. The N_s related numerical optimization problems are conjectured to be convex.

Decomposition of the Observer Gain Synthesis Problem

Consider Problem 2-4.3. Here, we derive an approach to find the swallow observer gain K, given the parameters K_{ii}, for $i = 1, \ldots, N_s$ and F_{cc}, based on a problem decomposition.

Theorem 3-3.15. *(Problem Decomposition for Swallow Kalman Filter) Given a decomposable problem, let a CLS with N_s subsystems be given, and let its observer system to be*

$$\begin{cases} \hat{x}(t+1) = (A + KC)\,\hat{x}(t) + Bu(t) - Ky(t) \\ \hat{y}(t) = C\hat{x}(t) + Du(t), \end{cases}$$

where every matrix is assumed to be in the swallow form. Assume that K_{ii}, for $i = 1, \dots, N_s$ and F_{cc} are fixed. Define

$$\begin{aligned}
A_i &= \begin{bmatrix} A_{ii} + K_{ii}C_{ii} & A_{ic} + B_{1c}F_{cc} + K_{ii}C_{ic} \\ 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix}; & \overline{C} &= \begin{bmatrix} 0 & C_{cc} \end{bmatrix}; \\
M_i &= \begin{bmatrix} M_{ii} & M_{ic} & K_{ii}N_{ii} & K_{ii}N_{ic} \\ 0 & M_{cc} & 0 & 0 \end{bmatrix}; & \overline{N} &= \begin{bmatrix} 0 & 0 & 0 & N_{cc} \end{bmatrix}; \\
A_{cc,pcl} &= A_{cc} + B_{cc}F_{cc}; \\
V_{i,tot} &= \begin{bmatrix} V_{x,ii} & V_{x,ic} & V_{xy,ii} & V_{xy,ic} \\ V_{x,ic}^T & V_{x,cc} & V_{xy,ic}^T & V_{xy,cc} \\ V_{xy,ii}^T & V_{xy,ic} & V_{y,ii} & V_{y,ic} \\ V_{xy,ic}^T & V_{xy,cc}^T & V_{y,ic}^T & V_{y,cc} \end{bmatrix}; & v_{i,tot}(t) &= \begin{bmatrix} v_{x,i}(t) \\ v_{x,c}(t) \\ v_{y,i}(t) \\ v_{y,c}(t) \end{bmatrix}; \\
A_{K,i} &= A_i - M_iV_{tot}\overline{N}^T\left(\overline{N}V_{tot}\overline{N}^T\right)^{-1}\bar{C}; & K_i &= \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix}; \\
L_{K,i}L_{K,i}^T &= M_iV_{tot}M_i^T - M_iV_{tot}\overline{N}^T\left(\overline{N}V_{tot}\overline{N}^T\right)^{-1}\overline{N}V_{tot}M_i^T; & e_{i,tot}(t) &= \begin{bmatrix} e_i(t) \\ e_c(t) \end{bmatrix}.
\end{aligned} \tag{3-8}$$

If

1) $(A_{ii} + K_{ii}C_{ii})$ *is asymptotically stable for* $i = 1, ..., N_s$,
2) (A_{cc}, C_{cc}) *is a detectable pair and*
3) $(A_{K,i}, L_{K,i})$ *is a stabilizable pair for* $i = 1, ..., N_s$,

then,

a) *the optimal value of* $K_i = \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix}$ *is given by*

$$K_i = \left[LQG\left(A_i^T, \overline{C}^T, M_iV_{i,tot}M_i^T, \overline{N}V_{i,tot}\overline{N}^T, M_iV_{i,tot}\overline{N}^T\right)\right]^T,$$

b) *and, particularly, the gain K_{cc} is computed independently as*

$$K_{cc} = \left[LQG\left(A_{cc,pcl}^T, C_{cc}^T, M_{cc}V_{x,cc}M_{cc}^T, N_{cc}V_{y,cc}N_{cc}^T, M_{cc}V_{xy,cc}N_{cc}^T\right)\right]^T.$$

Proof. The two claims are proved hereby.

a) As a result of Theorem 3-3.7, fixing K_{ii}, for $i = 1, \ldots, N_s$ and F_{cc}, the control synthesis problem decomposes into N_s independent subproblems, for which the separation property holds. To each of these problems, the following error process is associated

$$\begin{aligned}\begin{bmatrix} e_i(t+1) \\ e_c(t+1) \end{bmatrix} = & \begin{bmatrix} A_{ii}+K_{ii}C_{ii} & A_{ic}+B_{ic}F_{cc}+K_{ii}C_{ic} \\ 0 & A_{cc}+B_{cc}F_{cc} \end{bmatrix}\begin{bmatrix} e_i(t) \\ e_c(t) \end{bmatrix} + \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix}\begin{bmatrix} 0 & C_{cc} \end{bmatrix}\begin{bmatrix} e_i(t) \\ e_c(t) \end{bmatrix} + \\ & + \begin{bmatrix} M_{ii} & M_{ic} \\ 0 & M_{cc} \end{bmatrix}\begin{bmatrix} v_{x,i}(t) \\ v_{x,c}(t) \end{bmatrix} + \begin{bmatrix} K_{ii}N_{ii} & K_{ii}N_{ic} \end{bmatrix}\begin{bmatrix} v_{y,i}(t) \\ v_{yc}(t) \end{bmatrix} + \\ & + \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix}\begin{bmatrix} 0 & N_{cc} \end{bmatrix}\begin{bmatrix} 0 \\ v_{yc}(t) \end{bmatrix},\end{aligned}$$

for $i = 1, \ldots, N_s$. Collecting the disturbance matrices, we obtain

$$\begin{aligned}\begin{bmatrix} e_i(t+1) \\ e_c(t+1) \end{bmatrix} = & \begin{bmatrix} A_{ii}+K_{ii}C_{ii} & A_{ic}+B_{ic}F_{cc}+K_{ii}C_{ic} \\ 0 & A_{cc}+B_{cc}F_{cc} \end{bmatrix}\begin{bmatrix} e_i(t) \\ e_c(t) \end{bmatrix} + \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix}\begin{bmatrix} 0 & C_{cc} \end{bmatrix}\begin{bmatrix} e_i(t) \\ e_c(t) \end{bmatrix} \\ & + \begin{bmatrix} M_{ii} & M_{ic} & K_{ii}N_{ii} & K_{ii}N_{ic} \\ 0 & M_{cc} & 0 & 0 \end{bmatrix}\begin{bmatrix} v_i(t) \\ v_c(t) \\ v_{yi}(t) \\ v_{yc}(t) \end{bmatrix} + \begin{bmatrix} K_{ic} \\ K_{cc} \end{bmatrix}\begin{bmatrix} N_{cc} \end{bmatrix}\begin{bmatrix} v_{yc}(t) \end{bmatrix}.\end{aligned}$$

This is equivalent to

$$e_{i,tot}(t+1) = \left(\bar{A}_i + \bar{K}_i\bar{C}\right) e_{i,tot}(t) + \left(\bar{M}_i + \bar{K}_i\bar{N}\right) v_{i,tot}(t).$$

Since, by assumption, $(A_{ii} + K_{ii}C_{ii})$ is asymptotically stable for $i = 1, ..., N_s$, $(A_{K,i}, L_{K,i})$ is a stabilizable pair for $i = 1, ..., N_s$, and (A_{cc}, C_{cc}) is detectable, the optimal gain K_i can be derived by the LQG classical theory as the solution of

$$\begin{aligned} X_i = \; & A_iX_iA_i^T + M_iV_{i,tot}M_i^T + \\ & - \left[A_iX_i\overline{C}^T + M_iV_{i,tot}\overline{N}^T\right]\left[\overline{C}X_i\overline{C}^T + \overline{N}V_{i,tot}\overline{N}^T\right]^{-1}\left[A_iX_i\overline{C}^T + M_iV_{i,tot}\overline{N}^T\right]^T; \\ K_i = \; & -\left[A_iX_i\overline{C}^T + M_iV_{i,tot}\overline{N}^T\right]\left[\overline{C}X_i\overline{C}^T + \overline{N}V_{i,tot}\overline{N}^T\right]^{-1}, \end{aligned} \tag{3-9}$$

for $i = 1, \ldots, N_s$.

b) By mere expansion of the above expressions, we find

$$\begin{aligned} X_{cc} = \; & A_{cc,pcl}X_{cc}A_{cc,pcl}^T + M_{cc}V_{x,cc}M_{cc}^T + \\ & - \left[A_{cc,pcl}X_{cc}C_{cc}^T + M_{cc}V_{xy,cc}N_{cc}^T\right]\left[C_{cc}X_{cc}C_{cc}^T + N_{cc}V_{y,cc}N_{cc}^T\right]^{-1} \times \\ & \times \left[A_{cc,pcl}X_{cc}C_{cc}^T + M_{cc}V_{xy,cc}N_{cc}^T\right]^T; \\ K_{cc} = \; & -\left[A_{cc,pcl}X_{cc}C_{cc}^T + M_{cc}V_{xy,cc}N_{cc}^T\right]\left[C_{cc}X_{cc}C_{cc}^T + N_{cc}V_{y,cc}N_{cc}^T\right]^{-1}, \end{aligned}$$

which, by our notation, is nothing but

$$K_{cc} = \left[LQG\left(A_{cc,pcl}^T, C_{cc}^T, M_{cc}V_{x,cc}M_{cc}^T, N_{cc}V_{y,cc}N_{cc}^T, M_{cc}V_{xy,cc}N_{cc}^T\right)\right]^T.$$

This completes the proof. □

Remark 3-3.16. The fact that the observer gain K_{cc} can be computed independently does not follow from the problem decomposition resulting from Theorem 3-3.7. This is an important remark, as the proof of Theorem 3-3.7 is based on the independence of K_{cc}.

Theorem 3-3.8, introduced in the previous section, allowed part of the entries of F to be computed given an arbitrary value, of F_{cc}. Similarly, Theorem 3-3.15 proves that the whole swallow gain K can be optimally computed given arbitrary values of K_{ii}, $i = 1, \ldots, N_s$ and F_{cc} itself. As the parameters F_{ii}, for $i = 1, \ldots, N_s$, could be computed independently from the rest of the gain, the same holds here for the local observer gain K_{cc} of the coordinator.

Observer Gain Synthesis Procedure

In order to construct an algorithm to synthesize a swallow (locally) optimal observer gain K, we define the following subcost function of the cost function H. Each subcost function H_i is associated to a subsystem i, and it only depends on the parameters K_{ii} and F_{cc}, for $i = 1, \ldots, N_s$.

Definition 3-3.17. (Explicit Subcost Function H_i) Given a decomposable problem, the subcost function of H related to the i^{th} subsystem is defined as

$$H_i(K_{ii}, F_{cc}) = trace\left[W_{e,ii}(K_{ii}, F_{cc})\right],$$

for $i = 1, \ldots, N_s$, where $W_{e,ii}(K_{ii}, F_{cc})$, for $i = 1, \ldots, N_s$, are given by eq. (3-5), and $K_{ic}(K_{ii}, F_{cc})$ and $K_{cc}(F_{cc})$ are computed as indicated in Theorem 3-3.15.

The result of Theorem 3-3.15 can be exploited to synthesize a locally optimal observer gain K by numerical optimization of the cost functions H_i over the variables K_{ii}, for $i = 1, \ldots, N_s$. A synthesis procedure for the observer gain K follows.

Algorithm 3-3.18. *(Observer Gain Synthesis Procedure)*

1) *Fix values for $K_{ii} \in \mathbb{R}^{n_i \times o_i}$ such that $(A_{ii} + K_{ii}C_{ii})$ is asymptotically stable, for $i = 1, \ldots, N_s$, being n_i the number of states of the i^{th} subsystem and o_i its the number of outputs.*
2) *Fix the optimal gain $F_{cc} = F_{cc,opt}$ as computed by Algorithm 3-3.9.*
3) *For $i = 1, \ldots, N_s$, solve*

$$K_{ii,opt} = \arg\min_{K_{ii}} \; H_i(K_{ii}, F_{cc}).$$

4) *For $i = 1, \ldots, N_s$, determine K_{ic} and K_{cc} applying Theorem 3-3.15, fixing $K_{ii} = K_{ii,opt}$.*
5) *Construct the swallow observer gain K .*

Remark 3-3.19. This algorithm is not claimed to produce the globally optimal result, but the convexity of the involved numerical optimization problem is conjectured (see Section 3-3-4).

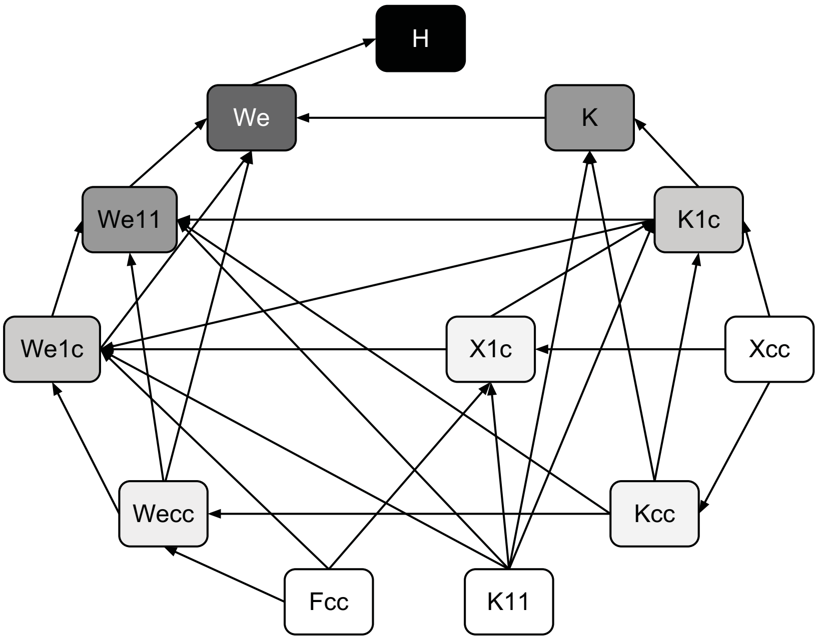

Figure 3-4: Variables interconnections and dependence on the parameter K_{ii} for the cost function H_i. The dark tonality of an element indicates the amount of previous (parallel) operations needed in order to compute its analytic expression.

Gradient and Hessian of the Cost Function H

The algorithm we have presented to determine $J(F_{cc})$ is here similarly reproduced to derive a symbolic expression for $H_i(K_{ii}, F_{cc})$, their gradients and the Hessians in terms of K_{ii}, for $i = 1, \ldots, N_s$. As for the previous case, we sketch in Figure 3-4 the interdependencies between the involved variables. This dependencies, although slightly more complex than that of Figure 3-3, also appear in a nested structure.

Algorithm 3-3.20. *(Analytic Expression for H_i, Gradient and Hessian in Terms of K_{ii}) Let $F_{cc} \in \mathbb{R}^{m_c \times n_c}$ and $K_{ii} \in \mathbb{R}^{n_i \times o_i}$, for $i = 1, \ldots, N_s$ be given, where n_i and n_c are the number of states of the i^{th} subsystem and the coordinator respectively; m_c is the number of inputs of the coordinator; o_i is the number of outputs of the i^{th} subsystem. For $i = 1, \ldots, N_s$, define the partially closed-loop and closed-loop state matrices*

$$\begin{aligned} & & A_{ii,cl}(K_{ii}) &= A_{ii} + K_{ii}C_{ii}; \\ A_{ic,pcl}(K_{ii}) &= A_{ic} + B_{1c}F_{cc} + K_{ii}C_{ic}; & A_{ic,cl}(K_{ii}) &= A_{ii} + B_{ic}F_{cc} + K_{ii}C_{ic} + K_{ic}C_{cc}; \\ A_{cc,pcl} &= A_{cc} + B_{cc}F_{cc}; & A_{cc,cl} &= A_{cc} + B_{cc}F_{cc} + K_{cc}C_{cc}. \end{aligned}$$

1) *Compute X_{cc} numerically from the Lyapunov equation*

$$\begin{aligned} X_{cc} = \; & A_{cc,pcl} X_{cc} A_{cc,pcl}^T + M_{cc} V_{x,cc} M_{cc}^T + \\ & - \left[A_{cc,pcl} X_{cc} C_{cc}^T\right] \left[C_{cc} X_{cc} C_{cc}^T + N_{cc} V_{y,cc} N_{cc}^T\right]^{-1} \left[A_{cc,pcl} X_{cc} C_{cc}^T\right]^T; \end{aligned}$$

note that this term does not depend on K_{ii}, $i = 1, \ldots, N_s$.

2) *Define* K_{cc} *as*

$$K_{cc} = -\left[A_{cc,pcl}X_{cc}C_{cc}^T\right]\left[C_{cc}X_{cc}C_{cc}^T + N_{cc}V_{y,cc}N_{cc}^T\right]^{-1};$$

note that this term does not depend on K_{ii}, $i = 1, \ldots, N_s$.

3) *solve the following Sylvester's equation for* $W_{e,cc}$ *analytically.*

$$\begin{aligned} W_{e,cc} = \;& A_{cc,cl}W_{e,cc}A_{cc,cl}^T + M_{cc}V_{x,cc}M_{cc}^T + M_{cc}V_{xy,cc}N_{cc}^TK_{cc}^T + \\ & + K_{cc}N_{cc}V_{xy,cc}^TM_{cc} + K_{cc}N_{cc}V_{y,cc}N_{cc}^TK_{cc}^T; \end{aligned}$$

note that this term does not depend on K_{ii}, $i = 1, \ldots, N_s$.

4) *For* $i = 1, \ldots, N_s$, *do*

4a) *solve the following Sylvester's equation for* $X_{ic}(K_{ii})$ *analytically,*

$$\begin{aligned} X_{ic}(K_{ii}) = \;& A_{ii,cl}(K_{ii})X_{ic}A_{cc,pcl}^T + A_{ic,pcl}(K_{ii})X_{cc}A_{cc,pcl}^T \quad + M_{ii}V_{x,ic}^TM_{cc}^T + M_{ic}V_{x,cc}M_{cc}^T \\ & + K_{ii}N_{ii}V_{y,ic}M_{cc}^T + K_{ii}N_{ic}V_{y,cc}M_{cc}^T + \\ & - \left[A_{ii,cl}(K_{ii})X_{ic}C_{cc} + A_{ic,pcl}(K_{ii})X_{cc}C_{cc} + M_{ii}V_{xy,ic}N_{cc}^T + \right. \\ & \left. M_{ic}V_{xy,cc}N_{cc}^T + K_{ii}N_{ii}V_{y,ic}N_{cc}^T + K_{ii}N_{ic}V_{y,cc}N_{cc}^T\right] \times \\ & \times \left[C_{cc}X_{cc}C_{cc}^T + N_{cc}V_{y,cc}N_{cc}^T\right]^{-1}\left[C_{cc}X_{cc}^TA_{cc,pcl}^T + N_{cc}V_{y,cc}M_{cc}^T\right]; \end{aligned}$$

notice that the term $X_{ic}(K_{ii})$ *is a rational function of the elements of* K_{ii} *because of its dependence from* $A_{ii,cl}(K_{ii})$, $A_{ic,pcl}(K_{ii})$, *and* K_{ii} *directly;*

4b) *determine* $K_{ic}(K_{ii})$,

$$\begin{aligned} K_{ic}(K_{ii}) = \;& -\left[A_{ii,cl}(K_{ii})X_{ic}(K_{ii})C_{cc} + A_{ic,pcl}(K_{ii})X_{cc}C_{cc} + M_{ii}V_{xy,ic}N_{cc}^T + M_{ic}V_{xy,cc}N_{cc}^T\right. \\ & \left. + K_{ii}N_{ii}V_{xy,ic}N_{cc}^T + K_{ii}N_{ic}V_{xy,cc}N_{cc}^T\right] \times \\ & \left[C_{cc}X_{cc}C_{cc}^T + N_{cc}V_{y,cc}N_{cc}^T\right]^{-1}. \end{aligned}$$

the term $K_{ic}(K_{ii})$ *is a rational function of the elements of* K_{ii} *because of its dependence from* $A_{ii,cl}(K_{ii})$, $A_{ic,pcl}(K_{ii})$ *and* K_{ii} *directly;*

4c) *solve the following Sylvester's equation for* $W_{e,ic}(K_{ii})$ *analytically,*

$$\begin{aligned} W_{e,ic}(K_{ii}) = \;& A_{ii,cl}(K_{ii})W_{e,ic}(K_{ii})A_{cc,cl}^T + A_{ic,cl}(K_{ii})W_{e,cc}A_{cc,cl}^T \\ & + M_{ii}V_{x,ic}M_{cc}^T + M_{ic}V_{x,cc}M_{cc}^T + M_{ii}V_{xy,ic}N_{cc}^TK_{cc}^T + M_{ic}V_{xy,cc}N_{cc}^TK_{cc}^T + \\ & + K_{ii}N_{ii}V_{xy,ic}M_{cc}^T + K_{ii}N_{ic}V_{xy,cc}M_{cc}^T + K_{ic}(K_{ii})N_{cc}V_{xy,cc}M_{cc}^T \\ & + K_{ii}N_{ii}V_{y,ic}N_{cc}K_{cc}^T + K_{ii}N_{ic}V_{y,cc}N_{cc}^TK_{cc}^T + K_{ic}(K_{ii})N_{cc}V_{y,cc}N_{cc}^TK_{cc}^T; \end{aligned}$$

the term $W_{e,ic}(K_{ii})$ *is a rational function of the elements of* K_{ii};

4d) *solve the following Sylvester's equation for* $W_{e,ii}(K_{ii})$ *analytically,*

$$\begin{aligned} W_{e,ii}(K_{ii}) = \;& A_{ii,cl}(K_{ii})W_{e,ii}(K_{ii})A_{ii,cl}^T(K_{ii}) + A_{ic,cl}(K_{ii})W_{e,ic}^T(K_{ii})A_{ii,cl}^T(K_{ii}) + \\ & + A_{ii,cl}(K_{ii})W_{e,ic}(K_{ii})A_{ic,cl}^T(K_{ii}) + A_{ic,cl}(K_{ii})W_{e,cc}A_{ic,cl}^T(K_{ii}) + \\ & + M_{ii}V_{x,ii}M_{ii}^T + M_{ii}V_{x,ic}^TM_{ic}^T + M_{ic}V_{x,cc}M_{ic}^T + \\ & + M_{ii}V_{xy,ii}N_{ii}^TK_{ii}^T + M_{ic}V_{xy,ic}^TN_{ii}^TK_{ii}^T + M_{ii}V_{xy,ic}N_{ic}^TK_{ii}^T + M_{ic}V_{xy,cc}N_{ic}^TK_{ii}^T + \\ & + M_{ii}V_{xy,ic}N_{cc}^TK_{ic}^T + M_{ic}V_{xy,cc}N_{cc}^TK_{ic}^T + K_{ii}N_{ii}V_{xy,ii}M_{ii}^T + K_{ii}N_{ic}V_{xy,ic}^TM_{ii}^T + \\ & + K_{ic}N_{cc}V_{xy,ic}M_{ii}^T + K_{ii}N_{ii}V_{xy,ic}M_{ic}^T + K_{ii}N_{ic}V_{xy,cc}M_{ic}^T + K_{ic}N_{cc}V_{xy,cc}M_{ic}^T + \\ & + K_{ii}N_{ii}V_{y,ii}N_{ii}^TK_{ii}^T + K_{ii}N_{ic}V_{y,ic}^TN_{ii}^T + K_{ic}N_{cc}V_{y,ic}^TN_{ii}^T + \\ & + K_{ii}N_{ii}V_{y,ic}N_{ic}^TK_{ii}^T + K_{ii}N_{ii}V_{y,ic}N_{cc}^TK_{ic}^T + K_{ii}N_{ic}V_{y,cc}N_{ic}^TK_{ii}^T + \\ & + K_{ic}N_{cc}V_{y,cc}N_{ic}^TK_{ii}^T + K_{ii}N_{ic}V_{y,cc}N_{cc}^TK_{ic}^T + K_{ic}N_{cc}V_{y,cc}N_{cc}^TK_{ic}^T; \end{aligned}$$

the term $W_{e,ii}(K_{ii})$ *is a rational function of the elements of* K_{ii};

5) For $i = 1, \ldots, N_s$, do

5a) determine the symbolic expression

$$H_i\left(K_{ii}, F_{cc}\right) = trace\left[W_{e,ii}\left(K_{ii}, F_{cc}\right)\right],$$

which results in a rational function of the elements of K_{ii};

5b) determine the gradient of $H_i\left(K_{ii}\right)$ as

$$Grad\left[H_i\left(K_{ii}, F_{cc}\right)\right] = \frac{\partial H_i\left(K_{ii}, F_{cc}\right)}{\partial K_{ii}},$$

where each element $\left(\frac{\partial H_i(K_{ii})}{\partial H_{ii}}\right)_{m,n} = \frac{\partial H_i(K_{ii})}{\partial H_{ii,mn}}$ for each m, n. The gradient of $H_i\left(K_{ii}\right)$ is a rational function of the elements of K_{ii}.

5c) Determine the Hessian of $H_i\left(K_{ii}\right)$ as

$$Hess\left[H_i\left(K_{ii}, F_{cc}\right)\right] = \frac{\partial^2 H_i\left(K_{ii}, F_{cc}\right)}{\partial K_{ii}^2},$$

where each element $\left(\frac{\partial^2 H_i(K_{ii})}{\partial K_{ii}^2}\right)_{m,n} = \frac{\partial^2 H_i(K_{ii})}{\partial K_{ii,mn}^2}$ for each m, n. The Hessian of $H_i\left(K_{ii}, F_{cc}\right)$ is a rational function of the elements of K_{ii}.

Remark 3-3.21. Recall that a procedure to solve Sylvester's equations analytically was given in Lemma 3-3.12. The validity of the algorithm can be confirmed by comparison with the proof of Theorem 3-3.15. As in the case of Algorithm 3-3.13, the concatenation of interdependent equations introduces a symbolic complexity that reflects into huge expressions for the gradient and the Hessian, even for the simplest cases.

With this we have concluded the solution to the control synthesis problem for decomposable systems. We will now make two important considerations about the results. The first one, in the next section, is about the convexity of the numerical minimization problems encountered in the solution process. Then, in Section 3-3-5, we will show how the constructed synthesis procedures can be decentralized.

3-3-4 About the Convexity of the Problems

Time and effort have been put on proving the convexity of the minimization problems

$$\inf_{F_{cc}} J\left(F_{cc}\right); \; \inf_{K_{ii}} H_i\left(K_{ii}, F_{cc}\right),$$

for $i = 1, \ldots, N_s$, with regards to the stability region of the closed-loop system. The reasons for which convexity is conjectured and some possible ways to prove it are discussed in this section.

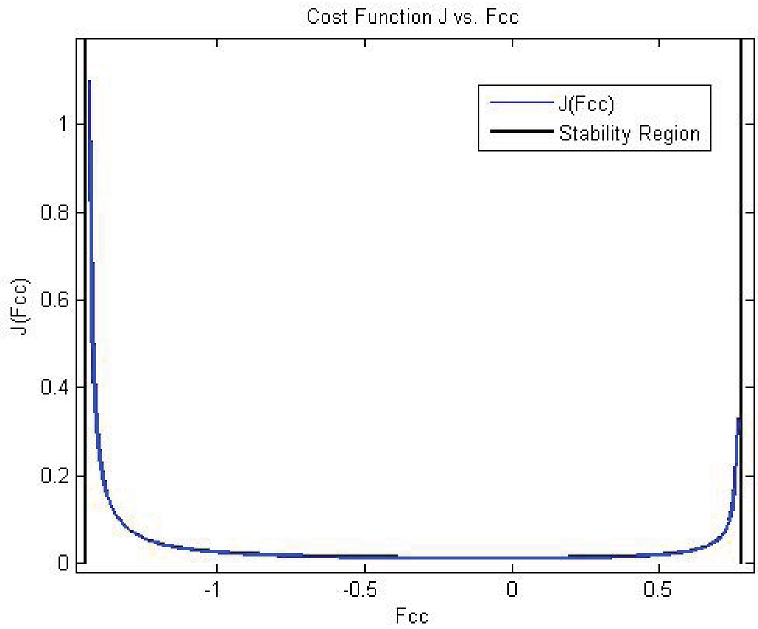

Figure 3-5: Typical shape of the cost function $J\left(F_{cc}\right)$ in the scalar case.

Conjecture of Convexity

Experimental evidence shows the convexity of the two problems. Three arguments stand behind this belief.

1. A typical plot of $J\left(F_{cc}\right)$ for the scalar case (every parameter is considered a scalar) is shown in Figure 3-5. Similar results were found for the n^{th}-order case, in which an arbitrary set of elements of F_{cc} were defined to be linearly dependent on a parameter p. The cost function $J\left(F_{cc}\left(p\right)\right)$ appeared to be convex in p.

2. Another study was conducted on the existence of more than one stability region for closed-loop systems of generic order. As we said, we in fact conjecture the convexity of the function on its stability region. The existence of more than one such regions would imply the incorrectness of the conjecture. This stability region appeared to be unique depending on parameters p_1, p_2 and p_3 in which the feedback gain $F_{cc}\left(p_1, p_2, p_3\right)$ was defined linearly. No more than three parameters per time were considered because of graphical representation reasons. The result of this experimental research never showed the existence of more than one stability region.

3. Numerical optimization methods always seem to converge to one global minimum. Table 3-1 summarizes some experimental results of numerical optimization starting from random, stabilizing initial values of F_{cc}, for randomly generated systems with increasing number of optimization variables n_v. It appears that, increasing the number of variables, the probability of failing to find the minimum of the function increases. However,

it seems that this failure is not due to the presence of more than one minimum, but to the numerical errors. In fact, by decreasing the tolerance of the optimization algorithm, the rate of convergence of the algorithm increases.

MATLAB *fmincon*	$n_v = 1$	$n_v = 2$	$n_v = 6$	$n_v = 15$	$n_v = 25$	$n_v = 50$
TolX= 10^{-5}	100%	100%	100%	86%	78%	60%
TolX= 10^{-10}	100%	100%	100%	92%	88%	76%
TolX= 10^{-15}	100%	100%	100%	96%	90%	82%

Table 3-1: Convergence rate of numerical optimizations using the MATLAB function *fmincon*. n_v is the number of optimization variables, given either by the elements of the F_{cc} gain for the minimization of J, or by the elements of K_{ii} for the minimization of H_i, for any $i \in [1, N_c]$; TolX is the tolerance on the minimal variation on a variable allowed in the numerical optimization. 50 randomly generated systems are considered for each case.

For the above reasons, the following conjecture is formulated.

Conjecture 3-3.22. *(Convexity of the Optimization Problems) The optimization problems*

$$\min_{F_{cc}} J\left(F_{cc}\right) \text{ and } \min_{K_{ii}} H_i\left(K_{ii}, F_{cc}\right),$$

for $i = 1, \ldots, N_s$, *are convex in the set of stabilizing values of the parameters* F_{cc} *and* K_{ii} *respectively.*

By the results formulated by Ho and Chu [24], the optimal control law for systems with nested information structure is linear. Thus, the convexity of this problem is conjectured to guarantee that the numerical computation of the globally optimal control law converges eventually.

About Numerical Errors

The numerical optimization problems to be solved for decomposable problems in Algorithm 3-3.9 and Algorithm 3-3.18 do not always converge to an optimal result. This failure could be attributed to two factors: the non-convexity of the function to be minimized, or to numerical errors. In turn, the numerical errors could be due to the following two different factors, whose occurrence is established by experimental evidence.

1. *Flat cost functions.* In Table 3-1, some numerical optimization attempts are summarized. There, the impression is given that the more optimization variables there are, the higher is the rate of failure in finding the optimal result. Because of space reasons, we do not provide with all the results obtained for numerical experiments. However, in multi-start optimizations, most of the converging results did not match perfectly, but were slightly different from each other. In the table, these results are still classified as successful. Different experiments showed how both the precision of the convergent results and the number of totally different results considerably decreased by improving

the tolerance parameters of the minimization algorithm (MATLAB's *fmincon*). This is explained by the presence of "flat" areas in the cost functions, where, therefore, the derivative computed on one or more variables of the minimization domain is almost zero. The "flatness" of the cost function can be observed in the typical plot produced in Figure 3-5.

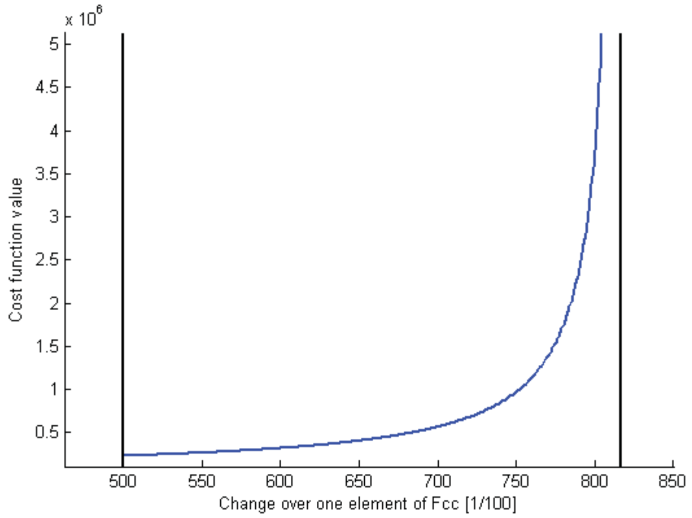

Figure 3-6: Example of "broken" cost function. The black lines determine the boundaries of the stability region. The minimum of the function with the stability region lies at the boundary just inside that region.

2. *Broken cost functions.* Not always the global minimum of the cost function lies in the stability region of the system. Often, the convex shape of the cost function is "broken", as it is shown in Figure 3-6. Line-search optimization methods encounter problems with this kind of functions. The result they seek is in fact outside the acceptable region, as we require the closed-loop system to be stable. Two solutions can be given to this problem. Both of them introducing other issues. The first one is to include a non-linear stability constraint, imposing the maximum eigenvalue of the closed-loop system to be located in the unitary circle. However, as we show in Figure 3-7, this is, in general, a non-convex constraint. By its implementation, the optimization problem would not be convex anymore, as to be convex, a constrained optimization problem requires the inequality constraint to be convex [10]. A second solution is that to include a barrier function, whose value increases as the maximum eigenvalues of the system approaches the instability boundaries. An intrinsic limitation of this approach is that the result obtained would not represent the global minimum of the cost function anymore, as it minimizes the sum of the cost function with the barrier function. Although the two solutions proposed have some limitations, they both worked well in experiments. In particular, the constrained minimization is a suggested solution.

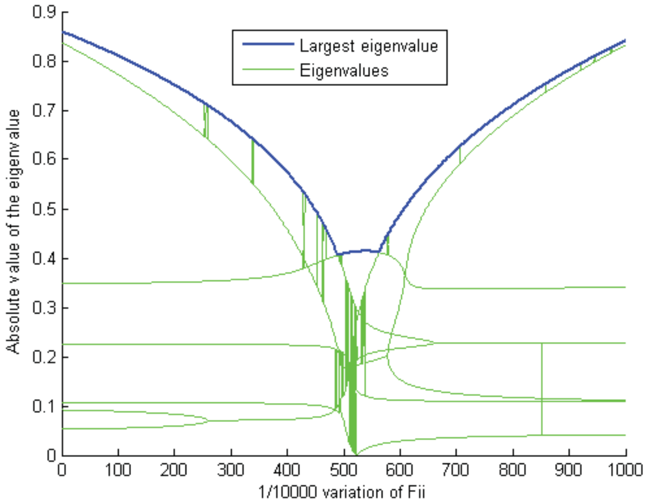

Figure 3-7: Non-convexity of the stability constraint. The absolute value of all the eigenvalues of a 10th order closed-loop system are plotted as a function of one (arbitrary) element of the parameter F. In this example we can observe that the maximum-eigenvalue function presents two local minima.

Directions for a Proof

We indicate three ways to prove the convexity of the two problems.

1. Show that the Hessian of the cost functions is positive-definite. A procedure to obtain a symbolic expression for the Hessians was given in Algorithm 3-3.13 and Algorithm 3-3.20.

2. Show that the subgradient inequality holds for both the cost functions (for a deeper insight on the subgradient inequality, see, for example, [10, 34]). A procedure to obtain a symbolic expression of the (sub)gradients was also given in Algorithm 3-3.13 and Algorithm 3-3.20.

3. The symbolic expression of each cost function is found by computing a nested set of expressions and solutions to Sylvester's equations. Figures 3-3 and 3-4 show how the cost functions depend on the optimization variables. As a convex function of a function which is convex in a parameter, is also convex in that parameter [10], the convexity of the cost function could be proved recursively.

Although efforts have been spent on these three directions, no results were achieved. The main problems with proving the convexity via the gradients or Hessians of the cost functions is the huge dimension of these expressions. Even for the simplest scalar case, if the Hessian of $J\left(F_{cc}\right)$ was to be reported on this document, it would occupy more than 20 pages, and the gradient more than 5. Operations with these expressions are hard to handle. Moreover, the algorithms we provide allow to compute gradient and Hessian for particular cases. No general results for every CLS and problem could be inferred from eventual proofs in particular cases.

The third way, which explores the elements of the functions step-by-step recursively, is more likely to produce a general result. We suggest this way to be pursued.

3-3-5 Decentralization of the Control Synthesis

A surprising result which was not sought, but that is very welcome, is treated hereby. The information constraints imposed to the control system can also be reflected to the control synthesis. We here explain how the synthesis of F and K can be done by allowing communication only from the coordinator to the subsystems, as the nested information pattern of the control system requires.

One of the main consequences of the gain synthesis procedures developed for decomposable problems is their decentralizability. In fact, the procedure of finding F and K can be done as indicated below.

Algorithm 3-3.23. *(Decentralized Control Synthesis) Let a decomposable problem be given.*

1) *At the level of the coordinator, run Algorithm 3-3.9 to find the gain F.*
2) *At the level of the coordinator, find the local observer gain K_{cc} as it is indicated in Theorem 3-3.15.*
3) *The coordinator communicates the gains F_{ii}, F_{ic} and F_{cc} to each subsystem i, for $i = 1, \ldots, N_s$.*
4) *Each subsystem i, for $i = 1 \ldots, N_s$, computes the observer's gains K_{ii} and K_{ic} using Algorithm 3-3.18, fixing the value of F_{cc} obtained from the coordinator.*

Remark 3-3.24. The validity of the algorithm follows from Theorem 3-3.1 and 3-3.15.

The decentralization of the computation is sketched in Figure 3-8.

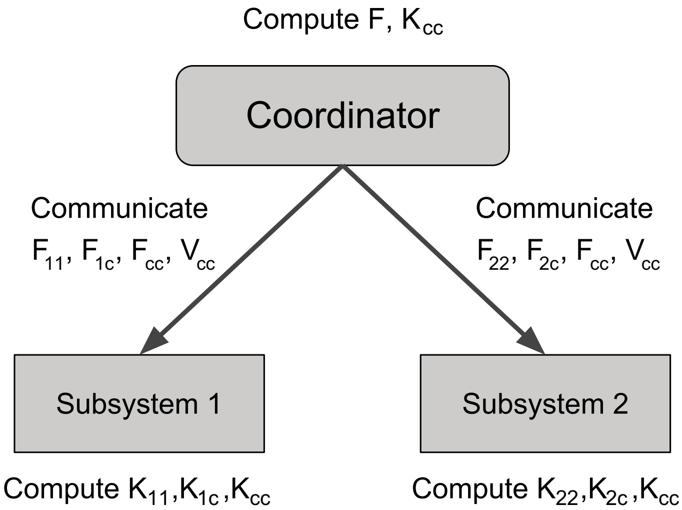

Figure 3-8: Decentralization of the control synthesis procedure.

While F depends on the selection of weighing matrices, K is merely determined by the variance of external disturbances. If these variances change at the coordinator's level, a new K_{cc} can be computed locally, and the data regarding the variance can be sent to the subsystem, which will carry a new optimization to find the new optimal values of K_{ii} and K_{ic}.

Note that if a change of variance is detected at the level of a subsystem, a new optimization is simply run locally to recompute the local observer's gains, without affecting the whole control system.

3-4 Virtual Coordination Problems

In this section we treat the problem of LQG coordination control for the case of *virtual coordination problems.*

Problem 3-4.1. (Virtual Coordination Problem Formulation) Solve problem 2-4.4 given the weighing matrices

$$Q_{tot} = Q_{tot}^T = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0; \qquad Q = Q^T = \begin{bmatrix} Q_{11} & 0 & Q_{1c} \\ 0 & Q_{22} & Q_{2c} \\ Q_{1c}^T & Q_{2c}^T & Q_{cc} \end{bmatrix};$$
$$S = \begin{bmatrix} S_{11} & 0 & S_{1c} \\ 0 & S_{22} & S_{2c} \\ 0 & 0 & S_{cc} \end{bmatrix}, \quad R = R^T = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{cc} \end{bmatrix} > 0.$$

and a disturbance vector $v_{tot}(t) = \left[v_x^T(t), v_y^T(t)\right]^T$ being a Gaussian zero-average noise with covariance matrices

$$V_{tot} = V_{tot}^T \quad \begin{bmatrix} V_x & V_{xy} \\ V_{xy}^T & V_y \end{bmatrix} \geq 0; \qquad V_x = V_x^T = \begin{bmatrix} V_{x,11} & 0 & 0 \\ 0 & V_{x,22} & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$V_{xy} = \begin{bmatrix} V_{xy,11} & 0 & 0 \\ 0 & V_{xy,22} & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad V_y = V_y^T = \begin{bmatrix} V_{y,11} & 0 & 0 \\ 0 & V_{y,22} & 0 \\ 0 & 0 & V_{y,cc} \end{bmatrix} > 0.$$

For this class of problems, we depict the coordinator as a virtual entity, whose state is unaffected by disturbances. Ideally, this class of problems means to represent computer-controlled distributed systems.

Notice that virtual coordination problems are a particular case of decomposable problems (see Section 3-3-1 for comparison). The results obtained for decomposable problems therefore naturally extend to virtual coordination problems.

The main issue for finding optimal control policies in decomposable problems was the fact that disturbances acting on the coordinator propagate into the subsystems both through the inputs and the states. Since this issue is absent in virtual coordination problems, the solution to the LQG coordination control becomes easier. In fact, we show that this matches with the solution of classical LQG control applied to each subsystem independently.

We follow the usual steps to solve the LQG coordination control problem: we start by a consideration on the separation property; then, we find the optimal swallow gains F and K separately.

3-4-1 Separation Property

We recall that we are basically dealing with a decomposable problem applied to a special class of systems. The decomposition of the control synthesis problem into N_s subproblems for which the separation property holds were proved in Theorem 3-3.7 for decomposable problems. This result holds for virtual coordination control problem as well.

The results related to the synthesis of the two gains are heavily improved for this class of systems. We show this in the following two sections.

3-4-2 Decentralization of the State-Feedback Gain Synthesis

By the following proposition, we show that the LQG coordination control problem of finding an optimal state-feedback gain in the swallow form reduces to the synthesis of local LQG control problems.

Proposition 3-4.2. *(State-Feedback Gain Synthesis for Virtual Coordination Problems) Let a virtual coordination problem be given. Define*

$$L_{F,ii}L_{F,ii}^T = Q_{ii} - S_{ii}R_{ii}^{-1}S_{ii}^T; \;\; A_{F,ii} = A_{ii} - B_{ii}R_{ii}^{-1}S_{ii}^T,$$

for $i = 1, \ldots, N_s$. *If*

1) F_{cc} is fixed arbitrarily such that $(A_{cc} + B_{cc}F_{cc})$ is asymptotically stable;

2) (A_{ii}, B_{ii}) is stabilizable, for $i = 1, ..., N$;

3) $(A_{F,ii}, L_{F,ii})$ is detectable, for $i = 1, ..., N$;

then,

a) for $i = 1, \ldots, N_s$ the optimal feedback gains F_{ii} are determined as

$$F_{ii} = LQG\left(A_{ii}, B_{ii}, Q_{ii}, R_{ii}, S_{ii}\right),$$

b) the rest of the gains have no influence on the optimality of the closed-loop system and can be chosen arbitrarily, as long as the closed-loop system is stable and F is in the swallow form,

$$F = \begin{bmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \\ 0 & 0 & F_{cc} \end{bmatrix}.$$

Proof. a) This follows from Theorem 3-3.8.

b) To show that the gains F_{ic}, $i = 1, \ldots, N_s$ and F_{cc} can be chosen arbitrarily, we proceed as follows. In a virtual coordination problem, we have $V_{x,cc} = 0$, and $V_{x,ic} = 0$ for $i = 1, \ldots, N_s$. Substituting these values in the closed-loop steady-state variance of the state W, we obtain

$$W_{ii} = \left(A_{ii} + B_{ii}F_{ii}\right)W_{ii}\left(A_{ii} + B_{ii}F_{ii}\right)^T + M_{ii}V_{ii}M_{ii}^T,$$

and $W_{ic} = 0$; $W_{cc} = 0$. The cost function becomes

$$J = \sum_{i=1}^{N_s} trace\left[W_{ii}\left(Q_{ii} + S_{ii}F_{ii} + F_{ii}^T S_{ii}^T + F_{ii}^T R_{ii}F_{ii}\right)\right].$$

Since the cost function only depends on F_{ii}, $i = 1, \ldots, N_s$, all the other gains are neutral to its optimality. Hence, they can be chosen arbitrarily.

This completes the proof. □

3-4-3 Decentralization of the Observer Gain Synthesis

We here show the dual result regarding the optimal swallow observer gain.

Proposition 3-4.3. *(Observer Gain Synthesis for Virtual Coordination Problems) Let a virtual coordination problem be given. Define*

$$\begin{aligned} A_{K,ii} = &\; A_{ii} - M_{ii}V_{xy,ii}N_{ii}^T\left(N_{ii}V_{y,ii}N_{ii}^T\right)^{-1}C_{ii}; \\ L_{K,ii}L_{K,ii}^T = &\; M_{ii}V_{x,ii}M_{ii}^T - M_{ii}V_{xy,ii}N_{ii}^T\left(N_{ii}V_{y,ii}N_{ii}^T\right)^{-1}N_{ii}V_{xy,ii}^T M_{ii}^T, \end{aligned}$$

for $i = 1, \ldots, N_s$. If

1) F_{cc} is fixed such that $(A_{cc} + B_{cc}F_{cc})$ is asymptotically stable;

2) (A_{ii}, C_{ii}), $i = 1, \dots, N_s$ are detectable pairs;

3) $(A_{K,ii}, L_{K,i})$, $i = 1, \dots, N_s$ are stabilizable pairs;

then,

a) for $i = 1, \dots N_s$, the optimal observer gain K_{ii} is given by

$$K_{ii} = \left[LQG\left(A_{ii}^T, C_{ii}^T, M_{ii}V_{x,ii}M_{ii}^T, N_{ii}V_{y,ii}N_{ii}^T, M_{ii}V_{xy,ii}N_{ii}^T\right)\right]^T,$$

b) the rest of the gains have no influence on the optimality of the closed-loop system and can be chosen arbitrarily, as long as the closed-loop system is stable and K is in the swallow form,

$$K = \begin{bmatrix} K_{11} & 0 & K_{1c} \\ 0 & K_{22} & K_{2c} \\ 0 & 0 & K_{cc} \end{bmatrix}.$$

Proof. a) Let us consider the algebraic Riccati equation leading to the Kalman gain in its general form,

$$\begin{aligned} X = \; & A_{pcl}X\bar{A}^T + MV_xM^T + \\ & - \left[A_{pcl}XC^T + MV_{xy}N^T\right]\left[CXC^T + NV_yN^T\right]^{-1}\left[A_{pcl}XC^T + MV_{xy}N^T\right]^T, \end{aligned}$$

where

$$A_{pcl} = \begin{bmatrix} A_{11} & 0 & A_{1c} + B_{1c}F_{cc} \\ 0 & A_{22} & A_{2c} + B_{2c}F_{cc} \\ 0 & 0 & A_{cc} + B_{cc}F_{cc} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{pcl,1c} \\ 0 & A_{22} & A_{pcl,2c} \\ 0 & 0 & A_{pcl,cc} \end{bmatrix}.$$

Following the assumptions, there exists a unique $X = X^T \geq 0$ that solves the above equation. We focus at first our attention on X_{cc}, where substituting $V_{x,cc} = 0$ and $V_{xy,cc} = 0$, we have

$$X_{cc} = A_{pcl,cc}X_{cc}A_{pcl,cc}^T - A_{pcl,cc}X_{cc}C_{cc}^T\left[C_{cc}X_{cc}C_{cc}^T + N_{cc}V_{y,cc}N_{cc}^T\right]^{-1}C_{cc}X_{cc}A_{pcl,cc}.$$

The unique solution to the equation is

$$X_{cc} = 0.$$

After substituting the known values of the variables, the term X_{ic} is given by

$$\begin{aligned} X_{ic} = \; & A_{ii}X_{ic}A_{pcl,cc}^T - \begin{bmatrix} A_{ii}X_{ii}C_{ii}^T + \bar{A}_{ic}X_{ic}^TC_{ii}^T + A_{ii}X_{ic}C_{ic}^T & 0 \end{bmatrix} \times \\ \times & \begin{bmatrix} C_{ii}X_{ii}C_{ii}^T + C_{ic}X_{ic}^TC_{ii}^T + C_{ii}X_{ic}C_{ic}^T + N_{ii}V_iN_{ii}^T & C_{ii}X_{ic}C_{cc}^T + C_{ic}X_{cc}C_{cc}^T + N_{ic}V_cN_{ic}^T \\ C_{cc}X_{ic}^TC_{ii}^T + N_{ic}^TV_cN_{ic} & N_{cc}^TV_cN_{cc} \end{bmatrix}^{-1} \times \\ \times & \begin{bmatrix} C_{ii}X_{ic}A_{pcl,cc}^T \\ 0 \end{bmatrix}, \end{aligned}$$

Thus, the only solution to the equation is

$$X_{ic} = 0,$$

for $i = 1, \ldots, N_s$. Substituting the two solutions $X_{cc} = 0$ and $X_{ic} = 0$, for $i = 1, \ldots, N_s$, in the algebraic Riccati equation for X_{ii}, we finally obtain

$$X_{ii} = A_{ii}X_{ii}A_{ii}^T + M_{ii}V_{x,ii}M_{ii}^T - \left[A_{ii}X_{ii}C_{ii}^T\right]\left[C_{ii}X_{ii}C_{ii}^T + N_{ii}V_{y,ii}N_{ii}^T\right]^{-1}\left[A_{ii}X_{ii}C_{ii}^T\right]^T,$$

for $i = 1 \ldots N_s$. Following from the LQG optimal control theory, the optimal observer gains K_{ii}, for $i = 1, \ldots, N_s$, are given by

$$K_{ii} = \ -A_{ii}X_{ii}C_{ii}^T\left[C_{ii}X_{ii}C_{ii}^T + N_{ii}V_{y,ii}N_{ii}^T\right]^{-1}.$$

b) To show that K_{cc} and K_{ic}, for $i = 1, \ldots, N_s$, can be chosen arbitrarily, we simply notice that these terms do not appear in the cost function to be minimized,

$$H = trace\left(W_e\right),$$

as, after substituting the known values of the covariance matrices, this becomes

$$\begin{aligned} H = \ & \sum_{i=1}^{N_s} trace\Big\{\left(A_{ii} + K_{ii}C_{ii}\right)W_{e,ii}\left(A_{ii} + K_{ii}C_{ii}\right)^T + M_{ii}V_{x,ii}M_{ii}^T + \\ & + M_{ii}V_{xy,ii}N_{ii}^TK_{ii}^T + K_{ii}N_{ii}V_{xy,ii}^TM_{ii}^T + K_{ii}N_{ii}V_{y,ii}N_{ii}^TK_{ii}^T\Big\}. \end{aligned}$$

This completes the proof. □

No ulterior research is necessary for the class of virtual coordination problems. The results of this section exhaust the solution to Problem 3-4.1.

3-5 Comments

Some general comments regarding the theory of LQG coordination control we developed in this chapter follow.

Off-line Computations For problems of known linear, time-invariant models and disturbance covariance matrix, the LQG coordination control synthesis can be run off-line and directly implemented into the CLS.

Transformations of the State-Space State-space transformations are useful to formulate control problems. Intuitively, any transformation leading to a state-space system composed of swallow matrices can be performed. Following from the conservation properties of the swallow matrix, any swallow transformation matrix T is therefore allowed for this purpose.

Separation Principle for General Problems We have stated that the biggest unsolved issue about general problems is the inapplicability of the separation principle. In Section 3-3 we have seen how, for decomposable problems, the separation property of the system is proved by allowing the gains F_{cc} and K_{ii}, for $i = 1, \ldots, N_s$ to be fixed. The same reasoning cannot be applied to general problems. In fact, since the weighing matrices and the covariance matrices are not limited to any structure, the cost functions cannot be decomposed in the same way as we have done for decomposable problems, and consequently the separation property does not hold.

Considerations on Block-Diagonal Gains By the results of Theorem 3-3.8 and Theorem 3-3.15, we notice that if we restrict our attention to block-diagonal gains F and K only, instead of the more general swallow matrix structure, the control synthesis procedure can be run in a much faster way (experimentally, it reveals to be about 6 times faster). In fact, no computations of LQG optimal gains would be required at each step, as the only gains to be found would be F_{ii}, for $i = 1, \ldots, N_s$ and K_{cc}, which can be computed independently, only once.

Comparison with Direct Numerical Approaches The largest part of this chapter is dedicated to the research of (locally) optimal results, either given a fixed parameter (as it is the case of decomposable problems) either in a more direct way (as it is the case for virtual coordination problems). An alternative, less thoughtful way to pursue the same result could be given by a numerical optimization over all the elements of the swallow gains. This approach was implemented by the author, and its results were used as a comparison to the results obtained through the proposed procedures. In Table 3-2 we compare the number of variables that would compose the optimization domain between the implementation of our approaches and that of direct line-search applied to the matrices as a whole. The direct numerical approach over the whole matrix not only proved to be way slower than our approach, but also hardly converged to the minimum because of numerical problems due to the larger number of variables.

Number of optimization variables	Finding F	Finding K
Direct line-search method	$\sum_{i=1}^{N_s} m_i (n_i + n_c) + m_c n_c$	$\sum_{i=1}^{N_s} n_i (o_i + o_c) + n_c o_c$
Decomposable Problems	$m_c n_c$	$\sum_{i=1}^{N_s} n_i o_i$
Virtual Coord. Problems	0	0
General Problems	Not Applicable	Not Applicable

Table 3-2: Comparison of the number of optimization variables between a direct line-search method and those proposed in this thesis. The parameters n_i, m_i and o_i represent the number of states, inputs, and outputs of each subsystem i and the coordinator respectively, for $i = 1, \ldots, N_s, c$. For virtual coordination problems no numerical optimization is required. No method is applicable to the general problems, as for this class of problems the separation property was not proved to hold.

Computational Time For decomposable problems, we offer a control synthesis that passes through a numerical optimization process. It may happen that the control synthesis is to be done while the controlled system is running. In these cases, it is necessary to know how much time the synthesis would require. Roughly, if the synthesis was run in a decentralized way following the instructions of Algorithm 3-3.23, for problems with about 15 optimization variables, running the synthesis on MATLAB, on a laptop of today's average computational power, the whole synthesis would require about 5 seconds. This value has to be considered as a mere estimate. Several parameters influence in fact the computational time. To mention some: the type and power of the processing unit, the number of optimization variables (over which the computations appear to require exponential time), the distance from the optimal value with respect to the starting point of the optimization, the communication delay of the information transmitted from the coordinator to the subsystems.

3-6 Conclusions

Throughout the chapter, we studied approaches to solve the LQG coordination control problem as it was formulated in chapter 2. We differentiated the results by following three different problem categories: the general problem, the decomposable problem, and the virtual coordination problem. The differences in the formulations of these problems lie in the assumptions made for the structures of the weighing matrices Q and R and on the covariance matrix of the disturbances acting on the system, V.

For the general problem, discussed in Section 3-2 where no limitations are given to the structure of these matrices, no results have been found. The main difficulty to be overcome for this class of problems is the applicability of the separation principle.

For decomposable problems, important results were found were found in Section 3-3. To begin, the separation property was proved to hold if part of the parameters to be found were fixed (F_{cc} and K_{ii}, $i = 1, \ldots, N_s$). After this result, a new approach was developed to obtain the optimal state-feedback gain and the optimal observer gain separately, starting from a problem decomposition. To find these two gains, a decentralizable technique has been developed. The optimization domain of the numerical problems to be solved is given by the elements of the parameters F_{cc} and K_{ii}, $i = 1, \ldots, N_s$. The convexity of these problems was conjectured. A procedure to obtain the analytical expression of the gradients and the Hessians of the cost functions to be minimized, based on the solution of Sylvester's equations, was also indicated. Since the two gains obtained by the procedure are in the swallow form, they comply with the nested information pattern restriction imposed.

In the end, in Section 3-4, special results have been achieved for the category of virtual coordination problems. For this class of problems, it was shown that the optimal controller can be obtained with no need of numerical optimization. The optimal gains are in fact proved to coincide with those locally obtained by the classical LQG control theory.

All the results presented in this section have been implemented in MATLAB. Interested readers may request the source codes to the author.

Chapter 4

Coordination Control of AUVs

In this chapter we present a case study. The LQG coordination control theory developed in this thesis is applied to control the position of a group of autonomous underwater vehicles (AUVs). The adopted approach is based on a previous paper by Kempker [27], where an LQ-optimal procedure was used to control a formation of AUVs. Objective of the control system is to guarantee a number of AUVs to track an external reference. The control system also has to take into account the possibility of AUVs to move in formation. The design of such control system is a straightforward application of the LQG coordination control theory. In particular, it follows from the results obtained for virtual coordination problems and those for the more generic class of decomposable problems.

The chapter is organized as follows. Section 4-1 introduces the importance of AUVs and the difficulties related to the relative control system to be synthesized; in Section 4-2, we indicate the approach used to control the vehicles; in Section 4-3 we formulate this problem in a mathematical way, while Section 4-4 solves the problem by implementation of the LQG coordination control theory. The resulting closed-loop system is simulated in Section 4-5. Then, Section 4-6 collects some comments on the control system developed, pointing out strengths and weaknesses of the approach, and in Section 4-7 suggestions for future work are listed. Conclusions are drawn in Section 4-8.

4-1 Introduction

We here provide the reader with a short introduction about AUVs. After motivating their importance by means of a small survey on their use in modern applications, we analyze the general difficulties in the coordination control of a groups of vehicles. After that, we show the general set-up of these vehicles, taking as example the AUVs used in the University of Porto, Portugal, for the Control for Coordination (C4C) Project [20].

4-1-1 Importance of AUVs

AUVs are already used, and planned to be used in future, for several tasks. The key of their success comes from the obvious fact that they are uninhabited vehicles. They can perform tasks in hostile environments [14]. Groups of AUVs can be used for mere supervision of ocean areas, pollution detection, and environmental purposes.

AUVs are also used for source localization of chemicals. Examples are given by the prospection of hydrothermal vents [18], by chemical plume tracing, or in general the tracing of a substance, down to its source [35, 36].

Furthermore, these vehicles have proved invaluable in oceanographic and environmental field studies, by providing levels of spatial-temporal sampling resolution which could have not be attained before [9]. In fact, persistent sampling over wide areas has the potential to revolutionize environmental field studies. This is done by collection of sensor readings over an operating area, so that a map can be generated. The map can then be used for other purposes such as model validation and mission re-planning (adaptive sampling) [19, 31].

Autonomous vehicles are capable of executing mission plans without the intervention of human operators, who can be a mere part of the planning of the vehicle. For the operator it is possible to simply generate plans for autonomous execution and to override them when necessary.

4-1-2 Difficulties

The main objective of the control system to be synthesized is to steer a group of underwater vehicles along predetermined trajectories, ensuring a few properties which may change from case to case. A number of *issues* are to be considered at this purpose.

1. Collision danger: the vehicles should not collide with obstacles, nor with each other.

2. Delays and packet-losses: communication stations need time to transmit data by sonar. Consequently, information travels with the speed of sound. Sometimes, this information is never received by the vehicles. Delays and packet-losses are to be taken into account when designing a control system for coordination of AUVs.

3. Stochasticity: water currents and turbulence act on the vehicles in a hardly measurable or predictable way. Stochastic properties of this kind of disturbances could be difficult to be modeled.

4. Energy storage limitation: the endurance of these vehicles is highly correlated with the limitations of current energy storage technologies, and by the size of the vehicles themselves. Energy consumption is not only affected by their actuation, but also by communication. The sonar, used by AUVs to communicate, is energetically expensive. Communication should therefore be limited.

5. Control for exceptional circumstances: in case of malfunctioning of either the control or the communication system, the AUV has to be recovered.

6. Computational power limitation: due to space, cost and energy reasons, the computational power of AUVs is limited.

7. Information constraint: because of the costs and the functionality of sensors underwater, the vehicles have access to limited information about their state.

The framework built in this thesis intrinsically takes into account some of the above problems. For instance, LQG coordination control does not require subsystems (AUVs) to communicate data to the coordinator (here considered as a surface vessel). This represents a significant energy saving, since communication underwater has a consistent energetic cost. Furthermore, the framework we developed implicitly allows to take into account biased disturbances, position tracking, and collision avoidance. A natural application of the selected approach could be that of formation-flying.

4-1-3 Set-up

There exist various different types of AUVs. To give the reader a coarse idea of their characteristics, we hereby report those of the AUVs used by the University of Porto, Portugal, where a team of researchers is conducting coordination control studies under grant of the EU [9, 8, 6, 7], within the C4C project [20]. A picture of the vehicle is shown in Figure 4-1, while a description of its hardware is reported on Figure 4-2. The data have been taken from a deliverable of the C4C project [8].

Figure 4-1: The AUV used at the University of Porto, Portugal. [8].

Each AUV is equipped with processing unit, camera, altimeter, leak and pressure sensor, and sonar. Although a GPS receiver is also implemented, the reader should be aware of the fact that the GPS signals do not pass the water surface. Therefore, the vehicles cannot rely on that to obtain an estimate of their own position. The positions of the vehicles can be estimated in different ways, as for instance by an acoustic localization network that makes use of beacons and external localization systems, as displayed in Figure 4-3.

	Integrated System
Batteries	Saft Li-Ion - 25,2V At 5.8Ah (X2)
CPU	Eurotech VIPER-M64-F32-I-V2I3
Imu	3Dm-Gx1
Ctd	Mark & Wedell
Sidescan	Marine Sonics Hds - 900Khz
Dvl/Adcp	Linkquest
Wifi	LiteStation2
Gps	Evk-5H
Acoustic Modem	WHOI Micro-Modem FH-FSK 25kHz
	Benthos ATM-885PCB
Gsm/Gprs	Telit Gm862
Altimeter	Imagenex 862
Pinger	Sonotronics
Pressure Sensor	Esterline (KPSI) KPSI
Acoustic Board	USTS Custom Made
Leak Sensor	USTL Custom Made
Fin Servos	Hitec HS-77BB(4 units)
Camera	Kongsberg Simrad Camera Simrad

Figure 4-2: Data-sheet of the AUV used at the University of Porto, Portugal. [8].

4-2 Approach to the Problem

For the coordination control of a group of AUVs, we will make use of the results developed for LQG coordination control theory. The approach with which these results are implemented are described in this section. In order for this approach to be implemented, a few assumptions are required. These are also discussed hereby.

4-2-1 Coordination Control Approach

We propose an approach that exploits the advantages of LQG coordination control. The control synthesis is divided in two passages.

1. A surface vessel, mounting a computation and communication system, acts as a virtual coordinator for one or more Master AUVs. Since we assume the system to be computer-controlled, this is a virtual coordination problem, for which optimal results have been presented in Section 3-3.

2. Each Master AUV is followed by a number of independent AUVs. Each Follower AUV simply maintains a fixed distance from the Master, which behaves like a coordinator to it. By means of some decoupling hypotheses on the covariance matrices of the disturbances acting on each AUV, we can treat this leader-followers problem with the LQG coordination control theory developed for decomposable problems. For this class of problems, only locally optimal results involving a numerical optimization have been obtained (see Section 3-4), but the global optimality of the results was conjectured.

In order to avoid an excessively large mathematical notational load, we choose to consider only one Master and two Follower AUVs. However, the approach can easily be extended to

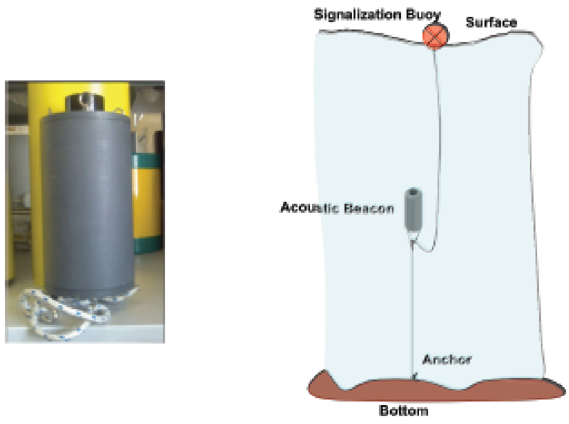

Figure 4-3: Sketch of a beacon used to track the position of the AUVs [9].

the case of any number of Master AUVs having any number of Follower vehicles. Figure 4-4 illustrates the case under focus.

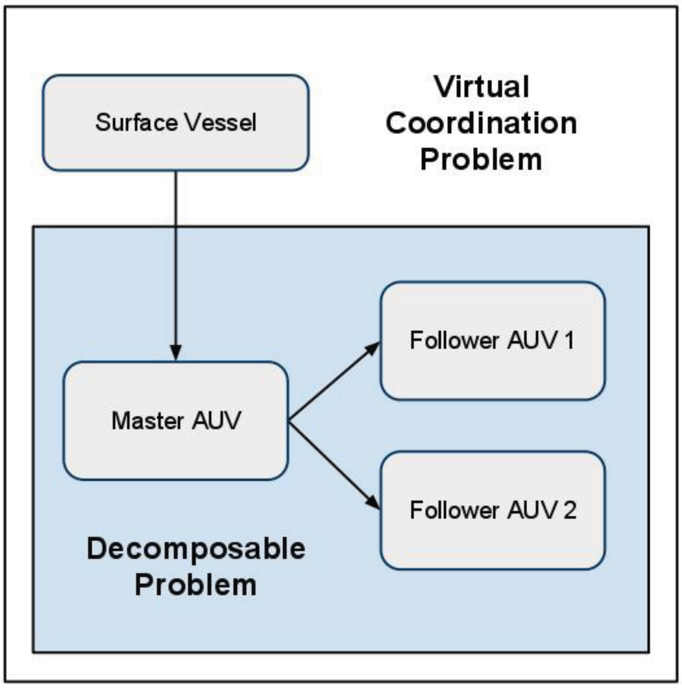

Figure 4-4: Scheme of coordination for three AUVs moving in formation.

4-2-2 Assumptions

In order to allow the implementation of LQG coordination control, a few assumptions are done about the problem.

1. The AUVs behave as linear systems, and they move on a plane. A linear model for each AUV is provided in [6]. This model is assumed to be reliable. Moreover, all the AUVs are assumed to be equal.
2. The vehicle has access to an estimate of its own position. Although GPS signals are not available underwater, position-estimation systems are available [8, 21]. This assumption is commonly made by researchers working on the same kind of problems [2, 19].
3. The disturbances acting on the states (water turbulence) and on the output (measurement noise) are assumed to have a Gaussian distribution.
4. The trajectories to be tracked by the vehicles are generated externally. Trajectory generation can be done in several different ways. An efficient linear-optimization-based way to do it can be found in [41].

4-3 Problem Formulation

For the control system to reflect the information constraints depicted in Figure 4-4, we formulate a nested coordination control problem. We hereby introduce the models we will use to represent AUVs' dynamics and disturbances. Then we formulate the problem.

4-3-1 Model for the AUVs

A linear model for one AUV was retrieved in [6]. Since the framework we built needs a discrete-time system, we apply a discretization to this model. Choosing $\Delta t = 1\ sec$, and adding stochastic disturbances, this becomes

$$\begin{cases} x_{AUV}(t+1) = A_{AUV}x(t) + B_{AUV}u_{AUV}(t) + M_{AUV}d_x(t) \\ y_{AUV}(t) = C_{AUV}x(t) + d_y(t). \end{cases}$$

where

$$\begin{aligned} x(t) &= \begin{bmatrix} p(t) \\ s(t) \end{bmatrix}; \\ A_{AUV} &= \begin{bmatrix} I & I \\ 0 & \left(\frac{\tau-1}{\tau}\right) I \end{bmatrix}; \quad B_{AUV} = \begin{bmatrix} 0 \\ \frac{1}{\tau} I \end{bmatrix}; \\ C_{AUV} &= \begin{bmatrix} I & 0 \end{bmatrix}; \qquad M_{AUV} = \begin{bmatrix} 0 \\ \frac{1}{\tau} M_x \end{bmatrix}; \end{aligned}$$

Here, $p(t) = [p_x(t), p_y(t), p_\alpha(t)]^T \in \mathbb{R}^3$ is the position vector including the coordinates of the vehicle over the plane (x and y represent the distances from a fixed origin, α represents the orientation); $s(t) \in \mathbb{R}^3$ is the corresponding velocity vector; $u(t) \in \mathbb{R}^3$ contains the generalized accelerations on the two directions and on the angle produced by the actuators; τ is a modeling parameter (for which the model is stable if $\tau > 0.5$). The uncontrollable inputs $d_x(t) \in \mathbb{R}^3$ and $d_y(t) \in \mathbb{R}$ represent the force disturbances acting on the vehicle and the measurement noise respectively. To model these disturbances in a realistic way, we consider them to be produced as the outputs of the following Gaussian system.

$$\begin{cases} x_d(t+1) = A_d x_d(t) + \begin{bmatrix} M_d & 0 \end{bmatrix} v_d(t) \\ \begin{bmatrix} d_x(t) \\ d_y(t) \end{bmatrix} = C_d \begin{bmatrix} d_x(t) \\ d_y(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D_d \end{bmatrix} v_d(t), \end{cases}$$

$v_d(t) \in \mathbb{R}^6$ being Gaussian white noise.

4-3-2 Coordination Control Problem

Since all the AUVs and the virtual coordinator (situated in the surface vessel) are dynamically independent, we represent the global state-space system as follows.

$$\begin{cases} \begin{bmatrix} x_{AUV1}(t+1) \\ x_{AUV2}(t+1) \\ x_{AUVM}(t+1) \\ x_{VESS}(t+1) \end{bmatrix} = \begin{bmatrix} A_{AUV} & 0 & 0 & 0 \\ 0 & A_{AUV} & 0 & 0 \\ 0 & 0 & A_{AUV} & 0 \\ 0 & 0 & 0 & A_{AUV} \end{bmatrix} \begin{bmatrix} x_{AUV1}(t) \\ x_{AUV2}(t) \\ x_{AUVM}(t) \\ x_{VESS}(t) \end{bmatrix} + \\ \qquad + \begin{bmatrix} B_{AUV} & 0 & 0 & 0 \\ 0 & B_{AUV} & 0 & 0 \\ 0 & 0 & B_{AUV} & 0 \\ 0 & 0 & 0 & B_{AUV} \end{bmatrix} \begin{bmatrix} u_{AUV1}(t) \\ u_{AUV2}(t) \\ u_{AUVM}(t) \\ u_{VESS}(t) \end{bmatrix} + \\ \qquad + \begin{bmatrix} M_{AUV} & 0 & 0 \\ 0 & M_{AUV} & 0 \\ 0 & 0 & M_{AUV} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{xAUV1}(t) \\ v_{xAUV2}(t) \\ v_{xAUVM}(t) \end{bmatrix}. \\ \begin{bmatrix} y_{AUV1}(t+1) \\ y_{AUV2}(t+1) \\ y_{AUVM}(t+1) \\ y_{VESS}(t+1) \end{bmatrix} = \begin{bmatrix} C_{AUV} & 0 & 0 & 0 \\ 0 & C_{AUV} & 0 & 0 \\ 0 & 0 & C_{AUV} & 0 \\ 0 & 0 & 0 & C_{AUV} \end{bmatrix} \begin{bmatrix} x_{AUV1}(t) \\ x_{AUV2}(t) \\ x_{AUVM}(t) \\ x_{VESS}(t) \end{bmatrix} + \\ \qquad + \begin{bmatrix} N_{AUV} & 0 & 0 & 0 \\ 0 & N_{AUV} & 0 & 0 \\ 0 & 0 & N_{AUV} & 0 \\ 0 & 0 & 0 & N_{AUV} \end{bmatrix} \begin{bmatrix} v_{yAUV1}(t) \\ v_{yAUV2}(t) \\ v_{yAUVM}(t) \\ v_{yVESS}(t) \end{bmatrix}. \end{cases}$$

Notice that the state-space matrices of the virtual coordinator are designed to reproduce the AUVs' dynamics. The reason for this choice are explained in Section 4-4-2, where the virtual coordination problem of steering the Master AUV is considered.

We assume the disturbances acting on the state and on the output to be uncorrelated. We also assume that there is no correlation between the disturbances acting on two agents which are not in communication with each other.

$$v_{tot}(t) = \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix}; \qquad V_{tot} = \begin{bmatrix} V_x & 0 \\ 0 & V_y \end{bmatrix};$$

$$v_x(t) = \begin{bmatrix} v_{xAUV1}(t) \\ v_{xAUV2}(t) \\ v_{xAUVM}(t) \end{bmatrix}; \quad V_x = \begin{bmatrix} V_{x,11} & 0 & V_{x,1M} \\ 0 & V_{x,22} & V_{x,2M} \\ V_{x,1M}^T & V_{x,2M}^T & V_{x,MM} \end{bmatrix};$$

$$v_y(t) = \begin{bmatrix} v_{yAUV1}(t) \\ v_{yAUV2}(t) \\ v_{yAUVM}(t) \\ v_{yVESS}(t) \end{bmatrix}; \quad V_y = \begin{bmatrix} V_{y,11} & 0 & V_{y,1M} & 0 \\ 0 & V_{y,22} & V_{y,2M} & 0 \\ V_{y,1M}^T & V_{y,2M}^T & V_{y,MM} & 0 \\ 0 & 0 & 0 & V_{y,VV} \end{bmatrix};$$

The weighing matrices Q and R can be chosen as follows.

$$Q = \begin{bmatrix} Q_{11} & 0 & Q_{1M} & 0 \\ 0 & Q_{22} & Q_{2M} & 0 \\ Q_{1M}^T & Q_{2M}^T & Q_{MM} & Q_{MV} \\ 0 & 0 & Q_{MV}^T & Q_{VV} \end{bmatrix}, \; R = \begin{bmatrix} R_{11} & 0 & 0 & 0 \\ 0 & R_{22} & 0 & 0 \\ 0 & 0 & R_{MM} & 0 \\ 0 & 0 & 0 & R_{VV} \end{bmatrix},$$

The goal to be achieved is the tracking of the signal coming from the surface vessel by the Master AUV, and the consequent tracking of the position of the Master AUV by the two Followers. Therefore, the feedback gain F and the optimal observer gain K are to be determined in the forms

$$F = \begin{bmatrix} F_{11} & 0 & F_{1M} & 0 \\ 0 & F_{22} & F_{2M} & 0 \\ 0 & 0 & F_{MM} & F_{MV} \\ 0 & 0 & 0 & F_{VV} \end{bmatrix} \; and \; \begin{bmatrix} K_{11} & 0 & 0 & 0 \\ 0 & K_{22} & 0 & 0 \\ 0 & 0 & K_{MM} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The structure of the two gains are chosen to reflect the information constraints of the closed-loop system. We assume in fact the two Follower AUVs not to communicate any data, and the position of the Master AUV to be known to the two Followers. This could imply a communication from the Master to the Followers, which we consider acceptable. Moreover, we assume the surface vessel to communicate data with the Master AUV only.

The observed states needed for the state-feedback action are retrieved locally. Each Follower estimates their own position and velocity, and those of the Master AUV, while the latter estimates its own, and receives the state of the virtual coordinator as an input.

4-4 LQG Coordination Control for AUVs

After having defined a linear model for the whole system, we proceed by applying the theory of LQG coordination control. Three separated problems are taken into account:

1. The decomposable problem of two Follower AUVs tracking the position of a Master AUV.
2. The virtual coordination problem of the Master tracking the coordinator's signal.
3. The disturbance rejection problem at the level of each AUV.

In the following pages, these three problems are solved.

4-4-1 LQG Coordination Control for Master-Followers

We use the theory of LQG coordination control for the problem of two AUVs following a Master AUVs having access to its position. As the two AUVs are autonomous, the state-space system we consider is the following.

$$\begin{cases} \begin{bmatrix} x_{AUV1}(t+1) \\ x_{AUV2}(t+1) \\ x_{AUVM}(t+1) \end{bmatrix} = \begin{bmatrix} A_{AUV} & 0 & 0 \\ 0 & A_{AUV} & 0 \\ 0 & 0 & A_{AUV} \end{bmatrix} \begin{bmatrix} x_{AUV1}(t) \\ x_{AUV2}(t) \\ x_{AUVM}(t) \end{bmatrix} + \\ \quad + \begin{bmatrix} B_{AUV} & 0 & 0 \\ 0 & B_{AUV} & 0 \\ 0 & 0 & B_{AUV} \end{bmatrix} \begin{bmatrix} u_{AUV1}(t) \\ u_{AUV2}(t) \\ u_{AUVM}(t) \end{bmatrix} + \\ \quad + \begin{bmatrix} M_{AUV} & 0 & 0 \\ 0 & M_{AUV} & 0 \\ 0 & 0 & M_{AUV} \end{bmatrix} \begin{bmatrix} v_{xAUV1}(t) \\ v_{xAUV2}(t) \\ v_{xAUVM}(t) \end{bmatrix} + \\ \begin{bmatrix} y_{AUV1}(t+1) \\ y_{AUV2}(t+1) \\ y_{AUVM}(t+1) \end{bmatrix} = \begin{bmatrix} C_{AUV} & 0 & 0 \\ 0 & C_{AUV} & 0 \\ 0 & 0 & C_{AUV} \end{bmatrix} \begin{bmatrix} x_{AUV1}(t) \\ x_{AUV2}(t) \\ x_{AUVM}(t) \end{bmatrix} + \\ \quad + \begin{bmatrix} N_{AUV} & 0 & 0 \\ 0 & N_{AUV} & 0 \\ 0 & 0 & N_{AUV} \end{bmatrix} \begin{bmatrix} v_{yAUV1}(t) \\ v_{yAUV2}(t) \\ v_{yAUVM}(t) \end{bmatrix}. \end{cases}$$

The related weighing and covariance matrices are given by

$$Q = \begin{bmatrix} Q_{11} & 0 & Q_{1M} \\ 0 & Q_{22} & Q_{2M} \\ Q_{1M}^T & Q_{2M}^T & Q_{MM} \end{bmatrix}; \qquad R = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{MM} \end{bmatrix};$$

$$V_x = \begin{bmatrix} V_{x,11} & 0 & V_{x,1M} \\ 0 & V_{x,22} & V_{x,2M} \\ V_{x,1M}^T & V_{x,2M}^T & V_{x,MM} \end{bmatrix}; \quad V_y = \begin{bmatrix} V_{y,11} & 0 & V_{y,1M} \\ 0 & V_{y,22} & V_{y,2M} \\ V_{y,1M}^T & V_{y,2M}^T & V_{y,MM} \end{bmatrix}; \quad V_{tot} = \begin{bmatrix} V_x & 0 \\ 0 & V_y \end{bmatrix}.$$

Because of the structure of the above matrices, we are facing a decomposable problem, of which the LQG coordination control solution was studied in Chapter 3. Therefore, we can use Algorithm 3-3.9 to obtain the feedback gain F in the swallow form, after applying the swallow state-space transformation

$$T = \begin{bmatrix} I & 0 & -I \\ 0 & I & -I \\ 0 & 0 & I \end{bmatrix}$$

to the system, since we desire the Followers to track the Master's position. We desire our observer to reconstruct the original state of the system, and therefore the gain K is built by using the original state-space matrices of the system, by means of Algorithm 3-3.18.

We obtain the optimal gains

$$F_{MasterFollowers} = \begin{bmatrix} F_{11} & 0 & F_{1M} \\ 0 & F_{22} & F_{2M} \\ 0 & 0 & F_{MM} \end{bmatrix}; \; K_{MasterFollowers} = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{MM} \end{bmatrix}.$$

4-4-2 LQG Coordination Control for Vessel-Master

The state-space model for the Vessel-Master coordination problem is given by

$$\left\{\begin{array}{l} \begin{bmatrix} x_{AUVM}(t+1) \\ x_{VESS}(t+1) \end{bmatrix} = \begin{bmatrix} A_{AUV} & 0 \\ 0 & A_{AUV} \end{bmatrix} \begin{bmatrix} x_{AUVM}(t) \\ x_{VESS}(t) \end{bmatrix} + \begin{bmatrix} B_{AUV} & 0 \\ 0 & B_{AUV} \end{bmatrix} \begin{bmatrix} u_{AUVM}(t) \\ u_{VESS}(t) \end{bmatrix} + \\ \qquad \begin{bmatrix} M_{AUV} \\ 0 \end{bmatrix} v_{xAUVM}(t) \\ \begin{bmatrix} y_{AUVM}(t+1) \\ y_{VESS}(t+1) \end{bmatrix} = \begin{bmatrix} C_{AUV} & 0 \\ 0 & C_{AUV} \end{bmatrix} \begin{bmatrix} x_{AUVM}(t) \\ x_{VESS}(t) \end{bmatrix} + \begin{bmatrix} N_{AUV} & 0 \\ 0 & N_{AUV} \end{bmatrix} \begin{bmatrix} v_{yAUVM}(t) \\ v_{yVESS}(t) \end{bmatrix}. \end{array}\right.$$

Weighing and covariance matrices for this problem are,

$$Q = \begin{bmatrix} Q_{MM} & Q_{MV} \\ Q_{MV}^T & Q_{VV} \end{bmatrix}; \quad R = \begin{bmatrix} R_{MM} & 0 \\ 0 & R_{VV} \end{bmatrix};$$

$$V_{tot} = \begin{bmatrix} V_x & 0 \\ 0 & V_y \end{bmatrix}; \qquad V_x = \begin{bmatrix} V_{x,MM} & 0 \\ 0 & 0 \end{bmatrix}; \qquad V_y = \begin{bmatrix} V_{y,MM} & 0 \\ 0 & V_{y,VV} \end{bmatrix};$$

Since no noise acts on the coordinator's state, this is a virtual coordination problem. For this class of systems, we have proven in Section 3-4 that the global LQG optimal control gains F and K can be computed locally at the level of the subsystems as if there was no coordinator. Thus, we have

$$\begin{aligned} F_{MM} = {} & LQG\left(A_{AUV}, B_{AUV}, Q_{MM}, R_{MM}\right); \\ K_{MM} = {} & \left[LQG\left(A_{AUV}^T, C_{AUV}^T, M_{AUV}V_{x,MM}M_{AUV}^T, N_{AUV}V_{y,MM}N_{AUV}^T\right)\right]^T. \end{aligned}$$

Notice that, because of the particular problem definition, these two gains result to be the same as those computed in the previous section.

The gain K_{VV} is not computed because, being virtual, the state $x_{VESS}(t)$ is assumed to be available. The gains F_{MV} and F_{VV} can be chosen arbitrarily, as they do not have an influence on the cost function (see Proposition 3-4.2). The coordinator's local gain F_{VV} is chosen to simulate the closed loop behavior of the Master AUV, while the gain F_{MV} is selected to guarantee the steady-state reference tracking of the surface vessel's output by the Master AUV. Therefore, we define

$$F_{VV} = -F_{MV} = F_{MM}.$$

About the state of the virtual coordinator, to track the external reference it receives a steady-state tracking gain is implemented,

$$G_V = \left[C_{AUV}\left(I - A_{AUV} - B_{AUV}F_{VV}\right)^{-1} B_{AUV}\right]^{\dagger},$$

where † indicates the pseudo-inverse of the matrix. In this way, the input to the vessel's virtual environment is defined as

$$u_{VESS}(t) = F_{MM}x_{VESS}(t) + G_V r(t),$$

$r(t)$ being the external reference for the Master AUV.

In summary, in this section we retrieved the two gains

$$F_{VesselMaster} = \begin{bmatrix} F_{MM} & F_{MV} \\ 0 & F_{VV} \end{bmatrix}; \; K_{VesselMaster} = \begin{bmatrix} K_{MM} & 0 \\ 0 & 0 \end{bmatrix}.$$

4-4-3 Disturbance Rejection

In LQG control, the mean value of the disturbances acting on the system is assumed to be zero. In AUVs coordination control, where the forces generated by water currents do not necessarily behave as Gaussian white noises, this assumption is not usually verified. In this case study, a realistic model for the disturbance is considered. Colored disturbances and modeling errors translate into steady-state tracking errors. To tackle this problem, a disturbance rejection filter is required. At this purpose, we build an observer of the disturbance acting on the AUV. We do this for the case of a Follower AUV, but the same process can be repeated for the other Follower and the Master AUV.

There are two different noises acting on the system: $v_x(t)$ plays the role of an external input while $v_y(t)$ is a measurement error. Assuming that neither the disturbances, nor the states of the AUV, are directly measurable, we analyze the effect of the disturbances over the output,

$$\begin{aligned} y_1(t) = \; & C_{AUV}\left(zI - A_{AUV} - B_{AUV}F_{11}\right)^{-1} B_{AUV}F_{1M}\hat{x}_{AUVM}(t) + \\ & + C_{AUV}\left(zI - A_{AUV} - B_{AUV}F_{11}\right)^{-1} M_{AUV}v_{xAUV1}(t) + N_{AUV}v_{yAUV1}(t). \end{aligned}$$

Now, let us define the generalized disturbance $d_1(t)$ which affects the output of the first Follower as

$$d_1(t) = C_{AUV}\left(zI - A_{AUV} - B_{AUV}F_{11}\right) M_1 v_{xAUV1}(t) + N_{AUV}v_{yAUV1}(t).$$

The input to the system is

$$u_{AUV1}(t) = F_{11}\hat{x}_{AUV1}(t) + F_{1M}\hat{x}_{AUVM}(t)$$

Let $\hat{d}_1(t)$ indicate the estimate of the disturbance $d_1(t)$. Implementing an additional input to the AUV, given by

$$u_{dAUV1}(t) = G_{dAUV1}\hat{d}_1(t) = -B^{\dagger}_{AUV}\hat{d}_1(t),$$

disturbance $d_1(t)$ is counteracted. Assuming the input $u_{dAUV1}(t)$ to balance the disturbance, and assuming this disturbance to be constant over time, we can build an observer of $d_i(t)$ as follows,

$$\hat{d}_1(t+1) = \hat{d}_1(t) + K_{dAUV1}\left[C_{AUV}\hat{x}_{AUV1}(t) - y_{AUV1}(t)\right].$$

The disturbance will converge to the actual value of $d(t)$ if and only if K_{dAUV1} is chosen such that the closed-loop system state matrix

$$\begin{bmatrix} \hat{x}_{AUV1}(t+1) \\ \hat{d}_{AUV1}(t+1) \\ x_{AUV1}(t+1) \end{bmatrix} = \begin{bmatrix} A_{AUV} + B_{AUV}F_{11} + K_{11}C_{AUV} & 0 & -K_{11}C_{AUV} \\ K_{dAUV1}C_{AUV} & I & -K_{dAUV1}C_{AUV} \\ B_{AUV}F_{11} & -B_{AUV}G_{dAUV1} & A_{AUV} \end{bmatrix} \begin{bmatrix} \hat{x}_{AUV1}(t) \\ \hat{d}_{AUV1}(t) \\ x_{AUV1}(t) \end{bmatrix}$$

is stable. The choice of this parameter is here done empirically. However, there might be a way to compute it optimally by use of the LQG theory.

4-4-4 Closed-Loop System

Implementing the controller obtained by this approach, the following closed-loop system is obtained.

$$
\left\{
\begin{aligned}
&\left[\begin{array}{c}
x_{AUV1}(t+1)\\ \hat{x}_{AUV1}(t+1)\\ \hat{d}_{AUV1}(t+1)\\ \hline
x_{AUV2}(t+1)\\ \hat{x}_{AUV2}(t+1)\\ \hat{d}_{AUV2}(t+1)\\ \hline
x_{AUVM}(t+1)\\ \hat{x}_{AUVM}(t+1)\\ \hat{d}_{AUVM}(t+1)\\ \hline
x_{VESS}(t+1)
\end{array}\right]
=
\left[\begin{array}{ccc}
A_{AUV} & B_{AUV}F_{11} & -B_{AUV}G_{dAUV1}\\
0 & A_{AUV}+B_{AUV}F_{11}+K_{11}C_{AUV} & 0\\
0 & K_{dAUV1}C_{AUV} & I\\ \hline
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\ \hline
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
x_{AUV1}(t)\\ \hat{x}_{AUV1}(t)\\ \hat{d}_{AUV1}(t)
\end{array}\right]
+\\
&+\left[\begin{array}{ccc}
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\ \hline
A_{AUV} & B_{AUV}F_{22} & -B_{AUV}G_{dAUV2}\\
0 & A_{AUV}+B_{AUV}F_{22}+K_{22}C_{AUV} & 0\\
0 & K_{dAUV2}C_{AUV} & I\\ \hline
0 & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
x_{AUV2}(t)\\ \hat{x}_{AUV2}(t)\\ \hat{d}_{AUV2}(t)
\end{array}\right]
+\\
&+\left[\begin{array}{ccc}
0 & B_{AUV}F_{1M} & 0\\
0 & K_{1M}C_{AUV}+B_{AUV}F_{1M} & 0\\
0 & 0 & 0\\ \hline
0 & B_{AUV}F_{1M} & 0\\
0 & K_{1M}C_{AUV}+B_{AUV}F_{1M} & 0\\
0 & 0 & 0\\ \hline
A_{AUV} & B_{AUV}F_{MM} & -B_{AUV}G_{dAUVM}\\
0 & A_{AUV}+B_{AUV}F_{MM}+K_{MM}C_{AUV} & 0\\
0 & K_{dAUVM}C_{AUV} & I\\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
x_{AUVM}(t)\\ \hat{x}_{AUVM}(t)\\ \hat{d}_{AUVM}(t)
\end{array}\right]
+\\
&+\left[\begin{array}{c}
0\\ 0\\ 0\\ \hline
0\\ 0\\ 0\\ \hline
B_{AUV}F_{MV}\\ B_{AUV}F_{MV}\\ 0\\ \hline
A_{AUV}+B_{AUV}F_{VV}
\end{array}\right]
x_{VESS}(t)+
\left[\begin{array}{c|c|c}
0 & 0 & 0\\
-K_{11} & 0 & 0\\
-K_{dAUV1} & 0 & 0\\ \hline
0 & 0 & 0\\
0 & -K_{22} & 0\\
0 & -K_{dAUV2} & 0\\ \hline
0 & 0 & 0\\
0 & 0 & -K_{MM}\\
0 & 0 & -K_{dAUVM}\\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
y_{AUV1}(t)\\ y_{AUV2}(t)\\ y_{AUVM}(t)
\end{array}\right]
+\\
&+\left[\begin{array}{c}
0\\ 0\\ 0\\ \hline
0\\ 0\\ 0\\ \hline
0\\ 0\\ 0\\ \hline
G_V
\end{array}\right]
r(t)+
\left[\begin{array}{c|c|c}
M_{AUV} & 0 & 0\\
0 & 0 & 0\\
0 & 0 & 0\\ \hline
0 & M_{AUV} & 0\\
0 & 0 & 0\\
0 & 0 & 0\\ \hline
0 & 0 & M_{AUV}\\
0 & 0 & 0\\
0 & 0 & 0\\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
v_{xAUV1}(t)\\ v_{xAUV2}(t)\\ v_{xAUVM}(t)
\end{array}\right]\\
&\left[\begin{array}{c}
y_{AUV1}(t)\\ y_{AUV2}(t)\\ y_{AUVM}(t)\\ y_{VESS}(t)
\end{array}\right]
=
\left[\begin{array}{cccc}
C_{AUV} & 0 & 0 & 0\\
0 & C_{AUV} & 0 & 0\\
0 & 0 & C_{AUV} & 0\\
0 & 0 & 0 & C_{AUV}
\end{array}\right]
\left[\begin{array}{c}
x_{AUV1}(t)\\ x_{AUV2}(t)\\ x_{AUVM}(t)\\ x_{VESS}(t)
\end{array}\right]
+\\
&+\left[\begin{array}{cccc}
N_{AUV} & 0 & 0 & 0\\
0 & N_{AUV} & 0 & 0\\
0 & 0 & N_{AUV} & 0\\
0 & 0 & 0 & N_{AUV}
\end{array}\right]
\left[\begin{array}{c}
v_{yAUV1}(t)\\ v_{yAUV2}(t)\\ v_{yAUVM}(t)\\ v_{yVESS}(t)
\end{array}\right].
\end{aligned}
\right.
$$

4-5 Simulation

The closed-loop system obtained in the previous section is simulated on MATLAB Simulink. Since position and velocity on every axis and angle is modeled as independent, we only simulate the movement over one degree of freedom. The following data is considered.

$$\tau = \; 0.8; \quad r(t) = \; 2\sin(0.01t) + 2; \quad c_{AUV1} = \; -3; \quad c_{AUV2} = \; -6;$$

where $r(t)$ is the reference signal to be tracked by the Master AUV, $c_{AUV1,2}$ are the distances to be kept between the Follower AUVs and the Master. The disturbance acting on the state is modeled as a colored noise, given as the output of the Gaussian system

$$\begin{cases} x_{dx}(t+1) = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.9 \end{bmatrix} x_{dx}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_x(t) \\ d_x(t) = \begin{bmatrix} 0.05 & 0 \end{bmatrix} x_{dx}(t) + 0.01 v_x(t). \end{cases}$$

The measurement disturbance $d_y(t)$ is also modeled as a colored noise, given by

$$d_y(t) = 1 + v_y(t),$$

with $v_x(t)$ and $v_y(t)$ Gaussian white noises with variances

$$V_x = 1; \; V_y = 0.01.$$

Considering the weighing matrices to be $Q = I$ and $R = I$, the following gains are computed through a collection of MATLAB functions.

$$F = \begin{bmatrix} -0.4629 & -0.2134 & 0 & 0 & 0.4622 & 0.2133 & 0 & 0 \\ 0 & 0 & -0.4629 & -0.2134 & 0.4622 & 0.2133 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4629 & -0.2134 & 0.4629 & 0.2134 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.4629 & -0.2134 \end{bmatrix};$$

$$K = \begin{bmatrix} -0.0938 & 0 & 0 & 0 \\ -0.0015 & 0 & 0 & 0 \\ 0 & -0.0938 & 0 & 0 \\ 0 & -0.0015 & 0 & 0 \\ 0 & 0 & -0.0938 & 0 \\ 0 & 0 & -0.0015 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A simulation of the closed-loop system is given in Figure 4-5. There, it is possible to notice how the variance of the position of the two Followers is higher than that of the Master. This is due to the simple fact that the two Followers track the position of the Master AUV, whose disturbances then propagate.

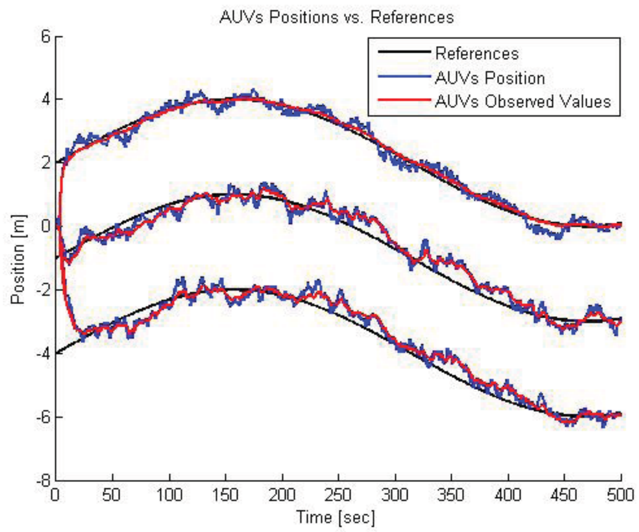

Figure 4-5: Closed-loop simulation of three AUVs moving in formation. A Master AUV tracks the output of a virtual coordinator situated in a surface vessel. Two other AUVs follow the Master.

The disturbances acting on the three AUVs with the respective observations are plotted in Figure 4-6.

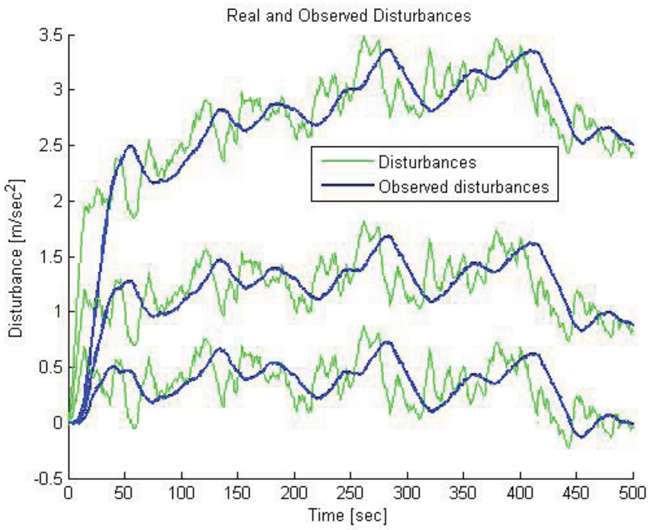

Figure 4-6: Stochastic disturbances acting on the three AUVs and their observed values.

The closed-loop behavior resulting from the LQG coordination control system shows how the one-way communication between coordinator and subsystems is sufficient for the tracking problem. Even if relatively strong, time-varying colored disturbances act on the three AUVs, the resulting tracking errors behave almost as a Gaussian white noises. This is shown in Figure

4-7. The slight sinusoidal behavior of the signal is due to a combination of the sinusoidal behavior of the disturbances, which are assumed to be constant.

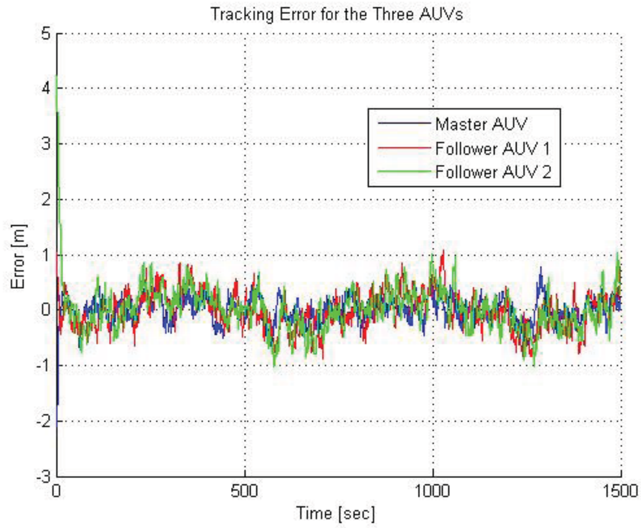

Figure 4-7: Tracking-error signals of the AUVs in the case of colored disturbances and sinusoidal reference.

If we consider the case in which only Gaussian white noises (with the given variances) affect the vehicles, and considering a constant reference to be tracked, the closed-loop tracking-errors we obtain are given in Figure 4-8.

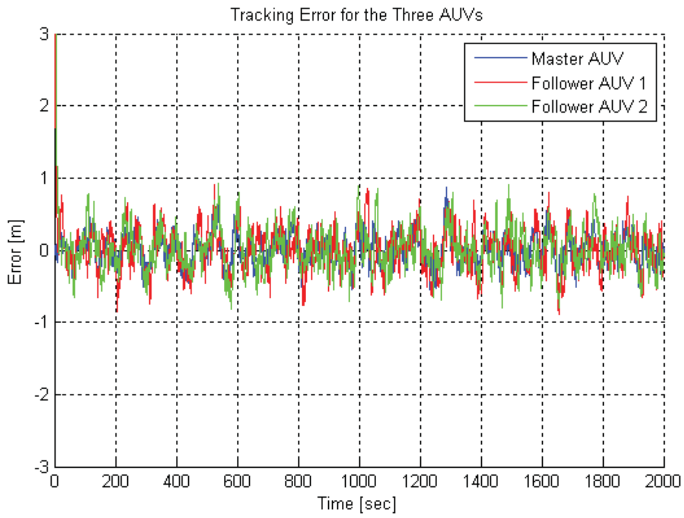

Figure 4-8: Tracking-error signals of the AUVs in the case of constant reference and Gaussian white noise disturbances.

A comparison of the computed variances in the two cases for the Master and the Followers is given in Table 4-1.

Tracking-Error Variances	Master AUV	Followers AUV
Colored Case	$0.0405\,m^2$	$0.1006\,m^2$
White Case	$0.0304\,m^2$	$0.0746\,m^2$

Table 4-1: Comparison of tracking-error variances in the case of Colored disturbances and sinusoidal reference to be tracked, and the case of Gaussian white noises and constant reference instead.

4-6 Comments

By means of the LQG coordination control theory, we have built a controller for the system which fits the information constraints imposed. A few comments on the control system are found below.

- Because of the nature of the developed LQG coordination control theory, the communication from the AUVs to their coordinator is not required for the closed-loop system to be stable.
- In the eventuality the models of the systems are time-invariant (no re-estimation of the model is necessary at each time-step) the control synthesis can be performed by respecting the communication constraints given to the control system itself. Therefore, no communication from AUVs to surface vessel would be required. See Section 3-3-5 for more details.
- For a correct functioning of the control system, the initial states of the AUVs and those of the disturbances should be estimated as well as possible. In fact, the observers have been optimally-built to guarantee the best steady-state behavior. They were not constructed to have a fast convergence to the real state of the system. In a sense, this is something that has to be avoided since the system is affected by stochastic disturbances, which we do not want to track.
- Setting $Q = I$ and $R = I$, the value of the cost function J with feedback gain computed by LQG coordination control is only 6% higher than that resulting from the classical LQG control synthesis, which is the best result possibly achievable [4], representing therefore the analytical minimum of the cost function. In Figure 4-9, we compare the values of the cost function J by changing the weighing factor $R = \epsilon I$, for different values of ϵ, both for the case of LQG coordination control and for the classical LQG control theory (for which no communication constraints are set). By decreasing the parameter ϵ, the difference between the results obtained with the two methods tends to reduce.
- The K gain computed by LQG coordination control is the same as that computed by the classical theory. This is due to the particular problem statement.
- Neither of the cost functions J and H take into account the energetical cost of communication, but only that related to the use of engines. Even if the cost of the function

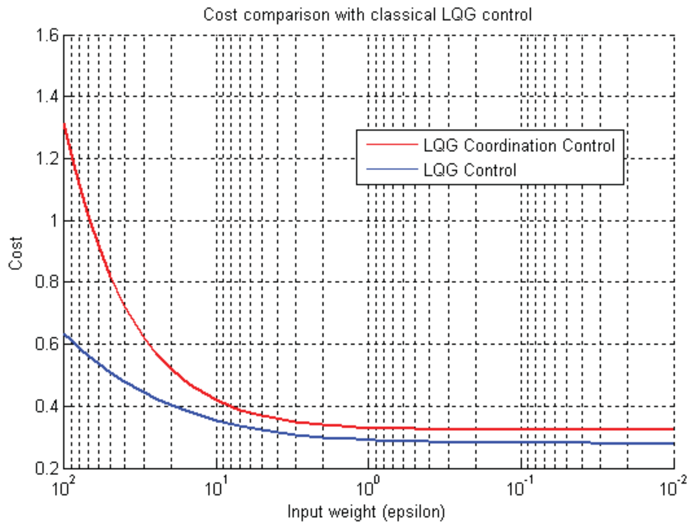

Figure 4-9: Values of the cost function J depending on the parameter ϵ, given the weighing matrices $Q = I$ and $R = \epsilon I$. The costs obtained with our LQG coordination control approach and those obtained with the classical LQG control theory are compared.

resulting from the implementation of classical LQG control is lower than that obtained by our LQG coordination control (see Figure 4-9) it is very likely that the latter actually implies a substantial reduction of energetic consumption.

- It is important to notice that, for those systems where no stringent time constraints are set, and where the references are precomputed, the communication from the vessel to the Master AUV may be sent in large packets. It is not necessary for them to be sent in real-time.

- The precision of the observers is based on how well the position of the AUVs is measured (either on the vehicles themselves or by an external measuring system). This precision is a crucial factor for the closed-loop system to work well, especially because the disturbances are observed using this data.

4-7 Suggestions for Future Work

The implemented control system leaves space for some improvements that can be applied in parallel. Some suggestions for future work are discussed in the following list.

1. A collision avoidance procedure could be implemented in every AUV. This could guarantee the AUVs not to collide with obstacles or with other vehicles. A possible approaches to collision avoidance is explained in [3, 39].

2. An event-based feedback can be implemented. By these we mean that where communication is avoided by the control system, this can still take place if particular event

happen. For example, in the case an unexpected obstacle is found or in case of malfunctioning.

3. Low-level non-linear controllers could be implemented in the AUVs for the actuation system. In this way, possible non-linearities could be taken into account by maintaining a linear model as a mask to the AUVs.

4. As the variance of the disturbance may vary, or the weights given to position errors and inputs may be changed, the control synthesis procedure could be recomputed in real-time in a distributed way. The procedure for doing so is explained in Section 3-3-5.

5. If the linear model describing the system is not accurate enough, a step-by-step linearization can take place. The control synthesis has to be done at every time-step. Unfortunately, in this occasion, the Follower AUVs will be required to share their model information with the Master AUV, and viceversa, possibly imposing a data communication. However, this is not a strict requirement for the Vessel-Master problem. In fact, in decomposable problems, such as the Master-Followers one, the model of the whole system is to be known to compute the optimal feedback gain F. This is not the same for virtual coordination problem (as it is the case of the Vessel-Master problem), where the coordinator's gains are chosen arbitrarily. Anyhow, for a good behavior of the whole system, the dynamics of the Master AUVs should be known to the coordinator.

6. No study has been done on communication delays and packet losses. This problem has not been treated in this thesis.

7. No optimal procedure has been studied to compute the disturbance observer gains K_{dAUV}. Knowing the properties of the disturbances, it is possible that these can be computed in an optimal way exploiting the classical LQG control theory. No research has been done in this direction.

8. Mechanical and safety constraints can be added to the control system. By the fact that the virtual coordinator emulates the dynamics of the Master AUVs, untrackable references are mediated by its action. Even if the external references change too fast, the virtual coordinator sends to the Master AUVs its simulated states, which are trackable if the model is correct.

4-8 Conclusions

In this chapter, we have implemented the theory of LQG coordination control developed in the thesis to coordinate a group of AUVs. To do this, we have separated the coordination problem in two parts.

The first part treated the coordination of Master AUVs by a surface vessel, that acts as coordinator for the vehicles, producing trajectories to be tracked. This was a virtual coordination problem. Optimal results were obtained in Section 3-4 for this class of problems, and therefore its solution provided the optimal gains for the control system.

The second part took into account the problem of a number of Follower AUVs tracking the position of a Master AUV. For this purpose, the theoretical framework generated for decomposable problems (proposed in Section 3-3) was used.

A comparison of the values of the costs resulting from our approach with the costs related to classical LQG control theory (the analytical minimum obtainable) showed a very small cost difference. Note that these costs did not include the energetical expenses coming from the communication channels, but only those from the engine of the vehicles. It is likely that the developed control system actually reduces, substantially, the energetic costs of the closed-loop system.

Chapter 5

Conclusions

In this chapter we delineate the conclusions about the theory of LQG coordination control that we have developed. In Section 5-1 we highlight the main contributions given by this thesis to the theory of linear coordination control. An objective analysis of the obtained results is found in Section 5-2, where strengths and weaknesses of the approach are discussed. At last, in Section 5-3 we present a list of suggestions for future work.

5-1 Contributions

A summary of all the contributions given by this thesis follows.

- The theory of LQG coordination control was developed for the class of *decomposable problems*. A number of important results results have been achieved.
  - By the results of Theorem 3-3.7, this category of problems is proved to be decomposable into N_s subproblems, for which the separation property holds if the parameters F_{cc} and K_{ii}, for $i = 1, \ldots, N_s$, are fixed.
  - On the footsteps of previous results obtained by Kempker, Ran and van Schuppen in [28], Theorem 3-3.8 was developed to obtain an optimal state-feedback gain F in the swallow form once the parameter F_{cc} is fixed. In particular, the gains F_{ii}, $i = 1, \ldots, N_s$, were proved to coincide with the solution to the local LQG control problem. Based on this result, an optimization procedure was defined to obtain a (locally) optimal swallow gain F varying the elements of the gain F_{cc} (see Algorithm 3-3.9). This optimization procedure was conjectured to be convex.
  - A similar, yet less trivial result was proved to hold, in Theorem 3-3.15, for the computation of the optimal observer gain K by fixing the parameters F_{cc} and K_{ii}, for $i = 1, \ldots, N_s$. In particular, the gain K_{cc} was proved to coincide with the solution to the local LQG control problem. Also for this case, an optimization procedure to

obtain a (locally) optimal swallow gain K was defined varying the elements of the parameters K_{ii}, for $i = 1, \ldots, N_s$ (Algorithm 3-3.18). This optimization procedure was also conjectured to be convex.

  - Based on the analytical solution of Sylvester's equations, a procedure to obtain a symbolic expressions for gradients and Hessians of the cost functions $J(F_{cc})$ and $H(F_{cc}, K_{11}, \ldots, K_{N_sN_s})$, in terms of their optimization parameters, was provided in Algorithm 3-3.13 and Algorithm 3-3.20.
  - A decentralization of the control synthesis complying with the communication constraints was developed in Algorithm 3-3.23. The only communication needed for the synthesis of the control gains goes from the coordinator to the subsystems.

- Particular results were obtained for the case of virtual coordination problems.
  - The optimal control gains F_{ii} and K_{ii}, for $i = 1, \ldots, N_s$, were proved to match with the solution to the local classical LQG control problem in Proposition 3-4.2 and Proposition 3-4.3.
  - All the rest of the gains were proven not have an influence on the cost functions J and H. As a consequence, they can be selected arbitrarily as long as the resulting closed-loop system is asymptotically stable.
- The theory was successfully implemented in a simulation environment to coordinate a group of AUVs, in Chapter 4.

5-2 Evaluation

An evaluation is here made with regards to the theoretical results achieved for LQG coordination control. We limit our comments to the results obtained for decomposable problems, as no achievements were presented for the general problem formulation. Strengths and weaknesses of the framework are considered.

5-2-1 Strengths

Some known strengths of the results obtained for decomposable problems are listed below.

Communication Avoidance Due to the structure of the control system, no communication is required from the subsystems to the coordinator. This represents the main strength of the approach. This important feature is also respected in some cases in which the control gains are to be recomputed on-line, as it is recalled in the next paragraph.

Decentralization of the Control Synthesis An important consequence of the problem decomposition studied in the chapter is the decentralization of the computation of the optimal gains F and K, which was described in Section 3-3-5. Whereas in more generic approaches (as for example those in [23, 44]) the whole optimal controller is computed in one single optimization problem, our approach separates the synthesis procedure. Two comments follow.

1. The more subsystems there are, the more useful this feature is, since N_s+1 optimization problems (1 for the synthesis of F, N_s for the syntheses of K_{ii}, for $i = 1, \dots, N_s$) are, in this way, solved by $N_s + 1$ agents instead of 1 only.
2. Throughout the whole procedure, no communication is required from the subsystems to the coordinator. Basically, this means that the information constraints imposed to the control system are preserved in the control synthesis itself.

Similarities with the Classical LQG Control Theory The same strengths of the classical LQG control theory reflect into LQG coordination control:

- the optimality of the control action;
- the possibility to select weights to define the importance of inputs and states.

Openness to Extensions Since we are working on linear systems, by merely defining optimal gains for feedback loop and observer, the control system is relatively open to modifications and additions. For instance, the following features can be implemented in parallel to our control system:

- reference-tracking feedforward gains;
- disturbance observer system and disturbance-rejection gains;
- an event-based state-update feedback to allow communication from the subsystems to the coordinator (in case, for example, of unpredicted behavior, malfunctioning or unexpected obstacles);
- limitations to inputs to avoid the mechanical systems to be harmed.

Some of the above parallel implementations were also considered in the case study.

Low Computational Burden Having the control gains computed separately, and in a decentralized manner, we obtain control systems which only require a small computational power. The most cumbersome operation which needs to be carried out is the minimization of a (conjectured to be convex) function everytime the gains are to be recomputed.

5-2-2 Weaknesses

Known weaknesses of LQG coordination control are discussed in this section.

Absence of Results for the General Case The results obtained in this thesis do not cover the general problem formulation, as it was defined in Section 3-2. For these problems, the separation principle has not been proved to hold, and the only information available is that the optimal control law has a linear form [24].

Dependence on Assumptions Intrinsic restrictions of the approach lie in its assumptions.

- The system is supposed to be linear and time-invariant.
- The stochastic disturbances acting on the states and on the outputs of the system are supposed to be white Gaussian noises, and their covariance matrix is required to be known.
- The categories of problems for which results are achieved (i.e., decomposable problems and virtual coordination problems) also require additional assumptions (see Section 3-4).

Whereas these requirements are not met, no optimality (not even local) can be inferred from the implementation of LQG coordination control.

Dependence on Numerical Optimization Although the control synthesis procedures proposed are conjectured to be convex, numerical optimization might fail to find the global minimum of the cost functions. In fact, depending on the precision set for a numerical algorithm to stop the computations, the results obtained may not be optimal. See Section 3-3-4 for more details about numerical problems.

A second problem introduced by numerical computation is related to the solution of discrete-time algebraic Riccati equations, which are to be solved everytime the cost functions are evaluated. Numerical problems occur when the poles of the system are too close to the unitary circle, and therefore to the instability boundary. Empiric solutions to this problem are given by the introduction of barrier functions and by the use of constrained optimization.

Local Optimality of the Solution to Decomposable Problems An entire section of this thesis is dedicated to conjecture the convexity of the numerical optimization problems leading to the control synthesis for decomposable problems (see Section 3-3-4). However, this proof was not presented. Although convexity could be proved in the future, this result is now missing, and it therefore represents a weakness of the theoretical framework. In fact, only local optimality can be inferred from the implementation of the present approach.

Step-by-Step Linearization Issue One of the strong points of our approach is that no communication is required from the subsystems to the coordinator. We also showed that this communication is not only avoided for the operation of the control system, but also in its eventual gains recomputation, which can be done in a decentralized way.

This is not ensured anymore when we are working with non-linear systems and we are approaching the problem with step-by-step linearizations, updating the state-space matrices at every time-step. If we assume the linearization to happen locally, the new state-space matrices have to be communicated to the coordinator for the computation of the gain F. The information of the whole model is in fact necessary for this computation to take place.

Prediction Inability The external reference to the coordinator is considered to be given. We did not put any effort on thinking about what this reference could be, or how to generate it. In the case this reference is known a-priori, our technique does not use any information of its future values to improve the performance of the control system.

This represents a little weakness with respect to other control techniques, as model predictive control, which instead uses this information to improve the closed-loop system's behavior [12].

5-3 Suggestions for Future Work

We suggest a list of possible future steps that can be taken in order to improve the current theoretical framework.

- The biggest piece missing to complete the mosaic of the present framework, is a proof for the convexity of the cost functions $J(F_{cc})$ and $H_i(K_{ii}, F_{cc})$, for $i = 1, \ldots, N_s$, within the stability region of the system. It is known that the two functions, which are rational in their variables, are not convex in their whole domain, but we conjecture this property within their stability region. Possibly, a proof could be constructed recursively by showing the convexity of each element appearing in the cost functions recursively. More detailed information regarding suggestions for proving the convexity of the functions are available in Section 3-3-4.
- An important issue to be considered is the control synthesis for nested CLSs. In fact, a CLS allows its subsystems to be CLSs themselves. However, the procedures developed in this thesis assume a classical information pattern for each subsystem. A new procedure to deal with nested CLSs is required. We indicate a possible direction to solve this issue. Hierarchy levels are to be considered: the coordinator of the system and its directly connected subsystems are considered the first level. All the other levels of coordination follow until the last, n^{th} level. The LQG coordination control problem is solved for the first level. Everytime a gain for a subsystem has to be computed, a new LQG coordination control problem is considered at the second level. This is done until the n^{th} level is reached, and the LQG coordination control problem is solved by the known procedures. This whole algorithm is run at every step of the first level's procedure. Notice that no problem arises in the case the first level is represented by a virtual coordination problem, as it was proved that the optimal controllers for this class of systems are computed locally at the level of the subsystems. The proposed direction is known to extend the computation time exponentially, and it was not explored well enough to ensure its effectiveness.
- About the general problem formulation, for which no results have been achieved in this thesis, the bottleneck is represented by the separation property. Proving its validity, or invalidity, would imply an important theoretical advance.
- No study has been done on the propagation delays of information. An important extension of the theoretical framework could be that of considering delays in the control action. Improvement in this direction can find valid foundations in the work of Rotkowitz, Cogill and Lall [43, 44].

- The performance of the LQG coordination control theory could be improved by allowing the coordinator to observe the state of the subsystems, maintaining the nested information pattern of the control system. The coordinator could, therefore, estimate the state of the subsystems by emulating their evolution starting from their (known) initial state. For this to happen, the coordinator must have access to the information of the models, the variances and the weighing matrices of the whole system, so that the two gains F and K can be computed. The observer system only has access to the coordinator's information. Therefore, the observed states would only converge to the real values if the closed-loop system is stable. Notice that drifts from the real subsystems' states and those observed by the coordinator could take place. Only superficial studies were done in this direction. Possible improvements of the control action could be achieved through this approach.
- The approach of LQG coordination control could be extended to other classes of systems than the CLSs. For examples, an extension of the theory could be thought for mammilary systems, which are defined in [50]. Basically, this class of systems is defined as the CLSs, but the swallow matrices are substituted by arrow matrices. Therefore, in a mammilary system the coordinator has access to the information coming from the subsystems, while the communication between subsystems is not allowed. This task might require some effort, as the separation property cannot be proved to hold by the same approach used in this thesis, and possibly it cannot be proved in general. Foundations for this kind of problem can be found in [44].

The author remains available to solve eventual doubts and to help improve the theoretical framework developed in this thesis.

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