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Integrated condition-based track maintenance planning and crew scheduling of railway networks

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ABSTRACT

We develop a multi-level decision making approach for optimal condition-based maintenance planning of a railway network divided into a large number of sections with independent stochastic deterioration dynamics. At higher level, a chance-constrained Model Predictive Control (MPC) controller determines the long-term section-wise maintenance plan, minimizing condition deterioration and maintenance costs for a finite planning horizon, while ensuring that the deterioration level of each section stays below the maintenance threshold with a given probabilistic guarantee in the presence of parameter uncertainty. The resulting large MPC optimization problem containing both continuous and discrete decision variables is solved using Dantzig-Wolfe decomposition to improve the scalability of the proposed approach. At a lower level, the optimal short-term scheduling of the maintenance interventions suggested by the high-level controller and the optimal routing of the corresponding maintenance crew is formulated as a capacitated arc routing problem, which is solved exactly by transforming it into a node routing problem. The proposed approach is illustrated by a numerical case study on the optimal treatment of squats of a regional Dutch railway network. Simulation results show that the proposed approach is robust, non-conservative, and scalable.

1. Introduction

Maintenance is crucial to guarantee the reliability, availability, and safety of a railway network. In this paper we focus on track maintenance, which takes more than 40% of the annual maintenance budget in the Dutch railway network (Zoeteman et al., 2014), and billions of dollars in the US (Peng and Ouyang, 2012). One important track maintenance intervention is grinding, which is applied to treat squats, a type of rolling contact fatigue, that first appear on rail surface and can lead to rail breakage if not treated properly. Early-stage squats can be effectively treated by grinding, while for severe squats, rail replacement is the only option. Another important track maintenance intervention is tamping, which is intended for ballast degradation and which repairs track irregularities by correcting track geometry parameters (Ling et al., 2014; He et al., 2015). An example of ballast defect is shown in Fig. 1.

Condition-based maintenance (Kobbacy and Murthy, 2008; Ben-Daya et al., 2016), where maintenance decisions are made according to the observed "condition" of the asset, has received growing popularity in various fields (Jardine et al., 2006; Fararooy and Allan, 1995). Unlike the time-based maintenance strategy (e.g. the current cyclic track maintenance strategy in the Netherlands),

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Fig. 1. Severely wore ballast stones under an old sleeper. Compared with the new ballast stones nearby with sharp edges, the wore ballast stones have become rounded in shape and covered by dusts. This problem is called the "foul" of the ballast, meaning that the ballast stones have been crushed, provide less drainage, move less and have less elasticity, leading to a reduction in their friction dissipation.

condition-based maintenance is efficient as it can avoid unnecessary maintenance, reducing the maintenance costs. The resources saved from unnecessary maintenance (e.g. available maintenance time) can then be allocated to perform necessary maintenance for severely deteriorated parts, improving the safety of the whole asset (Gebraeel et al., 2005). Condition-based maintenance is considered as the most promising maintenance strategy, as most system failures are preceded by one or more indicative signals (Ahmad and Kamaruddin, 2012). In this paper we consider condition-based maintenance optimization based on a mathematical model describing the deterioration process of the condition of the asset. The model can be either deterministic, e.g. (Wen et al., 2016; Famurewa et al., 2015), or stochastic, e.g. (Mercier et al., 2012; Vale and Ribeiro, 2014). Condition-based maintenance focuses on the time planning of maintenance interventions. How to optimally schedule the correspondent maintenance crew, including all necessary equipments and personnel, to perform the planned maintenance interventions on a railway network, taking into account the limited track possession time, is also of great concern for a maintenance contractor. The optimal scheduling of maintenance crew is usually formulated as a variant of the Vehicle Routing Problem (VRP)(Dantzig and Ramser, 1959), as by Heinicke et al. (2015) and Peng and Ouyang (2014).

1.1. Problem setting

We consider a railway network composed of multiple stations and lines, where a line is defined as the part of track between two stations. Each line is further divided into multiple sections. The degradation level of each section is represented by its condition, which can be quantified by performance indicators like reliability. The condition is usually evaluated from various parameters like visual lengths and crack depths of a squat, or track geometry parameters like alignment and longitudinal level. Many parameters are measured by maintenance contractors, but only a few of them are crucial in track maintenance decision making. For example, more than 30 track parameters are measured for the Swedish railway network, but only 5 track geometry parameters are considered significant for track irregularity (Al-Douri et al., 2016). Different types of maintenance interventions, with different effects and costs, can be applied to improve the condition of a section. In this paper, for each section, a discrete-time deterioration model is developed to describe the deterioration process of its condition. The sampling time is usually long (at least one month), due to the slow deterioration dynamics of railway infrastructures. Various parameter uncertainties (e.g. random degradation rate) are taken into account in the stochastic deterioration model.

Each type of maintenance intervention is performed by a specific maintenance crew. We define a maintenance operation as a round tour of the maintenance crew departing from and returning to a maintenance base, where the heavy machinery, like a grinding machine, can be stored. We also consider a fixed setup cost, including the cost of machinery and personnel, for each maintenance operation. Furthermore, we define a time period, which usually ranges from one week to one month, as the smallest time unit a maintenance operation can be performed. Each type of maintenance intervention has also a time budget, which specifies the maximal track possession time allocated to this specific maintenance intervention per time period. We consider flexible time budgets, i.e. in addition to the given time budgets, the maintenance contractor can request extra maintenance time with additional costs from the infrastructure manager.

We use the example of optimal treatment of squats on a small railway network (Fig. 2a) a) to illustrate the maintenance problem considered in this paper. The condition deterioration and effects of rail grinding and replacements of each section can be illustrated by the schematic plot in Fig. 2b. The sampling time is three months and the planning horizon is two years. Because of all the randomness in the deterioration dynamics, e.g. measurement errors and environmental influences, deterministic models that capture only the expected deterioration dynamics can lead to an over-optimistic maintenance plan with too few or too late maintenance intervention. As shown in Fig. 2b, if maintenance planning is only based on the average deterioration scenario, the maintenance agent might not see the urgency of grinding on the 6th month, as there is still some safety margin between the average degradation level and the maintenance threshold. However, the worst-case degradation level at that time step has almost reached the
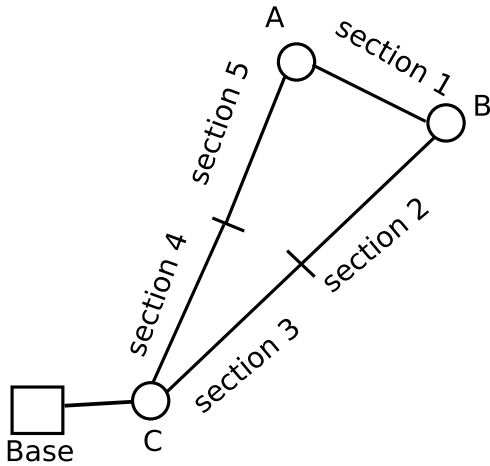
(a) A small railway network containing one maintenance base to store the machines, three stations, three lines and five sections.
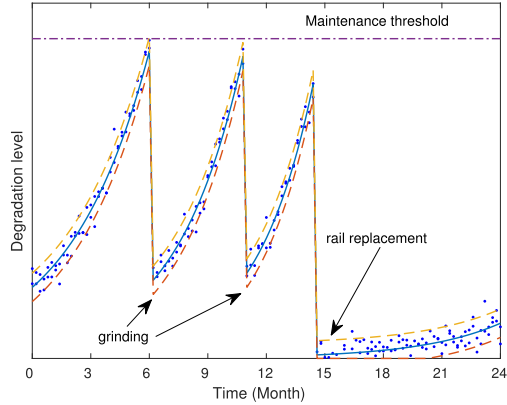
(b) Stochastic deterioration dynamics of one track section (section 1). The solid blue line represents the average scenario. The dashed orange and red lines represent the worst-case scenario and best-case scenario, respectively.
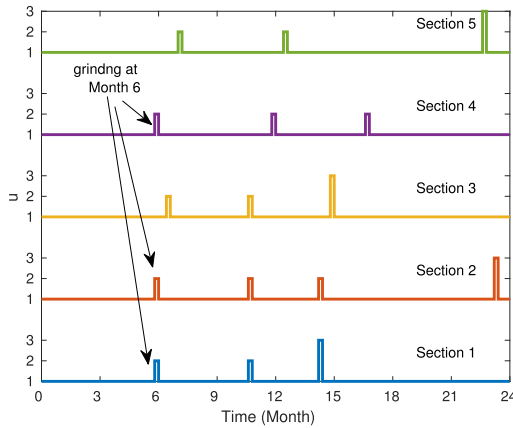
(c) Maintenance plan for the small network. Option 1 is no maintenance, option 2 is grinding, and option 3 is rail replacement.
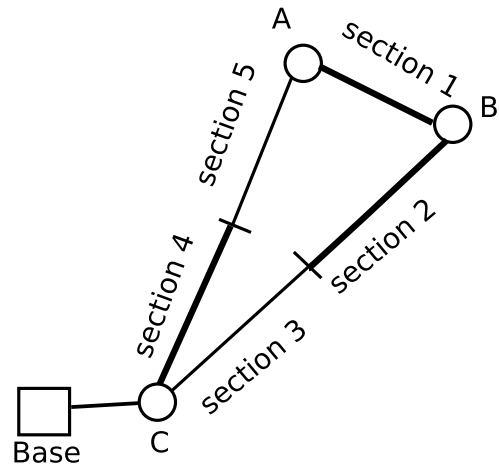
(d) Sections to be ground at the 6th month (marked in bold).

Fig. 2. Illustrative example for the problem description.

maintenance threshold, and the actual degradation level is also very close to the threshold. In practice, this might lead to hazards like rail breakage. An example of the maintenance plan within the two-year planning horizon for the small railway network is shown in Fig. 2c. At the 6th month, Sections 1, 2 and 4 need to be ground within the next three months. This short-term maintenance task is shown in Fig. 2d. For each month, a 6-h time slot is available for grinding. This time slot can be extended to 10 h with additional cost. A flat rate of 10 k€ must be paid by the maintenance contractor to rent the grinding machine for up to 10 h. The real-world optimization problem can then be stated as:

- Which maintenance option should be applied to each section of the network every three months within the two-year planning horizon, in order to minimize the total condition deterioration and maintenance costs, and to keep the degradation level of each section below the maintenance threshold?
- How to schedule the grinding crew, including the grinding machine and technicians, to complete all the grinding operations planned every three months, considering the trade-off between additional cost for extra maintenance time and the setup costs, e.g. the cost to rent the grinding machine?

Fig. 3. Schematic plot of the proposed multi-level approach.

1.2. Multi-level maintenance optimization

In this paper we develop an integrated multi-level approach that covers both the long-term condition-based maintenance planning, and the short-term scheduling of maintenance crews, for a large-scale railway network. This multi-level approach is illustrated by the schematic plot in Fig. 3. A maintenance intervention planning problem, and maintenance crew scheduling problems, ar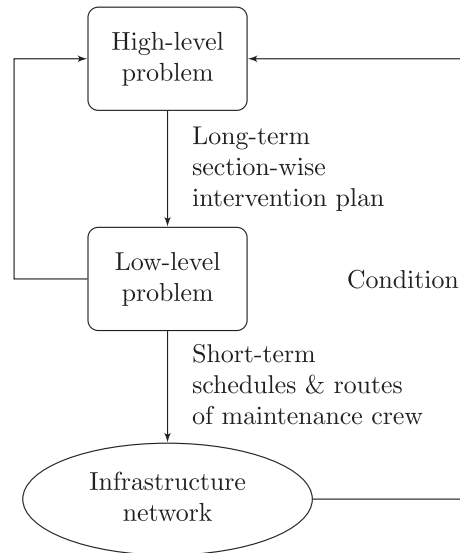e solved at the high level and low level, respectively. Based on the stochastic deterioration model, a Model Predictive Control (MPC) (Camacho and Alba, 2013; Rawlings and Mayne, 2009) approach is developed at the high level to determine the optimal maintenance intervention plan for the whole network, minimizing condition deterioration and maintenance costs over a prediction horizon. The MPC optimization problem is formulated as a chance-constrained optimization problem to keep the condition deterioration under a given threshold with a probabilistic guarantee. This probabilistic guarantee makes chance-constrained MPC much safer than deterministic MPC, where only the nominal dynamics (e.g. average scenario) is considered. It is also less conservative than robust MPC. A robust approach that considers the worst-case degradation scenario can indeed provide a theoretical guarantee that the degradation level of a section can never exceed the maintenance threshold. However, the resulting maintenance plan will be over-conservative as the probability of the worst-case scenario is in general small in practice. We propose the chance-constrained approach because it provides a balance between robustness and safety. It also provides a probabilistic guarantee on constraint satisfaction. Moreover, by adjusting the violation level the user can find the best trade-off between robustness and safety (as a smaller violation level leads to more conservative maintenance plan). In addition to the chance-constrained safety constraints on the condition of all the track sections, the MPC optimization problem also contains workload constraints that set a threshold on the maximal number of sections that can be treated by each type of maintenance intervention at one time step. The workload constraints are included out of concern of limited resources like available track possession time for maintenance. To improve the scalability of the proposed approach, distributed optimization scheme is applied to solve the MPC optimization problem containing both continuous and discrete decision variables.

At each decision step of the high-level problem, a maintenance crew scheduling problem is solved at the low level to obtain the optimal routes and scheduling of the maintenance crew in order to execute the maintenance plan determined at the high level. The low-level maintenance crew scheduling problem is triggered whenever the corresponding intervention is suggested by the high-level controller, and its planning horizon equals to the high-level sampling time. The objective of the low-level problem is to minimize the total setup costs of maintenance operations, the total travel costs of the maintenance crew, and the penalty costs associated with additional maintenance time (if there is any), over the whole network, while ensuring the planned intervention is completed before the next sampling time step. A feedback is sent from the low level to the high level when the low-level maintenance crew scheduling problem is infeasible, i.e. when the planned interventions obtained from the high level cannot be fulfilled before the next sampling time step due to lack of resources, e.g. available track possession time for maintenance. In this case we reduce the workload threshold on the corresponding maintenance intervention by one, and solve the high-level problem again. If the new high-level problem is feasible, the new intervention plan is applied to the low-level problem again. If the high-level problem becomes infeasible because of the tighter workload constraints, indicating that the available resources are not sufficient to keep the degradation levels of all the sections under a safety threshold, a new low-level problem is solved with increased resources, i.e. longer or more maintenance time slots. This iterative procedure between two levels continues until the low-level problem becomes feasible.

1.3. Contributions and structure of the paper

The major contributions of this paper include:

- An integrated multi-level approach for both long-term condition-based maintenance planning and short-term maintenance crew scheduling;
- A distributed optimization scheme to improve the scalability of the proposed approach for large-scale railway networks;
- A chance-constrained formulation to achieve a robust but non-conservative maintenance plan, in the presence of model uncertainties.

This paper is organized as follows: the state-of-the-art on railway maintenance optimization is presented in Section 2. The background on distributed MPC and chance-constrained optimization is provided in Section 3. We define the high-level maintenance intervention planning problem and the low-level maintenance crew scheduling problem, and propose methods to solve them, in Sections 4 and 5, respectively. The proposed multi-level approach is demonstrated by a numerical case study for optimal treatment of squats on a regional Dutch railway network in Section 6. Finally, we conclude our work and provide future directions in Section 7.

2. Maintenance optimization of railway infrastructures: state-of-the-art

Condition-based maintenance planning has been extensively discussed in literature and we restrict our scope to model-based approaches. A linear model is used by Wen et al. (2016) for the natural degradation of track quality, and a Mixed Integer Linear Programming (MILP) problem is formulated by Wen et al. (2016) to optimize tamping for a railway line over a finite planning horizon. An exponential model is employed for track geometry deterioration by Famurewa et al. (2015) to minimize total track possession time caused by tamping over a finite planning horizon, while keep the track geometry quality within a safe limit. The optimal condition-based tamping is formulated as an MILP problem by Gustavsson (2015), including the setup costs of tamping operations. Note that the deterioration models used in Wen et al. (2016), Famurewa et al. (2015) and Gustavsson (2015) are all deterministic models considering only nominal deterioration behaviour. The resulting maintenance strategies are not designed to be robust with the presence of various randomness like model uncertainties, measurement error, and missing data. In this case stochastic models, which describe the deterioration dynamics either by a stochastic process, or by a random-variable model (Frangopol et al., 2004), are preferred because of the robustness of the resulting maintenance strategy. A binary Mixed Integer Nonlinear Programming (MILNP) problem is developed by Vale and Ribeiro (2014) for optimal condition-based maintenance planning based on a stochastic deterioration model characterized by the Dagum probabilistic distributions. Other notable examples of condition-based track maintenance optimization approach based on stochastic deterioration models include (Mercier et al., 2012), where a bi-variate Gamma process describing the evolution of both the longitudinal and transverse levels is developed for the optimal planning of tamping operations for a French high-speed line, and (Quiroga and Schnieder, 2012), where a grey-box model is proposed to describe the ageing process of track geometry. A fuzzy Takagi–Sugeno internal model is used by Jamshidi et al. (2017) to capture the most important dynamics of squats evolution over time, and the effects of grinding and rail replacement are also modelled considering different representative scenarios. In our paper, we also use a stochastic deterioration model for condition-based maintenance planning. Similar to Vale and Ribeiro (2014), we capture the stochasticity of the deterioration process using a model with uncertain parameters. However, unlike (Vale and Ribeiro, 2014), we make no assumption on the probability distribution of the random parameters. This makes our approach applicable to more generic track defects.

In literature, the maintenance crew scheduling problem is usually formulated as a variant of VRP. For example, in Heinicke et al. (2015) the optimal scheduling of different maintenance tasks with various priorities over a railway network is formulated as a VRP with customer costs. In Peng and Ouyang (2014), the optimal clustering of small maintenance jobs into major projects is also recast as a VRP to minimize the total duration of all maintenance projects. Another popular approach for optimal scheduling of maintenance activities is the time–space network (Peng and Ouyang, 2012). A comparison between the VRP approach and the time–space network model for the scheduling of maintenance activities is provided in Gorman and Kanet (2010). Other approaches for the scheduling of maintenance activities over a railway network include the network-flow model proposed in Boland et al. (2013), and the MINLP formulation developed in Zhang et al. (2013). The maintenance schedule and the train timetable should be as compatible as possible to minimize the cost of traffic disruption. In most papers on maintenance scheduling, e.g. (Higgins, 1998; Budai et al., 2006), a timetable is already available, and the aim is to minimize disruption cost or timetable changes. On the other hand, (Boland et al., 2013 and Savelsbergh et al., 2015) start with a given maintenance plan, and adjust the train schedule accordingly to maximize the traffic throughput. Recently, an integrated approach has been developed by Lidén and Joborn (2017) for the joint optimization of train timetabling and maintenance scheduling. Its focus is to optimally schedule the traffic-free maintenance time windows that are sufficient for the regular maintenance activities and the desired amount of train traffic. However, the maintenance time windows can only be chosen from a set of available options, limiting the flexibility of the proposed approach.

Integrated approaches covering condition-based maintenance planning and crew scheduling are scarce in literature. One example is the travelling maintainer problem (Camci, 2014; Camci, 2015) for geographically distributed assets, e.g. railway switches, using prognostic information obtained from real-time condition-based monitoring. The resulting MINLP problem is solved by heuristics. However, the travelling maintainer problem is designed for general geographically distributed assets, and does not take into account practical issues related to railway track maintenance, e.g. the time to perform a maintenance intervention and the disruption to the train traffic. In addition, it does not consider uncertainties in the maintenance decision making. In our previous work (Su et al.,

2017), we have developed a multi-level approach for both condition-based maintenance planning and the clustering of individual track defects, taking into account the disruption cost to train traffic. However, the approach proposed there is only designed for a single railway line. From a computational perspective, most condition-based maintenance planning problems, e.g. (Gustavsson, 2015; Wen et al., 2016; Su et al., 2017), are solved by a centralized optimization scheme, which is not tractable for large-scale railway networks. As a consequence, the lack of an integrated, scalable approach for maintenance optimization under uncertainty for large-scale railway networks in literature is a key motivation for the current paper. In the literature, distributed control methods have been applied to distributed control methods applied to railway traffic control problem, e.g. the distributed optimal control method based on dual decomposition for automatic train regulation of large-scale metro networks (Li et al., 2018), the distributed MPC approach based on model-based partitioning for train traffic control (Kersbergen et al., 2016), and the augmented Lagrangian relaxation and the alternating direction method of multipliers applied to the cooperative planning of freight transport in Li et al. (2017). In the current paper, we investigate distributed optimization approach based on Dantzig-Wolfe decomposition to split the computational burden among many subproblems.

3. Preliminaries

In this section we introduce the methods used in the high-level problem. First, we provide a brief survey in Section 3.1 on MPC for hybrid systems with both continuous and discrete dynamics, and distributed solution methods for MILP problems. Chance-constrained optimization, which is used to address the stochastic deterioration dynamics, is explained in Section 3.2. Two solution methods for chance-constrained optimization problems, the widely-used scenario-based approach, and the robust scenario-based approach adapted in this paper, are also introduced.

3.1. MPC and distributed MILP

MPC is a popular control strategy that has been widely applied to several real-world problems like supply chain management (Schildbach and Morari, 2016; Nandola and Rivera, 2013), risk management of irrigation canals (Zafra-Cabeza et al., 2011), and drinking water network management (Grosso et al., 2014). MPC is an online control approach using a receding horizon principle. An optimization problem is solved at each sampling time step over a prediction horizon. Only the first entry of the resulting sequence of control action is applied, and a new optimization problem is solved at the next time step with updated information. The prediction horizon should be long enough to avoid myopic solutions, but not too long to avoid over conservative solutions. The generic deterioration of a railway infrastructure is one example of a hybrid system because the choice of maintenance activities can only take discrete values. In Su et al. (2016), the hybrid deterioration model is transformed into a standard Mixed Logical Dynamical (MLD) (Bemporad and Morari, 1999) system, resulting in a Mixed Integer Linear Programming (MILP) problem to be solved at every time step. Time Instant Optimization (TIO) (Schutter et al., 1998) is applied by Su et al. (2017) to transform the MPC optimization problem with both continuous and discrete decision variables into a nonsmooth nonlinear optimization problem with only continuous variables. Because an NP-hard problem must be solved at each time step, hybrid MPC is in general very computationally demanding for large-scale systems. For the sake of scalability, a distributed optimization scheme is usually adopted. However, there is a lack of distributed implementations of hybrid MPC in literature (Maestre and Negenborn, 2013). A distributed MPC method based on primal decomposition is developed in Luo et al. (2017) for a class of hybrid systems with discrete control inputs, global constraints, and limited information share between local controllers. Recently, a practical approach of a class of networked hybrid MPC is proposed by Mendes et al. (2017) who first determine the value of the binary decision variables in the local problem, and then transform the Mixed Integer Quadratic Programming (MIQP) MPC optimization problem into a set of Quadratic Programming (QP) problems through distributed coordination. Although the solutions of both approaches are suboptimal, numerical experiments show that the loss of optimality is small for the corresponding application. However, unlike the real-world problems mentioned in Luo et al. (2017) and Mendes et al. (2017), where some restrictions on the information exchange prohibit the implementation of a centralized optimization scheme, we adopt a distributed optimization scheme purely out of computational concerns. We apply the MLD framework to transform the hybrid deterioration dynamics, resulting in an MILP problem to be solved at each time step. Decomposition methods are used to divide the computational burden of the centralized MPC optimization problem among subproblems that are easier to solve. Benders decomposition (Benders, 1962), and Dantzig-Wolfe decomposition (Dantzig and Wolfe, 1960) are the most widely-used decomposition methods for large-scale Linear Programming (LP) and Mixed Integer Programming (MIP) problems. The choice of decomposition method depends on the structure of the original problem. Benders decomposition is more suitable for problems coupled through common variables (complicating variables), while Dantzig-Wolfe decomposition is designed for problems coupled through common constraints (complicating constraints).

Dantzig-Wolfe decomposition is widely used to solve large-scale LP problems. However, for MILP problems, Dantzig-Wolfe decomposition method only solves an LP relaxation of the original problem. Exact solutions to the original problem can be found by combining branch-and-bound with column generation, known as the branch-and-price (Barnhart et al., 1998) algorithm. One typical application of Dantzig-Wolfe decomposition is the vehicle routing problem and its variants (Feillet, 2010). The maintenance scheduling and routing problem of offshore wind farms has been formulated as a VRP with side constraints by Irawan et al. (2017) and solved efficiently using Dantzig-Wolfe decomposition. Dantzig-Wolfe decomposition has been used by Edlund et al. (2011) and Sokoler et al. (2014) for distributed implementation of the MPC optimization problem, which is an LP problem. Applications of Dantzig-Wolfe decomposition to MILP-MPC are relatively few. One example is (Gunnerud and Foss, 2010), where a suboptimal solution of the MILP problem is obtained through column generation.

3.2. Chance-constrained optimization

Let $(\Theta, \mathcal{B}(\Theta), \mathbb{P}_\theta)$ denote a probability space where Θ is a metric space with Borel σ-algebra $\mathcal{B}(\Theta)$ and probability distribution $\mathbb{P}_\theta$. We consider the following generic chance-constrained optimization problem

$$\min_{v \in \mathcal{V}} \mathbb{E}_\theta \left[J\left(v, \theta\right) \right] \tag{1}$$

$$\text{subject to:} \mathbb{P}_\theta[g(v, \theta) \leqslant 0] \geqslant 1 - \epsilon, \tag{2}$$

where $\theta \in \Theta \subseteq \mathbb{R}^{n_\theta}$ is a random vector containing all the uncertainties, and the parameter $\epsilon \in [0, 1]$ is the allowed violation level of the chance constraint. Moreover, we call a solution $v^* \in \mathcal{V}$ an ϵ-level solution if it is feasible for the chance constraint (2). For the moment, we only assume that the deterministic decision variable v is bounded, and make no assumption on the probability distribution of the random parameter θ.

3.2.1. Scenario-based approach

The standard scenario-based approach (Calafiore and Campi, 2006) approximates the chance constraint with a finite number of randomized scenarios, i.e. replacing the chance constraint (2) by the following set of deterministic hard constraints:

$$g(v, \theta^{(h)}) \leqslant 0 \quad \forall h \in \mathcal{H}, \tag{3}$$

where $\theta^{(h)}$ denotes the realization of uncertainties of the h-th scenario in the scenario set $\mathcal{H}$. The optimal solution of the scenario-based optimization problem (1), (3) is also a random variable as the scenarios are generated randomly. A confidence level β is associated with the scenario-based optimization problem to provide a probability bounds on its optimal solution. For a given β, the size of the scenario set $\mathcal{H}$ must be large enough to ensure that the optimal solution of (1), (3) is also an ϵ-level solution of the chance-constrained optimization problem (1) and (2) with a probability at least $1 - \beta$.

The focus of scenario-based approach is to find a lower bound on the size of the scenario set for a given violation and confidence level. Various scenario reduction techniques (e.g. (Henrion et al., 2009; Campi and Garatti, 2011; Li and Floudas, 2014; Li and Floudas, 2016; Chamanbaz et al., 2016)) have been applied to the standard scenario-based approach. A sampling-and-discarding approach is developed by Campi and Garatti (2011) that quantifies the trade-off between performance and feasibility. A mixed-integer programming problem is formulated by Li and Floudas (2014) for the optimal scenario-reduction problem to minimize the probabilistic distance and performance difference of the original and the reduced scenario distributions. Recently, some sequential scenario reduction techniques e.g. (Li and Floudas, 2016; Chamanbaz et al., 2016), have been developed for convex uncertain problems. The idea behind these sequential approaches is to verify the "temporary" scenario set at each time step against the given violation and confidence levels, and to increase the size of scenario set until it is validated.

Most proposed bounds on the size of the scenario set are only applicable to convex chance-constrained problems. For instance, it is assumed by Calafiore and Campi (2006), Calafiore (2010), Campi and Garatti (2008) and Zhang et al. (2015) that the chance constraint must be convex in the decision variable for any possible realization of the uncertainties. Even the scenario-based approach proposed by Grammatico et al. (2016) for non-convex control design also requires the chance constraints to be convex. Performance and feasibility bounds for a class of non-convex chance-constrained problems, including mixed-integer programming problems with integer decision variables in the chance constrains, are provided by Esfahani et al. (2015). However, the feasibility bound is very conservative and not applicable to large-scale non-convex chance-constrained problems.

3.2.2. Robust scenario-based approach

In this paper, we adopt the approach proposed by Margellos et al. (2014), which lies between a scenario-based method and a robust optimization approach. This two-phase approach first solves a scenario-based optimization problem to obtain a set $\mathcal{B}^*$ covering a given fraction (determined by the violation level) of the probability mass of the uncertainty with a certain confidence, and then solves a robust version of the original chance-constrained optimization problem, where the uncertainty lies in the intersection of $\mathcal{B}^*$ and the uncertainty space Θ. As in our case $\mathcal{B}^*$ is a strict subset of Θ in most situations, the result of this two-phase approach is less conservative than the direct robust approach, where the whole uncertainty space Θ is considered.

Here we briefly summarize this two-phase approach, which will be used in Section 4.2 to approximate the chance-constrained MPC problem by a deterministic optimization problem. First we solve the following standard scenario-based problem (as described in Section 3.2.1) for a given violation level ϵ and confidence level β:

$$\min_{\{(\underline{\tau}_i, \overline{\tau}_i)\}_{i=1}^{n_\theta}} \sum_{i=1}^{n_\theta} \overline{\tau}_i - \underline{\tau}_i \tag{4}$$

$$\text{subject to:} (\theta)_i^{(h)} \in [\underline{\tau}_i, \overline{\tau}_i] \quad \forall h \in \mathcal{H}, \forall i \in \{1, ..., n_\theta\}, \tag{5}$$

where $(\theta)_i$ represents the i-th entry of the random vector θ, and $\mathcal{H}$ is the set of random scenarios. The size of $\mathcal{H}$ is chosen according to the following condition (Alamo et al., 2010):

$$\left| \mathcal{H} \right| \geqslant \left\lceil \frac{1}{\epsilon} \cdot \frac{e}{e-1} \left(2n_\theta - 1 + \ln\frac{1}{\beta} \right) \right\rceil,$$

(6)

which ensures that the chance constraint $\mathbb{P}_\theta[(\theta)_i \in [\underline{\tau}_i, \overline{\tau}_i]] \geqslant 1 - \epsilon$ is satisfied with a confidence β for each $i \in \{1, ..., n_\theta\}$.

Let $\{(\underline{\tau}_i^*, \overline{\tau}_i^*)\}_{i=1}^{n_\theta}$ denote the optimal solution of the scenario-based problem (4) and (5). We can then construct a hyperbox $\mathcal{B}^* = \times_{i=1}^{n_\theta} [\underline{\tau}_i^*, \overline{\tau}_i^*] \subset \Theta$, and solve the following robust optimization problem:

$$\min_{v \in \mathcal{V}} \frac{1}{|\mathcal{H}|} \sum_{h=1}^{|\mathcal{H}|} J\left(v, \theta^{(h)} \right)$$

(7)

$$\text{subject to:} \max_{\theta \in \mathcal{B}^*} g\left(v, \theta \right) \leqslant 0.$$

(8)

The optimal solution of this robust optimization problem is an ϵ-feasible solution of the chance-constrained problem (1) and (2) with probability at least $1 - \beta$.

As discussed by Margellos et al. (2014), although this multi-level approach does not require convexity in the decision variable or in the uncertainty to be valid, it is tractable only when the associated robust optimization problem is tractable.

4. High-level maintenance intervention planning

We consider the optimal maintenance intervention planning for a network of railway track divided into n sections over a given planning horizon. Each section is viewed as a subsystem. The subsystems are coupled by global resource constraints, e.g. limited track possession hours.

Here we briefly outline the procedure of the high-level MPC approach. First, a deterioration model with uncertain parameters is developed for each subsystem in Section 4.1. Then for each subsystem, we formulated a local chance-constrained MPC optimization problem in Section 4.2. This chance-constrained optimization problem is then approximated by a much larger deterministic optimization problem using a robust scenario-based approach. The general nonlinear deterioration dynamics in each scenario is then approximated by a piecewise-affine model, which is transformed into a standard Mixed Logical Dynamical (MLD) (Bemporad and Morari, 1999) model in Section 4.3. Finally, the MLD-MPC optimization problem for the whole system is formulated in Section 4.4 by summing up all the local objective functions and constraints of all the sections, and including the global workload constraints related to limited resources. Dantzig-Wolfe decomposition is applied in Section 4.5 to solve the resulting large MLD-MPC optimization problem more efficiently.

4.1. Model of subsystems

In this section, we describe the deterioration model of a subsystem considering a generic defect, e.g. squats or ballast defects. Let the two-dimensional vector $x_{j,k} = [x_{j,k}^{\text{con}} \quad x_{j,k}^{\text{aux}}]^{\text{T}}$ denote the state of subsystem j at time step k of the planning horizon. In particular, the first entry $x_{j,k}^{\text{con}}$ represents the condition[1] of section j, while the second entry $x_{j,k}^{\text{aux}}$ is an auxiliary state to address the inefficiency of maintenance interventions, e.g. tamping becomes less effective the more it is applied to the same track. The condition $x_{j,k}^{\text{con}}$ and auxiliary state $x_{j,k}^{\text{aux}}$ lie within the bounded interval $[\underline{x}_j^{\text{con}}, \overline{x}_j^{\text{con}}]$ and $[\underline{x}_j^{\text{aux}}, \overline{x}_j^{\text{aux}}]$, respectively. Let $\mathcal{U}_j = \{1, ..., N\}$ denote the set containing the N possible maintenance options (including the "no maintenance" option represented by 1) that can be applied to subsystem j. The stochastic deterioration dynamics of each subsystem j can then be represented by the following generic model:

$$
\begin{aligned}
x_{j,k+1} &= f_j(x_{j,k}, u_{j,k}, \theta_{j,k}) \\
&= \begin{cases}
f_j^1\left(x_{j,k}, \theta_{j,k} \right) & \text{if } u_{j,k} = 1 \,(\text{no maintenance}) \\
f_j^q\left(x_{j,k}, \theta_{j,k} \right) & \text{if } u_{j,k} = q \quad \forall q \in \{2, ..., N-1\} \\
f_j^N\left(\theta_{j,k} \right) & \text{if } u_{j,k} = N \,(\text{full renewal})
\end{cases}
\end{aligned}
$$

(9)

$$\forall j \in \{1, ..., n\},$$

in which the vector $\theta_{j,k} \in \Theta_j \subset \mathbb{R}^{n_{\theta_j}}$ contains the realizations of the uncertain parameters (e.g. degradation rate) in subsystem j at time step k. Moreover, Θ_j is a bounded hyperbox.

As shown in (9), N independent functions are needed to describe the state dynamics corresponding to the N maintenance options. In particular, the function f_j^1, which corresponds to the "no maintenance" option, describes the natural degradation of the subsystem. The function f_j^N does not depend on the current state, as full renewal indicates replacing a section by a new section of track. The

---

[1] A higher value of $x_{j,k}^{\text{con}}$ indicates a worse condition.

effect of other maintenance interventions (e.g. tamping or grinding) on the subsystem in general depends on the current state of the track. For instance, grinding is effective only for early-stage squats. We consider each function f_j^q, $q = 1, ..., N$ to be in general nonlinear with respect to the state $x_{j,k}$. The deterioration model (9) can describe the stochastic degradation dynamics of the most typical railway track maintenance defects like squats and ballast defects. In practice, this general nonlinear model can be identified by a piecewise-affine model using historical data on track measurement (e.g. visual length of each squat in a track section, degradation trend from alignment measurements, etc.).

4.2. Chance-constrained MPC

In this section we present the chance-constrained MPC optimization problem for each subsystem, and apply the scenario-based robust approach developed in Margellos et al. (2014) to approximate each local chance-constrained MPC problem with a deterministic problem. Let N_P denote the prediction horizon, and define:

$$\widetilde{x}_{j,k} = [\widehat{x}_{j,k+1|k}^T ... \widehat{x}_{j,k+N_P|k}^T]^T$$
$$\widetilde{u}_{j,k} = [u_{j,k} ... u_{j,k+N_P-1}]^T$$
$$\widetilde{\theta}_{j,k} = [\theta_{j,k}^T ... \theta_{j,k+N_P-1}^T]^T,$$

where $\widehat{x}_{j,k+l|k} = [\widehat{x}_{j,k+l|k}^{con} \widehat{x}_{j,k+l|k}^{aux}]^T$ denotes the estimated state of subsystem j at time step $k + l$, based on information available at time step k. Vectors $\widetilde{x}_{j,k}^{con}$ and $\widetilde{x}_{j,k}^{aux}$ can be defined similarly as $\widetilde{x}_{j,k}$. The estimated state $\widehat{x}_{j,k+l|k}$ can be calculated recursively using (9), and $\widetilde{x}_{j,k}$ can be viewed as a function that depends on $\widetilde{u}_{j,k}$, $\widetilde{\theta}_{j,k}$ and is parametrized by the current state $x_{j,k}$, i.e.

$$\widetilde{x}_{j,k} = \widetilde{f}_j(\widetilde{u}_{j,k}, \widetilde{\theta}_{j,k}; x_{j,k}). \tag{10}$$

The objective of each local MPC controller is to minimize the trade-off between condition deterioration and maintenance costs within the prediction window, i.e.

$$J_j(\widetilde{x}_{j,k}, \widetilde{u}_{j,k}) = J_j^{Deg}(\widetilde{x}_{j,k}) + \phi_j J_j^{Maint}(\widetilde{u}_{j,k}), \tag{11}$$

in which

$$J_j^{Deg}(\widetilde{x}_{j,k}) = \|P\widetilde{x}_{j,k}\|_1, \tag{12}$$

and

$$J_j^{Maint}(\widetilde{u}_{j,k}) = \sum_{l=0}^{N_P-1} \sum_{q=1}^{N} c_{q,j}^{Maint} I_{u_{j,k+l}=q}. \tag{13}$$

The parameter ϕ_j captures the trade-off between condition deterioration and maintenance cost in subsystem j. Note that condition deterioration J_j^{Deg} and maintenance cost J_j^{Maint} should be properly transformed to the same scale. As condition of a track section is usually quantified by reliability or other performance indicators, one can scale condition deterioration by transforming performance loss into monetary loss, e.g. by setting a failure cost. The notation $\|\cdot\|_1$ represents the 1-norm. The parameter $c_{q,j}^{Maint}$ is the cost of the q-th maintenance intervention. The chance-constraint optimization problem for subsystem j at time step k can then be formulated as:

$$\min_{\widetilde{u}_{j,k}} \mathbb{E}_{\widetilde{\theta}_{j,k}}\left[J_j\left(\widetilde{x}_{j,k}, \widetilde{u}_{j,k}\right) \right] \tag{14}$$

$$\text{subject to:} \mathbb{P}_{\widetilde{\theta}_{j,k}}[\max_{l=1,...,N_P} \widehat{x}_{j,k+l|k}^{con}\left(\widetilde{u}_{j,k}, \widetilde{\theta}_{j,k}; x_{j,k}\right) \leqslant x_{max}^{con}] \geqslant 1 - \epsilon_j \tag{15}$$

$$\widetilde{x}_{j,k} = \widetilde{f}_j(\widetilde{u}_{j,k}, \widetilde{\theta}_{j,k}; x_{j,k}), \tag{16}$$

where ϵ_j is the violation level of the chance constraint of subsystem j, and the function $\widetilde{f}_j$ can be obtained by successive substitution of (9). The local cost function (14) is stochastic because $\widetilde{x}_{j,k}$, the estimated state in the prediction horizon, is dependent on the realizations of the uncertain parameters $\widetilde{\theta}_{j,k}$. The chance constraint (15) states that the probability that the worst estimated condition within the planning horizon does not exceed the maintenance threshold x_{max}^{con} is at least $1 - \epsilon_j$.

Note that the local chance constraint (14) is non-convex in the decision variables, as in the MPC formulation of the railway maintenance intervention planning problem, the control action is discrete because the available options for maintenance can only take discrete values. This is why we approximate the local chance-constrained problem (14) and (15) with a confidence level β_j using the two-phase scenario-based robust approach (Margellos et al., 2014) (see Section 3.2.2 for more details). Let $\mathcal{B}_j^*$ denote the hyperbox obtained by solving the scenario-based problem (4) and (5) for each dimension of $\widetilde{\theta}_{j,k}$. Let $\mathcal{H}_j$ denote the set of random scenarios of subsystem j, and define:

$$\widetilde{x}_{j,k}^{(h)} = \widetilde{f}_j\left(\widetilde{u}_{j,k}, \widetilde{\theta}_{j,k}^{(h)}; x_{j,k}\right) \tag{17}$$

for any $h \in \mathcal{H}_j$. The resulting robust optimization problem can then be written as:

$$\min_{\widetilde{u}_{j,k}, \widetilde{x}_{j,k}^{(h)}} \frac{1}{|\mathcal{H}_j|} \sum_{h \in \mathcal{H}_j} J_j\left(\widetilde{x}_{j,k}^{(h)}, \ \widetilde{u}_{j,k}\right) \tag{18}$$

$$\text{subject to:} \max_{\widetilde{\theta}_{j,k} \in \mathcal{B}_j^* \cap \widetilde{\Theta}_j} \max_{l=1, \ldots, N_P} \widehat{x}_{j,k+l|k}^{\text{con}}\left(\widetilde{u}_{j,k}, \ \widetilde{\theta}_{j,k}; x_{j,k}\right) \leqslant x_{\max}^{\text{con}} \tag{19}$$

$$\widetilde{x}_{j,k}^{(h)} = \widetilde{f}_j\left(\widetilde{u}_{j,k}, \ \widetilde{\theta}_{j,k}^{(h)}; x_{j,k}\right) \quad \forall \ h \in \mathcal{H}_j, \tag{20}$$

where (18) approximates the expectation of J_j. As proved by Margellos et al. (2014), any feasible solution of the robust optimization problem (18) and (19) is also an ϵ_j-solution of the chance-constrained MPC problem (14)–(16) with a probability of at least β_j.

We define the following worst-case scenario

$$\widetilde{\theta}_{j,k}^{(w)} \in \underset{\widetilde{\theta}_{j,k} \in \mathcal{B}_j^* \cap \widetilde{\Theta}_j}{\arg\max} \max_{l=1, \ldots, N_P} \widehat{x}_{j,k+l|k}^{\text{con}}\left(\widetilde{u}_{j,k}, \ \widetilde{\theta}_{j,k}; x_{j,k}\right). \tag{21}$$

The robust constraint (19) can then be replaced by:

$$P_j \widetilde{x}_{j,k}^{(w)}\left(\widetilde{u}_{j,k}, \ \widetilde{\theta}_{j,k}^{(w)}; x_{j,k}\right) \leqslant x_{\max}^{\text{con}}, \tag{22}$$

where P_j is a selecting matrix satisfying $P_j \widetilde{x}_{j,k} = \widetilde{x}_{j,k}^{\text{con}}$. We then define $\mathcal{S}_j = \mathcal{H}_j \cup \{w\}$ as the set containing all scenarios that need to be considered to approximate the chance-constrained MPC optimization problem (14)–(16) by the deterministic optimization problem (18), (20), (22).

As the convexity of the estimated condition $\widehat{x}_{j,k+l|k}^{\text{con}}$ is crucial in computing the worst-case scenario $\widetilde{\theta}_{j,k}^{(w)}$, and $\widehat{x}_{j,k+l|k}^{\text{con}}$ is obtained recursively using the system dynamics (9), we now provide a theorem to check the convexity of each $\widehat{x}_{j,k+l|k}^{\text{con}}$ for a given deterioration model. For convenience, we rewrite the vector-valued multi-variable function f_j in the following form:

$$f_j\left(x_{j,k}, \ u_{j,k}, \ \theta_{j,k}\right) = \begin{bmatrix} f_j^{\text{con}}\left(x_{j,k}^{\text{con}}, \ x_{j,k}^{\text{aux}}, \ u_{j,k}, \ \theta_{j,k}\right) \\ f_j^{\text{aux}}\left(x_{j,k}^{\text{con}}, \ x_{j,k}^{\text{aux}}, \ u_{j,k}, \ \theta_{j,k}\right) \end{bmatrix}. \tag{23}$$

We have the following theorem on the convexity of $\widehat{x}_{j,k+l|k}^{\text{con}}$ and $\widehat{x}_{j,k+l|k}^{\text{aux}}$:

Theorem 1. *If f_j^{con} and f_j^{aux} are convex in $\theta_{j,k}$ and convex and non-decreasing in $x_{j,k}^{\text{con}}$ and $x_{j,k}^{\text{aux}}$, then the functions $\widehat{x}_{j,k+l|k}^{\text{con}}$ and $\widehat{x}_{j,k+l|k}^{\text{aux}}$ are both convex in $\widetilde{\theta}_{j,k}$, for any $l \in \{2, \ldots, N_P\}$.*

The proof is given in Appendix A.

4.3. Mixed logical dynamical systems

We consider the following state dynamics for each scenario $s \in \mathcal{S}_j$:

$$x_{j,k+1}^{(s)} = f_j(x_{j,k}^{(s)}, \ u_{j,k}, \ \theta_{j,k}^{(s)}). \tag{24}$$

As stated in Section 4.1, for each maintenance option $q \in \{1, \ldots, N\}$, the function $f_j^q\left(\cdot, \theta_{j,k}^{(s)}\right)$ is in general nonlinear with respect to $x_{j,k}^{(s)}$. This nonlinear function can be approximated by function $f_j^{q,\text{PWA}}$, which is piecewise-affine with respect to $x_{j,k}^s$. In this way a piecewise-affine model f_j^{PWA} can be obtained to approximate (24). This approximation model can then be transformed into the following standard Mixed Logical Dynamical (MLD) system (Bemporad and Morari, 1999):

$$x_{j,k+1}^{(s)} = A_j^{(s)} x_{j,k}^{(s)} + B_{j,2}^{(s)} \delta_{j,k}^{(s)} + B_{j,3}^{(s)} z_{j,k}^{(s)} \tag{25}$$

$$E_{j,2}^{(s)} \delta_{j,k}^{(s)} + E_{j,3}^{(s)} z_{j,k}^{(s)} \leqslant E_{j,4}^{(s)} x_{j,k}^{(s)} + E_{j,5}^{(s)}, \tag{26}$$

where the vector $\delta_{j,k}^{(s)}$ contains all binary variables, and the vector $z_{j,k}^{(s)}$ contains all continuous auxiliary variables. In this way we obtain linear state dynamics with binary control actions for each scenario, as a preparation for applying decomposition methods for LP/MILP problems.

4.4. Centralized MLD-MPC problem

Define $\widetilde{\delta}_{j,k}^{(s)} = [(\delta_{j,k}^{(s)})^{\mathrm{T}}...(\delta_{j,k+N_{\mathrm{P}}-1}^{(s)})^{\mathrm{T}}]^{\mathrm{T}}$ and $\widetilde{\delta}_{j,k} = \left[(\widetilde{\delta}_{j,k}^{(1)})^{\mathrm{T}}...(\widetilde{\delta}_{j,k}^{(|S_j|)})^{\mathrm{T}}\right]^{\mathrm{T}}$. The vectors $\widetilde{z}_{j,k}^{(s)}$ and $\widetilde{z}_{j,k}$ are defined in the same way. The local robust scenario-based MPC optimization problem (18), (22), (20) for subsystem j can then be formulated as an MILP problem:

$$\min_{\widetilde{\delta}_{j,k}, \widetilde{z}_{j,k}} \left(\frac{1}{|\mathcal{H}_j|} \sum_{h \in \mathcal{H}_j} \left\| \widetilde{x}_{j,k}^{(h)} \right\|_1 \right) + w_j \left\| Q_j \widetilde{\delta}_{j,k} \right\|_1 \tag{27}$$

subject to: $\widetilde{x}_{j,k}^{(s)} = \widetilde{A}_j^{(s)} x_{j,k} + \widetilde{B}_{j,2}^{(s)} \widetilde{\delta}_{j,k} + \widetilde{B}_{j,3}^{(s)} \widetilde{z}_{j,k} \quad \forall s \in \mathcal{S}_j \tag{28}$

$$\widetilde{E}_{j,2}^{(s)} \widetilde{\delta}_{j,k} + \widetilde{E}_{j,3}^{(s)} \widetilde{z}_{j,k} \leqslant \widetilde{E}_{j,4}^{(s)} x_{j,k} + \widetilde{E}_{j,5}^{(s)} \quad \forall s \in \mathcal{S}_j \tag{29}$$

$$\widetilde{\delta}_{j,k} \in \{0, 1\}^{N_{\mathrm{P}} \cdot n_{\delta j}} \tag{30}$$

$$\widetilde{z}_{j,k} \in \mathbb{R}^{N_{\mathrm{P}} \cdot n_{zj}}, \tag{31}$$

and constraint (22),

where Q_j is a matrix with nonnegative entries. The first term in the objective function (27) corresponds to the mean of the accumulated condition deterioration within the prediction horizon, while the second term corresponds to the total maintenance cost. Constraints (28) and (29) are the N_{P}-prediction model derived from the MLD dynamics (25) and (26).

If we define $\widetilde{\delta}_k = [\widetilde{\delta}_{1,k}...\widetilde{\delta}_{n,k}]^{\mathrm{T}}$ and $\widetilde{z}_k = [\widetilde{z}_{1,k}...\widetilde{z}_{n,k}]^{\mathrm{T}}$, then the centralized MPC optimization problem can be formulated as:

$$\min_{\widetilde{\delta}_k, \widetilde{z}_k} \sum_{j=1}^{n} c_{j,1} \widetilde{\delta}_{j,k} + c_{j,2} \widetilde{z}_{j,k} \tag{32}$$

subject to: $\sum_{j=1}^{n} R_j \widetilde{\delta}_{j,k} \leqslant r \tag{33}$

$$F_{j,1} \widetilde{\delta}_{j,k} + F_{j,2} \widetilde{z}_{j,k} \leqslant l_j \quad \forall j \in \{1, ...,n\} \tag{34}$$

$$\widetilde{\delta}_k \in \times_{j=1}^{n} \{0, 1\}^{N_{\mathrm{P}} n_{\delta j}} \tag{35}$$

$$\widetilde{z}_k \in \times_{j=1}^{n} \mathbb{R}^{N_{\mathrm{P}} \cdot n_{zj}}. \tag{36}$$

Each cost vector in the objective function (32) can be obtained by substituting (28) into (27). Constraint (33) is the global constraint on the available resources, e.g. limited track possession time for maintenance, and constraints (34) summarize the local constraints (22), (28) and (29) for each subsystem.

4.5. Dantzig-Wolfe decomposition

The centralized MPC problem (32)–(36) is intractable for large-scale systems, e.g. railway network divided into many sections. This is why decomposition methods are used to improve the tractability of the MPC problem, thus improve the scalability of the proposed approach. Notice that without the global constraint (33), the centralized MPC problem can be solved by solving the n independent problems (27)–(31) for each subsystem j. This is typical for a railway maintenance intervention planning problem, where the maintenance decision for each section of track in a railway network is restricted by limited resources (e.g. available track possession time for maintenance or available machinery and personnel.) We apply Dantzig-Wolfe decomposition, which has initially been developed for LP problem with coupling constraints, to improve the scalability of the proposed MPC approach. First we define

$$\mathcal{P}_{j,k} = \{(\widetilde{\delta}_{j,k}, \widetilde{z}_{j,k}) \in \{0, 1\}^{N_{\mathrm{P}} n_{\delta j}} \times \mathbb{R}^{N_{\mathrm{P}} n_{zj}} : F_{j,1} \widetilde{\delta}_{j,k} + F_{j,2} \widetilde{z}_{j,k} \leqslant l_j\}, \tag{37}$$

which is the feasible region of the local MPC optimization problem for subsystem j. Let $\mathcal{G}_{j,k}$ denote the extreme points of the convex hull of $\mathcal{P}_{j,k}$. We call $\mathcal{G}_{j,k}$ the *generating set* of subsystem j at time step k. Let $\widetilde{\delta}_{j,k}^{[g]}$ and $\widetilde{z}_{j,k}^{[g]}$ denote the values of $\widetilde{\delta}_{j,k}$ and $\widetilde{z}_{j,k}$ of the extreme point $g \in \mathcal{G}_{j,k}$, respectively. According to Minkowski's theorem, each point in a compact polyhedron can be represented by a convex combination of its extreme points, which are called columns. Let $\mu_{j,g}$ denote the weight on the column $g \in \mathcal{G}_{j,k}$, and let μ_j denote the vector containing all the weights for columns in the generating set $\mathcal{G}_{j,k}$. Furthermore, define $\mu = [\mu_1^{\mathrm{T}}...\mu_n^{\mathrm{T}}]^{\mathrm{T}}$. The Dantzig-Wolfe reformulation of the centralized MPC problem (32)–(36) can then be written as:

$$\min_{\mu} \sum_{j=1}^{n} \sum_{g \in \mathcal{G}_{j,k}} \left(c_{j,1} \widetilde{\delta}_{j,k}^{[g]} + c_{j,2} \widetilde{z}_{j,k}^{[g]} \right) \mu_{j,g} \tag{38}$$

$$\text{subject to:} \sum_{j=1}^{n} \sum_{g \in \mathcal{G}_j} \left(R_j \widetilde{\delta}_{j,k}^{[g]} \right) \mu_{j,g} \leqslant r \tag{39}$$

$$\sum_{g \in \mathcal{G}_j} \mu_{j,g} = 1 \quad \forall j \in \left\{ 1, \ldots, n \right\} \tag{40}$$

$$\mu_{j,g} \geqslant 0 \quad \forall g \in \mathcal{G}_{j,k}, \forall j \in \{1, \ldots, n\} \tag{41}$$

$$\widetilde{\delta}_{j,k} = \sum_{g \in \mathcal{G}_{j,k}} \widetilde{\delta}_{j,k}^{[g]} \mu_{j,g} \in \times_{j=1}^{n} \{0, 1\}^{N_P n_{\delta_j}}. \tag{42}$$

Problem (38)–(42) is called the Dantzig-Wolfe reformulation by convexification. The disadvantage of this formulation is that the binary condition is imposed on the old decision variable $\widetilde{\delta}_{j,k}$, as shown in (42). However, as proved by Jans (2010), if the global constraint in the original binary MILP problem involve only binary variables, then the binary condition on the original variables can be directly transferred to the new variable μ. As the global constraint (33) contains only binary decision variables, we can then replace constraints (41) and (42) by the following equivalent binary condition on the new decision variable μ:

$$\mu_{j,g} \in \{0, 1\} \quad \forall g \in \mathcal{G}_{j,k}, \forall j \in \{1, \ldots, n\}. \tag{43}$$

Note that (43) is not included in the formulation (44)–(47) as the master problem is a linear relaxation of the Dantzig-Wolfe reformulation. The reformulated problem (38)–(40), (43) is still intractable, as the dimension of each generating set $\mathcal{G}_{j,k}$ grows exponentially with the dimension of the old variable $\widetilde{\delta}_{j,k}$. We use column generation to tackle this difficulty.

4.5.1. Column generation

Column generation (Vanderbeck and Wolsey, 2010) solves the linear relaxation of the Dantzig-Wolfe reformulation. We call this relaxed problem (38)–(41) the *master problem*. First we start with an initial partial generating set $\mathcal{G}_{j,k}^s \subset \mathcal{G}_{j,k}$ for each subsystem j and solve the following *restricted master problem*:

$$\min_{\mu} \sum_{j=1}^{n} \sum_{g \in \mathcal{G}_{j,k}^s} \left(c_{j,1} \widetilde{\delta}_{j,k}^{[g]} + c_{j,2} \widetilde{z}_{j,k}^{[g]} \right) \mu_{j,g} \tag{44}$$

$$\text{subject to:} \sum_{j=1}^{n} \sum_{g \in \mathcal{G}_{j,k}^s} \left(R_j \widetilde{\delta}_{j,k}^{[g]} \right) \mu_{j,g} \leqslant r \tag{45}$$

$$\sum_{g \in \mathcal{G}_{j,k}^s} \mu_{j,q} = 1 \quad \forall j \in \left\{ 1, \ldots, n \right\} \tag{46}$$

$$\mu_{j,g} \geqslant 0 \quad \forall g \in \mathcal{G}_{j,k}^s, \quad \forall j \in \{1, \ldots, n\}. \tag{47}$$

Each initial generating set $\mathcal{G}_{j,k}^s$ should be chosen to ensure the feasibility of the restricted master problem. This can be done by starting with the optimal solution of each subproblem j as the initial partial generating sets $\mathcal{G}_{j,k}^s$. If these initial partial generating set lead to an infeasible restricted master problem, i.e. violation of the global resource constraint (45), we use the big-M method and introducing artificial variables (similar to Phase I in the simplex method) to obtain a feasible initial solution for the restricted master problem. Let μ^* denote the optimal solution of this restricted master problem. The dual of problem (44)–(47) can be written as:

$$\max_{\lambda, \pi} - r\lambda + \sum_{j=1}^{n} \pi_j \tag{48}$$

$$\text{subject to:} \lambda \left(-R_j \widetilde{\delta}_{j,k}^{[g]} \right) + \pi_j \leqslant c_{j,1} \widetilde{\delta}_{j,k}^{[g]} + c_{j,2} \widetilde{z}_{j,k}^{[g]} \tag{49}$$

$$\forall g \in \mathcal{G}_{j,k}^s, \forall j \in \{1, \ldots, n\}$$
$$\lambda \geqslant 0 \tag{50}$$

$$\pi \in \mathbb{R}^n. \tag{51}$$

Let (λ^*, π^*) denote the optimal solution of the dual problem (48)–(51). We then define the following *pricing subproblem* for each subsystem j:

$$\rho_j = \min_{g \in \mathcal{G}_{j,k}} c_{j,1}\widetilde{\delta}_{j,k}^{[g]} + c_{j,2}\widetilde{z}_{j,k}^{[g]} + \lambda^* \left(R_j\widetilde{\delta}_{j,k}^{[g]} \right) - \pi_j^*$$

$$= \min_{(\widetilde{\delta}_{j,k}, \widetilde{z}_{j,k}) \in \mathcal{P}_{j,k}} c_{j,1}\widetilde{\delta}_{j,k} + c_{j,2}\widetilde{z}_{j,k} + \lambda^* \left(R_j\widetilde{\delta}_{j,k} \right) - \pi_j^*, \tag{52}$$

where ρ_j is called the *reduced cost* of subproblem j. The pricing problem (52) is an MILP because it determines the new column to enter the restricted master problem (44)–(47). It can be solved by state-of-the-art MILP solvers like Cplex or Gurobi. We add the new column, which is the optimal solution of (52), to the partial generating set $\mathcal{G}_{j,k}^s$ only if the corresponding reduced cost is negative. The restricted master problem (44)–(47) and its dual (48)–(51) are solved again including the new columns, and the new optimal dual solution (λ^*, π^*) is sent to each pricing subproblem. The iteration terminates when all reduced costs are 0, and the optimal solution of the restricted master problem corresponds to the optimal solution of the master problem.

4.5.2. Upper and lower bounds

Upper and lower bounds can be implemented to the basic column generation algorithm to achieve faster convergence. Any binary solution of the restricted master problem encountered in the column generation procedure provides an upper bound of the objective function value of the Dantzig-Wolfe reformulation and the centralized MPC optimization problem. A lower bound can be obtained by the Lagrangian dual function of the centralized MPC problem:

$$q(\lambda^*) = \inf_{\left(\widetilde{\delta}_k, \widetilde{z}_k\right) \in \times_{j=1}^n \mathcal{P}_{j,k}} \sum_{j=1}^n \left(c_{j,1}\widetilde{\delta}_{j,k} + c_{j,2}\widetilde{z}_{j,k} \right) + \lambda^* \left(\sum_{j=1}^n R_j\widetilde{\delta}_{j,k} - r \right)$$

$$= -\lambda^* r + \sum_{j=1}^n \left(\rho_j + \pi_j^* \right). \tag{53}$$

In addition to checking whether all reduced costs are 0, the upper and lower bounds provides another convergence criterion, i.e. whether the two bounds meet. The primal upper bounds are in general very weak, especially in the beginning of the column generation procedure, when the sets of columns are small. The dual bounds might oscillate, as the optimal solution of the dual of the restricted master problem might change drastically when a new column is added. Typical remedies for the erratic behaviour of the dual bounds include warm start e.g. (Sokoler et al., 2014), which provides a good dual bound at the beginning of the iteration, and stabilization techniques (Rousseau et al., 2007; Gschwind and Irnich, 2016), which add penalizing terms to (53) to avoid drastic change in the Lagrangian dual bounds. Another improvement of the standard column generation algorithm is the primal–dual column generation technique developed by Gondzio et al. (2013), which uses suboptimal primal and dual solutions of the restricted master problem to improve the stability of the iteration.

4.5.3. Inexact method

If the optimal solution of the master problem (38)–(41) obtained by column generation is also binary, then we have found the optimal solution of the Dantzig-Wolfe reformulation and the original MILP problem. However, the solution obtained through column generation is in general fractional. As stated by Gunnerud and Foss (2010), a feasible[2] suboptimal solution of the Dantzig-Wolfe reformulation can be obtained by solving the restricted master problem (44)–(47) as a binary MILP problem, using the sets of columns obtained at the end of column generation. Furthermore, a lower bound, and possibly an upper bound (depending on whether a binary solution is encountered during the iteration) are also provided by the column generation procedure. Exact solutions to the Dantzig-Wolfe formulation can be found by combining branch-and-bound with column generation, known as the branch-and-price (Barnhart et al., 1998) algorithm.

5. Low-level maintenance crew scheduling

The low-level problem is triggered whenever a maintenance intervention is suggested for at least one section of the whole network by the high-level MPC controller. Each type of maintenance intervention, e.g. grinding, is associated with a distinct low-level problem. Its goal is to find the optimal schedule to perform the planned maintenance activities, and the optimal route for the maintenance crew, minimizing the total setup costs of maintenance operations, the travelling costs of the maintenance crew, and the penalty cost on extra maintenance time (if any).

5.1. The arc routing problem

First we define the Capacitated Arc Routing Problem with Flexible Capacity (CARPFC), which is composed of:

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[2] Feasibility can be guaranteed, as long as the restricted master problem is binary feasible with initial sets of columns, or a binary solution is encountered during the iteration.

- a connected undirected graph $G = (\mathcal{V}, \mathcal{E})$;
- a depot node $0 \in \mathcal{V}$;
- a cost matrix C defining the travel cost associated with each edge;
- a set of required edges $\mathcal{R} \subseteq \mathcal{E}$ that must be serviced by a vehicle;
- a demand q_{ij} for each required edge $\{i, j\} \in \mathcal{R}$;
- a set $\mathcal{T}$ (fleet) containing all available vehicles;
- a fixed setup cost c_{Setup} associated with each vehicle;
- a flexible capacity $Q_t \in [\underline{Q}, \overline{Q}]$ associated with each vehicle $t \in \mathcal{T}$;
- a capacity-related cost $Q_{\text{Extra},t} = \nu(Q_t - \underline{Q})$ for any vehicle $t \in \mathcal{T}$, where ν is a positive parameter.

The CARPFC can then be defined as finding an optimal set of routes of the fleet starting and ending at the same depot, and the optimal capacities of the vehicles, that minimizes the total setup costs and travel costs of vehicles, and the costs related to the extra capacity, while ensuring that each required edge is serviced exactly once by a vehicle, and the edge demand is satisfied without exceeding the vehicle capacity.

To recast the low-level maintenance crew scheduling problem into a CARPFC, we map the physical network into a virtual graph $G = (\mathcal{V}, \mathcal{E})$. The stations are mapped into nodes in $\mathcal{V}$, and the lines are mapped into edges in $\mathcal{E}$. In particular, the maintenance base is mapped into the depot node 0. The travel cost of each edge is proportional to the length of the line. Furthermore, the lines in which at least one section is to be maintained before the next time step, are mapped into the required edges in $\mathcal{R}$. The demand of a required edge is interpreted as the estimated time to complete the maintenance activity on the corresponding line. Each time period in the low-level planning horizon corresponds to a vehicle in the CARPF, and the maintenance time budget per time period is translated as the capacity of the vehicle. The maintenance time budget is considered to be flexible within a given range, e.g. $\underline{Q} = 6$ hours and $\overline{Q} = 10$ hours per time period. This flexible maintenance time budget, which provides the maintenance contractor extra maintenance time at extra costs, further reduces the chance of having an infeasible low-level problem.

5.2. The node routing problem

We transform the arc routing problem described in Section 5.1 into an equivalent node routing problem because of the abundance of solution methods for node routing problems. We choose the approach developed by Baldacci and Maniezzo (2006). The transformed complete undirected graph is denoted by $\widehat{G} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}})$, with a new cost matrix $\widehat{C}$. Each endpoint of a required edge in $\mathcal{R}$ of the original graph becomes a customer node in $\widehat{\mathcal{V}}$ of the transformed graph, resulting in a node routing problem instance of $2|\mathcal{R}|$ customer nodes. We refer the readers to Baldacci and Maniezzo (2006) for the detailed transformation procedure. Furthermore, we partition $\widehat{\mathcal{V}}$ into the set of virtual depots $\mathcal{T}$, and the set of customer nodes $\widehat{C}$. Each virtual depot is a duplicate of the depot in the original graph, and corresponds to a vehicle $t \in \mathcal{T}$. We introduce the virtual depots to ensure that each tour is performed by one vehicle with a specific capacity. The demand of a customer node $i \in \widehat{C}$ in the transformed graph is denoted by $\hat{q}_i$. In this section we only provide the MILP formulation of the Capacitated Vehicle Routing Problem with Flexible Capacity (CVRPFC), which is a node routing counterpart of the CARPFC described in Section 5.1.

We define the binary decision variable:

$$x_{ij} = \begin{cases} 1 & \text{if node } j \text{ is visited directly after node } i; \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

for any $i, j \in \widehat{\mathcal{V}}$, and

$$z_{it} = \begin{cases} 1 & \text{if customer } i \text{ is visited by a vehicle from depot } t \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

for any node $i \in \widehat{C}$ and $t \in \mathcal{T}$.

We use the Miller-Tucker-Zemlin (MTZ) subtour elimination constraints (Miller et al., 1960), and define a continuous node potential variable u_i for each customer node $i \in \widehat{C}$. Because of the multiple virtual depots corresponding to heterogeneous vehicles, cycle imposement constraints are needed to ensure that each resulting route starts and ends at the same virtual depot, i.e. each round tour is made by the same vehicle. For this purpose, we choose the node-current based cycle imposement constraints (Burger et al., 2018), and define the continuous node current variable k_i for each node $i \in \widehat{\mathcal{V}}$.

The CVRPFC can then be expressed as:

$$\min \sum_{\{i,j\} \in \widetilde{\mathcal{E}}} \widetilde{c}_{ij} x_{ij} + \sum_{t \in \mathcal{T}} \sum_{j \in \widehat{\mathcal{V}}} (c_{\text{Setup}} + \nu(Q_t - \underline{Q})) x_{tj}, \tag{56}$$

where the first term corresponds to the travel costs, and the second term computes the total setup costs of the vehicles, including the costs related to the extra capacity, subject to the following assignment constraints:

$$\sum_{j \in \widehat{\mathcal{V}}} x_{ij} = \sum_{j \in \widehat{\mathcal{V}}} x_{ji} = 1 \quad \forall i \in \widehat{C} \tag{57}$$

$$\sum_{i \in \widetilde{\mathcal{V}}} x_{it} = \sum_{j \in \widetilde{\mathcal{V}}} x_{tj} \leqslant 1 \quad \forall \, t \in \mathcal{T}, \tag{58}$$

the following path continuity constraints

$$z_{it} - z_{jt} \leqslant 1 - x_{ij} - x_{ji} \quad \forall \, t \in \mathcal{T}, \, i, j \in \widehat{\mathcal{C}}, \, i \neq j \tag{59}$$

$$z_{jt} - z_{it} \leqslant 1 - x_{ij} - x_{ji} \quad \forall \, t \in \mathcal{T}, \, i, j \in \widehat{\mathcal{C}}, \, i \neq j \tag{60}$$

that ensure any two consecutive customers on a resulting tour are visited by the same vehicle; the labelling constraints

$$x_{tj} + x_{jt} - z_{jt} \leqslant 0 \quad \forall \, t \in \mathcal{T}, \, j \in \widehat{\mathcal{C}} \tag{61}$$

that ensure the first and last visited customer by a vehicle is associated with the corresponding virtual depot; the MTZ subtour elimination constraints

$$u_i - u_j + \overline{Q} x_{ij} + (\overline{Q} - \widehat{q}_i - \widehat{q}_j) x_{ji} \leqslant \overline{Q} - \widehat{q}_i \quad \forall \, i, j \in \widehat{\mathcal{C}}, \, i \neq j \tag{62}$$

the node-current based cycle imposement constraints

$$k_t = t \quad \forall \, t \in \mathcal{T} \tag{63}$$

$$k_i - k_j \leqslant (|\mathcal{T}| - 1)(1 - x_{ij}) \quad \forall \, i, j \in \widetilde{\mathcal{V}}, \, i \neq j \tag{64}$$

$$k_j - k_i \leqslant (|\mathcal{T}| - 1)(1 - x_{ij}) \quad \forall \, i, j \in \widetilde{\mathcal{V}}, \, i \neq j \tag{65}$$

the bounds for the continuous decision variables

$$\widehat{q}_i \leqslant u_i \leqslant \sum_{t \in \mathcal{T}} Q_t z_{it} \quad \forall \, i \in \widehat{\mathcal{C}} \tag{66}$$

$$1 \leqslant k_i \leqslant |\mathcal{T}| \quad \forall \, i \in \widetilde{\mathcal{V}}, \tag{67}$$

and finally the integrality constraints on the binary decision variables:

$$x_{ij} \in \{0, 1\} \quad \forall \, i, j \in \widehat{\mathcal{V}} \tag{68}$$

$$z_{it} \in \{0, 1\} \quad \forall \, i \in \widehat{\mathcal{C}}, \, \forall \, t \in \mathcal{T}. \tag{69}$$

Note that the problem (56)–(69) is a MINLP, as the flexible capacity Q_t is also a decision variable, leading to the nonlinear terms $Q_t x_{t,j}$ in the objective (56) and $Q_t z_{it}$ in constraints (66). Following the procedure developed by Bemporad and Morari (1999) and Williams (1993), we introduce the following continuous auxiliary decision variables:

$$y_{it} = z_{it} Q_t, \quad \phi_{tj} = x_{tj} Q_t \tag{70}$$

for $t \in \mathcal{T}, \, i \in \widehat{\mathcal{C}}, \, j \in \widetilde{\mathcal{V}}$, to eliminate the nonlinear terms. This results the following equivalent linear constraints (Bemporad and Morari, 1999):

$$y_{it} \leqslant \overline{Q} z_{it}, \quad \phi_{tj} \leqslant \overline{Q} x_{tj} \tag{71}$$

$$y_{it} \geqslant \underline{Q} z_{it}, \quad \phi_{tj} \geqslant \underline{Q} x_{tj} \tag{72}$$

$$y_{it} \leqslant Q_t - \underline{Q}(1 - z_{it}), \quad \phi_{tj} \leqslant Q_t - \underline{Q}(1 - x_{tj}) \tag{73}$$

$$y_{it} \geqslant Q_t - \overline{Q}(1 - z_{it}), \quad \phi_{tj} \geqslant Q_t - \overline{Q}(1 - x_{tj}) \tag{74}$$

that are equivalent to (70).

The MILP formulation of the CVRPFC can then be written as:

$$\min \sum_{\{i,j\} \in \widetilde{\mathcal{E}}} \widetilde{c}_{ij} x_{ij} + \sum_{t \in \mathcal{T}} \sum_{j \in \widetilde{\mathcal{V}}} \nu f_{tj} + \left(c_{\text{Setup}} - \nu \underline{Q} \right) x_{tj} \tag{75}$$

$$\text{subject to:} \, \widehat{q}_i \leqslant u_i \leqslant \sum_{t \in \mathcal{T}} y_{it} \quad \forall \, i \in \widehat{\mathcal{C}} \tag{76}$$

and constraints$(57) - (65), (67) - (69), (71) - (74).$

6. Case Study

6.1. Settings

A numerical case study on the optimal treatment of squats is performed on a part of the Dutch railway network containing Randstad Zuid and the middle-south region. This network contains 10 stations[3] and 13 lines, which are divided into 53 sections of 5 km, as shown in the schematic plot in Fig. 4. A squat (Fig. 5) is a type of rail contact fatigue, the evolution of which depends on the dynamic wheel-rail contact (Esveld, 2001). It first appears one the rail surface, and evolves into a network of cracks underneath the rail surface over time. If not treated properly, it can lead to hazards like rail breakage. For illustration purpose, two maintenance interventions, rail grinding and replacement, are considered for the treatment of squats. Jamshidi et al. (2017) report from field observation that grinding, which removes the irregularities on the rail surface, is effective for early-stage squats with visual length less than 20 mm, but for late-stage squat with visual length more than 50 mm, replacement is the only option. Effectiveness is also related to the grinding depth to reduce residual damages.

We adopt the big data analysis approach developed by Jamshidi et al. (2017) to calculate the failure probability of each squat. The failure probability, which is initially calculated from the visual length, estimates the probability that a given squat might lead to rail failure within the next million gross tons (MGT) step, which is 3 months in this case study. For this reason, the time step is also 3 months in the high-level MPC controller. The prediction horizon $N_P = 3$, i.e. 9 months, and the planning horizon is 20, i.e. 5 years.

A *simulation model* is developed to describe the evolution of failure probability of each individual squat. The details of the simulation model are given in Appendix B. The risk level, i.e. the condition, of a section of rail can then be calculated from the failure probabilities of all squats within the section[4]. By definition, the condition of each section is within the range $[0, 1]$. A *prediction model*, which describes the dynamics of the condition of a section, can be obtained by piecewise-affine identification based on the simulated data produced by the simulation model. Let $\mathcal{U} = \{1, 2, 3\}$ denote the set of all possible maintenance actions that can be applied to a section of rail, with 1, 2, 3 representing "no maintenance", "grinding", and "replacing", respectively. Replacing a section is 30 times as expensive as grinding. The parameter λ, which captures the trade-off between condition degradation and maintenance cost, takes a value of 100. Finally, we define the number of grinding operations on section j since the last replacement as the auxiliary variable $x_{j,k}^{\mathrm{aux}}$ in the prediction model.

The prediction model of section j, in accordance with the generic model (9), can then be expressed as:

$$x_{j,k+1}^{\mathrm{con}} = f_j^{\mathrm{con}}\left(x_{j,k}^{\mathrm{con}},\, u_{j,k},\, \theta_{j,k}\right)$$
$$= \begin{cases} f_j^{\mathrm{Deg}}\left(x_{j,k}^{\mathrm{con}},\, \theta_{j,k}\right) & \text{if } u_{j,k} = 1 \text{ (no maintenance)} \\ f_j^{\mathrm{Gr}}\left(x_{j,k}^{\mathrm{con}},\, \theta_{j,k}\right) & \text{if } u_{j,k} = 2 \text{ (grinding)} \\ 0 & \text{if } u_{j,k} = 3 \text{ (replacing)} \end{cases}$$

(77)

and

$$x_{j,k+1}^{\mathrm{aux}} = f_j^{\mathrm{aux}}\left(x_{j,k}^{\mathrm{aux}},\, u_{j,k}\right)$$
$$= \begin{cases} x_{j,k}^{\mathrm{aux}} & \text{if } u_{j,k} = 1 \text{ (no maintenance)} \\ x_{j,k}^{\mathrm{aux}} + 1 & \text{if } u_{j,k} = 2 \text{ (grinding)} \\ 0 & \text{if } u_{j,k} = 3 \text{ (replacing)}. \end{cases}$$

(78)

The threshold value $x_{\mathrm{max}}^{\mathrm{con}}$ in the chance constraint (15) is 0.95. As each grinding operation removes a certain depth of rail (e.g. 2 mm), grinding can only be applied consecutively to the same section for a limited number of times. So we have the following deterministic constraints on the auxiliary variable:

$$x_{j,k+l}^{\mathrm{aux}} \leqslant x_{\mathrm{max}}^{\mathrm{aux}} \quad \forall\, j \in \{1, ..., n\},\ \forall\, l \in \{1, ..., N_P\},$$

(79)

where $x_{\mathrm{max}}^{\mathrm{aux}} = 10$ in this case study.

The following global constraint is imposed on the maximal number of sections that can be ground at one time step:

$$\sum_{j=1}^{n} I_{u_{j,k}=1} \leqslant n_{\mathrm{max}}^{\mathrm{Gr}} \quad \forall\, l \in \left\{1, ..., N_P\right\},$$

(80)

where $n_{\mathrm{max}}^{\mathrm{Gr}} = 20$ in this case study.

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[3] Intermediate stations are out of the scope of this case study.
[4] We assume the failure of each squat is independent from that of other squats.
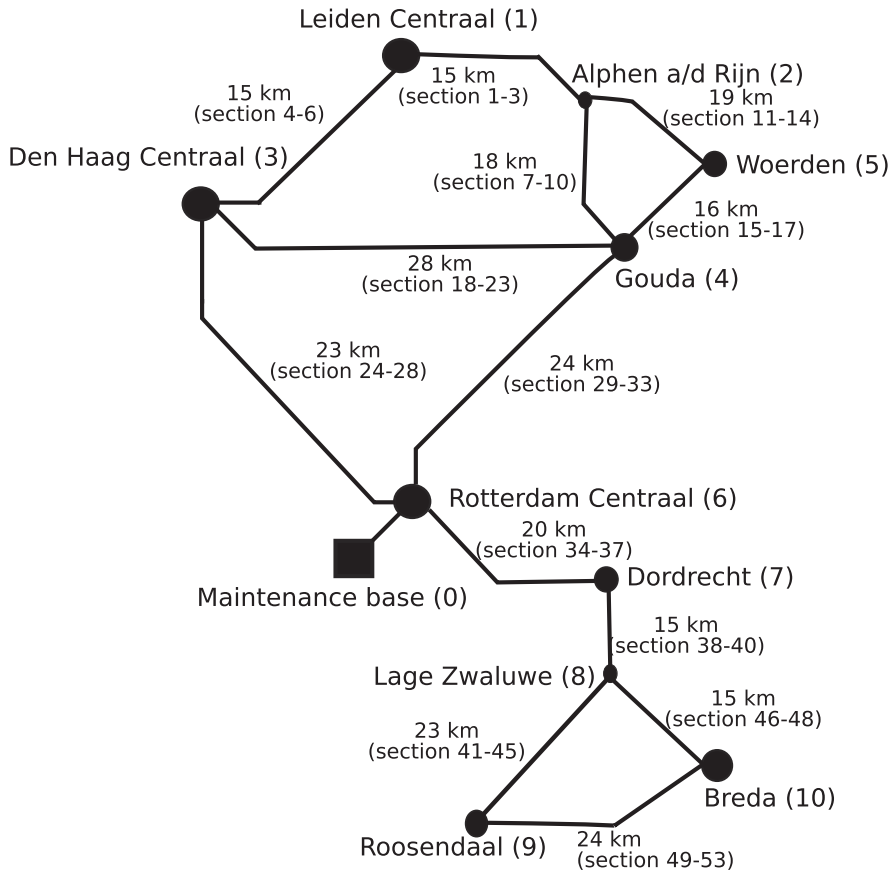
Fig. 4. The part of the Dutch railway network including Randstad Zuid and the middle-south region considered in the case study. The number next to a station is its index, while the maintenance base indexed as "0". The sections of each track line is indexed starting from the the station with the smaller index.
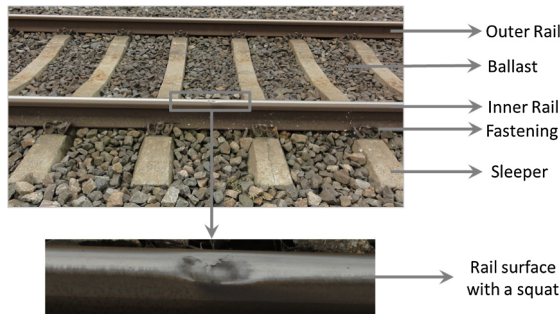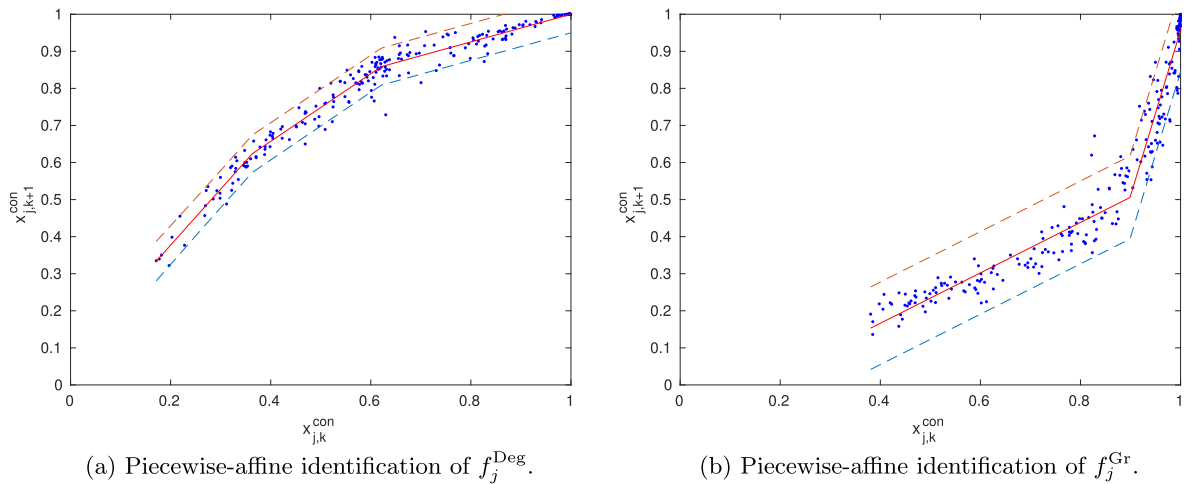

Fig. 5. A severe squat on the rail surface.

The function f_j^{Deg}, which describes the natural degradation of the condition, and the function f_j^{Gr}, which describes the effect of grinding on section j, are both piecewise-affine functions, as shown in the example in Fig. 6. To determine a piecewise-affine approximation of f_j^{Deg}, we partition the condition space of section j, i.e. $\mathcal{X}_j^{\mathrm{con}} = [0, 1]$, into three intervals $\mathcal{X}_{j,1}^{\mathrm{con}}$, $\mathcal{X}_{j,2}^{\mathrm{con}}$, and $\mathcal{X}_{j,3}^{\mathrm{con}}$. The natural degradation of condition can then be expressed by the following piecewise-affine function:

$$
f_j^{\mathrm{Deg}}(x_{j,k}^{\mathrm{con}}) = \begin{cases} y_{j,1}^{\mathrm{int}} + \frac{y_{j,2}^{\mathrm{int}} - y_{j,1}^{\mathrm{int}}}{x_{j,1}^{\mathrm{swi}}} x_{j,k}^{\mathrm{con}} & \text{if } x_{j,k}^{\mathrm{con}} \in \mathcal{X}_{j,1}^{\mathrm{con}} = [0, x_{j,1}^{\mathrm{swi}}) \\ y_{j,2}^{\mathrm{int}} + \frac{y_{j,3}^{\mathrm{int}} - y_{j,2}^{\mathrm{int}}}{x_{j,2}^{\mathrm{swi}} - x_{j,1}^{\mathrm{swi}}} (x_{j,k}^{\mathrm{con}} - x_{j,1}^{\mathrm{swi}}) & \text{if } x_{j,k}^{\mathrm{con}} \in \mathcal{X}_{j,2}^{\mathrm{con}} = [x_{j,1}^{\mathrm{swi}}, x_{j,2}^{\mathrm{swi}}) \\ y_{j,3}^{\mathrm{int}} + \frac{y_{j,4}^{\mathrm{int}} - y_{j,3}^{\mathrm{int}}}{1 - x_{j,2}^{\mathrm{swi}}} (x_{j,k}^{\mathrm{con}} - x_{j,2}^{\mathrm{swi}}) & \text{if } x_{j,k}^{\mathrm{con}} \in \mathcal{X}_{j,3}^{\mathrm{con}} = [x_{j,2}^{\mathrm{swi}}, 1], \end{cases}
$$

$$(81)$$

(a) Piecewise-affine identification of f_j^{Deg}.

(b) Piecewise-affine identification of f_j^{Gr}.

Fig. 6. Piecewise-affine identification of one prediction model with 95% nonsimultaneous observation confidence bound (indicated by the blue and red dashed lines). The data points for the identification are generated by aggregating the simulated failure probabilities of individual squats using the simulation model. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where $x_{j,1}^{\text{swi}}$ and $x_{j,2}^{\text{swi}}$ are the two switching points, and $y_{j,1}^{\text{int}}$, $y_{j,2}^{\text{int}}$, $y_{j,3}^{\text{int}}$, and $y_{j,4}^{\text{int}}$ are the four interpolation points.

The function f_j^{Gr} can also be represented by the following piecewise-affine function with three intervals:

$$f_j^{\text{Gr}}(x_{j,k}^{\text{con}}) = \begin{cases} 0 & \text{if } x_{j,k}^{\text{con}} \leqslant x_j^{\text{eff}} \\ \dfrac{y_j^{\text{sev}}}{x_j^{\text{sev}} - x_j^{\text{eff}}}(x_{j,k}^{\text{con}} - x_j^{\text{eff}}) & \text{if } x_j^{\text{eff}} < x_{j,k}^{\text{con}} \leqslant x_j^{\text{sev}} \\ y_j^{\text{sev}} + \dfrac{y_j^{\text{max}} - y_j^{\text{sev}}}{1 - x_j^{\text{sev}}}(x_{j,k}^{\text{con}} - x_j^{\text{sev}}) & \text{if } x_{j,k}^{\text{con}} > x_j^{\text{sev}}. \end{cases}$$

(82)

Five simulation models are used to generate the simulated data for piecewise-affine approximation, resulting in five different prediction models, the parameters of which are presented in Table 1. The uncertain parameters of the deterioration model (77) and (78) of section j are collected in the vector $\theta_{j,k} = \left[y_{j,1}^{\text{int}} \cdots y_{j,4}^{\text{int}} \, y_j^{\text{sev}} \, y_j^{\text{max}} \right]^{\text{T}}$. The confidence level β_j and violation level ϵ_j are both 0.1 for any section $j = 1, \ldots, n$, resulting in 591 random scenarios per section.

For illustration purpose we only trigger the low-level maintenance crew scheduling problem for grinding. The travel cost between any two station is 100€ per kilometer. Two 6-h maintenance time slots are available for grinding within one time step (3 months). A fixed setup cost of 100 k€ is associated with each maintenance time slot. Furthermore, extra time for grinding in addition to the given 6-h time slot can be requested with an hourly cost of 10 k€. However, a maximum length of 10 h is imposed on each maintenance time slot. The minimum amount of maintenance time spent on a railway line corresponds to the number of sections to be ground

Table 1
Parameters of the functions f_j^{Deg} and f_j^{Gr} for five different models. Both the nominal values and the 95% nonsimultaneous confidence bounds (given in the squar brackets) are provided for all uncertain parameters.

Parameter	Model				
	1	2	3	4	5
$x_{j,1}^{\text{swi}}$	0.512	0.526	0.543	0.363	0.563
$x_{j,2}^{\text{swi}}$	0.683	0.784	0.781	0.621	0.798
$y_{j,1}^{\text{int}}$	0.107 [0.086, 0.128]	0 [0,0]	0.051 [0.040, 0.063]	0.076 [0.036, 0.115]	0.058 [0.049,0.068]
$y_{j,2}^{\text{int}}$	0.783 [0.776, 0.790]	0.849 [0.845, 0.853]	0.815 [0.809. 0.821]	0.624 [0.615, 0.633]	0.805 [0.900, 0.809]
$y_{j,3}^{\text{int}}$	0.929 [0.924, 0.934]	0.975 [0.967, 0.983]	0.972 [0.966, 0.977]	0.859 [0.853, 0.865]	0.963 [0.958, 0.968]
$y_{j,4}^{\text{int}}$	1 [0.997, 1.003]	1 [0.997, 1.004]	1 [0.998, 1.002]	1 [0.994, 1.006]	1 [0.998, 1.002]
x_j^{eff}	0.156	0.177	0.172	0.141	0.106
x_j^{sev}	0.899	0.810	0.880	0.938	0.882
y_j^{sev}	0.506 [0.494, 0.518]	0.516 [0.505, 0.527]	0.502 [0.490, 0.514]	0.506 [0.490, 0.521]	0.443 [0.432, 0.455]
y_j^{max}	0.957 [0.944, 0.970]	0.991 [0.981, 1]	0.977 [0.965, 0.990]	0.922 [0.905, 0.939]	0.944 [0.931, 0.956]
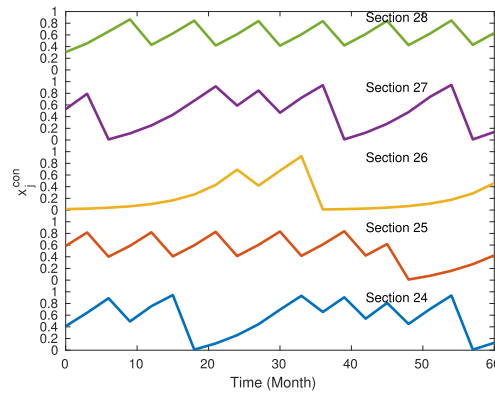
(a) Risk level per section calculated from the individual squat dynamics.
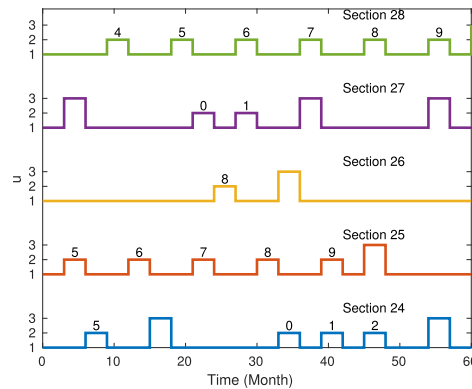
(b) Interventions suggested by the high-level MPC controller. The number above each grinding intervention indicates the number of previous grinding operations applied to the section since the last replacement.

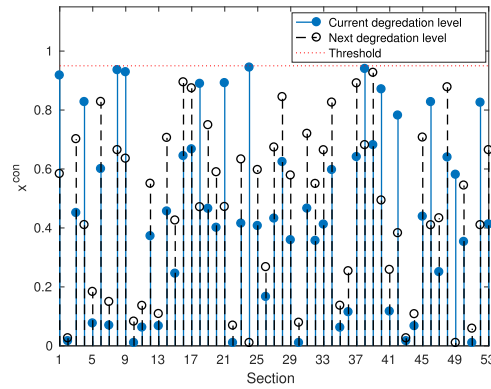**Fig. 7.** High-level simulation results for the line between Den Haag and Rotterdam.

before the next time step.

The multi-level approach is implemented in Matlab R2016b, on a desktop computer with an Intel Xeon E5-1620 eight-core CPU and 64 GB of RAM, running a 64-bit version of SUSE Linux Enterprise Desktop 12. The high-level MILP problem at each time step is solved distributely using the Dantzig-Wolfe decomposition method described in Section 4.5. If the solution obtained at the end of the column generation procedure is fractional, we solve the resulting restricted master problem as a binary MILP problem. The CVRPFC problem at the low level is solved directly. All the MILP and LP problems at the two levels are solved using CPLEX 12.7.

6.2. Discussion of results

A representative run is conducted to demonstrate how the proposed multi-level approach works. The partial results of the high-level MPC controller from the line between Den Haag and Rotterdam are shown in Fig. 7. As the deterioration dynamics and initial risk level of each section is different, the resulting intervention plan is also very different for different sections. For example, replacing is suggested at the first time step for section 27, because its initial risk level is already very high (almost 0.6). On the contrary, grinding is firstly suggested at time step 8, i.e. 24th month within the 5-year planning horizon for section 26, as its initial risk level is almost 0. The next grinding operation is suggested at least 18 months after a replacement, as the growth of the risk level is very slow for a newly replaced section of rail. A maintenance intervention is usually suggested when the risk level is sufficiently high, i.e. over 0.8, to justify the high cost of track maintenance operations. Unlike time-based cyclic maintenance approach, the interval between two consecutive interventions is flexible and ranges from 6 to 9 months.

The simulation results of the whole case study network at a representative time step are shown in Fig. 8. From Fig. 8a we can clearly see that no section in the whole network has a risk level exceeding the critical threshold at time step 7, indicating the network is safe at the current time step. Furthermore, the simulated risk levels of all the sections in the next time step are also below the

(a) Risk levels of the whole railway network at time step 6 (current time step) and time step 7 (next time step).
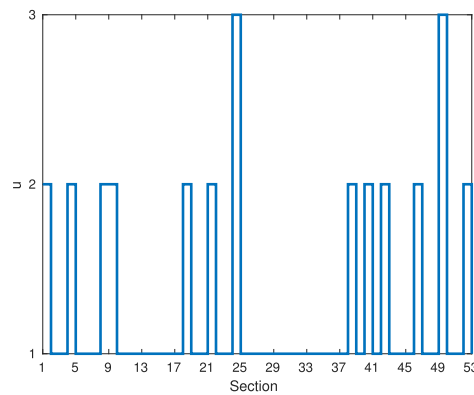
(b) Interventions suggested by the high-level MPC controller at time step 6 for the whole railway network.

Fig. 8. High-level simulation results for the whole railway network at time step 6.

threshold, ensuring the safety of the whole network three months later.

The risk level at time step 7 is the outcome of the intervention at time step 6. As shown in Fig. 8b, 11 sections are to be ground (as indicated by the number of sections where maintenance option $u = 2$) and 2 sections are to be replaced (as indicated by the number of sections where maintenance option $u = 3$) within the three months between time step 6 and 7. The results of the low-level crew scheduling problem to execute these planned grindings over the railway network are shown in Fig. 9. The planned grindings are performed in two different operations. In the first grinding operation, the maintenance crew starts from the maintenance base and drives to Dordrecht. The maintenance crew then spends 2 h grinding section 38 and 40 between Dordrecht and Lage Zwaluwe. It then traverses the "triangle" formed by Lage Zwaluwe, Roosendaal, and Breda, spending one hour grinding one section at each edge of the triangle. Finally the maintenance crew drives the same way back from Lage Zwaluwe to the maintenance base. The total maintenance time in this grinding operation is 5 h, which is less than the allocated 6-h maintenance slot. No additional cost for extra maintenance time is incurred for this operation. Similarly, a second tour is made by the maintenance crew to grind the remaining 6 sections in the other part of the network, as shown by the dotted line in Fig. 9. No additional cost for extra maintenance time is incurred for this operation neither.

6.3. Comparison with centralized MPC

A computational comparison is performed between the centralized MPC approach and the proposed distributed MPC approach. The only difference between the two approaches is that the MPC optimization problem is solved directly by an MILP solver in the centralized MPC approach, and distributedly using Dantzig-Wolfe decomposition method in the distributed MPC approach. We generate 14 MPC optimization problems for 14 fictional railway networks with a number of sections ranging from 10 to 140. The current states and values of uncertain parameters are randomly generated following a normal distribution. The trend of the CPU time
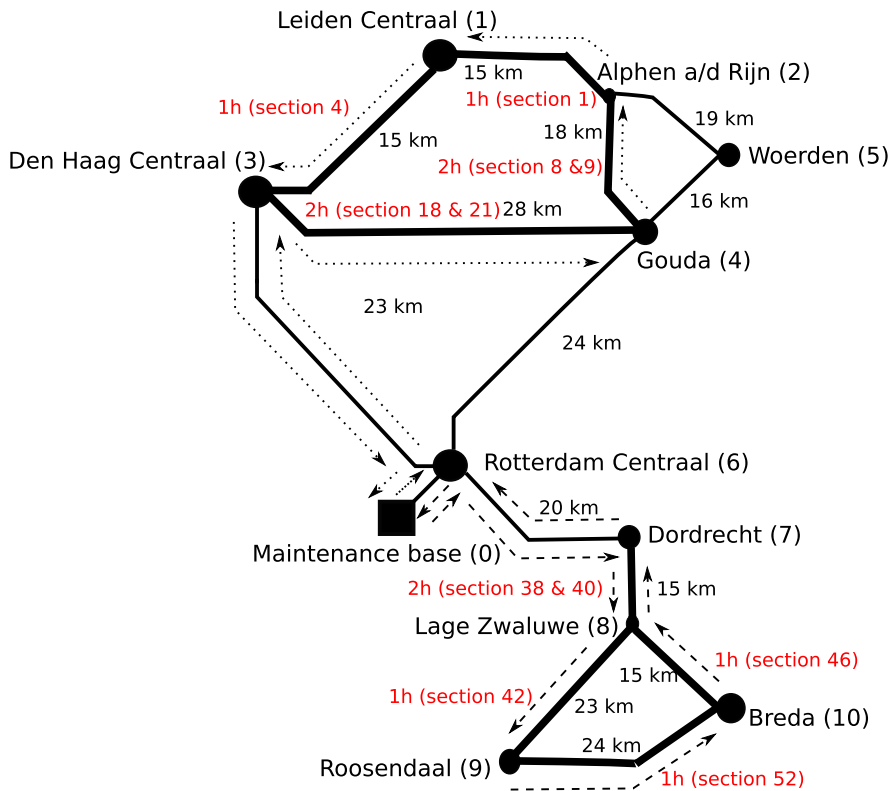
Fig. 9. Result of the low-level maintenance crew scheduling problem at the 6th time step. The railway lines that must be ground within the next time step (3 months) are marked in bold. For each railway line that requires grinding, the exact sections to be ground, and the minimum time to grinding them also provided. The dashed and the dotted arrows show the resulting itinerary of the first and second tour of the maintenance crew, respectively.
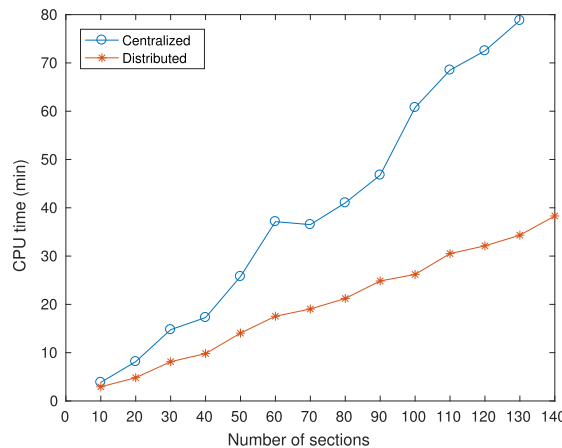

Fig. 10. Computational comparison of centralized MPC and distributed MPC based on the Dantzig-Wolfe decomposition for a prediction horizon $N_P = 3$.

with an increasing number of sections is plotted in Fig. 10. For the first 13 test instances, the distributed approach always takes shorter CPU time than the centralized approach. Moreover, the centralized approach fails when the number of sections becomes 140, due to memory related issues, while the distributed approach can still find a solution within 40 min.

As the distributed approach based on Dantzig-Wolfe decomposition is inexact, for each test instance we also check its relative loss of optimality, compared to the global optimum provided by the centralized approach. For the first 13 test instances, the distributed approach is able to achieve global optimality. As the centralized approach becomes intractable when the number of sections reaches 140, we cannot conclude whether the distributed approach finds the global optimum for the largest test instance.

6.4. Comparison with alternative approaches

In this section we compare the results of the proposed chance-constrained MPC approach to two alternative approaches, namely, the nominal MPC approach and the cyclic approach. The only difference between the nominal MPC approach and the chance-constrained MPC approach is that the nominal approach considers only the mean values of the uncertain parameters in the deterioration model. So it can be viewed as a deterministic counterpart of the chance-constrained MPC approach. The cyclic approach uses a time-based maintenance strategy, and performs grinding and replacing at a fixed optimal interval. Unlike the two MPC approaches, the cyclic approach is an offline approach, i.e. an optimal maintenance intervention plan for the whole planning horizon is computed beforehand and applied to the infrastructure network without updating it using real-time measurements or simulation. The formulation and solution approach of the cyclic approach is presented in Appendix C.

We created ten test instances in which the values of the uncertain parameters are randomly generated following a Gaussian distribution. Three criteria, safety, cost-effectiveness, and computational efficiency, are applied to evaluate the three approaches. Safety is measured by constraint violation v, which is calculated as following:

$$v = \max\left(\frac{x_{\text{worst}}^{\text{con}} - x_{\text{max}}^{\text{con}}}{x_{\text{max}}^{\text{con}}}, \, 0 \right), \tag{83}$$

where $x_{\text{worst}}^{\text{con}}$ is the highest risk level for all sections within the entire planning horizon. Cost-effectiveness is measured by closed-loop performance, which can be calculated by the summation of all the n local objectives (11) evaluated over the entire 5-year planning horizon. Finally, computational efficiency is measured by the CPU time required to solve all the high-level[5] optimization problems at all time steps. We only compare the CPU time of the two MPC approaches, since the cyclic approach is an offline approach in which only one optimization problem must be solved for the entire planning horizon. The summary of the comparison between the three approaches is presented in Table 2.

According to Table 2, the proposed chance-constrained approach is safe, as it has no constraint violation for the 10 test runs. It is also cost-effective, as the closed-loop performance is less than 40% of the reference cyclic approach in almost every test run. Note that in theory there is still a small probability (typically $\epsilon = 0.1$) that the degradation level of a section will exceed the maintenance threshold. A smaller violation level ϵ should be used if a more robust, but also more conservative, maintenance plan is desired by the practitioner. However, the chance-constrained MPC approach is also the slowest in terms of CPU time. It is almost 1700 times slower than the nominal MPC approach. This is because a much larger MILP problem (571 times as large as that of the nominal MPC approach) must be solved at each time step due to the consideration of high-dimensional parameter uncertainty. However, the long computation time does not impair its real-time implementability, as track degradation is a very slow process (3-month sampling time in the case study). The nominal MPC approach is fast and scores the best in closed-loop performance. However, as it does not take into account any uncertainty, the resulting intervention plan is unsafe, as shown by the constraint violations, which indicates degradation levels exceeding the maintenance threshold, in nine out of ten test runs. On the contrary, the cyclic maintenance approach results in very conservative intervention plans which tend to "over-maintain" the asset. The resulting intervention plans are safe (i.e. there is no constraint violation), but not cost-effective (i.e. they gave the worst closed-loop performance).

From the comparison with two alternative approaches, we can conclude that the proposed chance-constrained MPC approach is the most suitable one for track maintenance planning, as it is safe, cost-effective, and real-time implementable.

7. Conclusions and future work

In this paper we have developed an integrated approach for both long-term condition-based maintenance planning and short-term maintenance crew scheduling of a railway infrastructure network. Uncertainties in the deterioration dynamics are taken into account in condition-based maintenance planning, and distributed optimization scheme is adopted to improved the scalability of the proposed approach. An exact MILP formulation is proposed for the optimal scheduling and routing of maintenance crews with flexible maintenance time slot. This integrate approach can be applied to the optimal treatment of typical track defects like squats and ballast defects. The proposed approach has been illustrated by a numerical case study of the optimal treatment of squats for a regional Dutch railway network. Comparison with the centralized approach shows that the adopted distributed optimization scheme based on Dantzig-Wolfe decomposition is scalable. Comparison with two alternative approach shows that the proposed approach yields an excellent trade-off between safety and cost-effectiveness.

In this paper one performance indicator is used to describe the track condition, which is suitable for rail grinding operations. However, in practice multiple indicators might be required to capture all the important parameters related to track health, so that other maintenance activities could also be considered such as tamping to improve alignment, and maintenance of critical track components such as switches and crossings, and insulated rail joints. To manage different maintenance tasks at the same time will require the inclusion of additional constraints to avoid planning maintenance activities that exclude each other. In this case, the proposed approach can be extended by considering multiple deterioration functions for each track section, and combining all the performance indicators in the constraints and objective function. A challenge will be the definition of trade-offs between all performance indicators, so as to capture their relevance to maintenance planning (e.g. to guarantee the health condition of a crossing is more important than grinding a light surface defect). The definition of weights can be avoided by solving a multi-objective

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[5] This is because the same formulation for the low-level problems are used for all three approaches.

Table 2

A comparison between the proposed chance-constrained MPC approach (with subscript "CC"), the nominal approach (with subscript "Nom"), and the cyclic approach (with subscript "Cyc").

Run	Constraint violation			Closed-loop performance			CPU time (h)	
	v_{CC} (%)	v_{Nom} (%)	v_{Cyc} (%)	$\frac{J_{CC}}{J_{Cyc}}$ (%)	$\frac{J_{Nom}}{J_{Cyc}}$ (%)	J_{Cyc}	T_{CC}	T_{Nom}
1	0	0.063	0	39.335	34.148	670502	5.671	0.003
2	0	0.006	0	38.127	36.577	670504	5.075	0.003
3	0	0.353	0	37.635	35.043	670503	5.062	0.003
4	0	0.129	0	37.606	33.344	670502	5.703	0.003
5	0	0	0	36.354	34.536	670502	5.141	0.003
6	0	0.082	0	36.413	35.803	670502	5.802	0.003
7	0	0.021	0	39.425	36.250	670503	5.134	0.003
8	0	0.053	0	38.440	35.028	670500	5.126	0.003
9	0	0.0344	0	40.244	33.359	670503	5.088	0.003
10	0	0.172	0	38.902	34.656	670503	5.082	0.003

optimization version of the problem to address multiple performance criteria optimization. In the future, time-varying models can be considered to describe the changing deterioration process of the railway infrastructure in different seasons. Heterogeneous components, e.g. rail and switches, would also be considered in the maintenance optimization. Multiple types of typical track defects (e.g. head checks, corrugation, and ballast degradation) should also be considered to capture the critical interactions between different maintenance interventions (e.g. grinding and tamping) on the same railway network. Instead of formulating one low-level crew scheduling problem for each type of maintenance intervention, a more optimal schedule can be obtained by formulating the scheduling and routing of all maintenance interventions in the railway network as one single optimization problem, which can be a very large MILP if many maintenance interventions are considered. Distributed optimization methods or efficient heuristics/metaheuristics can be investigated for this challenging problem. Another improvement on the low-level problem is to consider multiple intermediate depots (Crevier et al., 2007) for the low-level planning horizon. This will make the proposed approach more applicable to large-scale railway networks, which in practice usually contain multiple maintenance bases, and the maintenance crew can depart from one base and stop in another base in one maintenance operation, and in the next maintenance period can start from the maintenance base where it stops previously. Finally, joint train scheduling and condition-based maintenance planning can be considered. A more optimal maintenance plan and time table can by obtained by solving a joint optimization problem. This is challenging not just because of the computational complexity, but also the short-term nature of train scheduling and the long-term effects of maintenance. To address these issues, a multi-level approach can be considered to incorporate the fast train traffic dynamics and the slow railway infrastructure degradation.

Acknowledgement

Appendix A. Proof of Theorem 1

Proof. We prove Theorem 1 by induction. First we prove that $x_{j,k+1|k}^{con}$ and $\hat{x}_{j,k+1|k}^{aux}$ are convex in $\widetilde{\theta}_{j,k}$. By definition we have

$$x_{j,k+1|k}^{con}\left(\widetilde{u}_{j,k}, \widetilde{\theta}_{j,k}\right) = f_j^{con}\left(x_{j,k}^{con}, x_{j,k}^{aux}, u_{j,k}, \theta_{j,k}\right)$$

$$\hat{x}_{j,k+1|k}^{aux}\left(\widetilde{u}_{j,k}, \widetilde{\theta}_{j,k}\right) = f_j^{aux}\left(x_{j,k}^{con}, x_{j,k}^{aux}, u_{j,k}, \theta_{j,k}\right).$$

Because f_j^{con} and f_j^{aux} are both convex in $\theta_{j,k}$, $\hat{x}_{j,k+1|k}^{con}$ and $\hat{x}_{j,k+1|k}^{aux}$ are also convex in $\theta_{j,k}$. Moreover, as $\hat{x}_{j,k+1|k}^{con}$ and $\hat{x}_{j,k+1|k}^{aux}$ have no dependence on $\theta_{j,k+2}, ..., \theta_{j,k+N_P}$, we have $\hat{x}_{j,k+1|k}^{con}$ and $\hat{x}_{j,k+1|k}^{aux}$ both convex on $\widetilde{\theta}_{j,k}$.

Then we prove that for any $p \leqslant l - 1$, if $\hat{x}_{j,k+p|k}^{con}$ and $\hat{x}_{j,k+p|k}^{aux}$ are both convex on $\widetilde{\theta}_{j,k}$, then $\hat{x}_{j,k+p+1|k}^{con}$ and $\hat{x}_{j,k+p+1|k}^{aux}$ are also convex on $\widetilde{\theta}_{j,k}$.

Since $\hat{x}_{j,k+p|k}^{con}$ and $\hat{x}_{j,k+p|k}^{aux}$ are both convex on $\widetilde{\theta}_{j,k}$, for any $w_1, w_2 \in \Theta^{N_P}$ and $\alpha \in [0, 1]$, we have:

$$\hat{x}_{j,k+p|k}^{con}(\widetilde{u}_{j,k}, \alpha w_1 + (1 - \alpha)w_2) \leqslant \alpha\hat{x}_{j,k+p|k}^{con}(\widetilde{u}_{j,k}, w_1) + (1 - \alpha)\hat{x}_{j,k+p|k}^{con}(\widetilde{u}_{j,k}, w_2)$$

$$\hat{x}_{j,k+p|k}^{aux}(\widetilde{u}_{j,k}, \alpha w_1 + (1 - \alpha)w_2) \leqslant \alpha\hat{x}_{j,k+p|k}^{aux}(\widetilde{u}_{j,k}, w_1) + (1 - \alpha)\hat{x}_{j,k+p|k}^{aux}(\widetilde{u}_{j,k}, w_2).$$

Denote $(v)_i$ as the i-th entry of vector v. Because f_j^{con} is non-decreasing in $x_{j,k}^{con}$ and $x_{j,k}^{aux}$, we have:

$$\hat{x}_{j,k+p+1|k}^{\text{con}}(\widetilde{u}_{j,k},\ \alpha w_1 + (1-\alpha)w_2)$$

$$= f_j^{\text{con}}\left(\hat{x}_{j,k+p|k}^{\text{con}}\left(\widetilde{u}_{j,k},\ \alpha w_1 + \left(1-\alpha\right)w_2\right),\ \hat{x}_{j,k+p|k}^{\text{aux}}\left(\widetilde{u}_{j,k},\ \alpha w_1 + \left(1-\alpha\right)w_2\right),\ u_{j,k+p+1},\ (\alpha w_1 + (1-\alpha)w_2)_{p+1}\right)$$

$$\leqslant f_j^{\text{con}}(\alpha \hat{x}_{j,k+p|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_1) + (1-\alpha)\hat{x}_{j,k+p|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_2),\ \alpha \hat{x}_{j,k+p|k}^{\text{aux}}(\widetilde{u}_{j,k},\ w_1) + (1-\alpha)\hat{x}_{j,k+p|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_2),$$

$$\widetilde{u}_{j,k},\ (\alpha w_1 + (1-\alpha)w_2)_{p+1})$$

Moreover, because $f_j^{\text{con}}\left(\cdot, \cdot, \widetilde{u}_{j,k},\ \cdot\right)$ is convex for any $\widetilde{u}_{j,k}$, and

$$(\alpha w_1 + (1-\alpha)w_2)_{p+1} = \alpha (w_1)_{p+1} + (1-\alpha)(w_2)_{p+1},$$

we have

$$f_j^{\text{con}}(\alpha \hat{x}_{j,k+p|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_1) + (1-\alpha)\hat{x}_{j,k+p|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_2),\ \alpha \hat{x}_{j,k+p|k}^{\text{aux}}(\widetilde{u}_{j,k},\ w_1) + (1-\alpha)\hat{x}_{j,k+p|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_2), \tag{84}$$

$$\widetilde{u}_{j,k},\ (\alpha w_1 + (1-\alpha)w_2)_{p+1}) \tag{85}$$

$$\leqslant \alpha f_j^{\text{con}}\left(\hat{x}_{j,k+p|k}^{\text{con}}\left(\widetilde{u}_{j,k},\ w_1\right),\ \hat{x}_{j,k+p|k}^{\text{aux}}\left(\widetilde{u}_{j,k},\ w_1\right),\ \widetilde{u}_{j,k},\ (w_1)_{p+1}\right) \tag{86}$$

$$\left(1-\alpha\right)f_j^{\text{con}}\left(\hat{x}_{j,k+p|k}^{\text{con}}\left(\widetilde{u}_{j,k},\ w_2\right),\ \hat{x}_{j,k+p|k}^{\text{aux}}\left(\widetilde{u}_{j,k},\ w_2\right),\ \widetilde{u}_{j,k},\ (w_2)_{p+1}\right) \tag{87}$$

$$= \alpha \hat{x}_{j,k+p+1|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_1) + (1-\alpha)\hat{x}_{j,k+p+1|k}^{\text{con}}(\widetilde{u}_{j,k},\ w_2) \tag{88}$$

Thus $\hat{x}_{j,k+p+1|k}^{\text{con}}$ is convex in $\widetilde{\theta}_{j,k}$. Similarly, we can prove $\hat{x}_{j,k+p+1|k}^{\text{aux}}$ is also convex in $\widetilde{\theta}_{j,k}$. □

Appendix B. Simulation model

The simulation model, that describes the evolution of the failure probability of one individual squat over time, is based on the big data analysis approach developed by Jamshidi et al. (2017). The probability that squat i at time step k might lead to rail failure can be calculated by:

$$\xi_{i,k} = (f_{\text{Prob}} \circ f_{\text{Cr}} \circ f_{\text{M}})(L_{i,k-1},\ L_{i,k}), \tag{89}$$

where $L_{i,k-1}$ and $L_{i,k}$ are two consecutive measurements on the visual length of squat i. The function f_{M} computes the estimated MGT from two consecutive measurements/simulated data on visual lengths, and the function f_{Cr} estimates the crack length growth from the estimated MGT. Finally, the function f_{Prob} calculates the failure probability from the crack growth length. We use the same functions as in the case study by Jamshidi et al. (2017).

For identification of function f_j^{Deg} and f_j^{Gr}, we create 200 pseudo sections, where the number of squats within a section is a random number with a mean value of 10 and standard deviation of 2. Let N_{Sq} denote the number of squats in a section of rail. The failure probability of one section of rail can then be calculated by:

$$x_k^{\text{con}} = 1 - \prod_{i=1}^{N_{\text{Sq}}} \left(1 - \xi_{i,k}\right). \tag{90}$$

The following squat evolution model is used to simulated the dynamics of the visual length of an individual squat i:

$$L_{i,k+1} = \begin{cases} aL_{i,k} + b & \text{if not treated} \\ \max(\phi(L_{i,k} - L_{\text{eff}}),\ 0) & \text{if ground} \end{cases}, \tag{91}$$

where a, b, and ϕ are all generated parameters from a normal distribution. For each squat i within a section, we can simulate three consecutive measurements of the visual length, i.e. $L_{i,k-1}$, $L_{i,k}$ and $L_{i,k+1}$, and calculate its failure probability at time step k and $k + 1$, namely, $\xi_{i,k}$ and $\xi_{i,k+1}$, respectively. The condition of the section at time step k and $k + 1$ can then be calculated using (90). Five sets of squat evolution models are used, resulting in five different condition deterioration models, which are randomly assigned to the 53 sections following a uniform discrete distribution in the case study.

Appendix C. Cyclic maintenance approach

In this section we describe the cyclic maintenance approach used in Section 6.4 as a comparison to the proposed MPC approach. Let $t_{0,j}$ and $T_{\text{Gr},j}$ denote the time instant of the first replacement and the fixed cycle of grinding for the j-th section, respectively. Furthermore, we assume that replacement is performed after r times of consecutive grinding since the last replacement on section j. Let k_{end} denote the planning horizon. The offline optimization problem the cyclic maintenance approach can be formulated as:

$$\min_{t_0, T_{Gr}, r} \sum_{k=1}^{k_{end}} \sum_{j=1}^{n} x_{j,k}^{con} + \lambda \sum_{q=2}^{3} c_{q,j}^{Maint} I_{u_{j,k}=q} \tag{92}$$

subject to

$$x_{j,k+1} = f_j(x_{j,k}, u_{j,k}; \mathbb{E}(\theta_{j,k})) \quad \forall j \in \{1, ..., n\}, \ \forall k \in \{0, ..., k_{end} - 1\} \tag{93}$$

$$x_{j,k}^{con} \leqslant x_{max}^{con}, \quad x_{j,k}^{aux} \leqslant x_{max}^{aux} \quad \forall j \in \{1, ..., n\}, \ \forall k \in \{1, ..., k_{end}\} \tag{94}$$

$$u_{j,k} = \begin{cases} 2, & \text{if } (k - t_{0,j}) \bmod \text{round}(T_{Gr,j}) = 0 \\ 3, & \text{if } k = t_{0,j} \text{ or } (k - t_{0,j}) \bmod \text{round}(rT_{Gr,j}) = 0 \\ 1, & \text{otherwise} \end{cases} \tag{95}$$

$$\forall j \in \{1,...,n\}, \ \forall k \in \{1,..., k_{end}\}$$
$$1 \leqslant t_{0,j} \leqslant T_{max} \quad \forall j \in \{1,...,n\} \tag{96}$$

$$1 \leqslant T_{j,Gr} \leqslant T_{max} \quad \forall j \in \{1, ..., n\} \tag{97}$$

$$1 \leqslant \mu_j \leqslant \mu_{max} \quad \forall j \in \{1, ..., n\}. \tag{98}$$

Appendix D. Supplementary material

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.trc.2019.05.045.

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