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SUMMARY

In Ref. 1 the author established the existence of a new class of exact subharmonic solutions to Duffing's equation, without damping, i.e., the term in $\dot{x}$ is absent. The present study is concerned with the full equation of Duffing, with damping present, and it is shown that, provided the damping coefficient, b, is sufficiently small, there exists a class of exact subharmonic solutions which stem from a sub-class of exact pure-subharmonic solutions.

CONTENTS

1. Introduction

The equation

$$\ddot{x} + b\dot{x} + c_1 x + c_3 x^3 = Q \operatorname{Sin} \omega t, \; b \neq 0, \tag{1.1}$$

is known as Duffing's equation, with damping. The only known broad class of subharmonic solutions to this equation is that related to the condition $c_3 << c_1$, in which the solution takes the form of a real Fourier series whose leading term is the dominant subharmonic. See for example Stoker, Ref. 2, pp. 103-109. Stoker's account is concerned with subharmonics of order 1/3 and it is demonstrated that these do not occur when $\omega^2 = 9c_1$. Further, the conclusion is drawn that these subharmonics cannot occur unless the damping coefficient, b, is of the same order as c_3, i.e. $b << c_1$.

In an earlier paper, Ref. 1, the author established the existence of a new class of pure subharmonic solutions of order 1/3 (i.e. with terms in $\operatorname{Sin} \frac{1}{3}\omega t$ and $\operatorname{Cos} \frac{1}{3}\omega t$ present, only) for Duffing's equation with $b = 0$ and went on to demonstrate the existence of an associated, broader, class of solutions which could be represented by a Fourier series whose minimum frequency was $\omega/3$. These solutions were shown to exist in intervals $-1 \leqslant 9c_1/\omega^2 < 1$, $1 < 9c_1/\omega^2 \leqslant 3$, with c_3 unrestricted. That no subharmonic solution of order 1/3 could be shown to exist when $\omega^2 = 9c_1$ agreed with Stoker's first conclusion. However, neither of the two treatments can be taken as proof that no subharmonic of order 1/3 exists when $\omega^2 = 9c_1$.

The present study is a natural continuation of the work in Ref. 1. Starting from a pure subharmonic solution of order 1/3 of equation (1.1), $b = 0$, the combined functional analytic, topological, method will be used to explore the existence of associated subharmonic solutions representable by Fourier series of minimum frequency $\omega/3$.

2. An Extended Class of Subharmonic Solutions

By writing $3\theta = \omega t$, equation (1.1) may be reduced to

$$x'' + hx' + g_1 x + g_3 x^3 - \Gamma \operatorname{Sin} 3\theta = 0, \; (h, g_3 \neq 0) \tag{2.1}$$

where

$$\left.\begin{aligned} h &= 3b/\omega \\ g_1 &= 9c_1/\omega^2 \\ g_3 &= 9c_3/\omega^2 \\ \text{and} \quad \Gamma &= 9Q/\omega^2 \end{aligned}\right\} \tag{2.2}$$

Starting with the case $h = 0$, $\Gamma = \Gamma_e$, there exist certain subharmonic solutions described in Ref. 1, equation (2.5). It should be noted that in Ref. 1, equation (2.16) should read $g_1 = 1 - 3g_3 a_e^2$. The family of straight lines in the g_1, g_3 plane are then as shown in Fig. 1. With g_1, g_3 and $\Gamma = \Gamma_e$ fixed and $h > 0$ it is anticipated that there will exist a subharmonic solution of (2.1) of the form

$$x = \sum_{n=0}^{\infty} \{a_{2n+1} \operatorname{Sin} (2n+1)\theta + b_{2n+1} \operatorname{Cos} (2n+1)\theta\} \tag{2.3}$$

As before the first approximation will be the pure-subharmonic solution

$$x = a_e \operatorname{Sin} \theta + b_e \operatorname{Cos} \theta, \tag{2.4}$$

whilst the second approximation will be

$$x = a_1 \operatorname{Sin} \theta + b_1 \operatorname{Cos} \theta + a_3 \operatorname{Sin} 3\theta + b_3 \operatorname{Cos} \theta \tag{2.5}$$

It then follows that

$$x^3 = \alpha_{10} \operatorname{Sin} \theta + \beta_{10} \operatorname{Cos} \theta + \alpha_{30} \operatorname{Sin} 3\theta + \dots + \beta_{90} \operatorname{Cos} 9\theta, \tag{2.6}$$

where

$$\alpha_{10} = \frac{3}{4} a_1(a_1^2 + b_1^2) - \frac{3}{4} a_3(a_1^2 - b_1^2) - \frac{3}{2} a_1 b_1 b_3 + \frac{3}{2} a_1(a_3^2 + b_3^2)$$

$$\beta_{10} = \frac{3}{4} b_1(a_1^2 + b_1^2) - \frac{3}{4} b_3(a_1^2 - b_1^2) + \frac{3}{2} a_1 b_1 a_3 + \frac{3}{2} b_1(a_3^2 + b_3^2)$$

$$\alpha_{30} = \frac{1}{4} a_1(3b_1^2 - a_1^2) + \frac{3}{2} a_3(a_1^2 + b_1^2) + \frac{3}{4} a_3(a_3^2 + b_3^2)$$

$$\beta_{30} = -\frac{1}{4} b_1(3a_1^2 - b_1^2) + \frac{3}{2} b_3(a_1^2 + b_1^2) + \frac{3}{4} b_3(a_3^2 + b_3^2)$$

$$\alpha_{50} = -\frac{3}{4} a_3(a_1^2 - b_1^2) + \frac{3}{2} a_1 b_1 b_3 + \frac{3}{4} a_1(a_3^2 - b_3^2) + \frac{3}{2} b_1 a_3 b_3$$

$$\beta_{50} = -\frac{3}{4} b_3(a_1^2 - b_1^2) - \frac{3}{2} a_1 b_1 a_3 - \frac{3}{4} b_1(a_3^2 - b_3^2) + \frac{3}{2} a_1 a_3 b_3$$

$$\alpha_{70} = -\frac{3}{4} a_1(a_3^2 - b_3^2) + \frac{3}{2} b_1 a_3 b_3$$

$$\beta_{70} = -\frac{3}{4} b_1(a_3^2 - b_3^2) - \frac{3}{2} a_1 a_3 b_3$$

$$\alpha_{90} = \frac{1}{4} a_3(3b_3^2 - a_3^2)$$

$$\beta_{90} = -\frac{1}{4} b_3(3a_3^2 - b_3^2)$$

Also

$$\left.\begin{aligned} x' &= a_1 \text{ Cos } \theta - b_1 \text{ Sin } \theta + 3a_3 \text{ Cos } 3\theta - 3b_3 \text{ Sin } 3\theta \\ &\text{and} \\ x'' &= -a_1 \text{ Sin } \theta - b_1 \text{ Cos } \theta - 9a_3 \text{ Sin } 3\theta - 9b_3 \text{ Cos } 3\theta \end{aligned}\right\} \quad (2.7)$$

Approximately x^3 by

$$x^3 = \alpha_{10} \text{ Sin } \theta + \beta_{10} \text{ Cos } \theta + \alpha_{30} \text{ Sin } 3\theta + \beta_{30} \text{ Cos } 3\theta$$

and substituting for x, x^3, x' and x'' in (2. 1) and equating coefficients of the distinct terms, i. e. Sin θ, Cos θ, Sin 3θ and Cos 3θ, respectively, to zero, gives rise to the following simultaneous equations:

$$(g_1 - 1)a_1 - hb_1 + g_3\alpha_{10} = 0 \quad (2.8)$$

$$(g_1 - 1)b_1 + ha_1 + g_3\beta_{10} = 0 \quad (2.9)$$

$$(g_1 - 9)a_3 - 3hb_3 + g_3\alpha_{30} = \Gamma_e \quad (2.10)$$

$$(g_1 - 9)b_3 + 3ha_3 + g_3\beta_{30} = 0 \quad (2.11)$$

The corresponding equations defining the pure subharmonic solution are, from Ref. 1,

$$(g_1 - 1)a_e + \frac{3}{4} g_3 a_e (a_e^2 + b_e^2) = 0 \quad (2.12)$$

$$(g_1 - 1)b_e + \frac{3}{4} g_3 b_e (a_e^2 + b_e^2) = 0 \quad (2.13)$$

$$\frac{1}{4} g_3 a_e (3b_e^2 - a_e^2) = \Gamma_e \quad (2.14)$$

$$-\frac{1}{4} g_3 b_e (3a_e^2 - b_e^2) = 0 \quad (2.15)$$

Writing $a_1 = (1 + \epsilon_1)a_e$, $b_1 = (k_1 + \epsilon_2)a_e$, $a_3 = \epsilon_3 a_e$, $b_3 = \epsilon_4 a_e$ and subtracting (2. 12) from (2. 8), (2. 13) from (2. 9), (2. 14) from (2. 10) and (2. 15) from (2. 11) gives, after division by $a_e \neq 0$ and the substitution

$$g_3 a_e^2 = \frac{1}{3} k_3 (g_1 - 1),$$

where $k_3 = -4$ or -1,

$$(g_1 - 1)\epsilon_1 - h\epsilon_2 + \frac{1}{4} k_3 (g_1 - 1)\{(3 + k_1^2)\epsilon_1 + 2k_1\epsilon_2 + (k_1^2 - 1)\epsilon_3 - 2k_1\epsilon_4 + \frac{4}{3} G_1(\ )\} = hk_1 \quad (2.16)$$

$$(g_1 - 1)\epsilon_2 + h\epsilon_1 + \frac{1}{4} k_3(g_1 - 1)\{2k_1\epsilon_1 + (1 + 3k_1^2)\epsilon_2 + 2k_1\epsilon_3 + (k_1^2 - 1)\epsilon_4 + \frac{4}{3} G_2(\ )\} = -h \qquad (2.17)$$

$$(g_1 - 9)\epsilon_3 - 3h\epsilon_4 + \frac{1}{4} k_3(g_1 - 1)\{(k_1^2 - 1)\epsilon_1 + 2k_1\epsilon_2 + 2(1 + k_1^2)\epsilon_3 + \frac{4}{3}G_3(\ )\} = 0 \qquad (2.18)$$

$$(g_1 - 9)\epsilon_4 + 3h\epsilon_3 + \frac{1}{4} k_3(g_1 - 1)\{-2k_1\epsilon_1 + (k_1^2 - 1)\epsilon_2 + 2(1 + k_1^2)\epsilon_4 + \frac{4}{3}G_4(\ )\} = 0 \qquad (2.19)$$

where

$$G_1(\epsilon_1, \dots, \epsilon_4) = \frac{3}{4}(1 + \epsilon_1)(\epsilon_1^2 + \epsilon_2^2) + \frac{3}{2}(1 + \epsilon_1)(\epsilon_3^2 + \epsilon_4^2) + \frac{3}{2}\epsilon_1(\epsilon_1 + k_1\epsilon_2 - \epsilon_3 - k_1\epsilon_4) + \frac{3}{2}\epsilon_2(k_1\epsilon_3 - \epsilon_4 - \epsilon_1\epsilon_4) + \frac{3}{4}\epsilon_3(k_1\epsilon_2^2 - \epsilon_1^2) \qquad (2.20)$$

$$G_2(\epsilon_1, \dots, \epsilon_4) = \frac{3}{4}(k_1 + \epsilon_2)(\epsilon_1^2 + \epsilon_2^2) + \frac{3}{2}(k_1 + \epsilon_2)(\epsilon_3^2 + \epsilon_4^2) + \frac{3}{2}\epsilon_1(\epsilon_2 + k_1\epsilon_3 - \epsilon_4) + \frac{3}{2}\epsilon_2(k_1\epsilon_2 + \epsilon_3 + k_1\epsilon_4 + \epsilon_1\epsilon_3) + \frac{3}{4}\epsilon_4(\epsilon_2^2 - \epsilon_1^2) \qquad (2.21)$$

$$G_3(\epsilon_1, \dots, \epsilon_4) = \frac{3}{4}(\epsilon_2^2 - \epsilon_1^2) + \frac{3}{2}\epsilon_3(\epsilon_1^2 + \epsilon_2^2) + \frac{3}{4}\epsilon_3(\epsilon_3^2 + \epsilon_4^2) + \frac{1}{4}\epsilon_1(3\epsilon_2^2 - \epsilon_1^2) + \frac{3}{2}\epsilon_1\epsilon_3 + 3k_1\epsilon_2\epsilon_3 \qquad (2.22)$$

$$G_4(\epsilon_1, \dots, \epsilon_4) = \frac{3}{4}k_1(\epsilon_2^2 - \epsilon_1^2) + \frac{3}{2}\epsilon_4(\epsilon_1^2 + \epsilon_4^2) + \frac{3}{4}\epsilon_4(\epsilon_3^2 + \epsilon_4^2) + \frac{1}{4}\epsilon_2(\epsilon_2^2 + 3\epsilon_1^2) - \frac{3}{2}\epsilon_1\epsilon_2 + 3\epsilon_1\epsilon_4 + 3k_1\epsilon_2\epsilon_4 \qquad (2.23)$$

The task of determining $\epsilon_1, \dots, \epsilon_4$ in terms of h from these four simultaneous cubic equations would be formidable, this, however, is not required. The reason for deriving this system of equations is firstly that they define a mapping M_o, used in the existence theorem, and secondly they provide a guide to the choice of a four-cell Λ, also used in the existence theorem. For the latter purpose let $\epsilon_1, \dots, \epsilon_4$ be of the first order of small quantities compared with unity, then $G_1, \dots, G_4$ contain terms of the second and higher orders of small quantities only. Under these conditions equations (2.16) to (2.19) may be adequately approximated by the linear equations

$$\{1 + \frac{1}{4}k_3(3 + k_1^2)\}\epsilon_1 + \{\frac{1}{2}k_1k_3 - \Omega_1\}\epsilon_2 + \frac{1}{4}k_3(k_1^2 - 1)\epsilon_3 - \frac{1}{2}k_1k_3\epsilon_4 = k_1\Omega_1 \qquad (2.24)$$

$$\left\{\frac{1}{2}k_1k_3 + \Omega_1\right\}\epsilon_1 + \left\{1 + \frac{1}{4}k_3(1 + 3k_1^2)\right\}\epsilon_2 + \frac{1}{2}k_1k_3\epsilon_3 + \frac{1}{4}k_3(k_1^2 - 1)\epsilon_4 = -\Omega_1 \tag{2.25}$$

$$\frac{1}{4}k_3(k_1^2 - 1)\epsilon_1 + \frac{1}{2}k_1k_3\epsilon_2 + \left\{\frac{1}{2}k_3(1 + k_1^2) + \Omega_2\right\}\epsilon_3 - 3\Omega_1\epsilon_4 = 0 \tag{2.26}$$

$$-\frac{1}{2}k_1k_3\epsilon_1 + \frac{1}{4}k_3(k_1^2 - 1)\epsilon_2 + 3\Omega_1\epsilon_3 + \left\{\frac{1}{2}k_3(1 + k_1^2) + \Omega_2\right\}\epsilon_4 = 0 \tag{2.27}$$

where

$$\Omega_1 = h/(g_1 - 1)$$

and

$$\Omega_2 = (g_1 - 9)/(g_1 - 1) \tag{2.28}$$

Consider the solution of these equations for $\epsilon_1, \ldots, \epsilon_4$ in the two cases, separately

Case (i) $k_1 = 0, \; k_2 = 1, \; k_3 = -4$

The equations become

$$\left.\begin{aligned} -2\epsilon_1 - \Omega_1\epsilon_2 + \epsilon_3 &= 0 \\ \Omega_1\epsilon_1 + \epsilon_4 &= -\Omega_1 \\ \epsilon_1 + (\Omega_2 - 2)\epsilon_3 - 3\Omega_1\epsilon_4 &= 0 \\ \epsilon_2 + 3\Omega_1\epsilon_3 + (\Omega_2 - 2)\epsilon_4 &= 0, \end{aligned}\right\} \tag{2.29}$$

with solutions

$$\epsilon_1 = -\Delta_1/\Delta_o, \; \epsilon_2 = \Delta_2/\Delta_o, \; \epsilon_3 = -\Delta_3/\Delta_o, \; \epsilon_4 = \Omega_1(1 - \epsilon_1)$$

where

$$\Delta_o = 2(\Omega_2 - 2) + (1 + 3\Omega_1)^2 - \Omega_1^2(\Omega_2 - 2)^2$$

$$\Delta_1 = -\Omega_1^2\{(\Omega_2 - 2)^2 + 9\Omega_1^2 - 3\Omega_1\}$$

$$\Delta_2 = -\Omega_1\{(\Omega_2 - 2) + 2(\Omega_2 - 2)^2 + 9\Omega_1^2\}$$

and

$$\Delta_3 = -\Omega_1^2\{6 + (\Omega_2 - 2) + 6\Omega_1^2(\Omega_2 - 2)\}$$

This means that $\epsilon_1, \ldots, \epsilon_4$ may be expressed in the form

$$\epsilon_1 = \Omega_1 \Phi_1(\Omega_1, \Omega_2), \ldots\ldots\ldots\ldots, \epsilon_4 = \Omega_1 \Phi_4(\Omega_1, \Omega_2) \qquad (2.30)$$

where the $\Phi_i(\Omega_1, \Omega_2)$, $i = 1, \ldots, 4$ all tend to finite values as $\Omega_1 \to 0$.

Thus $\epsilon_1, \ldots, \epsilon_4 \to 0$ as $\Omega_1 \to 0$ or $h \to 0$.

Case (ii) $k_1 = 3^{\frac{1}{2}}$, $k_2 = -2$, $k_3 = -1$

The equations become

$$-\frac{1}{2}\epsilon_1 - \left(\frac{1}{2} \times 3^{\frac{1}{2}} + \Omega_1\right)\epsilon_2 - \frac{1}{4}\epsilon_3 + \frac{1}{2} \times 3^{\frac{1}{2}}\epsilon_4 = 3^{\frac{1}{2}}\Omega_1$$

$$\left(-\frac{1}{2} \times 3^{\frac{1}{2}} + \Omega_1\right)\epsilon_1 - \frac{1}{2}\epsilon_2 - \frac{1}{2} \times 3^{\frac{1}{2}}\epsilon_3 - \frac{1}{4}\epsilon_4 = -\Omega_1$$

$$-\frac{1}{4}\epsilon_1 - \frac{1}{2} \times 3^{\frac{1}{2}}\epsilon_2 + \{-2 + \Omega_1\}\epsilon_3 - 3\Omega_1\epsilon_4 = 0$$

$$\frac{1}{2} \times 3^{\frac{1}{2}}\epsilon_1 - \frac{1}{4}\epsilon_2 + 3\Omega_1\epsilon_3 + \{-2 + \Omega_1\}\epsilon_4 = 0,$$

which again have solutions in the form

$$\epsilon_i = \Omega_1 \Phi_i(\Omega_1, \Omega_2)$$

and, therefore, $\epsilon_1, \ldots, \epsilon_4 \to 0$ as $\Omega_1 \to 0$ or $h \to 0$.

3. The Existence Theorem

The existence theorem, which will be used to prove the existence of the solution (2.3) of equation (2.1), is essentially the same as that described in Ref. 1. However, because of the presence of the term hx' in (2.1) it is now necessary to formulate the proof in terms of vectors and this gives rise to important differences of detail from the scalar technique set out in Ref. 1.

Consider the function space S of all real periodic vector functions $x(\theta) = (x_1, \ldots, x_j, \ldots, x_n)$ defined by Fourier series of the form

$$x_j(\theta) = \sum_{s=0}^{\infty} \{ a_{j,2s+1} \operatorname{Sin}(2s+1)\theta + b_{j,2s+1} \operatorname{Cos}(2s+1)\theta \}, \qquad (3.1)$$

and having a norm $\nu(x)$ defined by

$$\nu(x) = \max \nu(x_j), \quad j = 1,\ldots,n, \tag{3.2}$$

where

$$\nu(x_j) = \{ (2\pi)^{-1} \int_0^{2\pi} x_j^2(\theta) d\theta \}^{\frac{1}{2}} . \tag{3.3}$$

A projection operator P may be defined in S by the relations

$$Px = (P_1 x_1, \ldots, P_n x_n), \tag{3.4}$$

$$P_j x_j = \sum_{s=0}^{m} \{ a_{j,2s+1} \text{ Sin } (2s+1)\theta + b_{j,2s+1} \text{ Cos } (2s+1)\theta \}, \tag{3.5}$$

and, by definition, $P = P^2$.

The subspace $\widetilde{S}$ of S is defined by

$$\widetilde{S} = \{ x : x \epsilon S, \ Px = 0 \}, \tag{3.6}$$

and, thereby, if $x \epsilon \widetilde{S}$ then

$$x_j(\theta) = \sum_{s=m+1}^{\infty} \{ a_{j,2s+1} \text{ Sin } (2s+1)\theta + b_{j,2s+1} \text{ Cos } (2s+1)\theta \} \tag{3.7}$$

Define the operator H on $\widetilde{S}$ by the relation

$$Hx_j = \sum_{s=m+1}^{\infty} (2s+1)^{-1} \{ -a_{j,2s+1} \text{ Cos } (2s+1)\theta + b_{j,2s+1} \text{ Sin } (2s+1)\theta \}, \tag{3.8}$$

which corresponds to the integration of $x_j \epsilon \widetilde{S}$ with the constants of integration taken to be zero.

Write equation (2.1) in the system form

$$\left. \begin{aligned} x_1' &= q_1(x_2) = x_2 \\ x_2' &= q_2(x_1, x_2, \theta) = -g_1 x_1 - g_3 x_1^3 - hx_2 + \Gamma \text{ Sin } 3\theta, \end{aligned} \right\} \tag{3.9}$$

where q_1, q_2 are the components of the vector operator q on S. Further, consider the operators f, F and T on S defined by

$$fx = qx - Pqx \tag{3.10}$$

$$Fx = Hfx \tag{3.11}$$

and

$$y = Tx = Px + Fx; \tag{3.12}$$

or, in more detail,

$$fx = \begin{bmatrix} (fx)_1 \\ (fx)_2 \end{bmatrix} = \begin{bmatrix} x_2 - Px_2 \\ -g_1x_1 - g_3x_1^3 - hx_2 + \Gamma \text{ Sin } 3\theta + P(g_1x_1 + g_3x_1^3 + hx_2 - \Gamma \text{Sin } 3\theta) \end{bmatrix} \tag{3.13}$$

$$Fx = Hfx = \begin{bmatrix} H(x_2 - Px_2) \\ H\{-g_1x_1 - g_3x_1^3 - hx_2 + \Gamma \text{ Sin } 3\theta + P(g_1x_1 + g_3x_1^3 + hx_2 - \Gamma \text{Sin } 3\theta\} \end{bmatrix} \tag{3.14}$$

and

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (Tx)_1 \\ (Tx)_2 \end{bmatrix} = \begin{bmatrix} Px_1 + (Fx)_1 \\ Px_2 + (Fx)_2 \end{bmatrix}$$

$$= \begin{bmatrix} Px_1 + H(x_2 - Px_2) \\ Px_2 + H\{-g_1x_1 - g_3x_1^3 - hx_2 + \Gamma \text{ Sin } 3\theta + P(g_1x_1 + g_3x_1^3 + hx_2 - \Gamma \text{Sin } 3\theta)\} \end{bmatrix} \tag{3.15}$$

By placing certain bounds on $\nu(x)$, $|x|$, $\nu(x - Px)$ and $|x - Px|$ it is possible to define a subspace S* of S and, provided certain inequalities are satisfied, it will be shown that T : S* $\rightarrow$ S* and is also a contraction mapping. Because T is a contraction in S*, Banach's fixed point theorem (See Ref. 5, p.141) may be invoked to conclude that $y(\theta)$ exists uniquely in S* and is continuously dependent on the approximate solution (2.5). This means that a_5, b_5, a_7, ..., are uniquely determined by and continuously dependent on $a_1, \ldots, b_3$.

If $y(\theta) \epsilon S^*$ is a fixed element of T in S^*, then

$$y = Py + Fy$$

or

$$y - Py = \begin{bmatrix} y_1 - Py_1 \\ y_2 - Py_2 \end{bmatrix}$$

$$= \begin{bmatrix} H(y_2 - Py_2) \\ H\{-g_1 y_1 - g_3 y_1^3 - hy_2 + \Gamma \text{ Sin } 3\theta + P(g_1 y_1 + g_3 y_1^3 + hy_2 - \Gamma \text{ Sin } 3\theta)\} \end{bmatrix}$$

Differentiating this expression with respect to θ gives

$$y_1' = y_2 + P(y_1' - y_2)$$

$$y_2' = -g_1 y_1 - g_3 y_1^3 - hy_2 + \Gamma \text{ Sin } 3\theta + P(y_2' + g_1 y_1 + g_3 y_1^3 + hy_2 - \Gamma \text{ Sin } 3\theta)$$

Thus $y(\theta)$ will satisfy (3.9) provided

$$P\,(y_1' - y_2) = 0 \qquad (3.16)$$

$$P\,(y_2' + g_1 y_1 + g_3 y_1^3 + hy_2 - \Gamma \text{ Sin } 3\theta) = 0 \qquad (3.17)$$

If $y(\theta)$ is the fixed element described, then $y = x$ and $y_j(\theta)$ will be given by (3.1). Choosing $m = 1$, then

$$Py_j = a_{j1} \text{ Sin } \theta + b_{j1} \text{ Cos } \theta + a_{j3} \text{ Sin } 3\theta + b_{j3} \text{ Cos } 3\theta \qquad (3.18)$$

and

$$Py_j' = a_{j1} \text{ Cos } \theta - b_{j1} \text{ Sin } \theta + 3a_{j3} \text{ Cos } 3\theta - 3b_{j3} \text{ Sin } 3\theta \qquad (3.19)$$

Also, writing

$$y_1^3(\theta) = \alpha_1 \text{ Sin } \theta + \beta_1 \text{ Cos } \theta + \alpha_3 \text{ Sin } 3\theta + \beta_3 \text{ Cos } 3\theta + \alpha_5 \text{ Sin } 5\theta + \ldots \qquad (3.20)$$

then

$$Py_1^3(\theta) = \alpha_1 \text{ Sin } \theta + \beta_1 \text{ Cos } \theta + \alpha_3 \text{ Sin } 3\theta + \beta \text{ Cos } 3\theta \qquad (3.21)$$

Substituting from (3.18) and (3.19) into (3.16) and equating the coefficients of the distinct terms to zero then gives the relations

$$a_{11} = b_{21},\ b_{11} = -a_{21},\ 3a_{13} = b_{23} \text{ and } 3b_{13} = -a_{23} \tag{3.22}$$

A similar process, using (3.18), (3.19) and (3.21), applied to (3.17) then yields the relations

$$\left.\begin{aligned}
g_1a_{11} - b_{21} + ha_{21} + g_3\alpha_1 &= 0\\
g_1b_{11} + a_{21} + hb_{21} + g_3\beta_1 &= 0\\
g_1a_{13} - 3b_{23} + ha_{23} + g_3\alpha_3 - \Gamma &= 0\\
g_1b_{13} + 3a_{23} + hb_{23} + g_3\beta_3 &= 0
\end{aligned}\right\} \tag{3.23}$$

Substituting from (3.22) into (3.23) then gives the exact set of determining equations

$$\left.\begin{aligned}
V_1 &\equiv (g_1 - 1)a_{11} - hb_{11} + g_3\alpha_1 = 0\\
U_1 &\equiv (g_1 - 1)b_{11} + ha_{11} + g_3\beta_1 = 0\\
V_3 &\equiv (g_1 - 9)a_{13} - 3hb_{13} + g_3\alpha_3 - \Gamma = 0\\
U_3 &\equiv (g_1 - 9)b_{13} + 3ha_{13} + g_3\beta_3 = 0
\end{aligned}\right\} \tag{3.24}$$

The corresponding approximate set of determining equations (2.8) to (2.11) may be re-written in a similar form

$$\left.\begin{aligned}
v_1 &\equiv (g_1 - 1)a_{11} - hb_{11} + g_3\alpha_{10}\\
u_1 &\equiv (g_1 - 1)b_{11} + ha_{11} + g_3\beta_{10}\\
v_3 &\equiv (g_1 - 9)a_{13} - 3hb_{13} + g_3\alpha_{30} - \Gamma_e\\
u_3 &\equiv (g_1 - 9)b_{13} + 3ha_{13} + g_3\beta_{30}
\end{aligned}\right\} \tag{3.25}$$

where $a_{11} = a_1$, $b_{11} = b_1$, $a_{13} = a_3$ and $b_{13} = b_3$.

Denote by Λ the four-cell defined by $|a_1| \leqslant \mu_1|a_e|$, $|b_1| \leqslant \mu_2|a_e|$, $|a_3| \leqslant \mu_3|a_e|$, $|b_3| \leqslant \mu_4|a_e|$, $\mu_1, \mu_2, \mu_3, \mu_4 > 0$, in the Euclidean four-space of

Cartesian co-ordinates a_1, b_1, a_3, b_3. Let M and M_o be mappings, described by equations (3.24) and (3.25), respectively, from the vector space of components (a_1, b_1, a_3, b_3) to the space of components (V_1, U_1, V_3, U_3) and (v_1, u_1, v_3, u_3). These mappings are single-valued and continuous. Define C and C_o as the closed three-cells described by $M\Lambda_B$ and $M_o\Lambda_B$, respectively, where Λ_B is the boundary of Λ. It may be verified directly whether, or not, the origin of the image four-space lies in C_o, whether C_o has non-zero order, $v(C_o, 0)$, with respect to the origin (See Ref. 5, p.15 and p.30), and the distance $|(u, v) - 0|$ of the origin from C_o, may be determined. Further, using certain estimates for $|\alpha_1 - \alpha_{10}|$, $|\beta_1 - \beta_{10}|$, $|\alpha_3 - \alpha_{30}|$ and $|\beta_3 - \beta_{30}|$, the Euclidean distance between the three-cells C and C_o is given by

$$|(U, V) - (u, v)| = |\{(V_1 - v_1)^2 + (U_1 - u_1)^2 + (V_3 - v_3)^2 + (U_3 - u_3)^2\}^{\frac{1}{2}}|$$

$$= |g_3\{(\alpha_1 - \alpha_{10})^2 + (\beta_1 - \beta_{10})^2 + (\alpha_3 - \alpha_{30})^2 + (\beta_3 - \beta_{30})^2\}^{\frac{1}{2}}| \tag{3.26}$$

If it can be established that

$$\text{glb}|(U, V) - (u, v)| < \text{lub}|(u, v) - 0| \tag{3.27}$$

then by Rouché's theorem (See Ref. 7, Vol. 3, p.103) it follows that

$$v(C, 0) = v(C_o, 0) \neq 0 \tag{3.28}$$

or that

$$\gamma(M, \Lambda, 0) = \gamma(M_o, \Lambda, 0) \neq 0, \tag{3.29}$$

where $\gamma(M, \Lambda, 0)$ is the local topological degree of M at the origin relative to Λ. It then follows from Ref. 5, p.32, Theorem 6.6 that there is a point in the interior of Λ for which $v_1 = u_1 = v_3 = u_3 = 0$, and another point in the interior of Λ for which $V_1 = U_1 = V_3 = U_3 = 0$. This implies that the exact system of determining equations (3.24) are satisfied for certain values of a_1, b_1, a_3, b_3 contained in the cell Λ and, therefore, y = x, as given by equation (2.3), is an exact solution of (3.9) and, thereby, equation (2.1), for certain values a_1, b_1, a_3, b_3 contained in Λ.

4. Some Norms and Pseudo-norms

From (3.1) and (3.3) the norm

$$\nu(x_j) = \left\{ 2^{-1}(a_{j1}^2 + b_{j1}^2 + a_{j3}^2 + b_{j3}^2 + a_{j5}^2 + \ldots.)\right\}^{\frac{1}{2}} \qquad (4.1)$$

Thus for the components x_1 and x_2 described in (3.9)

$$\nu(x_1) = \left\{ 2^{-1}(a_{11}^2 + b_{11}^2 + a_{13}^2 + b_{13}^2 + \ldots.)\right\}^{\frac{1}{2}} \qquad (4.2)$$

and

$$\nu(x_2) = \left\{ 2^{-1}(a_{21}^2 + b_{21}^2 + a_{23}^2 + b_{23}^2 + \ldots.)\right\}^{\frac{1}{2}}$$

Now from (3.1)

$$x_1(\theta) = \sum_{s=0}^{\infty} \left\{ a_{1,2s+1} \operatorname{Sin}(2s+1)\theta + b_{1,\,2s+1} \operatorname{Cos}(2s+1)\theta \right\},$$

$$x_2(\theta) = \sum_{s=0}^{\infty} \left\{ a_{2,2s+1} \operatorname{Sin}(2s+1)\theta + b_{2,2s+1} \operatorname{Cos}(2s+1)\theta \right\}$$

and upon differentiation with respect to θ

$$x_1'(\theta) = \sum_{s=0}^{\infty} (2s+1)\left\{ a_{1,\,2s+1} \operatorname{Cos}(2s+1)\theta - b_{1,\,2s+1} \operatorname{Sin}(2s+1)\theta \right\}$$

From (3.9), $x_1' = x_2$ and, therefore, it must follow that

$$a_{2,\,2s+1} = -(2s+1)b_{1,\,2s+1} \text{ and } b_{2,\,2s+1} = (2s+1)a_{1,\,2s+1} \qquad (4.3)$$

Substituting (4.3) into the expression for $\nu(x_2)$ then gives

$$\nu(x_2) = \left\{ 2^{-1}(a_{11}^2 + b_{11}^2 + 9a_{13}^2 + 9b_{13}^2 + 25a_{15}^2 + \ldots.)\right\}^{\frac{1}{2}} \qquad (4.4)$$

It follows that

$$\nu(x_2) \geqslant \nu(x_1)$$

and, therefore,

$$\nu(x) = \nu(x_2) \qquad (4.5)$$

Also,

$$\nu(Px) = \nu(Px_2),$$

$$\nu(x - Px) = \nu(x_2 - Px_2)$$

$$= \left\{ 2^{-1}(25a_{15}^2 + 25b_{15}^2 + 49a_{17}^2 + \ldots.)\right\}^{\frac{1}{2}} \tag{4.6}$$

therefore

$$\left.\begin{aligned} &\nu(Px) \leqslant \nu(x) \\ \text{and} \quad &\nu(x - Px) \leqslant \nu(x) \end{aligned}\right\} \tag{4.7}$$

for all $x \epsilon S$.

Now

$$H(x_1 - Px_1) = -5^{-1}a_{15} \operatorname{Cos} 5\theta + 5^{-1}b_{15} \operatorname{Sin} 5\theta - 7^{-1}a_{17} \operatorname{Cos} 7\theta + \ldots.$$

and

$$H(x_2 - Px_2) = x_1 - Px_1 = a_{15} \operatorname{Sin} 5\theta + b_{15} \operatorname{Cos} 5\theta + a_{17} \operatorname{Sin} 7\theta \ldots. ,$$

thus

$$\nu H(x_1 - Px_1) = \left\{2^{-1}(5^{-2}a_{15}^2 + 5^{-2}b_{15}^2 + 7^{-2}a_{17}^2 + \ldots.)\right\}^{\frac{1}{2}} < \nu H(x_2 - Px_2)$$

and

$$\nu H(x - Px) = \nu H(x_2 - Px_2)$$

$$= \left\{ 2^{-1}(a_{15}^2 + b_{15}^2 + a_{17}^2 + \ldots.)\right\}^{\frac{1}{2}} = \nu(x_1 - Px_1).$$

It follows that

$$\nu H(x - Px) \leqslant 5^{-1}\nu(x - Px) \leqslant 5^{-1}\nu(x) \tag{4.8}$$

In addition to the norm ν, use will be made of the pseudonorm

$$|x_j(\theta)|, \quad 0 \leqslant \theta \leqslant 2\pi,$$

and the result that

$$|x_j| = \left|\sum_{s=0}^{\infty} \{a_{j,2s+1} \operatorname{Sin}(2s+1)\theta + b_{j,\ 2s+1} \operatorname{Cos}(2s+1)\theta\}\right|$$

$$\leqslant \sum_{s=0}^{\infty} \{|a_{j,2s+1}| + |b_{j,2s+1}|\} \tag{4.9}$$

will be used. Thus

$$|H(x_j - Px_j)| = \left|\sum_{s=2}^{\infty} (2s+1)^{-1} \{-a_{j,2s+1} \operatorname{Cos}(2s+1)\theta + b_{j,2s+1} \operatorname{Sin}(2s+1)\theta\}\right|$$

$$\leqslant \sum_{s=2}^{\infty} (2s+1)^{-1} \{ | a_{j,2s+1} | + | b_{j,2s+1} | \}$$

$$\leqslant \{ \sum_{s=2}^{\infty} (2s+1)^{-2} \}^{\frac{1}{2}} \{ \sum_{s=2}^{\infty} (a^2_{j,2s+1} + b^2_{j,2s+1}) \}^{\frac{1}{2}}$$

$$\leqslant 2^{\frac{1}{2}} \{ \sum_{s=2}^{\infty} (2s+1)^{-2} \}^{\frac{1}{2}} \nu(x_j - Px_j)$$

$$\leqslant 2^{\frac{1}{2}} \{ \sum_{s=2}^{\infty} (2s+1)^{-2} \}^{\frac{1}{2}} \nu(x)$$

Now

$\sum_{s=0}^{\infty} (2s+1)^{-2} = 1 + 3^{-2} + 5^{-2} + 7^{-2} + \ldots = \pi^2/8$, (See Ref. 3, p.219, or Ref. 4, p.167, Example 5 with $x = 0$,) therefore

$$|H(x_j - Px_j)| \leqslant 2^{\frac{1}{2}}(\pi^2/8 - 1 - 3^{-2})^{\frac{1}{2}} \nu(x_j - Px_j)$$

$$\leqslant 0.49516\ \nu(x_j - Px_j) \leqslant 0.49516\ \nu(x) \tag{4.10}$$

Similarly, if h is a real analytic operator in S, then

$$\nu H[hx - P(hx)] \leqslant 5^{-1} \nu[hx - P(hx)] \leqslant 5^{-1}\ \nu(hx) \tag{4.11}$$

and

$$| H[hx_j - P(hx_j)] | \leqslant 0.49516\ \nu[hx_j - P(hx_j)] \leqslant 0.49516\ \nu(hx_j) \tag{4.12}$$

5. <u>Conditions for T to be a Contraction Mapping in S*</u>

Consider the four cell Λ defined in Section 3. Then x^* is defined as

$$x^* = \begin{bmatrix} x^*_1 \\ x^*_2 \end{bmatrix} = \begin{bmatrix} a_{11} \text{ Sin } \theta + b_{11} \text{ Cos } \theta + a_{13} \text{ Sin } 3\theta + b_{13} \text{ Cos } 3\theta \\ a_{21} \text{ Sin } \theta + b_{21} \text{ Cos } \theta + a_{23} \text{ Sin } 3\theta + b_{23} \text{ Cos } 3\theta \end{bmatrix},$$

which from (4.3)

$$= \begin{bmatrix} a_{11} \text{ Sin } \theta + b_{11} \text{ Cos } \theta + a_{13} \text{ Sin } 3\theta + b_{13} \text{ Cos } 3\theta \\ -b_{11} \text{ Sin } \theta + a_{11} \text{ Cos } \theta - 3b_{13} \text{ Sin } 3\theta + 3a_{13} \text{ Cos } 3\theta \end{bmatrix} \tag{5.1}$$

with a_{11}, b_{11}, a_{13} and b_{13} contained in Λ. It follows that

$$\nu(x^*) = \nu(x_2^*) = \{2^{-1}(a_{11}^2 + b_{11}^2 + 9a_{13}^2 + 9b_{13}^2)\}^{\frac{1}{2}}$$

$$\leqslant c = \sigma(\mu)|a_e|, \quad (5.2)$$

$$|x_1^*| \leqslant |a_{11}| + |b_{11}| + |a_{13}| + |b_{13}|$$

$$\leqslant r_1 = \tau_1(\mu)|a_e|, \quad (5.3)$$

$$|x_2^*| \leqslant |a_{21}| + |b_{21}| + |a_{23}| + |b_{23}|$$

$$\leqslant |a_{11}| + |b_{11}| + |3a_{13}| + |3b_{13}|$$

$$\leqslant r_2 = \tau_2(\mu)|a_e|, \quad (5.4)$$

where

$$\sigma(\mu) = \{2^{-1}(\mu_1^2 + \mu_2^2 + 9\mu_3^2 + 9\mu_4^2)\}^{\frac{1}{2}}, \quad (5.5)$$

$$\tau_1(\mu) = \mu_1 + \mu_2 + \mu_3 + \mu_4 \quad (5.6)$$

and

$$\tau_2(\mu) = \mu_1 + \mu_2 + 3\mu_3 + 3\mu_4. \quad (5.7)$$

Define S* to be the set of all $x(\theta)$, as given by (3.1), which satisfy the conditions

$$\left.\begin{array}{l} P(x) = x^* \\ \nu(x) \leqslant d \\ |x_j| \leqslant R_j \\ \nu(x - Px) \leqslant \delta|a_e| \\ |x_j - Px_j| \leqslant \rho_j|a_e|, \end{array}\right\} \quad (5.8)$$

where d, R_j, δ and ρ_j will be chosen below. Then

$$\nu(Px) = \nu(x^*) \leqslant c$$

and

$$|Px_j| = |x_j| \leq r_j$$

for every x in S*.

Consider the mapping $T : S^* \rightarrow S$ defined in (3.12), then

$Py = PTx = P(Px + Fx) = PPx + PFx.$

But $PPx = Px$ and $PFx = PHfx = 0$,

thus

$$Py = Px = x^*.$$

Also $\quad y_1 - Py_1 = H(x_2 - Px_2) = x_1 - Px_1$

and $\quad y_2 - Py_2 = H\{-g_1x_1 - g_3x_1^3 - hx_2 + P(g_1x_1 + g_3x_1^3 + hx_2)\}$.

It is required now to obtain conditions for $T : S^* \rightarrow S^*$. For this purpose $\nu(y - Py)$ and $|y_j - Py_j|$ will be evaluated in terms of $\mu_1, \ldots, \mu_4$, $|a_e|$, δ and ρ_j. Write

$$x_1 = Px_1 + (x_1 - Px_1),$$

then

$$x_1^3 = (Px_1)^3 + 3(Px_1)^2(x_1 - Px_1) + 3Px_1(x_1 - Px_1)^2 + (x_1 - Px_1)^3.$$

Thus

$$\begin{aligned}
\nu(y - Py) &= \nu(y_2 - Py_2) \\
&= \nu H[g_1(x_1 - Px_1) + g_3(x_1^3 - Px_1^3) + h(x_2 - Px_2)] \\
&= \nu H\{g_1(x_1 - Px_1) + g_3[(Px_1)^3 - P(Px_1)^3] \\
&\quad + 3g_3[(Px_1)^2(x_1 - Px_1) - P(Px_1)^2(x_1 - Px_1)] \\
&\quad + 3g_3[(Px_1)(x_1 - Px_1)^2 - P(Px_1)(x_1 - Px_1)^2] \\
&\quad + g_3[(x_1 - Px_1)^3 - P(x_1 - Px_1)^3] + h(x_2 - Px_2)\} \\
&\leq |g_1|\nu H(x_1 - Px_1) + |g_3|\{\nu H[(Px_1)^3 - P(Px_1)^3]
\end{aligned}$$

$$+ 3\nu H[\ldots\ldots] + 3\nu H[\ldots\ldots] + \nu H[\ldots\ldots]\} + |h|\nu H(x_2 - Px_2). \tag{5.9}$$

Similarly

$$|y_2 - Py_2| \leqslant |g_1||H(x_1 - Px_1)| + |g_3|\{|H[(Px_1)^3 - P(Px_1)^3]| + 3|H[\ldots\ldots]|$$

$$+ 3|H[\ldots\ldots]| + |H[\ldots\ldots]|\} + |h||H(x_2 - Px_2)|. \tag{5.10}$$

Also, $|y_1 - Py_1| = |H(x_2 - Px_2)| \leqslant 0.49516\ \nu(x_2 - Px_2)$

$$\leqslant 0.49516\ \delta|a_e|. \tag{5.11}$$

Consider the terms in these inequalities in order, with $x \epsilon S^*$. Then

$$\left.\begin{aligned} &\nu H(x_1 - Px_1) \leqslant 5^{-1}\nu(x_1 - Px_1) \leqslant 5^{-2}\nu(x - Px) \leqslant 5^{-2}\delta|a_e| \\ &\text{and} \\ &|H(x_1 - Px_1)| \leqslant 0.49516\ \nu(x_1 - Px_1) \leqslant 0.09903\,\delta\,|a_e|. \end{aligned}\right\} \tag{5.12}$$

Now

$$(Px_1)^3 - P(Px_1)^3 = \alpha_{50}\ \text{Sin}\ 5\theta + \beta_{50}\ \text{Cos}\ 5\theta + \alpha_{70}\ \text{Sin}\ 7\theta + \beta_{70}\ \text{Cos} 7\theta$$

$$+ \alpha_{90}\ \text{Sin}\ 9\theta + \beta_{90}\ \text{Cos}\ 9\theta,$$

therefore

$$\nu[(PX_1)^3 - P(Px_1)^3] = [2^{-1}(\alpha_{50}^2 + \beta_{50}^2 + \alpha_{70}^2 + \beta_{70}^2 + \alpha_{90}^2 + \beta_{90}^2)]^{\frac{1}{2}}$$

$$= [2^{-1}(|\alpha_{50}|^2 + |\beta_{50}|^2 + |\alpha_{70}|^2 + |\beta_{70}|^2 + |\alpha_{90}|^2 + |\beta_{90}|^2)]^{\frac{1}{2}}$$

and

$$|(Px_1)^3 - P(Px_1)^3| \leqslant |\alpha_{50}| + |\beta_{50}| + \ldots\ldots + |\beta_{90}|.$$

In order to evaluate these, estimates for $|\alpha_{50}|$, etc. are required. These may be obtained from the expressions for α_{50}, etc. in Section 2 and the definition of Λ. Thus

$$|\alpha_{50}| \leqslant |a_e|^3\{\tfrac{3}{4}\mu_3(\mu_1^2 + \mu_2^2) + \tfrac{3}{2}\mu_1\mu_2\mu_4 + \tfrac{3}{4}\mu_1(\mu_3^2 + \mu_4^2) + \tfrac{3}{2}\mu_2\mu_3\mu_4\}.$$

It will be seen later that the values of $\mu_1, \ldots\ldots, \mu_4$ may be expressed in terms of $|\lambda|$, thus $|\alpha_{50}|, \ldots\ldots, |\beta_{90}|$ may be expressed in the form

$$\left.\begin{aligned} |\alpha_{50}| &\leq A_{50}(\lambda)|a_e|^3 \\ |\beta_{50}| &\leq B_{50}(\lambda)|a_e|^3 \\ &\cdots\cdots\cdots\cdots \\ |\beta_{90}| &\leq B_{90}(\lambda)|a_e|^3, \end{aligned}\right\} \quad (5.13)$$

where

$$\left.\begin{aligned} A_{50}(\lambda) &= \frac{3}{4}\left[\mu_3(\mu_1^2 + \mu_2^2) + \mu_1(\mu_3^2 + \mu_4^2)\right] + \frac{3}{2}\mu_2\mu_4(\mu_1 + \mu_3) \\ B_{50}(\lambda) &= \frac{3}{4}\left[\mu_4(\mu_1^2 + \mu_2^2) + \mu_2(\mu_3^2 + \mu_4^2)\right] + \frac{3}{2}\mu_1\mu_3(\mu_2 + \mu_4) \\ A_{70}(\lambda) &= \frac{3}{4}\mu_1(\mu_3^2 + \mu_4^2) + \frac{3}{2}\mu_2\mu_3\mu_4 \\ B_{70}(\lambda) &= \frac{3}{4}\mu_2(\mu_3^2 + \mu_4^2) + \frac{3}{2}\mu_1\mu_3\mu_4 \\ A_{90}(\lambda) &= \frac{1}{4}\mu_3(\mu_3^2 + 3\mu_4^2) \\ B_{90}(\lambda) &= \frac{1}{4}\mu_4(3\mu_3^2 + \mu_4^2). \end{aligned}\right\} \quad (5.14)$$

Thus

$$\nu\left[(Px_1)^3 - P(Px_1)^3\right] \leq \phi(\lambda)|a_e|^3 \quad (5.15)$$

and

$$\left|(Px_1)^3 - P(Px_1)^3\right| \leq \psi(\lambda)|a_e|^3, \quad (5.16)$$

where

$$\phi(\lambda) = \left[2^{-1}(A_{50}^2 + B_{50}^2 + \cdots + B_{90}^2)\right]^{\frac{1}{2}} \quad (5.17)$$

and

$$\psi(\lambda) = A_{50} + B_{50} + \cdots + B_{90} \quad (5.18)$$

It will be noted that

$$A_{50}, \ldots, B_{90}, \phi(\lambda), \psi(\lambda) \geq 0.$$

From (4.11) and (4.12) it now follows that

$$\nu H\left[(Px_1)^3 - P(Px_1)^3\right] \leq 5^{-1}\phi(\lambda)|a_e|^3 \quad (5.19)$$

and

$$|H[(Px_1)^3 - P(Px_1)^3]| \leqslant 0.49516\ \phi(\lambda)\ |a_e|^3 \tag{5.20}$$

Also from (4.11) and (4.12) it follows that

$$\begin{aligned} \nu H[(Px_1)^2(x_1 - Px_1) - P(Px_1)^2(x_1 - Px_1)] &< 5^{-1}\nu[(Px_1)^2(x_1 - Px_1) - P(Px_1)^2(x_1 - Px_1)] \\ &\leqslant 5^{-1}\nu[(Px_1)^2(x_1 - Px_1)] \\ &< 5^{-1}|Px_1|^2\nu(x_1 - Px_1) \\ &\leqslant 5^{-2}\tau_1^2(\lambda)\,\delta|a_e|^3 \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} |H[(Px_1)^2(x_1 - Px_1) - P(Px_1)^2(x_1 - Px_1)]| &\leqslant 0.49516\nu[(Px_1)^2(x_1 - Px_1) - P(Px_1)^2(x_1 - Px_1)] \\ &\leqslant 0.49516\nu[(Px_1)^2(x_1 - Px_1)] \\ &\leqslant 0.09903\ \tau_1^2(\lambda)\delta|a_e|^3 \end{aligned} \tag{5.22}$$

where, from (5.6) and the previous remarks concerning the expression of $\mu_1, \ldots, \mu_4$ in terms of λ, it is clear that τ_1 is a function of λ.

Similarly

$$\begin{aligned} \nu H[(Px_1)(x_1 - Px_1)^2 - P(Px_1)(x_1 - Px_1)^2] &\leqslant 5^{-1}\nu[(Px_1)(x_1 - Px_1)^2 - P(Px_1)(x_1 - Px_1)^2] \\ &< 5^{-1}\nu[(Px_1)(x_1 - Px_1)^2] \\ &\leqslant 5^{-1}|Px_1||x_1 - Px_1|\,\nu(x_1 - Px_1) \\ &\leqslant 5^{-2}\tau_1(\lambda)\rho_1\delta|a_e|^3, \end{aligned} \tag{5.23}$$

$$|H[(Px_1)(x_1 - Px_1)^2 - P(Px_1)(x_1 - Px_1)^2]| < 0.09903\ \tau_1(\lambda)\rho_1\,\delta|a_e|^3, \tag{5.24}$$

$$\begin{aligned} \nu H[(x_1 - Px_1)^3 - P(x_1 - Px_1)^3] &\leqslant 5^{-1}\nu[(x_1 - Px_1)^3 - P(x_1 - Px_1)^3] \\ &< 5^{-1}\nu[(x_1 - Px_1)^3] \\ &< 5^{-1}|x_1 - Px_1|^2\nu(x_1 - Px_1) \\ &\leqslant 5^{-2}\rho_1^2\,\delta\,|a_e|^3, \end{aligned} \tag{5.25}$$

$$|H[(x_1 - Px_1)^3 - P(x_1 - Px_1)^3]| \leqslant 0.09903\ \rho_1^2\delta\ |a_e|^3, \tag{5.26}$$

$$\nu H[(x_2 - Px_2)] = \nu(x_1 - Px_1) \leqslant 5^{-1}\nu(x_2 - Px_2) \leqslant 5^{-1}\delta|a_e| \tag{5.27}$$

and

$$|H(x_2 - Px_2)| \leqslant 0.49516\,\nu\,(x_2 - Px_2) \leqslant 0.49516\ \delta\,|a_e|. \tag{5.28}$$

Substituting the appropriate inequalities into (5.9) gives

$$\nu(y - Py) \leqslant 5^{-2}|a_e|\{(|g_1| + 5|h|)\delta + |g_3||a_e|^2[5\phi(\lambda) + 3\tau_1^2(\lambda)\delta + 3\tau_1(\lambda)\rho_1\delta + \rho_1^2\delta]\}.$$

or since $g_3|a_e|^2 = \frac{1}{3}|k_3||g_1 - 1|$, then

$$\nu(y - Py) \leqslant 5^{-2}|a_e|\{(|g_1| + 5|h|)\delta + \tfrac{1}{3}|k_3||g_1 - 1|[5\phi(\lambda) + 3\tau_1^2(\lambda)\delta + 3\tau_1(\lambda)\rho_1\delta + \rho_1^2\delta]\} \tag{5.29}$$

Similarly

$$|y_2 - Py_2| \leqslant 0.09903|a_e|\{(|g_1| + 5|h|)\delta + \tfrac{1}{3}|k_3||g_1 - 1|[5\phi(\lambda) + 3\tau_1^2(\lambda)\delta + 3\tau_1(\lambda)\rho_1\delta + \rho_1^2\delta]\} \tag{5.30}$$

and

$$|y_1 - Py_1| \leqslant 0.49516\ \delta\,|a_e|. \tag{5.31}$$

The conditions for $T : S^* \longrightarrow S^*$ may now be established. First it is required to ask whether, for $x \in S^*$,

$$\nu(y - Py) \leqslant \nu(x - Px) \leqslant \delta|a_e|,$$

$$|y_2 - Py_2| \leqslant |x_2 - Px_2| \leqslant \rho_2|a_e|$$

and

$$|y_1 - Py_1| \leqslant |x_1 - Px_1| < \rho_1|a_e|;$$

or, upon using (5.29), (5.30) and (5.31) and dividing throughout by $|a_e| \neq 0$, whether

$$5^{-2}\{(|g_1| + 5|h|)\delta + \tfrac{1}{3}|k_3||g_1 - 1|[5\phi(\lambda) + 3\tau_1^2(\lambda)\delta + 3\tau_1(\lambda)\rho_1\delta + \rho_1^2\delta]\} \leqslant \delta, \tag{5.32}$$

$$0.09903\{(|g_1| + 5|h|)\delta + \tfrac{1}{3}|k_3||g_1 - 1|[5\phi(\lambda) + 3\tau_1^2(\lambda)\delta + 3\tau_1(\lambda)\rho_1\delta + \rho_1^2\delta]\} < \rho_2 \tag{5.33}$$

and

$$0.49516\ \delta \leqslant \rho_1 . \tag{5.34}$$

Now for $x \epsilon S^*$, $Py = Px = Px^* = x^*$, and, therefore,

$$\nu(x) = \nu\left[Px + (x - Px)\right] \leqslant \nu(Px) + \nu(x - Px) \leqslant \left[\sigma(\lambda) + \delta\right]|a_e| \leqslant d, \tag{5.35}$$

$$|x_1| = |Px_1 + (x_1 - Px_1)| \leqslant |Px_1| + |x_1 - Px_1| \leqslant \left[\tau_1(\lambda) + \rho_1\right]|a_e| \leqslant R_1 \tag{5.36}$$

and

$$|x_2| = |Px_2 + (x_2 - Px_2)| \leqslant |Px_2| + |x_2 - Px_2| \leqslant \left[\tau_2(\lambda) + \rho_2\right]|a_e| \leqslant R_2 \tag{5.37}$$

Similarly

$$\nu(y) \leqslant \nu(Py) + \nu(y - Py) \leqslant \sigma(\lambda)|a_e| + \nu(y - Py), \tag{5.38}$$

$$|y_1| \leqslant |Py_1| + |y_1 - Py_1| \leqslant \tau_1(\lambda)|a_e| + |y_1 - Py_1| \tag{5.39}$$

and

$$|y_2| \leqslant |Py_2| + |y_2 - Py_2| \leqslant \tau_2(\lambda)|a_e| + |y_2 - Py_2| . \tag{5.40}$$

If the inequalities (5.32), (5.33) and (5.34) hold then from (5.35) to (5.40)

$$\nu(y) \leqslant \left[\sigma(\lambda) + \delta\right]|a_e| \leqslant d,$$

$$|y_1| \leqslant \left[\tau_1(\lambda) + \rho_1\right]|a_e| \leqslant R_1$$

and

$$|y_2| \leqslant \left[\tau_2(\lambda) + \rho_2\right]|a_e| \leqslant R_2,$$

and hence $y \epsilon S^*$. Choosing

$$\left.\begin{aligned} d &= \left[\sigma(\lambda) + \delta\right]|a_e| \\ &\text{and} \\ R_j &= \left[\tau_j(\lambda) + \rho_j\right]|a_e| , \end{aligned}\right\} \tag{5.41}$$

with λ, δ, ρ_1 and ρ_2 satisfying (5.32), (5.33) and (5.34), then $T : S^* \longrightarrow S^*$.

Conditions for T to be a contraction mapping in S* may be established in the following way. With x and $\bar{x}$ in S*,

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Px_1 + H(x_2 - Px_2) \\ Px_2 + H\{-g_1x_1 - g_3x_1^3 - hx_2 + P(g_1x_1 + g_3x_1^3 + hx_2)\} \end{bmatrix}$$

and

$$\bar{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} P\bar{x}_1 + H(\bar{x}_2 - P\bar{x}_2) \\ P\bar{x}_2 + H\{-g_1\bar{x}_1 - g_3\bar{x}_1^3 - h\bar{x}_2 + P(g_1\bar{x}_1 + g_3\bar{x}_1^3 + h\bar{x}_2)\} \end{bmatrix}$$

Also $Px = x^* = P\bar{x}$. Thus

$$y - \bar{y} = \begin{bmatrix} y_1 - \bar{y}_1 \\ y_2 - \bar{y}_2 \end{bmatrix}, \quad \text{where}$$

$$y_1 - \bar{y}_1 = H\left\{(x_2 - \bar{x}_2) - P(x_2 - \bar{x}_2)\right\}$$

and

$$y_2 - \bar{y}_2 = H\{-g_1[(x_1 - \bar{x}_1) - P(x_1 - \bar{x}_1)] - g_3[(x_1^3 - \bar{x}_1^3) - P(x_1^3 - \bar{x}_1^3)]$$
$$- h[(x_2 - \bar{x}_2) - P(x_2 - \bar{x}_2)]\}$$

Now

$\nu(y - \bar{y}) = \nu(y_2 - \bar{y}_2)$, which from equation (4.11)

$$\leqslant 5^{-1}\{|g_1|\nu(x_1 - x_1) + |g_3|\nu(x_1^3 - \bar{x}_1^3) + |h|\nu(x_2 - \bar{x}_2)\}$$

$$\leqslant 5^{-1}\{|g_1|\nu(x_1 - \bar{x}_1) + |g_3|\nu[(x_1 - \bar{x}_1)(x_1^2 + x_1\bar{x}_1 + \bar{x}_1^2)] + |h|\nu(x_2 - \bar{x}_2)\}$$

$$\leqslant 5^{-1}\{|g_1|\nu(x_1 - \bar{x}_1) + |g_3|(|x_1|^2 + |x_1||\bar{x}_1| + |\bar{x}_1|^2)\nu(x_1 - \bar{x}_1) + |h|\nu(x_2 - \bar{x}_2)\}$$

$$\leqslant 5^{-1}\{[|g_1| + |g_3|(|x_1|^2 + |x_1||\bar{x}_1| + |\bar{x}_1|^2)]\nu H(x_2 - \bar{x}_2) + |h|\nu(x_2 - \bar{x}_2)\}$$

$$\leqslant 5^{-1}\{5^{-1}(|g_1| + 3|g_3|R_1^2)\nu(x_2 - \bar{x}_2) + |h|\nu(x_2 - \bar{x}_2)\}$$

$$\leqslant 5^{-2}\{|g_1| + 5|h| + 3|g_3|R_1^2\}\nu(x - \bar{x}),$$

thus T is a contraction in S* provided that

$$|g_1| + 5|h| + 3|g_3|R_1^2 < 25.$$

Upon substituting for R_1 from (5.41) and then for $|a_e|$ this inequality becomes

$$|g_1| + 5|h| + [\tau_1(\lambda) + \rho_1]^2 |k_3||g_1 - 1| < 25 \qquad (5.42)$$

When the above conditions are satisfied for T to be a contraction mapping in S*, then it may be concluded, on the basis of Banach's fixed point theorem, Ref. 5, p.141, that the fixed element

$$y(\theta) = a_1 \text{ Sin } \theta + b_1 \text{ Cos } \theta + a_3 \text{ Sin } 3\theta + b_3 \text{ Cos } 3\theta + a_5 \text{ Sin } 5\theta + \ldots.$$

exists, is unique in S* and is continuously dependent on x*. Thus a_5, b_5, a_7, b_7, ..., are uniquely determined by and continuously dependent on a_1, a_3, b_1, b_3 for $a_1, \ldots, b_3$ in Λ.

6. Estimates for $|\alpha_1 - \alpha_{10}|, \ldots, |\beta_3 - \beta_{30}|$.

In order to be able to obtain the Euclidean distance between the cells C_o and C, estimates of a_{j5}, $b_{j5}, \ldots, b_{j9}$, $|\alpha_1 - \alpha_{10}|, \ldots, |\beta_3 - \beta_{30}|$ will be required. For x in S*, Py = Px = x* and it follows that

$$y_j - Py_j = a_{j5} \text{ Sin } 5\theta + b_{j5} \text{ Cos } 5\theta + a_{j7} \text{ Sin } 7\theta + \ldots, \qquad (6.1)$$

and, therefore, that a_{j5}, $b_{j5}, \ldots$ etc. are the Fourier coefficients of $(y_j - Py_j)$. Consider now a differentiable, periodic, function $G_j(\theta)$ of period 2π. Assuming that $G_j(\theta)$ has a Fourier series representation in both Sin $n\theta$ and Cos $n\theta$, then the Fourier coefficients will be given by

$$a_{jn} = \pi^{-1} \int_0^{2\pi} G_j(\theta) \text{ Sin } n\theta \, d\theta \text{ and } b_{jn} = \pi^{-1} \int_0^{2\pi} G_j(\theta) \text{ Cos } n\theta \, d\theta \qquad (6.2)$$

Upon integrating by parts there are produced the alternative relations

$$a_{jn} = \pi^{-1} \int_0^{2\pi} n^{-1} G_j'(\theta) \text{ Cos } n\theta \, d\theta \text{ and } b_{jn} = -\pi^{-1} \int_0^{2\pi} n^{-1} G_j'(\theta) \text{ Sin } n\theta \, d\theta \quad (6.3)$$

Now $|a_{jn}| = |\pi^{-1}\int_0^{2\pi} n^{-1}G_j'(\theta)\ \mathrm{Cos}\ n\theta\ d\theta|$.

$$\leqslant \pi^{-1}\int_0^{2\pi} |n^{-1}G_j'(\theta)\ \mathrm{Cos}\ n\theta|\ d\theta$$

which by the Schwarz inequality

$$\leqslant \pi^{-1}n^{-1}\left[\int_0^{2\pi} |G_j'(\theta)|^2 d\theta\right]^{\frac{1}{2}}\left[\int_0^{2\pi} |\mathrm{Cos}\ n\theta|^2 d\theta\right]^{\frac{1}{2}}$$

$$\leqslant 2^{\frac{1}{2}}n^{-1}\left[(2\pi)^{-1}\int_0^{2\pi} |G_j'(\theta)|^2 d\theta\right]^{\frac{1}{2}}\left[\pi^{-1}\int_0^{2\pi} |\mathrm{Cos}\ n\theta|^2 d\theta\right]^{\frac{1}{2}} = 2^{\frac{1}{2}}n^{-1}\nu(G_j').$$

Similarly

$$|b_{jn}| \leqslant 2^{\frac{1}{2}}n^{-1}\nu(G_j').$$

Identifying $(y_j - Py_j)$ with $G_j(\theta)$ then yields

$$|a_{jn}|,\ |b_{jn}| \leqslant 2^{\frac{1}{2}}n^{-1}\nu[(y_j - Py_j)'] \qquad (6.4)$$

Now

$$(y_2 - Py_2)' = -g_1x_1 - g_3x_1^3 - hx_2 + P(g_1x_1 + g_3x_1^3 + hx_2)$$

and

$$\nu[(y_2 - Py_2)'] = \nu\{g_1(x_1 - Px_1) + g_3(x_1^3 - Px_1^3) + h(x_2 - Px_2)\}$$

$$\leqslant |g_1|\nu(x_1 - Px_1) + |h|\nu(x_2 - Px_2) + |g_3|\{\nu[(Px_1)^3 - P(Px_1)^3]$$

$$+ 3\nu[\ldots.] + 3\nu[\ldots.] + \nu[\ldots.]\}$$

in a closely similar manner to (5.9). Thus

$$\nu[(y_2 - Py_2)'] \leqslant 5^{-1}N|a_e|$$

where

$$N = \left\{(|g_1| + 5|h|)\delta + \frac{1}{3}|k_3||g_1 - 1|\left[5\phi(\lambda) + 3\tau_1^2(\lambda)\delta + 3\tau_1(\lambda)\rho_1\delta + \rho_1^2\delta\right]\right\} \qquad (6.5)$$

Substitution into (6.4) then gives

$$|a_{2n}|, \ |b_{2n}| \leqslant 2^{\frac{1}{2}} 5^{-1} n^{-1} N |a_e| \qquad (6.6)$$

From (4.3) $a_{1n} = n^{-1} b_{2n}$ and $b_{1n} = -n^{-1} a_{2n}$,

thus

$$|a_{1n}|, \ |b_{1n}| \leqslant 2^{\frac{1}{2}} 5^{-1} n^{-2} N |a_e| \qquad (6.7)$$

As a preliminary to determining $|\alpha_1 - \alpha_{10}|$, etc. the following relations will be required:

$$(a_{11} \operatorname{Sin}\theta + b_{11} \operatorname{Cos}\theta + a_{13} \operatorname{Sin} 3\theta + b_{13} \operatorname{Cos} 3\theta)^2 \operatorname{Sin}\theta$$

$$= {}_s\gamma_1 \operatorname{Sin}\theta + {}_c\gamma_1 \operatorname{Cos}\theta + {}_s\gamma_3 \operatorname{Sin} 3\theta + \ldots + {}_c\gamma_7 \operatorname{Cos} 7\theta, \qquad (6.8)$$

$$(a_{11} \operatorname{Sin}\theta + b_{11} \operatorname{Cos}\theta + a_{13} \operatorname{Sin} 3\theta + b_{13} \operatorname{Cos} 3\theta)^2 \operatorname{Cos}\theta$$

$$= {}_s\xi_1 \operatorname{Sin}\theta + {}_c\xi_1 \operatorname{Cos}\theta + {}_s\xi_3 \operatorname{Sin} 3\theta + \ldots + {}_c\xi_7 \operatorname{Cos} 7\theta, \qquad (6.9)$$

$$(a_{11} \operatorname{Sin}\theta + b_{11} \operatorname{Cos}\theta + a_{13} \operatorname{Sin} 3\theta + b_{13} \operatorname{Cos} 3\theta)^2 \operatorname{Sin} 3\theta$$

$$= {}_s\eta_1 \operatorname{Sin}\theta + {}_c\eta_1 \operatorname{Cos}\theta + {}_s\eta_3 \operatorname{Sin} 3\theta + \ldots + {}_c\eta_9 \operatorname{Cos} 9\theta \qquad (6.10)$$

and

$$(a_{11} \operatorname{Sin}\theta + b_{11} \operatorname{Cos}\theta + a_{13} \operatorname{Sin} 3\theta + b_{13} \operatorname{Cos} 3\theta)^2 \operatorname{Cos} 3\theta$$

$$= {}_s\zeta_1 \operatorname{Sin}\theta + {}_c\zeta_1 \operatorname{Cos}\theta + {}_s\zeta_3 \operatorname{Sin} 3\theta + \ldots + {}_c\zeta_9 \operatorname{Cos} 9\theta, \qquad (6.11)$$

where

$${}_s\gamma_1 = \frac{1}{4}(3a_{11}^2 + b_{11}^2 + 2a_{13}^2 + 2b_{13}^2 - 2a_{11}a_{13} - 2b_{11}b_{13})$$

$${}_c\gamma_1 = \frac{1}{2}(a_{11}b_{11} + b_{11}a_{13} - a_{11}b_{13})$$

$${}_s\gamma_3 = a_{11}a_{13} - \frac{1}{4}(a_{11}^2 - b_{11}^2)$$

$$ {}_c\gamma_3 = -\frac{1}{2}a_{11}(b_{11} - 2b_{13}) $$

$$ {}_s\gamma_5 = \frac{1}{2}\left[-a_{11}a_{13} + b_{11}b_{13} + \frac{1}{2}(a_{13}^2 - b_{13}^2)\right] $$

$$ {}_c\gamma_5 = -\frac{1}{2}(a_{11}b_{13} + b_{11}a_{13} + a_{13}b_{13}) $$

$$ {}_s\gamma_7 = -\frac{1}{4}(a_{13}^2 - b_{13}^2) $$

$$ {}_c\gamma_7 = \frac{1}{2}a_{13}b_{13} $$

$$ {}_s\xi_1 = \frac{1}{2}(a_{11}b_{11} + b_{11}a_{13} - a_{11}b_{13}) $$

$$ {}_c\xi_1 = \frac{1}{4}(a_{11}^2 + 3b_{11}^2 + 2a_{13}^2 + 2b_{13}^2 + 2a_{11}a_{13} + 2b_{11}b_{13}) $$

$$ {}_s\xi_3 = \frac{1}{2}b_{11}(a_{11} + 2a_{13}) $$

$$ {}_c\xi_3 = b_{11}b_{13} - \frac{1}{4}(a_{11}^2 - b_{11}^2) $$

$$ {}_s\xi_5 = \frac{1}{2}(a_{11}b_{13} + b_{11}a_{13} + a_{13}b_{13}) $$

$$ {}_c\xi_5 = \frac{1}{2}\left[-a_{11}a_{13} + b_{11}b_{13} - \frac{1}{2}(a_{13}^2 - b_{13}^2)\right] $$

$$ {}_s\xi_7 = \frac{1}{2}a_{13}b_{13} $$

$$ {}_c\xi_7 = -\frac{1}{4}(a_{13}^2 - b_{13}^2) $$

$$ {}_s\eta_1 = a_{11}a_{13} - \frac{1}{4}(a_{11}^2 - b_{11}^2) $$

$$ {}_c\eta_1 = \frac{1}{2}b_{11}(a_{11} + 2a_{13}) $$

$$ {}_s\eta_3 = \frac{1}{4}(2a_{11}^2 + 2b_{11}^2 + 3a_{13}^2 - b_{13}^2) $$

$$ {}_c\eta_3 = \frac{1}{2}a_{13}b_{13} $$

$$ {}_s\eta_5 = \frac{1}{2}\left[a_{11}a_{13} + b_{11}b_{13} - \frac{1}{2}(a_{11}^2 - b_{11}^2)\right] $$

$$ {}_c\eta_5 = -\frac{1}{2}(a_{11}b_{11} + b_{11}a_{13} - a_{11}b_{13}) $$

$${}_s\eta_7 = \frac{1}{2}(-a_{11}a_{13} + b_{11}b_{13})$$

$${}_c\eta_7 = -\frac{1}{2}(a_{11}b_{13} + b_{11}a_{13})$$

$${}_s\eta_9 = -\frac{1}{4}(a_{13}^2 - b_{13}^2)$$

$${}_c\eta_9 = -\frac{1}{2}a_{13}b_{13}$$

$${}_s\zeta_1 = \frac{1}{2}a_{11}(2b_{13} - b_{11})$$

$${}_c\zeta_1 = b_{11}b_{13} - \frac{1}{4}(a_{11}^2 - b_{11}^2)$$

$${}_s\zeta_3 = \frac{1}{2}a_{13}b_{13}$$

$${}_c\zeta_3 = \frac{1}{4}(2a_{11}^2 + 2b_{11}^2 + a_{13}^2 + 3b_{13}^2)$$

$${}_s\zeta_5 = \frac{1}{2}(a_{11}b_{11} + b_{11}a_{13} - a_{11}b_{13})$$

$${}_c\zeta_5 = \frac{1}{2}\left[a_{11}a_{13} + b_{11}b_{13} - \frac{1}{2}(a_{11}^2 - b_{11}^2)\right]$$

$${}_s\zeta_7 = \frac{1}{2}(a_{11}b_{13} + b_{11}a_{13})$$

$${}_c\zeta_7 = \frac{1}{2}(-a_{11}a_{13} + b_{11}b_{13})$$

$${}_s\zeta_9 = \frac{1}{2}a_{13}b_{13}$$

$${}_c\zeta_9 = -\frac{1}{4}(a_{13}^2 - b_{13}^2)$$

From these, the following inequalities may be determined:

$$|{}_s\gamma_5|, |{}_c\xi_5| \leq \frac{1}{2}\left[\mu_1\mu_3 + \mu_2\mu_4 + \frac{1}{2}(\mu_3^2 + \mu_4^2)\right] |a_e|^2$$

$$|{}_c\gamma_5|, |{}_s\xi_5| \leq \frac{1}{2}(\mu_1\mu_4 + \mu_2\mu_3 + \mu_3\mu_4)|a_e|^2$$

$$|{}_s\gamma_7|, |{}_c\xi_7| \leq \frac{1}{4}(\mu_3^2 + \mu_4^2)|a_e|^2$$

$$|{}_c\gamma_7|, |{}_s\xi_7| \leqslant \frac{1}{2}\mu_3\mu_4|a_e|^2$$

$$|{}_s\eta_5|, |{}_c\zeta_5| \leqslant \left[\frac{1}{2}\mu_1\mu_3 + \mu_2\mu_4 + \frac{1}{2}(\mu_1^2 + \mu_2^2)\right]|a_e|^2$$

$$|{}_c\eta_5|, |{}_s\zeta_5| \leqslant \frac{1}{2}(\mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_4)|a_e|^2$$

$$|{}_s\eta_7|, |{}_c\zeta_7| \leqslant \frac{1}{2}(\mu_1\mu_3 + \mu_2\mu_4)|a_e|^2$$

$$|{}_c\eta_7|, |{}_s\zeta_7| \leqslant \frac{1}{2}(\mu_1\mu_4 + \mu_2\mu_3)|a_e|^2$$

$$|{}_s\eta_9|, |{}_c\zeta_9| \leqslant \frac{1}{4}(\mu_3^2 + \mu_4^2)|a_e|^2$$

$$|{}_c\eta_9|, |{}_s\zeta_9| \leqslant \frac{1}{2}\mu_3\mu_4|a_e|^2$$

From (3.20) the Fourier coefficients of $y_1^3(\theta)$ may be obtained by the relations

$$\alpha_n = \pi^{-1}\int_0^{2\pi} y_1^3(\theta)\,\mathrm{Sin}\,n\theta\,d\theta \text{ and } \beta_n = \pi^{-1}\int_0^{2\pi} y_1^3(\theta)\,\mathrm{Cos}\,n\theta\,d\theta$$

Also for x in S*, Py = Px = x*, and from (2.6) the Fourier coefficients of $(x_1^*)^3 = (Py_1)^3$ are

$$\alpha_{no} = \pi^{-1}\int_0^{2\pi} (Py_1)^3\,\mathrm{Sin}\,n\theta\,d\theta \text{ and } \beta_{no} = \pi^{-1}\int_0^{2\pi} (Py_1)^3\,\mathrm{Cos}\,n\theta\,d\theta$$

Thus

$$\alpha_n - \alpha_{no} = \pi^{-1}\int_0^{2\pi}\left[y_1^3 - (Py_1)^3\right]\,\mathrm{Sin}\,n\theta\,d\theta$$

$$= \pi^{-1}\int_0^{2\pi}(y_1 - Py_1)\left[(y_1 - Py_1)^2 + 3(y_1 - Py_1)(Py_1) + 3(Py_1)^2\right]\mathrm{Sin}\,n\theta d\theta$$

$$= {}_nJ_1 + 3\,{}_nJ_2 + 3\,{}_nJ_3 \qquad (6.12)$$

where

$$|{}_nJ_1| = |\pi^{-1}\int_0^{2\pi}(y_1 - Py_1)^3 \operatorname{Sin} n\theta \, d\theta$$

$$\leqslant 2|y_1 - Py_1|_{Max.}(2\pi)^{-1}\int_0^{2\pi}(y_1 - Py_1)^2 d\theta = 2|y_1 - Py_1|_{Max.}[\nu(y_1 - Py_1)]^2$$

$$\leqslant 2\rho_1|a_e|\,[5^{-1}\nu(y - Py)]^2 \leqslant 2 \times 5^{-2}\rho_1\delta^2|a_e|^3,$$

$$|{}_nJ_2| = |\pi^{-1}\int_0^{2\pi}(y_1 - Py_1)^2(Py_1) \operatorname{Sin} n\theta \, d\theta|$$

$$\leqslant 2|Py_1|_{Max.}(2\pi)^{-1}\int_0^{2\pi}(y_1 - Py_1)^2 d\theta = 2|Py_1|_{Max.}[\nu(y_1 - Py_1)]^2$$

$$\leqslant 2\tau_1(\lambda)|a_e|\,[5^{-1}\nu(y - Py)]^2 \leqslant 2 \times 5^{-2}\tau_1(\lambda)\delta^2|a_e|^3,$$

$$|{}_1J_3| = |\pi^{-1}\int_0^{2\pi}(y_1 - Py_1)(Py_1)^2 \operatorname{Sin}\theta \, d\theta|,$$

which from (6.8) becomes

$$|{}_1J_3| = |\pi^{-1}\int_0^{2\pi}(a_{15}\operatorname{Sin} 5\theta + b_{15}\operatorname{Cos} 5\theta + \ldots.)({}_s\gamma_1 \operatorname{Sin}\theta + \ldots. + {}_c\gamma_7 \operatorname{Cos} 7\theta)d\theta|$$

$$= |a_{15}\,{}_s\gamma_5 + b_{15}\,{}_c\gamma_5 + a_{17}\,{}_s\gamma_7 + b_{17}\,{}_c\gamma_7|$$

and upon substituting for $|a_{15}|$, $|{}_s\gamma_5|$, ... etc. this becomes

$$|{}_1J_3| \leqslant |{}_1j_3|\,|a_e|^3,$$

where

$$|{}_1j_3| = 2^{\frac{1}{2}}N\{5^{-2}[\tfrac{1}{2}(\mu_1\mu_3 + \mu_2\mu_4) + \tfrac{1}{4}(\mu_3^2 + \mu_4^2) + \tfrac{1}{2}(\mu_1\mu_4 + \mu_2\mu_3 + \mu_3\mu_4)]$$

$$+ 7^{-2}[\tfrac{1}{4}(\mu_3^2 + \mu_4^2) + \tfrac{1}{2}\mu_3\mu_4]\}; \qquad (6.13)$$

similarly

$$|{}_3J_3| = \left|\pi^{-1}\int_0^{2\pi}(y_1 - Py_1)(Py_1)^2 \text{ Sin } 3\theta \, d\theta\right|$$

$$= \left|\pi^{-1}\int_0^{2\pi}(a_{15}\text{ Sin } 5\theta + b_{15}\text{ Cos } 5\theta + \ldots)({}_s\eta_1 \text{ Sin } \theta + \ldots + {}_c\eta_9 \text{ Cos } 9\theta)d\theta\right|$$

$$= |a_{15}\,{}_s\eta_5 + b_{15}\,{}_c\eta_5 + a_{17}\,{}_s\eta_7 + b_{17}\,{}_c\eta_7 + a_{19}\,{}_s\eta_9 + b_{19}\,{}_c\eta_9|$$

or

$$|{}_3J_3| \leqslant |{}_3j_3|\,|a_e|^3,$$

where

$$|{}_3j_3| = 2^{\frac{1}{2}}N\Big\{5^{-2}\Big[\tfrac{1}{2}(\mu_1\mu_3 + \mu_2\mu_4) + \tfrac{1}{4}(\mu_1^2 + \mu_2^2) + \tfrac{1}{2}(\mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_4)\Big]$$

$$+ 7^{-2}\Big[\tfrac{1}{2}(\mu_1\mu_3 + \mu_2\mu_4) + \tfrac{1}{2}(\mu_1\mu_4 + \mu_2\mu_3)\Big] + 9^{-2}\Big[\tfrac{1}{4}(\mu_3^2 + \mu_4^2) + \tfrac{1}{2}\mu_3\mu_4\Big]\Big\}. \tag{6.14}$$

It follows that

$$|\alpha_1 - \alpha_{10}| \leqslant \{5^{-2}[2\rho_1 + 6\tau_1(\lambda)]\delta^2 + 3|{}_1j_3|\}|a_e|^3$$

and

$$|\alpha_3 - \alpha_{30}| \leqslant \{5^{-2}[2\rho_1 + 6\tau_1(\lambda)]\delta^2 + 3|{}_3j_3|\}|a_e|^3.$$

Also,

$$\beta_n - \beta_{no} = \pi^{-1}\int_0^{2\pi}\left[y_1^3 - (Py_1)^3\right]\text{ Cos } n\theta \, d\theta$$

$$= {}_nL_1 + 3\,{}_nL_2 + 3\,{}_nL_3, \tag{6.15}$$

where

$$|{}_nL_1| \leqslant 2 \times 5^{-2}\rho_1\delta^2|a_e|^3,$$

$$|{}_nL_2| \leqslant 2 \times 5^{-2}\tau_1(\lambda)\delta^2|a_e|^3,$$

$$|{}_1L_3| \leqslant |{}_1\ell_3|\,|a_e|^3$$

$$|{}_3L_3| \leqslant |{}_3\ell_3|\,|a_e|^3$$

$$|{}_1\ell_3| = |{}_1j_3| \text{ and } |{}_3\ell_3| = |{}_3j_3|.$$

Thus

$$\left.\begin{array}{l} |\alpha_1 - \alpha_{10}| \\ |\beta_1 - \beta_{10}| \end{array}\right\} \leqslant \left\{5^{-2}\left[2\rho_1 + 6\tau_1(\lambda)\right]\delta^2 + 3|{}_1j_3|\right\}|a_e|^3 \qquad (6.16)$$

and

$$\left.\begin{array}{l} |\alpha_3 - \alpha_{30}| \\ |\beta_3 - \beta_{30}| \end{array}\right\} \leqslant \left\{5^{-2}\left[2\rho_1 + 6\tau_1(\lambda)\right]\delta^2 + 3|{}_3j_3|\right\}|a_e|^3 \qquad (6.17)$$

7. <u>An Estimate for lub $|(u,v) - 0|$.</u>

Since $\mu_1, \ldots, \mu_4$ define the cell Λ, and hence S*, it is clear that if S* is to contain the proposed exact solution then Λ must contain the point (a_1, b_1, a_3, b_3) defined by the leading coefficients of the exact solution and $\mu_1, \ldots, \mu_4$ must be chosen so as to make this possible. Now

$$\left.\begin{array}{l} |a_1| = |(1 + \epsilon_1)a_e| \leqslant (1 + |\epsilon_1|)|a_e| \\ |b_1| = |(k_1 + \epsilon_2)a_e| \leqslant (|k_1| + |\epsilon_2|)|a_e| \\ |a_3| = |\epsilon_3 a_e| \leqslant |\epsilon_3|\,|a_e| \\ |b_3| = |\epsilon_4 a_e| \leqslant |\epsilon_4|\,|a_e| \end{array}\right\} \qquad (7.1)$$

and if $\mu_1, \ldots, \mu_4$ are chosen so that

$$\left.\begin{aligned} 1 + |\epsilon_1| &\leqslant \mu_1 \\ |k_1| + |\epsilon_2| &\leqslant \mu_2 \\ |\epsilon_3| &\leqslant \mu_3 \\ |\epsilon_4| &\leqslant \mu_4 \end{aligned}\right\} \qquad (7.2)$$

then

$$|a_1| \leqslant \mu_1 |a_e|, |b_1| \leqslant \mu_2 |a_e|, |a_3| \leqslant \mu_3 |a_e| \text{ and } |b_3| \leqslant \mu_4 |a_e|,$$

as required. The four-cell Λ is then defined as the set of points corresponding to all combinations of $\epsilon_1, \ldots, \epsilon_4$ in the intervals

$$1 - \mu_1 \leqslant \epsilon_1 \leqslant \mu_1 - 1,\ k_1 - \mu_2 \leqslant \epsilon_2 \leqslant \mu_2 - k_1,\ -\mu_3 \leqslant \epsilon_3 \leqslant \mu_3,\ -\mu_4 \leqslant \epsilon_4 \leqslant \mu_4 \qquad (7.3)$$

The set of points defining the boundary three-cell Λ_B are obtained by taking $\epsilon_1, \ldots, \epsilon_4$, in turn, to be their extreme values in the above definition. Thus Λ_B is made up of the following collection of three-cells:

$$\left.\begin{array}{l} \epsilon_1 = 1 - \mu_1,\ k_1 - \mu_2 \leqslant \epsilon_2 \leqslant \mu_2 - k_1,\ -\mu_3 \leqslant \epsilon_3 \leqslant \mu_3,\ -\mu_4 \leqslant \epsilon_4 \leqslant \mu_4, \\ \epsilon_1 = \mu_1 - 1,\ k_1 - \mu_2 \leqslant \epsilon_2 \leqslant \mu_2 - k_1,\ -\mu_3 \leqslant \epsilon_3 \leqslant \mu_3,\ -\mu_4 \leqslant \epsilon_4 \leqslant \mu_4, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 - \mu_1 \leqslant \epsilon_1 \leqslant \mu_1 - 1,\ k_1 - \mu_2 \leqslant \epsilon_2 \leqslant \mu_2 - k_1,\ -\mu_3 \leqslant \epsilon_3 \leqslant \mu_3,\ \epsilon_4 = \mu_4, \end{array}\right\} \qquad (7.4)$$

where $\epsilon_1, \ldots, \epsilon_4$ take on all values over their respective intervals.

From equations (3.25), (2.12), (2.13), (2.14) and (2.15) it follows that

$$v_1 = (g_1 - 1)(a_{11} - a_e) + g_3\{\alpha_{10} - \tfrac{3}{4} a_e(a_e^2 + b_e^2)\}$$

$$u_1 = (g_1 - 1)(b_{11} - b_e) + g_3\{\beta_{10} - \tfrac{3}{4} b_e(a_e^2 + b_e^2)\}$$

$$v_3 = (g_1 - 9)a_{13} + g_3\{\alpha_{30} - \tfrac{1}{4} a_e(3b_e^2 - a_e^2)\}$$

$$u_3 = (g_1 - 9)b_{13} + g_3\{\beta_{30} + \tfrac{1}{4} b_e(3a_e^2 - b_e^2)\}$$

which, from equations (2.16) to (2.19) become

$$v_1/a_e = (g_1 - 1)\epsilon_1 - h\epsilon_2 + \frac{1}{4} k_3(g_1 - 1)\{(3 + k_1^2)\epsilon_1 + 2k_1\epsilon_2 + (k_1^2 - 1)\epsilon_3 - 2k_1\epsilon_4 + \frac{4}{3} G_1\} - h k_1 \quad (7.5)$$

$$u_1/a_e = (g_1 - 1)\epsilon_2 + h\epsilon_1 + \frac{1}{4} k_3(g_1 - 1)\{2k_1\epsilon_1 + (1 + 3k_1^2)\epsilon_2 + 2k_1\epsilon_3 + (k_1^2 - 1)\epsilon_4 + \frac{4}{3} G_2\} + h \quad (7.6)$$

$$v_3/a_e = (g_1 - 9)\epsilon_3 - 3h\epsilon_4 + \frac{1}{4} k_3(g_1 - 1)\{(k_1^2 - 1)\epsilon_1 + 2k_1\epsilon_2 + 2(1 + k_1^2)\epsilon_3 + \frac{4}{3} G_3\} \quad (7.7)$$

$$u_3/a_e = (g_1 - 9)\epsilon_4 + 3h\epsilon_3 + \frac{1}{4} k_3(g_1 - 1)\{-2k_1\epsilon_1 + (k_1^2 - 1)\epsilon_2 + 2(1 + k_1^2)\epsilon_4 + \frac{4}{3} G_4\}, \quad (7.8)$$

where $G_1, \ldots, G_4$ are given by equations (2.20) to (2.23). Thus C_o is given by $M_o\Lambda_B$, where M_o is defined by equations (7.5) to (7.8) and the distance

$$|(u, v) - 0| = |\{v_1^2 + u_1^2 + v_3^2 + u_3^2\}^{\frac{1}{2}}| \quad (7.9)$$

In order to employ the inequality (3.27) it is required to determine $\text{lub}|(u, v) - 0|$. Formally it is not a difficult problem to determine the minima of $|(u, v) - 0|$, however, this process involves the solution of a set of simultaneous equations of third degree, a task which it is required to avoid. This difficulty may, to some extent, be overcome by the use of the inequality

$$\text{lub}|(u, v) - 0| \geqslant |\{(\text{lub } v_1)^2 + (\text{lub } u_1)^2 + (\text{lub } v_3)^2 + (\text{lub } u_3)^2\}^{\frac{1}{2}}| \quad (7.10)$$

which, when one of the components is dominant, usefully reduces to

$$\text{lub}|(u, v) - 0| \geqslant \text{lub} \begin{cases} |v_1| \\ |u_1| \\ |v_3| \\ |u_3| \end{cases} \quad (7.11)$$

The use of the right hand side of (7.10), in place of $\text{lub}|(u, v) - 0|$, in (3.27) will usually underestimate the size of the region in the g_1, g_3, h, Γ space for which the existence theorem can be proved to be satisfied. Nevertheless, valuable results can still be obtained by its use.

Consider the evaluation of lub v_1. It is first required to determine whether any minima exist in v_1 as $\epsilon_1, \ldots, \epsilon_4$ vary over Λ_B. Differentiating (7.5) gives

$$\left.\begin{aligned}
\frac{\partial v_1}{\partial \epsilon_1} &= (g_1 - 1)\left\{1 + \frac{1}{4} k_3(3 + k_1^2) + \frac{1}{3} k_3 \frac{\partial G_1}{\partial \epsilon_1}\right\} a_e \\
\frac{\partial v_1}{\partial \epsilon_2} &= \left\{-h + (g_1 - 1)\left[\frac{1}{2} k_1 k_3 + \frac{1}{3} k_3 \frac{\partial G_1}{\partial \epsilon_2}\right]\right\} a_e \\
\frac{\partial v_1}{\partial \epsilon_3} &= (g_1 - 1)\left\{\frac{1}{4} k_3(k_1^2 - 1) + \frac{1}{3} k_3 \frac{\partial G_1}{\partial \epsilon_3}\right\} a_e \\
\frac{\partial v_1}{\partial \epsilon_4} &= (g_1 - 1)\left\{-\frac{1}{2} k_1 k_3 + \frac{1}{3} k_3 \frac{\partial G_1}{\partial \epsilon_4}\right\} a_e
\end{aligned}\right\} \qquad (7.12)$$

where

$$\left.\begin{aligned}
\frac{\partial G_1}{\partial \epsilon_1} &= \frac{3}{2}\left\{3\epsilon_1 + k_1(\epsilon_2 - \epsilon_4) - \epsilon_3\right\} + \frac{3}{4}\left\{3\epsilon_1^2 + \epsilon_2^2 + 2(\epsilon_3^2 + \epsilon_4^2)\right\} \\
\frac{\partial G_1}{\partial \epsilon_2} &= \frac{3}{2}\left\{k_1(\epsilon_1 + \epsilon_3) + \epsilon_2 - \epsilon_4 + \epsilon_1(\epsilon_2 - \epsilon_4) + k_1\epsilon_2\epsilon_3\right\} \\
\frac{\partial G_1}{\partial \epsilon_3} &= \frac{3}{2}(-\epsilon_1 + k_1\epsilon_2 + 2\epsilon_3) + 3\epsilon_1\epsilon_3 + \frac{3}{4}(k_1\epsilon_2^2 - \epsilon_1^2) \\
&\text{and} \\
\frac{\partial G_1}{\partial \epsilon_4} &= \frac{3}{2}\left\{-k_1\epsilon_1 - \epsilon_2 + 2\epsilon_4 + \epsilon_1(2\epsilon_4 - \epsilon_2)\right\}
\end{aligned}\right\} \qquad (7.13)$$

Now in the subsequent analysis the values of $\mu_1, \ldots, \mu_4$ will be chosen so that the values of $\epsilon_1, \ldots, \epsilon_4$ on Λ are, at most, of the first order of small quantities compared with unity, and g_1 will be assumed to be other than unity.

Thus from (7.13), the derivatives $\partial G_1/\partial\epsilon_1, \ldots\ldots, \partial G_1/\partial\epsilon_4$ are, at most, of the first order of small quantities. It follows that the derivatives $\partial v_1/\partial\epsilon_1, \ldots, \partial v_1/\partial\epsilon_4$ can only be zero if the terms not dependent on $\epsilon_1, \ldots, \epsilon_4$ in (7.12) are zero. Taking these in turn

$$|g_1 - 1|\,|1 + \frac{1}{4} k_3(3 + k_1^2)| > 0$$

$$|-h + \frac{1}{2}(g_1 - 1)k_1 k_3| \gtrless 0$$

$$|g_1 - 1|\,|\frac{1}{4} k_3(k_1^2 - 1)| > 0$$

$$|g_1 - 1|\,|\frac{1}{2} k_1 k_2| > 0,$$

thus it is possible for $\partial v_1/\partial\epsilon_2$ and $\partial v_1/\partial\epsilon_4$ to be zero in this range of $\epsilon_1, \ldots, \epsilon_4$, but not $\partial v_1/\partial\epsilon_1$ or $\partial v_1/\partial\epsilon_3$. Now with $\epsilon_1 = 1 - \mu_1$ or $\epsilon_1 = \mu_1 - 1$ on Λ_B, the least condition for a minimum is

$$\frac{\partial v_1}{\partial\epsilon_2} = \frac{\partial v_1}{\partial\epsilon_3} = \frac{\partial v_1}{\partial\epsilon_4} = 0.$$

But $\partial v_1/\partial\epsilon_3 \neq 0$ in this range, so that there can be no minimum on the part of Λ_B defined by $\epsilon_1 = 1 - \mu_1$ or $\epsilon_1 = \mu_1 - 1$. Similarly with $\epsilon_2 = k_1 - \mu_2$ or $\epsilon_2 = \mu_2 - k_1$ on Λ_B, the least condition for a minimum is

$$\frac{\partial v_1}{\partial\epsilon_1} = \frac{\partial v_1}{\partial\epsilon_3} = \frac{\partial v_1}{\partial\epsilon_4} = 0.$$

But $\partial v_1/\partial\epsilon_1 \neq 0$ in this range, so that there can be no minimum on the part of Λ_B defined by $\epsilon_2 = \pm(k_1 - \mu_2)$. Similar arguments apply to the parts of Λ_B defined by $\epsilon_3 = \pm\mu_3$ and $\epsilon_4 = \pm\mu_4$. Since there are no minima in $v_1(\epsilon_1, \ldots, \epsilon_4)$ for $\epsilon_1, \ldots, \epsilon_4$ in Λ_B, it may be concluded that lub v_1 is the value of v_1 at one of the sixteen combinations of extreme values of $\epsilon_1, \ldots, \epsilon_4$. The values of $|v_1|$ at these combinations of extreme values may conveniently be called "corner values", $(v_1)_c$, as in Ref. 1. Direct comparison of these corner values then yields the least value

$$\min(v_1)_c \Big]_{\Lambda_B} = \text{lub } v_1 \tag{7.14}$$

Consider now lub u_1. Differentiating (7.6) gives

$$\left.\begin{aligned}
\frac{\partial u_1}{\partial \epsilon_1} &= \left\{h + (g_1 - 1)\left[\frac{1}{2}k_1k_3 + \frac{1}{3}k_3\frac{\partial G_2}{\partial \epsilon_1}\right]\right\} a_e \\
\frac{\partial u_1}{\partial \epsilon_2} &= (g_1 - 1)\left\{1 + \frac{1}{4}k_3(1 + 3k_1^2) + \frac{1}{3}k_3\frac{\partial G_2}{\partial \epsilon_2}\right\} a_e \\
\frac{\partial u_1}{\partial \epsilon_3} &= (g_1 - 1)\left\{\frac{1}{2}k_1k_3 + \frac{1}{3}k_3\frac{\partial G_2}{\partial \epsilon_3}\right\} a_e \\
\frac{\partial u_1}{\partial \epsilon_4} &= (g_1 - 1)\left\{\frac{1}{4}k_3(k_1^2 - 1) + \frac{1}{3}k_3\frac{\partial G_2}{\partial \epsilon_4}\right\} a_e
\end{aligned}\right\} \tag{7.15}$$

where $\partial G_2/\partial\epsilon_1, \ldots, \partial G_2/\partial\epsilon_4$ may be obtained from (2.21).

In Case (ii) the partial derivatives $\partial u_1/\partial\epsilon_2$, $\partial u_1/\partial\epsilon_3$ and $\partial u_1/\partial\epsilon_4$ cannot be zero, and therefore, lub u_1 corresponds to the least corner value.

In Case (i)

$$h + \frac{1}{2}(g_1 - 1)k_1k_3 \approx h, \quad 1 + \frac{1}{4}k_3(1 + 3k_1^2) \approx 0$$

$$\frac{1}{2}k_1k_3 \approx 0, \quad \frac{1}{4}k_3(k_1^2 - 1) > 0,$$

and if h can be of the first order of small quantities then only $\partial u_1/\partial\epsilon_4$, without further examination of the partial derivatives of G_2, can be guaranteed to be other than zero. For the parts of Λ_B defined by $\epsilon_1 = \pm(1 - \mu_1)$ $\epsilon_2 = \pm(k_1 - \mu_2)$ and $\epsilon_3 = \pm\mu_3$, respectively, it is clear that the least condition for a minimum cannot be satisfied, because $\partial u_1/\partial\epsilon_4 \neq 0$. When $\epsilon_4 = \pm\mu_4$ the least condition for a minimum in u_1 on the appropriate part of $M_o\Lambda_B$ is

$$\partial u_1/\partial\epsilon_1 = \left\{h - \frac{4}{3}(g_1 - 1)\frac{\partial G_2}{\partial\epsilon_1}\right\} a_e = 0$$

$$\partial u_1/\partial\epsilon_2 = -\frac{4}{3}(g_1 - 1)a_e\frac{\partial G_2}{\partial\epsilon_2} = 0$$

$$\partial u_1/\partial\epsilon_3 = -\frac{4}{3}(g_1 - 1)a_e\frac{\partial G_2}{\partial\epsilon_3} = 0$$

or, excluding the conditions $g_1 = 1$, $a_e = 0$, these become

$$h - 2(g_1 - 1)(\epsilon_2 - \epsilon_4 + \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) = 0 \tag{7.16}$$

$$\frac{3}{2}(\epsilon_1 + \epsilon_3) + \frac{3}{4}(\epsilon_1^2 + 3\epsilon_2^2 + 2\epsilon_3^2 + 2\epsilon_4^2) + \frac{3}{2}(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) = 0 \tag{7.17}$$

$$\frac{3}{2}\epsilon_2(1 + \epsilon_1 + 2\epsilon_3) = 0 \tag{7.18}$$

Now $1 + \epsilon_1 + 2\epsilon_3 \neq 0$, thus from (7.18), $\epsilon_2 = 0$ at the minima. Substituting $\epsilon_2 = 0$ in (7.16) reduces this to

$$h + 2(g_1 - 1)(1 + \epsilon_1)\epsilon_4 = 0$$

If now h is chosen sufficiently small for

$$\mu_4 > | h/2(g_1 - 1) | \tag{7.19}$$

then the above equation cannot be satisfied and there can be no minima for $\epsilon_1, \ldots, \epsilon_4$ in Λ_B defined by $\epsilon_4 = \pm\mu_4$. Thus lub u_1 is the least corner value.

From (7.7)

$$\left.\begin{aligned}
\frac{\partial v_3}{\partial \epsilon_1} &= (g_1 - 1)\left\{\frac{1}{4}k_3(k_1^2 - 1) + \frac{1}{3}k_3\frac{\partial G_3}{\partial \epsilon_1}\right\} a_e \\
\frac{\partial v_3}{\partial \epsilon_2} &= (g_1 - 1)\left\{\frac{1}{2}k_1k_3 + \frac{1}{3}k_3\frac{\partial G_3}{\partial \epsilon_2}\right\} a_e \\
\frac{\partial v_3}{\partial \epsilon_3} &= \left[(g_1 - 9) + (g_1 - 1)\left\{\frac{1}{2}k_3(1 + k_1^2) + \frac{1}{3}k_3\frac{\partial G_3}{\partial \epsilon_3}\right\}\right] a_e \\
\frac{\partial v_3}{\partial \epsilon_4} &= \left\{-3h + \frac{1}{3}k_3(g_1 - 1)\frac{\partial G_3}{\partial \epsilon_4}\right\} a_e
\end{aligned}\right\} \tag{7.20}$$

Now in Case (ii) both

$$\frac{1}{4}k_3(k_1^2 - 1) < 0 \text{ and } \frac{1}{2}k_1k_3 < 0$$

so that lub v_3 corresponds to the least corner value. In Case (i)

$$\frac{1}{4}k_3(k_1^2 - 1) > 0 \text{ and } \frac{1}{2}k_1k_3 = 0.$$

Thus $\partial v_3/\partial \epsilon_1 \neq 0$ and there can be no minima in v_3 for the parts of Λ_B defined by $\epsilon_2 = \pm(k_1 - \mu_2)$, $\epsilon_3 = \pm\mu_3$ and $\epsilon_4 = \pm\mu_4$, respectively. When $\epsilon_1 = \pm(1 - \mu_1)$ the least condition for a minimum in v_3 on the appropriate part of $M_o \Lambda_B$ is

$$\frac{\partial v_3}{\partial \epsilon_2} = -\frac{4}{3}(g_1 - 1)\frac{\partial G_3}{\partial \epsilon_2} a_e = 0 \tag{7.21}$$

$$\frac{\partial v_3}{\partial \epsilon_3} = \left[g_1 - 9 - (g_1 - 1)\left\{2 + \frac{4}{3}\frac{\partial G_3}{\partial \epsilon_3}\right\} \right] a_e = 0 \tag{7.22}$$

$$\frac{\partial v_3}{\partial \epsilon_4} = \left\{ -3h - \frac{4}{3}(g_1 - 1)\frac{\partial G_3}{\partial \epsilon_4} \right\} a_e = 0 \tag{7.23}$$

Now

$$\frac{\partial G_3}{\partial \epsilon_2} = \frac{3}{2}\epsilon_2(1 + \epsilon_1 + 2\epsilon_3),$$

therefore the condition (7.21) is satisfied by $\epsilon_2 = 0$. Also it may be seen that (7.22) cannot be satisfied for $-6 < g_1$, and there can be no minima. Thus lub v_3 corresponds to the least corner value.

A closely similar analysis shows that in Case (ii) lub u_3 corresponds to the least corner value, whilst in Case (i) this is again true provided that $-6 < g_1$.

Summarizing these results, it follows that if $\mu_1 - 1$, $\mu_2 - k_1$, μ_3 and μ_4 are chosen to be, at most, of the first order of small quantities, then in Case (ii)

$$\text{lub}\left|(u, v) - 0\right| > \left|\left\{\left[\min(v_1)_c\right]^2 + \ldots\ldots + \left[\min(u_3)_c\right]^2\right\}^{\frac{1}{2}}\right|_{\Lambda_B} \tag{7.24}$$

If, in Case (i), the additional conditions (7.19) and $-6 < g_1$ are met then $\text{lub}\left|(u, v) - 0\right|$ is again given by (7.24). The restriction on h and g_1 imposed in Case (i) may seem undesirable, however, it will be seen that these conditions are no more restrictive than further conditions imposed in Section 8.

8. Application of the Proof Existence When $|g_1 - 1|$ or h are Small

Some guidance to the choice of $\mu_1, \ldots, \mu_4$ may be obtained from the linearized second approximations obtained in Section 2. It can be seen from these that provided h is taken sufficiently small then $\epsilon_1, \ldots, \epsilon_4$ will always be small. These values may be written

$$|\epsilon_{i1}| = |\Omega_1 \Phi_i(\Omega_1, \Omega_2)| = E_{i1}|h|, \tag{8.1}$$

which define a four-cell, Λ_ϵ. A satisfactory choice for Λ is then a cell slightly larger than Λ_ϵ and such that Λ_ϵ is contained in the interior of Λ. For this purpose the values of $\mu_1, \ldots, \mu_4$ defining Λ may be chosen to be

$$\left.\begin{aligned} \mu_1 &= 1 + (1 + \zeta)E_{11}|h| \\ \mu_2 &= |k_1| + (1 + \zeta)E_{21}|h| \\ \mu_3 &= (1 + \zeta)E_{31}|h| \\ \mu_4 &= (1 + \zeta)E_{41}|h| \end{aligned}\right\} , \zeta > 0 \tag{8.2}$$

It then follows from (5.6) that

$$\tau_1 = (1 + |k_1|) + (1 + \zeta)|h| \Sigma E_{i1} = T_o + T_1|h| \tag{8.3}$$

and from (5.14) and (5.17) that ϕ may be expressed in the form

$$\phi = \phi_1|h| + \phi_2|h|^2 + \phi_3|h|^3, \quad \phi_1, \phi_2, \phi_3 > 0 \tag{8.4}$$

The simultaneous inequalities (5.32) to (5.34) may, alternatively be written as the equations

$$(|g_1| + 5|h|)\delta + \frac{1}{3}|k_3||g_1 - 1|(5\phi + 3\tau_1^2\delta + 3\tau_1\rho_1\delta + \rho_1^2\delta) = B\delta \tag{8.5}$$

$$(|g_1| + 5|h|)\delta + \frac{1}{3}|k_3||g_1 - 1|(5\phi + 3\tau_1^2\delta + 3\tau_1\rho_1\delta + \rho_1^2\delta) = D\rho_2 \tag{8.6}$$

$$\rho_1 = A\delta, \tag{8.7}$$

with

$$A \geqslant 0.4952, \quad 0 \leqslant B \leqslant 25, \quad 0 \leqslant D \leqslant 10.098.$$

Similarly the contraction condition (5.42) may be written as

$$|g_1| + 5|h| + |k_3||g_1 - 1|(\tau_1 + \rho_1)^2 = C \tag{8.8}$$

with

$$0 < C < 25$$

Substituting from (8.7) into (8.5) and (8.8) and writing

$$H = |k_3||g_1 - 1| \tag{8.9}$$

then gives

$$A^2\delta^3 + 3\tau_1 A\delta^2 + \{3(|g_1| + 5|h| - B)/H + 3\tau_1^2\}\delta + 5\phi = 0 \tag{8.10}$$

and

$$(|g_1| + 5|h| - C)/H + (\tau_1 + A\delta)^2 = 0 \tag{8.11}$$

From (8.11)

$$(|g_1| + 5|h| - B)/H = (C - B)/H - (\tau_1 + A\delta)^2,$$

which upon substitution into (8.10) gives

$$2A^2\delta^3 + 3A\tau_1\delta^2 + 3(B - C)\delta/H - 5\phi = 0 \tag{8.12}$$

The significance of equation (8.6) is that it defines ρ_2 and hence by (5.41), R_2.

Since by definition, δ must be real and positive, then only the real and positive roots of the cubic (8.12) are relevant. For the present purpose it is convenient to restrict the choice of A, B, C, H and hence τ_1 and ϕ to ranges of values which cause (8.12) to have only one positive real root and such that this root is small. This this choice is possible may be seen by writing (8.12) in the form

$$\delta = \frac{|H|}{3(B - C)}\{5\phi - 3A\tau_1\delta^2 - 2A^2\delta^3\} \tag{8.13}$$

and observing that, with A not too large and B > C, it is possible to obtain small positive values of δ which satisfy the expression by choosing $|h|$ and hence ϕ small, or by choosing H small. In the latter case, however, the choice of $|h|$ and hence ϕ is not unrestricted. Alternatively, the ranges of the variables which ensure that there is only one real and positive value of δ may be obtained more precisely from the relations given by Neumark in Ref. 6, p. 5, Case (A).

From (8.13) it follows that there exists a positive root

$$\delta < \frac{5H\phi}{3(B - C)} = \frac{5H}{3(B - C)} (\phi_1|h| + \phi_2|h|^2 + \phi_3|h|^3) \tag{8.14}$$

From (6.5), (8.5) and (8.14)

$$N = B\delta < \frac{5H\phi}{3(B - C)} \tag{8.15}$$

and from (6.13) $|{}_1j_3|$ may be expressed as

$$|{}_1j_3| = 5^{-1}N(J_1|h| + J_2|h|^2), \quad J_1, J_2 > 0$$

$$< \frac{B\ H\ \phi}{3(B - C)} (J_1|h| + J_2|h|^2) \tag{8.16}$$

Thus from (6.16), (8.7) and (8.3)

$$\left.\begin{array}{l} |\alpha_1 - \alpha_{10}| \\ |\beta_1 - \beta_{10}| \end{array}\right\} \leqslant \left\{5^{-2}\left[2A\delta + 6(T_o + T_1|h|)\right]\delta^2 + 3|{}_1j_3|\right\}|a_e|^3,$$

which upon further substitution from (8.16), (8.14) and (8.4) gives

$$\left.\begin{array}{l} |\alpha_1 - \alpha_{10}| \\ |\beta_1 - \beta_{10}| \end{array}\right\} \leqslant \frac{H}{B - C}\left\{X_2|h|^2 + \ldots + X_9|h|^9\right\}|a_e|^3, \quad X_2, \ldots, X_9 > 0 \tag{8.17}$$

Similarly

$$\left.\begin{array}{l} |\alpha_3 - \alpha_{30}| \\ |\beta_3 - \beta_{30}| \end{array}\right\} \leqslant \frac{H}{B - C}\left\{Y_2|h|^2 + \ldots + Y_9|h|^9\right\}|a_e|^3, \quad Y_2, \ldots, Y_9 > 0 \tag{8.18}$$

Substituting from (8.17) and (8.18) into (3.26) gives

$|(U, V) - (u, v)|$

$$< \left| \{2(X_2|h|^2 + \ldots + X_9|h|^9)^2 + 2(Y_2|h|^2 + \ldots + Y_9|h|^9)^2\}^{\frac{1}{2}} \right| \frac{|g_3||a_e|^3 H}{|B - C|}$$

Or, upon the further substitution

$$|g_3||a_e|^2 = H/3$$

the inequality becomes

$$|(U, V) - (u, v)| < \frac{H^2|a_e|}{|B - C|} \{Z_2|h|^2 + \ldots + Z_9|h|^9\}$$

or

$$\text{glb}|(U, V) - (u, v)| < \frac{H^2|a_e|}{|B - C|} \{ Z_2|h|^2 + \ldots + Z_9|h|^9\}, \qquad (8.19)$$

where $Z_2, \ldots, Z_9 > 0$. This quantity may be made as small as desired by taking H or $|h|$ sufficiently small.

To establish the proof when only h is small it will be observed that

$$(u_3)_c/a_e = \pm(g_1 - 9)\mu_4 \pm 3h\mu_3 + (g_1 - 1)\{\pm\tfrac{1}{2} k_1 k_3(\mu_1 - 1) \pm \tfrac{1}{4} k_3(k_1^2 - 1)(\mu_2 - k_1)$$
$$\pm\tfrac{1}{2} k_3(1 + k_1^2)\mu_4 + \tfrac{1}{3} k_3(G_4)_c\}$$

From (2.23) and (8.2) $(G_4)_c$ must have the form

$$(G_4)_c = G_{42}|h|^2 + G_{43}|h|^3,$$

therefore

$$(u_3)_c/a_e = \pm(g_1 - 9)(1 + \zeta)E_{41}|h| \pm 3h(1 + \zeta)E_{31}|h|$$
$$+(g_1 - 1)\{\pm\tfrac{1}{2} k_1 k_3(1 + \zeta)E_{11}|h| \pm\tfrac{1}{4} k_3(k_1^2 - 1)(1 + \zeta)E_{21}|h|$$
$$\pm\tfrac{1}{2} k_3(1 + k_1^2)(1 + \zeta)E_{41}|h| \pm\tfrac{1}{3} k_3(G_{42}|h|^2 + G_{43}|h|^3)\}$$

It follows that the minimum corner value of u_3 on Λ_B has the form

$$\left| \min (u_3)_c \right|_{\Lambda_B} = \{L_1|h| + L_2|h|^2 + L_3|h|^3\}|a_e|, \quad L_1, L_2, L_3 > 0$$

or

$$\text{lub}|(u, v) - 0| \geqslant \text{lub}|u_3| = L_1|h| + L_2|h|^2 + L_3|h|^3 \qquad (8.20)$$

Thus from (8.19) and (8.20), with $|h|$ chosen to be sufficiently small,

$$\text{glb}|(U, V) - (u, v)| < \text{lub}|(u, v) - 0|$$

and from (3.27) the desired result follows. This does not mean that the result holds for arbitrary finite g_1 since this is also governed by the contraction condition (5.42). The limiting values of g_1 may be obtained by putting $h \to 0$ and $C = 24.99 \ldots.$ in (5.42). Thus $\delta \to 0$, $\tau \to T_o$ and the equations for the limiting value become

$$|g_1| + 4|g_1 - 1| = 24.99 \ldots. \qquad \text{Case (i)}$$

and

$$|g_1| + (1 + 3^{\frac{1}{2}})^2|g_1 - 1| = 24.99 \ldots. \qquad \text{Case (ii)},$$

the solutions of which are

$$g_1 = 5.8 \text{ and } -4.2 \qquad \text{Case (i)}$$

and

$$g_1 = 3.83 \text{ and } -2.06 \qquad \text{Case (ii)}$$

This means that for h vanishly small the subharmonic solution (2.3) exists for g_1 in the interval $5.8 \geqslant g_1 \geqslant -4.2$, Case (i) and $3.83 \geqslant g_1 \geqslant -2.06$, Case (ii). With increasing values of h these intervals decrease in size.

To establish the proof with only $|g_1 - 1|$ small consider

$$\text{lub } v_1 = \min\left(\left[(g_1 - 1)\epsilon_1 - h(k_1 + \epsilon_2) + (g_1 - 1)\left\{\frac{1}{4} k_3(3 + k_1^2)\epsilon_1 + \frac{1}{2} k_1 k_3 \epsilon_2 + \frac{1}{4} k_3(k_1^2 - 1)\epsilon_3 - \frac{1}{2} k_1 k_3 \epsilon_4 + \frac{1}{3} k_3 G_1\right\}\right] a_e\right)_c \Big|_{\Lambda_B} \qquad (8.21)$$

Now as $(g_1 - 1) \to 0$, $H \to 0$ and

$$\text{lub}|u_3| \to \left|\min\{-h(k_1 + \epsilon_2)a_e\}_c\right|_{\Lambda_B} \geqslant (|k_1| - |\epsilon_2|)|h||a_e|$$

Provided that h is not large enough to make $|\epsilon_2| = |k_1|$ then it follows that for $(g_1 - 1)$ sufficiently small

$$\text{glb}|(U, V) - (u, v)| < \text{lub}|v_1| \leqslant \text{lub}|(u, v) - 0|$$

and the desired result follows from (3.27).

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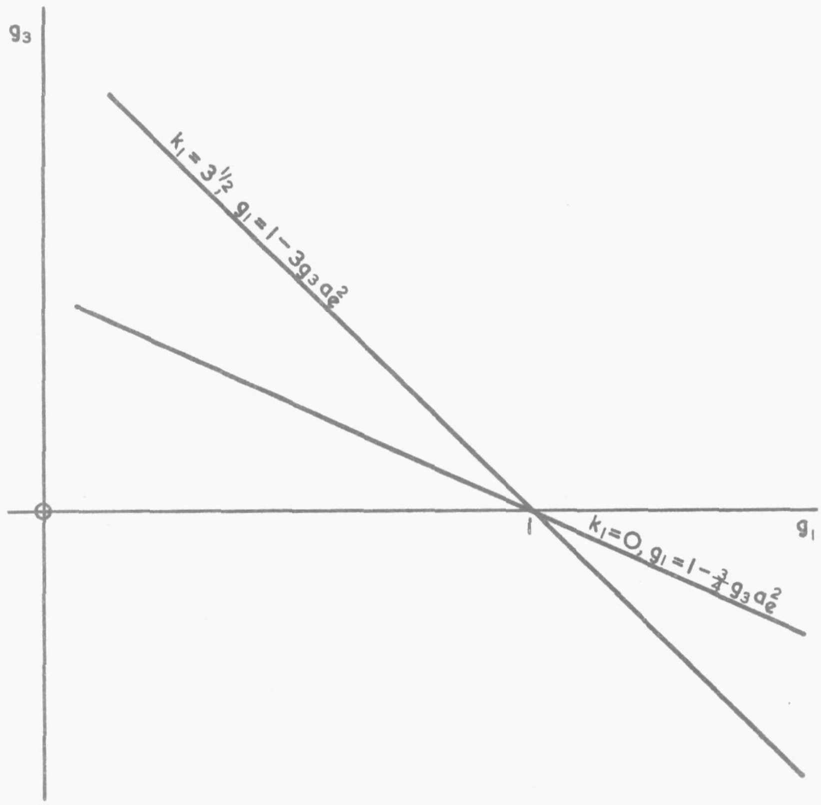

FIG.1. REGIONS OF EXISTENCE OF PURE SUBHARMONICS.
IN THE g_1, g_3 PLANE.