CoA Report Aero No. 175

THE COLLEGE OF AERONAUTICS

CRANFIELD

THE STABILITY OF THE SHORT-PERIOD MOTION OF AN AIRFRAME HAVING NON-LINEAR AERODYNAMIC CHARACTERISTICS SUBJECT TO A SINUSOIDAL ELEVATOR OSCILLATION

by

P.A.T. Christopher

THE COLLEGE OF AERONAUTICS

CRANFIELD

The stability of the short-period motion of an airframe having non-linear aerodynamic characteristics subject to a sinusoidal elevator oscillation

- by -

P.A.T. Christopher

CORRIGENDA

Page 1, Equation 1 should read:-

$$\ddot{w} - f_1(w).\,\dot{w} - f_2(w) = Q(\eta)$$

Page 10, last equation should read:-

$$Q(\rho_o, \xi_o) = \frac{d\psi}{d\tau} = \frac{1}{2\mu}\left\{1 + \frac{1}{\pi\omega^2}\int_0^2 \left[-\tfrac{1}{2}\omega b \operatorname{Sin}2\psi_o - c\operatorname{Cos}^2\psi_o + d\rho_o\operatorname{Cos}^4\psi_o + Q\rho_o^{-\frac{1}{2}}\operatorname{Sin}\psi_o\operatorname{Sin}\sigma\right]d\sigma\right\}$$

Page 14, second line from bottom should read:-

It should be noted that the term $(U_o+z_q)m_w$ in (52) has a different sign

176

CoA Report Aero No. 176

December, 1964

THE COLLEGE OF AERONAUTICS

CRANFIELD

The stability of the short-period motion of an airframe having non-linear aerodynamic characteristics and subject to a sinusoidal elevator oscillation

- by -

P.A.T. Christopher

SUMMARY

The analysis of the stability and response of second-order non-linear, non-autonomous systems by Minorsky's stroboscopic method is briefly presented. This theory is used to determine response curves and stability criteria for the short period motion of an airframe having non-linear normal force and pitching moment characteristics and subject to a sinusoidal elevator oscillation. These results are then compared with those obtained from quasi-linear theory. Some implications of these results in the synthesis of an automatic control system for an air-to-air missile are briefly discussed.

Contents

1. Introduction

In Ref. 1 non-linear expressions for the normal force and pitching moment have been introduced into the equations governing the short-period motion of an airframe. By confining attention to the motion in w, the perturbation velocity along the axis of yaw, the problem reduces to consideration of the stability of the non-autonomous equation

$$w - f_1(w).\ w - f_2(w) = Q(\eta) \tag{1}$$

where, for the case of the free motion, $Q(\eta) \equiv 0$. The stability of the free motion has been analysed by means of Poincaré's theory of singular points in the phase plane and various stability criteria obtained.

For the forced motion $Q(\eta) \not\equiv 0$ and in the general case the stability of (1) can no longer be decided in the same manner as for the degenerate case $Q(\eta) \equiv 0$. It is important to remember that for a linear system the stability is determined completely from the complementary function, provided $Q(\eta)$ is finite. For the non-linear case this result is no longer valid and the stability problem is different for each distinct form of $Q(\eta)$. It is worthy of note that the case $Q(\eta)$ constant, which can be interpreted as a step-function disturbance of the elevator, is an exception and can be treated by Poincaré's theory of singular points.

An important practical method for determining airframe response and the associated aerodynamic derivatives is the frequency response technique. This consists of oscillating the elevator sinusoidally and measuring, by instruments such as rate gyroscopes and accelerometers, the variations in the pitching motion It is of interest to consider the stability of an airframe when subject to a sinusoidal input in this way.

The problem reduces to solving (1) with $Q(\eta)$ of sinusoidal form and use is made of the stroboscopic method suggested by Minorsky (Refs. 2 and 3). This facilitates the transformation of the non-autonomous equation (1) in w and t to an autonomous form to which it is possible to apply Poincaré's theory of singular points and thereby obtain both response curves and stability criteria.

Notation

b, c, d	coefficients in Duffing's equation, see Section 4.
f =	$\dot{x}$, function in the general transformation to the stroboscopic plane
f_1, f_2	functions in equation (1)
m	airframe mass
q	angular velocity about axis of pitch
t	time
v =	$dx/d\sigma$
w	perturbation in velocity along axis of yaw
x	any dependent variable, see Sections 2, 3 and 4

$x_{c.g.}$		distance of centre of gravity aft of reference line, see Section 6 and Fig. 2.
A_1	=	$(U_o + z_q)\, m_w - m_q z_w$
A_3	=	$(U_o + z_q)\, m_3 - m_q z_3$
B		Moment of inertia about the axis of pitch
B_1	=	$(U_o + z_q)\, m_w + m_q + z_w$
B_3	=	$3z_3$
C_m		Pitching moment coefficient
C_n	=	C_z, normal force coefficient
D		the operator d/dt
D_r		reference diameter for moment coefficients
F		amplitude of sinusoidal oscillation in x or w
H	=	$(\eta_1^2 + \eta_2^2)^{\frac{1}{2}}$
J, K		functions in the short-period response equation (63)
M		moment about axis of pitch
P, Q		functions defining the stroboscopic system
$Q(\eta)$		forcing function of the elevator in equation (1)
S_r		reference area, body cross-sectional area
U		velocity tangential to flight path
U_o		velocity along the axis of roll (longitudinal body axis)
W		velocity along axis of yaw
Z	=	N, force along axis of yaw
z_w	=	$\frac{1}{m}\left(\frac{dZ}{dw}\right)_{w=o}$
z_q	=	$\frac{1}{m}\left(\frac{dZ}{dq}\right)_{q=o}$
z_η	=	$\frac{1}{m}\left(\frac{dZ}{d\eta}\right)_{\eta=o}$
m_w	=	$\frac{1}{B}\left(\frac{dM}{dw}\right)_{w=o}$
m_q	=	$\frac{1}{B}\left(\frac{dM}{dq}\right)_{q=o}$
m_η	=	$\frac{1}{B}\left(\frac{dM}{d\eta}\right)_{\eta=o}$
$z_3, z_5 \ldots$; $m_3, m_5 \ldots$		constants in the force and moment relations of equation (49)

α	=	angle of incidence in the pitching plane
α_F	=	F/U
δ	=	$\mathrm{Tan}^{-1}(\eta_2/\eta_1)$
ϵ		angle of downwash
η		elevator angle
η_a		amplitude of sinusoidal oscillation applied to elevator
η_1	=	$\eta_a \left[(U_o + z_q) m_\eta - z_\eta m_q \right]$
η_2	=	$\omega \eta_a z_\eta$
σ	=	ωt
θ		angle of elevation
τ, μ		dummy variables used in defining the stroboscopic system
ξ_o	=	$\psi_o + \sigma$
ω		frequency

Suffixes

o	refers to the zero order approximation to the stroboscopic system
t	refers to trimmed conditions
R	reference values

A dot over a variable indicates differentiation with respect to t, whilst a prime indicates differentiation with respect to σ.

2. The Stroboscopic Method

In order to get the stroboscopic method in perspective it is useful to refer to two papers by Poincaré, since between them they constitute the foundation of the theory of non-linear oscillations. The first, Ref. 4, is concerned with the theory of singular points and the topological configurations of the integral curves in the phase plane. These ideas have been explained and used in Ref. 1. The second, Ref. 5, is concerned with the "method of small parameters" or "perturbation method". This centres around the autonomous equation.

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}), \tag{2}$$

where μ is a small parameter and $f(x, \dot{x})$ is non-linear function of displacement and velocity. Obviously if μ is zero then the equation degenerates to that of the "harmonic oscillator" and in general the motion is considered to be near this degenerate case. Only periodic solutions are sought and the solution is written as a series in ascending powers of μ, with coefficients which are functions of the initial conditions and time, i.e.

$$x = A_o(x_o, \dot{x}_o, t) + \mu A_1(x_o, \dot{x}_o, t) + \mu^2 A_2(x_o, \dot{x}_o, t) + \ldots \text{ etc.} \tag{3}$$

Imposing the condition for periodidity then gives rise to a series of recurrence relations (these are in fact differential equations) which permit A_o, A_1, ... etc. to be determined. Equation (3) is an exact expression and could be thought to play a similar role as series solutions in the theory of linear differential equations and thereby define some higher transcendental function. Unfortunately, in practical problems, the analysis becomes exceedingly unwieldly if more than two or three terms are required.

For a system with a forcing term directly dependent on time, i.e. non-autonomous, the methods of Poincaré are not directly applicable. The problem may be formulated in a similar manner to (2) i.e.

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}, t), \tag{4}$$

where f is now explicitly dependent on time. If now it were possible to make a suitable co-ordinate transformation in order to reduce (4) to the autonomous form (2), then the solutions and their stability can be analysed by Poincaré's method. This, basically, is what Minorsky has done by introducing his concept of a "stroboscopic system".

Consider the non-autonomous system of differential equations

$$\frac{dx}{dt} = \dot{x} = v = Q(x, v, t) \; ; \; \ddot{x} = \dot{v} = P(x, v, t), \tag{5}$$

of which (4) is a particular case. Writing

and
$$\left.\begin{aligned} x &= \rho^{\frac{1}{2}} \operatorname{Cos} \psi \\ v &= \rho^{\frac{1}{2}} \operatorname{Sin} \psi \end{aligned}\right) \quad \ldots \tag{6}$$

then

$$\rho = x^2 + v^2 \tag{7}$$

and

$$\psi = \operatorname{Tan}^{-1}\left(\frac{v}{x}\right) \tag{8}$$

and (5) may be written

$$\frac{d\rho}{dt} = S(\rho, \psi, t); \; \frac{d\psi}{dt} = R(\rho, \psi, t) \tag{9}$$

It is now assumed that the system is forced by a periodic disturbance and that the problem is nearly linear such that the solution is a periodic motion of period 2π in the region of the linear solution.

For the linear case x will be of the form F Sin $(\sigma + \phi)$,

where $\sigma = \omega t$, and $x' = F_o \text{ Cos } (\sigma + \phi)$, where the prime refers to differentiation with respect to σ. This implies that

$$\rho = F_o^2 = \text{constant}$$

and differentiating x with respect to σ gives

$$x' = -\psi' \rho^{\frac{1}{2}} \text{Sin } \psi = v = \rho^{\frac{1}{2}} \text{Sin } \psi$$

or

$$\psi' = -1; \tag{10}$$

also

$$\rho' = 0 \tag{11}$$

The differential coefficients ψ' and ρ' may be looked upon as components of a field vector in the ρ, ψ plane and for the non-linear problem may be represented by

$$\rho' = f(\rho, \psi, \sigma)\ ; \quad \psi' = -1 + \mu g(\rho, \psi, \sigma) \tag{12}$$

where μ is a small parameter which expresses the difference between the linear and non-linear solutions. The transformed variables may, following Poincaré's method for the autonomous case, be written as power series

$$\left.\begin{aligned} \rho(\sigma) &= \rho_o(\sigma) + \mu\rho_1(\sigma) + \mu^2\rho_2(\sigma) + \\ \psi(\sigma) &= \psi_o(\sigma) + \mu\psi_1(\sigma) + \mu^2\psi_2(\sigma) + \end{aligned}\right) \tag{13}$$

in which

$$\left.\begin{aligned} &\rho_o(\sigma) = \rho_o = \text{constant} \\ &\text{and} \\ &\psi_o(\sigma) = \int -1 \,.\, d\sigma = \xi_o - \sigma \end{aligned}\right) \tag{14}$$

where ξ_o is constant.

Starting with the zero order approximation, (14), which is the solution to the linear problem, it is possible to build up a successive approximation according to (13). For instance the first order approximation is

$$\left.\begin{aligned} \rho(\sigma) &= \rho_o + \mu\rho_1(\sigma) \\ \psi(\sigma) &= \xi_o - \sigma + \mu\psi_1(\sigma) \end{aligned}\right) \quad (15)$$

where

$$\rho_1(\sigma) = \int_o^\sigma f(\rho_o, \psi_o, \sigma)d\sigma \quad (16)$$

and

$$\psi_1(\sigma) = \int_o^\sigma g(\rho_o, \psi_o, \sigma)d\sigma \quad (17)$$

In the phase plane the curve described by (14) is a circle of radius ρ_o, whereas (15) describes a spiral (in a moderate to lightly damped system) which may or may not converge to a circle, corresponding to the periodic solution assumed, as $\sigma \rightarrow \infty$. Rather than consider the continuous convergence or divergence of (15) with σ, Minorsky proposed studying the geometry of a set of points on the curve each separated discreetly by an interval equal to the period 2π. See Fig. 1(a). The increments in ρ and ψ due to the non-linearity and over the period 2π are

$$\left.\begin{aligned} \mu\rho_1(2\pi) &= \mu\int_o^{2\pi} f(\rho_o, \psi_o, \sigma)d\sigma = 2\pi\mu \quad P(\rho_o, \xi_o) \\ &\text{and} \\ \mu\psi_1(2\pi) &= \int_o^{2\pi} g(\rho_o, \psi_o, \sigma)d\sigma = 2\pi\mu \quad Q(\rho_o, \xi_o) \end{aligned}\right) \quad (18)$$

Putting $2\pi\mu = \Delta\tau$, $\Delta\rho = \mu\rho_1$, and $\Delta\xi = \mu\xi_1 \equiv \mu\psi_1$

then

$$\frac{\Delta\rho}{\Delta\tau} = P(\rho_o, \xi_o); \quad \frac{\Delta\xi}{\Delta\tau} = Q(\rho_o, \xi_o), \quad (19)$$

and when the period 2π is short enough in comparison with the time taken for the system to settle down to its steady periodic motion, then it is permissible to use continuous variables

$$\frac{d\rho}{d\tau} = P(\rho_o, \xi_o); \quad \frac{d\xi}{d\tau} = Q(\rho_o, \xi_o) \quad (20)$$

The study of (15) at discreet points separated by 2π in σ has an obvious analogy with the stroboscope and the resulting equations (20) are referred to as the "stroboscopic system" of equation (5). In Fig. 1(a) if the points A_1, A_2, ... etc. on the stroboscopic curve in the phase plane approach a fixed point A as $\sigma \rightarrow \infty$ then this implies that the system tends to a periodic motion with period 2π; each point A on the final circle is associated with a discreet stroboscopic curve. Since A is a fixed point then ρ_o and ξ_o must be constant and therefore must correspond to the conditions

$$\frac{d\rho}{d\tau} = P(\rho_o, \xi_o) = 0; \quad \frac{d\xi}{d\tau} = Q(\rho_o, \xi_o) = 0 \quad (21)$$

i.e. a singular point in the stroboscopic system. The following general conclusion may be drawn from the foregoing, "the existence of a stable singular point in the stroboscopic system (20) is indicative of the existence of a periodic solution of the original system (5)".

The previous conclusion is based on Minorsky's heuristic approach, a form of argument which is more likely to appeal to the engineer; a more rigorous development is given by Urabe in Ref. 6.

For specific applications of the method a relation is required which will transform a given d.e. in x, $\dot{x}$ and t to its corresponding stroboscopic system. Consider $x = \rho^{\frac{1}{2}}\text{Cos}\ \psi$ and differentiate with respect to σ, giving

$$x' = \tfrac{1}{2}\rho^{-\frac{1}{2}}\rho'\,\text{Cos}\ \psi - \rho^{\frac{1}{2}}\psi'\,\text{Sin}\ \psi = \rho^{\frac{1}{2}}\text{Sin}\ \psi$$

or

$$\rho'\ \text{Cos}\ \psi - \psi' 2\rho\ \text{Sin}\ \psi = 2\rho\ \text{Sin}\ \psi \tag{22}$$

Further,

$$x'' = v' = \tfrac{1}{2}\rho^{-\frac{1}{2}}\rho'\,\text{Sin}\ \psi + \rho^{\frac{1}{2}}\psi'\ \text{Cos}\ \psi$$

$$= \tfrac{1}{2}\rho^{-\frac{1}{2}}\left\{\rho'\text{Sin}\ \psi + 2\rho\psi'\ \text{Cos}\ \psi\right\} = f(x, x', \sigma),\ \text{say,}$$

or

$$\rho'\text{Sin}\ \psi + \psi'\, 2\rho\ \text{Cos}\ \psi = 2\rho^{\frac{1}{2}}\, f(x, x', \sigma) \tag{23}$$

Eliminating ρ' and ψ' in turn between (22) and (23) gives

$$\rho' = (2\rho)^2 \begin{vmatrix} \rho^{-\frac{1}{2}} f(x, x', \sigma) & \text{Cos}\ \psi \\ \text{Sin} & -\text{Sin}\ \psi \end{vmatrix} \div \begin{vmatrix} \text{Sin}\ \psi & 2\rho\text{Cos}\ \psi \\ \text{Cos}\ \psi & -2\rho\text{Sin}\ \psi \end{vmatrix}$$

or

$$\rho' = \rho\ \text{Sin}\ 2\psi + 2\rho^{\frac{1}{2}}\, f(x, x', \sigma)\ \text{Sin}\ \psi; \tag{24}$$

and

$$\psi' = 2\rho \begin{vmatrix} \text{Sin}\ \psi & \rho^{-\frac{1}{2}} f(x, x', \sigma) \\ \text{Cos}\ \psi & \text{Sin}\ \psi \end{vmatrix} \div \begin{vmatrix} \text{Sin}\ \psi & 2\rho\ \text{Cos}\ \psi \\ \text{Cos}\ \psi & -2\rho\ \text{Sin}\ \psi \end{vmatrix}$$

or

$$\psi' = -1 + \text{Cos}^2\psi + \rho^{-\frac{1}{2}} f(x, x', \sigma)\text{Cos}\ \psi \tag{25}$$

In the first approximation the increments in ρ and ψ are

$$\mu\rho_1(2\pi) = \int_0^{2\pi}\left\{\rho_0\ \text{Sin}\ 2\psi_0 + 2\rho_0^{\frac{1}{2}}\, f(x_0, x_0', \sigma)\ \text{Sin}\ \psi_0\right\} d\sigma$$

and

$$\mu\psi_1\,(2\pi) = \int_0^{2\pi}\left\{\mathrm{Cos}^2\psi_o + \rho_o^{-\frac{1}{2}}\, f(x_o, x_o', \sigma)\mathrm{Cos}\,\psi_o\right\} d\sigma.$$

Since

$$\int_0^{2\pi} \mathrm{Sin}\,2\,\psi_o\, d\sigma = \int_0^{2\pi} \mathrm{Sin}\,2(\xi_o - \sigma)d\sigma = 0$$

and

$$\int_0^{2\pi} \mathrm{Cos}^2\,\psi_o\, d\sigma = \int_0^{2\pi} \mathrm{Cos}^2\,(\xi_o - \sigma)d\sigma = \pi,$$

then

$$\mu\rho_1\,(2\pi) = 2\rho_o^{\frac{1}{2}} \int_0^{2\pi} f(x_o, x_o', \sigma)\,\mathrm{Sin}\,\psi_o\, d\sigma$$

and

$$\mu\psi_1\,(2\pi) = \pi + \rho_o^{-\frac{1}{2}} \int_0^{2\pi} f(x_o, x_o', \sigma)\,\mathrm{Cos}\,\psi_o d\sigma\,,$$

and the continuous stroboscopic system becomes

$$\frac{d\rho}{d\tau} = P(\rho_o, \xi_o) = \frac{\rho_o^{\frac{1}{2}}}{\mu\pi} \int_0^{2\pi} f(x_o, x_o', \sigma)\,\mathrm{Sin}\,(\xi_o - \sigma)d\sigma \tag{26}$$

and

$$\frac{d\xi}{d\tau} = Q(\rho_o, \xi_o) = \frac{1}{2\mu}\left\{1 + \frac{\rho_o^{-\frac{1}{2}}}{\pi} \int_0^{2\pi} f(x_o, x_o', \sigma)\mathrm{Cos}(\xi_o - \sigma)d\sigma\right. \tag{27}$$

3. Stability Criteria

Having transformed a given system into its stroboscopic counterpart, equations (26) and (27), Poincaré's theory of singular points may now be used to determine the character of the singular points in the stroboscopic plane and thereby determine the existence of periodic solutions of the original problem.

In general terms, assume that a system of differential equations

$$\frac{dx_i}{d\tau} = x_i\,(x_1, x_2, \ldots\ldots x_n) \quad i = 1, 2, \ldots\ldots n \tag{28}$$

has a set $x_{i_o}(\tau)$ of known periodic solutions which represent a closed curve in n-dimensional space (for $n = 2$ the curve lies in phase plane).

In order to discuss the general problem of equilibrium consider a neighbouring solution

$$x_i(\tau) = x_{i_o}(\tau) + \nu_i(\tau), \tag{29}$$

where $\nu_i(\tau)$ is a set of functions, called perturbations. Substituting from (29) into (28) and then developing the functions x_i in Taylor's series around x_{i_o}, retaining only the linear terms in ν_i, there is obtained a system of linear "variational equations"

$$\frac{d\nu_i}{d\tau} = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_{j_o}} \nu_i, \tag{30}$$

where $\partial X_i / \partial x_{j_o}$ are the partial derivatives of X_i with respect to x_j into which the known periodic solutions are replaced after differentiation. The coefficients of ν_i may be either constants or periodic functions of τ .

Using (29) as a basis, the "asymptotic stability" of the system can be defined as requiring $\nu_i(\tau) \to 0$ as $\tau \to \infty$ and the linear equations (30) permit of a more ready analysis of this condition than in the case of (28) which in its general form can be taken to be non-linear. This formulation is sufficiently broad to include both periodic motion and static equilibrium.

In the present application the system has one degree of freedom and is defined by

$$\frac{d\rho}{d\tau} = P(\rho, \xi) \;;\; \frac{d\xi}{d\tau} = Q(\rho, \xi) \tag{31}$$

If ρ_s, ξ_s is a singular point in the stroboscopic system then the variational equations become

$$\frac{d(\delta\rho)}{d\tau} = P_\rho\ (\rho_s, \xi_s)\delta\rho + P_\xi\ (\rho_s, \xi_s)\delta\xi \tag{32}$$

and

$$\frac{d(\delta\xi)}{d\tau} = Q_\rho\ (\rho_s, \xi_s)\delta\rho + Q_\xi\ (\rho_s, \xi_s)\delta\xi \tag{33}$$

where $\delta\rho$ and $\delta\xi$ correspond to ν_i and $P_\rho = \frac{\partial P}{\partial \rho}$, $P_\xi = \frac{\partial P}{\partial \xi}$, $Q_\rho = \frac{\partial Q}{\partial \rho}$ and $Q_\xi = \frac{\partial Q}{\partial \xi}$ correspond to $\frac{\partial x_i}{\partial x_{j_o}}$ in equation (30).

Equations (32) and (33) are of a similar form to equation (8a) of Ref. 1 and it follows that the type of singularity is governed by the characteristic equation

$$\lambda^2 - (P_\rho + Q_\xi)\lambda + (P_\rho Q_\xi - P_\xi Q_\rho) = 0 \tag{34}$$

Stable singularities occur when the inequalities

$$P_\rho + Q_\xi < 0 \tag{35}$$

and

$$\begin{vmatrix} P_\rho & P_\xi \\ Q_\rho & Q_\xi \end{vmatrix} > 0 \qquad (36)$$

are satisfied. These results can conveniently be represented on a diagram, Fig. 1(b), in which the conditions for the various types of singularity are given. It can be seen that the inequality (36) implies that the singular point is not a saddle (always unstable) and (35) that the real part of the roots $\lambda_{1,\,2}$ are negative.

This then establishes all the stability criteria required for the examples to be considered. For a rigorous development of stability theory of periodic motion the reader is referred to Ref. 7.

4. An Example - Duffing's Equation

Before proceeding to the problem of airframe short-period motion, it is of interest to apply the stroboscopic method to an example for which the solution, in the first approximation, is known. This should then give a more tangible interpretation to equations (26), (27), (35) and (36). The example chosen is Duffing's equation

$$\ddot{x} + b\dot{x} + cx - dx^3 = Q \operatorname{Sin} \omega t \qquad (37)$$

whose response and stability characteristics have been extensively discussed by Stoker in Ref. 8.

Write $\sigma = \omega t$ and (37) becomes

$$x'' = \frac{1}{\omega^2}\left\{ - \omega bx - cx + dx^3 + Q \operatorname{Sin} \sigma \right\}$$

which in terms of ρ and ψ is

$$x'' = \frac{1}{\omega^2}\left\{ -\omega b\rho^{\frac{1}{2}} \operatorname{Sin}\psi - c\rho^{\frac{1}{2}}\operatorname{Cos}\psi + d\rho^{3/2}\operatorname{Cos}^3\psi + Q \operatorname{Sin}\sigma \right\} \qquad (38)$$

From the general transformation (26) and (27) the stroboscopic system becomes:

$$P(\rho_o, \xi_o) = \frac{d\rho}{d\tau} = \frac{\rho_o}{\pi\mu\omega^2}\int_o^{2\pi}\left\{ -\omega b\operatorname{Sin}^2\psi_o - \tfrac{1}{2}c\operatorname{Sin}2\psi_o + d\rho_o\operatorname{Sin}\psi_o\operatorname{Cos}^3\psi_o \right.$$

$$\left. + Q\rho_o^{-\frac{1}{2}}\operatorname{Sin}\psi_o\operatorname{Sin}\sigma \right\} d\sigma$$

and

$$Q(\rho_o, \xi_o) = \frac{d\psi}{d\tau} = \frac{1}{2\mu}\left\{ 1 + \frac{1}{\pi\omega^2}\int_o^{2}\left[-\tfrac{1}{2}\omega b\operatorname{Sin}2\psi_o - c\operatorname{Cos}^2\psi_o + d\rho_o\operatorname{Cos}^4\psi_o \right.\right.$$

$$+ Q\rho_o^{-\frac{1}{2}}\operatorname{Sin}\psi_o\operatorname{Sin}\sigma\, d\sigma,$$

in which $\psi_o = \xi_o - \sigma$.

The definite integrals involved have the following values:

$$\int_o^{2\pi} \mathrm{Sin}^2\psi_o \, d\sigma = \int_o^{2\pi} \mathrm{Cos}^2\psi_o \, d\sigma = \pi,$$

$$\int_o^{2\pi} \mathrm{Sin}\,\psi_o \, d\sigma = \int_o^{2\pi} \mathrm{Sin}\,\psi_o \,\mathrm{Cos}^3\psi_o \, d\sigma = 0,$$

$$\int_o^{2\pi} \mathrm{Cos}^4\,\psi_o \, d\sigma = \frac{3\pi}{4},$$

$$\int_o^{2\pi} \mathrm{Sin}\,\psi_o \,\mathrm{Sin}\,\sigma \, d\sigma = -\pi\,\mathrm{Cos}\,\xi_o,$$

and

$$\int_o^{2\pi} \mathrm{Cos}\,\psi_o \,\mathrm{Sin}\,\sigma \, d\sigma = \pi\,\mathrm{Sin}\,\xi_o,$$

which upon substitution reduce the stroboscopic system to

$$\begin{aligned} P(\rho_o, \xi_o) &= -\frac{\rho_o}{\omega^2}\left(\omega b + Q\rho_o^{-\frac{1}{2}}\mathrm{Cos}\,\xi_o\right) \\ Q(\rho_o, \xi_o) &= \left\{\frac{1}{\omega^2}\left(-c + \tfrac{3}{4}\,d\rho_o + Q\rho_o^{-\frac{1}{2}}\mathrm{Sin}\,\xi_o\right) + 1\right\}\frac{1}{2\mu} \end{aligned} \qquad (39)$$

Imposing the condition $P(\rho_o, \xi_o) = Q(\rho_o, \xi_o) = 0$ for a singular point in the stroboscopic plane, and thereby periodic motion in the x, $\dot{x}$ plane, gives rise to the simultaneous equations

$$-\omega b = Q\rho_o^{-\frac{1}{2}}\mathrm{Cos}\,\xi_o \qquad (40)$$

$$c - \omega^2 - \tfrac{3}{4}d\,\rho_o = Q\rho_o^{-\frac{1}{2}}\mathrm{Sin}\,\xi_o \qquad (41)$$

Squaring and adding these equations gives

$$(c - \omega^2 - \tfrac{3}{4}d\rho_o)^2 + \omega^2 b^2 = \frac{Q^2}{\rho_o},$$

or since the periodic solution will be of the form $x = F\,\mathrm{Sin}\,(\sigma + \phi)$ then

$$\rho = x^2 + (x')^2 = F^2\mathrm{Sin}^2(\sigma + \phi) + F^2\,\mathrm{Cos}^2(\sigma + \phi) = F^2$$

or

$$(c - \omega^2 - \tfrac{3}{4}dF^2)^2 + \omega^2 b^2 = \left(\frac{Q}{F}\right)^2 \qquad (42)$$

which is the same result as that given by Stoker on page 91 of Ref. 8.

By definition

$$x = F \operatorname{Sin}(\sigma + \phi) = \rho^{\frac{1}{2}} \operatorname{Cos} \psi$$

and

$$v = F \operatorname{Cos}(\sigma + \phi) = \rho^{\frac{1}{2}} \operatorname{Sin} \psi$$

which when $\sigma = 0$ reduces to

$$F \operatorname{Sin} \phi = \rho^{\frac{1}{2}} \operatorname{Cos} \xi_o$$

and

$$F \operatorname{Cos} \phi = \rho^{\frac{1}{2}} \operatorname{Sin} \xi_o$$

or

$$\operatorname{Tan} \phi = \operatorname{Cot} \xi_o \qquad (43)$$

From (40), (41) and (43), the phase angle of the solution relative to the forcing term $Q \operatorname{Sin} \sigma$ is

$$\phi = \operatorname{Tan}^{-1}\left[\frac{-\omega b}{c - \omega^2 - \frac{3}{4}dF^2}\right] \qquad (44)$$

which again is in agreement with Ref. 8.

The stability boundaries of the motion described by (42) and (44) are given by (35) and (36). From the equations describing the stroboscopic system, (39),

$$P_\rho = -\frac{1}{\mu\omega^2}\left[\omega b + \tfrac{1}{2}Q\rho^{-\frac{1}{2}} \operatorname{Cos} \xi\right],$$

$$P_\xi = \frac{\rho^{\frac{1}{2}}Q}{\mu\omega^2} \operatorname{Sin} \xi ,$$

$$Q_\rho = \frac{1}{2\mu\omega^2}\left[\tfrac{3}{4}d - \tfrac{1}{2}Q\rho^{-3/2} \operatorname{Sin} \xi\right]$$

and

$$Q_\xi = \frac{Q\rho^{-\frac{1}{2}}}{2\mu\omega^2} \operatorname{Cos} \xi$$

The stability boundaries are then defined by

$$P_\rho + Q_\xi = 0 = -\frac{1}{\mu\omega^2}\left[\omega b + \tfrac{1}{2}Q\rho^{-\frac{1}{2}} \operatorname{Cos} \xi\right] + \frac{Q\rho^{-\frac{1}{2}}}{2\mu\omega^2} \operatorname{Cos} \xi$$

or $-\omega b = 0$; (45)

and

$$\begin{vmatrix} P_\rho & P_\xi \\ Q_\rho & Q_\xi \end{vmatrix} = 0 = -\frac{1}{\mu\omega^2}\left[\omega b + \tfrac{1}{2}Q_\rho^{-\frac{1}{2}}\mathrm{Cos}\,\xi\right]. \frac{Q\rho^{-\frac{1}{2}}}{2\mu\omega^2}\,\mathrm{Cos}\,\xi$$

$$-\frac{\rho^{\frac{1}{2}}Q}{\mu\omega^2}\,\mathrm{Sin}\,\xi\left[\tfrac{3}{4}d - \tfrac{1}{2}Q_\rho^{-3/2}\mathrm{Sin}\,\xi\right]\frac{1}{2\mu\omega^2}$$

or

$$\omega b\rho^{-\frac{1}{2}}\mathrm{Cos}\,\xi + \tfrac{3}{4}d\rho^{\frac{1}{2}}\mathrm{Sin}\,\xi + \frac{Q}{2\rho}\left[\mathrm{Cos}^2\xi - \mathrm{Sin}^2\xi\right] = 0 \qquad (46)$$

Now

$$\mathrm{Sin}\,\xi = \mathrm{Cos}\,\phi = \frac{F}{Q}\,(c - \omega^2 - \tfrac{3}{4}dF^2)$$

and

$$\mathrm{Cos}\,\xi = \mathrm{Sin}\,\phi = -\frac{F}{Q}\,\omega b,$$

therefore from (46) and upon expansion and manipulation

$$\frac{27}{16}\,d^2F^4 - 3(c - \omega^2)\,dF^2 + (c - \omega^2)^2 + b\omega^2 = 0 \qquad (47)$$

Since ω is in general not zero then (45) may be interpreted as a damping boundary corresponding to the disappearance of b in (37). When $b < 0$ the inequality (35) would no longer be satisfied and the system would be subject to a divergent oscillation.

Consider equation (42) which defines the response curves in the F, ω plane. Multiplying by F^2 and differentiating implicitly with respect to F gives

$$2\left[(c - \omega^2)\,F - \tfrac{3}{4}dF^3\right]\left[c - 2\omega F\,\frac{d\omega}{dF} - \omega^2 - \frac{9}{4}\,dF^2\right]$$

$$+\ b^2(2F\omega^2 + 2\omega F^2\,\frac{d\omega}{dF}) = 0.$$

Upon inserting the condition for vertical tangency, $\frac{d\omega}{dF} = 0$, this equation reduces to (47) i.e. the second stability boundary corresponds to the locus of the points of vertical tangency of the response curves in the F, ω plane. It will be seen that this result agrees with that of Stoker in Ref. 8 and all the subsequent discussion of "jump phenomena" is relevant to the present problem. It is worthy of note that the stroboscopic method avoids the use of the theory of Mathieu's equation, though be it very elegant, required in Stokers analysis. Further, the case treated in Ref. 8 is the conservative one, b = 0; in order to establish the boundaries defined by (45) and (47) by this method an additional co-ordinate transformation would be required to reduce the variational equation to Mathieu's form.

5. Equations of Longitudinal Motion of an Airframe

When synthesizing an automatic control system for an airframe it is often sufficient to consider only the short-period motion when formulating the transfer-functions in pitch. In this case the linear equations of motion become

$$\left.\begin{aligned} (D - z_w)W - (U_o + Z_q)\dot{\theta} &= z_\eta \ \eta \\ -(m_{\dot{w}}D + m_w)W + (D - m_q)\dot{\theta} &= m_\eta \ \eta \end{aligned}\right) \quad (48)$$

As indicated in Ref. 1, non-linear variations of the normal force Z and pitching moment M with vertical velocity W are introduced as power series in odd powers of W. Such series must be representative of anti-symmetrical normal force and pitching moment curves, these being characteristic of most configurations having aerodynamic symmetry. Thus

$$\left.\begin{aligned} \frac{Z(W)}{m} &= z_w W + z_3 W^3 + z_5 W^5 + \ldots\ldots \\ &\text{and} \\ \frac{M(W)}{B} &= m_w W + m_3 W^3 + m_5 W^5 + \ldots\ldots \end{aligned}\right) \quad (49)$$

For algebraic simplicity only two terms are retained in the remainder of the analysis, although this can readily extended to any reasonable number.

The non-linear equations of motion become

$$\dot{W} - (z_w W + z_3 W^3) - (U_o + z_q)\dot{\theta} = z_\eta \ \eta \quad (50)$$

and

$$- m_{\dot{w}} \dot{W} - (m_w W + m_3 W^3) + (D - m_q)\dot{\theta} = m_\eta \ \eta \quad (51)$$

Eliminating $\dot{\theta}$ between (50) and (51) then gives

$$(D - m_q)(\dot{W} - z_w W - z_3 W^3) - (U_o + z_q)(m_{\dot{w}}\dot{W} + m_w W + m_3 W^3)$$

$$= \left[z_\eta (D - m_q) + m_\eta (U_o + z_q) \right] \eta$$

or upon collecting terms

$$\ddot{W} - (B_1 + B_3 W^2)\dot{W} - (A_1 W + A_3 W^3) = \left[z_\eta D + (U_o + z_q)m_\eta - z_\eta m_q \right] \eta \quad (52)$$

where

$$A_1 = (U_o + z_q)m_w - m_q z_w, \qquad A_3 = (U_o + z_q)m_3 - m_q z_3,$$

$$B_1 = (U_o + z_q)m_{\dot{w}} + m_q + z_w \quad \text{and} \quad B_3 = 3z_3.$$

It should be noted that the term $(U_o + z_q)m$ in (52) has a different sign from the corresponding equation (15) of Ref. 1, the latter is incorrect.

In Ref. 1 only the free motion, $Q(\eta) \equiv 0$, is considered, in the present problem the airframe is assumed to be trimmed initially at some incidence α_t corresponding to a vertical velocity w_t and elevator angle η_t, and is then forced by a sinusoidal elevator motion of amplitude η_a, where η_a is taken to have the same sign as η_t. The elevator displacement is then

$$\eta = \eta_t + \eta_a \text{ Sin } \omega t \tag{53}$$

and the vertical velocity of the forced motion is

$$W = w_t + w \quad (w_t \text{ and } w \text{ both small}) \tag{54}$$

Since $\dot{W} = \dot{w}$ and $\ddot{W} = \ddot{w}$ then the equation of w motion becomes

$$\ddot{w} - \left[B_1 + B_3(w_t+w)^2\right]\dot{w} - \left[A_1(w_t+w) + A_3(w_t+w)^3\right]$$
$$= [z_\eta D + (U_o+z_q)m_\eta - z_\eta m_q](\eta_t + \eta_a \text{ Sin}\,\omega t).$$

Now the trimmed condition is defined by

$$-(A_1 w_t + A_3 w_t^3) = \left[(U_o+z_q)m_\eta - z_\eta m_q\right]\eta_t \tag{55}$$

therefore the equation of w motion reduces to

$$\ddot{w} - \left[(B_1 + B_3 w_t^2) + 2B_3 w_t w + B_3 w^2\right]\dot{w} - \left[(A_1 + 3A_3 w_t^2)w\right.$$
$$\left. + 3A_3 w_t w^2 + A_3 w^3\right] = \left[z_\eta D + (U_o+z_q)m_\eta - z_\eta m_q\right]\eta_a \text{ Sin } \omega_t,$$

and writing, as before, $\sigma = \omega t$ this equation finally becomes

$$\omega^2 w'' - \omega\left[(B_1 + B_3 w_t^2) + 2B_3 w_t w + B_3 w^2\right]w' - \left[(A_1 + 3A_3 w_t^2)w\right.$$
$$\left. + 3A_3 w_t . w^2 + A_3 w^3\right] = \eta_1 \text{Sin}\,\theta + \eta_2 \text{Cos}\,\theta = H \text{ Sin}(\sigma + \delta) \tag{56}$$

where

$$\eta_1 = \eta_a\left[(U_o+z_q)m_\eta - z_\eta m_q\right] = H \text{ Cos}\,\delta,$$

$$\eta_2 = \omega\eta_a z_\eta = H \text{ Sin}\,\delta,$$

$$H = (\eta_1^2 + \eta_2^2)^{\frac{1}{2}}$$

and $\delta = \text{Tan}^{-1}\left(\frac{\eta_2}{\eta_1}\right)$.

In order to determine the stroboscopic system corresponding to (56), write

$$w = \rho^{\frac{1}{2}}\mathrm{Cos}\,\psi, \quad w' = \rho^{\frac{1}{2}}\mathrm{Sin}\,\psi \text{ and } \rho = w^2 + (w')^2,$$

then

$$w'' = \frac{1}{\omega^2}\Big\{\omega\Big[(B_1 + B_3 w_t^2) + 2B_3 w_t \rho^{\frac{1}{2}}\mathrm{Cos}\,\psi + B_3\rho\,\mathrm{Cos}^2\psi\Big]\rho^{\frac{1}{2}}\mathrm{Sin}\,\psi$$

$$+ (A_1 + 3A_3 w_t^2)\rho^{\frac{1}{2}}\mathrm{Cos}\,\psi + 3A_3 w_t\,.\,\rho\,\mathrm{Cos}^2\psi + A_3\rho^{3/2}\mathrm{Cos}^3\psi$$

$$+ H\,\mathrm{Sin}\,(\sigma + \delta)\Big\},$$

and from the general transformation (26) and (27) the stroboscopic system becomes

$$P(\rho_o, \xi_o) = \frac{\rho_o^{\frac{1}{2}}}{\pi\mu\omega^2}\int_0^{2\pi}\Big\{\omega\Big[(B_1 + B_3 w_t^2) + 2B_3 w_t \rho_o^{\frac{1}{2}}\mathrm{Cos}\,\psi_o + B_3\rho_o\mathrm{Cos}^2\psi_o\Big]\rho_o^{\frac{1}{2}}\mathrm{Sin}^2\psi_o$$

$$+ (A_1 + 3A_3 w_t^2)\rho_o^{\frac{1}{2}}\mathrm{Sin}\,\psi_o\mathrm{Cos}\,\psi_o + 3A_3 w_t\rho_o\mathrm{Sin}\,\psi_o\mathrm{Cos}^2\psi_o$$

$$+ A_3\rho_o^{3/2}\mathrm{Sin}\,\psi_o\mathrm{Cos}^3\psi_o + H\,\mathrm{Sin}\,\psi_o\,\mathrm{Sin}(\sigma + \delta)\Big\}\,d\sigma$$

and

$$Q(\rho_o, \xi_o) = \frac{1}{2\mu} + \frac{\rho_o^{-\frac{1}{2}}}{2\pi\mu\omega^2}\int_0^{2\pi}\Big\{\omega\Big[(B_1 + B_3 w_t^2) + 2B_3 w_t \rho_o^{\frac{1}{2}}\mathrm{Cos}\,\psi_o + B_3\rho_o\mathrm{Cos}^2\psi_o\Big]$$

$$\times\ \rho_o^{\frac{1}{2}}\mathrm{Sin}\,\psi_o\,\mathrm{Cos}\,\psi_o$$

$$+ (A_1 + 3A_3 w_t^2)\,\rho_o^{\frac{1}{2}}\mathrm{Cos}^2\psi_o + 3A_3 w_t\rho_o\mathrm{Cos}^3\psi_o$$

$$+ A_3\rho_o^{3/2}\mathrm{Cos}^4\psi_o + H\,\mathrm{Sin}\,(\sigma + \delta)\,\mathrm{Cos}\,\psi_o\Big\}\,d\sigma$$

The definite integrals, additional to those already listed, are

$$\int_0^{2\pi}\mathrm{Sin}^2\psi_o\mathrm{Cos}\,\psi_o\,d\sigma = \tfrac{1}{2}\int_0^{2\pi}\mathrm{Sin}\,\psi_o\mathrm{Sin}\,2\psi_o\,d\sigma = 0$$

$$\int_0^{2\pi}\mathrm{Sin}^2\psi_o\,\mathrm{Cos}^2\psi_o\,d\sigma = \int_0^{2\pi}\mathrm{Cos}^2\psi_o(1 - \mathrm{Cos}^2\psi_o)d\sigma = \pi/4$$

$$\int_0^{2\pi}\mathrm{Sin}\,\psi_o\,\mathrm{Cos}^2\psi_o\,d\sigma = \tfrac{1}{2}\int_0^{2\pi}\mathrm{Sin}\,2\,\psi_o\,\mathrm{Cos}\,\psi_o\,d\sigma = 0$$

$$\int_0^{2\pi}\mathrm{Cos}^3\psi_o\,d\sigma = \int_0^{2\pi}\mathrm{Cos}\,\psi_o\,(1 - \mathrm{Sin}^2\psi_o)d\sigma = \int_0^{2\pi}[\mathrm{Cos}\,\psi_o - \tfrac{1}{2}\mathrm{Sin}\,\psi_o\mathrm{Sin}\,2\,\psi_o]d\sigma$$

$$= 0$$

$$\int_0^{2\pi} \operatorname{Sin}\, \psi_o \operatorname{Sin}(\sigma + \delta)\, d\sigma = -\pi \operatorname{Cos}(\delta + \xi_o)$$

and

$$\int_0^{2\pi} \operatorname{Cos}\psi_o \operatorname{Sin}(\sigma + \delta)\, d\sigma = \pi \operatorname{Sin}(\delta + \xi_o),$$

which upon substitution reduce the stroboscopic system to

$$P(\rho_o, \xi_o) = \frac{\rho_o}{\mu\omega^2}\left[\omega(B_1 + B_3 w_t^2) + \tfrac{1}{4}B_3\rho_o\omega - \rho_o^{-\frac{1}{2}} H \operatorname{Cos}(\delta + \xi_o)\right] \tag{57}$$

$$Q(\rho_o, \xi_o) = \frac{1}{2\mu\omega^2}\left[\omega^2 + (A_1 + 3A_3 w_t^2) + \tfrac{3}{4}A_3\rho_o + \rho_o^{-\frac{1}{2}} H \operatorname{Sin}(\delta + \xi_o)\right] \tag{58}$$

Imposing the condition $P(\rho_o, \xi_o) = 0 = Q(\rho_o, \xi_o)$ for a singular point in the stroboscopic plane gives rise to the simultaneous equations

$$\omega\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}\rho_o)\right] = H\rho_o^{-\frac{1}{2}} \operatorname{Cos}(\delta + \xi_o) \tag{59}$$

$$-\omega^2 - (A_1 + 3A_3 w_t) - \tfrac{3}{4}\rho_o A_3 = H\rho_o^{-\frac{1}{2}} \operatorname{Sin}(\delta + \xi_o) \tag{60}$$

Squaring and adding these equations gives

$$\omega^2\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}\rho_o)\right]^2 + \left[\omega^2 + (A_1 + 3A_3 w_t^2) + \tfrac{3}{4}\rho_o A_3\right]^2 = \frac{H^2}{\rho_o}$$

Now the resulting periodic oscillation may be expressed as

$$w = F \operatorname{Sin}(\sigma + \phi) \tag{61}$$

where $F = \rho_o^{\frac{1}{2}}$, and the response equation becomes

$$\omega^2\left[B_1 + B_3\,(w_t^2 + \tfrac{1}{4}F^2)\right]^2 + \left[\omega^2 + A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right]^2$$

$$= \left(\frac{H}{F}\right)^2 = \frac{1}{F^2}\left\{\eta_1^2 + \eta_2^2\right\} = \left(\frac{\eta_a}{F}\right)^2\left\{\left[(U_o + z_q)m_\eta - z_\eta m_q\right]^2 + z_\eta\omega^2\right\} \tag{62}$$

or

$$(\omega^2) + J\omega^2 + K = 0 \tag{63}$$

where

$$J = \left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right]^2 + 2\left[A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right] - \left(\frac{\eta_a}{F}\right)^2 z_\eta^2$$

and

$$K = \left[A_1 + 3A_3\,(w_t^2 + \tfrac{1}{4}F^2)\right]^2 - \left(\frac{\eta_a}{F}\right)^2\left[(U_o + z_q)m_\eta - z_\eta m_q\right]^2$$

Following the Duffing technique the amplitude, F, is considered to be prescribed and (63) is solved for ω.

For some configurations, in particular the rear controlled missile considered in Section 6.0, the magnitude of z_η is such that over the frequency range of interest (for a moderate centre of gravity margin the undamped natural frequency in pitch will be of the order of 2 to 5 radians per second) the value of η_2 is small compared with η_1 and $\delta = 0$. Further, the value of z_q is usually small compared with $U_o \simeq U$, $z_\eta\, m_q$ is small compared with Um_η and H may be taken as

$$H = \eta_a Um_\eta \tag{64}$$

and the coefficients in (63) become

$$J = \left[B_1 + U^2 B_3(\alpha_t^2 + \tfrac{1}{4}\alpha_F^2)\right]^2 + 2\left[A_1 + 3U^2 A_3(\alpha_t^2 + \tfrac{1}{4}\alpha_F^2)\right]$$

and

$$K = \left[A_1 + 3U^2 A_3(\alpha_t^2 + \tfrac{1}{4}\alpha_F^2)\right]^2 - \frac{\eta_a}{\alpha_F} \cdot m_\eta^{\;2}$$

where $\alpha_t \simeq \dfrac{w_t}{U}$ and $\alpha_F \simeq \dfrac{F}{U}$.

From equations (59) and (60)

$$\mathrm{Sin}(\delta + \xi_o) = -\frac{F}{H}\left[\omega^2 + A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right]$$

and

$$\mathrm{Cos}(\delta + \xi_o) = \frac{\omega F}{H}\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right];$$

also, from the definition of H and δ, $\mathrm{Sin}\,\delta = \eta_2/H$ and $\mathrm{Cos}\,\delta = \eta_1/H$.

Now

$$\mathrm{Sin}(\delta + \xi_o) = \mathrm{Sin}\,\delta\,\mathrm{Cos}\,\xi_o + \mathrm{Cos}\,\delta\,\mathrm{Sin}\,\xi_o = \frac{1}{H}\left[\eta_2\mathrm{Cos}\,\xi_o + \eta_1\mathrm{Sin}\,\xi_o\right]$$

and

$$\mathrm{Cos}(\delta + \xi_o) = \mathrm{Cos}\,\delta\,\mathrm{Cos}\,\xi_o - \mathrm{Sin}\,\delta\,\mathrm{Sin}\,\xi_o = \frac{1}{H}\left[\eta_1\mathrm{Cos}\,\xi_o - \eta_2\mathrm{Sin}\,\xi_o\right],$$

therefore

$$\eta_2\mathrm{Cos}\,\xi_o + \eta_1\mathrm{Sin}\,\xi_o = -F\left[\omega^2 + A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right] \tag{65}$$

and

$$\eta_1\mathrm{Cos}\,\xi_o - \eta_2\mathrm{Sin}\,\xi_o = \omega F\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right] \tag{66}$$

Eliminating $\mathrm{Sin}\,\xi_o$ and $\mathrm{Cos}\,\xi_o$ in turn between (65) and (66) then gives

$$(\eta_1^2 + \eta_2^2)\,\mathrm{Cos}\,\xi_o = \omega F\eta_1\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right] - \eta_2 F\left[\omega^2 + A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right]$$

and

$$(\eta_1^2 + \eta_2^2)\,\mathrm{Sin}\,\xi_o = -F\eta_1\left[\omega^2 + A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right] - \omega F\eta_2\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right];$$

further, for the steady periodic oscillation

$$w = \rho_o^{\frac{1}{2}}\,\mathrm{Cos}\,\xi_o = F\,\mathrm{Sin}\,\phi$$

and

$$w' = \rho_o^{\frac{1}{2}}\,\mathrm{Sin}\,\xi_o = F\,\mathrm{Cos}\,\phi,$$

implying that $\mathrm{Sin}\,\phi = \mathrm{Cos}\,\xi_o$ and $\mathrm{Cos}\,\phi = \mathrm{Sin}\,\xi_o$ and finally that

$$\mathrm{Sin}\,\phi = \frac{\omega F\eta_a}{H}\left\{\left[(U_o + z_q)m_\eta - z_\eta m_q\right]\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right] - z_\eta\left[\omega^2 + A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right]\right\} \tag{67}$$

and

$$\mathrm{Cos}\,\phi = -\frac{F\eta_a}{H^2}\left[(U_o + z_q)m_\eta - z_\eta m_q\right]\left[\omega^2 + A_1 + 3A_3(w_t^2 + \tfrac{1}{4}F^2)\right] + \omega^2 z_\eta\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right]\Big\} \tag{68}$$

The phase angle ϕ is that existing between the input sinusoid to the elevator and the output sinusoid describing the w or α motion. With the same approximations as those made in obtaining (63) and (64) the expressions become

$$\mathrm{Sin}\,\phi = \frac{\omega\,\alpha F}{\eta_a m_\eta}\left[B_1 + U^2 B_3(\alpha_t^2 + \tfrac{1}{4}\alpha^2 F)\right] \tag{69}$$

and

$$\mathrm{Cos}\,\phi = -\frac{\alpha F}{\eta_a m_\eta}\left[\omega^2 + A_1 + 3U^2 A_3(\alpha_t^2 + \tfrac{1}{4}\alpha^2 F)\right] \tag{70}$$

In calculating numerical values due regard must be taken of the sign of η_a, which will be the same as that of η_t and can be obtained from (55).

As shown in Section 3.0 the stability boundaries are defined by

$P_\rho + Q_\xi = 0$, $P_\rho Q_\xi - P_\xi Q_\rho = 0$. The partial derivatives are

$$P_\rho = \frac{1}{\mu\omega^2}\left\{\omega\left[B_1 + B_3(w_t^2 + \tfrac{1}{2}\rho_o)\right] - \tfrac{1}{2}H\rho_o^{-\frac{1}{2}}\mathrm{Cos}\,(\delta + \xi_o)\right.,$$

$$P_\xi = \frac{\rho_o^{\frac{1}{2}}H}{\mu\omega^2}\,\mathrm{Sin}\,(\delta + \xi_o),$$

$$Q_\rho = \frac{1}{2\mu\omega^2}\left[\tfrac{3}{4}A_3 - \tfrac{1}{2}\rho_o^{-3/2}H\,\mathrm{Sin}\,(\delta + \xi_o)\right.$$

and

$$Q_{\xi} = \frac{1}{2\,\mu\omega^2}\ \rho_o^{-\frac{1}{2}}\ H\ \mathrm{Cos}\ (\delta + \xi_o),$$

which give for the first boundary

$$P_{\rho} + Q_{\xi} = \frac{1}{\mu\omega^2}\left\{\omega\left[B_1 + B_3\,(w_t^2 + \tfrac{1}{2}\rho_o)\right]\right\} = 0,$$

or since ω is finite,

$$B_1 + B_3\ (w_t^2 + \tfrac{1}{2}F^2) = 0 \tag{71}$$

The second boundary is given by

$$P_{\rho}\,Q_{\xi} - P_{\xi}\,Q_{\rho} = \tfrac{1}{2}\left(\frac{1}{\mu\omega^2}\right)^2\left\{\left[\omega\left(B_1 + B_3\,(w_t^2 + \tfrac{1}{2}\rho_o)\right) - \tfrac{1}{2}\rho_o^{-\frac{1}{2}}H\ \mathrm{Cos}\ (\delta + \xi_o)\right]\right.$$
$$\left. \times\ \rho_o^{-\frac{1}{2}}\mathrm{Cos}(\delta + \xi_o) - \left[\tfrac{3}{4}A_3 - \tfrac{1}{2}\ {}_o^{-3/2}H\ \mathrm{Sin}\ (\delta + \xi_o)\right]\rho_o^{\frac{1}{2}}H\ \mathrm{Sin}\,(\delta + \xi_o)\right\} = 0$$

which upon substitution for Sin $(\delta + \xi_o)$ and Cos $(\delta + \xi_o)$ from equations (59) and (60) give

$$\omega^2\left[B_1 + B_3(w_t^2 + \tfrac{1}{2}F^2)\right]\left[B_1 + B_3\,(w_t^2 + \tfrac{1}{4}F^2)\right] - \tfrac{1}{2}\omega^2\left[B_1 + B_3\,(w_t^2 + \tfrac{1}{4}F^2)\right]^2$$
$$+ \tfrac{3}{4}A_3F^2\left[\omega^2 + A_1 + 3A_3\,(w_t^2 + \tfrac{1}{4}F^2)\right] + \tfrac{1}{2}\left[\omega^2 + A_1 + 3A_3\,(w_t^2 + \tfrac{1}{4}F^2)\right] = 0$$

or

$$(\omega^2)^2 + R\omega^2 + S = 0 \tag{72}$$

where

$$R = \left[B_1 + B_3(w_t^2 + \tfrac{3}{4}F^2)\right]\left[B_1 + B_3(w_t^2 + \tfrac{1}{4}F^2)\right] + 2\left[A_1 + 3A_3\,(w_t^2 + \tfrac{1}{2}F^2)\right]$$

and

$$S = \left[A_1 + 3A_3(w_t^2 + \tfrac{3}{4}F^2)\right]\left[A_1 + 3A_3\,(w_t^2 + \tfrac{1}{4}F^2)\right]$$

The stability boundary defined by (72) is in fact the locus of the points of vertical tangency of the response curves given by equation (62); a result which can readily be proved by differentiating (62) implicitly with respect fo F, remembering that H is a function of ω, and inserting the condition $d\omega/dF = 0$. The nature of the instability corresponds to the "jump phenomena" already discussed in relation to Duffing's equation and will be discussed in further detail in Section 6.0.

Equation (71) defines a damping boundary which may readily be compared with "quasi-linear" theory (i.e. stability theory based on the local slope of the aerodynamic force and moment characteristics). Quasi-linear theory indicates a disappearance of damping when the coefficient of W in (52) becomes zero. This implies that the greatest trimmed value of incidence is given by

$$B_1 + B_3\ w_t^2 = 0$$

or $$w_t = \left(-\frac{B_1}{B_3}\right)^{\frac{1}{2}} \tag{73}$$

a boundary which will exist only if B_3 is of opposite sign to B_1. Since B_1 is always negative then B_3 must be positive. From the definition of B_1, if z_w were the dominant term then the condition (73) would correspond to the stall. It is implied that stable pitching oscillations are possible at all values of W up to w_t given by (73), whereas (71) indicates that stable oscillations are possible only for values smaller than

$$w_t = \left[-\frac{B_1}{B_3} - \tfrac{1}{2}F^2\right]^{\frac{1}{2}} \tag{74}$$

This means that the value of w_t for which stable oscillations are possible is dependent on the amplitude of the oscillations, F, and reduces to the quasi-linear result for $F \to 0$.

6.0 A Numerical Example - Frequency Response of an Air-to-Air Missile Flying at High Altitude

The missile chosen for this example, which is hypothetical, is shown in Fig. 2. It is a cruciform, rear-controlled, air-to-air missile intended to be launched from a fighter aircraft and having a useful speed range of 1,500 to 3,500 f.p.s. The operational altitude is between 20 to 70 thousand feet. To avoid the missile's guidance system losing "sight" of the target, the airframe incidence is limited to ± 30 degrees. For the present purpose it has a weight of 500 lbs. and a moment of inertia in pitch (or yaw) of 4,000 lbs. feet^2.

The aerodynamic characteristics have been calculated using methods similar to those described by Nielsen in Ref. 9. Wing and control normal forces are based on linearised supersonic theory, whilst the inviscid contribution from the body has been obtained from an extended form of shock-expansion theory. Non-linear contributions arise from the body and downwash variations at the control. In the former viscous flow separation from the leading portion of the body produces a vortex sheet which rolls up into approximately streamwise vortices; these generate low pressures on the lee side of the body and produce normal force additional to that predicted by inviscid theory. In the latter the wing downwash at the control, which is not proportional to the geometric incidence of the missile, causes the control efficiency, when acting as a stabilizer (i.e. $\eta = 0$), to increase with missile incidence and thereby creates the non-linearity. For the present configuration the non-linear pitching moment produced by the downwash variation is three times that arising from viscous separation on the body.

At low incidence (< 5 degrees) and moderate centre of gravity margins, the non-linear pitching moment is small compared with the linear contribution. This implies that at relatively low altitudes, where the operating incidence will be small (this is usually the case, since the normal force imposed will have to be kept within structural limits), the airframe response will be linear to a good approximation. When flying at high altitude, even with comparatively small normal accelerations, the operating incidence is such that non-linear normal force and pitching moment contributions are of the same order as the linear and thereby introduce important changes in the frequency response (and in the transient response as well, although this problem will not be considered here) characteristics. The present example sets out to demonstrate these effects.

Choosing for example an altitude of 60,000 feet and a speed of 2,000 f.p.s. (Mach number = 2.066), then the aerodynamic characteristics are

$$\frac{dC_N}{d\alpha} = 20 \text{ per radian,}$$

and

$$\frac{dC_M}{d\alpha} = \frac{20(x_{c.g.} - 0.3)}{D_R},$$

where $x_{c.g.}$ is the centre of gravity position measured aft of the datum and the coefficients are based on a reference area $S_R = 0.785$ square feet and a reference diameter $D_R = 1$ foot.

Over the incidence range $\pm$ 30 degrees the non-linear normal force and pitching moment can be approximated by terms proportional to α^3. On this basis the normal force and pitching moment can be expressed as

$$C_N = 20\alpha + 40\alpha^3$$

and

$$C_M = \frac{20(x_{c.g.} - 0.3)}{D_R} - 100\alpha^3$$

Experimentally the normal force increment due to a change in η usually indicates a second control efficiency term dependent on η, however, this variation is normally small and for the present purpose is neglected. The elevator terms become

$$\frac{dC_N}{d\eta} = 3.82$$

and

$$\frac{dC_M}{d\eta} = 3.82 \frac{(4 - x_{c.g.})}{D_R},$$

where $(4 - x_{c.g.})$ is the distance between the hinge line of the elevator and the centre of gravity. This approximation for the "tailarm" is permissible if the aerodynamic centre of the elevator is near the hinge line; this is normally the case if the hinge moment is to be kept small.

The variation of control normal force with incidence is, of course, included in the value of $\frac{dC_N}{d\alpha}$ for the complete missile. In calculating certain aerodynamic derivatives the control contribution alone, and not in the presence of downwash, will be required. This value is

$$\frac{dC_{N_T}}{d\alpha} = 6.53$$

In addition a value of $\frac{d\epsilon}{d\alpha}$, the rate of change of downwash angle with incidence, will be required in determining $m_{\dot{w}}$. The control efficiency $(1 - \frac{d\epsilon}{d\alpha})$, varies between 0.4 at low incidence to about 0.8 at $\alpha = 25^\circ$, implying a variation of $\frac{d\epsilon}{d\alpha}$ from 0.6 to 0.2. To be consistent with other linear aerodynamic terms the value of $\frac{d\epsilon}{d\alpha}$ has been taken at $\alpha = 0$, an assumption which will overemphasize the importance of $m_{\dot{w}}$ at high incidence.

The stability derivatives z_w, m_w, z_η and m_η and the coefficients z_3 and m_3 stem directly from the primary aerodynamic coefficients and are defined by the expressions

$$z_w = \frac{1}{m}\left(\frac{\partial Z}{\partial w}\right)_{w=o} = -\frac{\rho U S_R}{2m}\left(\frac{dC_N}{d\alpha}\right)_{\alpha=o},$$

$$z_3 = \frac{1}{m}\left(\frac{\Delta Z}{\Delta w^3}\right) = -\frac{\rho U^2 S_R}{2m}\frac{\Delta C_N}{\Delta(U\alpha)^3} = \frac{\rho S_R}{2mU}\frac{\Delta C_N}{\Delta\alpha^3},$$

$$m_w = \frac{1}{B}\left(\frac{\partial M}{\partial w}\right)_{w=o} = \frac{\rho U S_R D_R}{2B}\left(\frac{dC_M}{d\alpha}\right)_{\alpha=o},$$

$$m_3 = \frac{1}{B}\left(\frac{\Delta M}{\Delta w^3}\right) = \frac{\rho S_R D_R}{2BU}\frac{\Delta C_M}{\Delta\alpha^3},$$

$$z_\eta = \frac{1}{m}\left(\frac{\partial Z}{\partial\eta}\right)_{\eta=o} = \frac{\rho U^2 SR}{2m}\left(\frac{dC_N}{d\eta}\right)_{\eta=o},$$

and

$$m_\eta = \frac{1}{B}\left(\frac{\partial M}{\partial\eta}\right)_{\eta=o} = -\frac{\rho U^2 S_R D_R}{2B}\left(\frac{d_{CM}}{d\eta}\right)_{\eta=o},$$

where U is the forward speed along the flight path, assumed to be equal to U_o, and α is taken to be approximately $\frac{W}{U}$.

For the given configuration the primary contribution to the derivatives z_q, m_q and $m_{\dot{w}}$ comes from the rear control and can be obtained from theory based on the quasi-static approximation (see Ref. 10). On this basis, and neglecting the control drag coefficient in comparison with $\frac{dC_{N_T}}{d\alpha}$, the derivatives become

$$z_q = \frac{1}{m}\left(\frac{\partial Z}{\partial q}\right)_{q=o} = -\frac{\rho U S_R}{2m}(4 - x_{c.g.})\left(\frac{dC_{N_T}}{d\alpha}\right)_{\alpha=o},$$

$$m_q = \frac{1}{B}\left(\frac{\partial M}{\partial q}\right)_{q=o} = -\frac{\rho U S_R}{2B}(4 - x_{c.g.})^2\left(\frac{dC_{N_T}}{d\alpha}\right)_{\alpha=o},$$

and

$$m_{\dot{w}} = \frac{1}{B}\left(\frac{\partial M}{\partial w}\right)_{w=o} = -\frac{\rho S_R}{2B}(4 - x_{c.g.})^2\left(\frac{dC_{N_T}}{d\alpha}\cdot\frac{d\epsilon}{d\alpha}\right)_{\alpha=o}$$

With m in slugs, S_R in square feet, D_R in feet, B in slugs feet^2, U in feet per second and ρ in slugs per cubic foot, the numerical values of the derivatives are

$$z_w = -0.227 \text{ sec}^{-1}, \quad z_3 = -1.133 \times 10^{-7} \text{ft.}^{-2}\text{sec.},$$

$$m_w = 0.0283\,(x_{c.g.} - 0.3)\ \text{ft}^{-1}\text{sec.}^{-1}, \quad m_3 = -\,0.354 \times 10^{-7}\text{ft.}^{3}\text{sec.},$$

$$z_\eta = -\,86.7\ \text{ft.sec.}^{-2}, \quad m_\eta = -\,10.83\,(4 - x_{c.g.})\text{sec.}^{-2},$$

$$z_q = -\,0.0741\,(4 - x_{c.g.})\ \text{ft.sec.}^{-1}, \quad m_q = -\,9.25 \times 10^{-3}\,(4 - x_{c.g.})^2\text{sec.}^{-1},$$

$$m_{\dot{w}} = -\,2.776 \times 10^{-6}\,(4 - x_{c.g.})^2\text{ft.}^{-1}.$$

It can be seen that the derivative z_q is very small when compared with U, to which it is added in all the relationships involved, and for convenience is neglected hereon.

Substituting these values in the expressions for A_1, A_3, etc. gives

$$A_1 = 56.6\,(x_{c.g.} - 0.3) - 2.1 \times 10^{-3}\,(4 - x_{c.g.})^2\ \text{sec.}^{-2}.$$

$$A_3 = -\,0.708 \times 10^{-4} - 1.048 \times 10^{-9}\,(4 - x_{c.g.})^2\text{ft.}^{-2},$$

$$B_1 = -\,0.227 - 14.80 \times 10^{-3}\,(4 - x_{c.g.})^2\ \text{sec.}^{-1},$$

and

$$B_3 = -\,3.399 \times 10^{-7}\text{ft.}^{-2}\text{sec.}$$

The equation of trim becomes

$$-\left[56.6\,(x_{c.g.} - 0.3) - 2.1 \times 10^{-3}\,(4 - x_{c.g.})^2\right] \times 2{,}000\alpha_t$$
$$+\left[0.708 \times 10^{-4} + 1.048 \times 10^{-9}\,(4 - x_{c.g.})^2\right] \times 2{,}000^3\alpha^3_t$$
$$= -\left[21.66 \times 10^3\,(4 - x_{c.g.}) + 0.802\,(4 - x_{c.g.})^2\right]\eta_t$$

and the 'trim curves' of α_t v η_t for various $x_{c.g.}$ are shown in Fig. 3, the curves being anti-symmetric about the η axis.

For the purpose of the example the missile is assumed to be flying level at 60,000 feet altitude and subject to a steady normal acceleration in the pitching plane of lg. The corresponding value of normal force coefficient is 1.42 at an incidence of 0.0703 radians (4.03 degrees), whilst the elevator angle to trim is marked on Fig. 3. Using the previous values the response curves for various positions of the centre of gravity and elevator amplitude, η_a, have been obtained and are shown in Figs. 4 and 5, whilst the associated phase angles are shown in Figs. 6 and 7.

Consider first the trim curves of Fig. 3. The aerodynamic centre at zero incidence (which is the same as the centre of pressure for the symmetrical airframe of the example) lies at 0.3 feet aft of the datum line and therefore moving the centre of gravity forward from 0.3 to 0.1, and further to - 0.1 produces an increase in the static stability, whereas moving the centre of gravity aft to 0.5 and 0.7 gives rise to static instability, i.e. it is statically unstable on the basis of conventional linear theory for equilibrium at $\alpha = 0$.

The value $x_{c.g.} = 0.1$, corresponding to a centre of gravity margin of 0.2 feet, would, on the basis of conventional static stability theory, be an acceptable figure and forward and backward movements of the centre of gravity from this position produce too great and too small amounts of static stability respectively. Taking $x_{c.g.} = 0.1$ feet as the optimum figure (in practice there would be a range of acceptable values about the optimum), it can be seen that the effect of the aerodynamic non-linearity is to cause a large increase in the value of the elevator angle to trim; so much so that the airframe is able to reach only 75 per cent of its limiting incidence before the elevators reach their mechanical stops. Such a restriction on the useful incidence range would limit the airframe manouvreability under conditions when it is at a premium.

The greater than linear increase of pitching moment with incidence causes the pitching motion to constitute a "hard" system and the response curves of Fig. 4 show the lean towards higher frequencies which is characteristic of such a system (see Ref. 8). For very small amplitudes, η_a, (less than about 0.5 degrees) the curves are close to those obtained from quasi-linear theory for small oscillations about the trimmed value. With increase of η_a points of vertical tangency occur in the curves and give rise to "jumps" in amplitude. For instance take the curve $\eta_a = 0.2$. Starting at a steady state value marked A, with increase of frequency the amplitude of oscillations about the trimmed incidence increase until the curve meets the locus of vertical tangency at C. This point is on the stability boundary defined by equation (72) and the resulting instability is the jump in amplitude from C to E. Further increase of frequency then gives rise to amplitude changes as depicted by the curve E to F. When the frequency is decreased from F to A another amplitude jump occurs from D to B, the point D lying on the other branch of the locus of vertical tangents.

It can be seen that the portion of the curve from C to D is never traversed, implying that the region between the two branches of the locus of vertical tangents is one of instability. The form of instability is that corresponding to unstable equilibrium since any small departures from C or D do not diminish with time. On the other hand the motion does not diverge indefinitely and is obviously a periodic motion in the neighbourhood of that existing prior to the jump. Such conditions call for extended definition of stability and has given rise to the concept of "orbital stability" which is discussed in Ref. 8.

The damping boundary, defined by equation (71), does not exist in the present problem since B_1 and B_3 are of the same sign throughout. Obviously the slope of the normal force curves will not increase indefinitely and will eventually have a maximum. All this implies is that the damping boundary lies outside the useful operating incidence range of the airframe.

Some measure of the accuracy of the response curves can be obtained by comparing the steady values of Fig. 4 with corresponding changes in trim on Fig. 3. Now the trimmed conditions defined by equation (55) are exact steady state solutions of (52) and the difference between these values, Fig. 3, and those from Fig. 4 are an indication of the inaccuracy of the amplitude of the fundamental and of the magnitude of the neglected higher harmonics i.e. 3ω, 5ω, etc. Both of these effects arise from the basic approximation made in establishing the first order stroboscopic system. In making the comparison it is important to remember that the region of maximum accuracy of the response curves is that embracing the resonance (this follows by direct comparison with the inverse iteration procedure used to solve Duffing's equation in Ref. 8) and therefore the comparison of steady state values is likely to be more pessimistic.

Another point which is likely to be of practical interest is that the ratio of the resonant peak amplitudes of the non-linear to linear values tends to decrease sharply with increase of η_a and would thereby tend to reduce the amount of operation on the incidence limits.

In quasi-linear theory the phase angle is closely related to the amplitude, the region of resonance corresponding to rapid changes in phase, as seen in Figs. 4 and 6. Increases of elevator amplitude produce an initial improvement in phase angle, but finally give rise to jumping. The locus of vertical tangency of the ϕ , ω curves corresponds exactly with that of the α_F, ω curves, a statement which can readily be demonstrated by differentiating equations (69) and (70) implicitly with respect to ϕ , imposing the condition $d\omega/d\phi = 0$, and thereby arriving at equation (72). It follows that the region between the branches of the locus of vertical tangents is a region of instability in a similar sense to that of Fig. 4 and jumps in phase angle occur between points such as C to E and D to B. It is worthy of note that similar jumps in phase angle are characteristic of the periodic solutions of Duffing's equation, a point which does not appear to have been made in the literature on this subject.

The discussion has, until now, been limited to explaining the effect of aerodynamic non-linearities on the frequency response of an airframe whose centre of gravity margin was optimized on the basis of conventional linear static stability theory. In assessing the relative importance of the non-linear phenomena it is necessary to remember that the airframe is only one part, albeit an important one, of the overall control loop. An essential feature of the loop will doubtless be the negative feedback of an output rate ($\dot{W}$ and/or $\dot{\theta}$) signal causing a considerable increase in the overall system damping. It can, therefore, be anticipated that the range of elevator amplitude for which jumping does not occur will be greatly increased. Quantitative evaluation of this effect must await further analysis, analysis which must be capable of taking into account the increase in order of the governing differential equation which is almost certain to arise when the other components of the control loop are included. The stroboscopic method, in the form given by Minorsky, is, of course, limited to systems of second order or lower.

One limitation which the control system will not conveniently be able to modify is the restriction of the useful incidence range brought about by the non-linear variation in pitching moment. In servomechanism parlance this corresponds to a reduction in aerodynamic gain or stiffness. The situation can be improved by reducing the static stability, as shown in Fig. 3. If static instability can be tolerated at low incidence very useful reductions in the elevator angle to trim can be obtained. It is of interest to investigate the response of the airframe under these conditions with a view to utilizing the previous improvements in a closed-loop control system. The case $x_{c.g.} = 0.7$ feet is typical of this condition.

In describing pitching motion it is often convenient to use the concepts of Poincaré's theory of singular points already employed in Ref. 1. With A_1 negative there exists only one trimmed incidence for a given value of η_t. The trimmed condition corresponds to a singular point of equation (52) and will be a stable spiral point.

For a given value of η_a there will be a single solution curve whose nature is related to the nature of the singularity. On Fig. 4, for instance, the ω axis corresponds to the basic singularity, whilst the values of α_F at $\omega = o$ correspond to changes in the position of the singularity due to the effective change in elevator angle, η_a. This implies that as $\omega \to o$ each solution curve degenerates towards the new singular point whose ordinates are $\eta_t + \eta_a$, $\alpha_t + (\alpha F)_{\omega = o}$, whilst when $\omega \to \infty$ the solution curves degenerate towards the basic singularity at the initial trimmed condition η_t, α_t.

When A_1 is positive, i.e. statically unstable at low incidence, three possible trimmed conditions can exist for a given value of η_t, as shown in Fig. 8. For the present example the point at A is appropriate. The conditions A and B correspond to saddle points, i.e. points of unstable equilibrium, whilst C is a stable spiral. Obviously any small disturbances will cause the airframe to depart from its trimmed value at A to the stable singularity at C. Nevertheless it is theoretically possible for a forced sinusoidal motion to be established about A. For small values of η_a three steady state values at A', B' and C' are indicated. These correspond with the typical end points A', B' and C' of the $\eta_a = 0.02$ radians response curves in Fig. 5. The angular displacement of A', B' and C' from A, on the trim curves, are then exact measures of the steady state values to which those on Fig. 5 approximate. Three distinct curves exist, the two lower curves representing motions which are in anti-phase at $\omega = o$, whilst the upper curve is initially in phase, as shown on Fig. 7. With increase of frequency the oscillation associated with A' degenerates to the singularity at A, with little change in phase. The other two response curves finally meet at the locus of vertical tangencies and there is a corresponding meeting of the phase curves. For larger values of η_a only one steady state value is indicated at D and a typical response curve for this case would be that for $\eta_a = 0.1$ radian.

The region between the branches of the locus of vertical tangency is again an unstable region and the curves lying in this region represent impossible motions. The intercepts of these loci on the α_F ordinate of Fig. 5 correspond to the points M (the maximum) and B on Fig. 8. This implies that for small values of η_a two steady oscillations are possible corresponding to the upper or lower response curves. Bearing in mind the initial transient required in order to move into the steady sinusoid it is clear that only the upper response curve is practically relevant at low frequency. With increase of frequency jumping from the upper to the lower curves occurs and for higher frequencies it is possible to maintain relatively small amplitude oscillations about A. With increasing values of η_a the lower and middle response curves finally meet at the locus of vertical tangency at $\omega = o$, corresponding to the meeting of the points A' and B', in Fig. 8, at the maximum, M. For greater values of η_a only a single response curve exists which now indicates the possibility of both upward and downward jumps in amplitude. Corresponding jumps in phase are also demonstrated by Fig. 7.

With additional damping provided by a rate feedback and not too large an amount of static instability, it can be seen that some possibility of successful operation of the complete control system exists. A full answer as to the practicality of the proposal must await an analysis of the complete system which must embrace both frequency and transient response.

The discussion has been deliberately focussed away from the purpose to which a knowledge of the frequency response can be put. Obviously the results will have different implications when taken in conjunction with control system synthesis than would be the case when applied to the analysis of aerodynamic derivatives from flight trials. In the former there is left the open question of what relevance the frequency response has to the transient response in deciding overall stability, while in the latter the non-linear distortion of amplitude and phase curves will have an important bearing on the frequency and elevator angles selected for the test and on the conditioning of the matrix used for extracting the derivatives from the response and phase curves. Finally, it is hoped that the results will have some intrinsic merit as solutions of a particular differential equation.

7.0 Conclusions

The important conclusions which may be drawn from this analysis are as follows:

(1) When the stroboscopic method of Minorsky is applied to obtain periodic solutions of Duffing's equation, it gives the same results as that obtained by Duffing using the method of inverse iteration, i.e. selecting the amplitude of the solution and solving for the frequency, rather than the reverse. Considerable simplification arises in determining the conditions for stability, thereby avoiding the need to resort to the stability theory of Mathieu's equation (see Ref. 8).

(2) The short-period motion of an airframe having non-linear aerodynamic characteristics and subject to a sinusoidal elevator deflection is shown to have a governing differential equation in W, the vertical velocity, which is closely allied to Duffing's form. Although the coefficient of W is of non-linear form, it is not such that the airframe experiences changes from negative to positive damping with increase of amplitude and thereby excludes the possibility of limit cycling. Resulting from this, the non-linear phenomena experienced during the pitching motion are similar in character to those associated with Duffing's equation; in particular, jumps occur in the amplitude and phase of the oscillations in incidence.

(3) In a similar way to Ref. 1, the analysis has been restricted to the stability and response of the equation governing the vertical velocity. Unlike the earlier problem it would seem that a solution for the angular rate of pitch, $\dot{\theta}$, is possible. The governing equation for the $\dot{\theta}$ motion can be obtained by substituting $w = F \operatorname{Sin}(\omega t + \phi)$, with ω , F and ϕ known, into equation (56).

(4) In assessing the relative importance of the effects arising from non-linear aerodynamic characteristics it is important to remember that the airframe is only one part, albeit an important one, of the overall control loop. Obviously feedback and shaping signals will have considerable influence on the overall stability and response, in particular the range of elevator amplitude for which jumping does not occur can be expected to increase. Quantitative assessment of this problem must await further analysis, analysis which must be capable of taking into account the increase in order of the governing d.e., which is almost certain to arise when the other components of the control loop are included.

(5) It will be noted that equation (56) is of a slightly different form from (37) and the stroboscopic method cannot, without reservation, be applied to it. The restoring term

$$C(w) = (A_1 + 3A_3 w_t^2)w + 3A_3 w_t . w^2 + A_3 w^3$$

is of assymetric form, except when $w_t = 0$. This assymetry, due to the term in w^2, will, when w_t is not small, produce what is known in electrical engineering terminology as a "rectification effect". This means that, in the first approximation, the response in w is not a simple sinusoid but takes the form

$$w = w_r + F \operatorname{Sin}(\sigma + \phi),$$

where the rectification term w_r is a function of frequency. Some discussion of this phenomena is given by McLachlan in Ref. 11 and it is clear that w_r can only be neglected if w_t is small. Distortion of an assymetric character also arises from the term in ww'. To avoid these complications it has been assumed that w_t is sufficiently small.

References

1. Christopher, P.A.T. — The Stability of the Short-Period Motion of an Airframe having Non-linear Normal Force and Pitching Moment Curves. The Aeronautical Quarterly, Vol. XI, August 1960.

2. Minorsky, N. — Dynamics and Non-linear Mechanics, Vol. II of Surveys in Applied Mathematics. Chapter 2, pp. 142-147. John Wiley, New York, 1958.

3. Lefschetz, S. — Differential Equations: Geometric Theory. Chapter XII, pp. 303-307. Interscience Publishers, New York, 1957.

4. Poincaré, H. — Mémoire sur les courbes définies par une équation différentielle. Journ. Math. Pures et Appl. See Vols. 3 and 4 for various papers on this topic, but in particular Vol. 4, (1) pp. 167 244 (1885).

5. Poincaré, H. — Les Méthodes Nouvelles de la Mécanique Céleste. Vol. 1, Chapter 4. Gauthiers-Villars, Paris (1892).

6. Urabe, M. — Reduction of periodic system to autonomous one by means of one-parameter group of transformations. Journal of Science of the Hiroshima University. Series A, Volume 20, No. 1, May, 1956, pp. 11 35.

7. Coddington, E.A., and Levinson, N. — Theory of Ordinary Differential Equations. Chapters 14 and 15. McGraw-Hill, New York, 1955.

8. Stoker, J.J. — Non-linear Vibrations in Mechanical and Electrical Systems. Chapters 4 and 6. Interscience Publishers, New York, 1950.

9. Nielsen, J.N. — Missile Aerodynamics. McGraw-Hill, New York, 1960.

10. Duncan, W.J. — The Principles of the Control and Stability of Aircraft. pp. 130-134. Cambridge University Press, 1952.

11. McLachlan, N.W. — Ordinary Non-Linear Differential Equations, pp. 48-52 and 66-69. Oxford University Press, 1955.

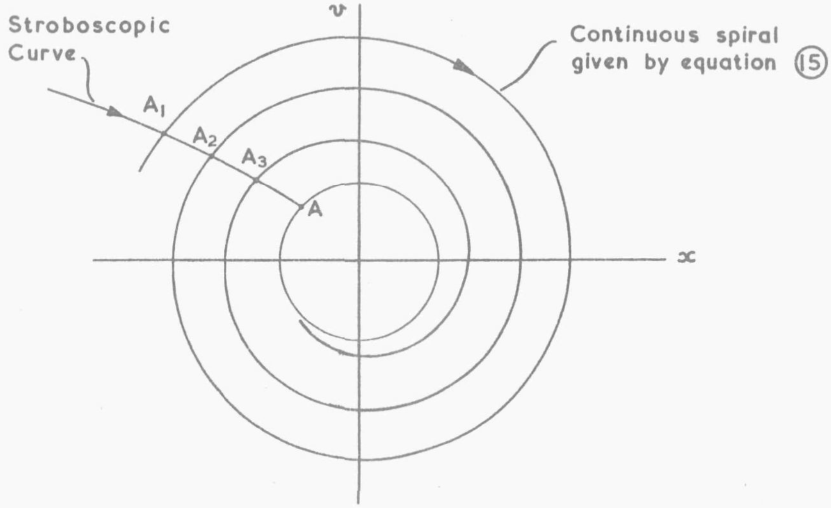

FIG. 1(a) THE STROBOSCOPIC SYSTEM

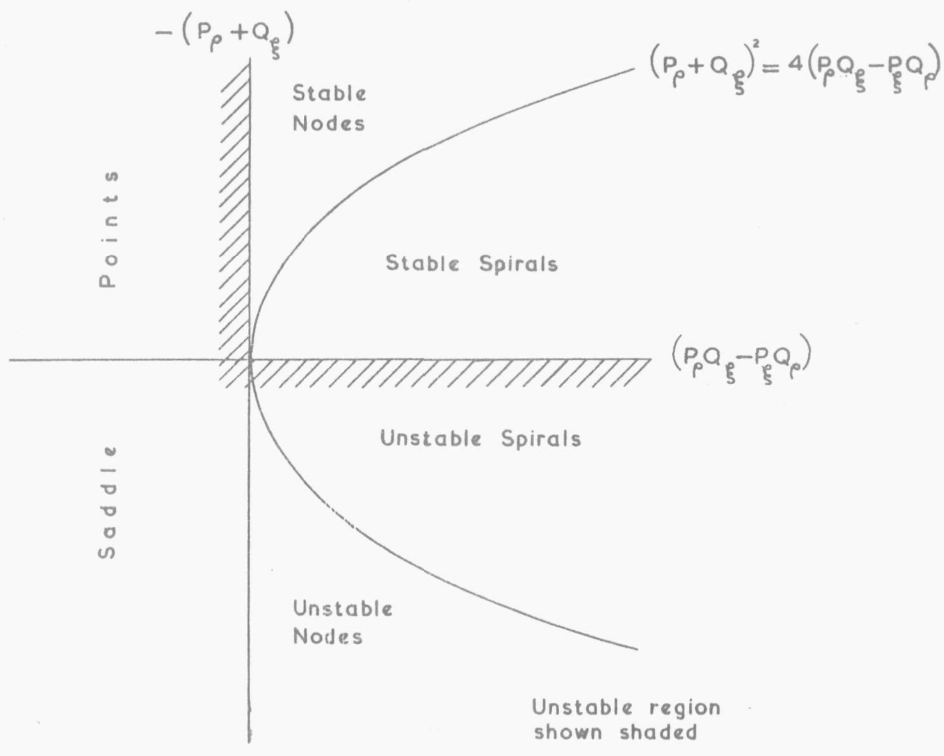

FIG. 1(b) CLASSIFICATION OF SINGULARITIES IN THE STROBOSCOPIC PLANE

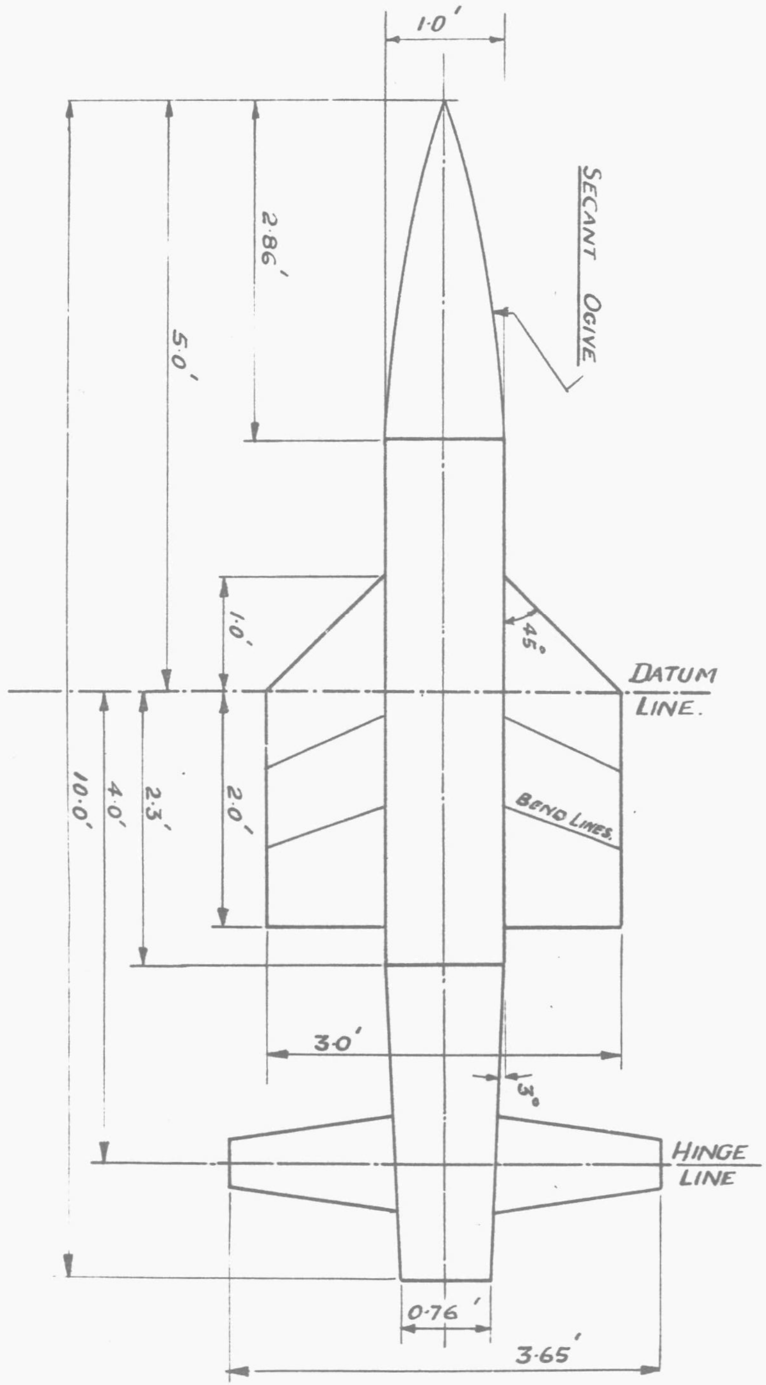

FIGURE 2

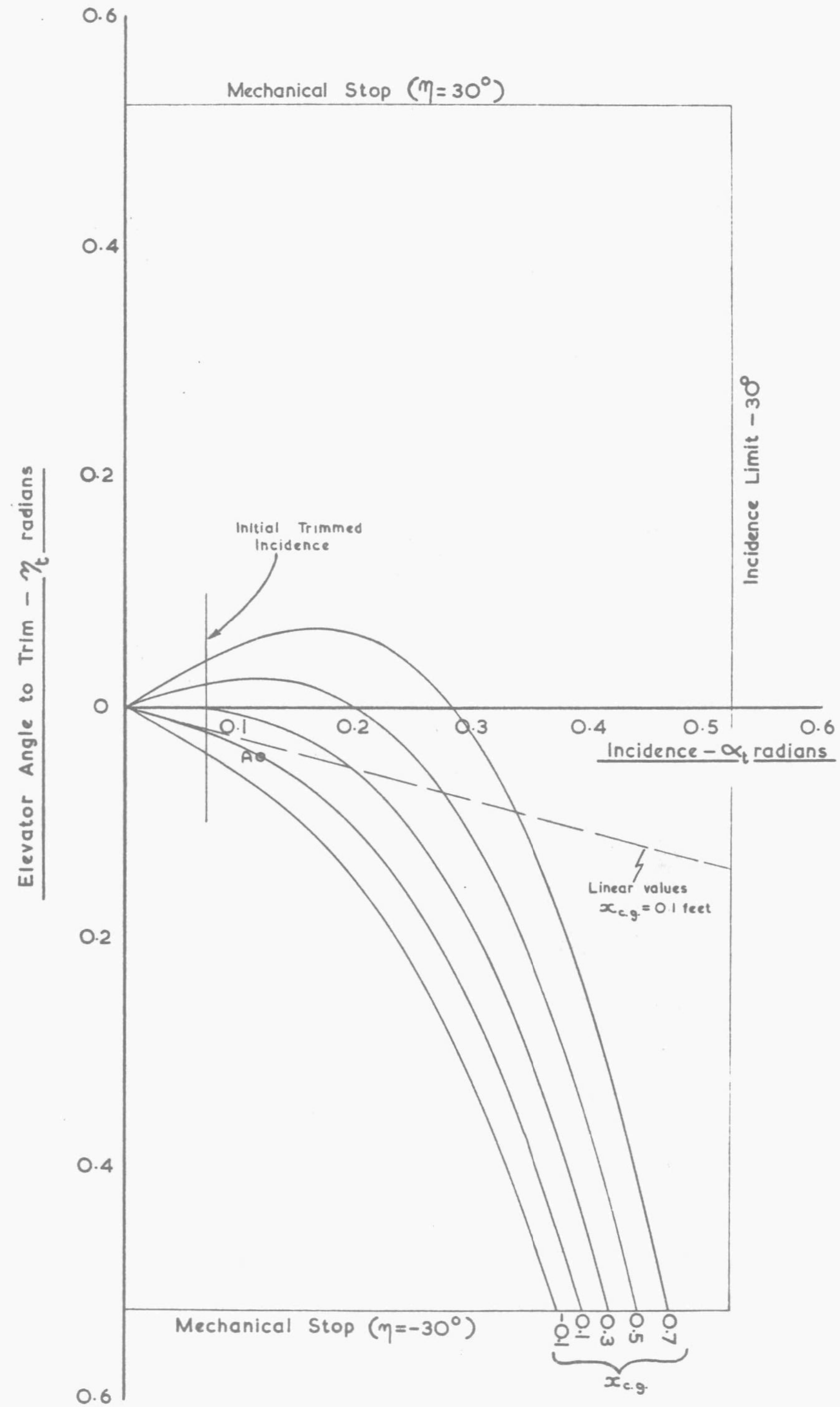

FIGURE 3

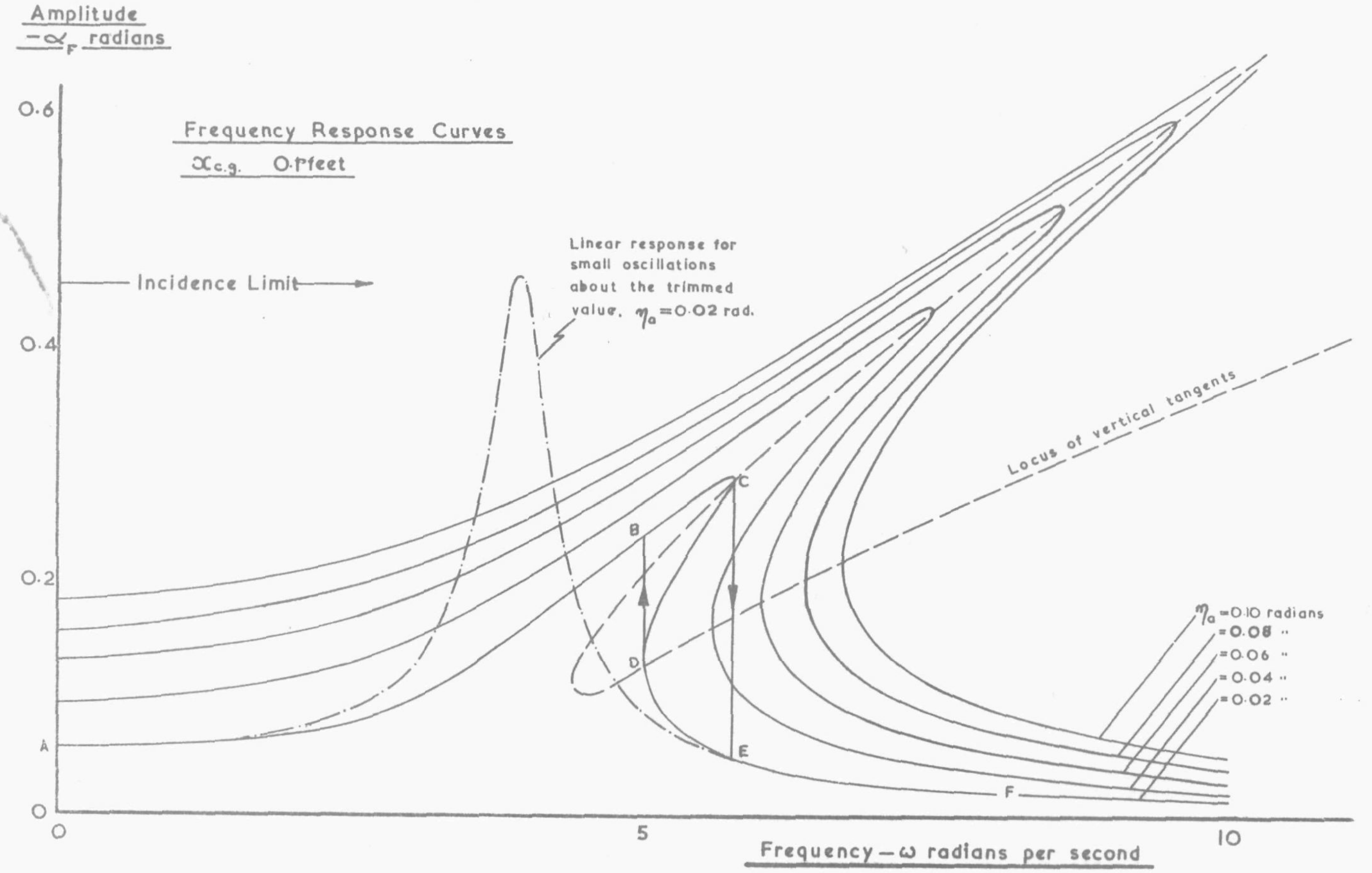

FIG. 4 FREQUENCY RESPONSE CURVES

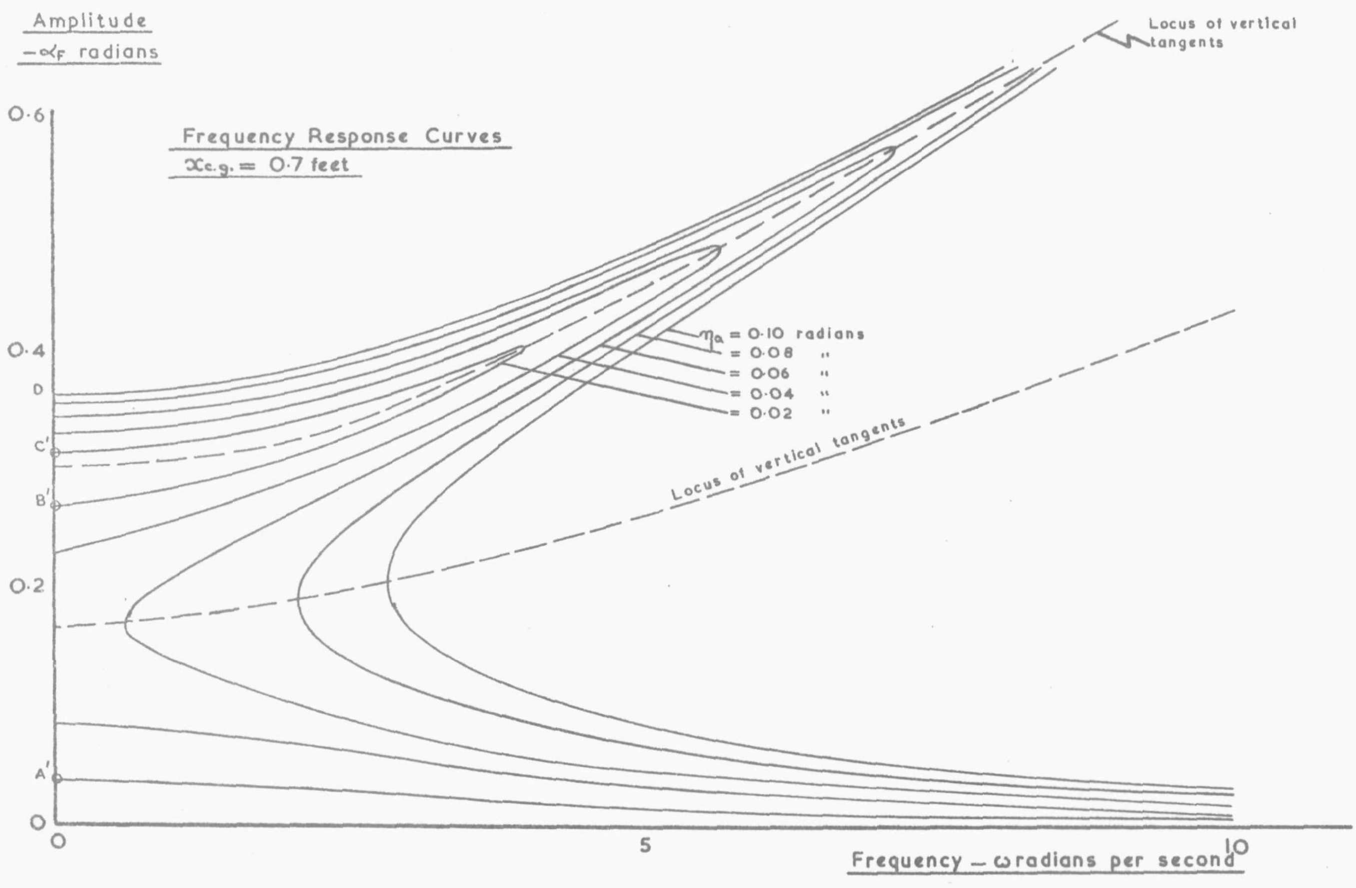

FIG. 5 FREQUENCY RESPONSE CURVES

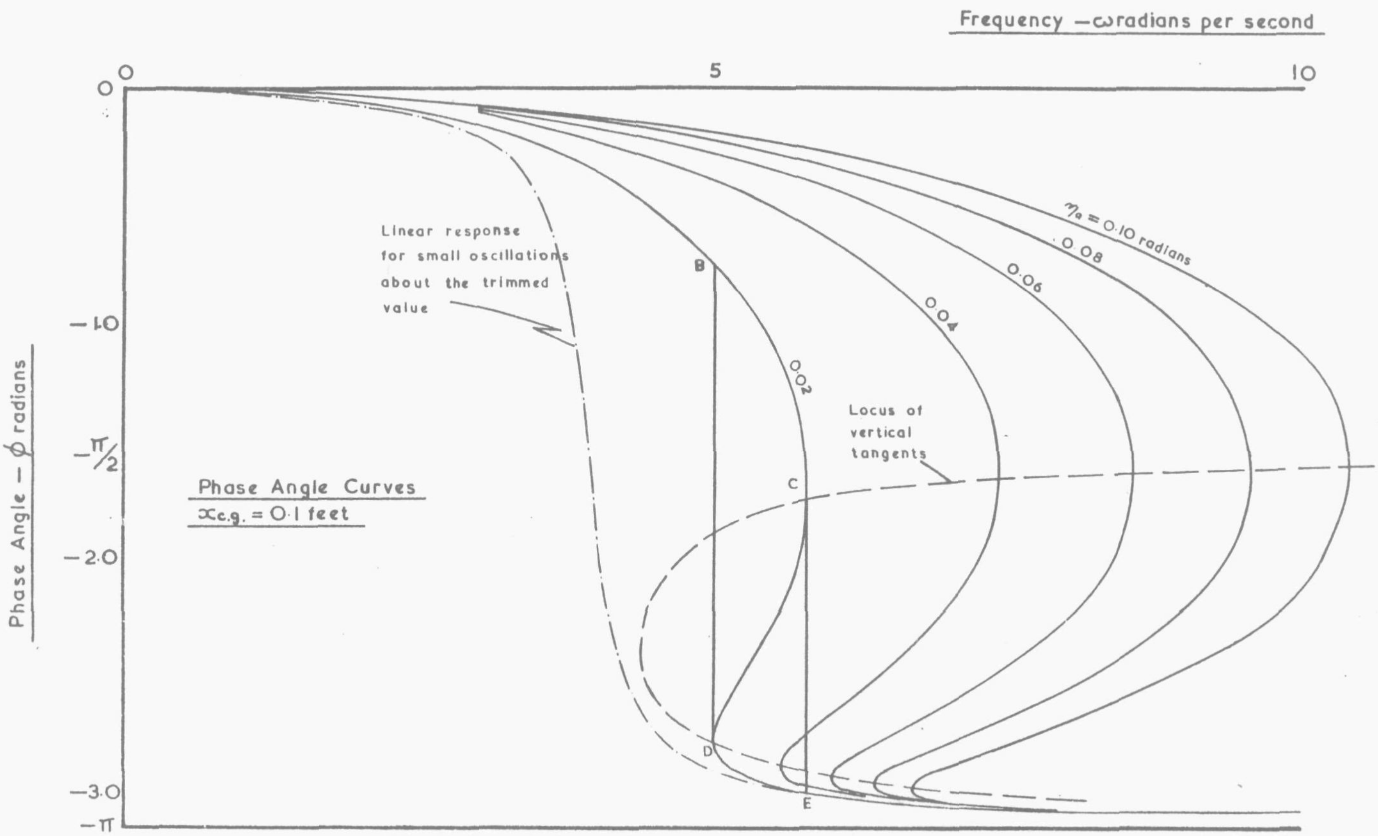

FIG. 6 PHASE ANGLE CURVES

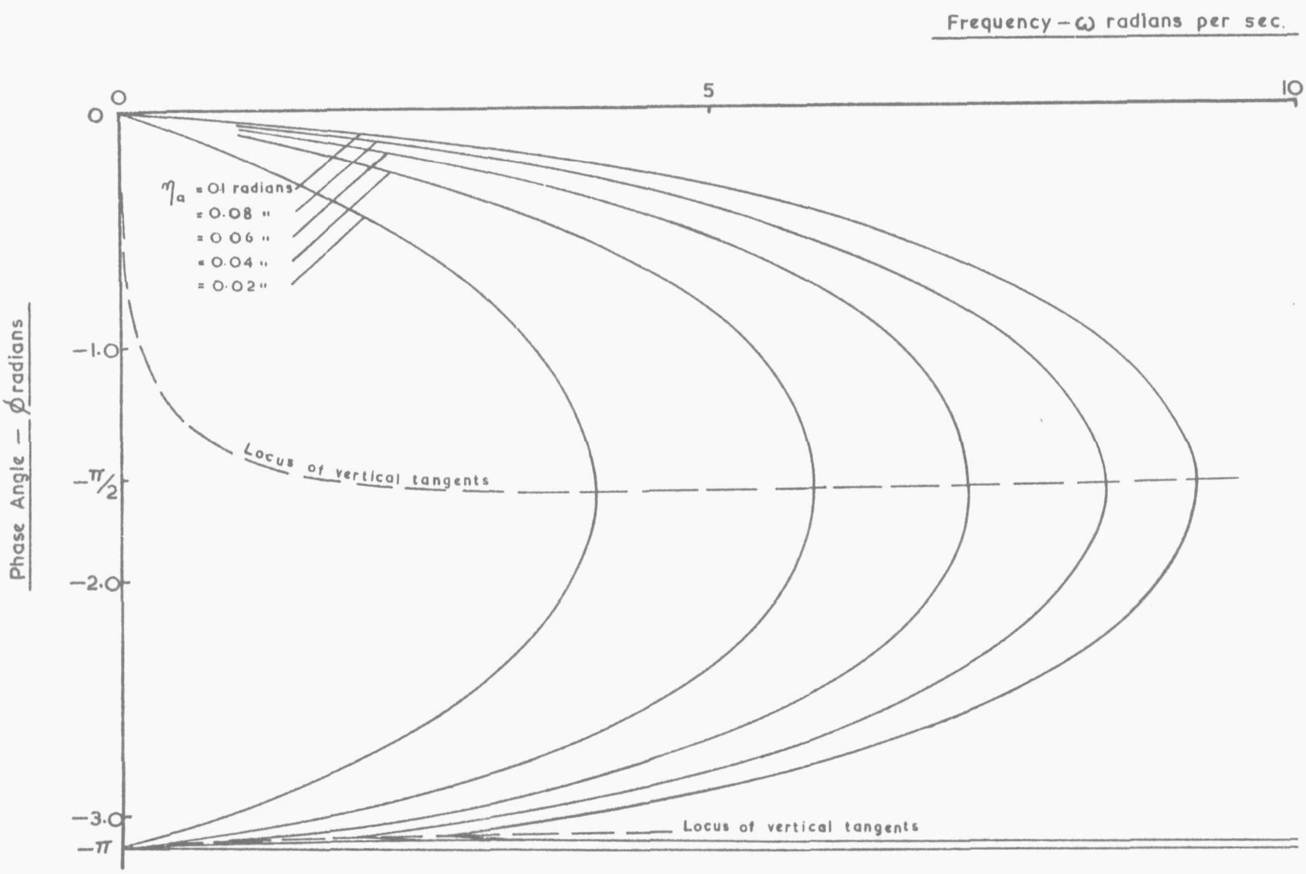

FIG. 7 PHASE ANGLE CURVES