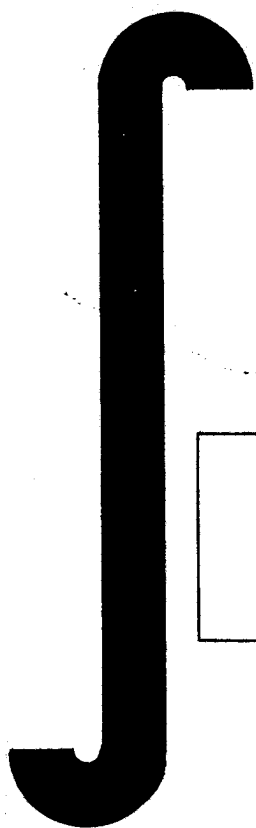


A CLASS OF INTEGRAL TRANSFORMS AND ASSOCIATED FUNCTION SPACES



A. SCHUITMAN

TR diss
1457

445229

717 J. 06

for diss. 14 87

**A CLASS OF INTEGRAL TRANSFORMS AND
ASSOCIATED FUNCTION SPACES**

A CLASS OF INTEGRAL TRANSFORMS AND ASSOCIATED FUNCTION SPACES



PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN
DOCTOR IN DE TECHNISCHE WETENSCHAPPEN
AAN DE TECHNISCHE HOGESCHOOL DELFT,
OP GEZAG VAN DE RECTOR MAGNIFICUS
PROF.DR.J.M.DIRKEN,
IN HET OPENBAAR TE VERDEDIGEN
TEN OVERSTAAN VAN HET COLLEGE VAN DEKANEN
OP DINSDAG 22 OKTOBER 1985 TE 16.00 UUR
DOOR

ADRIANUS SCHUITMAN
GEBOREN TE HALSTEREN

TR diss
1457

PROMOTOR:

PROF.DR.B.L.J.BRAAKSMA

LEDEN VAN DE COMMISSIE:

PROF.DR.H.J.A.DUPARC

PROF.DR.A.W.GROOTENDORST

PROF.DR.IR.A.J.HERMANS

PROF.DR.H.G.MEIJER

PROF.DR.C.DE VROEDT

DR.H.VAN HAERINGEN

Dit proefschrift is goedgekeurd door de promotor

INTRODUCTION

Among the many integral transforms which have been constructed since Laplace and Fourier both their nowadays familiar transforms introduced, so called Watson transforms play an important role in mathematics, not in the last place in applied analysis. A Watson transform is an integral transform, the kernel of which has the form $k(xt)$. As examples, Laplace's transform, Fourier's transform, Hankel's transform and many others belong to this class.

Another class of important integral transforms are those ones which have a kernel of the form $k(x/t)$. It is clear, by means of a substitution, that this transform is closely related to the Watson transform. In fact, properties of this transform follow directly from those of the corresponding properties of the Watson transform. However, transforms with kernel $k(x/t)$ occur so often that it is worthwhile to formulate theorems for this transform as well as for the Watson transform. We will denote an integral transform with kernel $k(x/t)$ by convolution transform.

An integral transform maps some function onto another one and as a consequence a function space into or onto another one. Operations in the original space are converted in general into operations in the image space. Integral transforms are therefore used in the first place if handling with the operations in the image space is easier to do or is better known as in the original space. As an example: the Laplace transform converts differentiation in the space of definition into a simple algebraic operation in the image space. It will be clear however that having obtained results in the image space, these results have to be put back into the original space in order to have the possibility to interpret them in relation with the problem one started with.

In applications the nature of the original space is more or less determined by the special properties of the functions under consideration. Operations in this space are often suggested by the

mathematical description of the problems to be solved. The following questions are then a natural consequence: which integral transform converts these operations into simpler ones; what are the essential properties of the functions in the original space, or better, how can one characterize the original space; what are the properties which determine the image space? Moreover, as already pointed out, there is the problem of the inversion: the nullspace of the transformation, does it consist of the null function only or not. If the null space contains only the zero function, the inverse exists and one can hope for a suitable inversion formula. As is well known, for many integral transforms inversion formulas have been constructed, some of them being so compact that they may be written down with a few number of symbols only, others being very complicated. We will make some remarks on the inversion within a moment.

There exists an important relation between Watson transforms on the one hand and Mellin transforms on the other. The Mellin transform of a function ϕ is defined by $\phi(s) = (M\phi)(s) = \int_0^\infty t^{s-1} \phi(t) dt$. Let A denote a Watson transform,

$$\psi(t) = (A\phi)(t) = \int_0^\infty k(xt) \phi(x) dx.$$

Let Ψ and K be the Mellin transforms of ψ and k respectively. Then we have (cf [67]):

$$\Psi(s) = K(s) \Phi(1-s).$$

This relation defines in fact a map $\phi \rightarrow \Psi$, which we denote by $m_1(K)$,

$$\Psi = m_1(K) \phi.$$

Combining these formulas we obtain

$$(*) \quad A = M^{-1} \circ m_1(K) \circ M$$

and this decomposition of A has to be given a sense since is obtained in a formal way only. To be more precise we again formulate in the form of a question: the nature of $m_1(K)$ being rather simple, are there function spaces which are of interest for the "user" of some Watson transform and which moreover is mapped onto another space by the Mellin transform, in such way that we can operate in this last space with $m_1(K)$.

The answer is in the affirmative as we will see in the next chapters. The relation (*) describes the method which we will use in a nutshell. First we will construct function spaces, T and S , which by the Mellin transform and its inverse are mapped onto each other. The spaces of type S are such that two of them are mapped onto each other by the map $m_1(K)$ and its inverse. We may translate the relation (*) into the following diagram, T_1, T_2, S_1 and S_2 being function spaces:

$$\begin{array}{ccc} & A & \\ T_1 & \xrightarrow{\quad} & T_2 \\ M \downarrow & & \uparrow M^{-1} \\ S_1 & \xrightarrow{m_1(K)} & S_2 \end{array} .$$

We will show that the diagram under certain conditions is commutative and that there are additional conditions which allow to reverse the direction of the arrows, thus giving a decomposition for A^{-1} . For the convolution transform a similar construction will be developed.

In applications first of all functions which are arbitrarily often differentiable are of great interest. Furthermore, if operations with functions are in the picture, things like "going to", "limit", "near" and so on are also important. Therefore we are concerned in the next chapters with function spaces of infinitely often differentiable functions, supplied with some topology. In fact we will construct the spaces S and T in such a way that they become locally convex vector spaces. Such spaces are widely used in pure and applied mathematics and extensions of our theory to distributions via the dual spaces and maps are easy to construct.

In chapter 1 we define locally convex vector spaces which will be denoted by $S^\theta(\lambda, \mu)$ and $T^\theta(\lambda, \mu)$. Here θ, λ and μ are real parameters, $\theta \geq 0, \lambda < \mu$. We prove that in the diagram we may substitute spaces of these types in such a way that M, M^{-1} and $m_1(K)$ are topological isomorphisms. The description of the topology will be given by means of a countable set of (semi)norms.

In chapter 2 we then consider the lifting-up of the map $m_1(K)$ to a Watson transform. A considerable part of this chapter is devoted to the inverse of the Watson transform. The theorems 10 and 11 give insight in the structure of the inversion for the Watson transform and theorems 14 and 15 give the corresponding inversion for the convolution transform. These theorems may be seen as the starting point from which many well known inversion formulas follow. Moreover, application of these theorems gives also several new inversion formulas. This is demonstrated in all the following chapters. Especially in this chapter we apply the inversion theorems in order to derive some new inversion formulas for the Laplace transform. Also (modifications of) classical inversion formulas for this transform are deduced.

Parallel to the treatment of the Watson transform we also consider in this chapter in detail the properties of the convolution transform.

In chapter 3 we consider the extension of the spaces of type $T^\theta(\lambda, \mu)$ to the case $\lambda \geq \mu$. Most of the theorems of chapter 2 may be extended to this case; the proofs of the theorems in this chapter are similar to those of chapter 2. In most cases we omit the details of the proofs and confine ourselves to indicating the differences in the proofs. However, spaces of type $S^\theta(\lambda, \mu)$ are not defined if $\lambda \geq \mu$. Therefore we need a special construction in order to apply the results of chapter 2.

The results of chapter 2 are derived under the condition that the Mellin transform K of the kernel k in the Watson or convolution transform is analytic in some subset, e.g. a vertical strip, of ϕ . In chapter 4 we consider the case in which K may have simple poles in this subset. The definition of the kernel k is then given by a Mellin-Barnes integral. The example at the end of this chapter is very general and contains many transforms which occur in literature if suitable values for the parameters are substituted.

Chapter 5, as well as the following chapters, may be seen as an example of the foregoing theory. In this chapter we consider as

an example of the convolution transform the fractional integration operators. Fractional integration has many applications, for example in solving integral equations. We give an example of this kind and also indicate the relation with Hankel's transform.

In chapter 6 we consider another example of the convolution transform. We give a treatment, in some detail, of the Stieltjes transform. In the same manner as in the other examples we pay much attention to inversion formulas.

Chapter 7 finally is devoted to examples of the Watson transform. We consider the transforms which have been constructed in 1940 and 1941 by Meijer, the so-called K- and W-transform. Here also we derive some inversion formulas.

The subjects of the chapters 5, 6 and 7 are chosen in a rather arbitrary way. Many other examples could be added and treated in the same manner. The example at the end of chapter 4 may also be given for convolution transforms and then contains several other transforms as special cases. A generalization to a kernel with Fox' H-function can be treated in a manner similar to the case of a G-function.

Each chapter is divided into sections, numbered 1,2,... . Formulas are given two numbers, the first one referring to the number of the section. For example, (3.7) is the seventh numbered formula of section three. If in some chapter we refer to a formula in another chapter, we add the number of the chapter; (II.3.7) refers to formula (3.7) in chapter 2.

CHAPTER 1	
CERTAIN FUNCTION SPACES	
MELLIN TRANSFORM	
1.0 INTRODUCTION	1
1.1 THE FUNCTION SPACE $T^\theta(\lambda, \mu)$	1
1.2 THE FUNCTION SPACE $S^\theta(\lambda, \mu)$	5
1.3 MELLIN TRANSFORM	12
1.4 BIBLIOGRAPHICAL NOTES	17
CHAPTER 2	
WATSON TRANSFORMS	
CONVOLUTION TRANSFORMS	
2.0 INTRODUCTION	18
2.1 WATSON TRANSFORMS ON SPACES OF TYPE $T^\theta(\lambda, \mu)$	20
2.2 INVERSION THEOREMS FOR WATSON TRANSFORMS	27
2.3 CONVOLUTION TRANSFORMS ON $T^\theta(\lambda, \mu)$ AND INVERSION THEOREMS	35
2.4 EXAMPLES AND APPLICATIONS	41
2.5 BIBLIOGRAPHICAL NOTES	56
CHAPTER 3	
EXTENSIONS TO THE CASE $\lambda \geq \mu$	
3.0 INTRODUCTION	58
3.1 DECOMPOSITION OF A FUNCTION $\phi \in T^\theta(\lambda, \mu)$	58
3.2 EXTENSION OF THE DEFINITION OF WATSON TRANSFORM AND OF CONVOLUTION TRANSFORM TO THE CASE	61
3.3 BIBLIOGRAPHICAL NOTES	66
CHAPTER 4	
KERNELS DEFINED BY MELLIN-BARNES INTEGRALS	
4.0 INTRODUCTION	67
4.1 THE SPACE $T_{P,M}^\theta(\lambda, \mu; s_1, \dots, s_p)$	67
4.2 EXTENSION OF WATSON TRANSFORMS	78
4.3 EXTENSION OF THE CONVOLUTION TRANSFORM	87
4.4 EXAMPLE	90
CHAPTER 5	
FRACTIONAL INTEGRATION OPERATORS AND HANKEL TRANSFORM	
5.0 INTRODUCTION	96
5.1 THE OPERATOR OF FRACTIONAL INTEGRATION $I(\eta, \alpha, H)$	98
5.2 THE OPERATOR OF FRACTIONAL INTEGRATION $K(\eta, \alpha, H)$	102
5.3 APPLICATION TO HANKEL TRANSFORM	108
5.4 EXAMPLES	112
5.5 BIBLIOGRAPHICAL NOTES	116

CHAPTER 6	
STIELTJES TRANSFORM	
6.0 INTRODUCTION	117
6.1 THE STIELTJES TRANSFORM S^α	118
6.2 INVERSION FORMULAS	121
6.3 EXTENSION TO THE CASE $\operatorname{Re} \alpha < 0$	126
6.4 INVERSION FORMULAS INVOLVING DIFFERENTIAL OPERATORS	127
6.5 EXAMPLE	132
6.6 BIBLIOGRAPHICAL NOTES	136
CHAPTER 7	
MEIJER TRANSFORMS	
7.0 INTRODUCTION	137
7.1 MEIJER'S K-TRANSFORM	138
7.2 MEIJER'S W-TRANSFORM	143
7.3 RELATIONS WITH OTHER TRANSFORMS	151
7.4 BIBLIOGRAPHICAL NOTES	155
REFERENCES	156
SAMENVATTING	163

CHAPTER 1

CERTAIN FUNCTION SPACES

MELLIN TRANSFORM

1.0 INTRODUCTION

In this chapter two types of function spaces are introduced. They will be denoted by $S^\theta(\lambda, \mu)$ and $T^\theta(\lambda, \mu)$ respectively. The parameters θ , λ and μ will be specified in the next section. The space $S^\theta(\lambda, \mu)$ is the image of a space of type $T^\theta(\lambda, \mu)$ under the Mellin transform. The spaces under consideration are equipped with a topology in such a way that the Mellin transform is a topological isomorphism between the vector space structures of both spaces. In section 1.1 we give the definition and some properties of $S^\theta(\lambda, \mu)$. Section 1.2 is devoted to spaces of type $T^\theta(\lambda, \mu)$ and some maps of these spaces. In section 1.3 we prove that the Mellin transform has the property just mentioned. It should be noted that several properties of $S^\theta(\lambda, \mu)$ follow from those of $T^\theta(\lambda, \mu)$ by means of the Mellin transform and vice versa.

1.1 THE FUNCTION SPACE $S^\theta(\lambda, \mu)$

At first we specify some parameters which occur frequently in the sequel. Let $\lambda, \mu \in \mathbb{R} \cup \{-\infty, \infty\}$ and let $(\lambda_n)_{n=0}^\infty$ and $(\mu_n)_{n=0}^\infty$ be sequences in $\mathbb{R}$. We assume the following relations between λ , μ , λ_n and μ_n .

$$(1.1) \quad \begin{cases} 1) \lambda < \mu \\ 2) \lambda_n > \lambda_{n+1}, \mu_n < \mu_{n+1}, \lambda_n < \mu_n, \quad (n \in \mathbb{N}) \\ 3) \lim_{n \rightarrow \infty} \lambda_n = \lambda, \lim_{n \rightarrow \infty} \mu_n = \mu. \end{cases}$$

We now give the definition of the space $S^\theta(\lambda, \mu)$.

DEFINITION 1. Assume (1.1) Let θ be a non negative real number. Then $S^\theta(\lambda, \mu)$ is the space of all functions ϕ with the properties

$$(1.2) \quad \phi \text{ is analytic on } \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\};$$

$$(1.3) \quad \text{for any } p, n \in \mathbb{N}, \phi(s) = O(s^{-p} e^{-\theta|s|}) \text{ as } s \rightarrow \infty, \\ \text{uniformly on } \{s \in \mathbb{C} \mid \lambda_n \leq \operatorname{Re} s \leq \mu_n\}.$$

In order to give $S^\theta(\lambda, \mu)$ the structure of a topological vectorspace we define norms in the following way. For any $n \in \mathbb{N}$

$$(1.4) \quad \sigma_n^\theta(\phi) = \sup_{\substack{p \leq n \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |s^p e^{\theta|s|} \phi(s)|, \quad p \in \mathbb{N}.$$

It is easily seen that the inequality

$$\sigma_n^\theta(\phi) \leq \sigma_{n+1}^\theta(\phi)$$

holds for any $n \in \mathbb{N}$. It follows that $S^\theta(\lambda, \mu)$ is a locally convex vectorspace.

REMARKS.

1. In (1.3), $\exp(-\theta|s|)$ can be replaced by $\exp(-\theta|\operatorname{Im} s|)$; the same could be done in (1.4). The resulting norms are equivalent.
2. The topology of $S^\theta(\lambda, \mu)$ is independent of the choice of the sequences (λ_n) and (μ_n) , provided they satisfy (1.1).
3. Contrary, as we will see in the next section, to the spaces $T^\theta(\lambda, \mu)$, the spaces $S^\theta(\lambda, \mu)$ are not defined if $\lambda \geq \mu$.

As we pointed out already in the introduction, the integral transforms which occur in the next chapters, can be lifted to rather simple maps on spaces of type $S^\theta(\lambda, \mu)$. The following theorems deal with those maps.

THEOREM 1. Assume (1.1). Let θ_1, θ_2 be non negative real numbers. Let K be an analytic function on $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$ such that

$$(1.5) \quad \left\{ \begin{array}{l} \text{for each } n \in \mathbb{N} \text{ there exists a real number } \gamma_n \text{ such that} \\ K(s) = O(s^{\gamma_n} e^{(\theta_1 - \theta_2)|s|}) \text{ as } s \rightarrow \infty, \text{ uniformly on the strip} \\ \{s \in \mathbb{C} \mid \lambda_n \leq \operatorname{Re} s \leq \mu_n\}. \end{array} \right.$$

Then the map

$$m(K) : S^{\theta_1}(\lambda, \mu) \rightarrow S^{\theta_2}(\lambda, \mu)$$

defined by

$$(1.6) \quad m(K)\phi(s) = K(s)\phi(s)$$

is continuous.

PROOF. Let $n \in \mathbb{N}$, $\phi_1 \in S^{\theta_1}$, $\phi_2 = m(K)\phi_1$. Then

$$\begin{aligned} \sigma_n^{\theta_2}(\phi_2) &= \sup_{\substack{k \leq n \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |s^k e^{\theta_2|s|} K(s) \phi_1(s)| \\ &= \sup_{\substack{k \leq n \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |K(s) e^{(\theta_2 - \theta_1)|s|}| \times |e^{\theta_1|s|} s^k \phi_1(s)|. \end{aligned}$$

If $|s| \leq 1$ the last expression is less than $M_1 \sigma_n^{\theta_1}(\phi_1)$, where

$$M_1 = \max_{|s| \leq 1} |K(s) e^{(\theta_1 - \theta_2)|s|}|.$$

If $|s| > 1$, we have

$$\begin{aligned} \sigma_n^{\theta_2}(\phi_2) &= \sup_{\substack{k \leq n \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |K(s) e^{(\theta_2 - \theta_1)|s|} s^{-\gamma_n} \times (e^{\theta_1|s|} s^{k+\gamma_n} \phi_1(s))| \\ &\leq M_2 \sigma_m^{\theta_1}(\phi_1), \end{aligned}$$

where $m \geq \max (n, n+\gamma_n)$ and M_2 has the value

$$M_2 = \sup_{\substack{|s| > 1 \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |K(s) e^{(\theta_2 - \theta_1)|s|} s^{-\gamma_n}|.$$

Combining the results, we see that indeed $m(K)$ is a map of S^{θ_1} into S^{θ_2} and moreover that for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ and a constant M_0 such that

$$\sigma_n^{\theta_2}(\phi_2) \leq M_0 \sigma_m^{\theta_1}(\phi_1),$$

M_0 being independent of ϕ_1 . Hence the continuity of $m(K)$. $\square$

Theorem 1 can be extended in the following way.

THEOREM 2. Let the conditions of theorem 1 be satisfied. Assume moreover that $(K(s))^{-1}$ exists and is analytic on $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$. Assume that for any $n \in \mathbb{N}$ there exists δ_n such that

$$(K(s))^{-1} = O(s^{\delta_n} e^{(\theta_2 - \theta_1)|s|}) \text{ as } s \rightarrow \infty,$$

uniformly on $\{s \in \mathbb{C} \mid \lambda_n \leq \operatorname{Re} s \leq \mu_n\}$, where δ_n is a real number. Then the map $m(K)$ defined by (1.6) is a topological isomorphism of $S^{\theta_1}(\lambda, \mu)$ onto $S^{\theta_2}(\lambda, \mu)$ and $\{m(K)\}^{-1} = m(K^{-1})$.

PROOF. The map $m(K^{-1}) : S^{\theta_2}(\lambda, \mu) \rightarrow S^{\theta_1}(\lambda, \mu)$, where $m(K^{-1})$ is defined as in (1.6), is continuous by the preceding theorem. Clearly it is the inverse map of $m(K)$. $\square$

For some purposes it is convenient to have a slightly different form of the foregoing theorems. We give the formulation in the next theorem. The proof of this theorem is a modification of the proofs of theorem 1 and theorem 2.

THEOREM 1'. Let the assumptions of theorem 1 be satisfied. Then the map

$$m_1(K) : S^{\theta_1}(1-\mu, 1-\lambda) \rightarrow S^{\theta_2}(\lambda, \mu)$$

defined by

$$(1.7) \quad m_1(K)\phi(s) = K(s)\phi(1-s)$$

is continuous. If moreover $H(s) = \frac{1}{K(1-s)}$ satisfies the conditions of theorem 2 then $\{m_1(K)\}^{-1} = m_1(H)$ and $m_1(K)$ is a topological isomorphism. $\square$

1.2 THE FUNCTION SPACE $T^\theta(\lambda, \mu)$

A short description of the space $T^\theta(\lambda, \mu)$ can be found in definition 4.

In order to get some insight in the structure of these spaces, we take the following approach.

Let $\theta \geq 0$. We define a subset of the complex numbers:

$$(2.1) \quad G_\theta = \{t \in \mathbb{C} \mid |\arg t| \leq \theta\}.$$

G_θ is to be understood as part of the Riemannian surface of the function $\log$. Note that $0 \notin G_\theta$ and $G_0 = \mathbb{R}^+$. G_θ° denotes the interior of G_θ with respect to the usual topology of $\mathbb{R}^2$.

Next we define a strictly positive weight function M_n for every $n \in \mathbb{N}$ by

$$M_n(t) = |t|^{\lambda_n} (1 + |t|)^{\mu_n - \lambda_n}, \quad t \in \mathbb{C} \setminus \{0\},$$

where the sequences (λ_n) and (μ_n) satisfy

$$(2.2) \quad \begin{cases} \lambda_0 > \lambda_1 > \dots > \lambda_n > \lambda_{n+1} > \dots, & \lim_{n \rightarrow \infty} \lambda_n = \lambda \\ \mu_0 < \mu_1 < \dots < \mu_n < \mu_{n+1} < \dots, & \lim_{n \rightarrow \infty} \mu_n = \mu \end{cases}$$

Our aim is to define the space $T^\theta(\lambda, \mu)$ as the projective limit of spaces with a rather simple topological structure. The next definition gives such spaces.

DEFINITION 2. Assume (2.2). T_n^θ is the generalized Sobolev space of all functions $\phi : G_\theta \rightarrow \mathbb{C}$ which satisfy

- 1) ϕ is analytic on G_θ^0 if $\theta > 0$; ϕ is n times continuously differentiable on G_0 if $\theta = 0$,
- 2) $\phi^{(j)}$ may be extended continuously from G_θ^0 to G_θ if $\theta > 0$, $j = 0, 1, \dots, n$,
- 3) $\tau_n^\theta(\phi) = \sup_{\substack{p \leq n \\ t \in G_\theta}} |M_n(t) t^p \phi^{(p)}(t)| < \infty$.

τ_n^θ is a norm and with this norm, T_n^θ is a Banach space. It is easy to see that the imbedding $T_{n+1}^\theta \rightarrow T_n^\theta$ is continuous. We prove even more.

THEOREM 3. For every $n \in \mathbb{N}$ the imbedding

$$i_n : T_{n+1}^\theta \rightarrow T_n^\theta$$

is compact.

PROOF. Let S_ℓ be the open set

$$S_\ell = \{t \in G_\theta^0 \mid \frac{1}{\ell+1} < |t| < \ell+1\}$$

for any $\ell \in \mathbb{N}$. S_ℓ is precompact, $S_\ell \subset S_{\ell+1}$, $\bigcup_{\ell=1}^\infty S_\ell = G_\theta^0$.

Assume that $(\phi_\nu)_{\nu=1}^\infty$ is a sequence in T_{n+1}^θ , for which

$$\tau_{n+1}^\theta(\phi_\nu) \leq 1, \quad \nu = 1, 2, \dots$$

We will construct a subsequence which converges in T_n^θ .

Firstly it follows from $\tau_m^\theta \leq \tau_{m+1}^\theta$, $m \in \mathbb{N}$, in other words from the continuity of the imbedding, that

$$(*) \quad (\phi^{(p)}(t))_{p=1}^\infty$$

is pointwise bounded for every $p = 1, \dots, n$ and every $t \in \bar{S}_\ell$. Now let $t \in S_1$ and let W_t be a convex neighbourhood of t which lies entirely in S_1 . If $t' \in W_t$ we have

$$(2.4) \quad \begin{aligned} |\phi_U^{(n)}(t) - \phi_U^{(n)}(t')| &= \left| \int_0^1 \frac{d}{d\xi} \phi_U^{(n)}(t' + \xi(t-t')) d\xi \right| \\ &\leq \int_0^1 |M_{n+1}(t_\xi) t_\xi^{n+1} \phi_U^{(n+1)}(t_\xi) \frac{t-t'}{M_{n+1}(t_\xi) t_\xi^{n+1}}| d\xi \end{aligned}$$

where $t_\xi = t' + \xi(t-t') \in W_t \subset S_1$. From (2.4) we deduce

$$(*) \quad \begin{aligned} |\phi_U^{(n)}(t) - \phi_U^{(n)}(t')| &\leq \tau_{n+1}^\theta(\phi_U) \frac{|t-t'|}{\inf_{t \in S_1} M_{n+1}(t)} \times 2^{n+1} \\ &\leq C_n |t-t'|, \end{aligned}$$

using the fact that $M_{n+1}(t_\xi) \geq \inf_{t \in S_1} M_{n+1}(t)$. C_n is a constant, independent of U and t . By a continuity argument it follows that (*) also

holds on $\bar{S}_1$. The relation (*) means that $(\phi_U^{(n)})$ is equicontinuous on $\bar{S}_1$. It follows from (*) and (**) that we may apply the Arzela-Ascoli theorem in order to select a subsequence $(\phi_{U_n})_{U=1}^\infty$ from the sequence (ϕ_U) such that the sequence $(\phi_{U_n}^{(n)})_{U=1}^\infty$ converges uniformly on $\bar{S}_1$ to some function ϕ_{0n} . Replacing n by $n-1$ in (2.4) and repeating the argument, we select a subsequence $(\phi_{U(n-1)})_{U=1}^\infty$ from $(\phi_{U_n})_{U=1}^\infty$ which converges to some function, $\phi_{0(n-1)}$ say, uniformly on $\bar{S}_1$. Proceeding in this way, after n steps we obtain a sequence $(\phi_{U0})_{U=1}^\infty$ which together with its derivatives up to the n th order converges uniformly on $\bar{S}_1$ to $\phi_{00}, \phi_{01}, \dots, \phi_{0n}$ respectively. By a standard

theorem $\phi_{0j} = \phi_{00}^{(j)}$ on S_1 , $j = 0, 1, \dots, n$ and ϕ_{0j} may be continuously extended to $\bar{S}_1$.

In the same manner we select a subsequence from $(\phi_{u0})_{u=1}^{\infty}$ which converges with its derivatives uniformly on $\bar{S}_2$, and so on. By the diagonalisation process finally we obtain a subsequence $(\psi_u)_{u=1}^{\infty}$ which converges uniformly on $\bar{S}_\ell$ for any $\ell = 1, 2, \dots$ to a limit function ψ_0 and also $\psi_u^{(j)} \rightarrow \psi_0^{(j)}$ on $\bar{S}_\ell$ as $u \rightarrow \infty$, $j = 1, 2, \dots, n$.

We prove that (ψ_u) converges in T_n^θ .

Choose $\varepsilon > 0$.

$$\begin{aligned}
 (2.5) \quad & \sup_{\substack{t \in G_\theta \\ p \leq n \\ |t| \leq 1/(\ell+1)}} \left| |t|^\lambda (1+|t|)^{\mu_n - \lambda} t^p (\psi_u^{(p)}(t) - \psi_\rho^{(p)}(t)) \right| \leq \\
 & \leq \sup_{|t| \leq 1/(\ell+1)} \left| |t|^{\lambda - \lambda_{n+1}} (1+|t|)^{\mu_n - \mu_{n+1} - \lambda_n + \lambda_{n+1}} \left(\tau_{n+1}^\theta(\psi_u) + \tau_{n+1}^\theta(\psi_\rho) \right) \right| \leq \\
 & \leq (\ell+1)^{\lambda_{n+1} - \lambda_n} (\tau_{n+1}^\theta(\psi_u) + \tau_{n+1}^\theta(\psi_\rho)) < \varepsilon \text{ if } \ell \geq \ell_0
 \end{aligned}$$

for some ℓ_0 , not depending on u and ρ . In a similar way we obtain

$$\begin{aligned}
 (2.6) \quad & \sup_{\substack{t \in G_\theta \\ p \leq n \\ |t| > \ell+1}} \left| |t|^\lambda (1+|t|)^{\mu_n - \lambda} t^p (\psi_u^{(p)}(t) - \psi_\rho^{(p)}(t)) \right| \leq \\
 & \leq \sup_{|t| > \ell+1} \left| \left(\frac{|t|}{1+|t|} \right)^{\lambda_n - \lambda_{n+1}} (1+|t|)^{\mu_n - \mu_{n+1}} \left(\tau_{n+1}^\theta(\psi_u) + \tau_{n+1}^\theta(\psi_\rho) \right) \right| \leq \\
 & \leq 2(\ell+2)^{\mu_n - \mu_{n+1}} < \varepsilon \text{ if } \ell \geq \ell_1,
 \end{aligned}$$

for some ℓ_1 , not depending on u and ρ . Fix some $\ell \geq \max(\ell_0, \ell_1)$ and for this ℓ consider S_ℓ . By uniform convergence there exists a u_0 such that

$$(2.7) \quad \sup_{\substack{t \in \overline{S}_\theta \\ p \leq n}} \left| |t|^{\lambda_n(1+|t|)} \mu_n^{-\lambda_n} t^p (\psi_\nu^{(p)}(t) - \psi_\rho^{(p)}(t)) \right| < \varepsilon$$

if $\nu, \rho > \nu_0$. Combining (2.5), (2.6), (2.7) and the completeness of T_n^θ , we obtain the desired convergence of (ψ_ν) in T_n^θ . $\square$

From the continuity of the maps i_n , defined in theorem 3, it follows that

$$T_0^\theta \xleftarrow{i_1} T_1^\theta \xleftarrow{i_2} T_2^\theta \xleftarrow{i_3} T_3^\theta \dots\dots$$

is a projective spectrum. By means of this spectrum we define the space $T^\theta(\lambda, \mu)$ as the projective limit.

DEFINITION 3. $T^\theta(\lambda, \mu) = \text{proj}_{n \leftarrow} T_n^\theta$.

The spaces T_n^θ are Banach spaces, hence $T^\theta(\lambda, \mu)$ is a Gelfand space or in other words a countably normed space. Moreover, from the fact that the maps i_n are compact, we deduce at once the following theorem.

THEOREM 4. $T^\theta(\lambda, \mu)$ is a (FM)-space (a Fréchet space which is also a Montel space). That is an F-space in which every bounded set is relatively compact. $\square$

It is not hard to see that the following theorem could be used as a definition of $T^\theta(\lambda, \mu)$, which is equivalent with definition 3.

THEOREM 5. Let $\theta > 0$. Assume (1.1). $T^\theta(\lambda, \mu)$ is the space of all functions ϕ for which

- 1) ϕ is analytic on G_θ° ,
- 2) $\phi^{(j)}$ may be continuously extended from G_θ° to G_θ ,
- 3) $\tau_n^\theta(\phi) = \sup_{\substack{p \leq n \\ t \in G_\theta}} |t|^{\lambda_n(1+|t|)} \mu_n^{-\lambda_n} t^p \phi^{(p)}(t) | < \infty$

In case of $\theta = 0$ we have: $T^0(\lambda, \mu)$ is the space of all functions ϕ for which

$$1) \quad \phi \in C^\infty(0, \infty),$$

$$2) \quad \tau_n^0(\phi) = \sup_{\substack{p \leq n \\ 0 < t < \infty}} |t^{\lambda_n(1+|t|)} t^{\mu_n - \lambda_n} t^p \phi^{(p)}(t)| < \infty.$$

□

REMARKS

4. It should be noticed that $T^\theta(\lambda, \mu)$ is defined for any pair of real numbers λ and μ in $\mathbb{R} \cup \{-\infty, \infty\}$.

5. The topology of $T^\theta(\lambda, \mu)$ is independent of the particular choice of the sequences (λ_n) and (μ_n) , provided they satisfy (2.2).

We now compare the spaces $T^\theta(\lambda, \mu)$ and $T^\theta(\alpha, \beta)$, where $\alpha < \lambda$, $\mu < \beta$.

Let the topology of $T^\theta(\lambda, \mu)$ be defined by the sequences (λ_n) and (μ_n) .

Let the sequences (α_n) and (β_n) which define the topology of the space $T^\theta(\alpha, \beta)$ be such that

$$\alpha_n \leq \lambda_n, \quad \beta_n \geq \mu_n, \quad n \in \mathbb{N}.$$

Let $\phi \in T^\theta(\alpha, \beta)$. Then

$$\begin{aligned} & \sup_{\substack{p \leq n \\ t \in G_\theta}} |t^{\lambda_n(1+|t|)} t^{\mu_n - \lambda_n} t^p \phi^{(p)}(t)| = \\ & \sup_{\substack{p \leq n \\ t \in G_\theta}} |t^{\alpha_n(1+|t|)} t^{\beta_n - \alpha_n} t^p \phi^{(p)}(t)| \times |t^{\lambda_n - \alpha_n(1+|t|)} t^{\mu_n - \beta_n - (\lambda_n - \alpha_n)}|. \end{aligned}$$

It follows that $\phi \in T^\theta(\lambda, \mu)$ and moreover that the imbedding

$T^\theta(\alpha, \beta) \rightarrow T^\theta(\lambda, \mu)$ is continuous.

Next we consider two spaces $T^\theta(\lambda, \mu)$ and $T^{\theta_0}(\lambda, \mu)$, where $0 < \theta_0 < \theta$.

If ϕ is analytic on G_θ^0 , it is also analytic on $G_{\theta_0}^0$ if $\theta_0 > 0$ and

all derivatives of ϕ are continuous on the boundary of $G_{\theta_0}^0$. If $\theta_0 = 0$

then for every $p \in \mathbb{N}$ the continuity of $\phi^{(p)}$ follows. From

$$\tau_n^{\theta_0}(\phi) \leq \tau_n^\theta(\phi), \quad n \in \mathbb{N}, \quad \phi \in T^\theta(\lambda, \mu),$$

one easily deduces the continuity of the imbedding $T^\theta(\lambda, \mu) \rightarrow T^{\theta_0}(\lambda, \mu)$. $\square$
Finally we prove a lemma which deals with a map of $T^\theta(\lambda, \mu)$ and which will be used in the sequel.

LEMMA. The map R , defined by

$$(2.8) \quad (R\phi)(t) = \frac{1}{t} \phi\left(\frac{1}{t}\right), \quad \phi \in T^\theta(\lambda, \mu)$$

is a topological isomorphism of $T^\theta(\lambda, \mu)$ onto $T^\theta(1-\mu, 1-\lambda)$ and

$$R \circ R = \text{id}.$$

PROOF. Let the topologies in $T^\theta(\lambda, \mu)$ and $T^\theta(1-\mu, 1-\lambda)$ be defined by the sequences (λ_n) , (μ_n) and $(1-\mu_n)$, $(1-\lambda_n)$ respectively. By straightforward differentiation one has

$$\left(\frac{1}{t} \phi\left(\frac{1}{t}\right)\right)^{(p)} = \sum_{j=0}^p c_{p,j} t^{-1-p-j} \phi^{(j)}\left(\frac{1}{t}\right),$$

where the constants $c_{p,j}$ are independent of ϕ . If we denote in both spaces the norms by τ_n , we obtain

$$\tau_n(R\phi) = \sup_{\substack{p \leq n \\ t \in G_\theta}} \left| t^{-\mu_n(1+|t|)} t^{\mu_n-\lambda_n} \sum_{j=0}^p c_{p,j} t^{-j} \phi^{(j)}\left(\frac{1}{t}\right) \right|,$$

and by the substitution $t \rightarrow t^{-1}$ this becomes

$$\begin{aligned} \tau_n(R\phi) &\leq \sup_{\substack{p \leq n \\ t \in G_\theta}} \sum_{j=0}^p |c_{p,j}| \times |t^{\lambda_n(1+|t|)} t^{\mu_n-\lambda_n} t^j \phi^{(j)}(t)| \\ &\leq M \tau_n(\phi) \end{aligned}$$

for some constant M , depending only on $c_{p,j}$ and n . It follows that $R\phi \in T^{\theta}(1-\mu, 1-\lambda)$ and moreover that R is continuous. Conversely, R considered as a map of $T^{\theta}(1-\mu, 1-\lambda)$ into $T^{\theta}(\lambda, \mu)$, is also seen to be continuous in the same manner as above. The fact that $R \cdot R = \text{id}$ is easy to prove. $\square$

1.3 MELLIN TRANSFORMS

The classical Mellin transform and its inverse are described by the pair of reciprocal formulas

$$(3.1) \quad (M\phi)(s) = \int_0^{\infty} t^{s-1} \phi(t) dt$$

$$(3.2) \quad (M^{-1}\phi)(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \phi(s) ds.$$

The function ϕ in (3.1) will be chosen in a space $T^{\theta}(\lambda, \mu)$ and it turns out that the image is a function in a space $S^{\theta}(\lambda, \mu)$. We firstly prove that in (3.1) it is possible to take in stead of the real axis another path of integration.

By the notation $\int_0^{\infty e^{i\theta}}$ we mean integration along a half line whose argument is θ .

THEOREM 6. Let $\lambda < \mu$. If $\phi \in T^{\theta}(\lambda, \mu)$, the Mellin transform of ϕ exists and

$$(3.3) \quad (M\phi)(s) = \int_0^{\infty e^{i\theta_1}} t^{s-1} \phi(t) dt$$

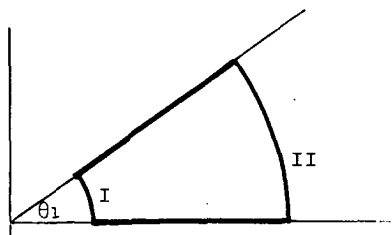
for any θ_1 for which $|\theta_1| \leq \theta$ and for any s with $\lambda < \text{Re } s < \mu$.

PROOF. Consider the contour in the figure. θ_1 is such that $-\theta \leq \theta_1 \leq \theta$. From

$$\phi(re^{i\xi}) = O(r^{-\lambda_n}) \text{ as } r \rightarrow 0$$

for any $\xi \in [-\theta, \theta]$ and $n \in \mathbb{N}$, it follows that

$$\int_I t^{s-1} \phi(t) dt = O(r^{\operatorname{Re} s - \lambda_n}) \text{ as } r \rightarrow 0.$$



For fixed s , $\lambda < \operatorname{Re} s < \mu$, there is $n \in \mathbb{N}$ such that $\lambda_n < \operatorname{Re} s$. We deduce

$$\int_I t^{s-1} \phi(t) dt \rightarrow 0 \text{ if } r \rightarrow 0.$$

In nearly the same way we prove that

$$\int_{II} t^{s-1} \phi(t) dt \rightarrow 0 \text{ if } R \rightarrow \infty.$$

The theorem now follows from Cauchy's theorem. □

The main tool for proving theorems in the next chapters is the fact that the Mellin transform provides an isomorphism between spaces $T^\theta(\lambda, \mu)$ and $S^\theta(\lambda, \mu)$. We state and prove the details in the following theorem.

THEOREM 7. Assume (1.1). Let $\theta \in \mathbb{R}$, $\theta \geq 0$. The Mellin transform M , defined by (3.3) is a topological isomorphism of $T^\theta(\lambda, \mu)$ onto $S^\theta(\lambda, \mu)$.

PROOF. Let $\phi \in T^\theta(\lambda, \mu)$ and $\Phi = M\phi$. By partial integration we obtain

$$(s)_p \Phi(s) = (-1)^p \int_0^\infty t^{s+p-1} \phi^{(p)}(t) dt, \quad \lambda < \operatorname{Re} s < \mu,$$

where $(s)_p = s(s+1)\dots(s+p-1)$.

from this we deduce, using a modified contour,

$$(3.4) \quad |e^{\theta|s}| (s)_p \phi(s) = \\ = \left| \int_0^{\infty} e^{i\theta_1} t^{\lambda_{n+1}} (1+|t|)^{\mu_{n+1}-\lambda_{n+1}} t^p \phi^{(p)}(t) \times \{t^{s-\lambda_{n+1}-1} (1+|t|)^{\lambda_{n+1}-\mu_{n+1}} e^{\theta|s|}\} dt \right|.$$

Assume $n \geq p$ and put $t = ue^{i\theta_1}$, $s = c+i\sigma$. Let $\lambda_n \leq c \leq \mu_n$.

The $\{...\}$ - part in the right hand side of (3.4) is in absolute value

$$u^{c-\lambda_{n+1}-1} (1+u)^{\lambda_{n+1}-\mu_{n+1}} e^{-\theta_1\sigma+\theta|c+i\sigma|}.$$

Consider the case $\sigma \geq 0$. We substitute $\theta_1 := \theta$ in (3.4) and obtain

$$(3.5) \quad |e^{\theta|s}| (s)_p \phi(s) < M_1 \tau_{n+1}^{\theta}(\phi),$$

uniformly on $\{s \in \mathbb{C} \mid \lambda_n \leq \operatorname{Re} s \leq \mu_n, \operatorname{Im} s \geq 0\}$.

Next consider the case $\sigma < 0$. Substituting now $\theta_1 := -\theta$, we deduce

(3.5) again, but now with some constant M_2 in stead of M_1 . Both M_1 and M_2 are independent of ϕ . Combining the results we have

$$\sigma_n^{\theta}(\phi) = \sup_{\substack{p \leq n \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |e^{\theta|s}| (s)_p \phi(s) < M \tau_{n+1}^{\theta}(\phi)$$

for some constant M which is independent of ϕ . From $(s)_p/s^p \rightarrow 1$ if $s \rightarrow \infty$ it follows that there exists $L > 0$ such that $|s^p| \leq 2|(s)_p|$ if $\lambda_n \leq \operatorname{Re} s \leq \mu_n$ and $|\operatorname{Im} s| > L$. Moreover there exists $M_0 > 2$ such that $|s^p| \leq M_0 (s)_0$, $p=1,2,\dots,n$ if $\lambda_n \leq \operatorname{Re} s \leq \mu_n$ and $|\operatorname{Im} s| \leq L$. Hence

$$\sigma_n^{\theta}(\phi) = \sup_{\substack{p \leq n \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |e^{\theta|s}| s^p \phi(s) < M_0 \sigma_n^{\theta}(\phi) < M_0 M \tau_{n+1}^{\theta}(\phi).$$

The continuity of M follows.

Conversely let $\phi \in S^\theta(\lambda, \mu)$ and $\phi = M^{-1}\phi$. Operating under the integral sign, which is allowed by absolute and uniform convergence of the resulting integral, we have

$$\phi^{(p)}(t) = \frac{(-1)^p}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s)_p \phi(s) t^{-s-p} ds, \quad |\arg t| \leq \theta.$$

It follows that

$$\begin{aligned} (3.6) \quad & |t^{\lambda_n(1+|t|)} \mu_n^{-\lambda_n} t^p \phi^{(p)}(t)| = \\ & = \frac{1}{2\pi} \left| \int_{c-i\infty}^{c+i\infty} e^{\theta|s|} (s)_p \phi(s) \{t^{-s+\lambda_n(1+|t|)} \mu_n^{-\lambda_n} e^{-\theta|s|}\} ds \right|. \end{aligned}$$

Put $t = ue^{i\theta_1}$, $s = c+i\sigma$. The absolute value of the $\{\dots\}$ -part of the integrand is

$$u^{-c+\lambda_n(1+u)} \mu_n^{-\lambda_n} e^{\theta_1\sigma-\theta|c+i\sigma|}.$$

$e^{\theta_1\sigma-\theta|c+i\sigma|}$ is uniformly bounded for $|\theta_1| \leq \theta$. Consider the case $0 < |t| \leq 1$. We substitute $c := \lambda_n$ in (3.6) in order to obtain

$$(3.7) \quad |t^{\lambda_n(1+|t|)} \mu_n^{-\lambda_n} t^p \phi^{(p)}(t)| \leq M_3 \sigma_{n+2}^\theta(\phi).$$

Here we have used the fact that there is a constant C such that

$$|e^{\theta|s|} (s)_p \phi(s)| < C |s|^{-2} \sigma_{n+2}^\theta(\phi);$$

we evade a threatening divergence of the integral because of the factor s^{-2} by bending the contour in such way that it does not pass through the origin.

In case of $|t| > 1$, put $c := \mu_n$, in order to get (3.7) again, but now M_3 replaced by M_4 . Both M_3 and M_4 are independent of ϕ . It follows that M^{-1} is continuous. □

REMARK

6. There is a simple relation between the Mellin transform M and the map R which is defined in (2.8):

$$(M \cdot R\phi)(s) = (M\phi)(1-s)$$

if $\phi \in T^\theta(\lambda, \mu)$.

1.4 BIBLIOGRAPHICAL NOTES

The function spaces $S^\theta(\lambda, \mu)$ and $T^\theta(\lambda, \mu)$ are extensions of the spaces $S(\lambda, \mu)$ and $T(\lambda, \mu)$ introduced in [4] by Braaksma and Schuitman. Locally convex vector spaces the elements of which have horizontal or vertical strips in the complex plane as domain of definition occur in the literature, however with different weight functions; see for example [60] and [66]. The notation of the norms on the spaces $T^\theta(\lambda, \mu)$ is related to Erdélyi, ([11]).

The imbedding theorem (theorem 3) is proved using well known techniques and is inspired by a theorem in [70], (cf §4, page 18). Basic ideas of many of the definitions and theorems in this chapter can be found in the monographs of Floret and Wloka ([19]), Wloka, ([70]), Köthe, ([33]) and Treves, ([68]). A short introduction with many examples to the manner in which we have defined the spaces $T^\theta(\lambda, \mu)$ is the expository paper of Wloka, ([71]).

The Mellin transform is a topological isomorphism of $T^\theta(\lambda, \mu)$ onto $S^\theta(\lambda, \mu)$ by theorem 7. Hence we could develop a theory for the Mellin transform in the distributional sense on the dual spaces.

The first paper on this subject is due to Fung Kang, ([23]) and his results are for example used by Perry, ([53]), to study Watson transforms in a manner similar to ours, (cf chapter 2). The paper of Fung Kang deals with modifications of Schwartz's spaces D' and S' , (cf [61]). Also Zemanian considers Mellin transforms, from a different point of view, of distributions, (cf [72]).

Finally we mention the fact that spaces of type $T^\theta(\lambda, \mu)$ also can be obtained as limits (in the projective sense) of spaces constructed by McBride, (cf [43] and [44]).

CHAPTER 2

WATSON TRANSFORMS

CONVOLUTION TRANSFORMS

2.0 INTRODUCTION

A transform $\phi \rightarrow A\phi$ where A is defined by

$$(0.1) \quad A\phi(x) = \int_0^{\infty} k(xt)\phi(t) dt$$

is called a Watson transform with kernel k . Many well known integral transforms are Watson transforms, e.g. Laplace transform, Hankel transform, Meijer transform and so on.

If in (0.1) we make the substitution $t \rightarrow 1/t$, we obtain in a formal way

$$A\phi(x) = \int_0^{\infty} k\left(\frac{x}{t}\right) \left[\frac{1}{t}\phi\left(\frac{1}{t}\right)\right] \frac{dt}{t}.$$

Hence, if the transform B is defined by

$$(0.2) \quad B\phi(x) = \int_0^{\infty} k\left(\frac{x}{t}\right)\phi(t) \frac{dt}{t}$$

we may write the transform A as the product $B \cdot R$, where the map R is the map defined in I, (2.8). From $R^{-1} = R$ it follows that we also have $B = A \cdot R$ and in this sense (0.1) and (0.2) may be seen as different forms of the same transform. However, several important transforms, e.g. fractional integration and Stieltjes transforms are of type (0.2). For this reason we will formulate theorems in the sequel for (0.1) as well as for (0.2). Proofs of related theorems may be derived from one another by means of the properties of R . The transform (0.2) is sometimes called a convolution transform. This

may be motivated as follows. The product convolution of two functions ϕ_1 and ϕ_2 is defined by

$$(0.3) \quad \int_0^x \phi_1\left(\frac{x}{t}\right) \phi_2(t) \frac{dt}{t}, \quad x > 0.$$

If ϕ_1 is defined to be zero in the interval $(0,1)$, the integral (0.3) may be written as

$$\int_0^{\infty} \phi_1\left(\frac{x}{t}\right) \phi_2(t) \frac{dt}{t}$$

which is the defining integral in (0.2) if $\phi_1 = k$ and $\phi_2 = \phi$.

In nearly all special cases of (0.1) and (0.2), e.g. the transforms already mentioned, the kernel k has a rather simple Mellin transform. Therefore we consider kernels k which are defined by

$$(0.4) \quad k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) t^{-s} ds.$$

We impose on K the condition of analyticity on some subset of $\mathbb{C}$, in most cases a vertical substrip of $\mathbb{C}$; furthermore K has to satisfy some special order relations if $s \rightarrow \infty$.

There is still another reason, even more important, why we choose (0.4) as the definition of the kernel. It is the relation which exists between Watson and Mellin transforms. From (0.1) we deduce at least in a formal way for the moment,

$$\begin{aligned} M \cdot A\phi(s) &= \int_0^{\infty} dx x^{s-1} \int_0^{\infty} k(xt) \phi(t) dt \\ &= \int_0^{\infty} dt \phi(t) t^{-s} \int_0^{\infty} k(u) u^{s-1} du \end{aligned}$$

by reversing the order of integration and a substitution. Using capitals to denote the Mellin transform this may be written as

$$(0.5) \quad \Psi(s) = K(s) \Phi(1-s), \quad A\phi = \psi.$$

The corresponding relation for (0.2) is even simpler:

$$(0.6) \quad \Psi(s) = K(s)\phi(s)$$

In the next section we will give these relations a precise meaning.

2.1 WATSON TRANSFORMS ON SPACES OF TYPE $T^\theta(\lambda, \mu)$

In this section we consider two types of order relations for the function K in the defining formula for the kernel of the Watson transform. The first theorem deals with an order relation of the form $K(s) = O(s^\delta)$ as $s \rightarrow \infty$, where δ is some real number.

THEOREM 8. Assume (I.1.1). Assume that the function K is analytic on $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$. Moreover let K satisfy the condition

$$(1.1) \quad \begin{cases} \text{for every } m \in \mathbb{N} \text{ there exists } \gamma_m \in \mathbb{R} \text{ such that} \\ K(s) = O(s^{\gamma_m}) \text{ as } s \rightarrow \infty, \\ \text{uniformly on } \{s \in \mathbb{C} \mid \lambda_m \leq \operatorname{Re} s \leq \mu_m\}. \end{cases}$$

Let $\lambda < c < \mu$, $n \in \mathbb{N}$ and

$$(1.2) \quad \frac{K(c+i\sigma)}{(1-c-i\sigma)_n} \in L(-\infty, \infty).$$

Define the function k_n by

$$(1.3) \quad k_n(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(1-s)_n} t^{n-s} ds, \quad t > 0.$$

Then the map A , defined by

$$(1.4) \quad (A\phi)(x) = \frac{d^n}{dx^n} \int_0^\infty t^{-n} k_n(xt) \phi(t) dt, \quad x > 0,$$

is a continuous map of $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$. (1.4) is equivalent to

$$(1.4)' \quad (A\phi)(x) = \frac{d^n}{dx^n} x^{n-1} \int_0^\infty u^{-n} k_n(u) \phi\left(\frac{u}{x}\right) du.$$

If $\theta > 0$, then (1.4)' defines a continuous map of $T^\theta(1-\mu, 1-\lambda)$ into $T^\theta(\lambda, \mu)$ and (1.4)' gives the analytic continuation of (1.4) to the sector G_θ . The map A satisfies the relation

$$(1.5) \quad A = M^{-1} \cdot m_1(K) \cdot M$$

where $m_1(K) : S^\theta(1-\mu, 1-\lambda) \rightarrow S^\theta(\lambda, \mu)$ is defined as in theorem 1' by

$$m_1(K)\phi(s) = K(s)\phi(1-s)$$

A is independent of the choice of n and c .

PROOF. Let $\phi \in T^\theta(1-\mu, 1-\lambda)$, $\psi = M\phi$. Then formally we have

$$\begin{aligned} (1.6) \quad (A\phi)(x) &= \frac{d^n}{dx^n} \int_0^\infty dt \, t^{-n} \phi(t) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(1-s)_n} (xt)^{n-s} ds \\ &= \frac{1}{2\pi i} \frac{d^n}{dx^n} \int_{c-i\infty}^{c+i\infty} ds \, x^{n-s} \frac{K(s)}{(1-s)_n} \int_0^\infty t^{-s} \phi(t) dt \\ &= \frac{1}{2\pi i} \frac{d^n}{dx^n} \int_{c-i\infty}^{c+i\infty} \frac{K(s)\phi(1-s)}{(1-s)_n} x^{n-s} ds. \end{aligned}$$

Hence

$$(1.7) \quad (A\phi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s)\phi(1-s) x^{-s} ds$$

The reversion of the order of integration is allowed if $x > 0$ because of the absolute convergence of the integrals. The transition from (1.6) to (1.7) is allowed by absolute and uniform convergence of the resulting integral. For $x > 0$ we therefore have (1.5) from (1.7).

Next consider the case of (1.4)'. Let $\arg x = \theta_1$, $|\theta_1| \leq \theta$.

Again in a formal way we obtain

$$\begin{aligned}
 (A\phi)(x) &= \frac{d^n}{dx^n} x^{n-1} \int_0^\infty du u^{-n} \phi\left(\frac{u}{x}\right) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(1-s)_n} u^{n-s} ds \\
 &= \frac{1}{2\pi i} \frac{d^n}{dx^n} x^{n-1} \int_{c-i\infty}^{c+i\infty} ds \frac{K(s)}{(1-s)_n} \int_0^\infty u^{-s} \phi\left(\frac{u}{x}\right) du \\
 (1.8) \quad &= \frac{1}{2\pi i} \frac{d^n}{dx^n} \int_{c-i\infty}^{c+i\infty} ds \frac{K(s)}{(1-s)_n} x^{n-s} \int_0^{\infty e^{-i\theta_1}} v^{-s} \phi(v) dv
 \end{aligned}$$

by the substitution $u = vx$. Hence, using theorem 6, we have

$$(1.7)' \quad (A\phi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) \phi(1-s) x^{-s} ds, \quad |\arg x| \leq \theta.$$

The reversion of the order of integration and the step from (1.8) to (1.7)' are motivated in the same manner as above. The integral in (1.7)' is absolutely convergent for any x for which $|\arg x| \leq \theta$. This follows at once from

$$K(s) \phi(1-s) x^{-s} = O(e^{|s|(\theta_1 - \theta)} s^{\gamma_m - p}) \text{ as } s \rightarrow \infty, \operatorname{Re} s = c,$$

for any $p \in \mathbb{N}$. Here m is such that $\lambda_m \leq c \leq \mu_m$. Moreover this integral is uniformly convergent on any compact subset of G_θ . Hence the integral in (1.7)' is analytic in x if $|\arg x| < \theta$ and is a continuous function of x if $|\arg x| \leq \theta$. So also in this case (1.5) makes sense. The continuity of A is now a consequence of the continuity of the factors in the representation (1.5). Moreover, (1.5) shows that A is independent of n and c . $\square$

REMARKS.

1. The numbers c and n , as indicated in the theorem, always exist

because of the order properties of K .

2. The kernel k_n may depend on the choice of c and n . If we have two cases which only differ in the value of c , say c_1 and c_2 , then the corresponding kernels $k_n^{(1)}$ and $k_n^{(2)}$ differ by a polynomial of degree $n-1$ at most. This immediately follows from (1.3). We see that this does lead to the same value of the right hand side of (1.6).

3. An equivalent definition of the map A in theorem 8 may be given as follows.

Let $n \in \mathbb{N}$; $\alpha_j \in \mathbb{C}$, $j=1, \dots, n$; $c \in (\lambda, \mu)$ such that

$$\frac{K(c+i\sigma)}{\prod_{j=1}^n (\alpha_j - c - i\sigma)} \in L(-\infty, \infty).$$

Define the function k_n^* and the operator D_n by

$$(1.3)^* \quad \begin{cases} k_n^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} [t^{-s} K(s) / \prod_{j=1}^n (\alpha_j - s)] ds, \quad t > 0; \\ D_n f(x) = \prod_{j=1}^n (x^{1-\alpha_j} \frac{d}{dx} x^{\alpha_j}) f(x) = \prod_{j=1}^n (\alpha_j + x \frac{d}{dx}) f(x), \quad f \in C^n. \end{cases}$$

(The operators in the right hand side of the last formula commute; so the order of the factors is immaterial).

Then (1.4) is equivalent to

$$(1.4)^* \quad (A\phi)(x) = D_n \int_0^\infty k_n^*(xt) \phi(t) dt$$

and (1.4)' is equivalent to

$$(1.4)^{**} \quad (A\phi)(x) = D_n \int_0^\infty k_n^*(u) \phi\left(\frac{u}{x}\right) \frac{du}{x}.$$

For example if $\alpha_j = p$ we get

$$k_n^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(p-s)_n} t^{-s} ds$$

$$(A\phi)(s) = x^{-p} \left(x \frac{d}{dx}\right)^n x^p \int_0^\infty k_n^*(xt) \phi(t) dt.$$

If we choose $\alpha_j = p+j$ we get

$$(1.3)^+ \quad k_n^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(1+p-s)_n} t^{-s} ds$$

$$(1.4)^+ \quad (A\phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p} \int_0^\infty k_n^*(xt) \phi(t) dt.$$

In the last formula, $p = 0$ gives another form for (1.4).

We next consider the case in which K is of negative exponential order. By this we mean an order relation of the form $K(s) = O(s^\delta e^{-\theta|s|})$, $s \rightarrow \infty$.

THEOREM 9. Assume (I.1.1). Let $\theta, \theta_0 \in \mathbb{R}$, $\theta_0 > 0$, $\theta \geq 0$.

Assume that the function K is analytic on $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$.

Let K satisfy the condition

$$(1.9) \quad \begin{cases} \text{for every } m \in \mathbb{N} \text{ there exists } \gamma_m \in \mathbb{R} \text{ such that} \\ K(s) = O(s^{\gamma_m} e^{-\theta_0|s|}) \text{ as } s \rightarrow \infty \\ \text{uniformly on } \{s \in \mathbb{C} \mid \lambda_m \leq \operatorname{Re} s \leq \mu_m\}. \end{cases}$$

Let k be defined by

$$(1.10) \quad k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) t^{-s} ds, \quad \lambda < c < \mu, \quad |\arg t| < \theta_0.$$

Then the map A defined by

$$(1.11) \quad (A\phi)(x) = \int_0^\infty k(xt) \phi(t) dt, \quad |\arg x| < \theta_0$$

is a continuous map of $T(1-\mu, 1-\lambda)$ into $T^{\theta_0}(\lambda, \mu)$. If $\theta > 0$ its restriction to $T^{\theta}(1-\mu, 1-\lambda)$ is a continuous map into $T^{\theta+\theta_0}(\lambda, \mu)$. In this case we have

$$(1.12) \quad (A\phi)(x) = x^{-1} \int_0^{\infty e^{i\theta_1}} k(u) \phi\left(\frac{u}{x}\right) du, \quad |\arg x| < \theta + \theta_0$$

for any θ_1 such that $|\theta_1| < \theta_0$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$.

Let $\lambda < c < \mu$ and $n \in \mathbb{N}$ be such that

$$(1.13) \quad \frac{K(c+i\sigma)}{(1-c-i\sigma)_n} e^{\theta_0 |\sigma|} \in L(-\infty, \infty).$$

Let k_n be defined by (1.3) if $|\arg t| \leq \theta_0$. Then we have

$$(1.12)' \quad (A\phi)(x) = \frac{d^n}{dx^n} x^{n-1} \int_0^{\infty e^{i\theta_1}} u^{-n} k_n(u) \phi\left(\frac{u}{x}\right) du, \quad |\arg x| \leq \theta + \theta_0,$$

if $\phi \in T^{\theta}(1-\mu, 1-\lambda)$, for any θ_1 such that

$$(1.14) \quad |\theta_1| \leq \theta_0, \quad \theta_1 - \theta \leq \arg x \leq \theta_1 + \theta.$$

In all cases (1.11), (1.12), (1.12)' we have the representation

$$(1.15) \quad A = M^{-1} \cdot m_1(K) \cdot M,$$

where $m_1(K)$ is the map defined in theorem 1'.

PROOF. From (1.9) it follows that $k(t)$ in (1.10) exists and is independent of the choice of c .

Repeating the first part of the proof of theorem 8 in case of $n=0$, we obtain for $(A\phi)(x)$ in (1.11)

$$(A\phi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) \phi(1-s) x^{-s} ds.$$

The integral is analytic in x if $|\arg x| < \theta_0$. It follows from $\phi \in S(1-\mu, 1-\lambda)$ that the integral converges absolutely if $|\arg x| \leq \theta_0$. The same holds for the derivatives of $(A\phi)(x)$. It follows that (1.15) holds. Hence the continuity of A .

Now consider the map A in (1.12)'. Let $\theta > 0$ and $\phi \in T^\theta(1-\mu, 1-\lambda)$. $k_n(u)$ and $\phi(\frac{u}{x})$ exist if $\arg u = \theta_1$ and (1.14) is satisfied.

We have

$$\begin{aligned} (A\phi)(x) &= \frac{d^n}{dx^n} x^{n-1} \int_0^{\infty e^{i\theta_1}} du u^{-n} \phi\left(\frac{u}{x}\right) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(1-s)_n} u^{n-s} ds \\ &= \frac{1}{2\pi i} \frac{d^n}{dx^n} \int_{c-i\infty}^{c+i\infty} ds \frac{K(s)}{(1-s)_n} x^{n-1} \int_0^{\infty e^{i\theta_1}} \phi\left(\frac{u}{x}\right) u^{-s} du. \end{aligned}$$

By the substitution $u := xt$ and using (1.14) we obtain

$$(1.16) \quad (A\phi)(x) = \frac{1}{2\pi i} \frac{d^n}{dx^n} \int_{c-i\infty}^{c+i\infty} ds \frac{K(s)}{(1-s)_n} x^{n-s} \int_C \phi(t) t^{-s} dt,$$

where C is the halfline $(0, \infty e^{i(\theta_1 - \arg x)})$. The reversion of the order of integration is allowed by absolute convergence of the integrals. From (1.16) we deduce, using theorem I.6,

$$(1.17) \quad (A\phi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) \phi(1-s) x^{-s} ds.$$

Since $\phi \in S^\theta(1-\mu, 1-\lambda)$, we may deduce from (1.13) that the last integral is absolutely convergent if $|\arg x| \leq \theta + \theta_0$ and is uniformly convergent on any compact subset of $G_{\theta+\theta_0}$. The same is true for the derivatives of $(A\phi)(x)$. It follows that the integral in (1.17) is an analytic function in x on $G_{\theta+\theta_0}$ and that this function and its derivatives can be extended continuously to $G_{\theta+\theta_0}$. (1.15) therefore also holds for this case which proves the continuity of A . From (1.15) we see that A in (1.12)' is the restriction of A in (1.11), defined on $T(1-\mu, 1-\lambda)$, to $T^\theta(1-\mu, 1-\lambda)$ and that its image is a subspace of $T^{\theta+\theta_0}(\lambda, \mu)$.

The relation (1.12) may be proven in the same manner. □

REMARKS

4. It should be noted that if $|\arg x| < \theta + \theta_0$ there exists θ_1 such that the conditions for the validity of (1.12) are satisfied. Indeed, it suffices to choose θ_1 such that

$$\max(-\theta_0, -\theta + \arg x) < \theta_1 < \min(\theta_0, \theta + \arg x)$$

and this is possible since the left hand side is less than the right hand side.

5. Analogous to remark 3 we may replace (1.12)' by

$$(1.12)^* \quad (A\phi)(x) = D_n \int_0^{\infty} e^{i\theta_1} k_n^*(u) \phi\left(\frac{u}{x}\right) \frac{du}{x}, \quad |\arg x| \leq \theta + \theta_0,$$

where D_n and k_n^* are defined as in remark 3 under the condition

$$(1.13)^* \quad \frac{K(c+i\sigma)}{\prod_{j=1}^n (\alpha_j - c - i\sigma)} e^{\theta_0 |\sigma|} \in L(-\infty, \infty)$$

As a special case we obtain

$$(1.12)^{**} \quad (A\phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p-1} \int_0^{\infty} e^{i\theta_1} k_n^*(u) \phi\left(\frac{u}{x}\right) du$$

if we choose $\alpha_j = p+j$.

2.2 INVERSION THEOREMS FOR WATSON TRANSFORMS

The proofs of the theorems in section 1 rest upon the fact that a map $m_1(K)$ defined on a space of type $S^\theta(\lambda, \mu)$ may be lifted to a Watson transform on a space of type $T^\theta(\lambda, \mu)$. In fact we used Mellin transforms to make the transition of $m_1(K)$ to A .

It is clear that if $m_1(K)$ is an isomorphism, then also A will be an isomorphism. This follows at once from theorem 7 where it was proved

that M is an isomorphism of $T^\theta(\lambda, \mu)$ onto $S^\theta(\lambda, \mu)$. In this case the inverse A^{-1} depends on the properties of $\{m_1(K)\}^{-1}$ and so it depends in fact on the behaviour of $\{K(s)\}^{-1}$. As will be made clear in the following, $m_1(K)$ is an isomorphism if $\{K(s)\}^{-1}$ is analytic on some vertical substrip of ϕ and satisfies order relations of the same type as those for K .

We first consider the inversion of the map A described in theorem 8.

THEOREM 10. Let the assumptions of theorem 8 be satisfied. Assume that K does not have zeros in the strip $\{s \in \phi \mid \lambda < \operatorname{Re} s < \mu\}$. Define the function H by

$$(2.1) \quad H(s) = \frac{1}{K(1-s)}, \quad s \in \{s \in \phi \mid 1-\mu < \operatorname{Re} s < 1-\lambda\}.$$

Assume that H satisfies the following condition:

$$(2.2) \quad \left\{ \begin{array}{l} \text{for every } m \in \mathbb{N} \text{ there exists } \delta_m \in \mathbb{R} \text{ such that} \\ H(s) = O(s^{\delta_m}) \text{ as } s \rightarrow \infty, \\ \text{uniformly on } \{s \in \phi \mid 1-\mu_m \leq \operatorname{Re} s \leq 1-\lambda_m\}. \end{array} \right.$$

Then the map A , defined by (1.4) and (1.4)' is an isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T^\theta(\lambda, \mu)$, $\theta \geq 0$. The inverse B possesses the following representation:

Let $\ell \in \mathbb{N}$, $1-\mu < d < 1-\lambda$ such that

$$(2.3) \quad \frac{H(d+i\sigma)}{(1-d-i\sigma)_\ell} \in L(-\infty, \infty).$$

Define $h_\ell(t)$ by

$$(2.4) \quad h_\ell(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{H(s)}{(1-s)_\ell} t^{\ell-s} ds \quad \text{if } t > 0.$$

Then if $\psi \in T(\lambda, \mu)$,

$$(2.5) \quad (B\psi)(x) = \frac{d^\ell}{dx^\ell} \int_0^\infty t^{-\ell} h_\ell(xt) \psi(t) dt, \quad x > 0.$$

If $\theta \geq 0$, $\psi \in T^\theta(\lambda, \mu)$, then

$$(2.5)' \quad (B\psi)(t) = \frac{d^\ell}{dx^\ell} x^{\ell-1} \int_0^\infty u^{-\ell} h_\ell(u) \psi\left(\frac{u}{x}\right) du, \quad |\arg x| \leq \theta.$$

PROOF. Since

$$A = M^{-1} \cdot m_1(K) \cdot M$$

we have

$$B = M^{-1} \cdot m_1(H) \cdot M.$$

The representations (2.5) and (2.5)' follow as in theorem 8. □

REMARK.

6. Analogous to remark 3 we define the operator D_ℓ by

$$D_\ell = \prod_{j=1}^{\ell} (x^{1-\beta_j} \frac{d}{dx} x^{\beta_j}), \quad \beta_j \in \phi.$$

We assume that H satisfies

$$H(d+i\sigma) / \prod_{j=1}^{\ell} (\beta_j - d - i\sigma) \in L(-\infty, \infty),$$

for some $d \in (1-\mu, 1-\lambda)$. We then define h_ℓ^* by

$$(2.4)^* \quad h_\ell^*(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} [t^{-s} H(s) / \prod_{j=1}^{\ell} (\beta_j - s)] ds, \quad t > 0.$$

Then (2.5) and (2.5)' respectively are equivalent to

$$(2.5)^* \quad (B\psi)(x) = D_\ell \int_0^\infty h_\ell^*(xt) \psi(t) dt$$

$$(2.5)^{**} \quad (B\psi)(x) = D_\ell \int_0^\infty h_\ell^*(u) \psi\left(\frac{u}{x}\right) \frac{du}{x}.$$

As special case we obtain putting for example $\beta_j = r+j$ in (2.5)**:

$$(2.5)^+ \quad (B\psi)(x) = x^{-r} \frac{d^\ell}{dx^\ell} x^{\ell+r-1} \int_0^\infty h_\ell^*(u) \psi\left(\frac{u}{x}\right) du$$

In the next theorem we consider the inversion of the map A defined in theorem 9. Here actually we may have that $\{K(s)\}^{-1}$ is of positive exponential order.

THEOREM 11. Let the assumptions of theorem 9 be satisfied. Assume that the function H defined by (2.1) is analytic in the strip $\{s \in \mathbb{C} \mid 1-\mu < \operatorname{Re} s < 1-\lambda\}$ and that H satisfies the following condition:

$$(2.6) \quad \begin{cases} \text{for every } m \in \mathbb{N} \text{ there exists } \delta_m \in \mathbb{R} \text{ such that} \\ H(s) = O(s^{\delta_m} e^{\theta_0 |s|}) \text{ as } s \rightarrow \infty, \\ \text{uniformly on } \{s \in \mathbb{C} \mid 1-\mu_m \leq \operatorname{Re} s \leq 1-\lambda_m\}. \end{cases}$$

Then the map A of theorem 9 is an isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T^{\theta+\theta_0}(\lambda, \mu)$ for any $\theta \geq 0$. The inverse B may be constructed as follows. Let $d \in (1-\mu, 1-\lambda)$ and let H_+ and H_- be functions defined on $\operatorname{Re} s = d$, such that

$$(2.7) \quad H_+(s) + H_-(s) = H(s) \text{ if } \operatorname{Re} s = d$$

and such that there exist $\ell \in \mathbb{N}$, $\alpha_+ \geq \theta_0$, $\alpha_- \geq \theta_0$ with the property

$$(2.8) \quad \frac{e^{\gamma \sigma} H_\pm(d+i\sigma)}{(1-d-i\sigma)_\ell} \in L(-\infty, \infty) \text{ if } \theta_0 \leq \pm \gamma \leq \alpha_\pm;$$

(here and in the following the upper and lower signs belong together).

Such functions exist if (2.6) is satisfied. Define

$$(2.9) \quad h_{\pm\ell}(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{H_{\pm}(s)}{(1-s)_{\ell}} t^{\ell-s} ds$$

if

$$(2.10) \quad \theta_0 \leq \pm \arg t \leq \alpha_{\pm}.$$

Then B may be characterized by:

if $\psi \in T^{\theta+\theta_0}(\lambda, \mu)$ and $|\arg x| \leq \theta$, then

$$(2.11) \quad (B\psi)(x) = \frac{d}{dx} \int_0^{\infty} h_{+\ell}(xt) t^{-\ell} \psi(t) dt + \int_0^{\infty} h_{-\ell}(xt) t^{-\ell} \psi(t) dt$$

for any $\gamma_{\pm} \in \mathbb{R}$ such that

$$(2.12) \quad \theta_0 \leq \pm(\gamma_{\pm} + \arg x) \leq \min(\alpha_{\pm}, \theta_0 + \theta \pm \arg x).$$

If $\alpha_{\pm} > \theta_0$, $\theta > 0$, $|\arg x| < \theta$, $|\gamma_{\pm}| \leq \theta + \theta_0$ and $\theta_0 < \pm(\gamma_{\pm} + \arg x) < \alpha_{\pm}$, then (2.11) holds with $\ell=0$; this gives another characterization of B on $T^{\theta+\theta_0}(\lambda, \mu)$.

PROOF. Using (1.15) we easily see that the inverse of A exists and is given by

$$(2.13) \quad B = M^{-1} \cdot m_1(H) \cdot M,$$

which is a continuous map of $T^{\theta+\theta_0}(\lambda, \mu)$ into $T^{\theta}(1-\mu, 1-\lambda)$, (cf theorem 2).

Assume there exist $H_{\pm}$ satisfying (2.7) and (2.8). If $\psi \in T^{\theta+\theta_0}(\lambda, \mu)$, $|\arg x| \leq \theta$ and (2.12) holds, then

$$\int_0^{\infty} h_{\pm\ell}(xt) t^{-\ell} \psi(t) dt = \frac{1}{2\pi i} \int_0^{\infty} dt \int_{d-i\infty}^{d+i\infty} \frac{H_{\pm}(s)}{(1-s)_{\ell}} x^{\ell-s} t^{-s} \psi(t) ds.$$

From the properties of $H_{\pm}$ and ψ it follows that we may change the order of integration. Thus we obtain for the right hand side

$$\begin{aligned} & \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} ds x^{\ell-s} \frac{H_{\pm}(s)}{(1-s)_{\ell}} \int_0^{\infty} e^{i\gamma_{\pm} t} t^{-s} \psi(t) dt \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} ds x^{\ell-s} \frac{H_{\pm}(s)}{(1-s)_{\ell}} \Psi(1-s); \end{aligned}$$

(in the last part we used theorem I.6). Substituting this result in the right hand side of (2.11) we get

$$\begin{aligned} & \frac{d^{\ell}}{dx^{\ell}} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} ds x^{\ell-s} \frac{H(s)}{(1-s)_{\ell}} \Psi(1-s) \\ &= (M^{-1} \cdot m_1(H) \cdot M\psi)(x) = (B\psi)(x) \end{aligned}$$

in view of (2.13). Hence we have proved (2.11).

If $\theta > 0$, $\alpha_{\pm} > \theta_0$ and $|\arg x| < \theta$, there exists $\gamma_{\pm}$ such that $|\gamma_{\pm}| \leq \theta + \theta_0$, $\theta_0 < \pm(\gamma_{\pm} + \arg x) < \alpha_{\pm}$. Now we may take $\ell = 0$ in (2.11). It is easy to see that for any x with $|\arg x| \leq \theta$ there exist $\gamma_{\pm}$ such that (2.12) holds. The proof of the existence of $H_{\pm}$ such that (2.7) and (2.8) hold will be given in corollary 1 to lemma 1. $\square$

REMARK

7. An equivalent description of B may be given analogous to remark 3.

Let β_j , $1 \leq j \leq \ell$ and D_{ℓ} be defined as in remark 6. Assume that $d \in (1-\mu, 1-\lambda)$ is such that

$$(2.8)^* \quad e^{\gamma\sigma} H_{\pm}(d+i\sigma) / \prod_{j=1}^{\ell} (\beta_j - d - i\sigma) \in L(-\infty, \infty) \quad \text{if } \theta_0 \leq \pm\gamma \leq \alpha_{\pm}.$$

Define $h_{\pm\ell}^*$ by

$$(2.9)^* \quad h_{\pm\ell}^*(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} [H_{\pm}(s) t^{-s} / \prod_{j=1}^{\ell} (\beta_j - s)] ds, \quad \theta_0 \leq \pm \arg t \leq \alpha_{\pm}.$$

Then (2.11) is equivalent to

$$(2.11)^* \quad (B\psi)(x) = D_\ell \left[\int_0^{\infty e^{i\gamma_+}} h_{+\ell}^*(xt) \psi(t) dt + \int_0^{\infty e^{i\gamma_-}} h_{-\ell}^*(xt) \psi(t) dt \right]$$

In the same manner as in remark 6 we obtain as a special case for $\beta_j = r+j$, $j=1, \dots, \ell$,

$$(2.11)^{**} \quad (B\psi)(x) = x^{-r} \frac{d^\ell}{dx^\ell} x^{\ell+r} \left[\int_0^{\infty e^{i\gamma_+}} h_{+\ell}^*(xt) \psi(t) dt + \int_0^{\infty e^{i\gamma_-}} h_{-\ell}^*(xt) \psi(t) dt \right].$$

In (2.11)** we may put $\ell=0$ if $\alpha_\pm > \theta_0$, $\theta_0 > 0$, $|\arg x| < \theta$, $\theta_0 < \pm(\gamma_\pm + \arg x) < \alpha_\pm$ and $|\gamma_\pm| \leq \theta + \theta_0$.

The existence of the functions H_+ and H_- is guaranteed by the assumption that H is analytic and satisfies (2.6). In order to prove this we use the following lemma.

LEMMA 1. Let λ, μ, p be real numbers, $p > 0$, $\lambda < \mu$. Then there exist analytic functions χ_1 and χ_2 defined on $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$ such that

$$\chi_1 + \chi_2 = 1,$$

$$\chi_1(s) = O(e^{-p \operatorname{Im} s}) \text{ as } \operatorname{Im} s \rightarrow \infty \text{ on } S,$$

$$\chi_2(s) = O(e^{p \operatorname{Im} s}) \text{ as } \operatorname{Im} s \rightarrow -\infty \text{ on } S.$$

PROOF. If $p(\mu - \lambda) < 2\pi$, we may choose

$$\chi_1(s) = [1 + e^{-pi(s-v)}]^{-1}, \quad \chi_2(s) = [1 + e^{pi(s-v)}]^{-1}$$

with $v = \frac{1}{2}(\lambda + \mu)$. Then χ_1 and χ_2 satisfy the requirements.

Now consider the general case. Choose $q > p$ and consider $1 + e^{qis}$.

This function has zeros $s_1, \dots, s_k$ in S . We may choose now

$$\chi_j(s) = \frac{1}{k!} \frac{d^k}{ds^k} [(1+e^{\pm q_1 s})^{-1} \prod_{h=1}^k (s-s_h)], \quad j = 1, 2,$$

where the upper and lower sign are to be chosen if $j=1$ and $j=2$ respectively. It is easily verified that these functions satisfy the requirements. $\square$

COROLLARY 1. Suppose H is analytic on $\{s \in \mathbb{C} \mid 1-\mu < \operatorname{Re} s < 1-\lambda\}$ and H satisfies (2.6). Then there exist H_+ , H_- , d which satisfy the conditions of theorem 11.

Given any closed subinterval $[d_1, d_2] \subset (1-\mu, 1-\lambda)$ with $d_2 - d_1 < 1$ and $\rho \geq 0$, there exist $r \in \mathbb{C}$, $\ell \in \mathbb{N}$, H_+ and H_- such that $H_{\pm}(s)$ and $H_{\pm}(s)/(1+r-s)_{\ell}$ are analytic on $\{s \in \mathbb{C} \mid d_1 \leq \operatorname{Re} s \leq d_2\}$, and for some $\varepsilon > 0$:

$$(2.14) \quad \begin{cases} H_+(s) = O(s^{\ell-1-\varepsilon} e^{i(\rho+\theta_0)s}), \\ H_-(s) = O(s^{\ell-1-\varepsilon} e^{-i\theta_0 s}). \end{cases}$$

as $\operatorname{Im} s \rightarrow \infty$ on $\{s \in \mathbb{C} \mid d_1 \leq \operatorname{Re} s \leq d_2\}$ and

$$(2.15) \quad \begin{cases} H_+(s) = O(s^{\ell-1-\varepsilon} e^{i\theta_0 s}), \\ H_-(s) = O(s^{\ell-1-\varepsilon} e^{-i(\rho+\theta_0)s}). \end{cases}$$

as $\operatorname{Im} s \rightarrow -\infty$ on $\{s \in \mathbb{C} \mid d_1 \leq \operatorname{Re} s \leq d_2\}$.

Then (2.8)* is satisfied for any $d \in [d_1, d_2]$, $\alpha_{\pm} = \rho + \theta_0$ and $\beta_j = r + j$.

PROOF. Apply lemma 1 with $p = 2\theta_0 + \rho$, $\rho \geq 0$, and with λ and μ replaced by d_1 and d_2 . Then choose $H_+ = \chi_1 H$ and $H_- = \chi_2 H$. It is not hard to see that now (2.14) and (2.15) hold for some $\ell \in \mathbb{N}$. This implies (2.8)* for any $d \in [d_1, d_2]$, $\alpha_{\pm} = \rho + \theta_0$ and $\beta_j = r + j$. $\square$

EXAMPLE. Let $v \geq 0$ and choose d such that $\frac{(\theta_0 + v)d}{\pi} \notin \mathbb{Z}$,

$$H_{\pm}(s) = \frac{\pm e^{\pm(\theta_0 + v)s}}{2i \sin(\theta_0 + v)s} H(s).$$

Then (2.7) and (2.8) hold with $\alpha_{\pm} = \theta_0 + 2\nu$. If $\nu = \theta$, we may choose $\gamma_{\pm} = \pm(\theta_0 + \theta)$ in (2.11).

REMARK.

8. The method of theorem 11 may also be used to describe the analytic continuation of $A\phi$, as defined by (1.4), theorem 8, with $\phi \in T^{\theta}(1-\mu, 1-\lambda)$. In stead of (1.4)' one may use

$$(2.17) \quad (A\phi)(x) = \frac{d^n}{dx^n} \left[\int_0^{\infty} e^{i\gamma_+} k_{+\ell}(xt) t^{-n} \phi(t) dt + \int_0^{\infty} e^{i\gamma_-} k_{-\ell}(xt) t^n \phi(t) dt \right],$$

where $k_{\pm\ell}$ are defined by (2.9) with the assumptions (2.7) and (2.8), however H replaced by K , h by k , ℓ by n and d by c , $c \in (\lambda, \mu)$.

Then (2.17) holds if (2.12) is satisfied with $\theta_0 = 0$.

A similar remark applies to theorem 9.

2.3 CONVOLUTION TRANSFORMS ON $T^{\theta}(\lambda, \mu)$ AND INVERSION THEOREMS

As already pointed out in the introduction of this chapter, the convolution transform may be obtained from the Watson transform by a simple substitution. In this section we give the theorems which are analogous to the theorems in the preceding chapter. The proofs may be given in nearly the same manner or follow from those of section 2.2 using the map R of chapter I, section 1.2. For this reason we will omit the proofs or only will give short sketches.

THEOREM 12. Assume the conditions of theorem 8 are satisfied. Then the map C , defined by

$$(C\phi)(x) = \frac{d^n}{dx^n} \int_0^{\infty} t^n k_n\left(\frac{x}{t}\right) \phi(t) \frac{dt}{t}, \quad x > 0$$

is continuous from $T(\lambda, \mu)$ into itself. (3.1) is equivalent to

$$(3.1)' \quad (C\phi)(x) = \frac{d^n}{dx^n} x^n \int_0^\infty u^{-n} k_n(u) \phi\left(\frac{x}{u}\right) \frac{du}{u}.$$

If $\theta > 0$, then (3.1)' defines a continuous map of $T^\theta(\lambda, \mu)$ into itself. In this case (3.1)' gives the analytic continuation of $C\phi$, defined in (3.1), to G_θ . The map C satisfies the relation

$$(3.2) \quad C = M^{-1} \cdot m(K) \cdot M,$$

where $m(K)$ is defined in theorem I.1. C is independent of c and n .

PROOF. A direct proof may be given using the techniques of the proof of theorem 8. Then we obtain

$$(C\phi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) \phi(s) x^{-s} ds$$

which is valid for (3.1) and (3.1)'. Hence (3.2) and the continuity follows.

We may also note that $C = A \cdot R$, where R is defined by (I.2.8). Then theorem 12 follows at once. We also have $C = R \cdot \tilde{A}$ if $\tilde{A}$ is the map of theorem 8 with (λ, μ) and $K(s)$ replaced by $(1-\mu, 1-\lambda)$ and $K(1-s)$ respectively.

REMARK

9. Let D_n and k_n^* be defined as in remark 3. For (1.4)* we write $(A\phi)(x) = (D_n A_n^* \phi)(x)$. We then have $C = A \cdot R = D_n \cdot A^* \cdot R$, from which it follows at once that

$$(3.1)^* \quad (C\phi)(x) = D_n \int_0^\infty k_n^*\left(\frac{x}{t}\right) \phi(t) \frac{dt}{t}$$

is equivalent to (3.1).

In the same manner we deduce the equivalence of (3.1)' and

$$(3.1)** \quad (C\phi)(x) = D_n \int_0^\infty k_n^*(u) \phi\left(\frac{x}{u}\right) \frac{du}{u}.$$

Special cases are obtained as in remark 3. Choosing $\alpha_j = p$, $j = 1, \dots, n$, we have

$$(C\phi)(x) = x^{-p} \left(x \frac{d}{dx}\right)^n x^p \int_0^\infty k_n^*\left(\frac{x}{t}\right) \phi(t) \frac{dt}{t},$$

where k_n^* is now defined by

$$k_n^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(p-s)^n} t^{-s} ds.$$

If we choose $\alpha_j = p+j$ then

$$(1.3)^+ \quad k_n^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(1+p-s)_n} t^{-s} ds$$

and

$$(3.1)^+ \quad (C\phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p} \int_0^\infty k_n^*(u) \phi\left(\frac{x}{u}\right) \frac{du}{u}.$$

Analogous to theorem 9 we next consider the case in which K is of negative exponential order.

THEOREM 13. Let the assumptions of theorem 9 be fulfilled. Then the map C , defined by

$$(3.3) \quad (C\phi)(x) = \int_0^\infty k\left(\frac{x}{t}\right) \phi(t) \frac{dt}{t}, \quad |\arg x| < \theta_0,$$

is continuous from $T(\lambda, \mu)$ into $T^{\theta_0}(\lambda, \mu)$. If $\theta > 0$, its restriction to $T^\theta(\lambda, \mu)$ is a continuous map into $T^{\theta+\theta_0}(\lambda, \mu)$. In this case we have

$$(3.4) \quad (C\phi)(x) = \int_0^{\infty e^{i\theta_1}} k(u) \phi\left(\frac{x}{u}\right) \frac{du}{u}, \quad |\arg x| < \theta + \theta_0$$

for any θ_1 such that $|\theta_1| < \theta_0$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$. If (1.13) holds for some $c \in (\lambda, \mu)$ and $n \in \mathbb{N}$, define k_n by (1.3) if $|\arg t| \leq \theta_0$. Then

$$(3.5) \quad (C\phi)(x) = \frac{d^n}{dx^n} x^n \int_0^{\infty e^{i\theta_1}} u^{-n} k_n(u) \phi\left(\frac{x}{u}\right) \frac{du}{u}, \quad |\arg x| \leq \theta + \theta_1,$$

if $\phi \in T^\theta(\lambda, \mu)$, for any θ_1 satisfying (1.14). C satisfies (3.2). $\square$

REMARKS

10. The existence of θ_1 satisfying the conditions for the validity of (3.4) and (3.5) follows from remark 4.

11. Analogous to remark 5 we now have as equivalent of (3.5)

$$(3.5)^* \quad (C\phi)(x) = D_n \int_0^{\infty e^{i\theta_1}} k_n^*(u) \phi\left(\frac{x}{u}\right) \frac{du}{u},$$

if $|\arg x| \leq \theta + \theta_0$. Here k_n^* is defined as in remark 5.

If we put $\alpha_j = p+j$ we have

$$k_n^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(1+p-s)_n} ds,$$

$$(3.5)^+ \quad (C\phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p} \int_0^{\infty e^{i\theta_1}} k_n^*(u) \phi\left(\frac{x}{u}\right) \frac{du}{u},$$

if $|\arg x| \leq \theta + \theta_0$ and $|\theta_1| \leq \theta_0$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$.

The following theorem gives the inverse of the map of theorem 12.

THEOREM 14. Let the assumptions of theorem 12 be satisfied. Assume that K does not have zeros in the strip $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$. Define the function H by $H(s) = (K(s))^{-1}$ if $s \in S$. Assume that H satisfies the condition

$$\left\{ \begin{array}{l} \text{for any } m \in \mathbb{N} \text{ there exists } \delta_m \in \mathbb{R} \text{ such that} \\ H(s) = O(s^{\delta_m}) \text{ as } s \rightarrow \infty \\ \text{uniformly on } \{s \in \mathbb{C} \mid \lambda_m \leq \operatorname{Re} s \leq \mu_m\}. \end{array} \right.$$

Then the map C defined in theorem 12 is an automorphism of $T^\theta(\lambda, \mu)$ for any $\theta \geq 0$. Its inverse D may be represented as follows. Let $\ell \in \mathbb{N}$, $\lambda < d < \mu$ be such that (2.3) holds. Define h_ℓ by (2.4). Then if $\psi \in T^\theta(\lambda, \mu)$,

$$(3.6) \quad (D\psi)(x) = \frac{d^\ell}{dx^\ell} \int_0^\infty t^\ell h_\ell\left(\frac{x}{t}\right) \psi(t) \frac{dt}{t} \quad \text{if } x > 0$$

and by

$$(3.7) \quad (D\psi)(x) = \frac{d^\ell}{dx^\ell} x^\ell \int_0^\infty u^{-\ell} h_\ell(u) \psi\left(\frac{x}{u}\right) \frac{du}{u} \quad \text{if } |\arg x| \leq \theta.$$

PROOF. In the proof of theorem 12 we observed $C = R \cdot \tilde{A}$. Hence $C^{-1} = \tilde{A}^{-1} \cdot R^{-1} = \tilde{B} \cdot R$, where $\tilde{B}$ is the map of theorem 10 with (λ, μ) and $K(s)$ replaced by $(1-\mu, 1-\lambda)$ and $K(1-s)$. $\square$

REMARK

12. Analogous to remark 6 the formulas (3.6) and (3.7) are equivalent to

$$(3.6)^* \quad (D\psi)(x) = D_\ell \int_0^\infty h_\ell^*\left(\frac{x}{t}\right) \psi(t) \frac{dt}{t}, \quad x > 0,$$

$$(3.7)^* \quad (D\psi)(x) = D_\ell \int_0^\infty h_\ell^*(u) \psi\left(\frac{x}{u}\right) \frac{du}{u}, \quad |\arg x| \leq \theta,$$

where D_ℓ and h_ℓ^* are defined as in remark 6, except that now $d \in (\lambda, \mu)$ instead of $d \in (1-\mu, 1-\lambda)$.

Again a special case is obtained by choosing $\beta_j = j+r$ as in remark 6. We have

$$h_{\ell}^*(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{H(s)}{(1+r-s)_{\ell}} t^{-s} ds, \quad t > 0,$$

$$(3.7)^+ \quad (D\psi)(x) = x^{-r} \frac{d^{\ell}}{dx^{\ell}} x^{\ell+r} \int_0^{\infty} h_{\ell}^*(u) \psi\left(\frac{x}{u}\right) \frac{du}{u}.$$

Our last theorem in this section gives the construction of the inverse of the map of theorem 13.

THEOREM 15. Let the assumptions of theorem 13 be satisfied. Assume that K does not have zeros in the strip $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$.

Define $H(s) = 1/K(s)$ if $s \in S$. Assume that H satisfies the condition

$$\left\{ \begin{array}{l} \text{For every } m \in \mathbb{N} \text{ there exists } \delta_m \in \mathbb{R} \text{ such that} \\ H(s) = O(s^{\delta_m} e^{\theta_0 |s|}) \text{ as } s \rightarrow \infty, \\ \text{uniformly on } \{s \in \mathbb{C} \mid \lambda_m \leq \operatorname{Re} s \leq \mu_m\}. \end{array} \right.$$

Then the map C of theorem 13 is an isomorphism of $T^{\theta}(\lambda, \mu)$ onto $T^{\theta+\theta_0}(\lambda, \mu)$ for any $\theta \geq 0$. The inverse D of C may be constructed as follows:

let $d \in (\lambda, \mu)$ and let H_+ and H_- be functions defined on $\operatorname{Re} s = d$ such that (2.7) and (2.8) hold as in theorem 11. Let $h_{\pm\ell}$ be defined by (2.9) if (2.10) holds. Then D may be characterized by: if $\psi \in T^{\theta+\theta_0}(\lambda, \mu)$ and $|\arg x| \leq \theta$ then

$$(3.8) \quad (D\psi)(x) = \frac{d^{\ell}}{dx^{\ell}} \left[\int_0^{\infty} e^{-i\gamma_+} h_{+\ell}\left(\frac{x}{t}\right) t^{\ell} \psi(t) \frac{dt}{t} + \int_0^{\infty} e^{-i\gamma_-} h_{-\ell}\left(\frac{x}{t}\right) t^{\ell} \psi(t) \frac{dt}{t} \right]$$

with $\gamma_{\pm} \in \mathbb{R}$ such that (2.12) holds.

If $\theta > 0$, $\alpha_{\pm} > \theta_0$ we may characterize D on $T^{\theta+\theta_0}(\lambda, \mu)$ by (3.8) with $\ell = 0$, $|\arg x| < \theta$ and any $\gamma_{\pm}$ with $\theta_0 < \pm(\gamma_{\pm} + \arg x) < \alpha_{\pm}$, $|\gamma_{\pm}| \leq \theta + \theta_0$. Now $h_{\pm 0}$ is defined by (2.9) if $\theta_0 < \pm \arg t < \alpha_{\pm}$.

REMARKS

13. Analogous to remark 7 here (3.8) is equivalent to

$$(3.8)^* \quad (D\psi)(x) = D_\ell \left[\int_0^{\infty} e^{-i\gamma_+} h_{+\ell}^* \left(\frac{x}{t} \right) \psi(t) \frac{dt}{t} + \int_0^{\infty} e^{-i\gamma_-} h_{-\ell}^* \left(\frac{x}{t} \right) \psi(t) \frac{dt}{t} \right],$$

where $h_{\pm\ell}^*$ and D_ℓ are defined as in remark 7.

If $\beta_j = j+r$ we have

$$(3.8)^+ \quad (D\psi)(x) = x^{-r} \frac{d^\ell}{dx^\ell} x^{\ell+r} \left[\int_0^{\infty} e^{-i\gamma_+} h_{+\ell}^* \left(\frac{x}{t} \right) \psi(t) \frac{dt}{t} + \int_0^{\infty} e^{-i\gamma_-} h_{-\ell}^* \left(\frac{x}{t} \right) \psi(t) \frac{dt}{t} \right],$$

where the kernels $h_{\pm\ell}^*$ are defined by

$$h_{\pm\ell}^*(t) = \frac{1}{2\pi i} \int_{-i\infty}^{d+i\infty} \frac{H_\pm(s)}{(1+r-s)_\ell} t^{-s} ds, \text{ if } \theta_0 \leq \pm \arg t \leq \alpha_\pm.$$

14. The existence of the functions $H_\pm$ in theorem 15 and in remark 7 follows from the corollary of lemma 1.

15. With necessary alteration of details remark 8 also applies to the map B of this section.

2.4 EXAMPLES AND APPLICATIONS

In this section we consider some rather simple applications of the foregoing theory. Some of these examples will be used in the next chapters.

First we apply theorem 9 to the Laplace transform. It turns out that the well known inversion formula for this transform may be derived with the inversion theorem 11. Moreover we also prove some interesting other inversion formula's for the Laplace transform.

EXAMPLE I.

In theorem 9 let $K(s) = \Gamma(s)$. $\Gamma(s)$ satisfies (1.9) with $\theta_0 = \frac{1}{2}\pi$ and (1.10) gives $k(t) = e^{-t}$ if $0 \leq \lambda < \mu \leq \infty$. In this case we have: the Laplace transform L ,

$$(L\phi)(x) = \int_0^{\infty} e^{-xt} \phi(t) dt$$

is a continuous map of $T^{\theta}(1-\mu, 1-\lambda)$ into $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$ for all $\theta \geq 0$.

Other cases following from theorem 9 are listed at the end of this example.

The function H , defined by $H(s) = (\Gamma(1-s))^{-1}$ satisfies (2.6) of theorem 11, again with $\theta_0 = \frac{1}{2}\pi$, so the Laplace transform is even a topological isomorphism of $T^{\theta}(1-\mu, 1-\lambda)$ onto $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$.

In order to construct inversion formulas, we apply theorem 11.

At first we derive the well known classical inversion formula and some related forms.

We put $H(s) = H_+(s) + H_-(s)$ with

$$(4.1) \quad H_{\pm}(s) = \frac{\pm 1}{2\pi i} e^{\pm i\pi s} \Gamma(s).$$

Let $\ell \in \mathbb{N}$ such that $\ell > \frac{3}{2} - \mu$. Choose $d \in (1-\mu, 1-\lambda)$ such that $d < \ell - \frac{1}{2}$ and $d \neq 1, 2, \dots, \ell$. Then $H_{\pm}(s)$ satisfy (2.8) with $\theta_0 = \frac{1}{2}\pi$ and $\alpha_{\pm} = \frac{3}{2}\pi$. Assume

$$\ell - k < d < \ell - k + 1 \text{ for some } k \in \mathbb{N}.$$

The corresponding functions $h_{\pm}(t)$, defined by (2.9) are

$$h_{\pm}(t) = \frac{\pm 1}{2\pi i} \left[e^t - \sum_{j=0}^{k-1} \frac{t^j}{j!} \right],$$

(we have omitted the subscript ℓ). By (2.11) with $\gamma_{\pm} = \pm \frac{1}{2}\pi$, we have the following inversion formula,

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i} \frac{d^{\ell}}{dx^{\ell}} \int_{-i\infty}^{i\infty} \left[e^{xt} - \sum_{j=0}^{k-1} \frac{(xt)^j}{j!} \right] t^{-\ell} \psi(t) dt,$$

which holds for $x > 0$, $\psi \in T^{\frac{1}{2}\pi+\theta}(\lambda, \mu)$.

In the general case $|\arg x| \leq \theta$ the inversion formula becomes

$$(4.2) \quad (L^{-1}\psi)(x) = \frac{1}{2\pi i} \frac{d^\ell}{dx^\ell} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) [e^{xt} - \sum_{j=0}^{k-1} \frac{(xt)^j}{j!}] t^{-\ell} \psi(t) dt,$$

where $\gamma_\pm$ satisfy (2.12) with $\alpha_\pm = \frac{3}{2}\pi$. As an example we may put $\gamma_\pm = \pm\frac{1}{2}\pi - \arg x$.

(4.2) may be simplified in the case in which there exists $k_0 \in \mathbb{N}$ such that $k_0 \leq k-1$ and $\ell - k_0 < 1 - \lambda$. Then for some positive ε ,

$$(4.3) \quad \begin{cases} \ell > \ell-1 > \dots > \ell-k+1 > d > 1-\mu+\varepsilon, \\ \ell-k+1 < \dots < \ell-k_0 < 1-\lambda-\varepsilon. \end{cases}$$

Any function $\psi \in T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$ satisfies

$$(4.4) \quad \begin{cases} \psi(t) = O(t^{-\mu+\varepsilon}) \text{ as } t \rightarrow \infty, & |\arg t| \leq \theta+\frac{1}{2}\pi, \quad \forall \varepsilon > 0, \\ \psi(t) = O(t^{-\lambda-\varepsilon}) \text{ as } t \rightarrow 0, & |\arg t| \leq \theta+\frac{1}{2}\pi, \quad \forall \varepsilon > 0. \end{cases}$$

We deduce that

$$\left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) t^{m-\ell} \psi(t) dt = 0, \quad m = k_0, k_0+1, \dots, k-1,$$

if $|\gamma_\pm| \leq \theta+\frac{1}{2}\pi$. Formula (4.2) becomes

$$(4.2)' \quad (L^{-1}\psi)(x) = \frac{1}{2\pi i} \frac{d^\ell}{dx^\ell} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) [e^{xt} - \sum_{j=0}^{k_0-1} \frac{(xt)^j}{j!}] t^{-\ell} \psi(t) dt.$$

If $\mu \leq \frac{3}{2}$ we may choose ℓ such that $\ell+\mu \in (\frac{3}{2}, 2]$ or $\ell+\mu \in (2, \frac{5}{2}]$.

In the first case we may choose $k_0 = 1$, in the second case we may choose $k_0 = 2$.

If $\mu > \frac{3}{2}$ and $0 < \lambda < 1$ we may choose $d < -\frac{1}{2}$, $\ell = 0$ and $k_0 = 0$. The resulting formula is

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) [e^{xt} \psi(t)] dt.$$

We may also simplify (4.2) in another way. By an analyticity (and continuity) argument the contour in (4.2) may be replaced by a contour which consists of the half line from $\infty e^{i\gamma_-}$ to $re^{i\gamma_-}$, ($r > 0$), the arc $t = re^{i\phi}$, $\gamma_- \leq \phi \leq \gamma_+$ and the half line from $re^{i\gamma_+}$ to $\infty e^{i\gamma_+}$. Here we assume that (2.12) is satisfied. As an example, if $\theta \leq \frac{1}{2}\pi$ we may choose $\gamma_{\pm} = \pm(\frac{1}{2}\pi + \theta)$.

From the first relations in (4.3) and (4.4) we deduce for any $\psi \in T^{\theta + \frac{1}{2}\pi}(\lambda, \mu)$

$$\int_C t^{j-\ell} \psi(t) dt = 0, \quad j = 0, \dots, k-1.$$

From Cauchy's theorem it now follows that we have

$$(4.5) \quad (L^{-1}\psi)(x) = \frac{1}{2\pi i} \frac{d^{\ell}}{dx^{\ell}} \int_C e^{xt} t^{-\ell} \psi(t) dt$$

if $|\arg x| \leq \theta$ and (2.12) holds. If $x > 0$ we may replace (4.5) by

$$(4.5)' \quad (L^{-1}\psi)(x) = \frac{1}{2\pi i} \frac{d^{\ell}}{dx^{\ell}} \int_{c-i\infty}^{c+i\infty} e^{xt} t^{-\ell} \psi(t) dt, \quad c > 0.$$

In order to construct another inversion formula we choose a different decomposition of $H(s)$. We define

$$H_{\pm}(s) = \frac{\pm 1}{2i} \frac{e^{\pm a\pi i(s+c)}}{\Gamma(1-s) \sin(a\pi(s+c))}, \quad a, c \in \mathbb{C}, \quad a \neq 0.$$

Then $H(s) = H_+(s) + H_-(s)$.

Let $\operatorname{Re} a \geq \frac{1}{2}$, $\ell \in \mathbb{N}$ and $\ell > d + \frac{1}{2}$. Then $H_{\pm}(s)$ satisfy (2.8) with $\theta_0 = \frac{1}{2}\pi$ and $\alpha_{\pm} = (2\operatorname{Re} a - \frac{1}{2})\pi$. Furthermore we obtain for the corresponding functions $h_{\pm}$, defined by (2.9), with $\ell = 0$ in that formula,

$$(4.6) \quad h_{\pm}(t) = \pm \frac{t^{\ell}}{2\pi i a} \sum_{k=k_0}^{\infty} \frac{t^{c+k/a}}{\Gamma(\ell+1+c+k/a)},$$

where k_0 is such that $-c - \frac{k_0}{a}$ is the first pole of $H_{\pm}$ to the left of the contour in (2.9).

We consider some special cases.

- a) Let $\mu > \frac{3}{2}$, $a = \frac{1}{2}$ and $\operatorname{Re} c > \frac{1}{2}$. Then we may choose $d < -\frac{1}{2}$ such that $k_0 = 0$. We may put now $\ell = 0$. Using $(1+c)_{2k} = 2^{2k} (\frac{1}{2} + \frac{1}{2}c)_k (1 + \frac{1}{2}c)_k$ we obtain

$$h_{\pm}(t) = \frac{\pm 1}{\pi i \Gamma(1+c)} t^c {}_1F_2(1; \frac{1}{2} + \frac{1}{2}c, 1 + \frac{1}{2}c; \frac{1}{2}t^2).$$

Hence we have the inversion formula for $L^{-1}: T^{\theta + \frac{1}{2}\pi}(\lambda, \mu) \rightarrow T^{\theta}(1-\mu, 1-\lambda)$

$$(L^{-1}\psi)(x) = \frac{1}{\pi i \Gamma(1+c)} \int_{-i\infty}^{i\infty} (xt)^c {}_1F_2(1; \frac{1}{2} + \frac{1}{2}c, 1 + \frac{1}{2}c; \frac{1}{2}x^2 t^2) \psi(t) dt, \quad x > 0,$$

which follows from (2.11) with $\gamma_{\pm} = \pm \frac{1}{2}\pi$.

In the general case we obtain

$$(L^{-1}\psi)(x) = \frac{1}{\pi i \Gamma(1+c)} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) (xt)^c {}_1F_2(1; \frac{1}{2} + \frac{1}{2}c, 1 + \frac{1}{2}c; \frac{1}{2}x^2 t^2) \psi(t) dt$$

if $|\arg x| \leq \theta$ and (2.12) is satisfied.

- b) Let $\mu > \frac{3}{2}$, $\frac{1}{2} < \operatorname{Re} c < \mu$, $a=1$, $\operatorname{Re} c > \lambda - 1$. Then we may choose $d < -\frac{1}{2}$ such that $-\operatorname{Re} c < d < 1 - \operatorname{Re} c$. Hence $k_0 = 0$ and we may put $\ell = 0$. The functions $h_{\pm}$ are

$$h_{\pm}(t) = \frac{\pm t^c}{2\pi i \Gamma(1+c)} {}_1F_1(1; 1+c; t), \quad \frac{1}{2}\pi \leq \arg t \leq \frac{3}{2}\pi.$$

The inversion formula (2.11) now becomes ($\gamma_{\pm} = \pm \frac{1}{2}\pi$):

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i \Gamma(1+c)} \int_{-i\infty}^{i\infty} (xt)^c {}_1F_1(1; 1+c; xt) \psi(t) dt$$

which holds if $x > 0$. If $|\arg x| \leq \theta$ and (2.12) holds we have

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i \Gamma(1+c)} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) (xt)^c {}_1F_1(1; 1+c; xt) \psi(t) dt.$$

- c) In case b) we put $\theta = \frac{1}{2}\pi$ and in (2.11) we take $\gamma_{\pm} = \pm\pi$. Then for $L^{-1} : T^{\pi}(\lambda, \mu) \rightarrow T^{\frac{1}{2}\pi}(1-\mu, 1-\lambda)$ we have

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i \Gamma(1+c)} \int_C (xt)^c {}_1F_1(1; 1+c; xt) \psi(t) dt,$$

where C is the contour along the negative axis, from $\infty e^{-\pi i}$ to 0 and from 0 to $\infty e^{\pi i}$ and the formula holds if $|\arg x| \leq \frac{1}{2}\pi$.

- d) For convenience let a be a real number, $a \geq \frac{1}{2}$. Let $c = 0$, $d < -\frac{1}{2}$. We may take $\ell = 0$. We assume $\lambda < 1$ and for this case we construct an inversion formula for the Laplace transform, which contains the Mittag-Leffler function $E_{\alpha}(t)$, (cf. [26a], II, p 64 and [58a], I, p 345-348).

The poles of the functions $H_{\pm}$ which are to the right of the contour are $-(k_0-1)/a, \dots, -1/a, 0$. It follows that for some $\varepsilon > 0$

$$1-\mu+\varepsilon < d < -\frac{k_0-1}{a} < \dots < -\frac{1}{a} < 0 < 1-\lambda-\varepsilon.$$

Using (4.4) we see that

$$(4.7) \quad \int_{-i\infty}^{i\infty} t^{m/a} \psi(t) dt = 0, \quad m = 0, 1, \dots, k_0-1.$$

Moreover

$$\begin{aligned} h_{\pm}(t) &= \pm \frac{1}{2\pi i a} \sum_{k=k_0}^{\infty} \frac{t^{k/a}}{\Gamma(1+k/a)} \\ &= \pm \frac{1}{2\pi i a} [E_{1/a}(t^{1/a}) - \sum_{k=0}^{k_0-1} \frac{t^{k/a}}{\Gamma(1+k/a)}]. \end{aligned}$$

Using (4.7) and (2.11) with $\gamma_{\pm} = \pm\frac{1}{2}\pi$ we now have the following inversion formula:

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i a} \int_{-i\infty}^{i\infty} E_{1/a}((xt)^{1/a}) \psi(t) dt, \quad x > 0.$$

If $|\arg x| \leq \theta$ and (2.12) is satisfied we have

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i a} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) E_{1/a}((xt)^{1/a}) \psi(t) dt.$$

The functions $h_{\pm}$ in (4.6) may be seen as special cases of the function ${}_p\psi_q$, introduced by E.M.Wright. These functions are generalizations of the hypergeometric functions. The asymptotic behaviour of the functions ${}_p\psi_q$ are treated in detail in [3a], section 12. It follows from an application of theorem 21 and of theorem 22 in [3a] that we have

$$h_{\pm}(t) = \frac{1}{2\pi i} e^t + O(t^{-1/a}),$$

as $t \rightarrow \infty$, $|\arg t| \leq \pi$ and if $a \in \mathbb{R}$, $a \geq \frac{1}{2}$.

We now compare the classical inversion formula (4.5) with $\ell = 0$ and the inversion formula following from (4.6). We again assume $a \geq \frac{1}{2}$. Hence in (4.6) we have $k_0 = 0$. We have

$$\begin{aligned} \int_C e^{xt} \psi(t) dt &= \int_{\infty e^{i\gamma_-}}^0 h_-(xt) \psi(t) dt - \int_0^{\infty e^{i\gamma_+}} h_+(xt) \psi(t) dt = \\ &= \int_C O(t^{-1/a}) \psi(t) dt. \end{aligned}$$

The integrand in the last integral is $O(t^{-1/a - \mu + \epsilon})$ as $t \rightarrow \infty$. From an application of Cauchy's theorem it now follows that this integral is zero.

This proves in another way the equivalence of the inversion formulas for this case. The equivalence of the classical inversion formula and the inversion formula involving the Mittag-Leffler function E_{α} may also be proven in the same way directly, using the asymptotic expansion of these function as treated in [58a], p 345-348.

For convenience we now list the main results on the Laplace transform.

LAPLACE TRANSFORM. LIST OF RESULTS.

If $\theta \geq 0$ and $0 \leq \lambda < \mu \leq \infty$ then the Laplace transform L defines a topological isomorphism of $T^{\theta}(1-\mu, 1-\lambda)$ onto $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$. Here L is given by

1) if $\phi \in T(1-\mu, 1-\lambda)$ and $|\arg x| < \frac{1}{2}\pi$ then

$$(L\phi)(x) = \int_0^{\infty} e^{-xt} \phi(t) dt;$$

2) if $\phi \in T^{\theta}(1-\mu, 1-\lambda)$, $|\arg x| < \frac{1}{2}\pi + \theta$ then

$$(L\phi)(x) = x^{-1} \int_0^{\infty e^{i\theta_1}} e^{-t} \phi\left(\frac{t}{x}\right) dt$$

where θ_1 is such that $|\theta_1| < \frac{1}{2}\pi$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$;

3) if $n \in \mathbb{N}$, $n > \lambda + \frac{1}{2} \geq n-1$,

$n_0 = 0$ if $n \leq \lambda + 1$, $n_0 = 1$ if $n > \lambda + 1$,

$\phi \in T(1-\mu, 1-\lambda)$, $|\arg x| \leq \frac{1}{2}\pi$ then

$$(L\phi)(x) = \frac{d^n}{dx^n} \int_0^{\infty} (-t)^{-n} [e^{-xt} - \sum_{j=0}^{n_0} \frac{(-xt)^j}{j!}] \phi(t) dt;$$

4) if n and n_0 are as in 3), $\phi \in T^{\theta}(1-\mu, 1-\lambda)$, $|\arg x| \leq \frac{1}{2}\pi + \theta$ then

$$(L\phi)(x) = \frac{d^n}{dx^n} x^{n-1} \int_0^{\infty e^{i\theta_1}} (-t)^{-n} [e^{-t} - \sum_{j=0}^{n_0} \frac{(-t)^j}{j!}] \phi\left(\frac{t}{x}\right) dt$$

where θ_1 is such that $|\theta_1| \leq \frac{1}{2}\pi$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$.

The inverse transform L^{-1} may be represented as follows:

1) if $\psi \in T^{\frac{1}{2}\pi}(\lambda, \mu)$ and

$\ell \in \mathbb{N}$, $\frac{3}{2} - \mu < \ell$,

$k_0 = 0$, $\ell = 0$ if $\mu > \frac{3}{2}$, $0 < \lambda < 1$,

$k_0 = 1$ if $\mu \leq \frac{3}{2}$, $\frac{3}{2} < \mu + \ell \leq 2$,

$k_0 = 2$ if $\mu \leq \frac{3}{2}$, $2 < \mu + \ell \leq \frac{5}{2}$, then

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i} \frac{d^\ell}{dx^\ell} \int_{-i\infty}^{i\infty} [e^{xt} - \sum_{j=0}^{k_0-1} \frac{(xt)^j}{j!}] t^{-\ell} \psi(t) dt, \quad x > 0;$$

2) if $\theta > 0$, $\psi \in T^{\frac{1}{2}\pi+\theta}(\lambda, \mu)$ and the conditions of 1) are satisfied then

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i} \frac{d^\ell}{dx^\ell} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) [e^{xt} - \sum_{j=0}^{k_0-1} \frac{(xt)^j}{j!}] t^{-\ell} \psi(t) dt,$$

$|\arg x| \leq \theta$ and $\gamma_{\pm}$ such that $\frac{1}{2}\pi \leq \pm(\gamma_{\pm} + \arg x) \leq \min(\frac{3}{2}\pi, \frac{1}{2}\pi + \theta \pm \arg x)$.
(cf. (4.5) for a related contour integral).

3) if $\theta > 0$, $\psi \in T^{\frac{1}{2}\pi+\theta}(\lambda, \mu)$, $\mu > \frac{3}{2}$, $\operatorname{Re} c > \frac{1}{2}$, then

$$(L^{-1}\psi)(x) = \frac{1}{\pi i \Gamma(1+c)} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) (xt)^c {}_1F_2(1; \frac{1}{2}+\frac{1}{2}c, 1+\frac{1}{2}c; \frac{1}{2}x^2 t^2) \psi(t) dt,$$

$|\arg x| \leq \theta$ and $\gamma_{\pm}$ as in 2).

4) if $\theta > 0$, $\psi \in T^{\frac{1}{2}\pi+\theta}(\lambda, \mu)$, $\mu > \frac{3}{2}$, $\frac{1}{2} < \operatorname{Re} c < \mu$,
then

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i \Gamma(1+c)} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) (xt)^c {}_1F_1(1; 1+c; xt) \psi(t) dt,$$

$|\arg x| \leq \theta$ and $\gamma_{\pm}$ as in 2). In particular, if $\theta = \frac{1}{2}\pi$, $\gamma_{\pm} = \pm\pi$,
 $\psi \in T^{\pi}(\lambda, \mu)$, then

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i \Gamma(1+c)} \int_C (xt)^c {}_1F_1(1; 1+c; xt) \psi(t) dt, \quad |\arg x| \leq \frac{1}{2}\pi,$$

where C is the contour along the negative axis, from $\infty e^{-\pi i}$ to 0
and from 0 to $\infty e^{\pi i}$.

5) if $\theta > 0$, $a \geq \frac{1}{2}$, $\lambda < 1$ and $\psi \in T^{\frac{1}{2}\pi+\theta}(\lambda, \mu)$ then

$$(L^{-1}\psi)(x) = \frac{1}{2\pi i a} \left(\int_{\infty e^{i\gamma_-}}^0 + \int_0^{\infty e^{i\gamma_+}} \right) E_{1/a}((xt)^{1/a}) \psi(t) dt, \quad |\arg x| \leq \theta,$$

where $\gamma_{\pm}$ are as in 2). Here E_{α} denotes the Mittag-Leffler function.

EXAMPLE II.

Let $K(s) = \frac{1}{(s-a)_n}$, $n \in \mathbb{N}$, $n \geq 2$, $a \in \mathbb{C}$. Assume $\operatorname{Re} a \leq \lambda < \mu$.

Define k by

$$k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) t^{-s} ds, \quad \operatorname{Re} a < c, \quad t > 0.$$

Using residue calculus it follows that

$$k(t) = \frac{1}{\Gamma(n)} t^{-a} (1-t)^{n-1} \quad \text{if } 0 < t < 1,$$

$$k(t) = 0 \quad \text{if } t > 1.$$

By theorem 8, formula (1.4)', the map A , defined by

$$(A\phi)(x) = \frac{1}{x\Gamma(n)} \int_0^1 t^{-a} (1-t)^{n-1} \phi\left(\frac{t}{x}\right) dt$$

is a continuous map of $T^\theta(1-\mu, 1-\lambda)$ into $T^\theta(\lambda, \mu)$ for any $\theta \geq 0$.

According to theorem 12, formula (3.1)' with $n = 0$, the map B , defined by

$$(B\phi)(x) = \frac{1}{\Gamma(n)} \int_0^1 t^{-a} (1-t)^{n-1} \phi\left(\frac{x}{t}\right) \frac{dt}{t}$$

is a continuous map of $T^\theta(\lambda, \mu)$ into itself. It will be seen in chapter V that B is a special case of a fractional integration operator.

It is obvious that $H(s) = (K(1-s))^{-1}$ satisfies the conditions of theorem 10, with $\ell = n+2$. Thus A is an isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T^\theta(\lambda, \mu)$. A representation for the continuous inverse may be given in the following way. Let $1-\mu < d < 1-\lambda$ and define h by

$$\begin{aligned} h(t) &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{H(s)}{(1-a-s)_{n+2}} t^{-s} ds \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{t^{-s}}{(n+1-a-s)(n+2-a-s)} ds, \quad t > 0. \end{aligned}$$

We obtain

$$\begin{aligned} h(t) &= t^{a-n-2} (t-1) \quad \text{if } t > 1, \\ &= 0 \quad \text{if } 0 < t < 1. \end{aligned}$$

Formula (2.5)⁺ gives the inversion ($D = \frac{d}{dx}$):

$$(A^{-1}\psi)(x) = x^{a-n+2} D_x^{-a+n+1} \int_1^\infty t^{a-n-2} (t-1) \psi\left(\frac{t}{x}\right) dt.$$

In a similar way we deduce an inversion formula for the map B , using theorem 14 and remark 11. In formula (3.7)⁺ we put $r = a-n-2$, $\ell = n+2$. We define h by

$$\begin{aligned} h(t) &= \frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{t^{-s}}{(s-a+n+1)(s-a+n)} ds \\ &= (-1)^n t^{-a+n} (1-t) \quad \text{if } 0 < t < 1 \\ &= 0 \quad \text{if } t > 1. \end{aligned}$$

B^{-1} is then given by

$$(B^{-1}\psi)(x) = x^{n+2-a} D_x^{n+2-a} \int_0^1 (-1)^n t^{-a+n} (1-t) \psi\left(\frac{x}{t}\right) \frac{dt}{t}.$$

EXAMPLE III.

Let K_1 and K_2 be functions which are analytic on $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$. Assume for every $n \in \mathbb{N}$

$$K_j(s) = O(s^{-1-\epsilon}) \text{ as } s \rightarrow \infty, \quad \lambda_n \leq \operatorname{Re} s \leq \mu_n, \quad \epsilon > 0, \quad j=1,2.$$

K_1 and K_2 satisfy the conditions of theorem 8 and theorem 12 respectively. Let k_1, k_2 be the corresponding kernels in these theorems,

$$k_j(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_j(s) t^{-s} ds, \quad \lambda < c < \mu, \quad j = 1, 2.$$

Let $A : T(1-\mu, 1-\lambda) \rightarrow T(\lambda, \mu)$ and $B : T(\lambda, \mu) \rightarrow T(\lambda, \mu)$ be the corresponding Watson and the convolution transform. One easily deduces

$$(B \cdot A)\phi(x) = \int_0^\infty dt \phi(t) \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_1(s) K_2(s) (xt)^{-s} ds \right].$$

Hence the composition of B and A is again a Watson transform from

$T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$. The kernel is the convolution product of k_1 and k_2 :

$$k(u) = \int_0^{\infty} k_1\left(\frac{u}{t}\right) k_2(t) \frac{dt}{t}.$$

EXAMPLE IV

Let A be the map defined in example II. Let B be the map B of example II with a replaced by $-a$. Assume $|\operatorname{Re} a| \leq \lambda < \mu$. We now apply example III with

$$K_1(s) = 1/(s-a)_n, \quad K_2(s) = 1/(s+a)_n.$$

We have

$$(B \cdot A)\phi(x) = \int_0^{\infty} k(u) \phi\left(\frac{u}{x}\right) x^{-1} du$$

where

$$k(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{(s-a)_n (s+a)_n} u^{-s} ds.$$

By the last formula of example III we have

$$k(u) = \frac{1}{(\Gamma(n))^2} \int_u^1 \left(\frac{u}{t}\right)^{-a} \left(1-\frac{u}{t}\right)^{n-1} t^a (1-t)^{n-1} \frac{dt}{t},$$

$0 < u < 1$ and $k(u) = 0$ if $u > 1$. By the substitution $\frac{t-u}{1-u} = z$ we have

$$(\dagger) \quad k(u) = \frac{1}{(\Gamma(n))^2} u^{a-n} (1-u)^{2n-1} \int_0^1 z^{n-1} (1-z)^{n-1} \left(1+\frac{1-u}{u} z\right)^{2a-n} dz.$$

On the other hand, using residue calculus we obtain

$$k(u) = \frac{1}{(\Gamma(n))^2} \sum_{k=0}^{n-1} \frac{(-1)^k u^k}{k!(n-k-1)!} \left[\frac{u^{-a}}{(2a-k)_n} + \frac{u^a}{(-2a-k)_n} \right], \quad 0 < u < 1,$$

which as a by product gives the evaluation of the integral in ($\dagger$), where $\pm 2a \neq 0, 1, \dots, n-1$.

EXAMPLE V.

Let K be defined on $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$ by

$$K(s) = \frac{\Gamma(s)\Gamma(\alpha-s)}{\Gamma(\gamma-s)}, \quad \alpha, \gamma \in \mathbb{C}.$$

Assume $0 \leq \lambda < \mu \leq \operatorname{Re} \alpha$. Then K satisfies the conditions of theorem 9 with $\theta_0 = \frac{1}{2}\pi$. We have using residue calculus,

$$k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s)t^{-s} ds = \frac{\Gamma(\alpha)}{\Gamma(\gamma)} {}_1F_1(\alpha; \gamma; -t), \quad |\arg t| < \frac{1}{2}\pi.$$

Hence by theorem 9, the map A , defined by

$$(A\phi)(x) = \frac{\Gamma(\alpha)}{\Gamma(\gamma)} \int_0^\infty {}_1F_1(\alpha; \gamma; -xt) \phi(t) dt,$$

is a continuous map of $T^\theta(1-\mu, 1-\lambda)$ into $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$, $\theta \geq 0$.

This example is in a slightly different form due to E.R.Love, T.R.Prabhakar and N.K.Kashyap, (cf [41]). They also considered the inverse of A . In chapter V we will give their inversion formula. Here we derive another inversion formula, using theorem 11. Let $H(s) = (K(1-s))^{-1}$. H satisfies (2.6) with $\theta_0 = \frac{1}{2}\pi$. Hence A is an isomorphism. Define $H_\pm(s)$ by

$$H_\pm(s) = \pm \frac{\Gamma(s+\gamma-1)\Gamma(s)e^{\pm\pi i s}}{2\pi i \Gamma(s+\alpha-1)}.$$

Assume $1-\mu < d < 1-\lambda$, $d < \operatorname{Re}(\alpha-\gamma)-\frac{1}{2}$, $d \neq 0, -1, \dots$, $d \neq 1-\gamma, -\gamma, \dots$. Then $H_\pm$ satisfy (2.7) and (2.8) with $\theta_0 = \frac{1}{2}\pi$, $\alpha_\pm = \frac{3}{2}\pi$, $\ell = 0$. Define $h_\pm$ by

$$h_\pm(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} H_\pm(s)t^{-s} ds, \quad \frac{1}{2}\pi \leq \arg t \leq \frac{3}{2}\pi.$$

Then by (2.11) with $\gamma_\pm = \pm\frac{1}{2}\pi$, $\ell = 0$ the inverse $A^{-1}: T^{\frac{1}{2}\pi}(\lambda, \mu) \rightarrow T(1-\mu, 1-\lambda)$ is given by

$$(A^{-1}\psi)(x) = \int_{-i\infty}^{i\infty} h(xt)\psi(t) dt, \quad x > 0,$$

where $h(xt) = \pm h_{\pm}(xt)$ if $\arg t = \pm \frac{1}{2}\pi$.

In the general case we have: $A^{-1} : T^{\frac{1}{2}\pi+\theta}(\lambda, \mu) \rightarrow T^{\theta}(1-\mu, 1-\lambda)$ is given by

$$(A^{-1}\psi)(x) = \int_{\infty e^{i\gamma-}}^0 h_{-}(xt)\psi(t) dt + \int_0^{\infty e^{i\gamma+}} h_{+}(xt)\psi(t) dt$$

if $|\arg x| \leq \theta$ and (2.12) is satisfied.

We consider the case in which d may be chosen in such a way that all poles of $H_{\pm}(s)$ are to the left of the contour. This is possible if $\operatorname{Re} \gamma > \lambda$, $\lambda < 1$, $\operatorname{Re} \alpha > \frac{3}{2}$ and $\operatorname{Re} \alpha > \operatorname{Re} \gamma + \frac{1}{2}$.

In terms of Meijers G-function we then have

$$h_{\pm}(t) = \frac{\pm 1}{2\pi i} G_{1,2}^{2,0}(e^{\mp \pi i} t \mid \alpha-1, \gamma-1, 0), \quad \frac{1}{2}\pi \leq \pm \arg t \leq \frac{3}{2}\pi.$$

This in turn may be written as

$$h_{\pm}(t) = \mp \frac{1}{2\pi i} e^{\mp \frac{1}{2}\pi i \gamma} t^{\frac{1}{2}\gamma-1} e^{\frac{1}{2}t} W_{\frac{1}{2}\gamma-\alpha+1, \frac{1}{2}\gamma-\frac{1}{2}}(e^{\mp \pi i} t),$$

where W denotes Whittaker's function.

We may also use residue calculus for the computation of $h_{\pm}$. We obtain

$$h_{\pm}(t) = \pm \frac{1}{2\pi i} \frac{\Gamma(\gamma-1)}{\Gamma(\alpha-1)} {}_1F_1(2-\alpha; 2-\gamma; t) \mp \frac{1}{2\pi i} t^{\gamma-1} e^{\mp \pi i \gamma} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma)} {}_1F_1(1+\gamma-\alpha; \gamma; t).$$

The equivalence of these expressions for $h_{\pm}$ may be seen from [12], p 216(6) and p 253(7) and from [26], p 1059, section 9.220.

If there are also poles of $H_{\pm}$ to the right of the contour, then of course there will occur additional terms in the kernels of the inversion formulas, due to the residues of these poles.

Also if the condition $d < \operatorname{Re}(\alpha-\gamma) - \frac{1}{2}$ is not satisfied, the inversion formulas will be more complicated since we then have $\ell \neq 0$.

EXAMPLE VI

In the same manner as in the preceding examples we may treat the following transform. We define the transform A by

$$(A\phi)(x) = \int_0^{\infty} \sqrt{xt} Y_{\nu}(xt) \phi(t) dt$$

where Y_{ν} is the Bessel function of second kind and of order ν . The Mellin transform of the kernel $\sqrt{t}Y_{\nu}(t)$ is given by

$$\begin{aligned} K(s) &= 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)} \cot \pi(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s) \\ &= 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(-\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}s)} \cdot \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}. \end{aligned}$$

We deduce that

$$\begin{aligned} H(s) &= 1/K(1-s) \\ &= 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}s)} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)} \\ &= 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}s)} \tan \pi(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s). \end{aligned}$$

From this it follows that

$$(M^{-1}H)(t) = \sqrt{t} \mathbb{H}_{\nu}(t)$$

where $\mathbb{H}_{\nu}$ is Struve's function. The inversion formula is

$$(A^{-1}\psi)(x) = \int_0^{\infty} \sqrt{xt} \mathbb{H}_{\nu}(xt) \psi(t) dt.$$

We omit the details. The reader is referred to [62a], p 278; cf also [12], I, p 219 (46) and p 220 (51).

2.5 BIBLIOGRAPHICAL NOTES

As pointed out in the introduction, many known integral transforms are of Watson or convolution type. Convolution also occurs in some applications to define for example new kernels for integral transforms (cf [1] and [6]). The theorems in this chapter are extensions of those in [4] where the theory of Watson and convolution transforms is given for spaces $T(\lambda, \mu)$. Watson transforms in distributional sense are also considered by Hsing-Yuan Hsu, ([27]), with applications to Hankel and Hilbert transforms and by Perry, ([53]). The last author considers the bilateral form of Watson transforms, i.e. $\int_{-\infty}^{\infty} k(xy) \phi(y) dy$. Rooney ([57]) considers both Watson and convolution transforms in combination with a simple form of our operator D_n , (cf remark 3 and remark 6), on certain Banach spaces of Lebesgue measurable functions and also products of these transforms. In the literature many applications can be found; an important one is the solution of (dual) integral equations, (for examples see [4], [38], [39], [69]).

Laplace transforms on topological vector spaces are intensively studied. The classical work of course is Schwartz's monograph; ([61]; confer also [24]). The paper of Boas deals with a "nearly Laplace kernel". Cooper ([5]) considers Laplace transforms in the sense of Schwartz. Krätzel ([36]) considers extensions of the Laplace transform. In his paper it is noted that Meijer's K-transform (see chapter 7 in this thesis), because of the asymptotic behaviour of the kernel, may be seen as an extension of the Laplace transform. He also gives further extensions to more general kernels. G and H functions as kernels of integral transforms occur in [4], [42], [45]. Also Swaroop considers a general kernel the behaviour of which resembles the Laplace kernel. As an example of the bilateral Laplace transform on a locally convex space see [60].

The product of two integral transforms of Watson or convolution type has been treated in a general context by Kober, ([31]).

As an example of the use of Watson transforms we also mention the work of Prabhakar (cf [55]). The integral equations in his paper originate from the field of physics. Later on in collaboration with Love and Kashyap other integral equations have been studied by the authors. Their results in [41] follow easily in chapter V. Another result with respect to their integral equation is given in example 5 of this chapter.

Many other applications of integral transforms can be found in [62a]. Confer also [67]. Hirschman and Widder [26b] consider a related convolution transform, the kernel being of the form $k(x-t)$. Finally it should be noted that the theory of this chapter is immediately applicable to kernels defined by the inverse Mellin transform of G - and H -functions. As for the latter ones this is already done for spaces of type $T(\lambda, \mu)$ in [4]. Confer also the example at the end of chapter IV. For general remarks see also the monograph of Marichev, ([42a]).

CHAPTER 3

EXTENSIONS THE CASE $\lambda \geq \mu$

3.0 INTRODUCTION

The integral transforms in chapter II have been defined on spaces $T^\theta(\lambda, \mu)$ with $\lambda < \mu$. However, the spaces $T^\theta(\lambda, \mu)$ also make sense for $\lambda \geq \mu$. The assumption $\lambda < \mu$ has been used in order to give a meaning to the following representations of the integral transforms:

$$M^{-1} \cdot m(K) \cdot M, \quad \text{resp.} \quad M^{-1} \cdot m_1(K) \cdot M,$$

where M is the Mellin transform and $m(K)$ and $m_1(K)$ are maps defined in chapter I. $m(K)$ is a map of a space of type $S^\theta(\lambda, \mu)$ onto another such one, and these spaces are only defined if $\lambda < \mu$. The same holds for $m_1(K)$. In this chapter we give an extension of the definitions of Watson and convolution transforms to spaces $T^\theta(\lambda, \mu)$ with $\lambda \geq \mu$. Properties of these extended maps like those already given in chapter II will be treated in the theorems of section III.2.

The extensions rest upon a certain decomposition of the functions under consideration. We begin in the next section with a study of these decompositions.

3.1 DECOMPOSITION OF A FUNCTION $\phi \in T^\theta(\lambda, \mu)$

Let $\lambda, \mu, \lambda', \mu'$ be real numbers such that

$$\lambda' < \mu', \quad \lambda, \mu \in (\lambda', \mu').$$

Futhermore, let $(\lambda_n), (\mu_n), (\lambda'_n), (\mu'_n), n \in \mathbb{N}$, be monotonic sequences of real numbers, such that

$$\lambda_n \uparrow \lambda, \quad \lambda'_n \uparrow \lambda', \quad \mu_n \uparrow \mu, \quad \mu'_n \uparrow \mu' \text{ as } n \rightarrow \infty.$$

We are going to construct a decomposition for any $\phi \in T^\theta(\lambda, \mu)$. This decomposition will have the following form:

$$\phi = \phi_1 + \phi_2, \quad \phi_1 \in T^\theta(\lambda', \mu), \quad \phi_2 \in T^\theta(\lambda, \mu').$$

Note that $\lambda' < \mu$, $\lambda < \mu'$. Hence the integral transforms of chapter II may be defined for both ϕ_1 and ϕ_2 .

THEOREM 16. Let $\lambda, \mu, \lambda', \mu'$ be as above. If $\phi \in T^\theta(\lambda, \mu)$, then there exist functions ϕ_1 and ϕ_2 such that $\phi_1 \in T^\theta(\lambda', \mu)$, $\phi_2 \in T^\theta(\lambda, \mu')$ and $\phi = \phi_1 + \phi_2$.

PROOF. We will construct functions ρ_1 and ρ_2 such that $\phi_1 = \rho_1\phi$, $\phi_2 = \rho_2\phi$ and $\phi = \phi_1 + \phi_2$. We take

$$\rho_1(t) = (1 - (1+t)^\alpha)^{-m_1}, \quad \rho_2(t) = 1 - \rho_1(t),$$

where $\alpha > 0$, $0 \leq \theta\alpha < \pi$, $m_1 = \frac{\lambda - \lambda'}{\alpha}$, $m_2 = \frac{\mu' - \mu}{\alpha}$. If $\theta > 0$, it follows from $\theta\alpha < \pi$ that ρ_1, ρ_2 are analytic on G_θ^0 and may be extended continuously to G_θ , including the derivatives of any order. By a Taylor series expansion it is seen that

$$\begin{aligned} \rho_1(t) &= O(t^{\lambda - \lambda'}) \text{ as } t \rightarrow 0, \quad t \in G_\theta, \\ \rho_1'(t) &= O(t^{\lambda - \lambda' - 1}) \text{ as } t \rightarrow 0, \quad t \in G_\theta \end{aligned}$$

and by induction

$$\rho_1^{(k)}(t) = O(t^{\lambda - \lambda' - k}) \text{ as } t \rightarrow 0, \quad t \in G_\theta, \quad k = 1, 2, \dots$$

Also $\rho_1(t) = 1 + O(t^{\mu - \mu'})$ as $t \rightarrow \infty$, $t \in G_\theta$,

$$\rho_1^{(k)}(t) = O(t^{\mu - \mu' - k}) \text{ as } t \rightarrow \infty, \quad t \in G_\theta, \quad k = 1, 2, \dots$$

Since $\rho_2 = 1 - \rho_1$, one obtains

$$\rho_2(t) = 1 + O(t^{\lambda-\lambda'}) \text{ as } t \rightarrow 0, t \in G_\theta,$$

$$\rho_2^{(k)}(t) = O(t^{\lambda-\lambda'-k}) \text{ as } t \rightarrow 0, t \in G_\theta, k = 1, 2, \dots,$$

$$\rho_2^{(k)}(t) = O(t^{\mu-\mu'-k}) \text{ as } t \rightarrow \infty, t \in G_\theta, k = 0, 1, \dots$$

Let τ'_n , $n = 0, 1, \dots$ denote the norms in $T^\theta(\lambda', \mu)$, associated with the sequences (λ'_n) and (μ_n) . Let $\phi \in T^\theta(\lambda, \mu)$, $\phi_1 = \rho_1 \phi$. Then we have

$$\tau'_n(\phi_1) = \sup_{\substack{p \leq n \\ t \in G_\theta}} |t^{\lambda'_n} (1+|t|)^{\mu_n - \lambda'_n} t^p (\rho_1(t) \phi(t))^{(p)}|.$$

For some $p \in \mathbb{N}$, $0 \leq p \leq n$, consider

$$(1.2) \quad t^{\lambda'_n} (1+|t|)^{\mu_n - \lambda'_n} t^p \sum_{k=0}^p \binom{p}{k} \rho_1^{(k)}(t) \phi^{(p-k)}(t).$$

If $t \rightarrow 0$, then

$$\rho_1^{(k)}(t) \phi^{(p-k)}(t) = O(t^{\lambda-\lambda'-k} t^{k-p-\lambda_\ell}), \quad k = 0, 1, \dots, p,$$

for any $\ell = 0, 1, 2, \dots$. Hence we have that the expression (1.2) is

$$(1.3) \quad O(t^{\lambda'_n + \lambda - \lambda' - \lambda_\ell}) \text{ as } t \rightarrow 0.$$

The exponent in (1.3) is positive for ℓ sufficiently large. Hence (1.2) is bounded in a neighbourhood of 0.

If $t \rightarrow \infty$, we have

$$(1.4) \quad \rho_1^{(k)}(t) \phi^{(p-k)}(t) = O(t^{\mu-\mu'-k} t^{k-p-\mu_\ell}), \quad k = 1, 2, \dots, p,$$

$$\rho_1(t) \phi^{(p)}(t) = O(t^{-p-\mu_\ell}),$$

for any $\ell = 0, 1, \dots$. For the expression (1.2) we have the order relation

$$(1.5) \quad O(t^{\mu_n + \mu - \mu' - \mu_\ell}) + O(t^{\mu_n - \mu_\ell}) \text{ as } t \rightarrow \infty.$$

Both exponents are negative if ℓ is sufficiently large. It follows that (1.2) is also bounded in a neighbourhood of ∞ . We now deduce that $\tau'_n(\phi_1) < \infty$, hence ϕ_1 is in $T^\theta(\lambda', \mu)$.

Analogously we may show that $\phi_2 \in T^\theta(\lambda, \mu')$, and the assertion of the theorem follows. $\square$

3.2 EXTENSION OF THE DEFINITION OF WATSON TRANSFORMS AND OF CONVOLUTION TRANSFORMS TO THE CASE $\lambda > \mu$

In this section we give the extension of the definition of Watson transforms and convolution transforms on spaces $T^\theta(\lambda, \mu)$, λ and μ arbitrary real numbers, by means of the decomposition of the function $\phi \in T^\theta(\lambda, \mu)$, as described in the foregoing section.

The proofs of the theorems concerning the extension of the convolution transforms are modifications of the proofs of the theorems which give the extension of the Watson transforms. For this reason we do not give the proofs of all theorems in this section.

Our first theorem gives the extension in the case of the map of theorem 8.

THEOREM 17. Let $\lambda, \mu \in \mathbb{R}$. Suppose $K(s)$ is analytic on $S = \{s \in \mathbb{C} \mid c_1 < \operatorname{Re} s < c_2\}$ and $K(s) = O(s^\gamma)$ as $s \rightarrow \infty$ on S , where $c_1 < \min(\lambda, \mu)$, $c_2 > \max(\lambda, \mu)$, $\gamma \in \mathbb{R}$. Choose $n \in \mathbb{N}$, $n > \gamma + 1$, $p \in \mathbb{C}$, such that $K(s)/(1+p-s)_n$ is analytic on S . Define k_n^* by (II.1.3)⁺ with $c \in (c_1, c_2)$. Then the map A defined by (II.1.4)⁺ is a continuous map of $T^\theta(1-\mu, 1-\lambda)$ into $T^\theta(\lambda, \mu)$ for any $\theta \geq 0$. This map is independent of the choice of p, n, c_1, c_2, c . It is an extension of the map A of theorem 8 with $\lambda=c_1, \mu=c_2$.

PROOF. Because of theorem 8 and remark II.3 we only need to consider the case $\lambda \geq \mu$. In the definition of k_n^* in (II.1.3)⁺ we may replace c by any number $c \in (c_1, c_2)$ without changing k_n^* , because the integrand in (II.1.3)⁺ is analytic and $O(s^{\gamma-n})$ as $s \rightarrow \infty$ on S . Let $\phi \in T^\theta(1-\mu, 1-\lambda)$. Choose a decomposition $\phi = \phi_1 + \phi_2$, where

$$\phi_1 \in T^\theta(1-c_2, 1-\lambda), \phi_2 \in T^\theta(1-\mu, 1-c_1),$$

(cf. theorem 16). Now apply (II.1.4)⁺ to ϕ_j , $j = 1, 2$. According to theorem 8 and remark II.3, $A\phi_j$ exists, $j = 1, 2$, $A\phi_1 \in T^\theta(\lambda, c_2) \subset T^\theta(\lambda, \mu)$ and $A\phi_2 \in T^\theta(c_1, \mu) \subset T^\theta(\lambda, \mu)$. Hence (II.1.4)⁺ makes sense for the ϕ we started with and $\phi \in T^\theta(\lambda, \mu)$. From theorem 8 and remark II.3 it follows that the map A is an extension of the map A of theorem 8 with $\lambda = c_1$ and $\mu = c_2$.

Choosing $\phi_1 = \rho_1\phi$, $\phi_2 = \rho_2\phi$ as in the proof of theorem 16 it is easy to see that the maps defined by $\phi \rightarrow A(\rho_j\phi)$, $j = 1, 2$, are continuous from $T^\theta(1-\mu, 1-\lambda)$ into $T^\theta(\lambda, \mu)$. Hence A is continuous. It is also easy to see that A is independent of the choice of c_1 and c_2 . As in theorem 8 and remark II.3 it is evident that A is independent of the choice of n , p and c . □

Similarly we may prove

THEOREM 18. Let $\lambda, \mu \in \mathbb{R}$. Suppose $K(s)$ is analytic on $S = \{s \in \mathbb{C} \mid c_1 < \operatorname{Re} s < c_2\}$ and

$$K(s) = O(s^\gamma e^{-\theta_0|s|}) \text{ as } s \rightarrow \infty, s \in S,$$

where $c_1 < \min(\lambda, \mu)$, $c_2 > \max(\lambda, \mu)$, $\gamma \in \mathbb{R}$, $\theta_0 > 0$.

Let k be defined by (II.1.10) with the condition $\lambda < c < \mu$ replaced by $c \in (c_1, c_2)$. Then the map A defined by (II.1.12) with $|\theta_1| < \theta_0$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$ is continuous from $T^\theta(1-\mu, 1-\lambda)$ into $T^{\theta+\theta_0}(\lambda, \mu)$ for any $\theta \geq 0$. This map is independent of the choice of c , c_1 , c_2 and it is an extension of the map A of theorem 9 with $\lambda = c_1$, $\mu = c_2$. If k_n^* is defined as in theorem 17 for $|\arg t| \leq \theta_0$, then (II.1.12)** holds for any $\phi \in T^\theta(1-\mu, 1-\lambda)$, $\theta \geq 0$ and any θ_1 such that (II.1.14) is satisfied.

REMARK

1. Theorem 17 and 18 respectively may also be formulated in such a way that (II.1.4)*, respectively (II.1.12)* apply.

The conditions for the maps A in theorems 17 and 18 to be isomorphisms are very similar to the conditions in theorems 10 and 11, chapter II. The next theorems give the inversion in case of A being an isomorphism.

THEOREM 19. Let the assumptions of theorem 17 be satisfied. Let H be defined by (II.2.1) and H be analytic on $S_1 = \{s \in \mathbb{C} \mid 1-c_2 < \operatorname{Re} s < 1-c_1\}$. Suppose

$$H(s) = O(s^{\delta}) \text{ as } s \rightarrow \infty, s \in S_1$$

for some $\delta \in \mathbb{R}$. Then the map A of theorem 17 is an isomorphism of $T^{\theta}(1-\mu, 1-\lambda)$ onto $T^{\theta}(\lambda, \mu)$ for any $\theta \geq 0$. Its continuous inverse B may be constructed as follows.

Let $\ell \in \mathbb{N}$, $r \in \mathbb{C}$, such that $\ell > \delta+1$, and

$$H(s)/(1+r-s)_{\ell} \text{ is analytic on } S_1.$$

Then define h_{ℓ}^* by (II.2.4)* with $\beta_j = r+j$ and $d \in (1-c_2, 1-c_1)$. If $\psi \in T^{\theta}(\lambda, \mu)$, then B satisfies (II.2.5)⁺ for $|\arg x| \leq \theta$.

PROOF. We only consider the case $\lambda \geq \mu$ because of theorem 10 and remark 6. Consider the map B of theorem 10 on $T^{\theta}(\lambda, c_2)$ and $T^{\theta}(c_1, \mu)$. Let $\psi \in T^{\theta}(\lambda, \mu)$. Choose a decomposition $\psi = \psi_1 + \psi_2$, where $\psi_1 \in T^{\theta}(\lambda, c_2)$ and $\psi_2 \in T^{\theta}(c_1, \mu)$, (cf. theorem 16). Define $B\psi = B\psi_1 + B\psi_2$. This is equivalent to the definition of B via (II.2.5)⁺, (cf. the proof of theorem 17). Then $B\psi \in T^{\theta}(1-\mu, 1-\lambda)$, $B\psi_1 \in T^{\theta}(1-c_2, 1-\lambda)$, $B\psi_2 \in T^{\theta}(1-\mu, 1-c_1)$. Hence $AB\psi = AB\psi_1 + AB\psi_2$ as in theorem 17 and $AB\psi_j = \psi_j$, $j = 1, 2$. So $AB = \operatorname{Id}$ and similarly $BA = \operatorname{Id}$. $\square$

THEOREM 20. Let the assumptions of theorem 18 be satisfied. Assume that the function H defined by (II.2.1) is analytic in the strip $S_1 = \{s \in \mathbb{C} \mid 1-c_2 < \operatorname{Re} s < 1-c_1\}$. Suppose that

$$H(s) = O(s^{\delta} e^{\theta_0 |s|}) \text{ as } s \rightarrow \infty, s \in S_1$$

for some $\delta \in \mathbb{R}$. Then the map A of theorem 18 is an isomorphism of

$T^{\theta}(1-\mu, 1-\lambda)$ onto $T^{\theta+\theta_0}(\lambda, \mu)$ for any $\theta \geq 0$. The continuous inverse B may be constructed as follows.

Choose $\ell \in \mathbb{N}$, $r \in \mathbb{C}$, $\rho > 0$ and functions $H_{\pm}$ which satisfy (II.2.7) on S_1 such that

$$H_{\pm}(s)/(1+r-s)_{\ell} \text{ is analytic on } S_1$$

and

$$(II.2.14) \text{ holds if } \operatorname{Im} s \rightarrow \infty, s \in S_1$$

$$(II.2.15) \text{ holds if } \operatorname{Im} s \rightarrow -\infty, s \in S_1$$

with $\varepsilon > 0$. (The existence of these numbers follow from the corollary of lemma 1). Then define $h_{\pm\ell}^*$ by (II.2.9)* if $\theta_0 \leq \pm \arg t \leq \rho + \theta_0$, with $\beta_{\pm} = r + j$ and $d \in (1-c_2, 1-c_1)$. Then (II.2.11)** holds for any $\psi \in T^{\theta+\theta_0}(\lambda, \mu)$ and $|\arg x| \leq \theta$ if $\gamma_{\pm}$ satisfies (II.2.12) with $\alpha_{\pm} = \rho + \theta_0$. In case of $\theta > 0$, $\rho > 0$ we may characterize B on $T^{\theta+\theta_0}(\lambda, \mu)$ by (II.2.11) with $\ell = 0$, $|\arg x| < \theta$ and any $\gamma_{\pm}$ with $\theta_0 < \pm(\gamma_{\pm} + \arg x) < \rho + \theta_0$, $|\gamma_{\pm}| \leq \theta_0 + \theta$. Now $h_{\pm 0}$ is defined by (II.2.9) with $\ell=0$ if $\theta_0 < \pm \arg t < \rho + \theta_0$. $\square$

Finally we consider the extensions of the convolution map in theorems 12 - 15.

THEOREM 21. Let the assumptions of theorem 17 be satisfied. Then the map C defined by (II.3.1)⁺ is continuous from $T^{\theta}(\lambda, \mu)$ into itself for any $\theta \geq 0$. Here k_n^* is defined as in theorem 17. C does not depend on the choice of c , c_1 , c_2 , n and p . It is an extension of the map C of theorem 12 with $\lambda=c_1$, $\mu=c_2$. $\square$

THEOREM 22. Let the assumptions of theorem 18 be satisfied and let k be defined as in theorem 18. Then the map C defined by (II.3.4) is continuous from $T^{\theta}(\lambda, \mu)$ into $T^{\theta+\theta_0}(\lambda, \mu)$ for any $\theta \geq 0$. In (II.3.4), θ_1 has to be chosen such that $|\theta_1| < \theta_0$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$. This map C is an extension of the map C of theorem 13 with $\lambda=c_1$, $\mu=c_2$.

It does not depend on the choice of c, c_1, c_2 . If k_n^* is defined as in theorem 17 for $|\arg t| \leq \theta_0$, then $(II.3.5)^+$ holds for any $\phi \in T^\theta(\lambda, \mu)$, $\theta \geq 0$ and θ_1 satisfying the condition following $(II.3.5)^+$. $\square$

THEOREM 23. Let the assumptions of theorem 17 be satisfied. Assume $H(s) = 1/K(s)$ is analytic on S . Suppose

$$H(s) = O(s^\delta) \text{ as } s \rightarrow \infty \text{ on } S$$

for some $\delta \in \mathbb{R}$. Then the map C of theorem 21 is an automorphism of $T^\theta(\lambda, \mu)$ for any $\theta \geq 0$. Its continuous inverse D is given by $(II.3.7)^+$ with h_ℓ^* defined as in theorem 19, $|\arg x| \leq \theta$. $\square$

THEOREM 24. Let the assumptions of theorem 18 be satisfied. Assume $H(s) = 1/K(s)$ is analytic on S . Suppose

$$H(s) = O(s^{\delta} e^{\theta_0 |s|}) \text{ as } s \rightarrow \infty \text{ on } S$$

for some $\delta \in \mathbb{R}$. Then the map C of theorem 22 is an isomorphism of $T^\theta(\lambda, \mu)$ onto $T^{\theta+\theta_0}(\lambda, \mu)$ for any $\theta \geq 0$. Its continuous inverse D is given by $(II.3.8)^+$ where $h_{\pm\ell}^*$ are defined as in theorem 20. If $\theta > 0$ and $\rho > 0$ we may characterize D on $T^{\theta+\theta_0}(\lambda, \mu)$ by $(II.3.8)^+$ with $\ell = 0$, $|\arg x| < \theta$ and any $\gamma_\pm$ with $\theta_0 < \pm(\gamma_\pm + \arg x) < \theta_0 + \rho$, $|\gamma_\pm| \leq \theta + \theta_0$. Now $h_{\pm 0}$ is defined by $(II.2.9)^*$ if $\theta_0 < \pm \arg t < \theta_0 + \rho$. $\square$

3.3 BIBLIOGRAPHICAL NOTES

Spaces of testing functions which are more or less related to the spaces $T^\theta(\lambda, \mu)$ are often used in the theory of integral transforms.

The Zemanian approach (cf [72]; in this monograph also a detailed list of references is included), has as starting point countably normed spaces which contain the kernel of the integral transform under consideration. The construction of the describing seminorms differs from case to case. It depends on the special case of the integral transform. The concept of the spaces $T^\theta(\lambda, \mu)$ has more generality. However there are examples in which one can prove that testing function spaces which are constructed in the Zemanian sense are identical to a space $T^\theta(\lambda, \mu)$ or to a space directly derived from it. An example can be found in [59].

As another example we mention the fact that inserting particular values for the parameters, spaces $T^\theta(\lambda, \mu)$ are identical to some spaces introduced by Schwartz, ([61]).

CHAPTER 4

KERNELS DEFINED BY MELLIN-BARNES INTEGRALS

4.0 INTRODUCTION

In the foregoing chapters the kernel of the integral transforms, Watson transforms as well as convolution transforms, have been defined by means of the inverse Mellin transform of some function K . It was assumed in all cases that there exists a vertical strip $\{s \in \mathbb{C} \mid p < \operatorname{Re} s < q\}$ in which K is analytic.

In this chapter we consider an extension of the Watson and convolution transform by defining the kernel k by a Mellin-Barnes integral,

$$k(t) = \frac{1}{2\pi i} \int_C K(s) ds$$

where C is a contour from $a-i\infty$ to $b+i\infty$, and K may have a finite number of first order poles in the strip under consideration. In order to do so we will define in the first place a new function space which is related to spaces of type $T^\theta(\lambda, \mu)$. This will be done in section 1. Also in this section we will define an auxiliary operator and its inverse which provides a topological isomorphism between the new spaces and the old ones. In section 2 we then consider the extensions of the integral transforms.

4.1 THE SPACE $T_{p,m}^\theta(\lambda, \mu; s_1, \dots, s_p)$

At first we repeat the definition of G_θ :

$$G_\theta = \{t \in \mathbb{C} \mid |\arg t| \leq \theta\}.$$

Next we define two other sets,

$$G_{1\theta} = \{t \in \mathbb{C} \mid 0 < |t| \leq 1 \text{ and } |\arg t| \leq \theta\},$$

$$G_{2\theta} = \{t \in \mathbb{C} \mid 1 \leq |t| \text{ and } |\arg t| \leq \theta\}.$$

We define two function spaces for which $G_{1\theta}$ and $G_{2\theta}$ are the sets of definition respectively.

DEFINITION 1. Let $\theta \geq 0$, $\lambda \in \mathbb{R}$. Let $\{\lambda_n\}$ be a sequence in $\mathbb{R}$ such that $\lambda_n \uparrow \lambda$ if $n \rightarrow \infty$. Then $T_1^\theta(\lambda)$ is the space of functions $\phi: G_{1\theta} \rightarrow \mathbb{C}$ with the properties:

ϕ is analytic on $G_{1\theta}^O$ if $\theta > 0$ and $\phi \in C^\infty(G_{1\theta})$ if $\theta = 0$;

$\phi^{(j)}$ may be extended continuously from $G_{1\theta}^O$ to $G_{1\theta}$ if $\theta > 0$, $j \in \mathbb{N}$;

$$(1.1) \quad \tau_{n1}(\phi) = \sup_{\substack{p \leq n \\ t \in G_{1\theta}}} |t^{\lambda_n + p} \phi^{(p)}(t)| < \infty, \quad n \in \mathbb{N}. \quad \square$$

With the multinorm (1.1), $T_1^\theta(\lambda)$ is a Fréchet space.

DEFINITION 2. Let $\theta \geq 0$, $\mu \in \mathbb{R}$. Let $\{\mu_n\}$ be a sequence in $\mathbb{R}$ such that $\mu_n \uparrow \mu$ if $n \rightarrow \infty$. Then $T_2^\theta(\mu)$ is the space of functions $\phi: G_{2\theta} \rightarrow \mathbb{C}$ with the following properties:

ϕ is analytic on $G_{2\theta}^O$ if $\theta > 0$ and $\phi \in C^\infty(G_{2\theta})$ if $\theta = 0$;

$\phi^{(j)}$ may be extended continuously to $G_{2\theta}$ if $\theta > 0$, $j \in \mathbb{N}$;

$$(1.2) \quad \tau_{n2}(\phi) = \sup_{\substack{p \leq n \\ t \in G_{2\theta}}} |t^{\mu_n + p} \phi^{(p)}(t)| < \infty, \quad n \in \mathbb{N}. \quad \square$$

Also $T_2^\theta(\mu)$ is a Fréchet space with the multinorm (1.2)

DEFINITION 3. Assume $\lambda < \mu$. Let $p \in \mathbb{N}$. Let $s_1, \dots, s_p$ be different complex numbers. Assume

$$\lambda < \operatorname{Re} s_j < \mu, \quad j = 1, \dots, p.$$

Let $m \in \mathbb{N}$, $m \leq p$. Then $T_{p,m}^\theta(\lambda, \mu; s_1, \dots, s_p) = T_{p,m}^\theta(\lambda, \mu; \underline{s})$ is the space of all functions ϕ with the properties

there exist $c_j(\phi) \in \mathbb{C}$, $j = 1, \dots, p$, such that

$$\chi_1(x) = \phi(x) - \sum_{j=1}^m c_j(\phi) x^{-s_j} \in T_1^\theta(\lambda),$$

$$\chi_2(x) = \phi(x) - \sum_{j=m+1}^p c_j(\phi) x^{-s_j} \in T_2^\theta(\mu). \quad \square$$

Clearly

$$(1.3) \quad \tau_n(\phi) = \sup_{j \leq n} |c_j(\phi)| + \tau_{n1}^\theta(\chi_1) + \tau_{n2}^\theta(\chi_2) < \infty,$$

and $T_{p,m}^\theta(\lambda, \mu; \underline{s})$ is a Fréchet space with the multinorm (1.3).

$T_{p,m}^\theta(\lambda, \mu; \underline{s})$ is an extension of the space $T^\theta(\lambda, \mu)$.

We consider a differential operator which is connected to this space:

$$D = \prod_{j=1}^p \left(t \frac{d}{dt} + s_j \right) = \prod_{j=1}^p \left(t^{1-s_j} \frac{d}{dt} t^{s_j} \right)$$

The factors in the right hand side commute, so the ordering of the factors is immaterial.

It turns out that the map

$$(1.4) \quad D : T_{p,m}^\theta(\lambda, \mu; \underline{s}) \rightarrow T^\theta(\lambda, \mu)$$

where D is defined by $D\phi = D\phi$, is an isomorphism.

THEOREM 25. Let $\lambda < \mu$ and s_j as before. Then the map D , defined by (1.4) is a continuous isomorphism. Its inverse may be defined as follows. If $\psi \in T^\theta(\lambda, \mu)$, $|\arg x| \leq \theta$, then

$$(1.5) \quad (D^{-1}\psi)(x) = - \sum_{j=1}^m \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \int_x^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} + \\ + \sum_{j=m+1}^p \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \int_0^x \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t}.$$

Equivalently,

$$(1.6) \quad (D^{-1}\psi)(x) = \frac{1}{2\pi i} \int_C \left\{ \prod_{j=1}^p (s_j - s) \right\}^{-1} \Psi(s) x^{-s} ds$$

where $\Psi = M\psi$ and C is a contour in the strip $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$ which runs from $a-i\infty$ to $b+i\infty$ such that $s_1, \dots, s_m$ are to the left of C and $s_{m+1}, \dots, s_p$ are to the right of C .

PROOF. The theorem is proved as is easily seen, if we can show that the following assertions are true.

Assertion 1: D maps $T_{p,m}^\theta(\lambda, \mu, \underline{s})$ continuously into $T^\theta(\lambda, \mu)$;

assertion 2: D^{-1} , defined by (1.6) maps $T^\theta(\lambda, \mu)$ continuously into $T_{p,m}^\theta(\lambda, \mu, \underline{s})$;

assertion 3: $D \cdot D^{-1} = Id$ on $T^\theta(\lambda, \mu)$, D^{-1} defined by (1.6);

assertion 4: $D^{-1} \cdot D = Id$ on $T_{p,m}^\theta(\lambda, \mu, \underline{s})$, D^{-1} defined by (1.6);

assertion 5: (1.5) and (1.6) imply one another.

Proof of assertion 1. Since $Dx^{-s_j} = 0$ it follows by straightforward computation that if $\phi \in T_{p,m}^\theta(\lambda, \mu, \underline{s})$ there exist constants M_1 and M_2 independent of ϕ , such that

$$\sup_{\substack{j \leq n \\ t \in G_\theta}} |t^{\lambda_{n(1+|t|)}} t^{\mu - \lambda_{n,j}} t^{j(D\phi)^{(j)}}(t)| < M_1 \tau_{n+p,1}(\phi) + M_2 \tau_{n+p,2}(\phi).$$

Hence $\phi \in T^\theta(\lambda, \mu)$ and moreover

$$D : T_{p,m}^\theta(\lambda, \mu, \underline{s}) \rightarrow T^\theta(\lambda, \mu)$$

is continuous.

Proof of assertion 2. Consider D^{-1} defined by (1.6). Let $\psi \in T^\theta(\lambda, \mu)$ and $\Psi = M\psi$. Choose r such that $\operatorname{Re} s_j \in (\lambda_r, \mu_r)$, $j = 1, \dots, p$. Here (λ_n) and (μ_n) are the defining sequences for the norms in both spaces under consideration. By residue calculus we get

$$\begin{aligned} (1.7) \quad \tilde{\phi}(x) &= (D^{-1}\psi)(x) = - \sum_{j=1}^m \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \psi(s_j) x^{-s_j} + \\ &\quad + \frac{1}{2\pi i} \int_{\lambda_r - i\infty}^{\lambda_r + i\infty} \left\{ \prod_{j=1}^p (s_j - s) \right\}^{-1} \Psi(s) x^{-s} ds \\ &= \sum_{j=1}^m d_j(\psi) x^{-s_j} + \chi_1(x), \end{aligned}$$

where

$$d_j = - \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \psi(s_j), \quad j = 1, \dots, m.$$

In the same manner

$$\begin{aligned} (1.8) \quad \tilde{\phi}(x) &= \sum_{j=m+1}^p \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \psi(s_j) x^{-s_j} + \\ &\quad + \frac{1}{2\pi i} \int_{\mu_r - i\infty}^{\mu_r + i\infty} \left\{ \prod_{j=1}^p (s_j - s) \right\}^{-1} \Psi(s) x^{-s} ds \\ &= \sum_{j=m+1}^p d_j(\psi) x^{-s_j} + \chi_2(x), \end{aligned}$$

where

$$d_j = \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \psi(s_j), \quad j = m+1, \dots, p.$$

According to the definition of the multinorm in $S^\theta(\lambda, \mu)$ and the continuity of the inverse Mellin transform and by virtue of (I.1.4) and theorem I.7, there exist constants C_1 and C_2 such that

$$|d_j(\psi)| < C_1 \sigma_r^\theta(\psi), \quad j = 1, \dots, p$$

and

$$\sigma_r^\theta(\psi) < C_2 \tau_{r+1}(\psi)$$

where C_1 and C_2 are independent of ψ and σ_r^θ denotes a norm in $S^\theta(\lambda, \mu)$. It follows that for some constant C_3 , independent of ψ , we have

$$(1.9) \quad \sup_{j \leq p} |d_j(\psi)| < C_3 \tau_{r+1}(\psi)$$

χ_1 is an analytic function of x on G_θ . Choose $n > r$. We have

$$\tau_{n,1}(\chi_1) = \sup_{\substack{j \leq n \\ t \in G_\theta}} |t^{\lambda_n + j} \chi_1^{(j)}(t)|.$$

Inserting the defining integral for χ_1 we obtain

$$\tau_{n,1}(\chi_1) \leq \sup_{\substack{j \leq n \\ t \in G_\theta}} \left| \int_{\lambda_r - i\infty}^{\lambda_r + i\infty} \frac{(s)_j \Psi(s)}{(s_1 - s) \dots (s_p - s)} t^{\lambda_n - s} ds \right|$$

Using arguments similar to those which lead from (I.3.6) to (I.3.7) we see that for some constant C_4 which again is independent of ψ

$$\tau_{n,1}(\chi_1) \leq C_4 \sigma_{n+2}^\theta(\psi).$$

From the proof of theorem I.7 it now follows that for some constant C_5 ,

$$(1.10) \quad \tau_{n,1}(\chi_1) \leq C_5 \tau_{n+3}^\theta(\psi).$$

In the same manner

$$(1.11) \quad \tau_{n,2}(\chi_2) \leq C_6 \tau_{n+3}(\psi).$$

Combining (1.9), (1.10) and (1.11) we have

$$(1.12) \quad \tau_n(D^{-1}\psi) \leq C \tau_m(\psi)$$

for some $C \in \mathbb{R}^+$ and some $m > n$, C being independent of ψ . From (1.12) we see $D^{-1}\psi \in T_{p,m}^\theta(\lambda, \mu, \underline{s})$ and also the continuity of D^{-1} follows.

Proof of assertion 3. If $\psi \in T^\theta(\lambda, \mu)$ then

$$D \frac{1}{2\pi i} \int_C \frac{\Psi(s)}{(s_1-s) \dots (s_p-s)} x^{-s} ds = \frac{1}{2\pi i} \int_C \Psi(s) x^{-s} ds = \psi(x).$$

Hence $D \cdot D^{-1} = Id$.

Proof of assertion 4. Let $\phi \in T_{p,m}^\theta(\lambda, \mu, \underline{s})$, $D\phi = \psi$ and $\tilde{\phi} = D^{-1}\psi$, where D^{-1} is defined by (1.6). The assertion is proved if we can show $\phi = \tilde{\phi}$. Because of assertions 1 and 3 we have $D\tilde{\phi} = \psi$.

From (1.7) it follows that

$$\begin{aligned} d_j(\psi) &= - \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \int_0^\infty t^{s_j-1} (D\tilde{\phi})(t) dt \\ &= - \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \int_0^\infty \frac{d}{dt} t^{s_j} \left(\prod_{\substack{h=1 \\ h \neq j}}^p t^{1-s_h} \frac{d}{dt} t^{s_h} \right) \tilde{\phi}(t) dt \\ &= - \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \left[t^{s_j} \left(\prod_{\substack{h=1 \\ h \neq j}}^p t^{1-s_h} \frac{d}{dt} t^{s_h} \right) \tilde{\phi}(t) \right]_0^\infty. \end{aligned}$$

Since we have

$$(1.13) \quad \phi(t) = \sum_{h=m+1}^p c_h t^{-s_h} + \chi_2(t), \quad \chi_2 \in T_2(\mu),$$

we may deduce that the contribution at $t = \infty$ only comes from the function χ_2 . Using the definition of $T_2(\mu)$ a simple computation shows that this contribution is zero. From the representation

$$(1.14) \quad \phi(t) = \sum_{h=1}^m c_h t^{-s_h} + \chi_1(t), \quad \chi_1 \in T_1(\lambda)$$

we deduce that the contribution at $t = 0$ in the limit term equals

$$c_j \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} t^{s_j} \left(\prod_{\substack{h=1 \\ h \neq j}}^p t^{1-s_h} \frac{d}{dt} t^{s_h} \right) t^{-s_j} \Big|_0.$$

Hence $d_j = c_j$, $j = 1, \dots, m$. Similarly $d_j = c_j$, $j = m+1, \dots, p$.

Using the representation (1.13) and (1.14) for ϕ and the corresponding representations for $\tilde{\phi}$ it easily follows that

$$(1.15) \quad \phi - \tilde{\phi} \in T^\theta(\lambda, \mu).$$

Moreover, $D(\phi - \tilde{\phi}) = 0$ and x^{-s_j} , $j=1, \dots, p$ being a fundamental system for $Dw = 0$, we have

$$\phi - \tilde{\phi} = \sum_{j=1}^p a_j x^{-s_j}, \quad a_j \in \mathbb{C}.$$

Combining this with (1.15) we deduce $\phi = \tilde{\phi}$.

Proof of assertion 5. We have

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \left\{ \prod_{j=1}^p (s_j - s) \right\}^{-1} \Psi(s) x^{-s} ds = \\ & = \frac{1}{2\pi i} \int_C \left\{ \prod_{j=1}^p (s_j - s) \right\}^{-1} \int_0^x \left(\frac{t}{x} \right)^s \psi(t) \frac{dt}{t} ds + \end{aligned}$$

$$+ \frac{1}{2\pi i} \int_C \left\{ \prod_{j=1}^p (s_j - s) \right\}^{-1} \int_x^\infty \left(\frac{t}{x}\right)^s \psi(t) \frac{dt}{t} ds$$

$$= I_1 + I_2.$$

First consider the case $p > 1$. Since

$$\int_0^x \left(\frac{t}{x}\right)^s \psi(t) \frac{dt}{t}$$

is an analytic function of s if $\operatorname{Re} s > \lambda$ and $|\arg x| \leq \theta$ and since the integral is $O((\operatorname{Re} s - \lambda_r)^{-1})$ as $\operatorname{Re} s \rightarrow \infty$ for any $r \in \mathbb{N}$, we may apply residue calculus to obtain

$$I_1 = \sum_{j=m+1}^p \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \int_0^x \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t}.$$

Similarly

$$I_2 = - \sum_{j=1}^m \left\{ \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j) \right\}^{-1} \int_x^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t}.$$

So (1.6) implies (1.5).

Now consider the case $p = 1$. Then

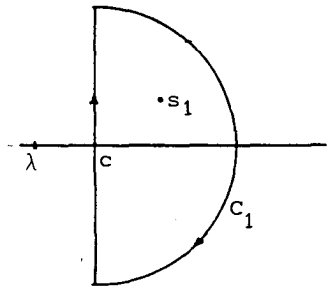
$$I_1 = \frac{1}{2\pi i} \int_C \frac{1}{s_1 - s} ds \int_0^x \left(\frac{t}{x}\right)^s \psi(t) \frac{dt}{t}.$$

Consider the contour in the figure.

We have

$$\int_{C_1} \frac{1}{s_1 - s} ds \int_0^x \left(\frac{t}{x}\right)^s \psi(t) \frac{dt}{t} \Big|_{s=c+Re^{i\alpha}} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} O\left(\frac{1}{R}\right) O\left(\frac{1}{R \cos \alpha + k}\right) R i e^{i\alpha} d\alpha, \quad R \rightarrow \infty,$$

where k is some positive constant. For the right hand side we may write



$$O(1) \int_0^{\frac{1}{2}\pi} \frac{1}{R \sin \alpha + k} d\alpha = O(1) \int_0^{\frac{1}{2}\pi} \frac{1}{R\alpha + k} d\alpha = o(1) \text{ as } R \rightarrow \infty.$$

It follows that

$$I_1 = \int_0^x \left(\frac{t}{x}\right)^{s_1} \psi(t) dt$$

and for I_2 we obtain a similar expression. Also here it follows that (1.6) implies (1.5).

Conversely, let D^{-1} be defined by (1.5). Using $\psi = M^{-1}\Psi$ we have

$$\int_x^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_x^\infty \left(\frac{t}{x}\right)^{s_j} \frac{dt}{t} \int_{C_0} \Psi(s) t^{-s} ds, \quad j=1, \dots, m,$$

where C_0 is a contour such that $\operatorname{Re}(s_j - s) < 0$ if $s \in C_0$, $j=1, \dots, m$.

Reversing the order of integration in the right hand side, we obtain

$$\int_x^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} = -\frac{1}{2\pi i} \int_C \frac{\Psi(s)}{s_j - s} x^{-s} ds, \quad j = 1, \dots, m,$$

where we have replaced C_0 by C . In the same manner we obtain

$$\int_0^x \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_C \frac{\Psi(s)}{s_j - s} x^{-s} ds.$$

Finally a straightforward computation shows the equivalence of the right hand side of (1.5) and the right hand side of (1.6). It follows that (1.5) implies (1.6) □

We will indicate a second proof of theorem 25. Without mentioning all details we may proceed as follows.

The differential equation $Dy = 0$ has as a fundamental system

$\{x^{-s_1}, \dots, x^{-s_p}\}$, which follows from $D = \prod_{j=1}^p \left(t^{1-s_j} \frac{d}{dt} t^{s_j}\right)$. A particular

solution of $Dy = \psi$, $\psi \in T^\theta(\lambda, \mu)$ is given by

$$y_0(x) = \sum_{j=1}^p \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j)^{-1} \int_0^x \left(\frac{t}{x}\right)^{s_j} \psi(t) dt.$$

This may be proven directly by the method of the variation of the parameters, or alternatively by induction $p \rightarrow p+1$, using the definition of D . The general solution of $Dy = \psi$ is

$$y(x) = \sum_{j=1}^p \gamma_j x^{-s_j} + y_0(x).$$

We are looking for a solution in $T_{p,m}^\theta(\lambda, \mu; \underline{s})$. From $\psi \in T^\theta(\lambda, \mu)$ it follows that

$$i_1(x) = \int_0^x \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} = x^{-s_j} \int_0^x O(t^{s_j-1-\lambda-\varepsilon}) dt = O(x^{-\lambda-\varepsilon}), \quad x \rightarrow 0,$$

for any $\varepsilon > 0$. It follows that $i_1(x) \in T_1^\theta(\lambda)$. From the definition of $T_{p,m}^\theta(\lambda, \mu; \underline{s})$ and the fact that x^{-s_j} is not a function in $T_1^\theta(\lambda)$, we may conclude that $\gamma_j = 0$ if $j > m$.

In the same way we obtain

$$i_2(x) = \int_x^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} = x^{-s_j} \int_x^\infty O(t^{s_j-1-\mu+\varepsilon}) dt = O(x^{-\mu+\varepsilon}), \quad x \rightarrow \infty,$$

for any $\varepsilon > 0$. Hence $i_2(x) \in T_2^\theta(\mu)$. Write

$$\int_0^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} = d_j x^{-s_j}.$$

From $\int_x^\infty = \int_0^\infty - \int_0^x$, $x^{-s_j} \notin T_2^\theta(\mu)$ and the definition of $T_{p,m}^\theta(\lambda, \mu; \underline{s})$ we see

that we have to put $d_j = \gamma_j$ if $j \leq m$. Then

$$\gamma_j x^{-s_j} + \int_0^x \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} = - \int_0^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t}$$

is a function in $T_2^\theta(\mu)$. We may conclude that the following solution is in $T_{p,m}^\theta(\lambda, \mu; \underline{s})$:

$$y(x) = - \sum_{j=1}^m \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j)^{-1} \int_x^\infty \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t} + \sum_{j=m+1}^p \prod_{\substack{h=1 \\ h \neq j}}^p (s_h - s_j)^{-1} \int_0^x \left(\frac{t}{x}\right)^{s_j} \psi(t) \frac{dt}{t}$$

Moreover it is seen that this is the only solution in that space.

This proves assertion 1 to 4 of the foregoing proof. The equivalence of (1.6) and (1.5) follows as in the preceding proof.

4.2 EXTENSION OF WATSON TRANSFORMS

In this section we define a kernel k for a Watson transform by means of a Mellin-Barnes integral. Here the integrand may have poles in the strip $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$, contrary to the case of chapter II.

It turns out that the corresponding Watson transform can be factorized as the product of two continuous maps, the map D^{-1} of theorem 25 and a Watson transform as described in theorem (II.8).

THEOREM 26. Assume (I.1.1). Let K be an analytic function on the strip $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$ except for simple poles $s_1, \dots, s_p$. Assume (II.1.1), $0 \leq m \leq p$, $c_1, c_2 \in (\lambda, \mu)$. Let C be a contour in S from $c_1 - i\infty$ to $c_2 + i\infty$ such that $s_1, \dots, s_m$ are to the left of C and $s_{m+1}, \dots, s_p$ are to the right of C . Choose $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$ such that

$$(2.1) \quad \frac{K(s)}{(1+\alpha-s)_n} \text{ is analytic on } C,$$

$$\frac{K(s)}{(1+\alpha-s)_n} = O(s^{-1-\delta}) \text{ as } s \rightarrow \infty \text{ on } C,$$

for some $\delta > 0$. Define k_n by

$$(2.2) \quad k_n(t) = \frac{1}{2\pi i} \int_C \frac{K(s)}{(1+\alpha-s)_n} t^{-s} ds, \text{ if } t > 0.$$

Then the map A defined by

$$(2.3) \quad (A\phi)(x) = x^{-\alpha \frac{d^n}{dx^n}} x^{n+\alpha-1} \int_0^\infty k_n(t) \phi\left(\frac{t}{x}\right) dt$$

is a continuous map of $T^\theta(1-\mu, 1-\lambda)$ into $T_{p,m}^\theta(\lambda, \mu; \underline{s})$ for any $\theta \geq 0$. It is independent of the choice of n and α .

Let

$$\tilde{K}(s) = \left\{ \prod_{j=1}^p (s_j - s) \right\} K(s)$$

and with this $\tilde{K}$ let $\tilde{A}^\gamma$ be defined by (II.1.5):

$$\tilde{A} = M^{-1} \cdot m_1(\tilde{K}) \cdot M,$$

as map from $T^\theta(1-\mu, 1-\lambda)$ into $T^\theta(\lambda, \mu)$. Then A admits the decomposition

$$A = D^{-1} \cdot \tilde{A},$$

where D and D^{-1} are defined as in theorem 25.

PROOF. Since there exists $r \in \mathbb{N}$ such that C lies entirely in the strip $\{s \in \mathbb{C} \mid \lambda_r \leq \operatorname{Re} s \leq \mu_r\}$ we may choose $n > \gamma_r + 1$, $n \in \mathbb{N}$. Then we may choose $\alpha \in C$ such that the assertions concerning K and k_n are satisfied.

It is easy to see that if $\phi \in T^\theta(1-\mu, 1-\lambda)$ and $|\arg x| \leq \theta$ then

$$\begin{aligned} \int_0^\infty k_n(t) \phi\left(\frac{t}{x}\right) dt &= \frac{1}{2\pi i} \int_C ds \frac{K(s)}{(1+\alpha-s)_n} \int_0^\infty t^{-s} \phi\left(\frac{t}{x}\right) dt \\ &= \frac{1}{2\pi i} \int_C \frac{K(s)}{(1+\alpha-s)_n} \phi(1-s) x^{1-s} ds. \end{aligned}$$

Hence $A\phi(x)$ exists and

$$(2.4) \quad (A\phi)(x) = \frac{1}{2\pi i} \int_C K(s) \phi(1-s) x^{-s} ds.$$

The right hand side may be written as

$$\frac{1}{2\pi i} \int_C \left\{ \prod_{j=1}^p (s_j - s) \right\}^{-1} K(s) \phi(1-s) x^{-s} ds$$

and using (1.6) we see that this is precisely $(D^{-1} \cdot \tilde{A})\phi(x)$, where $\phi \in T^\theta(1-\mu, 1-\lambda)$ and $|\arg x| \leq \theta$. Hence

$$A = D^{-1} \cdot \tilde{A}.$$

From theorem 8 it follows that $\tilde{A}: T^\theta(1-\mu, 1-\lambda) \rightarrow T^\theta(\lambda, \mu)$ is continuous. From theorem 25 it follows that $D^{-1}: T^\theta(\lambda, \mu) \rightarrow T_{p,m}^\theta(\lambda, \mu; \underline{s})$ is continuous and the theorem is proved. $\square$

REMARK

1. In the same manner as in remark (II.3) we have the following equivalent definition of the map A of theorem 26. Let $\alpha_j \in \mathbb{C}$, $j = 1, \dots, n$ be such that

$$\frac{K(s)}{\prod_{j=1}^n (\alpha_j - s)} \quad \text{is analytic on } \mathbb{C},$$

$$\frac{K(s)}{\prod_{j=1}^n (\alpha_j - s)} = O(s^{-1-\delta}) \quad \text{as } s \rightarrow \infty \text{ on } \mathbb{C}, \quad \delta > 0.$$

Define k_n^* by

$$k_n^*(t) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{K(s)}{(\alpha_1 - s) \dots (\alpha_n - s)} t^{-s} ds.$$

Define the operator D_n as in (II.1.3)*.

Then (2.3) is equivalent to

$$(A\phi)(x) = D_n \int_0^\infty k_n^*(t) \phi\left(\frac{t}{x}\right) \frac{dt}{x}, \quad |\arg x| \leq \theta.$$

If $\alpha_j = j + \alpha$ this reduces to (2.3).

In order to construct an inversion formula if the map A of theorem 26 is one to one, we consider the case in which K has possibly a finite number of simple zeros in the strip $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$. In that case the function H defined by $H(s) = \{K(1-s)\}^{-1}$ has simple poles. We then may expect that there exists a symmetrical extension of theorem 26. The details can be found in the next theorem.

THEOREM 27. Assume (I.1.1). Let the function K be analytic on $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$ except for simple poles $a_1, \dots, a_p$, $p \in \mathbb{N}$. Assume that the zeros of K are simple and that these zeros are $b_1, \dots, b_q$, $q \in \mathbb{N}$. Let C be a contour in S from $c_1 - i\infty$ to $c_2 + i\infty$

such that

$$\left. \begin{matrix} a_1, \dots, a_m \\ b_{n+1}, \dots, b_q \end{matrix} \right\} \text{ are to the left of } C,$$

$$\left. \begin{matrix} a_{m+1}, \dots, a_p \\ b_1, \dots, b_n \end{matrix} \right\} \text{ are to the right of } C,$$

where $0 \leq m \leq p$, $0 \leq n \leq q$. Assume

$$(2.5) \quad \begin{cases} \operatorname{Re} b_j > \operatorname{Re} a_h & \text{if } j=1, \dots, n; h=1, \dots, m, \\ \operatorname{Re} b_j < \operatorname{Re} a_h & \text{if } j=n+1, \dots, q; h=m+1, \dots, p. \end{cases}$$

Define H by $H(s) = \{K(1-s)\}^{-1}$. Let $K(s)$ and $H(s)$ satisfy conditions of type (II.1.1) and (II.2.2) respectively.

Choose $\alpha \in \mathbb{C}$, $g \in \mathbb{N}$ such that

$$\frac{K(s)}{(1+\alpha-s)_g} \text{ is analytic on } C,$$

$$\frac{K(s)}{(1+\alpha-s)_g} = O(s^{-1-\delta_1}) \text{ as } s \rightarrow \infty \text{ on } C, \delta_1 > 0.$$

Define $k_g(t)$ by

$$k_g(t) = \frac{1}{2\pi i} \int_C \frac{K(s)}{(1+\alpha-s)_g} t^{-s} ds \text{ if } t > 0.$$

Then the map A defined by

$$(2.6) \quad (A\phi)(x) = x^{-\alpha} \frac{d^g}{dx^g} x^{g+\alpha-1} \int_0^\infty k_g(t) \phi\left(\frac{t}{x}\right) dt, \quad |\arg x| \leq \theta,$$

is a continuous isomorphism of $T_{q,n}^\theta(1-\mu, 1-\lambda; 1-\underline{b})$ onto $T_{p,m}^\theta(\lambda, \mu; \underline{a})$, where $1-\underline{b} = (1-b_1, \dots, 1-b_q)$ and $\underline{a} = (a_1, \dots, a_p)$. A is independent of the choice of C , g and α .

If (2.5) is not satisfied then the assertion remains true if in (2.6) the integral is replaced by its Hadamard finite part, (cf [61], p 38).

The inverse may be characterized as follows.

Choose $\ell \in \mathbb{N}$, $\beta \in \mathbb{C}$ such that

$$\{K(s)(\beta+s)_\ell\}^{-1} \text{ is analytic on } \mathbb{C},$$

$$\{K(s)(\beta+s)_\ell\}^{-1} = O(s^{-1-\delta_2}) \text{ as } s \rightarrow \infty \text{ on } \mathbb{C}, \delta_2 > 0.$$

Define $h_\ell(t)$ by

$$h_\ell(t) = \frac{1}{2\pi i} \int_{\mathbb{C}} \{K(s)(\beta+s)_\ell\}^{-1} t^{s-1} ds \text{ if } t > 0.$$

Then

$$(A^{-1}\psi)(x) = x^{-\beta} \frac{d^\ell}{dx^\ell} x^{\ell+\beta-1} \int_0^\infty h_\ell(t) \psi\left(\frac{t}{x}\right) dt$$

if $\psi \in T_{p,m}^\theta(\lambda, \mu; \underline{a})$ and $|\arg x| \leq \theta$. Here again the integral has to be replaced by its Hadamard principal part if (2.5) is not satisfied.

PROOF. Define the following differential operators:

$$D_1 = \prod_{j=1}^q x^{b_j} \frac{d}{dx} x^{1-b_j} = \prod_{j=1}^q \left(x \frac{d}{dx} + 1 - b_j\right),$$

$$D_1^* = (-1)^q \prod_{j=1}^q x^{1-b_j} \frac{d}{dx} x^{b_j} = (-1)^q \prod_{j=1}^q \left(x \frac{d}{dx} + b_j\right),$$

$$D_2 = \prod_{j=1}^p x^{1-a_j} \frac{d}{dx} x^{a_j} = \prod_{j=1}^p \left(x \frac{d}{dx} + a_j\right).$$

Let $\tilde{k}_g(t)$ be defined by

$$\tilde{k}_g(t) = \frac{1}{2\pi i} \int_C \frac{K(s)}{(1+\alpha-s)_g} \prod_{j=1}^q (s-b_j)^{-1} t^{-s} ds, \quad t > 0.$$

Then we have

$$k_g(t) = D_1^* \tilde{k}_g(t).$$

Let $\phi \in T_{q,n}^\theta(1-\mu, 1-\lambda; 1-b)$ and $\phi_1 = D_1 \phi$. By theorem 25 $\phi_1 \in T^\theta(1-\mu, 1-\lambda)$.
Let

$$\alpha_1 = \max_{h=1, \dots, m} \operatorname{Re} a_h \quad \text{if } m > 0, \quad \alpha_1 = \lambda \quad \text{if } m = 0,$$

$$\alpha_2 = \min_{h=m+1, \dots, p} \operatorname{Re} a_h \quad \text{if } m < p, \quad \alpha_2 = \mu \quad \text{if } m = p,$$

$$\beta_1 = \min_{j=1, \dots, n} \operatorname{Re} b_j \quad \text{if } n > 0, \quad \beta_1 = \mu \quad \text{if } n = 0,$$

$$\beta_2 = \max_{j=n+1, \dots, q} \operatorname{Re} b_j \quad \text{if } n < q, \quad \beta_2 = \lambda \quad \text{if } n = q.$$

It follows from (2.5) that $\alpha_1 < \beta_1$, $\alpha_2 > \beta_2$. Moreover, for every $\varepsilon > 0$,

$$(x \frac{d}{dx})^j k_g(x), (x \frac{d}{dx})^j \tilde{k}_g(x) = O(x^{-\alpha_1 - \varepsilon}) \quad \text{if } x \rightarrow 0, \quad x \in \mathbb{R},$$

$$(x \frac{d}{dx})^j k_g(x), (x \frac{d}{dx})^j \tilde{k}_g(x) = O(x^{-\alpha_2 + \varepsilon}) \quad \text{if } x \rightarrow \infty, \quad x \in \mathbb{R},$$

$$(x \frac{d}{dx})^j \phi(x) = O(x^{\beta_1 - 1 - \varepsilon}) \quad \text{if } x \rightarrow 0, \quad |\arg x| \leq \theta,$$

$$(x \frac{d}{dx})^j \phi(x) = O(x^{\beta_2 - 1 + \varepsilon}) \quad \text{if } x \rightarrow \infty, \quad |\arg x| \leq \theta.$$

Hence $\int_0^\infty k_g(t) \phi(\frac{t}{x}) dt$ is absolutely convergent if $|\arg x| \leq \theta$. Now

$$(*) \quad \int_0^\infty k_g(t) \phi(\frac{t}{x}) dt = \int_0^\infty \tilde{k}_g(t) \phi_1(\frac{t}{x}) dt.$$

Indeed, by partial integration

$$\begin{aligned}
\int_0^\infty k_g(t) \phi\left(\frac{t}{x}\right) dt &= (-1)^q \int_0^\infty \left\{ \prod_{j=1}^q (t^{1-b_j} \frac{d}{dt} t^{b_j}) \tilde{k}_g(t) \right\} \phi\left(\frac{t}{x}\right) dt = \\
&= (-1)^q \left\{ \prod_{j=1}^{q-1} (t^{1-b_j} \frac{d}{dt} t^{b_j}) \tilde{k}_g(t) \right\} t \phi\left(\frac{t}{x}\right) \Big|_{t=0}^\infty + \\
&+ (-1)^{q-1} \int_0^\infty \left\{ \prod_{j=1}^{q-1} (t^{1-b_j} \frac{d}{dt} t^{b_j}) \tilde{k}_g(t) \right\} \left[u^q \frac{d}{du} \{ u^{1-b_q} \phi(u) \} \right]_{u=t/x} dt,
\end{aligned}$$

where the limit term is zero. Repeating this q times the assertion follows. If (2.5) does not hold, the right hand side of (*) is the Hadamard principal part of the left hand side, (cf [61]).

We have

$$A\phi = A_1\phi_1 = A_1 D_1 \phi,$$

where A_1 is defined by

$$(A_1 \psi)(x) = x^{-\alpha} \frac{d^q}{dx^q} x^{g+\alpha-1} \int_0^\infty \tilde{k}_g(t) \psi\left(\frac{t}{x}\right) dt$$

if $\psi \in T^\theta(1-\mu, 1-\lambda)$, $|\arg x| \leq \theta$. By theorem 26 A_1 is a continuous map of $T^\theta(1-\mu, 1-\lambda)$ into $T_{p,m}^\theta(\lambda, \mu; \underline{a})$, which may be seen by replacing in this theorem $K(s)$ by $K(s) \prod_{j=1}^q (s-b_j)^{-1}$. From theorem 26 it also follows that

$$A_1 = D_2^{-1} \cdot \gamma$$

where $\gamma\psi = M^{-1}(\tilde{K}(s)\Psi(1-s))$ for every $\psi \in T^\theta(1-\mu, 1-\lambda)$ with

$$\tilde{K}(s) = K(s) \prod_{j=1}^p (a_j - s) \prod_{j=1}^q (s - b_j)^{-1}.$$

Hence

$$A = D_2^{-1} \cdot \gamma \cdot D_1.$$

From theorem 25 it follows that

$$D_1 : T_{q,n}^{\theta}(1-\mu, 1-\lambda; 1-\underline{b}) \rightarrow T^{\theta}(1-\mu, 1-\lambda).$$

$$D_2^{-1} : T^{\theta}(\lambda, \mu) \rightarrow T_{p,m}^{\theta}(\lambda, \mu; \underline{a})$$

are continuous isomorphisms.

$\tilde{K}(s)$ is analytic on $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$, $\tilde{H}(s) = \{\tilde{K}(1-s)\}^{-1}$ is analytic on $\{s \in \mathbb{C} \mid 1-\mu < \operatorname{Re} s < 1-\lambda\}$. Moreover $\tilde{H}$ satisfies an order relation of the form (II.2.2). This follows from the assumptions on H . Hence we may apply theorem 10 in order to conclude that also

$$\gamma : T^{\theta}(1-\mu, 1-\lambda) \rightarrow T^{\theta}(\lambda, \mu)$$

is a continuous isomorphism. It follows that

$$A : T_{q,n}^{\theta}(1-\mu, 1-\lambda; 1-\underline{b}) \rightarrow T_{p,m}^{\theta}(\lambda, \mu; \underline{a})$$

is a continuous isomorphism and

$$A^{-1} = D_1^{-1} \cdot \gamma^{-1} \cdot D_2.$$

In the same manner as above we can prove that the map defined by the right hand side of (2.7) is a continuous isomorphism of $T_{p,n}^{\theta}(\lambda, \mu; \underline{a})$ onto $T_{q,n}^{\theta}(1-\mu, 1-\lambda; 1-\underline{b})$. Using

$$\tilde{H}(s) = H(s) \prod_{j=1}^q (1-b_j-s) \prod_{j=1}^p (s+a_j-1)^{-1}$$

we see that (2.6) indeed is a representation of the inverse of A . $\square$

REMARKS

2. In the same manner as in remark 1 an equivalent definition of the map A of theorem 27 may be given.
3. We may also give an equivalent definition of the map A^{-1} . Let D_{ℓ} be defined as in remark (II.6). Assume

$$\frac{H(s)}{(\beta_1-s)\dots(\beta_\ell-s)} = O(s^{-1-\delta}) \text{ as } s \rightarrow \infty \text{ on } C_1, \delta > 0,$$

where C_1 is the contour which arises from the contour C by replacing s by $1-s$. Define $h_\ell^*(t)$ by

$$h_\ell^*(t) = \frac{1}{2\pi i} \int_{C_1} \frac{H(s)}{(\beta_1-s)\dots(\beta_\ell-s)} t^{-s} ds.$$

Then A^{-1} of theorem 27 is also given by

$$(A^{-1}\psi)(x) = D_\ell \int_0^\infty h_\ell^*(t) \psi\left(\frac{t}{x}\right) \frac{dt}{x}, \quad |\arg x| \leq \theta.$$

4. As special cases in remark 2 and remark 3 respectively, we can choose some of the points α_j equal to b_j , respectively some of the points β_j equal to $1-a_j$. If $q < g$ even all points α_j may be chosen equal to q of the points b_j . Similarly, if $p < \ell$ we may choose $\beta_j = 1-a_j$, $j=1, \dots, p$.

4.3 EXTENSION OF THE CONVOLUTION TRANSFORM

In this section we give the theorems for the extension of the convolution transforms, corresponding to the theorems of the foregoing section. The proofs of these theorems may be given directly in the same manner as the proofs of the theorems of section 2. It is also possible to use the properties of the map R which was defined in the lemma of chapter I.

We first have the following analogue of theorem 26.

THEOREM 28. Let the conditions of theorem 26 be satisfied. Let $k_n(t)$ be defined as in that theorem. Then the map C , defined by

$$(3.1) \quad (C\phi)(x) = x^{-\alpha} \frac{d^n}{dx^n} x^{\alpha+n} \int_0^\infty k_n(t) \phi\left(\frac{x}{t}\right) \frac{dt}{t}$$

if $\phi \in T^\theta(\lambda, \mu)$, $|\arg x| \leq \theta$, is a continuous map of $T^\theta(\lambda, \mu)$ into $T_{p,m}^\theta(\lambda, \mu; \underline{s})$ for any $\theta \geq 0$.

Let $\tilde{\mathcal{C}}$ be the operator defined in theorem 12, (II.3.2) with $K(s)$ replaced by $\tilde{K}(s)$, where

$$\tilde{K}(s) = \left\{ \prod_{j=1}^p (s_j - s) \right\} K(s).$$

Then

$$C = D^{-1} \cdot \tilde{\mathcal{C}}$$

if D^{-1} is given by (1.6).

PROOF. Using the method described in the introduction of chapter II we have

$$\mathcal{C} = \tilde{\mathcal{A}} \cdot R$$

where $\tilde{\mathcal{A}}$ is as in theorem 26. Hence

$$D^{-1} \cdot \mathcal{C} = D^{-1} \cdot \mathcal{A} \cdot R = A \cdot R$$

and (3.1) now immediately follows from (2.3) by the substitution $\phi := R\phi$. $\square$

In order to prove the next theorem, which is the convolution version of theorem 27, we mention the following lemma.

LEMMA. The map R , defined in the lemma of section I.2 is a continuous isomorphism of $T_1^\theta(\lambda)$ onto $T_2^\theta(1-\lambda)$ for any λ and $\theta \geq 0$. Its continuous inverse is given by $R^{-1} = R$.

PROOF. Apart from some details, the proof is identical with the proof of the lemma of chapter I. $\square$

Using the definition of $T_{p,m}^\theta(\lambda, \mu; \underline{s})$ and the lemma above, it is not hard to see the validity of the following corollary.

COROLLARY. The map R is a continuous isomorphism of $T_{p,m}^\theta(\lambda, \mu; \underline{s})$ onto $T_{p,p-m}^\theta(1-\mu, 1-\lambda; 1-\underline{s})$ for any λ, μ and $\theta \geq 0$, where $\underline{s}$ and $1-\underline{s}$ are given by

$$\begin{aligned}\underline{s} &= (s_1, \dots, s_p) \\ 1-\underline{s} &= (1-s_p, \dots, 1-s_1).\end{aligned}$$

The continuous inverse is given by $R^{-1} = R$. $\square$

THEOREM 29. Let the assumptions of theorem 27 be satisfied. Then the map $C: T_{q,q-n}^\theta(\lambda, \mu; \underline{\hat{b}}) \rightarrow T_{p,m}^\theta(\lambda, \mu; \underline{a})$ given by

$$(3.3) \quad (C\phi)(x) = x^{-\alpha \frac{d^g}{dx^g}} x^{\alpha+g} \int_0^\infty k_g(t) \phi\left(\frac{x}{t}\right) \frac{dt}{t}, \quad |\arg x| \leq \theta,$$

is a continuous isomorphism. Here $\underline{\hat{b}} = (b_q, \dots, b_1)$. If (2.5) is not satisfied then the integral in the right hand side has to be replaced by its Hadamard finite part. C is independent of C , g and α . The continuous inverse may be characterized as follows.

Choose $\ell \in \mathbb{N}$, $\beta \in \mathbb{C}$ such that

$$\frac{1}{K(s)(1+\beta-s)_\ell} \text{ is analytic on } C,$$

$$\frac{1}{K(s)(1+\beta-s)_\ell} = O(s^{-1-\delta_2}) \text{ if } s \rightarrow \infty \text{ on } C, \delta_2 > 0.$$

Define $h_\ell(t)$ by

$$h_\ell(t) = \frac{1}{2\pi i} \int_C \frac{1}{K(s)(1+\beta-s)_\ell} t^{-s} ds.$$

Then C^{-1} is given by

$$(3.4) \quad (C^{-1}\psi)(x) = x^{-\beta} \frac{d^\ell}{dx^\ell} x^{\beta+\ell} \int_0^\infty h_\ell(t) \psi\left(\frac{x}{t}\right) \frac{dt}{t}$$

if $\psi \in T_{p,m}^\theta(\lambda, \mu; \underline{a})$ and $|\arg x| \leq \theta$. It is independent of C, β and ℓ .

PROOF. From (3.3) and (2.6) we deduce $C = A \cdot R$. By the corollary of the lemma, the map

$$R : T_{q,q-n}^\theta(\lambda, \mu; \underline{b}) \rightarrow T_{q,n}^\theta(1-\mu, 1-\lambda; 1-\underline{b})$$

is a continuous isomorphism. Combining this with theorem 27 we get theorem 29. □

REMARKS

5. The map C of theorem 28 may be given in an equivalent way.

Let $\alpha_j, j=1, \dots, n, k_n^*(t)$ and D_n be defined as in remark 1. Then the map C of theorem 28 is also given by

$$(C\phi)(x) = D_n \int_0^\infty k_n^*(t) \phi\left(\frac{x}{t}\right) \frac{dt}{t}$$

if $\phi \in T^\theta(\lambda, \mu)$ and $|\arg x| \leq \theta$.

6. Also the map C of theorem 29 may be described alternatively as in remark 5. The inverse has a representation analogous with the representation in remark 3.

Let D_ℓ , β_j , $j=1, \dots, \ell$ and $h_\ell^*(t)$ be defined as in remark 3. Then

$$(C^{-1}\psi)(x) = D_\ell \int_0^\infty h_\ell^*(t) \psi\left(\frac{x}{t}\right) \frac{dt}{t}, \quad |\arg x| \leq \theta.$$

In the same manner as in remark 4 we may choose some or all of the points α_j or β_j in the same special way.

4.4 EXAMPLES

In this section we first give some simple special cases of theorem 25, (examples I, II and III). Example IV is a more complicated one.

EXAMPLE I.

Consider the function K of example II of chapter II:

$$K(s) = \frac{1}{(s-a)^n}, \quad n \geq 2, \quad a \in \mathbb{C}.$$

Assume

$$(4.1) \quad \operatorname{Re} a - m \leq \lambda < \operatorname{Re} a - m + 1, \quad \operatorname{Re} a < \mu, \quad m \in \mathbb{N}.$$

Let k be defined by

$$(4.2) \quad k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) t^{-s} ds$$

where $\operatorname{Re} a < c < \mu$. As an extension of example II.II we now have: the map A , defined by

$$(A\phi)(x) = \frac{1}{\Gamma(n)} x^{-1} \int_0^1 t^{-a} (1-t)^{n-1} \phi\left(\frac{t}{x}\right) dt$$

is a continuous isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T_{m_0, m_0}^\theta(\lambda, \mu; \underline{a})$.

Here $m_0 = \min(m, n)$ and $\underline{a} = (a - m_0 + 1, \dots, a)$.

The assertion follows at once from theorem 27. It should be noted that we have from theorem 25 that $A \circ R = D^{-1}$ if $p = m_0 = n$.

EXAMPLE II.

Let $a \in \mathbb{C}$, $2a \notin \mathbb{Z}$, $n \in \mathbb{N}$, $n \geq 1$. Consider the function K ,

$$K(s) = \frac{1}{(s-a)_n (s+a)_n}.$$

Assume $\operatorname{Re} a \geq 0$. Moreover let (4.1) be satisfied and

$$(4.3) \quad -\operatorname{Re} a - \ell \leq \lambda < -\operatorname{Re} a - \ell + 1, \quad \ell \in \mathbb{N}.$$

Let k be defined by (4.2), again with $\operatorname{Re} a < c < \mu$. Then the map A ,

$$(4.4) \quad (A\phi)(x) = \frac{1}{x} \int_0^\infty k(t) \phi\left(\frac{x}{t}\right) \frac{dt}{t}, \quad |\arg x| \leq \theta,$$

is a continuous isomorphism of $T^\theta(\lambda, \mu)$ onto $T_{m_0+\ell_0, m_0-\ell_0}^\theta(\lambda, \mu; \underline{a})$ for any $\theta \geq 0$. Here $m_0 = \min(m, n)$, $\ell_0 = \min(\ell, n)$ and

$$\underline{a} = (a - m_0 + 1, \dots, a, -a - \ell_0 + 1, \dots, -a)$$

The assertion follows from theorem 28. An explicit expression for k can be found in example IV, chapter II.

REMARK.

Inversion formulas for A may be deduced from theorem 27 in case of example I and from theorem 28 in case of example II.

EXAMPLE III.

Let $K(s)$ be as in example I. Assume

$$\lambda < c < \operatorname{Re} a - n + 1 < \dots < \operatorname{Re} a < \mu.$$

Define k by (4.2). Then

$$k(t) = 0 \text{ if } 0 < t < 1,$$

$$k(t) = \frac{(-1)^n}{\Gamma(n)} t^{-a} (t-1)^{n-1} \text{ if } t > 1$$

and the map A , defined by

$$(A\phi)(x) = \frac{(-1)^n}{x\Gamma(n)} \int_1^\infty t^{-a} (t-1)^{n-1} \psi\left(\frac{t}{x}\right) dt$$

is a continuous isomorphism of $T^\theta(\lambda, \mu)$ onto $T_{n,0}^\theta(\lambda, \mu; \underline{a})$, where

$$\underline{a} = (a-n+1, \dots, a).$$

If $\tilde{K}(s)$ and $\tilde{A}$ are defined as in theorem 26, i.e.

$$\tilde{K}(s) = (s-a)_n K(s) = 1,$$

$$\tilde{A} = M^{-1} \cdot m_1(1) \cdot M,$$

we obtain

$$(\tilde{A}\phi)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(1-s) x^{-s} ds.$$

It follows that $\tilde{A} = R$, R being the map defined in the lemma of chapter I. Hence A has the decomposition

$$(4.5) \quad A = D^{-1} \cdot R$$

where D^{-1} is defined in (1.5) or (1.6) for this special case. From (4.5) it follows at once that A^{-1} exists. The latter fact may also be deduced by an application of theorem 27. A formula for the inverse A^{-1} may be derived in the same manner as in the corresponding example of chapter II.

EXAMPLE IV

Let $\rho, \nu, p, q \in \mathbb{N}$, $0 \leq \rho \leq q$, $0 \leq \nu \leq p$. Let

$$K(s) = \frac{\prod_{j=1}^{\rho} \Gamma(b_j + s) \prod_{j=1}^{\nu} \Gamma(1 - a_j - s)}{\prod_{j=1}^q \Gamma(1 - b_j - s) \prod_{j=1}^p \Gamma(a_j + s)}.$$

Assume

$$p+q \leq 2(\rho+\nu);$$

$$a_j - b_{j_0} \neq 1, 2, \dots \text{ if } j=1, \dots, \nu \text{ and } j_0=1, \dots, \rho.$$

Moreover let

$$\lambda < 1 - \operatorname{Re} a_j, \quad j=1, \dots, \nu;$$

$$\mu > -\operatorname{Re} b_j, \quad j=1, \dots, \rho.$$

Let S be the strip $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$.

Choose in S a contour from $c_1 - i\infty$ to $c_2 + i\infty$ such that

$-b_j, -b_j - 1, \dots$ are to the left of C if $j=1, \dots, \rho$;

$1-a_j, 2-a_j, \dots$ are to the right of C if $j=1, \dots, \nu$.

We have

$$K(s) = O(1)e^{\frac{1}{2}\pi i \operatorname{Im} s \{p+q-2(\rho+\nu)\}} \times \\ \times |s|^{(q-p)\operatorname{Re} s - \frac{1}{2}(q-p) + \operatorname{Re}\{\sum_{j=1}^q b_j - \sum_{j=1}^p a_j\}},$$

if $s \rightarrow \infty$, $s \in S$.

Case I. Assume $p+q = 2(\rho+\nu)$.

Let

$$(*) \quad (q-p)c_j - \frac{1}{2}(q-p) + \operatorname{Re}\left(\sum_{j=1}^q b_j - \sum_{j=1}^p a_j\right) < -1, \quad j=1, 2.$$

Then we may apply theorem 26 with $n=0$. In (2.3) the kernel k is now, (cf [12] I, p 207),

$$k(t) = G_{p,q}^{\rho,\nu}(t \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.).$$

The inverse in this case follows from theorem 27. With $H(s) = (K(1-s))^{-1}$ we obtain for the kernel of the inversion formula

$$h(t) = G_{p,q}^{q-\rho, p-\nu}(t \left| \begin{matrix} -a_{\nu+1}, \dots, -a_p, -a_1, \dots, -a_\nu \\ -b_{\rho+1}, \dots, -b_q, -b_1, \dots, -b_\rho \end{matrix} \right.).$$

If (*) is not satisfied we may apply theorem 26 with $n > 0$. In order to get a simple expression for k_n in (2.3) it is preferable to choose $1+\alpha$ in (2.1) equal to one of the $-a_j$. If this is not possible one can try to apply remark 1 with suitable values of α_j , corresponding to the Γ -factors in the denominator of $K(s)$, (cf [4], section 4, example 3).

Case II. If $p+q < 2(\rho+\nu)$ we may apply theorem 26. A formula for the inverse may be derived in the same manner as in chapter II, using $H(s) = H_+(s) + H_-(s)$.

However we may also proceed as follows.

$$\text{Put } H(s) = H_0(s) \pi^{-\rho-\nu} \prod_1^\rho \sin \pi(s-b_j) \prod_1^\nu \sin \pi(s-a_j)$$

where

$$H_0(s) = \prod_{j=1}^q \Gamma(s-b_j) \prod_{j=1}^p \Gamma(1-s+a_j).$$

The Mellin transform of H_0 is given by (cf [14])

$$(MH_0)(t) = G_{p,q}^{q,p}(t \left| \begin{matrix} -a_1, \dots, -a_p \\ -b_1, \dots, -b_q \end{matrix} \right.)$$

Now for some $c_g \in \mathbb{C}$, $g = 0, \dots, \rho+\nu$ we obtain

$$\pi^{-\rho-\nu} \prod_{j=1}^{\rho} \sin \pi(s-b_j) \prod_{j=1}^{\nu} \sin \pi(s-a_j) = \sum_{g=0}^{\rho+\nu} c_g e^{\pi i s(g-\rho-\nu)}$$

and hence

$$H(s) = \sum_{g=0}^{\rho+\nu} c_g e^{\pi i s(g-\rho-\nu)} H_0(s)$$

Furthermore

$$M\{e^{\pi i s g} H_0(s)\}(x) = G_{p,q}^{q,p}(x e^{-\pi i g} \left| \begin{matrix} -a_1, \dots, -a_p \\ -b_1, \dots, -b_q \end{matrix} \right.)$$

For some $r \in \mathbb{R}$ we have

$$H_0(s) = O(s^r e^{-\frac{1}{2}\pi(p+q)|\operatorname{Im} s|}) \text{ as } s \rightarrow \infty,$$

uniformly on any vertical strip, r depending on that strip. It follows that there exists $\ell \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ such that, (cf (II.2.8)),

$$\frac{1}{(1-d-i\sigma)_{\ell}} H_0(d+i\sigma) e^{\pi i(d+i\sigma)g} e^{\gamma\sigma} \in L(-\infty, \infty),$$

$|\gamma - g\pi| \leq \frac{1}{2}(p+q)\pi$, $g = -\rho-\nu, \dots, \rho+\nu$. As an example we may choose

$$\pm\gamma_{\pm} = (\rho+\nu - \frac{p+q}{2})\pi \text{ if } \pm g \geq 0.$$

In the simple case that we may put $\ell=0$, we obtain the inversion formula

$$\begin{aligned}
 (I) \quad (A^{-1}\psi)(x) = & \int_0^{\infty} e^{i\gamma t} \sum_{g=-\rho-\nu}^0 c_{g,p,q} G_{p,q}^{q,p}(xte^{-\pi ig} \left| \begin{smallmatrix} -a_1, \dots, -a_p \\ -b_1, \dots, -b_q \end{smallmatrix} \right.) \psi(t) dt + \\
 & + \int_0^{\infty} e^{i\gamma t} \sum_{g=1}^{\rho+\nu} c_{g,p,q} G_{p,q}^{q,p}(xte^{-\pi ig} \left| \begin{smallmatrix} -a_1, \dots, -a_p \\ -b_1, \dots, -b_q \end{smallmatrix} \right.) \psi(t) dt
 \end{aligned}$$

where the sums contain in both cases only terms for which $g+\rho+\nu = 0 \pmod{2}$. This formula holds if $\arg x = 0$. If $\ell > 0$ we have to modify the formula in the same manner as for example formula (II.2.11). The proof of (I) is similar to the proof of theorem 11.

REMARKS

1. If $g \neq \pm(\rho+\nu)$ one may take other contours in the corresponding integrals in (I). This follows from the fact that for such values of g we may take other values for γ , satisfying $|\gamma - g\pi| \leq \frac{1}{2}(\rho+q)\pi$.
2. In case of the Laplace transform and Meijer's K-transform we only have the terms with the coefficients c_{-1} and c_1 ; in this case $\rho+\nu=1$ and $c_0=0$.

CHAPTER 5

FRACTIONAL INTEGRATION OPERATORS AND HANKEL TRANSFORM

5.0 INTRODUCTION

The fractional integrals which we will consider in this chapter are generalizations of the Riemann-Liouville integrals and the Weyl integrals.

The Riemann-Liouville integral of order α is defined by

$$(R^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0, \quad \operatorname{Re} \alpha > 0.$$

The Weyl fractional integral of order α is defined by

$$(W^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t) dt, \quad x > 0, \quad \operatorname{Re} \alpha > 0.$$

Kober and Erdélyi have introduced the following generalizations, (cf [8], [30]).

$$(0.1) \quad (I_m^{\eta, \alpha} \phi)(x) = \frac{m}{\Gamma(\alpha)} x^{-m(\alpha+\eta)} \int_0^x (x-t^m)^{\alpha-1} t^{m\eta+m-1} \phi(t) dt, \quad x > 0,$$

$$(0.2) \quad (K_m^{\eta, \alpha} \phi)(x) = \frac{m}{\Gamma(\alpha)} x^{m\eta} \int_x^{\infty} (t^m-x^m)^{\alpha-1} t^{-m\eta-m\alpha+m-1} \phi(t) dt, \quad x > 0,$$

where $m > 0$, $\alpha, \eta \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$.

It is clear that the Riemann-Liouville and the Weyl integrals are special cases of (0.1) and (0.2) respectively. The integrals (0.1) and (0.2) may be written in the following form.

$$(0.1)' \quad (I_m^{\eta, \alpha} \phi)(x) = \int_0^{\infty} I_m^{\eta, \alpha} \left(\frac{x}{t} \right) \phi(t) \frac{dt}{t} = \int_0^{\infty} I_m^{\eta, \alpha}(u) \phi \left(\frac{x}{u} \right) \frac{du}{u},$$

$$(0.2) \quad (K_m^{\eta, \alpha} \phi)(x) = \int_0^\infty K_m^{\eta, \alpha}\left(\frac{x}{t}\right) \phi(t) \frac{dt}{t} = \int_0^\infty K_m^{\eta, \alpha}(u) \phi\left(\frac{x}{u}\right) \frac{du}{u},$$

where

$$(0.3) \quad I_m^{\eta, \alpha}(t) = \begin{cases} \frac{m}{\Gamma(\alpha)} (t^m - 1)^{\alpha-1} t^{-m(\eta+\alpha)} & \text{if } t > 1 \\ 0 & \text{if } 0 < t < 1 \end{cases}$$

$$(0.4) \quad K_m^{\eta, \alpha}(t) = \begin{cases} \frac{m}{\Gamma(\alpha)} (1-t^m)^{\alpha-1} t^{m\eta} & \text{if } 0 < t < 1 \\ 0 & \text{if } t > 1 \end{cases}.$$

$I_m^{\eta, \alpha}$ and $K_m^{\eta, \alpha}$ have simple Mellin transforms. In fact we have (cf[13]),

$$(0.5) \quad I_m^{\eta, \alpha}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1+\eta-s/m)}{\Gamma(1+\eta+\alpha-s/m)} t^{-s} ds, \quad c < m(1+\operatorname{Re} \eta),$$

$$(0.6) \quad K_m^{\eta, \alpha}(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(\eta+s/m)}{\Gamma(\alpha+\eta+s/m)} t^{-s} ds, \quad d > -m\operatorname{Re} \eta.$$

It follows from (0.1)' and (0.2)' that fractional integration may be seen as a special case of a convolution transform, where the kernels are defined by (0.5) and (0.6). We will treat some properties of these transforms in the next sections. Without loss of generality we may consider the case $m = 1$ only. This follows from the relations

$$(I_m^{\eta, \alpha} \phi)(x) = (I_1^{\eta, \alpha} \phi_m)(x^m),$$

$$(K_m^{\eta, \alpha} \phi)(x) = (K_1^{\eta, \alpha} \phi_m)(x^m),$$

where $\phi_m(x) = \phi(x^{1/m})$.

In section 5.1 we consider the operator $I_1^{\eta, \alpha}$, which will be denoted by $I(\eta, \alpha, 0)$ or by $I(\eta, \alpha)$ for short. In the same section we also consider the cut fractional integration operator, $I(\eta, \alpha, h)$, $h=1, 2, \dots$. In the section 5.2 we then consider the analogous properties of the operator $K_1^{\eta, \alpha}$, denoted by $K(\eta, \alpha, 0)$ or $K(\eta, \alpha)$ and its cut form.

There is a vast literature on the subject. This is partly due to the many cases in which fractional integrals are applicable to solve differential and (dual) integral equations. For literature see [44]. Historical notes may be found in [58].

We will give some applications of the fractional integration operators in connection with other integral transforms.

5.1 THE OPERATOR OF FRACTIONAL INTEGRATION $I(\eta, \alpha, h)$

In this section we define the operators of fractional integration and of cut fractional integration $I(\eta, \alpha, h)$, $h=0$ and $h \geq 1$ respectively, and deduce some of their properties.

We consider the following cases:

- I $\lambda < \mu \leq \operatorname{Re} \eta + 1$, $h = 0$
- II $\mu \leq \lambda < \operatorname{Re} \eta + 1$, $h = 0$
- III $h \in \mathbb{N} \setminus \{0\}$ and $\operatorname{Re} \eta + h \leq \lambda < \mu \leq \operatorname{Re} \eta + h + 1$
- IV $h \in \mathbb{N} \setminus \{0\}$ and $\operatorname{Re} \eta + h < \mu \leq \lambda < \operatorname{Re} \eta + h + 1$
- V $\lambda < \mu$, λ, μ arbitrary.

Let $\theta \geq 0$. Then we define the fractional integration operator $I(\eta, \alpha, h)$ on $T^\theta(\lambda, \mu)$ as follows. If $\operatorname{Re} \alpha > 1$ and $\phi \in T^\theta(\lambda, \mu)$,

$$\begin{aligned}
 (1.1) \quad I(\eta, \alpha, h)\phi(x) &= \\
 &= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \left[\int_0^x \{(x-t)^{\alpha-1} - \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-t/x)^j x^{\alpha-1}\} t^\eta \phi(t) dt + \right. \\
 &\quad \left. - \int_x^\infty \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-t/x)^j x^{\alpha-1} t^\eta \phi(t) dt \right], \quad x > 0.
 \end{aligned}$$

If $\operatorname{Re} \alpha \leq 1$ we choose some $n \in \mathbb{N}$ such that $\operatorname{Re} \alpha + n > 1$ and define

$$(1.2) \quad I(\eta, \alpha, h)\phi(x) = x^{-\alpha-\eta} \frac{d^n}{dx^n} \{x^{\alpha+\eta+n} I(\eta, \alpha+n, h)\phi(x)\}.$$

Then $I(\eta, \alpha, h)$ is a continuous operator of $T^\theta(\lambda, \mu)$ into itself in cases

I-IV and it does not depend on the choice of n . This follows from theorem II.12 and remark II.9 in the cases I and III and from theorem II.21 in the cases II and IV, if we apply these theorems with

$$(1.3) \quad K(s) = \frac{\Gamma(1+\eta-s)}{\Gamma(1+\eta+\alpha-s)}.$$

The function K defined by (1.3) is analytic in any strip $\{s \in \mathbb{C} \mid c_1 < \operatorname{Re} s < c_2\}$ which does not contain the points $\eta+1+m$, $m \in \mathbb{N}$ and on such strips we have

$$K(s) = O(s^{-\alpha}) \text{ as } s \rightarrow \infty.$$

(II.1.3)⁺ becomes with $p = \alpha + \eta$, $n \in \mathbb{N}$ such that $n + \operatorname{Re} \alpha > 1$:

$$(1.4) \quad k_n^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1+\eta-s)}{\Gamma(1+n+\eta+\alpha-s)} t^{-s} ds, \quad t > 0,$$

where $c \in (\lambda, \mu)$ in cases I and III and $c \in [\mu, \lambda]$ in cases II and IV.

By residue calculus we get

$$k_n^*(t) = \sum_{j=h}^{\infty} \frac{(-1)^j t^{-\eta-1-j}}{\Gamma(n-j+\alpha) j!} = \frac{t^{-\eta-1}}{\Gamma(\alpha+n)} \left\{ (1-1/t)^{\alpha+n-1} - \sum_{j=0}^{h-1} \binom{\alpha+n-1}{j} (-t)^{-j} \right\}, \quad t \geq 1$$

$$k_n^*(t) = \sum_{j=0}^{h-1} \frac{(-1)^{j+1} t^{-\eta-1-j}}{\Gamma(n-j+\alpha) j!} = - \frac{t^{-\eta-1}}{\Gamma(\alpha+n)} \sum_{j=0}^{h-1} \binom{\alpha+n-1}{j} (-t)^{-j}, \quad 0 \leq t \leq 1.$$

If $\phi \in T^{\theta}(\lambda, \mu)$, then define

$$(\dagger) \quad I(\eta, \alpha, h) \phi(x) = x^{-\alpha-\eta} \frac{d^n}{dx^n} x^{\alpha+\eta+n} \int_0^{\infty} k_n^*(t) \phi\left(\frac{x}{t}\right) \frac{dt}{t}.$$

Because of theorem II.12 and remark II.9, (II.3.1)⁺, the map $I(\eta, \alpha, h)$ is continuous from $T^{\theta}(\lambda, \mu)$ into itself in the cases I and III. Similarly applying theorem II.21 we have the same result in cases II and IV. Now $(\dagger)$ with $n = 0$ is equivalent to (1.1) if $\operatorname{Re} \alpha > 1$. In the general case we get (1.2).

(1.2) is also valid if $\operatorname{Re} \alpha + n \leq 1$. For let m be such that $\operatorname{Re} \alpha + m > 1$. Then using (1.2), $I(\eta, \alpha + n, h)$ is well defined by

$$I(\eta, \alpha + n, h) \phi(x) = x^{-\alpha - n - \eta} \frac{d^{m-n}}{dx^{m-n}} \{x^{\alpha + \eta + m} I(\eta, \alpha + m, h) \phi(x)\}.$$

If we insert this in the right hand side of (1.2) we obtain

$$x^{-\alpha - \eta} \frac{d^m}{dx^m} \{x^{\alpha + \eta + m} I(\eta, \alpha + m, h) \phi(x)\},$$

which again by (1.2) is precisely $I(\eta, \alpha, h)$.

From (1.1) and (1.2) or directly from (1.3) it now follows that

$$(1.5) \quad I(\eta, 0, h) \phi(x) = \phi(x)$$

and we may deduce from (1.2) and (1.5) that if $m \in \mathbb{N}$,

$$(1.6) \quad I(\eta, -m, h) \phi(x) = x^{m-\eta} \frac{d^m}{dx^m} \{x^\eta \phi(x)\}.$$

This operator does not depend on h and is defined on any space $T^\theta(\lambda, \mu)$, $\theta \geq 0$. (Now $K(s) = (\eta - s) \dots (\eta + 1 - m - s)$ is a polynomial).

It coincides with the operator D in (IV.1.4) with $s_j = \eta + 1 - j$, $j = 1, \dots, m$.

The inversion of $I(\eta, \alpha, h)$ follows from theorems 14 and 23 with

$$H(s) = \frac{1}{K(s)} = \frac{\Gamma(1 + \eta + \alpha - s)}{\Gamma(1 + \eta - s)}.$$

H arises from K by replacing η and α by $\eta + \alpha$ and $-\alpha$ respectively.

So $I(\eta, \alpha, h)$ is an automorphism of $T^\theta(\lambda, \mu)$ and

$$(1.7) \quad [I(\eta, \alpha, h)]^{-1} = I(\eta + \alpha, -\alpha, h_1)$$

if in combination with one of the cases I-IV also one of the cases VI-IX holds, where

$$\text{VI} \quad \lambda < \mu \leq 1 + \operatorname{Re}(\eta + \alpha), \quad h_1 = 0$$

VII $\mu \leq \lambda < 1 + \operatorname{Re}(\eta + \alpha)$, $h_1 = 0$

VIII $h_1 \in \mathbb{N} \setminus \{0\}$, $h_1 + \operatorname{Re}(\eta + \alpha) \leq \lambda < \mu \leq h_1 + 1 + \operatorname{Re}(\eta + \alpha)$

IX $h_1 \in \mathbb{N} \setminus \{0\}$, $h_1 + \operatorname{Re}(\eta + \alpha) < \mu \leq \lambda < h_1 + 1 + \operatorname{Re}(\eta + \alpha)$.

Note that if α is a negative integer, the inverse is given by the operator D^{-1} of (IV.1.6).

We now turn to the case V. Let $\lambda < \mu$. Assume that α is not an integer.

The function K in (1.3) has simple poles in $s = \eta + 1 + j$, $j \in \mathbb{N}$ and simple zeros in $s = 1 + \eta + \alpha + j$, $j \in \mathbb{N}$. Assume

$$(1.8) \quad \begin{cases} \lambda < \operatorname{Re} \eta + 1 + j < \mu \text{ if } j = j_0, \dots, j_0 + p - 1, j_0, p \in \mathbb{N} \\ \lambda < 1 + \operatorname{Re} \eta + \operatorname{Re} \alpha + k < \mu \text{ if } k = k_0 - 1, \dots, k_0 - q, k_0 \geq q, \\ \qquad \qquad \qquad k_0, q \in \mathbb{N} \end{cases}$$

and the other zeros and poles of K are outside the strip

$S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$. Let c be such that

$$(1.9) \quad \begin{cases} \operatorname{Re} \eta + j_0 + m < c < \operatorname{Re} \eta + j_0 + m + 1 \\ \operatorname{Re} \alpha + \operatorname{Re} \eta + k_0 - r < c < \operatorname{Re} \eta + \operatorname{Re} \alpha + k_0 - r + 1 \end{cases}$$

where $m, r \in \mathbb{N}$, $1 \leq m \leq p$ if $p > 0$, $1 \leq r \leq q$ if $q > 0$.

Let k_n^* and $I(\eta, \alpha, h)$ be defined by (1.4) and (†) for this value of c if $\operatorname{Re}(\eta + \alpha) > 1$. Then we may apply theorem IV.29 with

$$K(s) = \Gamma(1 + \eta - s) / \Gamma(1 + \eta + \alpha - s).$$

Write

$$(1.10) \quad \begin{aligned} \underline{b} &= (\eta + \alpha + k_0 + 1 - q, \dots, \eta + \alpha + k_0) \\ \underline{a} &= (\eta + j_0 + 1, \dots, \eta + j_0 + p). \end{aligned}$$

Then $I(\eta, \alpha, m + j_0)$ is an isomorphism of $T_{q, q-r}^\theta(\lambda, \mu; \underline{b})$ onto $T_{p, m}^\theta(\lambda, \mu; \underline{a})$.

If $p = q = 0$ this reduces to the previous case. Since

$$1/K(s) = \Gamma(1 + \eta + \alpha - s) / \Gamma(1 + \eta - s)$$

and (1.9) holds, its inverse is given by

$$I^{-1}(\eta, \alpha, m + j_0) = I(\eta + \alpha, -\alpha, k_0 - r).$$

If (1.9) is not satisfied we may apply theorem 29 in the following way. Let C be a contour from $c_1 - i\infty$ to $c_2 + i\infty$, which lies entirely

in the strip S . Assume that

$$\left. \begin{array}{l} \eta+j_0+m+1, \dots, \eta+j_0+p \\ \eta+\alpha+k_0-r+1, \dots, \eta+\alpha+k_0-q+1 \end{array} \right\} \text{ are to the right of } C$$

and the other poles and zeros are to the left of C . Then we may apply IV (3.3) to this case if in that formula the integral is replaced by its Hadamard finite part.

Finally we mention the fact that we may derive relations between fractional integrals of different order in the same manner as is done in [4]. As an example, if all operators make sense:

$$I(\eta, \alpha, h) \cdot I(\eta + \alpha, \beta, h) = I(\eta + \alpha, \beta, h_1) \cdot I(\eta, \alpha, h) = I(\eta, \alpha + \beta, h).$$

5.2 THE OPERATOR OF FRACTIONAL INTEGRATION $K(\eta, \alpha, h)$

The second operator of fractional integration may be treated in the same manner as the operator $I(\eta, \alpha, h)$ in the foregoing section. However we will take a different approach which will enable us to use the results of section 1. As before we put $m = 1$ and in stead of $K_m^{\eta, \alpha}$ we use the notation $K(\eta, \alpha)$. The cut fractional form will be denoted by $K(\eta, \alpha, h)$, $h \geq 1$, $h \in \mathbb{N}$.

From (0.2) we have

$$(2.1) \quad K(\eta, \alpha) \phi(x) = \frac{1}{\Gamma(\alpha)} x^\eta \int_x^\infty (t-x)^{\alpha-1} t^{-\eta-\alpha} \phi(t) dt, \quad x > 0.$$

By the substitution $t := 1/t$, $x := 1/x$, $\psi = K(\eta, \alpha) \phi$, we obtain

$$(2.2) \quad \frac{1}{x} \psi\left(\frac{1}{x}\right) = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha} \int_0^x (x-t)^{\alpha-1} t^\eta \frac{1}{t} \phi\left(\frac{1}{t}\right) dt.$$

Using the map R which was defined in chapter I by (I.2.8) we may write in stead of (2.2)

$$(2.3) \quad R\psi = I(\eta, \alpha) \cdot R\phi.$$

This in turn may be read as

$$(2.4) \quad K(\eta, \alpha) = R \cdot I(\eta, \alpha) \cdot R.$$

In the same manner we may deduce from (1.1), for the moment in a formal way,

$$(2.5) \quad \begin{aligned} R \cdot I(\eta, \alpha, h) \cdot R\phi(x) = \\ = \frac{x^\eta}{\Gamma(\alpha)} \left[\int_x^\infty \{ (t-x)^{\alpha-1} t^{1-\alpha} - \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-xt^{-1})^j \} t^{-\eta-1} \phi(t) dt + \right. \\ \left. - \int_0^x \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-xt^{-1})^j t^{-\eta-1} \phi(t) dt \right], \operatorname{Re} \alpha > 1. \end{aligned}$$

The right hand side gives the definition of the cut fractional integration operator as given by Erdélyi (cf [8]).

Denoting this operator by $K(\eta, \alpha, h)$ we have

$$(2.6) \quad K(\eta, \alpha, h) = R \cdot I(\eta, \alpha, h) \cdot R, \quad h = 1, 2, \dots, \operatorname{Re} \alpha > 1.$$

From the lemma of chapter I and the preceding section we see that the right hand side of (2.6) represents a continuous map of $T^\theta(\lambda, \mu)$ into itself if α and η satisfy one of the conditions I-IV with λ and μ replaced by $1-\mu$ and $1-\lambda$ respectively. We therefore use (2.6) to derive properties of the operator $K(\eta, \alpha, h)$ from those of $I(\eta, \alpha, h)$. First we mention some properties of R .

By direct computation we may show

$$(2.7) \quad R x^\alpha \phi(x) = x^{-\alpha} R \phi(x), \quad \alpha \in \mathbb{C},$$

$$(2.8) \quad R \frac{d}{dx} \phi(x) = -x \frac{d}{dx} x R \phi(x).$$

From (2.7) and (2.8) we deduce

$$(2.9) \quad R(x^2 \frac{d}{dx}) \phi(x) = x^{-1} (-\frac{d}{dx}) x R \phi(x).$$

Using (2.9) we may derive a simple expression for $R(\frac{d}{dx})^m$, $m \in \mathbb{N}$.

We have

$$\begin{aligned} R(\frac{d}{dx})^m &= R \prod_{j=1}^m (x^{1-j} \frac{d}{dx} x^j) x^{-m} \\ &= R \prod_{h=1}^m (x^{h-m} \frac{d}{dx} x^{m+1-h}) x^{-m} \\ &= R x^{-m-1} (x^2 \frac{d}{dx})^m x^{1-m} \\ &= x^{m+1} (\frac{1}{x} (-\frac{d}{dx}) x)^m x^{m-1} R, \end{aligned}$$

hence we obtain

$$(2.10) \quad R(\frac{d}{dx})^m = x^m (-\frac{d}{dx})^m x^m R, \quad m \in \mathbb{N}.$$

Consider again (2.6); (if $h = 0$, (2.6) has to be replaced by (2.4)).

The effect of the right hand side on a space $T^\theta(\lambda, \mu)$ is given by the following scheme (in case I'-IV' below; cf lemma 1 in § I.1):

$$(2.11) \quad T^\theta(\lambda, \mu) \xrightarrow{R} T^\theta(1-\mu, 1-\lambda) \xrightarrow{I(\eta, \alpha, h)} T^\theta(1-\mu, 1-\lambda) \xrightarrow{R} T^\theta(\lambda, \mu).$$

From section 1 it follows that $I(\eta, \alpha, h)$ is a continuous map if one of the following relations I'-V' holds, which arise from I-V, section 1, by replacing λ by $1-\mu$ and μ by $1-\lambda$:

$$\text{I}' \quad -\operatorname{Re} \eta \leq \lambda < \mu$$

$$\text{II}' \quad -\operatorname{Re} \eta < \mu \leq \lambda$$

$$\text{III}' \quad -\operatorname{Re} \eta - h \leq \lambda < \mu \leq -\operatorname{Re} \eta - h + 1, \quad h \in \mathbb{N} \setminus \{0\}$$

$$\text{IV}' \quad -\operatorname{Re} \eta - h < \mu \leq \lambda < -\operatorname{Re} \eta - h + 1, \quad h \in \mathbb{N} \setminus \{0\}$$

$$\text{V}' \quad \lambda < \mu, \quad \lambda, \mu \text{ arbitrary.}$$

Hence we have: if $\operatorname{Re} \alpha > 1$, then $K(\eta, \alpha, h)$, $h=0, 1, \dots$ is a continuous operator of $T^\theta(\lambda, \mu)$ if one of the conditions I' - IV' is satisfied.

We now extend to the case $\operatorname{Re} \alpha \leq 1$.

Let $n \in \mathbb{N}$ be such that $n + \operatorname{Re} \alpha > 1$. Apply (2.7) and (2.10) to the right hand side of (1.2). We obtain

$$R \cdot I(\eta, \alpha, h) \cdot R = x^{\alpha+\eta+n} \left(-\frac{d}{dx}\right)^n x^{-\alpha-\eta} R \cdot I(\eta, \alpha+n, h) \cdot R.$$

Hence we may consider

$$(2.12) \quad K(\eta, \alpha, h) = x^{\alpha+\eta+n} \left(-\frac{d}{dx}\right)^n x^{-\alpha-\eta} K(\eta, \alpha+n, h)$$

as the extension of $K(\eta, \alpha, h)$ to the case $\operatorname{Re} \alpha \leq 1$.

As a special case we derive from (1.6), (2.6) and (2.10)

$$(2.13) \quad K(\eta, -m, h) = x^\eta \left(-\frac{d}{dx}\right)^m x^{m-\eta}, \quad m \in \mathbb{N}.$$

Assume now that in combination with one of the cases I' - IV' also one of the cases VI' - IX' hold, where

$$\text{VI}' \quad -\operatorname{Re}(\eta+\alpha) \leq \lambda < \mu, \quad h_1=0$$

$$\text{VII}' \quad -\operatorname{Re}(\eta+\alpha) < \mu \leq \lambda, \quad h_1=0$$

$$\text{VIII}' \quad -\operatorname{Re}(\eta+\alpha) - h_1 \leq \lambda < \mu \leq -\operatorname{Re}(\eta+\alpha) - h_1 + 1, \quad h_1 \in \mathbb{N} \setminus \{0\}$$

$$\text{IX}' \quad -\operatorname{Re}(\eta+\alpha) - h_1 < \mu \leq \lambda < -\operatorname{Re}(\eta+\alpha) - h_1 + 1, \quad h_1 \in \mathbb{N} \setminus \{0\}.$$

These relations follow from VI - IX by replacing λ and μ by $1-\mu$ and $1-\lambda$ respectively. Then $I(\eta, \alpha, h)$ is an automorphism of $T^\theta(1-\mu, 1-\lambda)$ and from (2.6) and the properties of R it follows that $K(\eta, \alpha, h)$ is an automorphism of $T^\theta(\lambda, \mu)$.

From (1.7) we deduce

$$(2.14) \quad [K(\eta, \alpha, h)]^{-1} = K(\eta + \alpha, -\alpha, h_1).$$

From (2.5), (2.6) and (2.12) the following explicit expression for $K(\eta, \alpha, h)\phi(t)$ may be deduced directly. Let $\phi \in T^\theta(\lambda, \mu)$. Then if one of the conditions I'-IV' is satisfied we have

$$\begin{aligned} K(\eta, \alpha, h)\phi(x) &= \frac{1}{\Gamma(\alpha+n)} x^{\eta+\alpha+n} \left(-\frac{d}{dx}\right)^n x^{-\eta-\alpha} \times \\ &\times \left[\int_0^{\frac{x}{t}} \left(\frac{x}{t}\right)^\eta \left\{ \left(1 - \frac{x}{t}\right)^{\alpha+n-1} - \sum_{j=0}^{h-1} \binom{\alpha+n-1}{j} \left(-\frac{x}{t}\right)^j \right\} \phi(t) \frac{dt}{t} + \right. \\ &\left. - \int_0^{\frac{x}{t}} \left(\frac{x}{t}\right)^\eta \sum_{j=0}^{h-1} \binom{\alpha+n-1}{j} \left(-\frac{x}{t}\right)^j \phi(t) \frac{dt}{t} \right], \quad |\arg x| \leq \theta. \end{aligned}$$

If $n = 0$ this reduces to (2.5).

REMARK

From the representation

$$I(\eta, \alpha, h) = M^{-1} \cdot m(K) \cdot M$$

where K is defined by (1.3) and $m(K)$ by (I.1.6), we have by straightforward computation

$$\begin{aligned} K(\eta, \alpha, h) &= R \cdot M^{-1} \cdot m(K) \cdot M \cdot R \\ &= M^{-1} \cdot m_1(1) \cdot m_1(K) \cdot M \\ &= M^{-1} \cdot m(\tilde{K}) \cdot M \end{aligned}$$

where $m_1(\cdot)$ is defined by (I.1.7) and $\tilde{K}(s) = K(1-s)$. It follows that the Mellin transform of the kernel of $K(\eta, \alpha, h)$ is given by

$$(11) \quad H(s) = \frac{\Gamma(\eta+s)}{\Gamma(\eta+\alpha+s)}$$

if $\lambda < \operatorname{Re} s < \mu$ in case I' or III' and $\mu \leq \operatorname{Re} s \leq \lambda$ in case of II' or IV'.

We now consider the case V' .

Assume

$$(2.18) \quad \begin{cases} \lambda < -\operatorname{Re} \eta - j < \mu \text{ if } j = j_0, \dots, j_0 + p - 1, j_0 \in \mathbb{N}, p \in \mathbb{N} \\ \lambda < -\operatorname{Re} \eta - \operatorname{Re} \alpha - k < \mu \text{ if } k = k_0 - 1, \dots, k_0 - q, q \in \mathbb{N}, k \in \mathbb{N}, \\ k_0 \in \mathbb{N} \setminus \{0\}, k_0 \geq q \end{cases}$$

and let the other poles and zeros be outside the strip $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$, (cf. (II)).

Assume $m, r \in \mathbb{N}$, $c \in \mathbb{R}$ and

$$(2.19) \quad \begin{cases} -\operatorname{Re} \eta - j_0 - m < c < -\operatorname{Re} \eta - j_0 - m + 1, & 0 \leq m \leq p \\ -\operatorname{Re} \eta - \operatorname{Re} \alpha - k_0 + r < c < 1 - \operatorname{Re} \eta - \operatorname{Re} \alpha - k_0 + r, & 0 \leq r \leq q. \end{cases}$$

By the corollary of the lemma of chapter IV we obtain that the map

$$R : T_{q,r}^{\theta}(\lambda, \mu; -\eta - \alpha - k_0 + 1, \dots, -\eta - \alpha + q - k_0) \rightarrow \\ \rightarrow T_{q,q-r}^{\theta}(1 - \mu, 1 - \lambda; 1 + \eta + \alpha + k_0 - q, \dots, \eta + \alpha + k_0)$$

is a continuous isomorphism. From section 1 we deduce that

$$I(\eta, \alpha, m + j_0) : T_{q,q-r}^{\theta}(1 - \mu, 1 - \lambda; 1 + \eta + \alpha + k_0 - q, \dots, \eta + \alpha + k_0) \rightarrow \\ \rightarrow T_{p,m}^{\theta}(1 - \mu, 1 - \lambda; \eta + j_0 + 1, \dots, \eta + j_0 + p)$$

is a continuous isomorphism. Again an application of the lemma already mentioned gives the continuity of the isomorphism

$$R : T_{p,m}^{\theta}(1 - \mu, 1 - \lambda; \eta + j_0 + 1, \dots, \eta + j_0 + p) \rightarrow \\ \rightarrow T_{p,p-m}^{\theta}(\lambda, \mu; 1 - \eta - j_0 - p, \dots, 1 - \eta - j_0).$$

Hence if

$$\underline{a} = (-\eta - \alpha - k_0 + 1, \dots, -\eta - \alpha - k_0 + q)$$

$$\underline{b} = (1 - \eta - j_0 - p, \dots, -\eta - j_0)$$

the map

$$(2.20) \quad K(\eta, \alpha, m+j_0) : T_{q,r}^{\theta}(\lambda, \mu; \underline{a}) \rightarrow T_{p,p-m}^{\theta}(\lambda, \mu; \underline{b})$$

is a continuous isomorphism. Its continuous inverse may be derived using (2.14),

$$[K(\eta, \alpha, m+j_0)]^{-1} = K(\eta+\alpha, -\alpha, k_0-r).$$

If (2.19) is not satisfied, let C be a contour in S from $c_1-i\infty$ to $c_2+i\infty$. Assume that

$$\left. \begin{array}{l} -\eta-j_0-m+1, \dots, -\eta-j_0 \\ -\eta-\alpha-k_0+r+1, \dots, -\eta-\alpha-k_0+q \end{array} \right\} \text{ are to the right of } C$$

and the other poles and zeros are to the left of C .

Then the assertion remains valid if the integrals are interpreted as the Hadamard principal part.

5.3 APPLICATIONS TO HANKEL TRANSFORMS

The Hankel transform H_{ν} is defined by

$$(3.0) \quad \psi(x) = (H_{\nu}\phi)(x) = \int_0^{\infty} (xt)^{\frac{1}{2}} J_{\nu}(xt) \phi(t) dt$$

where J_{ν} is the Bessel function of the first kind and of order ν . If we substitute $x:=\sqrt{2x}$, $t:=\sqrt{2t}$ we obtain (with a slightly different notation in the index)

$$(3.1) \quad \tilde{\psi}(x) = (H_a \tilde{\phi})(x) = \int_0^{\infty} J_{2a}(2\sqrt{xt}) \tilde{\phi}(t) dt,$$

where $\tilde{\psi}(x) = (2x)^{-\frac{1}{2}} \psi(\sqrt{2x})$ and $\tilde{\phi}(t) = (2t)^{-\frac{1}{2}} \phi(\sqrt{2t})$. We will denote (3.1) also as Hankel transform. It has the advantage that foregoing theorems are applicable in a more direct way. In the sequel we omit the $\sim$.

The Mellin transform of the kernel in (3.1) is given by (cf [46], p 196)

$$K(s) = \frac{\Gamma(a+s)}{\Gamma(a+1-s)}, \quad \operatorname{Re} s > -\operatorname{Re} a.$$

Assume

$$(3.2) \quad -\operatorname{Re} a \leq \lambda < \mu \leq 0.$$

K satisfies

$$K(s) = O(s^{2\operatorname{Re} s - 1}) \text{ as } s \rightarrow \infty,$$

uniformly on $\{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$. $K(s)$ and $[K(1-s)]^{-1} = K(s)$ satisfy the conditions of theorem 8 and of theorem 10 with $n=0$. Hence

$$(3.3) \quad H_a : T^\theta(1-\mu, 1-\lambda) \rightarrow T^\theta(\lambda, \mu)$$

where

$$(H_a \phi)(x) = \int_0^\infty J_{2a}(2\sqrt{u}) \phi\left(\frac{u}{x}\right) \frac{du}{x}$$

is a continuous isomorphism and $H_a \cdot H_a = Id$, if (3.2) is satisfied. If (3.2) is weakened to

$$(3.5) \quad -\operatorname{Re} a \leq \lambda < \mu$$

we choose $n \in \mathbb{N}$ and $c \in \mathbb{R}$ such that

$$(3.6) \quad \lambda < c < \mu, \quad c \neq \operatorname{Re} a + 1, \dots, \operatorname{Re} a + n,$$

$$(3.7) \quad \frac{K(s)}{(a+1-s)_n} \in L(-\infty, \infty) \text{ if } \operatorname{Re} s = c.$$

Define $k(t)$ by

$$k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{K(s)}{(a+1-s)_n} t^{-s} ds, \quad t > 0.$$

We have

$$k(t) = t^{-\frac{1}{2}n} J_{2a+n}(2\sqrt{t}).$$

The map $H_{a,n}$, defined by

$$(3.8) \quad (H_{a,n}\phi)(x) = x^{-a} \frac{d^n}{dx^n} x^{n+a-1} \int_0^\infty t^{-\frac{1}{2}n} J_{2a+n}(2\sqrt{t}) \phi\left(\frac{t}{x}\right) dt$$

is a continuous isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T^\theta(\lambda, \mu)$ if (3.5), (3.6) and (3.7) are satisfied. This follows from (II.1.3)⁺ and (II.1.4)⁺ with $p = a$. Both (3.1) and (3.8) may be factorized by $M^{-1} \cdot m_1(K) \cdot M$, where $K(s) = \Gamma(a+s)/\Gamma(a+1-s)$. The inverse is given by $H_{a,n}^{-1} = H_{a,n}$. Using the decomposition of $H_{a,n}$ in terms of the Mellin transform and the corresponding decomposition of $I(\eta, \alpha, h)$, (cf (1.3)), we have formally

$$(3.9) \quad (H_{a,n}\phi)(x) = x^{-\alpha} I(a+\alpha, -2\alpha, h) H_{a+\alpha, n} x^{-\alpha} \phi(x),$$

if $\phi \in T^\theta(1-\mu, 1-\lambda)$ and $\alpha \in \mathbb{C}$. As an example of the validity of (3.9) let $h = 0$ and assume

$$-\operatorname{Re} a \leq \lambda < \mu \leq \operatorname{Re}(a+2\alpha) + 1.$$

Then condition I of section 1 is satisfied for the operator $I(a+\alpha, -2\alpha, 0)$ of $T^\theta(\lambda-\alpha, \mu-\alpha)$ and hence (3.9) holds.

We next consider an example of a more general case $\lambda < \mu$. The (simple) poles of K are $-a, -a-1, \dots$. K has simple zeros $a+1, a+2, \dots$. Let $p, q \in \mathbb{N}$ and assume $\lambda < c < 0$. Assume furthermore that the strip $S = \{s \in \mathbb{C} \mid \lambda < \operatorname{Re} s < \mu\}$ contains the points $-a-p+1, \dots, -a$ and the points $a+1, \dots, a+q$ and no other zeros and poles of K . Let C be the straight line from $c-i\infty$ to $c+i\infty$. Assume

$$(3.10) \quad \left. \begin{array}{l} -a-p+j, \quad j=1, \dots, m \\ a+h, \quad h=1, \dots, q-n \end{array} \right\} \text{are to the left of } C$$

$$\left. \begin{array}{l} -a-p+j, \quad j=m+1, \dots, p \\ a+h, \quad h=q-n+1, \dots, q \end{array} \right\} \text{are to the right of } C,$$

where $0 \leq m \leq p$ and $0 \leq n \leq q$. We have

$$\frac{\Gamma(a+s)}{\Gamma(a+1-s)} = O(s^{2c-1}) \text{ as } s \rightarrow \infty, \operatorname{Re} s = c.$$

It follows that the conditions of theorem 27 are satisfied. Let

$$k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) t^{-s} ds \text{ if } t > 0.$$

By residue calculus we obtain

$$(3.11) \quad k(t) = J_{2a}(2\sqrt{t}) - \sum_{k=0}^{p-m-1} \frac{(-1)^k t^{a+k}}{k! \Gamma(2a+k+1)}, \quad t > 0.$$

Hence the cut-Hankel transform H_a^C , defined by

$$(3.12) \quad (H_a^C \phi)(x) = \int_0^\infty [J_{2a}(2\sqrt{t}) - \frac{1}{\Gamma(2a+1)} \sum_{k=0}^{p-m-1} \frac{(-1)^k t^{a+k}}{k! \Gamma(2a+k+1)}] \phi\left(\frac{t}{x}\right) \frac{dt}{x},$$

$|\arg x| \leq \theta$, is a continuous isomorphism of $T_{q,n}^\theta(1-\mu, 1-\lambda; 1-a-q, \dots, -a)$ onto $T_{p,m}^\theta(\lambda, \mu; -a-p+1, \dots, -a)$ if $\lambda < c < 0$ and (3.10) is satisfied.

If it is not possible to choose a straight line such that (3.10) holds we may proceed as follows. Assume $\lambda < c_j < 0$, $j = 1, 2$. Let now C be a contour from $c_1 - i\infty$ to $c_2 + i\infty$ which lies entirely in S such that (3.10) holds for this contour. Then (3.12) remains valid if the integral is interpreted as its principal part.

An inversion formula for the transform in (3.12) may be constructed as indicated in theorem 27.

Finally we give an example of the cut-Hankel transform if the condition $\lambda < c < 0$ is omitted. Assume p, q, m, n, S and C as before. Moreover in this example let

$$(3.13) \quad -\operatorname{Re} a + 1 > \mu.$$

Let $g \in \mathbb{N}$ be such that

$$\frac{\Gamma(a+s)}{\Gamma(a+1-s)} \frac{1}{(1-a-s)_g} = O(s^{-1-\delta}) \text{ as } s \rightarrow \infty, s \in C, \delta > 0.$$

Define $k_g(t)$ as in theorem 27 with α replaced by $-a$.

Using residue calculus we obtain

$$(3.15) \quad k_g(t) = (-)^g t^{-\frac{1}{2}g} J_{2a-g}(2\sqrt{t}) - \sum_{k=0}^{p-m+g-1} \frac{(-1)^k t^{a-g+k}}{k! \Gamma(2a-g+k+1)}, \quad t > 0.$$

From theorem 27 it now follows that if (3.10), (3.13) and (3.14) are satisfied the cut-Hankel transform H_a^C , defined by

$$(H_a^C \phi)(x) = x^a \frac{d^g}{dx^g} x^{g-a-1} \int_0^\infty k_g(t) \phi\left(\frac{t}{x}\right) dt, \quad |\arg x| \leq \theta,$$

is a continuous isomorphism of $T_{q,n}^\theta(1-\mu, 1-\lambda; 1-a-q, \dots, -a)$ onto $T_{p,m}^\theta(\lambda, \mu; -a-p+1, \dots, -a)$.

The conditions under which (3.9) is valid in the last cases are much more complicated than in the first example. We omit details.

5.4 EXAMPLES

EXAMPLE I

Let α be a complex number, $\operatorname{Re} \alpha > 1$. We define the map L_α by

$$(4.1) \quad (L_\alpha \phi)(x) = \int_0^\infty t^{-\alpha} e^{-t} \phi\left(\frac{t}{x}\right) \frac{dt}{x}, \quad \phi \in T(1-\mu, 1-\lambda), \quad x > 0.$$

Assume $\operatorname{Re} \alpha \leq \lambda < \mu$ or $\operatorname{Re} \alpha < \mu \leq \lambda$. If $c > \operatorname{Re} \alpha$ we have

$$(4.2) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s-\alpha) t^{-s} ds = t^{-\alpha} e^{-t}, \quad t > 0$$

and we deduce from theorem 11 and theorem 20 that L_α is a continuous isomorphism of $T(1-\mu, 1-\lambda)$ onto $T^{\frac{1}{2}\pi}(\lambda, \mu)$. On the other hand I' and II' of section 2 are satisfied respectively with $\eta = -\alpha$. Hence the fractional integration operator $K(-\alpha, \alpha)$ is a continuous automorphism of $T^{\frac{1}{2}\pi}(\lambda, \mu)$. Hence the composition $K(-\alpha, \alpha) \cdot L_\alpha$ is a continuous isomorphism of $T(1-\mu, 1-\lambda)$ onto $T^{\frac{1}{2}\pi}(\lambda, \mu)$. Using the representations (cf example I of chapter II; also note that $L = L_0$):

$$L = M^{-1} \cdot m_1(K_1) \cdot M,$$

$$K(-\alpha, \alpha) = M^{-1} \cdot m(K_2) \cdot M,$$

where $K_1(s) = \Gamma(s)$ and $K_2(s) = \Gamma(s-\alpha)/\Gamma(s)$ we deduce

$$(4.3) \quad L_\alpha = K(-\alpha, \alpha) \cdot L.$$

From (4.3) we may deduce L_α^{-1} in terms of the inverses of the factors. It is not hard to extend this example to the case $\operatorname{Re} \alpha \leq 1$. We may also weaken the condition $\operatorname{Re} \alpha \leq \lambda < \mu$ to $\lambda < \mu$, using the theorems of chapter IV.

EXAMPLE II

We consider the map B of example II of chapter II with $\operatorname{Re} a \leq \lambda < \mu$ or $\operatorname{Re} a < \mu \leq \lambda$.

$$K(s) = \frac{1}{(s-a)_n} = \frac{\Gamma(s-a)}{\Gamma(s+n-a)}$$

we see that the conditions I' or II' of section 2 are satisfied with $\eta = -a$.

$$B = K(-a, -a+n).$$

We give the following extension of B . Assume

$$\lambda < \operatorname{Re} a - n, \quad \operatorname{Re} a < c < \mu.$$

From (2.20) with $\eta = -a$, $\alpha = -a+n$, $q=r=m=j_0=0$, $p=n$ it follows that

$$K(-a, -a+n) : T^\theta(\lambda, \mu) \rightarrow T_{n,n}^\theta(\lambda, \mu; a-n+1, \dots, a)$$

is a continuous isomorphism.

This map is a special case of the map D^{-1} of theorem 25. This theorem gives direct information for the map B in cases where I' or II' are not satisfied.

In our last example of this chapter we turn again to an example of chapter II. We will derive another inversion formula for the map A in example V of chapter II, i.e.

$$(A\phi)(x) = \frac{\Gamma(\alpha)}{\Gamma(\gamma)} \int_0^\infty {}_1F_1(\alpha; \gamma; -xt) \phi(t) dt.$$

EXAMPLE III

The map A of example V, chapter II, has the representation

$$A = M^{-1} \cdot m_1(K) \cdot M$$

where

$$K(s) = \frac{\Gamma(s)\Gamma(\alpha-s)}{\Gamma(\gamma-s)}.$$

Let $\lambda < \mu \leq \operatorname{Re} \alpha$, $\theta \geq 0$. Then A is a continuous map of $T^\theta(1-\mu, 1-\lambda)$ into $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$. Consider the fractional integration operator $K(\alpha-1, \gamma-\alpha)$ of $T^\theta(1-\mu, 1-\lambda)$. It is a continuous automorphism which follows from the fact that I' of section 2 is satisfied for this case. If moreover $0 \leq \lambda < \mu$, by example II, chapter II, the Laplace transform

$$L : T^\theta(1-\mu, 1-\lambda) \rightarrow T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$$

is a continuous isomorphism. Hence the composition $L \cdot K(\alpha-1, \gamma-\alpha)$ is a continuous isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$ if $0 \leq \lambda < \mu \leq \operatorname{Re} \alpha$. Now let $K_1(s) = \Gamma(s)$ and $K_2(s) = \Gamma(\alpha-1+s)/\Gamma(\gamma-1+s)$. L and $K(\alpha-1, \gamma-\alpha)$ have the representations

$$L = M^{-1} \cdot m_1(K_1) \cdot M$$

$$K(\alpha-1, \gamma-\alpha) = M^{-1} \cdot m(K_2) \cdot M$$

from which we deduce

$$L \cdot K(\alpha-1, \gamma-\alpha) = M^{-1} \cdot m_1(K) \cdot M = A.$$

Assume $\mu \leq \operatorname{Re} \gamma$. Then VI' is satisfied and it follows that (cf (2.14))

$$(4.4) \quad A^{-1} = K(\gamma-1, \alpha-\gamma) \cdot L^{-1}$$

which is the inversion formula of Love, Prabhakar and Kashyap, (cf [55]). Using the techniques of chapter III we may also prove (4.4) if $0 < \mu \leq \lambda < \operatorname{Re} \alpha$.

In the same manner as above we may prove

$$(4.5) \quad A = I(\alpha-1, \gamma-\alpha) \cdot L$$

if $0 \leq \lambda < \mu \leq \operatorname{Re} \alpha$ or $0 < \mu \leq \lambda < \operatorname{Re} \alpha$. If moreover $\mu \leq \operatorname{Re} \gamma$ then

$$A^{-1} = L^{-1} \cdot I(\gamma-1, \alpha-\gamma).$$

REMARKS

1. If in the last example the condition $\mu \leq \operatorname{Re} \gamma$ is not satisfied we may derive modified inversion formulas if VIII or IX are valid in case of the operator I and VIII' or IX' in case of the operator K .

2. Let A be an integral transform with kernel k . If k is the inverse Mellin transform of a G -function, it is possible in many cases to write A as the product of fractional integrals and (inverse) Laplace transforms. Example V above is a simple case. Another example may be found in [29]. Confer also [67a] for an example.

5.5 BIBLIOGRAPHICAL NOTES

Fractional integration in connection with various integral transforms or in connection with integral equations have been applied by many authors.

In this chapter the basic ideas developed in [4] are extended to spaces $T^{\theta}(\lambda, \mu)$, $\theta > 0$. The generalizations of the Weyl integrals and of the Riemann-Liouville integrals mentioned in the introduction (section 5.0, formulas (0.1) and (0.2)) occur in [8] and [30]. There exist many formulas which give relations between fractional integrals of different order, negative order, order half an integer and so on. Formulas of this kind can be found in [3], [4], [8], [30], [40], [48], [56]. Several of these papers also contain relations between the fractional integrals (0.1) and (0.2).

Further generalizations have been proposed. As an example, Parashar ([50]) extends to G functions. Sethi and Banerji, ([62]), also consider an extension. In their paper Lowndes' form of fractional integration is used to solve certain integral equations; (references to Lowndes' work can be found in this paper also). Other applications to integral equations occur in [4], [15], [47], [58]. The monograph of Ross, (Ed., [58]), contains several applications to other fields in mathematics; the first lecture in this volume is a good introduction to the history of fractional integration. An application to differential operators can be found in [9]. Applications may also be found in [62a].

Fractional integration with respect to distribution theory has been treated in many studies. Examples of papers which deal with the subject are [4], [10], [14], [44].

Osler ([48]) considers applications to infinite series. In his paper an extensive table of fractional integrals is included.

Applications of fractional integration to Hankel transforms in order to extend some well known results can be found in papers of Koh, ([32]), Lee, ([35]), Fenyő, ([18]), McBride, ([43]), Sneddon, ([62a]). Related forms of the Hankel transforms occur in [34], [59]. Cf also [42a].

CHAPTER 6

STIELTJES TRANSFORMS

6.0 INTRODUCTION

The Stieltjes transform of a function ϕ is defined by

$$(0.1) \quad (S\phi)(x) = \int_0^{\infty} \frac{\phi(t)}{t+x} dt, \quad x > 0,$$

whenever the integral converges. A more general form is

$$(0.2) \quad (S^{\alpha}\phi)(x) = \int_0^{\infty} \frac{\phi(t)}{(t+x)^{\alpha}} dt, \quad \alpha > 0, \quad x > 0.$$

For some purposes it is convenient to consider the transform

$$(0.3) \quad \psi(x) = S(m, \alpha, \beta)\phi(x) = m\Gamma(\alpha)x^{m\beta} \int_0^{\infty} (x^m + t^m)^{-\alpha} t^{m(\alpha-\beta)} \phi(t) \frac{dt}{t},$$

where $m > 0$, $\alpha, \beta \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$.

If in (0.3) we make the substitution $x := x^{1/m}$, $t := t^{1/m}$, $x^{-\beta}\psi(x^{1/m}) = \tilde{\psi}(x)$, $t^{-\beta}\phi(t^{1/m}) = \tilde{\phi}(t)$, we obtain

$$(0.4) \quad \tilde{\psi}(x) = \Gamma(\alpha) \int_0^{\infty} \frac{t^{\alpha-1}\tilde{\phi}(t)}{(t+x)^{\alpha}} dt.$$

We see that it suffices to consider the case (0.4), which without the tilde we may write as

$$(0.5) \quad \psi(x) = \Gamma(\alpha) \int_0^{\infty} \left(1 + \frac{x}{t}\right)^{-\alpha} \phi(t) \frac{dt}{t}.$$

Apart from the factor $\Gamma(\alpha)$, (0.5) is the same as (0.2) when the latter is applied to $t^{\alpha-1}\phi(t)$.

In the literature two types of inversion formulas are considered.

The first type of inversion formulas, most often based on a paper of Sumner, (cf. [64]) are formulas using contour integration in the

complex plane. Examples of such formulas may be found in [11],

Another type of inversion formulas uses (limits of) differential operators. Examples may be found in [51],

In section 2 and 4 respectively we will treat some examples of these inversion formulas for the Stieltjes transform.

6.1 THE STIELTJES TRANSFORM S^α

We define the Stieltjes transform S^α by

$$(1.1) \quad (S^\alpha \phi)(x) = \Gamma(\alpha) \int_0^\infty \left(1 + \frac{x}{t}\right)^{-\alpha} \phi(t) \frac{dt}{t}, \quad |\arg x| < \pi.$$

We consider first the case $\operatorname{Re} \alpha > 0$. In section 3 we will make some remarks about the case $\operatorname{Re} \alpha \leq 0$. Since

$$M\{\Gamma(\alpha)(1+t)^{-\alpha}\}(s) = \Gamma(s)\Gamma(\alpha-s)$$

if $0 < \operatorname{Re} s < \operatorname{Re} \alpha$, we consider the function

$$K(s) = \Gamma(s)\Gamma(\alpha-s).$$

It has poles in $s = -1, -2, \dots$ and in $s = \alpha, \alpha+1, \dots$.

Moreover

$$(1.2) \quad K(s) = \frac{\Gamma(\alpha-s)}{\Gamma(1-s)} \frac{\pi}{\sin \pi s}.$$

It follows that

$$K(s) = O(s^{\alpha-1} e^{-\pi |\operatorname{Im} s|}) \text{ as } s \rightarrow \infty,$$

uniformly on any strip $\{s \in \mathbb{C} \mid a \leq \operatorname{Re} s \leq b\}$. So we may apply theorems 13 and 22. Let

$$\begin{aligned}
 S_0 &= \{s \in \mathbb{C} \mid 0 < \operatorname{Re} s < \operatorname{Re} \alpha\} \\
 (1.3) \quad S_h &= \{s \in \mathbb{C} \mid \operatorname{Re} \alpha + h - 1 < \operatorname{Re} s < \operatorname{Re} \alpha + h\}, \quad h = 1, 2, \dots \\
 S_{-h} &= \{s \in \mathbb{C} \mid -h < \operatorname{Re} s < 1 - h\}, \quad h = 1, 2, \dots
 \end{aligned}$$

We define for any $h \in \mathbb{Z}$

$$(1.4) \quad k_h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \Gamma(\alpha-s) t^{-s} ds, \quad c \in S_h, \quad |\arg t| < \pi.$$

By means of residue calculus we deduce if $|t| < 1$,

$$(1.5) \quad k_h(t) = \Gamma(\alpha) \left\{ (1+t)^{-\alpha} - \sum_{m=0}^{-h-1} \binom{-\alpha}{m} t^m - \sum_{m=0}^{h-1} \binom{-\alpha}{m} t^{-m-\alpha} \right\},$$

which by analytic continuation is also valid if $|\arg t| < \pi$. Here

$\sum_0^{-h-1} = 0$ if $h \geq 0$ and $\sum_0^{h-1} = 0$ if $h \leq 0$.

It is clear that the kernel in (1.1) is $k_0(t)$. As an extension of (1.1) we define for $h \in \mathbb{Z}$

$$(1.6) \quad (S_h^\alpha \phi)(x) = \Gamma(\alpha) \int_0^\infty \left[(1+\frac{x}{t})^{-\alpha} - \sum_{m=0}^{-h-1} \binom{-\alpha}{m} \left(\frac{x}{t}\right)^m - \sum_{m=0}^{h-1} \binom{-\alpha}{m} \left(\frac{x}{t}\right)^{-m-\alpha} \right] \phi(t) \frac{dt}{t},$$

$|\arg x| < \pi$, whenever the integral makes sense. According to theorem 13, remark II.11, theorem 22 and theorem 24 we have

THEOREM 30. Let $\operatorname{Re} \alpha > 0$, $h \in \mathbb{Z}$, $\lambda, \mu \in \mathbb{R}$ and either $\lambda < \mu$, $\lambda, \mu \in \bar{S}_h$, or $\lambda \geq \mu$, $\lambda, \mu \in S_h$.

Then for any $\phi \in T(\lambda, \mu)$ the integral (1.6) converges if $|\arg x| < \pi$ and defines a function in $T^\pi(\lambda, \mu)$. The map S_h^α is an isomorphism of $T(\lambda, \mu)$ onto $T^\pi(\lambda, \mu)$. If $\theta \geq 0$ we may extend S_h^α to a continuous isomorphism of $T^\theta(\lambda, \mu)$ onto $T^{\theta+\pi}(\lambda, \mu)$ by means of

$$(1.7) \quad (S_h^\alpha \phi)(x) = \Gamma(\alpha) \int_0^{\infty e^{i\theta_1}} \left[(1+u)^{-\alpha} - \sum_{m=0}^{-h-1} \binom{-\alpha}{m} u^m - \sum_{m=0}^{h-1} \binom{-\alpha}{m} u^{-m-\alpha} \right] \phi\left(\frac{x}{u}\right) \frac{du}{u},$$

where $\phi \in T^\theta(\lambda, \mu)$, $|\arg x| < \pi + \theta$ and $\theta_1 \in \mathbb{R}$ such that $|\theta_1| < \pi$,

and $|\arg x - \theta_1| \leq \theta$.

A representation of $(S_h^\alpha \phi)(x)$ valid for $\phi \in T^\theta(\lambda, \mu)$, $|\arg x| \leq \pi + \theta$ may be given as follows.

Choose $n \in \mathbb{N}$, $p \in \mathbb{C}$ such that $\operatorname{Re} p + 1 > \max(\lambda, \mu)$, $n > \operatorname{Re} \alpha$. Define for $c \in S_h$ and $|\arg t| \leq \pi$

$$(1.8) \quad k_{hn}^*(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\alpha-s)}{(1+p-s)_n} t^{-s} ds.$$

Then

$$(1.9) \quad (S_h^\alpha \phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p} \int_0^{\infty e^{i\theta_1}} k_{hn}^*(u) \phi\left(\frac{x}{u}\right) \frac{du}{u}$$

if $\phi \in T^\theta(\lambda, \mu)$, $|\arg x| \leq \pi + \theta$, $\theta_1 \in \mathbb{R}$ such that $|\theta_1| \leq \pi$, $|\arg x - \theta_1| \leq \theta$. In particular, if $|\arg x| \leq \pi$ and $\phi \in T^\theta(\lambda, \mu)$ for any $\theta \geq 0$, then

$$(1.10) \quad (S_h^\alpha \phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p} \int_0^\infty k_{hn}^*\left(\frac{x}{t}\right) \phi(t) \frac{dt}{t}.$$

□

We may evaluate k_{hn}^* by means of residue calculus. We carry out the calculation in the case $h=0$; the other cases lead to similar results. Choose $0 < c < 1$, $c < \operatorname{Re} \alpha$, $p \in \mathbb{N}$. Then we get

$$\begin{aligned} k_n^*(t) &= \frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s-1) \dots (s-p) \Gamma(s-p-n) \Gamma(\alpha-s) t^{-s} ds \\ &= \frac{(-1)^{p+n}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{d^p}{dt^p} t^{p-s} \Gamma(s-p-n) \Gamma(\alpha-s) ds \\ &= (-1)^{n+p} \frac{d^p}{dt^p} \left\{ \Gamma(\alpha-p-n) (1+t)^{p+n-\alpha} t^{-n} + \right. \\ &\quad \left. - \sum_{j=0}^{p+n-1} (-1)^j \frac{\Gamma(\alpha-p-n+j)}{j!} t^{j-n} \right\}. \end{aligned}$$

Hence, carrying out the differentiation

$$(1.11) \quad k_n^*(t) = (-1)^n t^{-n} \left\{ \sum_{h=0}^p \binom{p}{h} \Gamma(\alpha-n-h) (n)_h (1+t)^{n+h-\alpha} t^{-h} + \right. \\ \left. - \sum_{j=0}^{n-1} (-1)^j \frac{\Gamma(\alpha-p-n+j)}{j!} (n-j)_p t^{j-p} \right\}.$$

Here we assumed $\alpha \neq p+1, \dots, p+n-1$. Otherwise we have to interpret the right hand side as the limit for α tending to one of these values.

6.2 INVERSION FORMULAS

In this section we give several inversion theorems for the Stieltjes transform. We assume that the conditions of the previous section are satisfied and deduce representations for $(S_h^\alpha)^{-1}$ from theorems 15 and 24, where $H(s) = 1/K(s) = 1/\{\Gamma(s)\Gamma(\alpha-s)\}$.

1) Choose in theorems 15 and 24

$$(2.1) \quad H_{\pm}(s)/(\alpha-s)_\ell = \frac{\pm \Gamma(1-s)}{2\pi i \Gamma(\alpha+\ell-s)} e^{\pm \pi i s}$$

where $\ell \in \mathbb{N}$, $\ell > 2 - \operatorname{Re} \alpha$. Define

$$(2.2) \quad h_{\pm \ell}(t) = \frac{\pm 1}{(2\pi i)^2} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(1-s)}{\Gamma(\alpha+\ell-s)} e^{\pm \pi i s} t^{-s} ds$$

if $\arg t = \pm \pi$, $d \neq 1, 2, \dots$ and $d \in (\lambda, \mu)$ if $\lambda < \mu$, $d \in [\mu, \lambda]$ if $\mu \leq \lambda$ and $[\mu, \lambda]$ does not contain any of the points $1, 2, \dots$.

Then we have, (cf. (II.3.8) ⁺),

$$(2.3) \quad (S_h^\alpha)^{-1} \psi(x) = x^{1-\alpha} \frac{d^\ell}{dx^\ell} x^{\ell+\alpha-1} \left[\int_0^{\infty} e^{-(\pi-\arg x)i} h_{+\ell}\left(\frac{x}{t}\right) \psi(t) \frac{dt}{t} + \right. \\ \left. + \int_0^{\infty} e^{(\pi+\arg x)i} h_{-\ell}\left(\frac{x}{t}\right) \psi(t) \frac{dt}{t} \right]$$

if

$$(2.4) \quad \psi \in T^{\theta+\pi}(\lambda, \mu), \quad \theta \geq 0, \quad |\arg x| \leq \theta.$$

We may evaluate $h_{\pm\ell}$ by means of residue calculus. The most simple inversion formula is found if $d < 1$. So we assume $\lambda < 1$ if $\lambda < \mu$ or if $\mu \leq \lambda$. Let ℓ be such that $\ell > 2 - \operatorname{Re} \alpha$. Then we have

$$(2.5) \quad \begin{aligned} h_{\pm\ell}(t) &= 0 \text{ if } 0 < |t| < 1, \arg t = \pm\pi, \\ h_{\pm\ell}(t) &= \frac{\mp 1}{2\pi i \Gamma(\ell + \alpha - 1)} t^{-1} \left(1 + \frac{1}{t}\right)^{\ell + \alpha - 2} \text{ if } |t| > 1, \arg t = \pm\pi. \end{aligned}$$

From (2.3) we obtain

$$\begin{aligned} (S_h^\alpha)^{-1} \psi(x) &= x^{1-\alpha} \frac{d^\ell}{dx^\ell} x^{\ell + \alpha - 1} \left[\int_0^{xe^{-\pi i}} h_{+\ell}\left(\frac{x}{t}\right) \psi(t) \frac{dt}{t} + \right. \\ &\quad \left. + \int_0^{xe^{\pi i}} h_{-\ell}\left(\frac{x}{t}\right) \psi(t) \frac{dt}{t} \right]. \end{aligned}$$

The integrands are $O(1)t^{-\lambda-\varepsilon}$, $t \rightarrow 0$, $|\arg t| \leq \theta + \pi$, for any $\varepsilon > 0$. From this, using Cauchy's theorem we see that we may replace the contour by another one, which runs from $xe^{-\pi i}$ to $xe^{\pi i}$ and lies in the sector $\{t \in \mathbb{C} \mid |\arg t| \leq \theta + \pi\}$. This contour may pass through the origin. We summarize in

THEOREM 31. Let the assumptions of theorem 30 be satisfied.

Assume $\lambda < 1$ if $\lambda < \mu$ or $\mu \leq \lambda$. Let $\ell \in \mathbb{N}$ be such that $\ell > 2 - \operatorname{Re} \alpha$. Then the continuous inverse of S_h^α , $(S_h^\alpha)^{-1} : T^{\theta+\pi}(\lambda, \mu) \rightarrow T^\theta(\lambda, \mu)$ is given by

$$(2.6) \quad (S_h^\alpha)^{-1} \psi(x) = \frac{x^{1-\alpha}}{2\pi i \Gamma(\ell + \alpha - 1)} \frac{d^\ell}{dx^\ell} \int_C (x+t)^{\ell + \alpha - 2} \psi(t) dt$$

$|\arg x| \leq \theta$, where C is a contour in the sector $|\arg t| \leq \pi + \theta$, which runs from $xe^{-\pi i}$ to $xe^{\pi i}$ and which may pass through the origin. By the substitution $t := vx$ we obtain

$$(2.6)' \quad (S_h^\alpha)^{-1} \psi(x) = \frac{x^{1-\alpha}}{2\pi i \Gamma(\ell + \alpha - 1)} \frac{d^\ell}{dx^\ell} \{x^{\ell + \alpha - 1} \int_{C_0} (1+v)^{\ell + \alpha - 2} \psi(xv) dv\},$$

$|\arg x| \leq \theta$, C_0 being a contour in the sector $|\arg v| \leq \pi$ which runs from $e^{-\pi i}$ to $e^{\pi i}$ and which again may pass through the origin. $\square$

2) Choose in theorems 15 and 24

$$(2.7) \quad \frac{H_{\pm}(s)}{(\alpha-s)_{\ell}} = \frac{\mp(-1)^{\ell} \Gamma(1-\alpha-\ell+s)}{2\pi i \Gamma(s)} e^{\pm \pi i(s-\alpha)},$$

where $\ell \in \mathbb{N}$, $\ell > 2 - \operatorname{Re} \alpha$. Define $h_{\pm\ell}$ by

$$(2.8) \quad h_{\pm\ell}(t) = \frac{\mp(-1)^{\ell}}{(2\pi i)^2} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(1-\alpha-\ell+s)}{\Gamma(s)} e^{\pm \pi i(s-\alpha)} t^{-s} ds$$

if $\arg t = \pm\pi$ and $\mu > \operatorname{Re} \alpha + \ell - 1$ if $\lambda < \mu$ or if $\mu \leq \lambda$. Then (2.3) is valid for $\psi \in T^{\theta+\pi}(\lambda, \mu)$, $|\arg x| \leq \theta$. Again in this case we $h_{\pm\ell}$ may be evaluated by residue calculus. We choose $d > \operatorname{Re} \alpha + \ell - 1$. Then

$$(2.9) \quad h_{\pm\ell}(t) = 0 \text{ if } |t| > 1, \arg t = \pm\pi,$$

$$h_{\pm\ell}(t) = \frac{\pm 1}{2\pi i} \frac{t^{1-\alpha-\ell}}{\Gamma(\ell+\alpha-1)} (1+t)^{\ell+\alpha-2} \text{ if } |t| \leq 1, \arg t = \pm\pi.$$

From (2.3) we obtain

$$(S_h^{\alpha-1})\psi(x) = \frac{1}{2\pi i} x^{1-\alpha} \frac{d}{dx} \int_{xe^{-\pi i}}^{\infty e^{i(\arg x - \pi)}} (x+t)^{\alpha+\ell-2} \psi(t) dt -$$

$$+ \int_{xe^{\pi i}}^{\infty e^{i(\arg x + \pi)}} (x+t)^{\alpha+\ell-2} \psi(t) dt]$$

The behaviour of the integrands at infinity is given by the order relation $O(1)t^{\alpha+\ell-2-\mu+\varepsilon}$, $t \rightarrow \infty$, $|\arg t| \leq \theta+\pi$. From $\operatorname{Re} \alpha + \ell - 1 - \mu < 0$ and Cauchy's theorem we deduce that we may replace the contour in (+) by another one in the sector $|\arg t| \leq \theta+\pi$. We formulate these results in

THEOREM 31'. Let the assumptions of theorem 30 be satisfied. Assume $\mu > \operatorname{Re} \alpha + \ell - 1$ if $\lambda < \mu$ or $\mu \leq \lambda$. Let $\ell \in \mathbb{N}$, $\ell > 2 - \operatorname{Re} \alpha$. Then the continuous inverse $(S_h^{\alpha-1}) : T^{\theta+\pi}(\lambda, \mu) \rightarrow T^{\theta}(\lambda, \mu)$ is given by (2.6),

where C is a contour in the sector $|\arg t| \leq \theta + \pi$, which runs from $xe^{-\pi i}$ to $xe^{\pi i}$, which may be unbounded but does not pass through the origin. Slightly different forms of the inversion formula are obtained as in theorem 31 by suitable substitutions. $\square$

3) Choose in theorems 15 and 24

$$(2.11) \quad \frac{H_+(s)}{(\alpha-s)_\ell} = \frac{e^{\pm \pi i(s+\gamma)}}{2i\Gamma(s)\Gamma(\alpha+\ell-s)\sin\pi(s+\gamma)}$$

where $\ell \in \mathbb{N}$, $\ell > 2 - \operatorname{Re} \alpha$, $\gamma \in \mathbb{C}$. Define

$$(2.12) \quad h_{\pm\ell}(t) = \frac{\pm 1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{e^{\pm \pi i(s+\gamma)} t^{-s}}{2i\Gamma(s)\Gamma(\alpha+\ell-s)\sin\pi(s+\gamma)} ds, \quad \arg t = \pm\pi$$

where $d + \operatorname{Re} \gamma \notin \mathbb{Z}$ and $d \in (\lambda, \mu)$ if $\lambda < \mu$, and $d \in [\mu, \lambda]$ if $\mu \leq \lambda$ and $[\mu, \lambda]$ does not contain any of the points $-\operatorname{Re} \gamma$, $-\operatorname{Re} \gamma + 1$, $\dots$.

Then (2.3) holds if (2.4) is satisfied.

Since $h_{\pm\ell}$ is the same for any two values of γ which differ an integer, we may choose γ such that

$$\left. \begin{array}{l} \lambda < 1 - \operatorname{Re} \gamma \\ -\operatorname{Re} \gamma < \mu \end{array} \right\} \text{ if } \lambda < \mu \text{ or if } \mu \leq \lambda.$$

Then we may choose d such that

$$-\operatorname{Re} \gamma < d < 1 - \operatorname{Re} \gamma.$$

and we easily obtain by means of residue calculus

$$(2.13)' \quad h_{\pm\ell}(t) = \pm \frac{1}{2\pi i} \frac{t^\gamma}{\Gamma(-\gamma)\Gamma(\alpha+\ell+\gamma)} {}_2F_1(1, \gamma+1; \alpha+\gamma+\ell; -t) \quad \text{if } |t| \leq 1, \arg t = \pm\pi,$$

$$(2.13)'' \quad h_{\pm\ell}(t) = \mp \frac{1}{2\pi i} \frac{t^{\gamma-1}}{\Gamma(1-\gamma)\Gamma(\alpha+\ell-1+\gamma)} {}_2F_1(1, 2-\alpha-\ell-\gamma; 1-\gamma; -t^{-1}) \quad \text{if } |t| > 1, \arg t = \pm\pi.$$

Here we use appropriate limiting values in case $-\alpha-\gamma-\ell \in \mathbb{N}$ or $\gamma-1 \in \mathbb{N}$.

We summarize this case in the following theorem.

THEOREM 31". Let the assumptions of theorem 30 be satisfied.

Let $\ell \in \mathbb{N}$, $\ell > 2 - \operatorname{Re} \alpha$, $\gamma \in \mathbb{C}$. Assume

$$\left. \begin{array}{l} \lambda < 1 - \operatorname{Re} \gamma \\ -\operatorname{Re} \gamma < \mu \end{array} \right\} \text{ if } \lambda < \mu \text{ or if } \mu \leq \lambda.$$

Let $h_{\pm\ell}(t)$ be defined by (2.12) with $-\operatorname{Re} \gamma < d < 1 - \operatorname{Re} \gamma$. Then the inverse $(S_h^\alpha)^{-1}$ is given by (2.3) if (2.4) is satisfied. Here $h_{\pm\ell}(t)$ is explicitly given by (2.13)' and (2.13)". $\square$

The inversion formula in theorem 31" follows from the inversion formula of theorem 31' if $\mu > \operatorname{Re} \alpha + \ell - 1$. This may be seen in the following way. The function $h_{\pm\ell}(t)$ in (2.13)' has an analytic continuation on the Riemann surface of the log function for $|t| > 1$. The analytic continuation to $|t| > 1$ may be given using [12], I, p 63 (17). We obtain (we denote the analytic continuation also by the r.h.s. of (2.13)'):

$$\begin{aligned} & \pm \frac{1}{2\pi i} \frac{t^\gamma}{\Gamma(-\gamma)\Gamma(\alpha+\ell+\gamma)} {}_2F_1(1, \gamma+1; \alpha+\gamma+\ell; -t) = \\ & = \mp \frac{1}{2\pi i} \frac{t^{\gamma-1}}{\Gamma(1-\gamma)\Gamma(\alpha+\ell+\gamma-1)} {}_2F_1(1, 2-\alpha-\ell-\gamma; 1-\gamma; -t^{-1}) + \\ (*) \quad & \pm \frac{(1+t^{-1})^{\alpha+\ell-2} t^{-1}}{\Gamma(\alpha+\ell-1)}. \end{aligned}$$

The first term in the right hand side is (2.13)", the second term is the kernel which occurs in theorem 31'. Hence the inversion formulas of theorem 31' and of theorem 31" are in accordance with each other if we can show that

$$\int_{\infty e^{i(\arg x - \pi)}}^{(0^+)} \left(\frac{x}{t}\right)^{\gamma-1} {}_2F_1\left(1, 2-\alpha-\ell-\gamma; 1-\gamma; \frac{-t}{x}\right) \psi(t) \frac{dt}{t} = 0;$$

(the integrand is to be understood in the sense of the analytic continuation). But this follows from an application of Cauchy's theorem, using the asymptotic behaviour of the integrand near zero and near ∞ , (which may be deduced from (*)).

6.3 EXTENSION TO THE CASE $\operatorname{Re} \alpha \leq 0$

Assume that the conditions of theorem 30 are satisfied for the transforms $S_h^{\alpha'}$ and S_h^α , in such a way that they are both defined on some space $T^\theta(\lambda, \mu)$. It follows from chapter V, section 1, that then also the conditions are satisfied under which the operator of fractional integration $I(\alpha-1, \alpha'-\alpha, h)$ is defined on $T^\theta(\lambda, \mu)$ and on $T^{\theta+\pi}(\lambda, \mu)$.

We have the following representations for the operators:

$$S_h^{\alpha'} = M^{-1} \cdot m(K_1) \cdot M,$$

$$S_h^\alpha = M^{-1} \cdot m(K_2) \cdot M,$$

$$I(\alpha-1, \alpha'-\alpha, h) = M^{-1} \cdot m(K_3) \cdot M,$$

where

$$K_1(s) = \Gamma(s) \Gamma(\alpha'-s),$$

$$K_2(s) = \Gamma(s) \Gamma(\alpha-s),$$

$$K_3(s) = \frac{\Gamma(\alpha-s)}{\Gamma(\alpha'-s)}.$$

It follows that in a formal way we have

$$(3.1) \quad S_h^{\alpha'} \cdot I(\alpha-1, \alpha'-\alpha, h) = I(\alpha-1, \alpha'-\alpha, h) \cdot S_h^{\alpha'} = S_h^\alpha.$$

We specify the conditions for (3.1) in the case $\alpha' = \alpha + n$, $n \in \mathbb{N}$.

Consider the conditions (cf theorem 30)

$$(3.2) \quad \left. \begin{aligned} 0 &\leq \lambda < \mu \leq n + \operatorname{Re} \alpha, h' = 0 \\ 0 &< \mu \leq \lambda < n + \operatorname{Re} \alpha, h' = 0 \\ \operatorname{Re} \alpha + n + h' - 1 &\leq \lambda < \mu \leq \operatorname{Re} \alpha + n + h' \\ \operatorname{Re} \alpha + n + h' - 1 &< \mu \leq \lambda < \operatorname{Re} \alpha + n + h' \\ -h' &\leq \lambda < \mu \leq 1 - h' \\ -h' &< \mu \leq \lambda < 1 - h' \end{aligned} \right\} h' \in \mathbb{N} \setminus \{0\},$$

and the conditions (cf section 5.1)

$$\begin{aligned}
 (3.3) \quad & 0 \leq \lambda < \mu \leq \operatorname{Re} \alpha, h=0 \\
 & 0 < \mu \leq \lambda < \operatorname{Re} \alpha, h=0 \\
 & \operatorname{Re} \alpha + h - 1 \leq \lambda < \mu \leq \operatorname{Re} \alpha + h, h=1, \dots, n-1 \text{ or } \operatorname{Re} \alpha + n - 1 \leq \lambda < \mu \\
 & \operatorname{Re} \alpha + h - 1 < \mu \leq \lambda < \operatorname{Re} \alpha + h, h=1, \dots, n-1 \text{ or } \operatorname{Re} \alpha + n - 1 < \mu \leq \lambda.
 \end{aligned}$$

Let $\operatorname{Re} \alpha > 0$, $n \in \mathbb{N}$. Assume that one of the conditions (3.2) is satisfied. Assume moreover that one of the conditions of (3.2) with $n = 0$ and $h' = h$ is also satisfied. Then $S_h^{\alpha+n}$ and S_h^α are continuous isomorphisms of $T^\theta(\lambda, \mu)$ onto $T^{\theta+\pi}(\lambda, \mu)$. From chapter 5, section 1 it follows that $I(\alpha-1, n, h)$ is a continuous automorphism of $T^\theta(\lambda, \mu)$ and of $T^{\theta+\pi}(\lambda, \mu)$. Hence we have

$$(3.4) \quad S_h^{\alpha+n} \cdot I(\alpha-1, n, h) = I(\alpha-1, n, h) \cdot S_h^{\alpha+n} = S_h^\alpha.$$

We use (3.4) to extend the definition of S_h^α .

Assume $\operatorname{Re} \alpha \leq 0$. Let $n \in \mathbb{N}$ be such that $n + \operatorname{Re} \alpha > 0$. Assume that one of the conditions (3.2) and one of the conditions (3.3) are satisfied. Then both compositions in (3.4) are isomorphisms of $T^\theta(\lambda, \mu)$ onto $T^{\theta+\pi}(\lambda, \mu)$, (cf case V, page). Hence (3.4) defines $S_h^\alpha : T^\theta(\lambda, \mu) \rightarrow T^{\theta+\pi}(\lambda, \mu)$ in case $\operatorname{Re} \alpha \leq 0$ and this definition is independent of the choice of n provided $n + \operatorname{Re} \alpha > 0$. The inverse map is easily given (cf. theorem 25):

$$(3.4)' \quad (S_h^\alpha)^{-1} \psi(x) = x^{1-\alpha} \frac{d^n}{dx^n} x^{n+\alpha-1} (S_h^{\alpha+n})^{-1} \psi(x),$$

$$\psi \in T^{\theta+\pi}(\lambda, \mu).$$

Formulas for the inverse $(S_h^{\alpha+n})^{-1}$ may be taken from section 2.

6.4 INVERSION FORMULAS INVOLVING DIFFERENTIAL OPERATORS

In this section we derive inversion formulas for the Stieltjes transform which make use of differential operators. In fact we construct operators which contain a parameter n and which for

large n are "nearly" the inverse for the Stieltjes transform.

We define the function K as in section 1, $K(s) = \Gamma(s)\Gamma(\alpha-s)$. We

assume $\operatorname{Re} \alpha > 0$ and $\lambda < \mu$. The map $m(K)$, defined in chapter 1, may be lifted to the map $S_h^\alpha: T^\theta(\lambda, \mu) \rightarrow T^{\theta+\pi}(\lambda, \mu)$ by means of the decomposition

$$S_h^\alpha = M^{-1} \cdot m(K) \cdot M.$$

The continuous inverse of $m(K)$ is defined by

$$m^{-1}(K)\Psi(s) = \frac{\Psi(s)}{\Gamma(s)\Gamma(\alpha-s)}, \quad \Psi \in S^{\theta+\pi}(\lambda, \mu).$$

We are going to construct a sequence of linear continuous maps

$(B_n)_{n=1}^\infty$ with the property

$$\lim_{n \rightarrow \infty} B_n = m^{-1}(K)$$

in the space $L(S^{\theta+\pi}, S^\theta)$, the space of all continuous linear maps of $S^{\theta+\pi}$ into S^θ . By means of the Mellin transform again the B_n are lifted to continuous linear maps $D_n: T^{\theta+\pi}(\lambda, \mu) \rightarrow T^\theta(\lambda, \mu)$, with the property

$$\lim_{n \rightarrow \infty} D_n = (S_h^\alpha)^{-1}.$$

Here the limit is taken in $L(T^{\theta+\pi}, T^\theta)$ in the sense of the strong operator topology. Hence we shall have to prove

$$\lim_{n \rightarrow \infty} B_n \Psi = m^{-1}(K) \Psi$$

for all $\Psi \in S^{\theta+\pi}(\lambda, \mu)$.

First we mention two lemmas.

LEMMA 1. Let G be a compact set, in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Then uniformly in G

$$\frac{\Gamma(n+\zeta)\Gamma(n+\alpha-\zeta)}{\Gamma(n)\Gamma(n+\alpha)} = 1 + O\left(\frac{1}{n}\right) \text{ as } n \rightarrow \infty, \quad \zeta \in G.$$

LEMMA 2. Let $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$. There exist a constant $M > 0$ and $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow \left| \frac{\Gamma(n+\zeta)\Gamma(n+\alpha-\zeta)}{\Gamma(n)\Gamma(n+\alpha)} \right| < M$$

uniformly on $\{\zeta \in \mathbb{C} \mid d_1 \leq \operatorname{Re} \zeta \leq d_2\}$.

Proof. The first lemma follows easily from [12], I, p.49.

For the proof of the second lemma we observe that by definition

$$|\Gamma(s)| = \left| \int_0^\infty e^{-t} t^{s-1} dt \right| \leq \int_0^\infty e^{-t} t^{\operatorname{Re} s - 1} dt \leq \Gamma(\operatorname{Re} s)$$

if $\operatorname{Re} s > 0$. For n sufficiently large

$$\begin{aligned} \left| \frac{\Gamma(n+\zeta)\Gamma(n+\alpha-\zeta)}{\Gamma(n)\Gamma(n+\alpha)} \right| &\leq \frac{\Gamma(n+\operatorname{Re} \zeta)\Gamma(n+\operatorname{Re}(\alpha-\zeta))}{|\Gamma(n)\Gamma(n+\alpha)|} \\ &\leq M |n|^{\operatorname{Re} \zeta + \operatorname{Re}(\alpha-\zeta) - \alpha} \\ &\leq M, \end{aligned}$$

for some $M \in \mathbb{R}^+$. □

We now turn to the definition of B_n . Let $\theta \geq 0$, $\operatorname{Re} \alpha > 0$. For any $\Psi \in S^{\theta+\pi}(\lambda, \mu)$ we define

$$(B_n \Psi)(s) = (-1)^n \frac{n^{(s)} n^{(s-n-\alpha+1)}}{\Gamma(n)\Gamma(n+\alpha)} \Psi(s).$$

Clearly B_n defines a map from $S^{\theta+\pi}(\lambda, \mu)$ onto $S^\theta(\lambda, \mu)$, which by theorem 1, chapter I, is continuous. We prove that $B_n \Psi(s)$ converges pointwise to $m^{-1}(K)\Psi(s)$. Indeed, putting $m^{-1}(K)\Psi(s) = \Phi(s)$, we have

$$\begin{aligned} B_n \Psi(s) - m^{-1}(K)\Psi(s) &= [(-1)^n \frac{n^{(s)} n^{(s-n-\alpha+1)}}{\Gamma(n)\Gamma(n+\alpha)} - \frac{1}{\Gamma(s)\Gamma(\alpha-s)}] \Psi(s) \\ &= [\frac{\Gamma(n+s)\Gamma(n+\alpha-s)}{\Gamma(n)\Gamma(n+\alpha)} - 1] \Phi(s) \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

by lemma 1.

Next we prove the convergence $B_n \Psi \rightarrow m^{-1}(K) \Psi$ in the topology of $S^\theta(\lambda, \mu)$.

Denote $B_n \Psi$ by ϕ_n . We have

$$s^p e^{\theta|s|} |[\phi_n(s) - \phi(s)]| = \left[\frac{\Gamma(s+n)\Gamma(n+\alpha-s)}{\Gamma(n)\Gamma(n+\alpha)} - 1 \right] s^p e^{\theta|s|} |\phi(s)|.$$

For some constant $\tau_0 > 0$, to be specified within a moment, we define

$$V_0 = \{s \in \mathbb{C} \mid \lambda_n \leq \operatorname{Re} s \leq \mu_n \text{ and } |\operatorname{Im} s| < \tau_0\},$$

$$V_1 = \{s \in \mathbb{C} \mid \lambda_n \leq \operatorname{Re} s \leq \mu_n \text{ and } |\operatorname{Im} s| \geq \tau_0\}.$$

Let $k \in \mathbb{N}$, $\varepsilon > 0$. From lemma 2 it follows that there exist M_1 and $n_1 \in \mathbb{N}$ such that

$$(4.2) \quad \left| \frac{\Gamma(n+s)\Gamma(n+\alpha-s)}{\Gamma(n)\Gamma(n+\alpha)} - 1 \right| < M_1, \quad \lambda_k \leq \operatorname{Re} s \leq \mu_k, \quad n > n_1,$$

where M_1 is independent of s . Of course, in general M_1 may depend on the choice of k .

From the definition of $S^\theta(\lambda, \mu)$ we deduce that there exists τ_0 such that

$$(4.3) \quad \sup_{\substack{p \leq k \\ s \in V_1}} |s^p e^{\theta|s|} |\phi(s)| < \frac{\varepsilon}{M_1}.$$

From lemma 1 it follows that there exists $n_2 \in \mathbb{N}$ such that

$$(4.4) \quad \left| \frac{\Gamma(n+s)\Gamma(n+\alpha-s)}{\Gamma(n)\Gamma(n+\alpha)} - 1 \right| < \frac{\varepsilon}{\sigma_k(\Phi)} \quad \text{if } n > n_2,$$

uniformly on V_0 . (σ_k belongs to the multinorm on $S^\theta(\lambda, \mu)$).

Combining (4.2), (4.3) and (4.4) we obtain

$$\sup_{\substack{p \leq k \\ \lambda_n \leq \operatorname{Re} s \leq \mu_n}} |s^p e^{\theta|s|} |(B_n \Psi - \Phi)(s)| < 2\varepsilon \text{ if } n > \max(n_1, n_2).$$

ε being arbitrary, the assertion follows.

From the foregoing results we may conclude that in the sense of the strong operator topology in $L(S^{\theta+\pi}, S^\theta)$ we have

$$B_n \rightarrow m^{-1}(K) \text{ as } n \rightarrow \infty.$$

We now lift the maps B with the aid of the Mellin transform to the spaces $T^\theta(\lambda, \mu)$ and $T^{\theta+\pi}(\lambda, \mu)$. We give the exact formulation in the next theorem.

THEOREM 32. Assume $\operatorname{Re} \alpha > 0$, $\lambda < \mu$. Let D denote ordinary differentiation. Define the map D_n by

$$(4.5) \quad D_n \psi(x) = \frac{(-1)^n}{\Gamma(n)\Gamma(n+\alpha)} x^{1-\alpha} D^n x^{2n+\alpha-1} D^n \psi(x)$$

if $\psi \in T^{\theta+\pi}(\lambda, \mu)$. Then D_n is a continuous isomorphism of $T^{\theta+\pi}(\lambda, \mu)$ onto $T^\theta(\lambda, \mu)$ for any $\theta \geq 0$. Assume moreover that one of the following conditions is satisfied (cf. theorem 30):

$$0 \leq \lambda < \mu \leq \operatorname{Re} \alpha, h=0,$$

$$\operatorname{Re} \alpha + h - 1 \leq \lambda < \mu \leq \operatorname{Re} \alpha + h, h=1, 2, \dots$$

Then in the sense of the strong operator topology in $L(T^{\theta+\pi}(\lambda, \mu), T^\theta(\lambda, \mu))$, we have

$$\lim_{n \rightarrow \infty} D_n = (S_h^\alpha)^{-1}.$$

Proof. It is easy to show that $D_n \psi(x) = M^{-1} \cdot B_n \cdot M \psi(x)$ for any $n \in \mathbb{N}$ and $\psi \in T^{\theta+\pi}(\lambda, \mu)$. Hence the continuity of D_n .

We also have

$$B_n \cdot M \psi(x) \rightarrow m^{-1}(K) \cdot M \psi(x) = M \cdot (S_h^\alpha)^{-1} \psi(x),$$

where " $\rightarrow$ " is to be understood on one hand as pointwise convergence and on the other hand as convergence in $S^\theta(\lambda, \mu)$. Lifting with the continuous isomorphism M^{-1} to the space $T^\theta(\lambda, \mu)$ one obtains

$$D_n \psi(x) \rightarrow (S_h^\alpha)^{-1} \psi(x) \text{ if } n \rightarrow \infty.$$

As before we now have the validity of the theorem. □

REMARKS

1. The isomorphism D_n in (4.5) is related to the map which we have introduced in chapter IV, (1.4). If in (IV.1.4) we take

$$\underline{s} = (\alpha, \alpha+1, \dots, \alpha+n-1, 1-n, 2-n, \dots, 0)$$

then besides the factor $(-1)^n / (\Gamma(n)\Gamma(n+\alpha))$, D_n coincides with the map D in (IV.1.4). From this also the continuity of D_n follows.

2. The order of the factors in $(s)_n (s-n-\alpha+1)_n$ in the definition of B_n is irrelevant. Therefore in changing the order of the operations in the defining relation for D_n one may construct other inversion formulas. As examples we mention Pollard's formula (cf[54]),

$$D_{n1}\psi(x) = \frac{(-1)^n}{\Gamma(n)\Gamma(n+\alpha)} x^{-\alpha} D^n (x^{2n+\alpha-1} D^n \psi(x)),$$

and a formula which is related to (4.5)

$$D_{n2}\psi(x) = \frac{(-1)^n}{\Gamma(n)\Gamma(n-\alpha)} x^n D^n x^{1-\alpha} D^n x^{n+\alpha-1} \psi(x).$$

Here as before D denotes ordinary differentiation.

6.5 EXAMPLE

In this example we consider a different approach of example V of chapter II. As in that example we assume

$$(5.1) \quad 0 \leq \lambda < \mu \leq \operatorname{Re} \alpha.$$

We define

$$H_{\pm}(s) = \pm \frac{1}{2\pi i} e^{\pm i\pi(s-\gamma+1)} \Gamma(s-\gamma+1).$$

Then $H_+(s) + H_-(s) = 1/\Gamma(\gamma-s)$.

Let $\ell \in \mathbb{N}$ be such that $\mu > -(\operatorname{Re} \gamma + \ell) + \frac{5}{2}$. Then we may choose c such that

$$(5.2) \quad 1 - \mu < c < 1 - \lambda, \quad c < \operatorname{Re} \gamma + \ell - \frac{3}{2}.$$

Define $h_{\pm \ell}$ by

$$h_{\pm \ell}(t) = \frac{\pm 1}{(2\pi i)^2} (-1)^\ell \int_{c-i\infty}^{c+i\infty} e^{\pm i\pi(s-\gamma+1)} \Gamma(s-\gamma-\ell+1) t^{-s} ds.$$

From the last inequality of (5.2) it follows that the integral is absolutely convergent if $\frac{1}{2}\pi \leq \pm \arg t \leq \frac{3}{2}\pi$.

We evaluate the integral in the case in which c is chosen in such a way that $\gamma + \ell - 1$ is the only pole of the integrand to the right of the contour. Using residue calculus we obtain

$$h_{\pm \ell}(t) = \pm \frac{t^{-\gamma-\ell+1}}{2\pi i} (e^t - 1), \quad \frac{1}{2}\pi \leq \pm \arg t \leq \frac{3}{2}\pi.$$

We now use the results of theorem II.11 and remark II.7. We put $\beta_j = \gamma + j - 1$, $j=1, \dots, \ell$, $\theta = \frac{3}{2}\pi$, $\alpha_{\pm} = \frac{3}{2}\pi$. Moreover we replace λ and μ by $1-\mu$ and $1-\lambda$ respectively. The map B , defined by

$$(B\psi)(x) = x^{1-\gamma} D^{\ell} x^{\ell+\gamma-1} \frac{1}{2\pi i} \left(\int_{\infty e^{i\gamma-}}^0 + \int_0^{\infty e^{i\gamma+}} \right) (xt)^{-\ell-\gamma+1} (e^{xt} - 1) \psi(t) dt,$$

$|\arg x| \leq \theta + \frac{1}{2}\pi$, is a continuous isomorphism of $T^{\theta+\pi}(1-\mu, 1-\lambda)$ onto $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$. Here $\gamma_{\pm}$ is such that $|\gamma_{\pm}| \leq \theta + \pi$ and $\frac{1}{2}\pi \leq \pm(\arg x + \gamma_{\pm}) \leq \frac{3}{2}\pi$. For B we have the representation

$$(5.3) \quad B = M^{-1} \cdot m_1(K) \cdot M$$

if $K(s) = 1/\Gamma(\gamma-s)$.

Next we consider a special form of the Stieltjes transform:

$$(5.4) \quad (S^{\alpha} \phi)(x) = \int_0^{\infty} \left(\frac{x}{t}\right)^{\alpha-1} \left(1 + \frac{x}{t}\right)^{-\alpha} \phi(t) \frac{dt}{t}.$$

Apart from the factor $\Gamma(\alpha)$ the kernel is given by

$$(5.5) \quad \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s+\alpha-1) \Gamma(\alpha-(s+\alpha-1)) t^{-s} ds, \quad 1-\operatorname{Re} \alpha < d < 1.$$

From (5.1) we deduce

$$(5.6) \quad -\operatorname{Re} \alpha \leq 1-\mu < 1-\lambda \leq 1.$$

From section 1 we now may deduce that the map $\tilde{S}^\alpha$ is a continuous isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T^{\theta+\pi}(1-\mu, 1-\lambda)$ if (5.1) holds for any $\theta \geq 0$. A representation for $\tilde{S}^\alpha$ is

$$(5.7) \quad \tilde{S}^\alpha = M^{-1} \cdot m(\tilde{K}) \cdot M$$

where $\tilde{K}(s) = \Gamma(s+\alpha-1) \Gamma(1-s)$.

Combining (5.3) and (5.7) we obtain the following factorization of the map A of example V, chapter II:

$$(5.8) \quad A = B \cdot \tilde{S}^\alpha$$

Applying theorem II.11 again it is not hard to see that the inverse of B is the Laplace transform L_γ , given by

$$(L_\gamma \phi)(x) = \int_0^\infty (xt)^{\gamma-1} e^{-xt} \phi(t) dt$$

if $\operatorname{Re} \gamma > \mu$ and a related expression if the last condition is not satisfied. The inverse of $\tilde{S}^\alpha$ may be constructed along the pattern of section 2. From (5.8) we deduce

$$A^{-1} = [\tilde{S}^\alpha]^{-1} \cdot L_\gamma$$

and this gives a solution of the integral equation of Love, Prabhakar and Kashyap, (cf [41]),

$$\int_0^\infty \frac{t^{\gamma-1}}{\Gamma(\gamma)} {}_1F_1(\alpha; \gamma; -xt) \phi(t) dt = \frac{g(x)}{\Gamma(\alpha)}$$

where $g(x)$ is a given function.

REMARK.

Under suitable conditions we may factorize the Stieltjes transform S_h^α using the general form of the Laplace transform as in the example above. From the Mellin transform of the kernel of the Stieltjes transform we may deduce

$$S_h^\alpha = L \cdot L_{-1+\alpha}.$$

We omit the details.

6.6 BIBLIOGRAPHICAL NOTES

Stieltjes transforms and its inverses are important in applied mathematics. As an example of an application see the integral equation in section 6.5 where the solution is given in terms of the Stieltjes transform.

In the last ten years also generalizations of the kernel have been considered. Joshi ([29]) gives an extension to a hypergeometric kernel, in fact following earlier work of Swaroop, ([65]). He proves complex inversion formulas, which are in the form of a repeated integral. Also Srivastava suggests various extensions, (cf [63]). His inversion formulas are constructed along the same line, having the form of a repeated integral. Finally a general kernel using a G function is given by Pathak, ([52]). Stieltjes transforms for distributions are treated in [11], [49], and [51] and this is only a choice. Especially Erdélyi's paper [11] has given impulses to our approach. The advantage of our treatment consists mainly in the fact that the the inversion is straightforward. This is due to the homeomorphic character of the transform on the function spaces.

Inversion formulas using complex integration have been firstly proposed by Sumner, ([64]). Inversion formulas using differential operators are known for a long time, (cf [14]). They are used also for the inversion of the generalised forms of the Stieltjes transform, (for example in [7]), and for the inversion in the distributional sense, (cf [52]).

Several remarks on the subject may also be found in the books of Hirschman and Widder ([26b]) and of Marichev ([42a]).

CHAPTER 7

MEIJER TRANSFORMS

7.0 INTRODUCTION

Fractional integration and Stieltjes transforms, which are the subject of chapter V and of chapter VI respectively, are examples of the (product) convolution transform. In this chapter we consider as examples of Watson transforms two transforms which are due to C.S.Meijer; (cf [44a] and [44b]). In fact these transforms are special cases of the "G-transform" in the last example of chapter IV. The first transform which in the sequel we will denote by K-transform, is defined by

$$\psi(x) = (K_\nu \phi)(x) = \int_0^\infty \sqrt{xt} K_\nu(xt) \phi(t) dt$$

where K_ν is the Bessel function of third kind and of order ν . In the same manner as in the case of the Hankel transform in section V.3 we substitute $x := \sqrt{2x}$ and $t := \sqrt{2t}$. The result is a slightly different form,

$$\tilde{\psi}(x) = \int_0^\infty K_\nu(2\sqrt{xt}) \tilde{\phi}(t) dt$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are defined as in section V.3, (3.1). It follows that without loss of generality we may take as definition of the K-transform

$$(0.1) \quad (K_a \phi)(x) = \int_0^\infty K_{2a}(2\sqrt{xt}) \phi(t) dt.$$

This definition has the advantage that theorems of the foregoing chapters are easier to apply. The reason for the index $2a$ in stead of ν is that the Mellin transform of the kernel is slightly easier to handle.

Meijer's second transform, which we will denote by W-transform, is defined by

$$(W_{a,b}\phi)(x) = \int_0^\infty e^{-\frac{1}{2}xt} (xt)^{-b-\frac{1}{2}} W_{b+\frac{1}{2},a}(xt) \phi(t) dt.$$

Here $W_{\dots}$ is Whittaker's function.

In order to make the Mellin transform of the kernel more simple to calculate with we add the factor t^b in the kernel of the transform. Of course there is no loss of generality if we adopt the following formula as the defining relation for the W-transform:

$$(0.2) \quad (W_{a,b}\phi)(x) = \int_0^\infty e^{-\frac{1}{2}xt} (xt)^{-\frac{1}{2}} W_{b+\frac{1}{2},a}(xt) \phi(t) dt.$$

In section 1 we consider the map K_a . In section 2 some properties of $W_{a,b}$ will be derived. In section 3 we will see that there are some simple relations with other integral transforms.

7.1 MEIJER'S K-TRANSFORM

Consider the function K ,

$$K(s) = \frac{1}{2} \Gamma(s-a) \Gamma(s+a), \quad \operatorname{Re} a \geq 0.$$

Assume $2a \notin \mathbb{N}$. Then K has first order poles in $\pm a - j$, $j=0,1,\dots$.

If $\operatorname{Re} s = c$ we have $K(s) = O(|s|^{2c-1} \exp(-\pi |\operatorname{Im} s|))$ as $s \rightarrow \infty$. Hence to any pair of real numbers α_1 and β_1 , $\alpha_1 < \beta_1$, there exists $\gamma_1 \in \mathbb{R}$ such that

$$K(s) = O(|s|^{\gamma_1} \exp(-\pi |\operatorname{Im} s|)) \text{ as } s \rightarrow \infty,$$

uniformly on $\{s \in \mathbb{C} \mid \alpha_1 \leq \operatorname{Re} s \leq \beta_1\}$. Moreover, if H is defined by $H(s) = 1/K(1-s)$, to any pair of real numbers α_2 and β_2 , $\alpha_2 < \beta_2$, there exists $\gamma_2 \in \mathbb{R}$ such that

$$H(s) = O(|s|^{\gamma_2} \exp(\pi |\operatorname{Im} s|)) \text{ as } s \rightarrow \infty,$$

uniformly on $\{s \in \mathbb{C} \mid \alpha_2 \leq \operatorname{Re} s \leq \beta_2\}$.

It follows that K and H satisfy the conditions of theorems II.9 and II.11 if $\operatorname{Re} a \leq \lambda < \mu$ and of theorems III.18 and III.20 if $\operatorname{Re} a < \mu \leq \lambda$. Let $k(t)$ be defined by (II.1.10) if $\lambda < c < \mu$ or $\mu \leq c \leq \lambda$. Then

$$k(t) = K_{2a}(2\sqrt{t}), \quad |\arg t| < \pi,$$

where K_{2a} is the Bessel function of third kind and of order $2a$, (cf [13], I, p 349, [46] p 196). We then have from the theorems mentioned above and the remarks II.5, III.1:

THEOREM 33. Assume $0 \leq \operatorname{Re} a \leq \lambda < \mu$ or $0 \leq \operatorname{Re} a < \mu \leq \lambda$. Then the map K_a , defined by (0.1) for $|\arg x| < \pi$ is a continuous isomorphism of $T(1-\mu, 1-\lambda)$ onto $T^\pi(\lambda, \mu)$. For any $\theta \geq 0$ its restriction to $T^\theta(1-\mu, 1-\lambda)$ is a continuous isomorphism onto $T^{\theta+\pi}(\lambda, \mu)$. It is given by

$$(1.1) \quad (K_a \phi)(x) = x^{-1} \int_0^{\infty e^{i\theta_1}} K_{2a}(2\sqrt{u}) \phi\left(\frac{u}{x}\right) du, \quad |\arg x| < \theta + \pi,$$

where θ_1 is such that $|\theta_1| < \pi$, $\theta_1 - \theta \leq \arg x \leq \theta_1 + \theta$.

We may construct a representation which is also valid if $|\arg x| \leq \theta + \pi$ as follows. Let $n \in \mathbb{N}$, $n > 2\max(\lambda, \mu)$ and $p \in \mathbb{C}$, $\operatorname{Re} p + 1 > \max(\lambda, \mu)$.

Define $k_n(t)$ by

$$(1.2) \quad k_n(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s-a) \Gamma(s+a)}{2(1+p-s)_n} t^{-s} ds, \quad |\arg t| \leq \pi.$$

Then we have

$$(1.3) \quad (K_a \phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p-1} \int_0^{\infty e^{i\theta_1}} k_n(u) \phi\left(\frac{u}{x}\right) du$$

where $|\arg x| \leq \theta + \pi$, $|\theta_1| \leq \pi$, $|\arg x - \theta_1| \leq \theta$.

We also have the following decomposition of the map K_a ,

$$(1.4) \quad K_a = M^{-1} \cdot m_1(K) \cdot M$$

with $K(s) = \frac{1}{2} \Gamma(s-a) \Gamma(s+a)$ and $m_1(K)$ defined as in chapter I. □

REMARK

Let the assumptions of theorem 33 be satisfied. We take a slightly different kernel in stead of (1.2). For $m \in \mathbb{N}$ and $|\arg t| \leq \pi$ we define

$$\begin{aligned} k_{2m}(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s-a)\Gamma(s+a)}{2(1-a-s)_m(1+a-s)_m} t^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{2} \Gamma(s-a-m)\Gamma(s+a-m) t^{-s} ds. \end{aligned}$$

Assume

$$\left. \begin{aligned} c < m, \lambda < c < \mu \\ c \neq \pm \operatorname{Re} a + m, \pm \operatorname{Re} a + m - 1, \dots \end{aligned} \right\} \text{ if } \lambda < \mu$$

$$\left. \begin{aligned} \mu \leq c \leq \lambda, c < m \\ \text{there are no poles of the} \\ \text{integrand in } \{s \in \mathbb{C} \mid \mu \leq \operatorname{Re} s \leq \lambda\} \end{aligned} \right\} \text{ if } \lambda \geq \mu.$$

Then we have

$$k_{2m}(t) = t^{-m} \left[K_{2a}(2\sqrt{t}) - \sum_{j=0}^{j_1-1} \frac{\frac{1}{2}\Gamma(2a)t^{k-a}}{k!(1-2a)_k} - \sum_{j=0}^{j_2-1} \frac{\frac{1}{2}\Gamma(-2a)t^{k+a}}{k!(1+2a)_k} \right]$$

if $a+m, \dots, a+m-j_1+1$ and $-a+m, \dots, -a+m-j_2+1$ are those poles of the integrand which are to the right of the contour.

It follows from theorems 9, 11, 18, 20 and remarks II.5 and III.1 that in this case we have

$$(K_a \phi)(x) = x^{-a} {}_D^m x^{m+2a} {}_D^m x^{m-a-1} \int_0^{\infty} e^{i\theta_1} t^{-m} k_{2m}(t) \phi\left(\frac{t}{x}\right) dt$$

if $|\arg x| \leq \pi + \theta$, $|\theta_1| \leq \pi$, $|\arg x - \theta_1| \leq \theta$ and $\phi \in T^\theta(1-\mu, 1-\lambda)$.

In order to derive inversion formulas for K_a we apply theorem II.11 and theorem III.20. The function H of these theorems is now

$$H(s) = 2[\Gamma(1+a-s)\Gamma(1-a-s)]^{-1}.$$

H is an entire function and

$$H(s) = O(|s|^{2d-1} e^{\pi |\operatorname{Im} s|}), \text{ as } s \rightarrow \infty, \operatorname{Re} s = d.$$

Let the assumptions of theorem 33 be satisfied.

1. Choose

$$H_{\pm}(s) = \pm \frac{1}{\pi i} e^{\pm \pi i(s+a)} \cdot \frac{\Gamma(s+a)}{\Gamma(1+a-s)}.$$

Let $1-\mu < d < 1-\lambda$ if $\lambda < \mu$ or $1-\lambda \leq d \leq 1-\mu$ if $\mu \leq \lambda$. Assume $d \neq -\operatorname{Re} a, -\operatorname{Re} a - 1, \dots$ if $\lambda < \mu$ and if $\mu \leq \lambda$ assume that no poles are in the strip $\{s \in \mathbb{C} \mid 1-\lambda \leq \operatorname{Re} s \leq 1-\mu\}$. Let $\ell > 2(1-\mu)$. Then $\ell > 2d$ if $1-\lambda \leq d \leq 1-\mu$ and if $\lambda < \mu$ we may choose d such that $\ell > 2d$.

Define $h_{\pm\ell}(t)$ as in (II.2.9)* with $\beta_j = a+j, j=1, \dots, \ell$:

$$(1.5) \quad h_{\pm\ell}(t) = \frac{\pm 1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{e^{\pm \pi i(s+a)} \Gamma(s+a)}{\pi i \Gamma(\ell+1+a-s)} t^{-s} ds, \arg t = \pm \pi.$$

We have

$$h_{\pm\ell}(t) = \pm \frac{1}{\pi i} \left\{ t^{-\frac{1}{2}\ell} I_{2a+\ell}(2\sqrt{t}) - \sum_{j=0}^{g-1} \frac{t^{a+j}}{j! \Gamma(\ell+2a+j+1)} \right\}$$

where $g \in \mathbb{N}$ is such that $-\operatorname{Re} a - g < d < -\operatorname{Re} a - g + 1$ if $d < -\operatorname{Re} a$ and otherwise $g = 0$. Here $I_{\cdot}$ denotes the modified Bessel function of the first kind and of order $2a+\ell$.

From (II.2.11)** we deduce with $\gamma_{\pm} = \pi$

$$(1.6) \quad (K_a^{-1} \psi)(x) = \frac{x^{-a}}{\pi i} \frac{d^{\ell}}{dx^{\ell}} x^{a+\ell} \int_C \{ (xt)^{-\frac{1}{2}\ell} I_{2a+\ell}(2\sqrt{xt}) - \sum_{j=0}^{g-1} \frac{(xt)^{a+j}}{j! \Gamma(\ell+2a+j+1)} \} \psi(t) dt,$$

$\psi \in T^{\pi}(\lambda, \mu)$, $x > 0$, where C is the contour which runs from $\infty e^{-\pi i}$ to zero and from zero to $\infty e^{i\pi}$.

As a special case, if $\mu > 1$ and $\lambda - 1 < \operatorname{Re} a$ we may choose $-\operatorname{Re} a < d < 0$ if $\lambda < \mu$ and hence $\ell = 0$ and $g = 0$. Then we have if $\lambda < \mu$ or $\mu \leq \lambda$,

$$(1.6)' \quad (K_a^{-1} \psi)(x) = \frac{1}{\pi i} \int_C I_{2a}(2\sqrt{t}) \psi\left(\frac{t}{x}\right) \frac{dt}{x}, \quad \psi \in T^{\pi}(\lambda, \mu), \quad x > 0.$$

REMARK

In order to derive the foregoing inversion formulas we have applied theorems II.11 and III.20 with $h_{\pm\ell}$ defined by (1.5). It should be noted that we have $\alpha_{\pm} = \pm\pi$. Hence formula (1.6) can not be extended to the case $\psi \in T^{\theta+\pi}(\lambda, \mu)$ if $\theta > 0$.

2. Choose

$$(1.7) \quad H_{\pm}(s) = \frac{e^{\pm\pi i(s+b)}}{i\Gamma(1-a-s)\Gamma(1+a-s)\sin\pi(s+b)}, \quad b \in \mathbb{C}.$$

We make the following assumptions:

$$\left. \begin{aligned} 1-\mu < d < 1-\lambda \\ d \neq -\operatorname{Re} b + k, \quad k \in \mathbb{Z} \end{aligned} \right\} \text{if } \lambda < \mu,$$

$$\left. \begin{aligned} 1-\lambda \leq d \leq 1-\mu \\ \{s \in \mathbb{C} \mid 1-\lambda \leq \operatorname{Re} s \leq 1-\mu\} \text{ does not} \\ \text{contain any of the points } -b+k, k \in \mathbb{Z} \end{aligned} \right\} \text{if } \mu \leq \lambda.$$

Let $\ell > 2(1-\mu)$. Then as in case 1. we may choose $\ell > 2d$. Define $h_{\pm\ell}(t)$ by

$$(1.8) \quad h_{\pm\ell}(t) = \frac{\pm 1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{e^{\pm\pi i(s+b)}}{i\Gamma(1-a-s)\Gamma(\ell+a-s+1)\sin\pi(s+b)} t^{-s} ds,$$

if $\arg t = \pm\pi$. Then the inverse is given by

$$(1.9) \quad (K_a^{-1}\psi)(x) = x^{-a} \frac{d^{\ell}}{dx^{\ell}} x^{a+\ell} \int_C h_{\pm\ell}(xt) \psi(t) dt,$$

if $\psi \in T^{\pi}(\lambda, \mu)$ and $x > 0$. Here C is the same contour as in the case above. The assertion follows as in case 1. from theorem II.11 and theorem III.20.

As a simple example we calculate $h_{\pm\ell}$ in a special case. Assume $\min(\lambda, \mu) > 1$. Then we may choose $d < 0$ and hence $\ell = 0$. Assume moreover that for some b we have

$$\lambda - 1 < \operatorname{Re} b < \mu.$$

Then we have, using residue calculus,

$$(1.10) \quad h_{\pm}(t) = \frac{\pm t^b}{\pi i \Gamma(1-a+b) \Gamma(1+a+b)} {}_1F_2(1; 1-a+b, 1+a+b; t), \quad \arg t = \pm \pi.$$

The corresponding inversion formula is now

$$(1.11) \quad (K_a^{-1} \psi)(x) = \frac{1}{\pi i \Gamma(1-a+b) \Gamma(1+a+b)} \times \\ \times \int_C {}_1F_2(1; 1-a+b, 1+a+b; xt) (xt)^b \psi(t) dt,$$

where $\psi \in T^{\pi}(\lambda, \mu)$, $x > 0$ and C is the contour in (1.9). If $b=a$ this reduces to formula (1.6)'.

REMARK

Also here we have to put $\alpha_{\pm} = \pi$ in the application of theorem II.11 and III.20 if $h_{\pm, \ell}$ is defined by (1.8). Hence formula (1.9) and formula (1.11) can not be extended to the case $\psi \in T^{\theta+\pi}(\lambda, \mu)$ for positive θ .

7.2 MEIJER'S W-TRANSFORM

The kernel of the integral transform in (0.2) is

$$(2.1) \quad k(t) = e^{-\frac{1}{2}t} t^{-\frac{1}{2}} W_{b+\frac{1}{2}, a}(t),$$

where $W_{\dots}$ denotes Whittaker's function. Its Mellin transform is

$$(2.2) \quad K(s) = \frac{\Gamma(s+a) \Gamma(s-a)}{\Gamma(s-b)}, \quad \operatorname{Re} s > \operatorname{Re} a \geq 0,$$

(cf [12] I p 216 and [46] p 146). If $\operatorname{Re} s$ remains bounded we have

$$K(s) = O(s^{-\frac{1}{2}\pi |\operatorname{Im} s|} |s|^{\operatorname{Re}(s+b)-\frac{1}{2}}) \quad \text{as } |\operatorname{Im} s| \rightarrow \infty,$$

$$(K(1-s))^{-1} = O(s^{\frac{1}{2}\pi |\operatorname{Im} s|} |s|^{\operatorname{Re}(s-b)-\frac{1}{2}}) \quad \text{as } |\operatorname{Im} s| \rightarrow \infty.$$

It follows that to any pair of real numbers α_1 and β_1 there exists $\gamma_1 \in \mathbb{R}$ such that

$$K(s) = O(s^{-\frac{1}{2}\pi|\operatorname{Im} s|} |s|^{\gamma_1}) \text{ as } s \rightarrow \infty,$$

uniformly on $\{s \in \mathbb{C} \mid \alpha_1 \leq \operatorname{Re} s \leq \beta_1\}$. Also to any pair of real numbers α_2 and β_2 there exists γ_2 such that

$$H(s) = 1/K(1-s) = O(s^{\frac{1}{2}\pi|\operatorname{Im} s|} |s|^{\gamma_2}) \text{ as } s \rightarrow \infty,$$

uniformly on $\{s \in \mathbb{C} \mid \alpha_2 \leq \operatorname{Re} s \leq \beta_2\}$.

We may apply theorem II.11 and III.20. We then have the following theorem.

THEOREM 34. Let $a, b \in \mathbb{C}$, $\operatorname{Re} a \geq 0$. Assume $\max(\operatorname{Re} a, \operatorname{Re} b) \leq \lambda < \mu$ or $\max(\operatorname{Re} a, \operatorname{Re} b) < \mu \leq \lambda$. Then the W -transform, defined in (0.2) for $|\arg x| < \frac{1}{2}\pi$, $\phi \in T(1-\mu, 1-\lambda)$, is a continuous isomorphism of $T(1-\mu, 1-\lambda)$ onto $T^{\frac{1}{2}\pi}(\lambda, \mu)$. For any $\theta > 0$ its restriction to $T^{\theta}(1-\mu, 1-\lambda)$ is a continuous isomorphism onto $T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$. This extension may be described by

$$(2.3) \quad (W_{a,b}\phi)(x) = \frac{1}{x} \int_0^{\infty} e^{i\theta_1 t} t^{-\frac{1}{2}} W_{b+\frac{1}{2},a}(t) \phi\left(\frac{t}{x}\right) dt, \quad |\arg x| < \theta + \frac{1}{2}\pi,$$

where θ_1 is such that

$$(2.4) \quad |\theta_1| < \frac{1}{2}\pi \text{ and } \theta_1 - \theta \leq \arg x \leq \theta_1 + \theta.$$

A representation which is valid if $|\arg x| \leq \theta + \frac{1}{2}\pi$ is given by

$$(2.5) \quad (W_{a,b}\phi)(x) = x^{-p} \frac{d^n}{dx^n} x^{n+p-1} \int_0^{\infty} e^{i\theta_1 t} k_n(t) \phi\left(\frac{t}{x}\right) dt,$$

where $|\theta_1| \leq \frac{1}{2}\pi$, $|\arg x - \theta_1| \leq \theta$. The kernel in (2.5) is defined as in (II.1.3)⁺:

$$k_n(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s-a)\Gamma(s+a)}{\Gamma(s-b)(1+p-s)_n} t^{-s} ds, \quad |\arg t| \leq \frac{1}{2}\pi$$

if p and n are such that $\operatorname{Re} p + 1 > \max(\lambda, \mu)$, $n > c + \frac{1}{2} + \operatorname{Re} b$. □

In the same manner as in the case of the K -transform we derive inversion formulas. We use again the decomposition of the function

H , namely $H(s) = H_+(s) + H_-(s)$, according to theorem II.11 and theorem III.20. We assume $\max(\operatorname{Re} a, \operatorname{Re} b) \leq \lambda < \mu$ or $\max(\operatorname{Re} a, \operatorname{Re} b) < \mu \leq \lambda$ and $d \in (1-\mu, 1-\lambda)$ or $d \in [1-\lambda, 1-\mu]$.

1. Define

$$H_{\pm}(s) = \pm \frac{1}{2\pi i} \frac{\Gamma(s+a)\Gamma(1-s-b)e^{\pm\pi i(s+a)}}{\Gamma(1-s+a)}.$$

We have

$$H_{\pm}(s) = O(|s|^{d-\frac{1}{2}-\operatorname{Re} b} e^{-\frac{1}{2}\pi|\operatorname{Im} s|} e^{\pm\pi\operatorname{Im} s}) \text{ as } s \rightarrow \infty, \operatorname{Re} s = d.$$

Let $\ell \in \mathbb{N}$ be such that $\ell > \frac{3}{2} - \mu - \operatorname{Re} b$; if $\lambda < \mu$ we may choose d such that $\ell > d + \frac{1}{2} - \operatorname{Re} b$; if $\lambda \geq \mu$ then for all $d \in [1-\lambda, 1-\mu]$ we have $\ell > d + \frac{1}{2} - \operatorname{Re} b$. Define $h_{\pm}(t)$ by (II.2.9)* with $\beta_j = a+j$, $j=1, \dots, \ell$ and $\alpha_{\pm} = \frac{3}{2}\pi$:

$$(2.6) \quad h_{\pm\ell}(t) = \frac{\pm 1}{(2\pi i)^2} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s+a)\Gamma(1-s-b)}{\Gamma(\ell+1-s+a)} e^{\pm\pi i(s+a)} t^{-s} ds$$

if $\frac{1}{2}\pi \leq \pm \arg t \leq \frac{3}{2}\pi$. We assume that the line $\operatorname{Re} s = d$ does not contain any of the poles of the integrand. From (II.2.11)** with $\gamma_{\pm} = \pm\frac{1}{2}\pi$ we obtain the inversion formula

$$(2.7) \quad (W_{a,b}^{-1}\psi)(x) = x^{-a} \frac{d^{\ell}}{dx^{\ell}} x^{a+\ell} \left[\int_0^{i\infty} h_{+\ell}(xt)\psi(t) dt + \int_0^{-i\infty} h_{-\ell}(xt)\psi(t) dt \right], \quad x > 0, \psi \in T^{\frac{1}{2}\pi}(\lambda, \mu),$$

which formula holds if $\ell + \operatorname{Re} b + \mu > \frac{3}{2}$.

If $\theta > 0$ and $\psi \in T^{\frac{1}{2}\pi+\theta}(\lambda, \mu)$ the inversion formula becomes

$$(W_{a,b}^{-1}\psi)(x) = x^{-a} \frac{d^{\ell}}{dx^{\ell}} x^{a+\ell} \left[\int_0^{\infty e^{i\gamma_+}} h_{+\ell}(xt)\psi(t) dt + \int_0^{\infty e^{i\gamma_-}} h_{-\ell}(xt)\psi(t) dt \right]$$

if $|\arg x| \leq \theta$ and $\gamma_{\pm}$ satisfy (II.2.12) with $\alpha_{\pm} = \frac{3}{2}\pi$ and $\theta_0 = \frac{1}{2}\pi$. We consider a special case. Assume $\lambda < 1 + \operatorname{Re} a$. If $\lambda < \mu$, we may choose d such that $d > -\operatorname{Re} a$; so the poles $s = -a - j$, $j \in \mathbb{N}$ are to the left of $\operatorname{Re} s = d$. If $\lambda \geq \mu$, the poles $s = -a - j$, $j \in \mathbb{N}$ are to the left of $\operatorname{Re} s = 1 - \lambda$. Using residue calculus we obtain

$$(2.8) \quad h_{\pm \ell}(t) = \frac{\pm 1}{2\pi i} \frac{\Gamma(1+a-b)}{\Gamma(2a+\ell+1)} t^a {}_1F_1(1+a-b; 2a+\ell+1; t),$$

$\frac{1}{2}\pi \leq \arg t \leq \frac{3}{2}\pi$. From this we deduce the inversion formula

$$(2.9) \quad (W_{a,b}^{-1}\psi)(x) = \frac{\Gamma(1+a-b)}{2\pi i \Gamma(2a+\ell+1)} x^{-a} \frac{d^{\ell}}{dx^{\ell}} x^{a+\ell} \times \\ \times \int_{-i\infty}^{i\infty} (xt)^a {}_1F_1(1+a-b; 2a+\ell+1; xt) \psi(t) dt$$

if $x > 0$ and $\psi \in T^{\frac{1}{2}\pi}(\lambda, \mu)$, $\lambda < 1 + \operatorname{Re} a$, $\ell + \operatorname{Re} b + \mu > \frac{3}{2}$ and the assumptions of theorem 34 are fulfilled. In terms of Whittaker's function this formula is, (cf [12] I, p 264)

$$(W_{a,b}^{-1}\psi)(x) = \frac{\Gamma(1+a-b)}{2\pi i \Gamma(2a+\ell+1)} x^{-a} \frac{d^{\ell}}{dx^{\ell}} x^{a+\ell} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}xt} (xt)^{-\frac{1}{2}-\frac{1}{2}\ell} M_{b-\frac{1}{2}+\frac{1}{2}\ell, a+\frac{1}{2}\ell}(xt) \psi(t) dt$$

if $x > 0$ and $\psi \in T^{\frac{1}{2}\pi}(\lambda, \mu)$. If $\psi \in T^{\theta+\frac{1}{2}\pi}(\lambda, \mu)$, $\theta > 0$, we have to modify the contour as already indicated.

The last inversion formula in the case $\ell = 0$ and formula (1.6) with $\ell = 0$ and $g = 0$ are due to Meijer, (cf [44a] and [44b]).

REMARK

If $b = a$ the kernel K reduces to $K(s) = \Gamma(s+a)$. The W -transform then reduces to a special form of the Laplace transform and the foregoing inversion formulas for this case are modifications of the inversion formulas which we have obtained for the Laplace transform in chapter II.

2. We define the functions $H_{\pm}$ by

$$H_{\pm}(s) = \frac{\pm \Gamma(1-s-b) e^{\pm \frac{1}{2}\pi(s+p)i}}{2i \Gamma(1-s-a) \Gamma(1-s+a) \sin \frac{1}{2}\pi(s+p)}, \quad p \in \phi.$$

We have

$$H_{\pm}(s) = O(|s|^{d-b-\frac{1}{2}} e^{\mp \frac{1}{2} \pi \operatorname{Im} s}) \text{ as } s \rightarrow \infty, \operatorname{Re} s = d.$$

Let $\lambda < \mu$, $\ell \in \mathbb{N}$, $\ell > \frac{3}{2} - \mu - \operatorname{Re} b$. Then choose d such that $\ell > d + \frac{1}{2} - \operatorname{Re} b$, $1 - \mu < d < 1 - \lambda$ and such that the line $\operatorname{Re} s = d$ does not contain any pole of the functions $H_{\pm}$. Then

$$\frac{e^{\pm \frac{1}{2} \pi \sigma} H_{\pm}(d + i\sigma)}{(1 - d - i\sigma + a)_{\ell}} \in L(-\infty, \infty).$$

Define $h_{\pm \ell}(t)$ by

$$(2.11) \quad h_{\pm \ell}(t) = \frac{\mp 1}{4\pi} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(1-s-b) e^{\pm \frac{1}{2} \pi (s+p)i}}{\Gamma(1-s-a) \Gamma(\ell+1-s+a) \sin \frac{1}{2} \pi (s+p)} t^{-s} ds,$$

$\arg t = \pm \pi$.

Then with these $h_{\pm \ell}$ the inversion formula (2.7) holds.

In order to give a simple example of calculating the right hand side of (2.11) we assume $\lambda < 1 + \operatorname{Re} p < 2 + \mu$. Then we may choose d such that $1 - \mu < d < 1 - \lambda$ and $-\operatorname{Re} p < d < 2 - \operatorname{Re} p$. Using residue calculus we obtain

$$(2.12) \quad h_{\pm \ell}(t) = \frac{\pm 1}{\pi i} \frac{\Gamma(1+p-b) t^p}{\Gamma(1+p-a) \Gamma(\ell+p+a+1)} \sum_{m=0}^{\infty} \frac{(1+p-b)_{2m}}{(1+p-a)_{2m} (\ell+p+a+1)_{2m}} t^{2m}.$$

As an example of the use of this expression for $h_{\pm \ell}$ we consider a simple case. Assume $\frac{3}{2} - \operatorname{Re} b < \mu$. Then we may choose $d < \operatorname{Re} b - \frac{1}{2}$ and $\ell = 0$. For p we take the value a . Then the Σ -part in (2.12) becomes, if $\operatorname{Re}(a+b) > \frac{1}{2}$:

$$\sum_{m=0}^{\infty} \frac{(\frac{1}{2} + \frac{1}{2} a - \frac{1}{2} b)_m (1 + \frac{1}{2} a - \frac{1}{2} b)_m}{(\frac{1}{2})_m (a+1)_m (a+\frac{1}{2})_m m!} \left(\frac{t^2}{4}\right)^m.$$

We obtain the inversion formula

$$(2.13) \quad (W_{a,b}^{-1}\psi)(x) = \frac{1}{\pi i} \frac{\Gamma(1+a-b)}{\Gamma(2a+1)} \int_{-i\infty}^{i\infty} (xt)^a {}_2F_3\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}a-\frac{1}{2}b, 1+\frac{1}{2}a-\frac{1}{2}b; \frac{1}{2}x^2t^2 \\ \frac{1}{2}, a+\frac{1}{2}, a+1 \end{matrix}\right] \psi(t) dt,$$

$x > 0$, $\psi \in T^{\frac{1}{2}\pi}(\lambda, \mu)$, which holds if the following conditions are satisfied: $\max(\operatorname{Re} a, \operatorname{Re} b) < \lambda < \mu$, $\frac{3}{2} - \operatorname{Re} b < \mu$, $\lambda < 1 + \operatorname{Re} a < 2 + \mu$, $\operatorname{Re}(a+b) > \frac{1}{2}$. If $\ell \neq 0$ or $\mu \leq \lambda$ similar results can be derived.

REMARKS

1. For the derivation of formula (2.13) we have used the formula

$$(1+q)_{2m} = 2^{2m} \left(\frac{1}{2} + \frac{1}{2}q\right)_m (1 + \frac{1}{2}q)_m.$$

2. Adding and subtracting odd terms in (2.12) we see that the right hand side may be written as

$$\begin{aligned} & \frac{\pm 1}{2\pi i} \frac{\Gamma(1+p-b)t^p}{\Gamma(1+p-a)\Gamma(\ell+p+a+1)} [{}_2F_2(1, 1+p-b; 1+p-a, \ell+p+a+1; t) + \\ & \quad + {}_2F_2(1, 1+p-b; 1+p-a, \ell+p+a+1; -t)] \end{aligned}$$

which for $p = a$ becomes

$$(*) \quad \frac{\pm 1}{2\pi i} \frac{\Gamma(1+a-b)t^a}{\Gamma(2a+\ell+1)} [{}_1F_1(1+a-b; 2a+\ell+1; t) + {}_1F_1(1+a-b; 2a+\ell+1; -t)].$$

From [35a], p 201, (1) and (2) we deduce

$$\begin{aligned} t^p {}_2F_2(1, 1+p-b; 1+p-a, \ell+p+a+1; -t) = \\ O(t^{-b-\ell}) + O(t^{p-1}) + O(t^{b-1}), \quad t \rightarrow \infty, \quad |\arg t| \leq \frac{1}{2}\pi. \end{aligned}$$

Hence if the following conditions are satisfied:

$$\max(\operatorname{Re} a, \operatorname{Re} b) \leq \lambda < \mu,$$

$$\lambda < 1 + \operatorname{Re} p < 1 + \mu, \quad \ell < \frac{3}{2} - \mu - \operatorname{Re} b,$$

we obtain, using Cauchy's theorem

$$\int_{-i\infty}^{i\infty} (xt)^p {}_2F_2(1, 1+p-b; 1+p-a, \ell+p+a+1; -xt) \psi(t) dt = 0, \quad x > 0,$$

$\psi \in T^{\frac{1}{2}\pi}(\lambda, \mu)$. For this case the inversion formula (2.7) becomes

$$\begin{aligned} (W_{a,b}^{-1} \psi)(x) &= \frac{1}{2\pi i} \frac{\Gamma(1+p-b)}{\Gamma(1+p-a)\Gamma(\ell+p+a+1)} x^{-a} \frac{d^\ell}{dx^\ell} x^{a+\ell} \times \\ &\times \int_{-i\infty}^{i\infty} (xt)^p {}_2F_2(1, 1+p-b; 1+p-a, \ell+p+a+1; xt) \psi(t) dt. \end{aligned}$$

In case $p=a$ this reduces to (2.9).

(2.9) also follows directly from (*), using

$${}_1F_1(1+a-b; 2a+\ell+1; -t) = O(t^{b-a-1}) + O(t^{-a-b-\ell}).$$

as $t \rightarrow \infty$, $|\arg t| \leq \frac{1}{2}\pi$, (cf [35a], p 201; [4a], p 330, p 322).

3. We choose

$$H_{\pm}(s) = \frac{\pm \Gamma(1-s-b) e^{\pm \pi(s+p)i}}{2i \Gamma(1-s-a) \Gamma(1-s+a) \sin \pi(s+p)}, \quad p \in \mathbb{C}.$$

We have

$$H_{\pm}(s) = O(|s|^{d-b-\frac{1}{2}} e^{-\frac{1}{2}\pi |\operatorname{Im} s|} \mp \pi \operatorname{Im} s) \text{ as } s \rightarrow \infty, \operatorname{Re} s = d.$$

Let $\ell \in \mathbb{N}$, $\ell > \frac{3}{2} - \mu - \operatorname{Re} b$. Then if $\lambda < \mu$ we may choose d such that $\ell > d - \operatorname{Re} b + \frac{1}{2}$. Assume that the line $\operatorname{Re} s = d$ does not contain any pole of $H_{\pm}(s)$. Then

$$\frac{e^{\pm \frac{1}{2}\pi \operatorname{Im} s} H_{\pm}(s)}{(1-s+a)_{\ell}} \in L(-\infty, \infty), \quad \operatorname{Re} s = d.$$

We define $h_{\pm\ell}(t)$ as in the previous case,

$$(2.14) \quad h_{\pm\ell}(t) = \frac{\mp 1}{4\pi} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(1-s-b) e^{\pm \pi(s+p)i}}{\Gamma(1-s-a) \Gamma(\ell+1-s+a) \sin \pi(s+p)} t^{-s} ds$$

if $\arg t = \pm \frac{1}{2}\pi$. Then (2.7) holds if (2.7)⁺ is satisfied.

Consider the following simple example. Assume

$$\max(\operatorname{Re} a, \operatorname{Re} b) \leq \lambda < \mu, \quad \lambda < 1 + \operatorname{Re} p < 1 + \mu.$$

Then we may choose d such that $-p < d < 1-p$, $1-\mu < d < 1-\lambda$. Using residue calculus we obtain with $\ell=0$:

$$h_{\pm}(t) = \frac{\pm t^p \Gamma(1+p-b)}{2\pi i \Gamma(1+p-a) \Gamma(1+p+a)} \times \\ \times {}_2F_2(1, 1+p-b; 1+p-a, 1+p+a; t).$$

We thus have obtained the same inversion formula as in Remark 2 above, for the special case $\ell=0$. In the same manner of course we can derive the formula for $\ell > 0$.

Without writing out the details we remark that we may also use the decomposition

$$H_{\pm}(s) = \frac{\pm \Gamma(1-s-b)}{2i \Gamma(1-s-a) \Gamma(1-s+a)} \frac{e^{\pm r\pi(s+p)i}}{\sin r\pi(s+p)}, \quad p \in \mathbb{C}$$

with $r \geq \frac{1}{2}$ in order to derive other inversion formulas. In this case (II.2.8) is satisfied with

$$\frac{1}{2}\pi \leq \pm \gamma \leq (2r - \frac{1}{2})\pi, \quad \alpha_{\pm} = (2r - \frac{1}{2})\pi.$$

7.3 RELATIONS WITH OTHER TRANSFORMS

In all cases of integral transforms we have considered in this and the previous chapter there is a simple factorization of the form

$$(3.1) \quad M^{-1} \circ m(K) \circ M \text{ or } M^{-1} \circ m_1(K) \circ M$$

where K denotes the Mellin transform of the kernel of the transform. A similar factorization holds for the inverse transforms. If we make adequate assumptions in order to satisfy the conditions of the underlying theorems, we may use (3.1) to consider products of integral transforms. We have already seen some examples in the foregoing chapters, among others in those cases where we have extended the definition of an integral transform.

For the K -transform and the W -transform of this chapter we can also consider extensions and relations with other transforms. Far from being complete we choose some examples in order to indicate the method.

EXAMPLE I.

Assume $\lambda < \mu$, $\operatorname{Re} a \geq 0$, $2a \notin \mathbb{Z}$. Let $k, \ell \in \mathbb{N}$ be such that

$$\operatorname{Re} a - k \leq \lambda < \operatorname{Re} a - k + 1, \quad \operatorname{Re} a < \mu,$$

$$-\operatorname{Re} a - \ell \leq \lambda < -\operatorname{Re} a - \ell + 1.$$

Choose $n \in \mathbb{N}$ such that $\operatorname{Re} a - n \leq \lambda$ and $n \geq 1$. Consider the functions H_1 and H_2 ,

$$H_1(s) = \Gamma(s+n+a)\Gamma(s+n-a);$$

$$H_2(s) = \frac{1}{(s+a)_n (s-a)_n}.$$

Define k_1 and k_2 by

$$k_j(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} H_j(s) t^{-s} ds, \quad \operatorname{Re} a < c < \mu, \quad j = 1, 2,$$

$|\arg t| < \pi$ if $j=1$, $t > 0$ if $j=2$. We then have: the map A_1 , defined by

$$(A_1 \phi)(x) = \int_0^\infty (xt)^n k_{2a}(2\sqrt{xt}) \phi(t) dt$$

is a continuous isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T^{\theta+\pi}(\lambda, \mu)$ for any $\theta \geq 0$. Furthermore the map A_2 , defined by

$$(A_2 \phi)(x) = \int_0^\infty k_2(t) \phi\left(\frac{x}{t}\right) \frac{dt}{t}$$

is a continuous isomorphism of $T^{\theta_0}(\lambda, \mu)$ onto $T_{k+\ell, k+\ell}^{\theta_0}(\lambda, \mu; \underline{a})$ if $\ell \leq n$, for any $\theta_0 \geq 0$, where

$$\underline{a} = (a-k+1, \dots, a, -a-\ell+1, \dots, -a)$$

and of $T^{\theta_0}(\lambda, \mu)$ onto $T_{k+n, k+n}^{\theta_0}(\lambda, \mu; \underline{\tilde{a}})$ if $\ell > n$, where

$$\underline{\tilde{a}} = (a-k+1, \dots, a, -a-n+1, \dots, -a).$$

This assertion follows from example II, chapter IV.

We consider the product $A_2 \circ A_1$. It defines a continuous isomorphism of $T^\theta(1-\mu, 1-\lambda)$ onto $T_{k+\ell, k+\ell}^{\theta+\pi}(\lambda, \mu; \underline{a})$ if $\ell \leq n$.

The maps A_1 and A_2 have also the representations

$$A_1 = M^{-1} \circ m_1(H_1) \circ M, \quad A_2 = M^{-1} \circ m(H_2) \circ M.$$

It follows that

$$A_2 \circ A_1 = M^{-1} \circ m_1(K) \circ M,$$

where $K(s) = H_1(s)H_2(s) = \Gamma(s+a)\Gamma(s-a)$. Hence the K -transform defined by (0.1) is a continuous map of $T^\theta(1-\mu, 1-\lambda)$ onto $T_{k+\ell, k+\ell}^{\theta+\pi}(\lambda, \mu; \underline{a})$. This may also be proven directly using theorems 26 and 27. Confer also the last example in chapter IV.

If $\ell > n$ we obtain similar results.

EXAMPLE II

We give a decomposition of K_a into the product of two transforms.

Assume $\operatorname{Re} a \geq 0$, $\operatorname{Re} a \leq \lambda < \mu$ or $\operatorname{Re} a < \mu \leq \lambda$. For any $\theta \geq 0$ we define

$A : T^{\theta}(1-\mu, 1-\lambda) \rightarrow T^{\theta+\frac{1}{2}\pi}(1-\mu, 1-\lambda)$ by

$$(A\phi)(x) = \int_0^{\infty} e^{i\theta_1} t^{a-1} \exp(-t^{-1}) \phi\left(\frac{x}{t}\right) \frac{dt}{t}, \quad |\arg x| < \theta + \frac{1}{2}\pi,$$

where θ_1 is such that $|\theta_1| < \frac{1}{2}\pi$ and $\theta_1 - \theta \leq \arg x \leq \theta + \theta_1$. Then we have

$$(3.2) \quad A = M^{-1} \circ m(K) \circ M$$

where $K(s) = \Gamma(1-s-a)$, (cf. theorem II.13 and theorem III.22).

Consider also the Laplace transform $L_a : T^{\theta+\frac{1}{2}\pi}(1-\mu, 1-\lambda) \rightarrow T^{\theta+\pi}(\lambda, \mu)$, defined by

$$(L_a \phi)(x) = \int_0^{\infty} e^{i\theta_2} t^a e^{-t} \phi\left(\frac{t}{x}\right) \frac{dt}{x}, \quad |\arg x| < \theta + \pi,$$

where θ_2 is such that $|\theta_2| < \frac{1}{2}\pi$ and $\theta_2 - \theta - \frac{1}{2}\pi \leq \arg x \leq \theta + \frac{1}{2}\pi + \theta_2$.

Then L_a may also be represented by

$$(3.3) \quad L_a = M^{-1} \circ m_1(K) \circ M$$

where now $K(s) = \Gamma(s+a)$. Using (3.2) and (3.3) we obtain

$$(3.4) \quad K_a = \frac{1}{2} L_a \circ A$$

if K_a is the K-transform defined in (1.1).

EXAMPLE III

Let $\operatorname{Re} a \geq 0$ and $\operatorname{Re} b \geq 0$. Assume $\max(\operatorname{Re} a, \operatorname{Re} b) \leq \lambda < \mu$ or $\max(\operatorname{Re} a, \operatorname{Re} b) < \mu \leq \lambda$. Then both K_a and K_b are isomorphisms of $T^{\theta}(1-\mu, 1-\lambda)$ onto $T^{\theta+\pi}(\lambda, \mu)$, $\theta \geq 0$. In order to give a relation between

K_a and K_b we consider the operators of fractional integration $K(a, b-a)$ and $K(-a, a-b)$. The conditions I' or II' and VI' or VII', chapter V, section 2 are satisfied. Hence both the operators are automorphisms of $T^{\theta+\pi}(\lambda, \mu)$ for any $\theta \geq 0$. Using decompositions of all four transforms we obtain in the same manner as in the foregoing examples

$$K_a = K(a, b-a) \cdot K(-a, a-b) \cdot K_b.$$

It is not hard to see that the order of the fractional integration operators in the right hand side may be reversed.

EXAMPLE IV

In this example we consider a factorization of the W-transform. Let $\operatorname{Re} a \geq 0$ and $\operatorname{Re} a \leq \lambda < \mu$ or $\operatorname{Re} a < \mu \leq \lambda$. Then the fractional integration operator $I(-a, a-b)$ is a continuous operator of $T^{\theta}(1-\mu, 1-\lambda)$, $\theta \geq 0$. This follows immediately by inspection of the conditions I and II of chapter V. We have as in the previous examples

$$(3.5) \quad I(-a, a-b) = M^{-1} \cdot m(K) \cdot M$$

where $K(s) = \Gamma(1-a-s)/\Gamma(1-b-s)$. Let L_a be the Laplace transform defined in example II. Combining (3.3) and (3.5) we obtain

$$W_{a,b} = L_a \cdot I(-a, a-b).$$

7.4 BIBLIOGRAPHICAL NOTES

The K- and W-transforms which are the subject of this chapter were given by C.S.Meijer in 1940, 1941, (cf [44a],[44b]). Several related forms have been proposed in later years, some of them leading to generalizations of the kernel to special forms of the G function. The papers of Krätzel ([36], [37]) and of Swaroop ([56]) may be seen as examples of such extensions. Their results for spaces of type $T^0(\lambda, \mu)$ may be derived directly using the methods of chapter 2 and chapter 4. They also follow as special cases of the example in chapter 4 which has a G-function as kernel. Fox ([22]) has given a decomposition of the inverse K-transform, the factors being the inverse Laplace transform and a fractional integral. Bora and Saxena ([3]) mention the relation between Meijer transforms, Hankel transforms and fractional integrals. Their results may be deduced for spaces $T^0(\lambda, \mu)$ in the same manner as the results of example IV (section 7.3). Joshi's work ([29]) may be seen in some sense as closely related to Swaroop's paper. Zemanian ([72] and earlier papers) considers Meijer transforms in the distributional sense.

REFERENCES

- [1] ARORA,J.L. Certain theorems in the generalised Fourier transforms.
Kyungpook Math.J. 15 (1975), 271-280.
- [2] BOAS,R.P. Generalized Laplace integrals.
Bull.Am.Math.Soc. 48 (1942), 286-294
- [3] BORA,S.L. and SAXENA,R.K. On fractional integration.
Publ. Institut Math. 25 (1971) 19-22.
- [4] BRAAKSMA,B.L.J. and SCHUITMAN,A. Some classes of Watson transforms
and related integral equations for generalized functions.
Siam J.Math.Anal. 7 (1976), 771-798.
- [4a] BRAAKSMA,B.L.J. Asymptotic expansions and analytic
continuations for a class of Barnes-integrals.
Groningen, 1963.
- [5] COOPER,J.L.B. Laplace transforms of distributions.
Canad.J.Math. 18 (1966), 1325-1332.
- [6] DAHIYA,R.S. Operational calculus and self-reciprocal functions.
Indag.Math. 38 (1976), 17-21.
- [7] DAHIYA,R.S. On Stieltjes transforms.
Istanbul Tek.Univ.Bül. 26 (1973) 2, 50-54.
- [8] ERDELYI,A. On fractional integration and its applications
to the theory of Hankel transforms.
Quart.J.Math.Oxford Ser. 11 (1940), 293-303.
- [9] ERDELYI,A. An application of fractional integrals.
J.Analyse Math. 14 (1965), 113

- [10] ERDELYI, A. Fractional integrals of generalized functions.
J. Austral. Math. Soc. 14 (1972) 30-37.
- [11] ERDELYI, A. Stieltjes transforms of generalised functions.
Proc. Roy. Soc. Edinburgh Sect A 76 (1977), 231-249.
- [12] ERDELYI, A. Higher transcendental functions, I, II.
New York-Toronto-London, 1953.
- [13] ERDELYI, A. (Ed.) Tables of integral transforms I, II.
New York-Toronto-London, 1954.
- [14] ERDELYI, A. and McBRIDE, A.C. Fractional integrals of distributions.
Siam J. Math. Anal. 1 (1970) 547-557.
- [15] ERDELYI, A. and SNEDDON, I.N. Fractional integration and dual
integral equations.
Canad. J. Math. 14 (1962) 685-693.
- [16] EIJNDHOVEN, S.J.L., van. A theory of generalized functions based
on one parameter groups of unbounded self adjoint operators.
TH-Report 81-WSK-03 (1981), Techn. University Eindhoven.
- [17] EIJNDHOVEN, S.J.L., van. On Hankel invariant distribution spaces.
EUT-Report 82-WSK-01 (1982), Eindhoven University of Technology.
- [18] FENYÖ, S. Hankel Transformationen verallgemeinerter Funktionen.
Mathematica vol. 8 (31) (1966), 235-242.
- [19] FLORET, K and WLOKA, J. Einführung in die Theorie der lokal-
konvexen Räume.
Lecture Notes in Mathematics 56 (1966), Berlin-Heidelberg-N.Y.
- [20] FOX, C. The G and H functions as symmetrical Fourier kernels.
Trans. Amer. Math. Soc. 98 (1961), 395-429.
- [21] FOX, C. A formal solution of certain dual integral equations.
Trans. Amer. Meth. Soc. 98 (1961), 395-429.

- [22] FOX, C. An inversion formula for the kernel $K_{\nu}(x)$.
Proc. Camb.Philos.Soc. 61 (1965), 457-467.
- [23] FUNG KANG. Generalized Mellin transforms I.
Sci.Sinica 7 (1958) 582-605.
- [24] GELFAND, I.M. and SCHILOW, G.E. Verallgemeinerte Funktionen II, III.
Berlin, (II 1969, III 1964).
- [25] GRAAF, J. de. A theory of generalized functions based on
holomorphic semigroups.
TH-Report 79-WSK-02 (1979). Techn. University Eindhoven.
- [26] GRADSHTEYN, I.S. and RYZHIK, I.M. Table of integrals, series
and products.
New York-London-Toronto-Sydney-San Francisco, (1980).
- [26a] HILLE, E. Analytic Function Theory, I, II.
Massachusetts-Toronto-London, 1962.
- [26b] HIRSCHMAN, I.I. and WIDDER, D.V. The convolution transform.
Princeton, 1955.
- [27] HSING-YUAN HSU. Distributional Watson transforms.
Can.J.Math. XXIV (1972), 1191-1197.
- [28] JONES, D.S. Generalised functions.
London-New York-Toronto-Sydney, (1966).
- [29] JOSHI, J.M.C. On Joshi's generalized Stieltjes transform.
Ganita 28 (1977), 15-24.
- [30] KOBER, H. On fractional integrals and derivatives.
Quart.J.Math.Oxford Ser. 11 (1940), 193-211.
- [31] KOBER, H. Eine Verallgemeinerung der Transformation vom
Fourier-Typ.
Quart.J.Math.Oxford. Ser 8 (1973), 172-185.

- [32] KOH, E.L. The Hankel transformation of negative order for distributions of rapid growth.
Siam J.Math.Anal. 1 (1970), 322-327.
- [33] KÖTHE, G. Topologische Lineare Räume.
Berlin-Heidelberg-New York, (1966).
- [34] LEE, W.Y.K. On spaces of type H_{μ} and their Hankel transformations.
Siam J.Math.Anal. 5 (1974), 336-348.
- [35] LEE, W.Y.K. On Schwartz's Hankel transformation of certain spaces of distributions.
Siam J.Math.Anal. 6 (1975), 427-432.
- [35a] LUKE, Y.L. The special functions and their approximations.
New York, 1969.
- [36] KRÄTZEL, E. Eine Verallgemeinerung der Laplace- und Meijer-Transformation.
Wissenschaftliche Z.Friedrich-Schiller Univ. Jena 14 (1965), 369-381.
- [37] KRÄTZEL, E. Bemerkungen zur Meijer-Transformation und Anwendungen.
Math.Nachrichten 30 (1965) 327-334.
- [38] LOVE, E.R. Some integral equations involving hypergeometric functions.
Proc.Edinburgh Math.Soc. (II) 15 (1976), 169-198.
- [39] LOVE, E.R. Two more hypergeometric integral equations.
Proc.Camb.Philos.Soc. 63 (1976), 1055-1076.
- [40] LOVE, E.R. Two index laws for fractional integrals and derivatives.
J.Australian Math.soc. 14 (1972) 358-410.
- [41] LOVE, E.R., PRABHAKAR, T.R., KASHYAP, N.K. A confluent hypergeometric integral equation.
Glasgow Math.J. 23 (1982), 31-40.
- [42] MALGONDE, S.P. and SAXENA, R.K. An inversion formula for the distributional H-transformation.
Math.Ann. 258 (1981/1982) 4, 409-417.

- [42a] MARICHEV, O.I. Handbook of integral transforms of higher transcendental functions. Chichester, (1983).
- [43] McBRIDE, A.C. The Hankel transform of some classes of generalized functions and connections with fractional integration.
Proc. Roy. Soc. Edinburgh Sect A 81 (1978), 95-117.
- [44] McBRIDE, A.C. Fractional integral transforms of generalized functions.
Research Notes in Mathematics, San Francisco-London-Melbourne.
- [44a] Meijer, C.S. Über eine Erweiterung der Laplace-Transformation, I und II.
Proc. Akad. Wet. Amsterdam, 43 (1940), 599-608, 702-711.
- [44b] Meijer, C.S. Eine neue Erweiterung der Laplace-Transformation, I und II.
Proc. Akad. Wet. Amsterdam, 44 (1941), 727-737, 831-839.
- [45] MISRA, O.P. Generalised convolution for G-transformation.
Bull. Cal. Math. Soc. 64 (1972) 137-142.
- [46] OBERHETTINGER, F. Tables of Mellin transforms.
Berlin-Heidelberg-New York, (1974).
- [47] ORTON, M. The generalized Abel equations for Schwartz distributions.
Siam J. Math. Anal. 11 (1980), 596-611.
- [48] OSLER, T.J. Leibniz rule for fractional derivatives generalized and an application to infinite series.
Siam J. Appl. Math. 18 (1970), 658-673.
- [49] PANDEY, J.N. On the Stieltjes transform of generalized functions.
Proc. Cambridge Philos. Soc. 71 (1972) 85-96.
- [50] PARASHAR, B.P. Domain and range of fractional integration operators.
Mathematicae Japonicae 12 (1967) 141-145.
- [51] PATHAK, R.S. A distributional generalised Stieltjes transformation.
Proc. Edinburgh Math. Soc. (2) 20 (1976) 1, 15-22.

SAMENVATTING

Van de vele integraaltransformaties die in de loop der jaren zijn ontstaan, nemen transformaties van het Watson type, (kern van de vorm $k(xt)$) en van het convolutie type (kern van de vorm $k(x/t)$) een belangrijke plaats in. Zeker wanneer men klassen van functies tot zijn beschikking heeft die door zulke integraaltransformaties een-aan-een op elkaar worden afgebeeld en men dus over de inverse kan beschikken, hebben ze veelal een breed toepassingsgebied, niet in het minst in de toegepaste analyse. In dit proefschrift wordt een klasse van lokaal convexe ruimten ingevoerd die door integraaltransformaties van het genoemde type homeomorf op elkaar worden afgebeeld. De definiërende eigenschappen zijn zodanig algemeen gekozen dat door specialisatie vele bekende transformaties ontstaan. De revue passeren achtereenvolgens de Laplace transformatie (hoofdstuk II), fractionele integratie (hoofdstuk V), Stieltjes transformatie (hoofdstuk VI) en Meijer's transformaties (hoofdstuk VII). Daarnaast komen in diverse voorbeelden nog andere transformaties aan de orde.

De integraalrepresentatie van de inverse transformaties wordt in hoofdstuk II behandeld. Toepassing op de genoemde specialisaties geeft naast bekende inversie formules in (bijna) alle gevallen ook nieuwe inversie formules. In diverse gevallen wordt het verband van deze nieuwe inversie formules onderling en de relatie tot klassieke inversie formules aangegeven.

In één geval, namelijk bij de Stieltjes transformatie, worden ook enkele inversie formules afgeleid die berusten op (de limieten van) differentiaal operatoren.

- [52] PATHAK, R.S. A representation theorem for a class of Stieltjes transformable generalised functions.
J. Indian Math. Soc. 38 (1974) 339-344.
- [53] PERRY, W.L. Inversion of general integral transforms.
Siam J. Math. Anal. 5 (1974), 764-773.
- [54] POLLARD, H. Studies on the Stieltjes transform.
Bull. Amer. Math. Soc. 48 (1942), 214.
- [55] Prabhakar, T.R. Some integral equations with Kummer's function in the kernels.
Canad. Math. Bull. 14 (1971), 391-404.
- [56] ROONEY, P.G. On the range of certain fractional integrals.
Can. J. Math. XXIV (1972) 1198-1216.
- [57] ROONEY, P.G. A technique for studying the boundedness and extendability of certain types of operators.
Can. J. Math. XXV (1973), 1090-1102.
- [58] ROSS, B. (Ed.) Fractional calculus and its applications.
Lecture notes in Mathematics 457 (1975), Berlin-Heidelberg-New York.
- [58a] SANSONE, G. and GERRETSEN, J. Lectures on the theory of a complex variable, I.
Groningen, 1960.
- [59] SCHUITMAN, A. On a certain test function space for Schwartz's Hankel transform.
Delft Progr. Rep. 2 (1977), 193-206.
- [60] SCHUITMAN, A. A testing function space for the two sided Laplace transform.
Delft Progr. Rep. 3 (1978) 187-197.
- [61] SCHWARTZ, L. Théorie des distributions I, II.
Paris, (I 1950, II 1951).
- [62] SETHI, P.L. and BANERJI, P.K. Fractional integration and dual integral equations.
J. Indian Math. Soc. 38 (1974), 359-363.

- [62a] SNEDDON, I.A. The Use of Integral Transforms.
New York, 1972.
- [63] SRIVASTAVA, H.M. Some remarks on a generalization of the
Stieltjes transform.
Publ. Math. Debrecen 23 (1976) 1-2, 119-122.
- [64] SUMNER, D.B. An inversion formula for the generalized Stieltjes
transform.
Bull. Amer. Math. Soc. 55 (1949), 174-183.
- [65] SWAROOP, R. On a generalization of the Laplace- and the
Stieltjes transformation.
Ann. Soc. Sc. Bruxelles 78 (1964), 105-112.
- [66] SZNAJDER, S. On some classes of distributions.
Coll. Math. XXII (1971), 133-139.
- [67] TITCHMARSH, E.C. Introduction to the theory of Fourier integrals.
London, (1948).
- [67a] TITCHMARSH, E.C. Complex Fourier Bessel transforms.
Quart. J. Math., Oxford Ser 19 (1948), 164-176.
- [68] TREVES, F. Topological vector spaces, distributions and kernels.
New York-London, (1967).
- [69] WALTON, J.R. A distributional approach to dual integral
equations of Titchmarsh type.
Siam J. Math. Anal. 6 (1975), 628-643.
- [70] WLOKA, J. Grundrume und verallgemeinerte Funktionen.
Lecture Notes in Mathematics 82 (1969). Berlin-Heidelberg-
-New York.
- [71] WLOKA, J. Verallgemeinerte Funktionen.
Jahresbericht D.M.V. 71 (1969), 25-47.
- [72] ZEMANIAN, A.H. Generalized integral transforms. New York, (1968).