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Fourieranalyse op eindige groepen en de irreduceerbare representaties van de symmetrische groep.

(Fourier analysis on finite groups and the irreducible representations of the symmteric group)

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Abstract

In this paper we will naturally extend the concept of Fourier analysis to functions on arbitrary groups. We will generalise the idea of a convolution and try to find a formula for Fourier coefficients in such way that the coefficients of the convolution can easily be calculated.

In the first section we will start off in familiar territory as we work our way through the Abelian groups. On the cyclic groups the comparison with the torus and the Fourier series is easily made and this enables us to easily copy the functions from the Fourier series and use them on our group. We then expand this idea by comparing the other groups to Fourier series on multiple variables. Here we can again copy the functions over and after some calculations we end up with our desired theorems.

Then we will continue working on groups in general but sadly for the non-Abelian groups the idea of comparing it to the Fourier series does not work. To remedy this problem we will introduce representations, homomorphisms between the group and invertible matrices. After introducing the concept of a representation we will show some remarkable theorems from Representation theory, such as Maschke's theorem and Schur's lemma. With the help of these theorems we can find the irreducible representations, whose matrix entries from an orthogonal basis. These representations are what we will use to transform the convolution into matrix multiplication.

In the last chapter we will go into more specifics on the representations of the symmetric group. The representations on this group can be found with the help of the Young tableaux. Among these tableaux we will find the Specht Modules, on which the group action of S_n action will give rise to the irreducible representations. To conclude we will show how to turn these irreducible representations of the symmetric group into matrices.

Contents

1 Introduction

The trick of decomposing a function into a sum of more manageable functions has existed for a long while. During his analysis of the heat equation, Joseph Fourier claimed that all periodic functions could be made out of sines and cosines. Although this statement is not true (take for instance the indictator function of $\mathbb{Q}$), it was eventually shown to be true for Hölder continuous functions. Ever since then mathematicians have been working on generalising and expanding on this idea under the name of harmonic analysis. The intention of this paper is to give the reader an overview of how to generalise the idea of the Fourier series to functions on arbitrary finite groups and to work out these functions for the symmetric group.

2 Abelian groups

We start by looking at the Abelian groups. These are groups are quite closely related to the torus and the real line. They also are a lot easier to deal with in most cases, making it an excellent starting point for our exploration into Fourier analysis on groups. But before we start we should quickly go over what we actually mean by Fourier analysis.

2.1 Fourier series

We'll start by first defining what exactly we are trying to find and prove for the functions of the group by stating some important theorems about Fourier series. When working on $L^2(\mathbb{T})$ with the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\overline{g(x)}\,dx$ the functions $\phi_n(x) = e^{inx}$ form a complete orthonormal basis which have the following additional properties:

- The Fourier coefficients can be calculated using the following formula $\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{inx}\,dx$
- The convolution, which is defined in the following way $(f * g)(x) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x-t)g(t)\,dt$, has easy to calculate Fourier coefficients namely $\widehat{(f * g)}(n) = \hat{f}(n) \cdot \hat{g}(n)$

We will try and generalise this idea to the functions on our group. Most of the work surrounding Fourier series is about the convergence of the series to the function. This won't play a big roll when looking at our groups, since they will be finite. In general we will denote the set of functions with complex values on a group G as $L(G)$. We will define the inner product on this group analogous to the one on $L^2(\mathbb{T})$ by replacing our integral with the mean over the entire group. This gives us for $f, g \in L(G)$

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{h\in G} f(h)\overline{g(h)}.$$

With the same idea in mind we can also define the convolution on these functions as

$$(f * g)(k) = \frac{1}{|G|}\sum_{l\in G} f(kl^{-1})g(l)$$

Throughout this chapter we will stick to writing the group operator as addition but in chapter 3 we will switch to the more general case and start working with multiplicative notation for groups.

2.2 The cyclic group

The cyclic group, denoted by C_n, is in many ways very similar to the one dimensional torus, $\mathbb{T}$. The cyclic group is even isomorphic to a subgroup of the Torus, namely the evenly spaced out points $\{0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, \frac{(n-1)2\pi}{n}\}$. With this in mind, we can simply restrict $\phi_n(x)$ to our subgroup and check if this already satisfies our needs.

Theorem 2.1. The set $\{\phi_0(x), \phi_1(x), \cdots, \phi_{n-1}(x)\}$ restricted to C_n forms an orthonormal basis for $L(C_n)$.

Proof. First we check to see that the functions indeed are of unit length,

$$||\phi_j(x)||^2_{L(C_n)} = \langle \phi_j(x), \phi_j(x) \rangle = \frac{1}{n}\sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}j}\overline{e^{i\frac{2\pi k}{n}j}} = \frac{1}{n}\sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}j}e^{-i\frac{2\pi k}{n}j} = \frac{1}{n}\sum_{k=0}^{n-1} 1 = 1.$$

Now we need to check if the functions are independent. Let $j \neq l$

$$\langle \phi_j(x), \phi_l(x) \rangle = \frac{1}{n}\sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}j}\overline{e^{i\frac{2\pi k}{n}l}} = \frac{1}{n}\sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}j}e^{-i\frac{2\pi k}{n}l} = \frac{1}{n}\sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}(j-l)} = \frac{1-e^{i\frac{2\pi n}{n}(j-l)}}{1-e^{i\frac{2\pi}{n}(j-l)}} = 0$$

In the last step we use that our sum is a partial geometric series and the fact $j-l$ is not divisible by n. It should be noted that this step would not work if $(j-l)$ would be a multiple of n. Since the size of the basis corresponds to the dimension $L(C_n)$, it follows that the functions are indeed an orthonormal basis of $L(C_n)$ □

Remark. It should be noted that we did not need all the ϕ_n for our basis. The reason for this is that they coincide with other functions already in our basis. For example when restricting ϕ_{n+1} to our subgroup we find that $\phi_{n+1} = \phi_1$.

Now that we have shown that $\{\phi_0(x), \phi_1(x), \cdots, \phi_{n-1}(x)\}$ form an orthonormal basis, it becomes clear on how we should define the formula for the Fourier coefficients. We take as the formula for the Fourier coefficients the inner product between the function and ϕ_m. So the formula becomes

$$\hat{f}(m) = \langle f, \phi_m \rangle = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\overline{\phi_m(k)} = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e^{-i\frac{2\pi k}{n}m}.$$

Theorem 2.2. For $f, g \in L(C_n)$ the Fourier coefficients of the convolution are given by $\widehat{(f*g)}(n) = \hat{f}(n) \cdot \hat{g}(n)$

Proof.

$$
\begin{aligned}
\widehat{(f*g)}(m) &= \frac{1}{n}\sum_{k=0}^{n-1} e^{-i\frac{2\pi k}{n}m}(f*g)(k) \\
&= \frac{1}{n}\sum_{k=0}^{n-1} e^{-i\frac{2\pi k}{n}m}\frac{1}{n}\sum_{l=0}^{n-1} f(k-l)g(l) \\
&= \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} e^{-i\frac{2\pi k}{n}m} f(k-l)g(l) \\
&= \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} e^{-i\frac{2\pi(k-l)}{n}m} f(k-l)e^{-i\frac{2\pi l}{n}m}g(l) \\
&= \frac{1}{n^2}\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} e^{-i\frac{2\pi(k-l)}{n}m} f(k-l)e^{-i\frac{2\pi l}{n}m}g(l) \\
&= \frac{1}{n}\sum_{l=0}^{n-1} e^{-i\frac{2\pi l}{n}m}g(l)\frac{1}{n}\sum_{k=0}^{n-1} e^{-i\frac{2\pi(k-l)}{n}m} f(k-l) \\
&= \frac{1}{n}\sum_{l=0}^{n-1} e^{-i\frac{2\pi l}{n}m}g(l)\hat{f}(m) \\
&= \hat{f}(m)\cdot\hat{g}(m).
\end{aligned}
$$

In the second to last line we used that on a sum over a finite group the translation does not effect the sum. □

As we can see the restricted exponential functions did exactly what we had hoped for! Now we continue our work into arbitrary Abelian groups.

2.3 Other Abelian groups

Now that we have seen how to work with the cyclic group, we can generalise our findings to arbitrary Abelian groups. To do this we will need the fundamental theorem for Abelian groups. This theorem has several different version but we will state it as follows.

Theorem 2.3 (The fundamental theorem of Abelian groups)**.** Every Abelian group is isomorphic to the direct product of cyclic groups.

We will not proof this theorem since that would take too much space but most introductory books on algebra will contain a proof of this theorem. [1]

So for now we can assume that we are working with a direct product of cyclic groups. Using this information we can once again use the fact that cyclic groups look like one dimensional tori and consequently their direct product, and also the original Abelian group, looks like a higher dimensional torus. When working with Fourier analysis in n dimension we find the following basis:

$$\phi_{\mathbf{m}}(\mathbf{x}) = \prod_{j=1}^{n} e^{x_j m_j i}$$

[1]the reader for Algebra 1 AM1060 is one of these books

where $\mathbf{m} = (m_1, \cdots, m_n)$ stands for a vector of frequency in each variable and $\mathbf{x} = (x_1, \cdots, x_n)$ is the location on the torus. By once again restricting the functions to our direct product of cyclic groups, we find that this basis is a product of the functions on C_{k_i} and look like

$$\phi_{\mathbf{m}}(\mathbf{x}) = \prod_{j=1}^{n} \phi_{m_j}(x_j)$$

To simplify our proofs we will first show that the inner product of $L(G)$ on these functions is nothing more than just the product of the inner product of the cyclic groups.

Lemma 2.4. For any Abelian group $G \cong C_{k_1} \times C_{k_2} \times \cdots \times C_{k_l}$ the inner product of the function $\phi_{\mathbf{n}}$ and $\phi_{\mathbf{m}}$ simplifies to

$$\langle \phi_{\mathbf{n}}, \phi_{\mathbf{m}} \rangle_G = \prod_{i=1}^{l} \langle \phi_{n_i}, \phi_{m_i} \rangle_{C_{k_i}}.$$

Proof. We will prove this lemma by induction. As our base case take $l = 1$. Because we use the same inner product on both groups, this equality holds.

Now assume the equality holds for $l - 1$, we will show that it also must hold for l.

$$\begin{aligned}
\langle \phi_{\mathbf{n}}, \phi_{\mathbf{m}} \rangle_G &= \frac{1}{|G|} \sum_{h \in G} \phi_{\mathbf{n}}(h) \overline{\phi_{\mathbf{m}}(h)} \\
&= \frac{1}{\prod_{i=1}^{l} k_i} \sum_{c_1 \in C_{k_1}} \sum_{c_2 \in C_{k_2} \times \cdots \times C_{k_l}} \phi_{\mathbf{n}}(c_1, c_2) \overline{\phi_{\mathbf{m}}(c_1, c_2)} \\
&= \frac{1}{k_1} \sum_{c_1 \in C_{k_1}} \phi_{n_1}(c_1) \overline{\phi_{m_1}(c_1)} \frac{1}{\prod_{i=2}^{l} k_i} \sum_{c_2 \in C_{k_2} \times \cdots \times C_{k_l}} \phi_{(n_2, \cdots, n_l)}(c_2) \overline{\phi_{(m_2, \cdots, m_l)}(c_2)} \\
&= \frac{1}{k_1} \sum_{c_1 \in C_{k_1}} \phi_{n_1}(c_1) \overline{\phi_{m_1}(c_1)} \langle \phi_{(n_2, \cdots, n_l)}, \phi_{(m_2, \cdots, m_l)} \rangle_{C_{k_2} \times \cdots \times C_{k_l}} \\
&= \frac{1}{k_1} \sum_{c_1 \in C_{k_1}} \phi_{n_1}(c_1) \overline{\phi_{m_1}(c_1)} \prod_{i=2}^{l} \langle \phi_{n_i}, \phi_{m_i} \rangle_{C_{k_i}} \\
&= \prod_{i=1}^{l} \langle \phi_{n_i}, \phi_{m_i} \rangle_{C_{k_i}}
\end{aligned}$$

By induction our lemma holds. ☐

With this lemma the question of orthogonality becomes a lot easier

Theorem 2.5. On the Abelian group $G \cong C_{k_1} \times C_{k_2} \times \cdots \times C_{k_l}$ the set $\{\phi_{\mathbf{n}} : 0 \leq n_i < k_i\}$ restricted to G forms an orthonormal basis for $L(G)$.

Proof. First we check to see that the functions indeed are of unit length.

$$||\phi_{\mathbf{n}}||^2_{L(G)} = \langle \phi_{\mathbf{n}}, \phi_{\mathbf{n}} \rangle = \prod_{i=1}^{l} \langle \phi_{n_i}, \phi_{n_i} \rangle_{C_{k_i}} = \prod_{i=1}^{l} 1 = 1.$$

Now we need to check if the functions are independent. Let $n_j \neq m_j$ for some j, then

$$\langle \phi_{\mathbf{n}}, \phi_{\mathbf{m}} \rangle_G = \prod_{i=1}^{l} \langle \phi_{n_i}, \phi_{m_i} \rangle_{C_{k_i}} = \langle \phi_{n_j}, \phi_{m_j} \rangle_{C_{k_j}} \prod_{i=1 \wedge i \neq j}^{l} \langle \phi_{n_i}, \phi_{m_i} \rangle_{C_{k_i}} = 0.$$

In the second step we used theorem 2.1. Since the size of the basis corresponds with the dimension of $L(G)$, it follows that the function are indeed an orthonormal basis for $L(G)$ □

Because the $\phi_{\mathbf{n}}$'s form an orthonormal basis, we can again define our definition for the Fourier coefficients using the inner product.

$$\hat{f}(\mathbf{n}) = \langle f, \phi_{\mathbf{n}} \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{\phi_{\mathbf{n}}(g)}.$$

Theorem 2.6. The Fourier coefficients of the convolution are given by $\widehat{(f * g)}(\mathbf{n}) = \hat{f}(\mathbf{n}) \cdot \hat{g}(\mathbf{n})$

Proof.

$$\begin{aligned}
\widehat{(f * g)}(\mathbf{n}) &= \frac{1}{|G|} \sum_{k \in G} \overline{\phi_{\mathbf{n}}(k)} (f * g)(k) \\
&= \frac{1}{|G|} \sum_{k \in G} \overline{\phi_{\mathbf{n}}(k)} \frac{1}{|G|} \sum_{l \in G} f(k-l) g(l) \\
&= \frac{1}{|G|^2} \sum_{k \in G} \sum_{l \in G} \overline{\phi_{\mathbf{n}}(k)} f(k-l) g(l) \\
&= \frac{1}{|G|^2} \sum_{k \in G} \sum_{l \in G} \overline{\phi_{\mathbf{n}}(k-l)} f(k-l) \overline{\phi_{\mathbf{n}}(l)} g(l) \\
&= \frac{1}{|G|^2} \sum_{l \in G} \sum_{k \in G} \overline{\phi_{\mathbf{n}}(k-l)} f(k-l) \overline{\phi_{\mathbf{n}}(l)} g(l) \\
&= \frac{1}{|G|} \sum_{l \in G} \overline{\phi_{\mathbf{n}}(l)} g(l) \frac{1}{|G|} \sum_{k \in G} \overline{\phi_{\mathbf{n}}(k-l)} f(k-l) \\
&= \frac{1}{|G|} \sum_{l \in G} \overline{\phi_{\mathbf{n}}(l)} g(l) \hat{f}(\mathbf{n}) \\
&= \hat{f}(\mathbf{n}) \cdot \hat{g}(\mathbf{n})
\end{aligned}$$

In the second to last line we used that on a sum over a finite group the translation does not effect the sum. □

It should be clear to the reader that the Fourier series on Abelian groups are clearly related to the original idea of Fourier analysis and how the functions still correspond with the ones in the continuous case. What is not immediately obvious, is how to continue from this point into non-Abelian groups. They do not have the same resemblance to the torus as the Abelian groups do. In the next section we shall generalise further using the idea of unitary irreducible representations of a group, which behave similarly to the $\phi_{\mathbf{n}}$ that we found in the Abelian case.

3 Representation theory

In this chapter we will go through the basics of representation theory and show some proofs for several important theorems, like Maschke's theorem and Schur's lemma. After we have established a good basis for representation theory, we will continue by showing that the irreducible representations form the functions we are looking for to transform the convolution into matrix multiplication.

3.1 Representations

One of the problems with groups is that they are inherently quite abstract in nature. Matrix representations try to remedy this problem by transforming the elements of a group into invertible matrices, which have been extensively studied in subjects like linear algebra.

Let GL_d denote the group of invertible $d \times d$ matrices with complex matrix elements and matrix multiplication as group operation.

Definition 3.1 (Matrix representation)**.** Let G be a group. A matrix representation of G of degree d is a homomorphism $X : G \to GL_d$,

That is. a function $X : G \to GL_d$ such that

$$X(\epsilon) = I$$
$$X(gh) = X(g)X(h)$$

for all $g, h \in G$

There are loads of options for making representations. An easy one to start with would be the trivial representations: its function corresponds with the trivial homomorphism from the group to $GL_1 \cong \mathbb{C}^*$ with multiplication.

Example 3.1. The trivial representation of a group G is the homomorphism that maps each element to $1 \in GL_1$. The identity element is clearly mapped to 1 and it follows that

$$X(g)X(h) = 1 \cdot 1 = 1 = X(gh)$$

and thus this is indeed a representation.

The functions $\phi_{\mathbf{n}}$ that we used in the last chapter for Abelian groups are all representations too. Since filling the zero vector will just give you

$$\phi_{\mathbf{n}}(\mathbf{0}) = \prod_{j=1}^{n} \phi_{n_j}(0) = \prod_{j=1}^{n} e^{\frac{2\pi i \cdot 0}{n_j}} = 1$$

and for $g, h \in G$ it holds that

$$\phi_{\mathbf{n}}(g+h) = \prod_{j=1}^{n} \phi_{n_j}(g+h) = \prod_{j=1}^{n} e^{\frac{2\pi i\cdot(g+h)}{n_j}} = \prod_{j=1}^{n} e^{\frac{2\pi i\cdot g}{n_j}} e^{\frac{2\pi i\cdot h}{n_j}} = \phi_{\mathbf{n}}(h) \cdot \phi_{\mathbf{n}}(g).$$

Another obvious representation for groups that work on points, such as S_n or D_n, would be the matrices that permute the entries of the vectors with n coordinates. This representations on S_n is called *the defining representation* and its function sends $\pi \in S_n$ to the matrix A with entries $a_{i,j} = \delta_{i,\pi(j)}$, where δ denotes the Kronecker delta. These matrices permute vectors in such a way that $x_i = (Ax)_{\pi(i)}$.

Example 3.2. For clarity let's write out the matrices for S_3.

$$X(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad X((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$X((13)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad X((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$X((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad X((132)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now that we have seen some matrix representations, it is time to continue to representations in general. Let V be a vector space, then the general linear group, $GL(V)$, is the group of all linear invertible transformations on V with composition as group action.

Definition 3.2 (Representation)**.** Let G be a group and V be a vector space. A representation of G on V is a homomorphism, $\rho : G \to GL(V)$. For the ease of notation we will usually simply drop the ρ simply writing $g\mathbf{v}$ instead of $\rho(g)\mathbf{v}$.

At first this definition can seem a bit much, but with the help of a basis for the vector space V one can translate $GL(V)$ back into matrices. After this transformation we have essentially ended up again with a matrix representation.

An alternative method for defining representations is by letting a group act on a vector space in a linear way.

Definition 3.3. Let V be a vector space and let G act on V such that

$$\begin{aligned} g\mathbf{v} &\in V \\ g(c\mathbf{v} + d\mathbf{w}) &= c(g\mathbf{v}) + d(g\mathbf{w}) \\ g(h\mathbf{v}) &= (gh)\mathbf{v} \\ \epsilon\mathbf{v} &= \mathbf{v} \end{aligned}$$

for all $g, h \in G$; $\mathbf{v}, \mathbf{w} \in V$; $c, d \in \mathbb{C}$

Essentially this definition requires that the actions are all linear and that the group operation can be interchanged with composition. You could identify all the action of each element to a linear operator and end up with a function between the group and the linear operators, which satisfies the first definition.

This second definition also allows us to work with group actions on sets too. We can simply expand a finite set S, on which G already has an action, to a vector space. Simply let the elements of S be the basis for the vector space. Then you will get a vector space $\mathbb{C}S$ of dimension $|S|$ which has elements that look like

$$\mathbf{v} = c_1 \cdot s_1 + c_2 \cdot s_2 + \cdots + c_n \cdot s_n$$

The action is already defined for the elements of the basis so this action naturally extends to the vector space using linearity in the following way

$$g\mathbf{v} = c_1 \cdot g(s_1) + c_2 \cdot g(s_2) + \cdots + c_n \cdot g(s_n)$$

Example 3.3. Take for example $A = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ and take as group S_3, then we can make it into a vector space and add them in the following way,

$$\begin{aligned} \mathbf{v} &= 3 \cdot \mathbf{1} + \mathbf{2} \\ \mathbf{w} &= \mathbf{1} + \mathbf{2} \\ \mathbf{v} + \mathbf{w} &= 3 \cdot \mathbf{1} + \mathbf{2} + \mathbf{1} + \mathbf{2} = 4 \cdot \mathbf{1} + 2 \cdot \mathbf{2} \end{aligned}$$

Now the actions of S_3 on this vector space work as follows

$$(13)\mathbf{v} = 3 \cdot (13)\mathbf{1} + (13)\mathbf{2} = 3 \cdot \mathbf{3} + \mathbf{2} = \mathbf{2} + 3 \cdot \mathbf{3}$$

With this definition and using the individual elements as basis, we find the matrices from example 3.2

One set that groups can act on is themselves using the group operator. When the group is expanded into a vector space as before it is usually referred to as the group algebra $\mathbb{C}[G]$. This gives rise to a very important representation.

Definition 3.4 (The regular representation)**.** Let G be a group. Then the regular representation is defined by letting the group G act on the its own group algebra $\mathbb{C}[G]$.

Example 3.4. Take as example the group $S_3 = \{\epsilon, (12)\}$. To work out what $\rho(g)$ looks like for an element we check how it combines with the basis. Let's work out $\rho((12))$.

$$\begin{aligned} (1,2)\epsilon &= (\mathbf{1}, \mathbf{2}) \\ (1,2)(\mathbf{1}, \mathbf{2}) &= \epsilon \end{aligned}$$

So we find the matrix

$$\rho((1,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Beside coming up with representations, it is also possible to make representations out of existing representations. Take for example the direct product.

Definition 3.5 (Direct sum of two representations)**.** Let ρ and μ be representations of G on vectors space V and W. Then the direct sum of V and W, $V \bigoplus W$, has a representation by letting two representations work on their corresponding vector space. The matrices of this new representation look as follows

$$\left(\begin{array}{c|c} \rho(g) & 0 \\ \hline 0 & \mu(g) \end{array}\right).$$

This new representation is also usually denoted with $\rho \oplus \mu$

In addition to combining representations, we can also rewrite our matrices in a more comprehensible form using a transformations.

Example 3.5. Let T be an invertible $d \times d$ matrix and X be a matrix representation of degree d. Then $T^{-1}XT$ is also a representation since

$$\begin{aligned} T^{-1}X(\epsilon)T &= T^{-1}I_dT = I_d \\ T^{-1}X(gh)T &= T^{-1}X(g)X(h)T = T^{-1}X(g)T^{-1}TX(h)T \end{aligned}$$

In a more abstract sense these functions are called *isomorphisms* between two representations with the following definition

Definition 3.6 (Homomorphism)**.** A function $\theta : V \to W$ is called an homomorphism between two representations ρ and μ on V and W respectively, if it is a linear operation between V and W and the following holds for all $g \in G$ and $v \in V$

$$\theta(\rho(g)v) = \mu(g)\theta(v)$$

If in addition the function is bijective, it is called an *isomorphism.*

3.2 Reducibility

Now that we have seen some representations and how we combine them, one could wonder if we could rewrite a representation as a direct product of smaller representations. Take for example the regular representation of S_2 from example 3.4, it turns out that all the matrices share eigenvectors, namely

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Now if we rewrite the matrices using with the eigenvectors as basis we find the matrices:

$$\rho(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad \rho((1,2)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As you might know $S_2 \cong C_2$, so it should not be a surprise that along the diagonal we find again our functions ϕ_0 and ϕ_1 which are both representations themselves. A key part in the reduction here is that there is a non-trivial (not the entire vector space or the subspace $\{\mathbf{0}\}$) subspace that is not affected by the operations of $\rho(g)$. These kind of subspaces are called invariant.

Definition 3.7 (Invariant subspace)**.** Let ρ be a representation of G on V, then W is called an invariant subspace if it is a linear subspace of V and for all $g \in G$ and $\mathbf{w} \in W$ it holds that $g\mathbf{w} \in W$.

Definition 3.8 (Reducibility)**.** A representation is called *reducible* if it has a non-trivial invariant subspace and consequently all matrices can be rewritten to have the shape

$$\left(\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array} \right)$$

A representation that is not reducible is referred to as *irreducible*

Reducibility already gives us a clear indication that the representation contains other representations but it does not explicitly give us the two representations that were combined. One way to find invariant subspaces is through the use of the homomorphism θ from earlier. This is because homomorphisms preserve some sense of the structure so we can show that $\ker \theta$ and $\operatorname{im} \theta$ are invariant since $\{\mathbf{0}\}$ and the entire vector space are invariant subspaces.

Theorem 3.1. Let θ be an homomorphism from the representation ρ on V to the representation μ on W. Then $\ker \theta$ and $\operatorname{im} \theta$ are invariant subspaces.

Proof. Let $v \in \ker\theta$ then we have to show that for all $g \in G$, $\rho(g)v \in \ker\theta$ or equivalently $\theta(\rho(g)v) = \mathbf{0})$.

$$\theta(\rho(g)v) = \mu(g)\theta(v) = \mu(g)\mathbf{0} = \mathbf{0}$$

Now for the image we take $w \in \operatorname{im}\theta$ then we have to show that $\forall g \in G$, $\mu(g)w \in \operatorname{im}\theta$ or that there exists a $v \in V$ such that $\theta(v) = \mu(g)w$. Since $w \in \operatorname{im}$ there exists a $v' \in V$ such that $\theta(v') = w$. Now let take $v = \rho(g)v'$ which is still in V, then

$$\theta(v) = \theta(\rho(g)v') = \mu(g)\theta(v') = \mu(g)w \quad \square$$

This theoren also has a direct consequence on the homomorphisms between irreducible representations.

Corollary 3.2 (Schur's lemma)**.** A homomorphism θ between two irreducible representations ρ and μ on V and W is either an isomorphism or it maps all vectors to the zero vector

Proof. Because θ is an homomorphism it follows that $\ker\theta$ is a invariant subspace on V. But since ρ is irreducible it must follow that $\ker\theta = V$ or $\ker\theta = \{\mathbf{0}\}$. If $V = \ker\theta$ we end up mapping all vectors to the zero vector

If $\ker\theta = \{\mathbf{0}\}$ we find that V is isomorphic to $\operatorname{im}\theta$ because the kernel is just the zero vector. Since μ is irreducible it follows that $\operatorname{im}\theta = W$ or $\operatorname{im}\theta = \{\mathbf{0}\}$. This either means that $V = \{\mathbf{0}\}$ which again maps all vectors to the zero vector or $V \cong W$ in which case the homomorphism is an isomorphism. $\square$

The structure of the matrices of a reducible representations

$$\left(\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right)$$

already shows us some useful information. We can already see that the $A(g)$ part of the matrices already forms a representation but the $C(g)$ gets muddied down when $B(g) \neq 0$. We could solve this problem if the basis vectors outside of the invariant subspace were to also form an invariant subspace. With some help of an inner product we can find these vectors.

Definition 3.9 (Invariant inner product)**.** Let V be an inner product space with a representation of G. The inner product is called invariant if for for all $g \in G$ and $u, v \in V$ it holds that

$$\langle u, v\rangle = \langle gu, gv\rangle.$$

Now with an invariant inner product we find that the orthogonal complement of the invariant subspace also forms an invariant subspace and that the representation can be split into two.

Theorem 3.3. Let W be an invariant subspace of V which has an invariant inner product, then $W^\perp$ is also an invariant subspace. Representations on V can be written as the direct sum of two representations on W and $W^\perp$.

Proof. Let $\mathbf{v} \in W^\perp$, then we have to show that for all $g \in G$, that $g\mathbf{v} \in W^\perp$. Take any $\mathbf{w} \in W$ then

$$\langle g\mathbf{v}, \mathbf{w}\rangle = \langle g^{-1}g\mathbf{v}, g^{-1}\mathbf{w}\rangle = \langle \mathbf{v}, g^{-1}\mathbf{w}\rangle = 0.$$

So $g\mathbf{v} \in W^\perp$ and consequently $W^\perp$ is an invariant subspace. Now take a representation on V and take as basis the union of a basis for W and $W^\perp$. We then find that the matrices of this representation have the shape

$$\left(\begin{array}{c|c} \rho(g) & 0 \\ \hline 0 & \mu(g) \end{array}\right)$$

Because the subspaces are invariant we end up with two sections of zeroes and two square quadrants ρ and μ, which are representations on W and $W^\perp$ □

With this new insight we can prove Maschke's theorem

Theorem 3.4 (Maschke's Theorem)**.** Let G be a group, then every representation of G on a vector space V can be written as the direct sum of irreducible representations.

Proof. In this proof we will use induction on the degree of the representation. For representations of degree one we find that all the representations are irreducible and therefore the theorem holds for this degree. Now assume that we know that the theorem holds for representations with degree lesser than d and take a representation of degree d. If this representations is irreducible then our theorem holds. If our representation is reducible, then we can find an invariant non-trivial subspace W. Now we can take a basis for W and expand this basis to V and write it as $\{b_1, \cdots, b_d\}$. Now we impose an inner product on V. Take two vectors $u = \sum_{i=1}^d c_i b_i$ and $v = \sum_{i=1}^d d_i b_i$ then their in product becomes

$$\langle u, v\rangle = \sum_{i=1}^{d} c_i \overline{d_i}.$$

This inner product is not necessarily invariant so we want new construct one that is. We define our new inner product as follows

$$\langle u, w\rangle = \sum_{g\in G} \langle gu, gw\rangle.$$

This new product is invariant, since the action of $h \in G$ will only permute the elements of the group withint the sum. Now we can apply theorem 3.3 and so we split our representation in to the direct sum of two smaller representations, which by our assumption can be written as a direct sum of irreducible representations. Consequently our representation is equivalent to a sum of these two sums and thus by induction on the degree we have shown the theorem for all representations. □

Maschke's theorem indicates how important the irreducible representations are in representation theory. The proof also gives us a method for finding an inner product on the vector spaces which makes our representations unitary. The benefit of having our representation in unitary form is that finding inverse can be done quickly by taking the Hermitian adjoint, after finding an orthonormal basis using the Gram-Schmidt process. However there is not a clear indication on how we find this decomposition, because finding the invariant subspaces is hard. in the next section we will find a method for determining whether a representation is irreducible and which irreducible representations make up a reducible representation.

3.3 Character theory

The character of a representations is defined in the following way.

Definition 3.10 (Character)**.** The character of the representation ρ is defined as a function $\chi_\rho : G \to \mathbb{C}$ and is calculated as follows

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g))$$

This function is surprisingly simple yet still very powerful. Since using the trace requires us to write our representation in terms of matrices we should first check that this is well defined. Suppose V has two bases and write out the representation in one of them. Then in the other basis the representation can be written as $T^{-1}\rho T$ where T is the transformation between the two bases.

$$\mathrm{Tr}(T^{-1}\rho(g)T) = \mathrm{Tr}(T^{-1}T\rho(g)) = \mathrm{Tr}(\rho(g))$$

where we used that the matrices within the trace can be commuted. As a direct consequence we also find that two isomorphic representations have the same character.

Since the characters are also functions on the group, we can reuse the inner product on $L(G)$ as follows

$$\langle \chi_\rho, \chi_\mu \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_\rho(g)\overline{\chi_\mu(g)}.$$

Since two isomorphic representations have the same character, we can use the fact that every representation is isomorphic to a unitray representation.

$$\overline{\chi_{\mu(g)}} = \mathrm{Tr}(\overline{\mu(g)}) = \mathrm{Tr}(\overline{\mu(g)}^T) = \mathrm{Tr}(\mu(g)^\dagger) = \mathrm{Tr}(\mu(g)^{-1}) = \mathrm{Tr}(\mu(g^{-1})) = \chi_\mu(g^{-1})$$

Where the dagger denotes the Hermitian adjoint. Now we can rewrite the inner product to

$$\langle \chi_\rho, \chi_\mu \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_\rho(g)\chi_\mu(g^{-1}).$$

Now that we have all the definitions out of the way we will show some important results for the character functions.

Theorem 3.5. Let ρ and μ be representations with degree d_ρ and d_μ, then the following holds

1. $\chi_\rho(\epsilon) = d_\rho$
2. if $g, h \in G$ are conjugates, then $\chi_\rho(g) = \chi_\rho(h)$
3. $\rho \oplus \mu$ has the character $\psi = \chi_\rho + \chi_\mu$
4. if ρ and μ are irreducible, then $\langle \chi_\rho, \chi_\mu \rangle = \delta_{\rho,\mu}$

Proof.

1. $\chi_\rho(\epsilon) = \mathrm{Tr}(\rho(\epsilon)) = \mathrm{Tr}(I_{d_\rho}) = d_\rho$
2. There is a $k \in G$ such that $g = k^{-1}hk$. It follows that

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\rho(k^{-1}hk)) = \mathrm{Tr}(\rho(k^{-1})\rho(h)\rho(k)) = \mathrm{Tr}(\rho(k)^{-1}\rho(k)\rho(h)) = \chi_\rho(h)$$

3. $\psi(g) = \text{Tr}(\rho(g) \oplus \mu(g)) = \rho(g)_{1,1} + \cdots + \rho(g)_{d_\rho,d_\rho} + \mu(g)_{1,1} + \cdots + \mu(g)_{d_\mu,d_\mu} = \chi_\rho + \chi_\mu$

4. To aid us in our proof we will construct some homomorphisms between these two representations as follows
$$T = \frac{1}{|G|}\sum_{g\in G}\rho(g)X\mu(g^{-1})$$
Here X can be any matrix of shape $d_\rho \times d_\mu$. For this to qualify as a homomorphism it must hold that $\rho(g)T = T\mu(g)$, for all $g \in G$ or alternatively $\rho(g)T\mu(g^{-1}) = T$, for all $g \in G$.
$$\begin{aligned}\rho(g)T\mu(g^{-1}) &= \rho(g)\frac{1}{|G|}(\sum_{h\in G}\rho(h)X\mu(h^{-1}))\mu(g^{-1}) \\ &= \frac{1}{|G|}\sum_{h\in G}\rho(g)\rho(h)X\mu(h^{-1})\mu(g^{-1}) \\ &= \frac{1}{|G|}\sum_{h\in G}\rho(gh)X\mu((gh)^{-1}) \\ &= T\end{aligned}$$
So we have indeed made a whole collection of homomorphisms. From corollary 3.2 we know that either our homomorphism is a zero matrix or an invertible matrix.

Let's assume the two representations are not isomorphic, then T must be a zero matrix. If we zoom in on one entry of T we find that
$$T_{i,j} = \frac{1}{|G|}\sum_{g\in G}\sum_{k=1}^{d_\rho}\sum_{l=1}^{d_\mu}\rho(g)_{i,k}X_{k,l}\mu(g^{-1})_{l,j} = 0$$
By just setting only $X_{k,l}$ to one it follows that for all i, j, k, l
$$\frac{1}{|G|}\sum_{g\in G}\rho(g)_{i,k}\mu(g^{-1})_{l,j} = 0$$
Now it turns out the inner product can be written as
$$\begin{aligned}\langle\chi_\rho, \chi_\mu\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\chi_\mu(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\text{Tr}(\rho(g))\text{Tr}(\mu(g^{-1})) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{k=1}^{d_\rho}\rho(g)_{k,k}\sum_{l=1}^{d_\mu}\mu(g^{-1})_{l,l} \\ &= \sum_{k=1}^{d_\rho}\sum_{l=1}^{d_\mu}\frac{1}{|G|}\sum_{g\in G}\rho(g)_{k,k}\mu(g^{-1})_{l,l} \\ &= 0\end{aligned}$$

To simplify our proof we will look at the inner product of the character of an irreducible representation with itself since two representations have the same character anyway if they are isomorphic. We will first narrow down the possibilities for T. Since both cI_d with $c \in \mathbb{C}$ and T commute with our representation, their difference must also commute with the representation.

$$\rho(g)(T - cI_d) = \rho(g)T - \rho(g)cI_d = T\rho(g) - cI_d\rho(g) = (T - cI_d)\rho(g)$$

Now if we take c to be an eigenvalue of T we find that $T - cI_d$ is not invertible and can't be an isomorphism. By corollary 3.2 $T - cI_d$ has to be a zero matrix, and consequently $T = cI_d$. From the trace we find that

$$c = \frac{1}{d}cd = \frac{1}{d}\mathrm{Tr}(T) = \frac{1}{d}\frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\rho(g)X\rho(g^{-1})) = \frac{1}{d}\mathrm{Tr}(X)$$

Now the entries on the diagonal are

$$\frac{1}{d}\sum_{k=1}^{d} X_{k,k} = T_{i,i} = \sum_{k=1}^{d}\sum_{l=1}^{d}\frac{1}{|G|}\sum_{g\in G}\rho(g)_{i,k}X_{k,l}\rho(g^{-1})_{l,i}$$

Since this holds for any choice X it follows that the coefficients in front of the entries of X on both sides must be equal

$$\frac{1}{|G|}\sum_{g\in G}\rho(g)_{i,k}\rho(g^{-1})_{l,j} = \frac{1}{d}\delta_{k,l}\delta_{i,j}$$

Now we rewrite the inner product and find

$$\begin{aligned}
\langle \chi_\rho, \chi_\rho \rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\chi_\rho(g^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\rho(g))\mathrm{Tr}(\rho(g^{-1})) \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{k=1}^{d}\rho(g)_{k,k}\sum_{l=1}^{d}\rho(g^{-1})_{l,l} \\
&= \sum_{k=1}^{d}\sum_{l=1}^{d}\frac{1}{|G|}\sum_{g\in G}\rho(g)_{k,k}\rho(g^{-1})_{l,l} \\
&= \sum_{k=1}^{d}\sum_{l=1}^{d}\frac{1}{d}\delta_{k,l} \\
&= 1
\end{aligned}$$

□

This theorem together with Maschke's theorem has some strong implications on the decomposition of representations.

Corollary 3.6. Let ρ be a representation of degree d and it can be written as

$$\rho(g) = \bigoplus_{i=1}^{k} m_i \rho^i$$

where ρ_i are non-isomorphic irreducible representations. then the following holds

1. $\chi_\rho(g) = \sum_{i=1}^{k} m_i \chi_{\rho^i}$
2. $\langle \chi_\rho, \chi_{\rho^i} \rangle = m_i$
3. $\langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^{k} m_i^2$
4. $\langle \chi_\rho, \chi_\rho \rangle = 1$ implies that ρ is irreducible
5. Two representations ρ and μ are isomorphic if and only if there characters coincide

Proof.

1. This follows directly from induction on the third part of theorem 3.5.
2. This is a consequence of the linearity of the inner product, the orthoongality of the characters and the previous statement.
3. This is a consequence of the linearity of the inner product and the previous statement.
4. Assume $\langle \chi_\rho, \chi_\rho \rangle = 1$. From our previous statement we can deduce that $\sum_{i=1}^{k} m_i^2 = 1$. Since m_i are natural numbers it follows that there is one j such that, $m_j = 1$. Thus ρ is made up of a single irreducible representations and thus is itself irreducible.
5. The left implication has already been shown in the beginning of this section. For the right implication we start by applying Maschke's theorem to ρ and finding the irreducible representations. Since the character of μ is identical, it follows the inner product of χ_μ with the irreducible representations of ρ is equal to that of $\chi_r ho$ with the irreducible representations. This means that μ is isomorphic to the same sum of irreducible representation as ρ. Since being isomorphic is transitive it follows that ρ and μ are isomorphic.

□

What we have seen from these theorems is that the character of a representation is a very useful tool in representation theory.

3.4 The regular representation

The regular representation, that we defined in definition 3.4, is one of the more special representations since it is very closely linked to the functions on the group. It also has an important function within representation theory since it codifies the entire group bijectively into matrices. Let's take a closer look at the character of the regular representation.

Since the regular representation of G is essentially a collection of matrices that permute the basis vectors it follows that then entries on the diagonal of $X(g)$ are one if the basis vector remains the same, which would mean $gh = h$, and otherwise are zero. So the character of this representation is the equal to the number of stable points. But since $gh = h$ would imply $g = \epsilon$, it must follow that the regular representation has the character

$$\chi(g) = \begin{cases} |G|, & \text{if } g = \epsilon \\ 0, & \text{else} \end{cases}.$$

This unique character has some remarkable consequence when applying corollary 3.6 namely that for each irreducible representation ρ^i

$$\langle \chi, \chi_{\rho^i} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi_{\rho^i}(g)} = \frac{1}{|G|} |G| d_{\rho^i} = d_{\rho^i}.$$

This also leads to the immediate consequence that there are is a finite number of irreducible representations up to isomorphism and that $|G| = \sum d_{\rho^i}^2$. With this new information we can prove that the matrix entries of the irreducible representations form an orthonormal basis for the functions on the group G.

Theorem 3.7. The set of functions $\phi_{\rho^i,k,l} := \sqrt{d_{\rho^i}} \rho^i_{k,l}$, where ρ^i is a unitary irreducible representations of G, from an orthonormal basis on $L(G)$

Proof. Most of the work we have already done in the proof of theorem 3.5! Namely we already showed that

$$\langle \rho^i_{k,l}, \rho^j_{m,n} \rangle = \frac{1}{|G|} \sum_{g \in G} \rho(g)_{i,k} \overline{\rho(g)_{m,n}} = \frac{1}{|G|} \sum_{g \in G} \rho(g)_{i,k} \rho(g^{-1})_{n,m} = \frac{1}{d_{\rho^i}} \delta_{i,j} \delta_{k,m} \delta_{l,n},$$

which already gives an orthogonal set. The only thing left to do is make sure that they have unit length. We can multiply our functions by $\sqrt{d_{\rho^i}}$ and we then find that our set is orthonormal. Because the set is orthonormal and has the size $\sum d_{\rho^i}^2 = |G|$ it is an orthonormal basis for $L(G)$ □

Using some more tricks we can also show that the characters of the irreducible representations form an orthonormal basis for class functions, which are functions such that they are constant on the conjugacy classes.

Corollary 3.8. The set of characters of the irreducible representations form an orthonormal basis for the class functions.

Proof. We have already shown that the characters of irreducible representations form an orthonormal set. To show it is also a basis we need to show that any class function can be decomposed into a linear combination of these characters. Let f be a class function on the group G. Since $f \in L(G)$, it can be written as $f(g) = \sum c_{i,k,l} \phi_{\rho^i,k,l}(g)$. Because f is a class function it is not affected by conjugation so

$$\begin{aligned} f(g) = \frac{1}{|G|} \sum_{h \in G} f(hgh^{-1}) &= \frac{1}{|G|} \sum_{h \in G} \sum_{i,k,l} c_{i,k,l} \phi_{\rho^i,k,l}(hgh^{-1}) \\ &= \sum_{i,k,l} \sqrt{d_{\rho^i}} c_{i,k,l} \frac{1}{|G|} \sum_{h \in G} \frac{1}{\sqrt{d_{\rho^i}}} \phi_{\rho^i,k,l}(hgh^{-1}) \\ &= \sum_{i,k,l} \sqrt{d_{\rho^i}} c_{i,k,l} \frac{1}{|G|} \sum_{h \in G} \rho^i(hgh^{-1})_{k,l} \\ &= \sum_{i,k,l} \sqrt{d_{\rho^i}} c_{i,k,l} \frac{1}{|G|} \sum_{h \in G} (\rho^i(h) \rho^i(g) \rho^i(h^{-1}))_{k,l} \end{aligned}$$

The last sum should again look familiar as

$$T = \frac{1}{|G|} \sum_{g \in G} \rho(g) X \mu(g^{-1})$$

with in this case $X = \rho(g)$ from the proof of theorem 3.5. There we found the formula

$$T_{i,j} = \delta_{i,j}\frac{1}{d}\mathrm{Tr}(X)$$

for isomorphic representations. Filling in these formulae then we find

$$\sum_{i,k,l}\sqrt{d_{\rho^i}}c_{i,k,l}\delta_{k,l}\frac{1}{d_{\rho^i}}\mathrm{Tr}(\rho^i(g)) = \sum_{i,k}\frac{1}{\sqrt{d_{\rho^i}}}c_{i,k,k}\chi_{\rho^i}$$

So we can write any class function as a linear combination of the characters of the irreducible representations and thus the characters form a orthonormal basis for the class functions. □

3.5 The Fourier transform

We have now seen that the matrix elements of the irreducible representations do form a basis for our functions. Now with this information we can define the Fourier coefficients as follows

$$\hat{f}(\rho^i, j, k) = \langle f, \phi_{\rho^i,j,k}\rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{\phi_{\rho^i,j,k}}.$$

There is also another definition in addition to this pretty standard formula. Since our functions ϕ directly come from the irreducible representations we could define the Fourier coefficients as a matrix for each representation,

$$\hat{\mathbf{f}}(\rho^i) = \frac{1}{|G|}\sum_{g\in G} f(g)\rho^i(g)^\dagger.$$

When explicitly written out we just find that this matrix Fourier coefficient contains all "normal" coefficients with the coordinates the other way around.

$$\hat{\mathbf{f}}(\rho^i)_{j,k} = \frac{1}{\sqrt{d_{\rho^i}}}\hat{f}(\rho^i, k, j)$$

The inversion formula for the matrix coefficients uses some tricky notation, but we find again that is just the sum over all the functions.

Theorem 3.9 (Fourier inversion formula)**.** Let $f \in L(G)$ then we find that

$$f(g) = \sum_i \mathrm{Tr}(\hat{\mathbf{f}}(\rho^i)d_{\rho^i}\rho^i(g))$$

Proof. We will simply write out the trace to find

$$\sum_i \mathrm{Tr}(\hat{\mathbf{f}}(\rho^i)d_{\rho^i}\rho^i(g)) = \sum_i\sum_{k=1}^{d_{\rho^i}}\sum_{l=1}^{d_{\rho^i}}\sqrt{d_{\rho^i}}\hat{\mathbf{f}}(\rho^i)_{k,l}\sqrt{d_{\rho^i}}\rho^i(g)_{l,k} = \sum_i\sum_{k=1}^{d_{\rho^i}}\sum_{l=1}^{d_{\rho^i}}\hat{f}(\rho^i, l, k)\phi_{i,l,k} = f(g). \quad \square$$

These Fourier matrix coefficients allow us to rewrite the convolution into matrix multiplication

Theorem 3.10. The matrix Fourier coefficients of the convolution $(f*g)(x) = \frac{1}{|G|}\sum_{h\in G} f(xh^{-1})g(h)$ can be written as

$$\widehat{\mathbf{f}*\mathbf{g}}(\rho^i) = \hat{\mathbf{g}}(\rho^i)\hat{\mathbf{f}}(\rho^i)$$

Proof.

$$\begin{aligned}
\widehat{\mathbf{f} * \mathbf{g}}(\rho^i) &= \frac{1}{|G|}\sum_{x\in G}(f*g)(x)\rho^i(x)^\dagger \\
&= \frac{1}{|G|}\sum_{x\in G}\frac{1}{|G|}\sum_{h\in G} f(xh^{-1})g(h)\rho^i(x)^\dagger \\
&= \frac{1}{|G|}\sum_{x'\in G}\frac{1}{|G|}\sum_{h\in G} f(x')g(h)\rho^i(x'h)^\dagger \\
&= \frac{1}{|G|}\sum_{x'\in G} f(x')\left(\frac{1}{|G|}\sum_{h\in G} g(h)\rho^i(h)^\dagger\right)\rho^i(x')^\dagger \\
&= \hat{\mathbf{g}}(\rho^i)\left(\frac{1}{|G|}\sum_{x'\in G} f(x')\rho^i(x')^\dagger\right) \\
&= \hat{\mathbf{g}}(\rho^i)\hat{\mathbf{f}}(\rho^i)
\end{aligned}$$

□

4 The Symmetric Group

We have seen that finding the irreducible representations for a non-Abelian group can be very hard but in the case of the symmetric group we are in luck. For the symmetric group there exists a method for finding an irreducible representations for each conjugacy class. In this chapter we will show how to find the representations and how to write them out in terms of matrices if needed.

4.1 S_n, Young tableaux and partial orders

Before we start with the construction of the irreducible representations, we first need to establish the vector space that we are going to work on and some of its properties. In addition to this, we also will discuss some other concepts that are needed for proofs along the way.

Let's find out how many irreducible representations we are looking for by finding the conjugacy classes of S_n. Recall that two elements g and h are in the same conjugacy class if there is an element k such that:

$$kgk^{-1} = h.$$

In the case of the symmetric group we find that the conjugacy classes are based on the cycles. To show that this is indeed true, we will first show that conjugation does not affect the cycles of the g. Take an element $g = (i_1^1, i_2^1, \cdots, i_{l_1}^1)\cdots(i_1^m, i_2^m, \cdots, i_{l_m}^m)$. To see how the conjugate behaves we let it act on $k(i_1^1)$.

$$kgk^{-1}(k(i_1^1)) = kg(i_1^1) = k(i_2^1).$$

After doing this for some more points, it should become clear that

$$kgk^{-1} = (k(i_1^1), k(i_2^1), \cdots, k(i_{l_1}^1))\cdots(k(i_1^m), k(i_2^m), \cdots, k(i_{l_m}^m)),$$

which has the same cycles as g. With this insight it should also be apparent how one finds a k for two arbitrary permutations with the same cycles.

Since the sum of the length of the cycles is always n, we can characterise the conjugacy classes with partitions of n. A partition of n, usually denoted by $\lambda \vdash n$, is a decreasing vector of natural numbers, which sum up to n. Take for example the conjugacy class of $(134)(26) \in S_6$ since it has a cycle of length 3, 2 and 1, its conjugacy class has corresponds to the partition $\lambda = (3, 2, 1)$.

Another important concept on the symmetric group is the sign of a permutation. The sign of permutation σ with partition $(\lambda_1, \cdots, \lambda_k)$ is defined as

$$sgn(\sigma) = (-1)^{\sum_{i=1}^{l}(\lambda_i - 1)}.$$

It turns out that this function is also a representation of degree one and therefore a unitary irreducible representation.

Sometimes we only want to have permutations, which only work on a subset of our numbers. This subgroups is usually denoted by S_A. Take for example $A = \{1, 3, 5\}$, then $S_A = \{\epsilon, (1, 3), (1, 5)(3, 5)(1, 3, 5), (1, 5, 3)\}$.

The vector spaces we will be working on are based on Young tableau. A Young tableau looks as follows

Definition 4.1 (Young tableau)**.** Let $\lambda \vdash n$ be a partition, then a Young tableaux with shape λ is a table with rows of length λ_i boxes, which is filled with the numbers $\{1, \cdots, n\}$

Here is an example with $\lambda = (3, 3, 2)$:

<table><tr><td>4</td><td>8</td><td>3</td></tr><tr><td>1</td><td>5</td><td>2</td></tr><tr><td>7</td><td>6</td><td></td></tr></table>

A natural way for the permutations to act on these tableaux is by letting it work on the entries of the tableau. This would give the following.

$$(25)(713)\begin{array}{|c|c|c|}\hline 4 & 8 & 3 \\ \hline 1 & 5 & 2 \\ \hline 7 & 6 & \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 4 & 8 & 7 \\ \hline 3 & 2 & 5 \\ \hline 1 & 6 & \\ \hline \end{array}$$

There are two special sets of permutations related to a tableau, namely the column stabilizers and row stabilizers. These are permutations that permute the specific entries of a tableau within a row/column. For a tableau t they are written as R_t and C_t.

Example 4.1. Take the tableau

$$t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Then we find as stabilizers $R_t = S_{\{1,3\}} \times S_{\{2,4\}} = \{\epsilon, (13), (24), (13)(24)\}$ and $C_t = S_{\{1,2\}} \times S_{\{3,4\}} = \{\epsilon, (12), (34), (12)(34)\}$

Based on these sets we can also define a row equivalence relation as follows

Definition 4.2 (Row equivalence on tableaux)**.** Two tableaux t and s with the same shape are called row equivalent if they have the same row stabilizers. The equivalence class is usually denoted by $\{t\}$

Let's take a look at an example of an equivalence class.

$$t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

$$\{t\} = \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline\end{array} \right\}$$

It should be noted that the curly brackets in the last equation carried two meanings. Around the t represented that it was the equivalence class and on the right hand side of the equation it represents the set. Luckily this is not a reoccurring problem, since we don't usually write out the set.

The definition for column equivalence is very similar. To avoid confusing the two we will write the column equivalence classes with square brackets $[t]$.

We can transfer the action from the tableaux to the equivalence classes. This would mean that

$$\pi\{t\} := \{\pi t\}$$

Now that we have a set with an action, we can generalise the set of tableaux of shape λ up to equivalence classes to a vector space in the same we did in example 3.3. This vector space is called the permutation representation corresponding to the partition λ and is denoted by M^λ.

In some of the examples it is convenient to write down the row equivalence classes as a tableau In these cases we will write the tableau with only horizontal lines. For example

$$\begin{array}{cc}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} = \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline\end{array} \right\}$$

The stablizer sets also have some nice properties.

Theorem 4.1. Let t be a tableau and σ a permutation then

1. $C_{\sigma t} = \sigma^{-1} C_t \sigma = \{\sigma^{-1}\pi\sigma : \pi \in C_t\}$
2. $R_{\sigma t} = \sigma^{-1} R_t \sigma = \{\sigma^{-1}\pi\sigma : \pi \in R_t\}$

Proof. With some set theory we can show that

$$\pi \in C_{\sigma t} \Leftrightarrow \pi\{\sigma t\} = \{\sigma t\} \Leftrightarrow \pi\sigma\{t\} = \sigma\{t\} \Leftrightarrow \sigma^{-1}\pi\sigma\{t\} = \{t\} \Leftrightarrow \sigma^{-1}\pi\sigma\{t\} \in C_t.$$

This reasoning also holds for the row stabilizers □

In some later proofs we will also need more of a structure on the partitions. In this case we will establish a partial ordering on the partition. A set with a partial ordering is usually referred to as a poset and has the following properties.

Definition 4.3. A poset is a set S with a partial order $(\geq)$ such that for all $a, b, c \in S$.

$$a \geq a$$
$$a \geq b \wedge b \geq a \Rightarrow a = b$$
$$a \geq b \wedge b \geq c \Rightarrow a \geq c$$

We also use $a > b$ to denote $a \geq b \wedge a \neq b$.

In case of the partitions we define the order as $\lambda \trianglerighteq \mu$ when for all $i \in \mathbb{N}$

$$\sum_{k=1}^{i} \lambda_k \geq \sum_{k=1}^{i} \mu_k.$$

We extend our partition with zeroes if there are no more entries.

Example 4.2. $(3, 2, 1) \trianglerighteq (2, 2, 1, 1)$ since for the partial, $3 \geq 2$,$5 \geq 4$, $6 \geq 5$ and $6 \geq 6$, the condition holds.

Important to notice is that this relation does not always give an indication which is the greater of the two. Take as an example $(3, 3)$ and $(4, 1, 1)$ both at some point have a larger partial sum. Another way to show that there is an order relation is with the dominance theorem.

Theorem 4.2 (Dominance in partitions)**.** Let t and s be tableaux of shape λ and μ respectively. If for each row of s all elements are in different columns of t, then $\lambda \trianglerighteq \mu$.

Proof. We can sort the elements in the columns of t in the order in which row they appear in s. This would mean that the set of the elements of the first row of s is a subset of the elements in the first row of t, because they are all in different columns and after the sorting they should be on top. This can even be generalised to the set of the first i rows of s since it is not possible with our current sorted t for an element of s to be lower than i in the column of t since that would imply that a column contains two elements from a row of s. Since the number of elements in the first i corresponds to the partial sum of i, we find that

$$\begin{aligned}\sum_{k=1}^{i} \lambda_k &= \text{ the number of elements in the first } i \text{ rows of } t \\ &\geq \text{ the number of elements in the first } i \text{ rows of } s \\ &= \sum_{k=1}^{i} \mu_k.\end{aligned}$$ □

This proof is quite abstract so I will also give an example. Take the tableaux

$$t = \begin{array}{|c|c|c|}\hline 5 & 3 & 6 \\ \hline 1 & 2 & 4 \\ \hline\end{array} \text{ and } s = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 5 \\ \hline 6 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

We can see that s and t fit the requirements of our theorem. So now we will sort the elements in the columns of t. Since 1 comes in a lower row than 5 in s we need to switch them. Same goes for 4 and 6. We find the following new tableau

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 5 & 2 & 6 \\ \hline\end{array}.$$

This new tableau has the first i rows of s in its first i rows. So now we can transfer this to partial sums and find $\lambda \trianglerighteq \mu$

Posets also have natural definition for the concept of a maximum or largest element. But since not all pairs elements need have a larger element we also define an additional concept.

Definition 4.4. An element of a poset is called maximum if it is larger than all other elements. An element is called maximal if there are no elements that are larger than it. It is possible for a poset to have multiple maximal elements.

In our case it turns out that the partition (n) is the maximum of the partitions of n.

4.2 Specht modules

The Specht modules are the irreducible representations of S_n we are looking for. They are denoted by S^λ and they are subspaces of their corresponding M^λ. In this section we will prove that they indeed are all irreducible representations of the symmetric group. Afterwards we will find a basis for the Specht modules allowing us to calculate the matrices.

Specht modules are formed by so called polytabloids. Polytabloids are vectors in M^λ and are made as follows from a tableau t

$$e_t = \sum_{\sigma \in C_t} sgn(\sigma)\sigma\{t\}$$

Example 4.3. Take the tableau

$$t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

We already found the set of stabilizers earlier as $\{\epsilon, (12), (34), (12)(34)\}$. For the polytabloid we find

$$\begin{aligned} e_t &= \epsilon \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} - (12)\begin{array}{cc}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} - (34)\begin{array}{cc}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} + (12)(34)\begin{array}{cc}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \\ &= \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{cc}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \end{aligned}$$

Because the polytabloids are defined using the stabilizers we can leverage some properties from it.

Theorem 4.3. Let t be a tableau and σ be a permutation then

$$e_{\sigma t} = \sigma e_t.$$

Proof. With some help of theorem 4.1 we find that

$$e_{\sigma t} = \sum_{\pi \in C_{\sigma t}} sgn(\pi)\pi\{\sigma t\} = \sum_{\pi \in C_t} sgn(\sigma\pi\sigma^{-1})\sigma\pi\sigma^{-1}\sigma\{t\} = \sigma \sum_{\pi \in C_t} sgn(\pi)\pi\{t\} = \sigma e_t. \quad \square$$

These polytabloids together form the Specht modules, which as it turns out turn the actions of S_n into an irreducible representation.

Definition 4.5 (Specht module)**.** Let λ be a partition, then $S^\lambda \subseteq M^\lambda$ is the subspace spanned by all the polytabloids e_t, where t is a tableau of shape λ

Now we will first show that this subspace is invariant and irreducible.

Theorem 4.4. The Specht modules are invariant subspaces and the representation defined with the action is irreducible.

Proof. Since the Specht module is generated by the polytabloids, it suffices to show that the set of polytabloids is invariant. This we have already shown in theorem 4.3, and consequently it follows for all linear combinations of the polytabloids.

We will show that the Specht modules are irreducible by showing all invariant subspaces are trivial. Take a tableau t of shape λ, then for another tableau s with shape λ there is a $\pi \in C_t$ such that $\{s\} = \pi\{t\}$. It follows then

$$\sum_{\sigma \in C_t} sgn(\sigma)\sigma\{s\} = \sum_{\sigma \in C_t} sgn(\sigma)\sigma\pi\{t\} = \sum_{\sigma \in C_t} sgn(\sigma\pi^{-1})\sigma\{t\} = sgn(\pi^{-1})e_t = sgn(\pi)e_t$$

We see that with this method we can always construct a multiple of a polytabloid from any tableau. Now assume that there is an invariant subspace V in the Specht module S^λ, then take an element $v \in V$ and express it as $v = \sum c_{\{s\}}\{s\}$. Since this subspace is invariant we find that

$$\sum_{\sigma \in C_t} \sigma sgn(\sigma)v = \sum_{\sigma \in C_t} \sigma sgn(\sigma) \sum c_{\{s\}}\{s\} = \sum c_{\{s\}} \sum_{\sigma \in C_t} \sigma sgn(\sigma)\{s\} = \sum_s \pm c_{\{s\}} e_t,$$

Where the $\pm$ is determined by the $sgn(\pi)$ we found in the last equation. This new sum must also be in V, since V is invariant to the actions of S_n and is closed under addition and scalar multiplication. Now if there are v and t such that $\sum_s \pm c_{\{s\}} \neq 0$, then e_t is part of the subset and because the subspace is invariant, $\pi e_t = e_{\pi t}$ is also in the subspace V. This means that $S^\lambda \subseteq V$ and that $V = S^\lambda$.

In the case that $\sum_s \pm c_{\{s\}} = 0$ for all the vectors and tableaux, we can show with an inner product that $V \subseteq S^{\lambda\perp}$. We take the inner product such the set of row equivalence classes forms an orthonormal basis. The inner product is invariant, since the actions of S_n permute the equivalence classes. It follows for $\pi \in S_n$ that $\langle u, \pi v\rangle = \langle \pi^{-1}u, v\rangle$. Now we can show that V is indeed orthogonal to our module. Take $u \in V$ and a tableau t with shape λ

$$\langle u, e_t\rangle = \langle u, \sum_{\sigma \in C_t} sgn(\sigma)\sigma\{t\}\rangle = \langle \sum_{\sigma \in C_t} sgn(\sigma)\sigma^{-1}u, \{t\}\rangle = \langle \sum_{\sigma \in C_t} sgn(\sigma^{-1})\sigma u, \{t\}\rangle = \langle \mathbf{0}, \{t\}\rangle = 0$$

Thus $V \subseteq S^{\lambda\perp}$ and consequently $V = \{\mathbf{0}\}$. Thus we have shown that every invariant subspace has to be trivial and the Specht modules are irreducible. □

So we found that the Specht modules indeed are irreducible representations. Since we have the same number of representations as partitions and therefore conjugacy classes, we only need to show that they are not isomorphic to each other.

Theorem 4.5. The Specht modules are pairwise non-equivalent.

Proof. We will prove this theorem by contradiction. Assume that λ and μ are two different partition and let θ be an isomorphism between the two Specht modules. Then we will extend this isomorphism to a homomorphism θ' from M^λ to M^μ by taking $\theta'(v) = 0$ for $v \in S^{\lambda\perp}$ and defining the rest in accordance with linearity. Then take tableaux t and s with shape λ and μ. Since our original θ was an isomorphism $\theta'(e_{\{t\}}) \neq \mathbf{0}$. Now we will rewrite the polytabloid using the isomorphism

$$\theta'(e_t) = \theta'(\sum_{\sigma \in C_t} sgn(\sigma)\sigma\{t\}) = \sum_{\sigma \in C_t} sgn(\sigma)\sigma\theta'(\{t\})$$

From this and $\theta'(e_t) \neq \mathbf{0}$ we can deduce that there is a $\{s\}$ such that $\sum_{\sigma \in C_t} sgn(\sigma)\sigma\{s\} \neq \mathbf{0}$. Now this is only possible if for each two elements a and b in a row of s, are not in the same column of t, because if this were the case then (a, b) would be in C_t and since $\{\epsilon, (a, b)\}$ forms a subgroup of C_t, it is possible to rewrite the transforming sum as

$$\sum_{\sigma \in C_t} sgn(\sigma)\sigma = (\epsilon - (a, b)) \sum_{\pi \in S_n/\{\epsilon,(a,b)\}} \pi.$$

These sums and brackets might look a little strange since we see them as group elements but remember that we are treating them as operators in this case and this allows us to add and subtract all we want. Since a and b are in the same row of s it follows that $(a, b)\{s\} = \{s\}$ and thus $\sum_{\sigma \in C_t} sgn(\sigma)\sigma s = 0$. So we can conclude that each pair of two elements a and b in a row of s, can not be in the same column of t. By theorem 4.2 it follows that $\lambda \trianglerighteq \mu$. Now we can do this entire proof again for $\mu \trianglerighteq \lambda$ which would mean that $\mu = \lambda$. Thus it must be that all the Specht modules are pairwise non-equivalent □

We now know that the Specht modules are indeed the sought after irreducible representations of the symmetric group, but we still have no idea what this vector space looks like and what kind of matrices these representations give rise to. To get a better idea let's try to find a basis.

Definition 4.6 (standard tableaux)**.** A tableau is called standard if both its columns and rows are increasing. Its polytabloid is also called standard.

As it turns out the polytabloids of standard tableaux form the basis of the Specht module. All the standard tableaux of shape λ can be made with the following procedure. First write down a tableau of shape (1). Then keep adding boxes with the next number, in such a way that it remains a tableau and you do not exceed the lines of the shape.

Example 4.4. Let's try finding the standard tableaux for the partition $(2, 2)$.

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

But in step two we could have also chosen the other corner for the 2 giving us

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \rightarrow \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

You could also write this down in a tree but this can quickly get out of hand with larger groups. Now that we have seen how to find the basis let's prove it is actually independent. To aid us in the proof we will need to establish an partial order on the row equivalence classes. The order will be based on how much a tableau looks like a standard tableau. We do this by looking at the shape of the partial tableaux t_i, which is simply the tableau with only the first i elements.

Example 4.5. Lets write out the partial tableaux of this tableau,

$$\begin{array}{cc}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$$

With these partials

$$\begin{array}{c}\hline \\ \hline 1 \\ \hline\end{array} \to \begin{array}{c}\hline 2 \\ \hline 1 \\ \hline\end{array} \to \begin{array}{cc}\hline 2 & \\ \hline 1 & 3 \\ \hline\end{array} \to \begin{array}{cc}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$$

And this gives the rise to the following sequence of shapes. $(0,1) \to (1,1) \to (1,2) \to (2,2)$

Definition 4.7 (Partial order on row equivalence classes)**.** $\{s\} \trianglerighteq \{t\}$ if for all shapes of the partial tableaux it holds that $s_i \trianglerighteq t_i$.

It should be clear that having lower numbers in the lower rows will get you get you higher on the order. Or more concretely put.

Lemma 4.6. Take a tableau t and let $k < l$ be natural numbers such that k is in a higher row than l. then

$$(kl)t \triangleright t$$

Proof. The shapes λ and μ of $(kl)t$ and t up until k and after l remain the same so we only need to look at the cases in between. In these cases the entry for the row of l is increased by one and the row of k is decreased by one. Since the row of l is early it follows that $\lambda_i \trianglerighteq \mu_i$ and consequently $(kl)t \triangleright t$. □

Now with this order we can make some arguments on the linear Independence of the set of polytabloids.

Theorem 4.7. The set of polytabloids of the standard row equivalence classes is linearly independent

Proof. From lemma 4.6 we can tell that a standard tableau is the maximum within its own polytabloid. Now within the set of standard tableaux we can find an tableau s such that it is maximal. In this proof we will show Independence by showing that the only linear combination for the zero vector is setting all coefficients to zero.

Take an arbitrary linear combination, which leads to the zero vector. For it to add up to the zero vector the coefficient in front of s would need to be zero because no other standard polytabloid contains $\{s\}$ because that would mean that $\{t\} \triangleright \{s\}$, which violates are assumption that s is maximal.

We can repeat this logic until we have determined that all the coefficients would have to be zero and thus the only linear combination for the zero vector is all zeroes. Thus the set must be independent. □

The only thing left to show that the span of this set is also the entire Specht module. We will use something call the straightening algorithm. This algorithm takes a polytabloid and returns it as a sum of polytabloids which have a higher order in the column equivalence classes. For this algorithm to work we will first need Garnir elements.

Definition 4.8 (Garnir elements)**.** Let A B be two finite disjoint sets of natural numbers and let π be representatives of the cosets such that

$$S_{A \cup B} = \bigcup_{\pi} \pi(S_A \times S_B)$$

Then the corresponding Garnir element is

$$g_{A,B} = \sum_{\pi} sgn(\pi)\pi$$

One should note that the Garnir element is operator on our vector space and not a group and element. It also dependents on the the chosen traversal. For the algorithm we will standardise our choice in traversal. Let's first calculate a Garnir element.

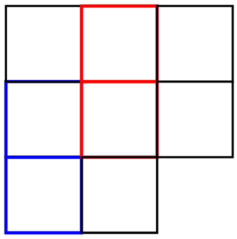
Example 4.6. Take $A = \{1, 4\}$ and $B = \{2, 3\}$. Then we can take as traversal

$$\{\epsilon, (1, 2), (1, 3), (4, 2), (4, 3), (4, 3), (1, 2)(4, 3)\}$$

Now we can sum over these elements and find the Garnir element

$$g_{A,B} = \epsilon - (1, 2) - (1, 3) - (4, 2) - (4, 3) - (4, 3) + (1, 2)(4, 3)$$

In the straightening algorithm we will work on with tableaux t where there is a row i where two adjacent elements $t_{i,j}$ and $t_{i,j+1}$ are not increasing. The sets for the Garnir elements will be $A = \{t_{a,j} : a \geq i\}$ and $B = \{t_{a,j+1} : a \leq i\}$. Together these sets form a skewed tetromino shape. Now for our Garnir element we take the traversal in such a way that the columns j and $j + 1$ are still increasing. If we had a tableau of shape $(3, 3, 2)$ with out of order elements in row 2 in the first and second column the the blue boxes represent A and the red ones B.



Example 4.7. Let's practice by finding the element for this tableau

1	2	4
5	3	
6		

The out of order elements are 5 and 3. So we find the sets $A = \{5, 6\}$and $B = \{2, 3\}$ and the traversal must be $\{\epsilon, (3, 5), (2, 5, 3), (2, 5, 6, 3), (3, 5, 6), (2, 5)(3, 6)\}$. Applying one of these elements to the original tableau removes the out of order elements but in some cases only moves it to a lower row.

Now we will show that applying this Garnir element to a polytabloid actually gives us a linear combination of polytabloids.

Theorem 4.8. Let t be a tableau with out of order elements in a row then,

$$g_{A,B}e_t = \mathbf{0}$$

Proof. we will first show that for $\pi \in C_t$, $\sum_{\sigma \in S_{A \cup B}} sgn(\sigma)\sigma\{\pi t\} = \mathbf{0}$. Since the size of the set $A \cup B$ is bigger than the column, it must follow that πt has atleast two elements $n, m \in A \cup B$ such that n and m are next to each other in a row of πt. Now we can rewrite this sum as

$$\begin{aligned} \sum_{\sigma \in S_{A \cup B}} sgn(\sigma)\sigma\{\pi t\} &= \sum_{\sigma \in S_{A \cup B}/\{\epsilon,(a,b)\}} sgn(\sigma)\sigma(\epsilon - (n, m))\{\pi t\} \\ &= \sum_{\sigma \in S_{A \cup B}/\{\epsilon,(a,b)\}} sgn(\sigma)\sigma(\{\pi t\} - \{\pi t\}) \\ &= \mathbf{0} \end{aligned}$$

Since e_t is simply a sum of these permutations it follows that $\sum_{\sigma \in S_{A\cup B}} sgn(\sigma)\sigma e_t = \mathbf{0}$

Since $S_A \times S_B \subseteq C_t$, it follows that for $\pi \in S_A \times S_B$

$$sgn(\pi)\pi e_t = sgn(\pi)\pi \sum_{\sigma \in C_t} sgn(\sigma)\sigma\{t\} = \sum_{\sigma \in C_t} sgn(\pi\sigma)\pi\sigma\{t\} = \sum_{\sigma \in C_t} sgn(\sigma)\sigma\{t\} = e_t$$

Now we can let π be the traversal for the Garnir element, then

$$\begin{aligned}
\mathbf{0} &= \sum_{\sigma \in S_{A\cup B}} sgn(\sigma)\sigma e_t \\
&= \sum_{\pi} \sum_{\sigma \in S_A \times S_B} sgn(\pi\sigma)\pi\sigma e_t \\
&= \sum_{\pi} \sum_{\sigma \in S_A \times S_B} sgn(\pi)\pi sgn(\sigma)\sigma e_t \\
&= \sum_{\pi} \sum_{\sigma \in S_A \times S_B} sgn(\pi)\pi e_t \\
&= (|A|! + |B|!) g_{A,B} e_t.
\end{aligned}$$

□

Since the identity is part of the sum in the Garnir element, we can after applying it to a vector, transfer the original vector to the other side to find that a linear combination for our polytabloid made out of other polytabloids, which do not have the mixed elements in that row. With this linear combination we can prove that we have indeed found the basis

Theorem 4.9. The set of polytabloids from the standard tableaux span the entirety of the Specht module

Proof. Let t be a tableau and take $\pi \in C_t$, Then

$$e_{\pi t} = \sum_{\sigma \in C_{\pi t}} sgn(\sigma)\sigma\{\pi t\} = \sum_{\sigma \in C_{\pi t}} sgn(\sigma)\sigma\pi\{t\} = \sum_{\sigma \in C_{\pi t}} sgn(\sigma\pi^{-1})\sigma\{t\} = sgn(\pi)e_t$$

This means that if we can show it for one tableau in a column equivalence class, that we are already finished for the entire equivalence class. To simplify our work we will always choose the tableau with increasing columns to represent each class. We will now work with induction on the partial order of the column equivalence classes. This order is analogous to the one on the row equivalence classes. The tableau made by filling in the numbers by working from top to bottom in each column and then left to right, is the maximum in this order and is standard.

Now assume for all $[s] \rhd [t]$ that this is in the span of our basis. If this tableau is standard then it is in the span of our basis. If it is not standard, then we can find an out of order pair in a row and constuct a Garnir element. In theorem 4.8 we saw that $g_{A,B}e_t = \mathbf{0}$ so we find that

$$e_t = -\sum_{\pi} sgn(\pi)\pi e_t$$

Now since we assumed all the columns were increasing we can label the elements of A and B that $b_1 < b_2 < \cdots < b_i < a_i < \cdots a_l$. Thus all the elements in the traversal except for the identity switch around a natural number which is greater with a smaller one from a lower column. By the analogous lemma 4.6 we find that $\pi e_t \rhd e_t$. So by our assumptions e_t is in the span of our basis. Our original theorem follows from this by induction over the order. □

We know now how to find a basis so let's work out a larger example.

4.3 The matrix representation of $S^{(2,2)}$

We have just seen a whole lot of theory but now we will actually show how to construct the irreducible representations. In this section we will work with the group S_4 and we will be working with the partition $(2,2)$. The standard tableaux of shape $(2,2)$ we have already found in example 4.4, namely

$$t_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \text{ and } t_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

Now these tableaux give rise to the following polytabloids

$$e_{t_1} = \begin{array}{c}\hline 1 \;\; 2 \\ \hline 3 \;\; 4 \\ \hline \end{array} - \begin{array}{c}\hline 2 \;\; 3 \\ \hline 1 \;\; 4 \\ \hline \end{array} - \begin{array}{c}\hline 1 \;\; 4 \\ \hline 2 \;\; 3 \\ \hline \end{array} + \begin{array}{c}\hline 3 \;\; 4 \\ \hline 1 \;\; 2 \\ \hline \end{array}$$

and

$$e_{t_2} = \begin{array}{c}\hline 1 \;\; 3 \\ \hline 2 \;\; 4 \\ \hline \end{array} - \begin{array}{c}\hline 2 \;\; 3 \\ \hline 1 \;\; 4 \\ \hline \end{array} - \begin{array}{c}\hline 1 \;\; 4 \\ \hline 2 \;\; 3 \\ \hline \end{array} + \begin{array}{c}\hline 2 \;\; 4 \\ \hline 1 \;\; 3 \\ \hline \end{array}.$$

Becasue working out the linear combination of polytabloids is strenuous and laborious task, it is best to use some the properties of the representation. Because $\rho(gh) = \rho(g)\rho(h)$, it suffices to just find the matrices for a set that generates the entire group. In the case of the symmetric group this is the set of all adjacent transpositions. For this group that is the set $\{(1,2),(2,3),(3,4)\}$.

Since a transposition $(k, k+1)$ do not drastically alter the tableau t we can split what happens to the tableau in three cases:

1. If k and $k+1$ are in the same column then $(k, k+1)e_t = -e_t$
2. If k and $k+1$ share a row then we can use Garnir elements to find the decomposition
3. If k and $k+1$ are in different rows and columns then $(k, k+1)t$ is also standard

Let's start of by seeing how $(1,2)$ acts on our polytabloids. For t_1 we are in case 2 and we have to find the Garnir elements with $A = \{2,3\}$ and $B = \{1\}$. In order to preserve the ascending order in the columns, we take the set $\{\epsilon, (1,2), (1,2,3)\}$ and we find that $e_{(1,2)t_1} = (1,2)e_{(1,2)t_1} - (1,2,3)e_{(1,2)t_1} = e_{t_1} - e_{t_2}$. For the polytabloid e_{t_2} we are in the case 1 and so we can just write $e_{(1,2)t_2} = -e_{t_2}$. So in total this gives

$$\rho((1,2)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Now we will continue with $(2,3)$. Since this leads to case 2 with both polytabloids we find that

$$\rho((2,3)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

When applying (34) to e_{t_1} we end in case 2 again with $A = \{4\}$ and $B = \{2,3\}$ and find the Garnir elements $\{\epsilon, (3,4), (4,3,2)\}$. Thus $e_{(3,4)t_1} = (3,4)e_{(3,4)t_1} - (4,3,2)e_{(3,4)t_1} = e_{t_1} - e_{t_2}$. Now with e_{t_2} we end in case 1 and find $e_{(3,4)t_2} = -e_{t_2}$. So for $(3,4)$ we find.

$$\rho((3,4)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Now we have calculated the matrices for the transpositions, it is time to find how we should combine these transpositions to make an arbitrary element. To find the smallest combination of transpositions for σ, we will find its inverse by sorting the array $\sigma([1, 2, \cdots, n])$ in ascending order. We will sort in the following manner

1. Find the largest i such that i is not in place
2. Switch i with its right neighbour until it is in place and denote all switches made with adjacent transpositions
3. If the array is sorted stop, else return to 1

By performing all the switches in reverse order we have found a combination for our σ. Let's do this once by hand

Example 4.8. Take as example $\sigma = (1, 2, 4)$ from S_4. When we apply this to the sorted array we find $\sigma([1, 2, 3, 4,]) = []$. Now let's start sorting

$$[2, 4, 3, 1] \to [2, 3, 4, 1] \to [2, 3, 1, 4]$$

So far we have used $(2, 3)(3, 4)$. Let's continue sorting.

$$[2, 3, 1, 4] \to [2, 1, 3, 4].$$

We are almost there. We only need to out the 2 in its place

$$[2, 3, 1, 4] \to [1, 2, 3, 4].$$

So we find that $\sigma^{-1} = (2, 3)(3, 4)(2, 3)(1, 2)$ and by simply reversing the order we find $\sigma = (1, 2)(2, 3)(3, 4)(2, 3)$

The algorith does not only give us a decomposition, it even gives the smallest decomposition. When there elements i and j, such that $i > j$ and $\sigma(j > \sigma(i))$, then the permutation σ has an inversion. In our sorting method we resolve an inversions with each switch, and thus the total number of adjacent transpositions is equal to the number of inversions. The total number of inversion also gives us a minimal amount of transpositions needed for our decomposition into adjacent transpositions. Since every transposition can at most make at most one inversion, the total number of adjacent transposition needs to be greater or equal to the the number of inversion. What we have just shown is that our algorithm always gives an optimal decomposition.

Now that we have found the matrices in the representation for all the elements of S_4, the only part left to do is to make sure that they are unitary. We have seen earlier how to construct an invariant inner product in Maschke's theorem. We could start out with an inner product which has the basis of standard polytabloids as orthonormal basis and is then expanded through linearity and conjugate symmetry. Then one could use the Gram-Schmidt process to find a basis for which the representation is unitary. This is would be a lot of work and I would not recommend doing this by hand but using MATLAB for assistance. In case you do not want to do any of the work by hand, you could use the GAP library[2] to quickly find the representations you are looking for.

Using the following code for GAP we can find the matrices.

[2]https://www.gap-system.org/index.html

```
LoadPackage("repsn");;
G := SymmetricGroup(4);; #the 4 can be replaced for other n
Chi := Irr(G)[3];; #The 3 indicates which irr. rep. you would want
rep := IrreducibleAffordingRepresentation(chi);;
(1,2)^rep; #you replace (1,2) with other elements
```

Using the code we get these matrices

$$\begin{aligned}
\rho((1,2)) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\rho((2,3)) &= \begin{pmatrix} 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i & 0 \end{pmatrix} \\
\rho((3,4)) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

And thus we have found our unitary irreducible representation for $(2,2)$ on S_4

References

[1] Sagan, B.E. (1991) *The Symmetric Group, Combinatorial Algorithms, and Symmetric Functions* (2nd edition). Springer

[2] Steinberg, B. (2011) *Representation Theory of Finite Groups: An Introductory Approach* (1st edition) . Springer

[3] Ceccherini-Silberstein, T., Scarabotti, F., Tolli, F. (2010) *Representation Theory of the Symmetric Groups: The Okounkov-Vershik Approach, Character Formulas, and Partition Algebras.* Cambridge university press