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Dirichlet problems associated to abstract nonlocal space–time differential operators

JOSHUA WILLEMS

Abstract. Let the abstract fractional space–time operator $(\partial_t + A)^s$ be given, where $s \in (0, \infty)$ and $-A\colon \mathsf{D}(A) \subseteq X \to X$ is a linear operator generating a uniformly bounded strongly measurable semigroup $(S(t))_{t \geq 0}$ on a complex Banach space X. We consider the corresponding Dirichlet problem of finding $u\colon \mathbb{R} \to X$ such that

$$\begin{cases} (\partial_t + A)^s u(t) = 0, & t \in (t_0, \infty), \\ u(t) = g(t), & t \in (-\infty, t_0], \end{cases}$$

for given $t_0 \in \mathbb{R}$ and $g\colon (-\infty, t_0] \to X$. We define the concept of L^p-solutions, to which we associate a mild solution formula which expresses u in terms of g and $(S(t))_{t \geq 0}$ and generalizes the well-known variation of constants formula for the mild solution to the abstract Cauchy problem $u' + Au = 0$ on (t_0, ∞) with $u(t_0) = x \in \overline{\mathsf{D}(A)}$. Moreover, we include a comparison to analogous solution concepts arising from Riemann–Liouville and Caputo type initial value problems.

1. Introduction

1.1. Background and motivation

Space–time nonlocal problems involving fractional powers of a parabolic operator arise in physics, biology, probability theory and statistics. The flat parabolic Signorini problem and certain models for semipermeable membranes can be formulated as obstacle problems for the fractional heat operator $(\partial_t - \Delta)^s$, where $s \in (0, 1)$ and Δ denotes the Laplacian, acting on functions $u\colon J \times \mathcal{D} \to \mathbb{R}$ for a given time interval $J \subseteq \mathbb{R}$ and a connected non-empty open spatial domain $\mathcal{D} \subseteq \mathbb{R}^d$ (see, e.g., [1,28]). In the context of continuous time random walks, equations of the form $(\partial_t - \Delta)^s u = f$ for $f\colon J \times \mathcal{D} \to \mathbb{R}$ are considered examples of *master equations* governing the (non-separable) joint probability distribution of jump lengths and waiting times [6]. The case where f is replaced by spatiotemporal Gaussian noise $\dot{\mathcal{W}}$ has applications to the statistical modeling of spatial and temporal dependence in data: The resulting class of fractional parabolic *stochastic* partial differential equations (SPDEs) has been

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proposed and analyzed in [14,16] as a spatiotemporal generalization of the *SPDE approach* to spatial statistical modeling, which was initiated by Lindgren, Rue and Lindström [18] and has subsequently gained widespread popularity [17].

After [28] and [22] independently generalized the Caffarelli–Silvestre extension approach from the fractional elliptic to the parabolic setting, there has been a surge of literature on space–time nonlocal problems involving fractional powers of $\partial_t + L$ for more general elliptic operators L acting on functions $u : \mathcal{D} \to \mathbb{R}$ (see for instance [2–5,7,15,19]). In particular, in [28, Remark 1.2], the *natural Dirichlet problem* for the nonlocal space–time operator $(\partial_t + L)^s$ is introduced, given by

$$\begin{cases} (\partial_t + L)^s u(t,x) = f(t,x), & (t,x) \in J \times \mathcal{D}, \\ u(t,x) = g(t,x), & (t,x) \in \mathbb{R}^{d+1} \setminus (J \times \mathcal{D}), \end{cases} \tag{1.1}$$

where $g : \mathbb{R}^{d+1} \setminus (J \times \mathcal{D}) \to \mathbb{R}$ is a given function prescribing the values of u outside of the spatiotemporal region $J \times \mathcal{D}$. The definition of $(\partial_t + L)^s$, given in Sect. 2.4, generalizes that of the Riemann–Liouville fractional time derivative ∂_t^s (i.e., the case where $L = 0$) using the theory of semigroups. Equations involving only a fractional time derivative have been studied widely; see for instance the monographs [8,13,25, 26] for an introduction to the subject.

In the integer-order case, the space–time differential operator is local, so that the analog to (1.1) is an initial boundary value problem. Identifying any $u : J \times \mathcal{D} \to \mathbb{R}$ with $u : J \to X$, where $J := (t_0, \infty)$ and X is a Banach space to be thought of as containing functions from $\mathcal{D}$ to $\mathbb{R}$, the corresponding infinite-dimensional initial value problem for $s = 1$ is the *abstract Cauchy problem*

$$\begin{cases} (\partial_t + A)u(t) = f(t), & t \in J, \\ u(t_0) = x \in X. \end{cases} \tag{1.2}$$

Here, $A : \mathsf{D}(A) \subseteq X \to X$ is a linear operator, whose domain $\mathsf{D}(A)$ can be used to encode (Dirichlet) boundary conditions, and $f : J \to X$ is a given forcing function. Although there exist various definitions of solutions to (1.2) (e.g., mild, strong and L^p-solutions (see Sect. 2.3)), the main focus of this article is on mild solutions. The mild solution to (1.2) can only be defined under the assumption that $-A$ is the infinitesimal generator of a suitably regular semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on X, see Sect. 2.2. Intuitively, the relation between $(S(t))_{t\geq 0}$ and A can be viewed as a generalization of the way in which the matrix exponential $S(t) = e^{-tA}$ is associated to matrix A. If, moreover, the right-hand side f is sufficiently (Bochner) integrable, then the *mild solution* of (1.2) is defined by

$$u(t) := S(t - t_0)x + \int_{t_0}^{t} S(t - \tau) f(\tau) \mathrm{d}\tau, \quad t \in J, \tag{1.3}$$

which is commonly known as the *variation of constants formula*, again by analogy with the finite-dimensional (matrix) case.

In this work we consider an abstract counterpart of (1.1) in the setting of (1.2), namely the following Dirichlet problem for $(\partial_t + A)^s$ with $s \in (0, \infty) \setminus \mathbb{N}$:

$$\begin{cases} (\partial_t + A)^s u(t) = 0, & t \in (t_0, \infty), \\ u(t) = g(t), & t \in (-\infty, t_0], \end{cases} \tag{1.4}$$

where $g\colon (-\infty, t_0] \to X$. We restrict ourselves to $J = (t_0, \infty)$ since $(\partial_t + A)^s u(t)$ depends only on the values of u to the left of $t \in \mathbb{R}$ (see Sect. 2.4). Moreover, we only consider $f \equiv 0$ since the problem is linear in u and the mild solution formula for $f \not\equiv 0$ and $g \equiv 0$ (or $J = \mathbb{R}$) is known to be given by a Riemann–Liouville type fractional parabolic integral, cf. [28, Theorem 1.17]. We will define the concept of an L^p-*solution* to (1.4) and show that it can be expressed in terms of g and $(S(t))_{t\geq 0}$ in the following way: For all $t \in (t_0, \infty)$,

$$\begin{aligned} u(t) := {} & \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1)) g(t_0 - (t-t_0)\tau)\mathrm{d}\tau \\ & + \sum_{k=1}^{\lfloor s \rfloor} \frac{(t-t_0)^{\{s\}+k-1}}{\Gamma(\{s\}+k)} S(t-t_0)[(\partial_t + A)^{\{s\}+k-1} g](t_0), \end{aligned} \tag{1.5}$$

where Γ denotes the gamma function and $s = \lfloor s \rfloor + \{s\}$ for $\lfloor s \rfloor \in \mathbb{N}_0$ and $\{s\} \in (0, 1)$. This formula generalizes (1.3) to fractional orders and is therefore taken as the definition of the *mild solution* to (1.4).

1.2. Contributions

The main contribution of this work is the introduction and motivation of (1.5) as the definition of the mild solution to (1.4) for $s \in (0, \infty) \setminus \mathbb{N}$ and bounded continuous g, rigorously formulated in Definition 4.2. This definition is motivated by Theorem 4.5, which shows that L^p-*solutions* to (1.4) satisfy (1.5) under certain natural conditions. Although its proof relies on the uniform exponential stability of $(S(t))_{t\geq 0}$, the resulting formula is well-defined under the more general assumption that $(S(t))_{t\geq 0}$ is uniformly bounded. In particular, this includes the case $A = 0$, so that (1.5) with $S(\,\cdot\,) \equiv \mathrm{Id}_X$ can be viewed as a solution to the Dirichlet problem associated to the fractional time derivative ∂_t^s. Likewise, if $(S(t))_{t\geq 0}$ is uniformly exponentially stable, then the integral in (1.5) also converges for $\{s\} = 0$, so that (1.5) remains meaningful for integers $s = n \in \mathbb{N}$ and reduces to the integer-order solution formula:

$$u(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0)[(\partial_t + A)^k g](t_0), \quad t \in (t_0, \infty).$$

If $(S(t))_{t\geq 0}$ is merely uniformly bounded, then we can still show that the first term of (1.5) converges to $S(t-t_0)g(t_0)$ as $\{s\} \uparrow 1$ for all $t \in (t_0, \infty)$ (see Proposition 4.7). For constant initial data $g \equiv x \in X$, we find that (1.5) can be conveniently expressed in terms of an operator-valued incomplete gamma function (see Corollary 4.8).

In addition to (1.5), we define solution concepts for the Cauchy problems associated to fractional parabolic Riemann–Liouville and Caputo type derivative operators (see Proposition 5.1 and Definitions 5.2 and 5.3) for comparison. The higher-order terms comprising the summation in (1.5) turn out to be analogous to the corresponding terms in the Riemann–Liouville solution. The integral term in (1.5), however, is continuous at t_0 under mild conditions on $(S(t))_{t\geq 0}$ or g, in contrast to the lowest-order term in the Riemann–Liouville formula, which has a singularity there. As opposed to the Caputo type initial value problem, the solutions to (1.4) are in general different for distinct $s_1, s_2 \in (n, n+1)$ for $n \in \mathbb{N}_0$.

To the best of the author's knowledge, the solution formula given by (1.5) is new even in the scalar-valued case $X := \mathbb{C}$, $A := a \in \overline{\mathbb{C}}_+$ and $(S(t))_{t\geq 0} = (e^{-at})_{t\geq 0}$, as are the Riemann–Liouville and Caputo type solutions for $a \in \overline{\mathbb{C}}_+ \setminus \{0\}$.

1.3. Outline

This article is structured as follows. In Sect. 2, we establish some notation and collect preliminary results regarding semigroups, fractional calculus, first-order abstract Cauchy problems and the Phillips functional calculus associated to semigroup generators. These notions are first used in Sect. 3 to investigate problem (1.4) for $t_0 = -\infty$, i.e., in the absence of prescribed initial data. Sect. 4 is concerned with the rigorous definition of mild and L^p-solutions to (1.4); after establishing the relation between these two concepts, we focus on the mild solution and establish some of its most important properties. The comparison with the solution concepts associated to Riemann–Liouville and Caputo type initial value problems is presented in Sect. 5.

2. Preliminaries

2.1. Notation

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the sets of positive and non-negative integers, respectively. We write $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ for the floor and ceiling functions; the fractional part of $\alpha \in [0, \infty)$ is defined by $\{\alpha\} := \alpha - \lfloor \alpha \rfloor$. The maximum (respectively, minimum) of two real numbers $\alpha, \beta \in \mathbb{R}$ is denoted by $\alpha \vee \beta$ (respectively, $\alpha \wedge \beta$). The function $t \mapsto t_+^\beta$ is defined by $t_+^\beta := t^\beta$ if $t \in (0, \infty)$ and $t_+^\beta := 0$ otherwise. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are, respectively, denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$; if $z \neq 0$, then its argument is written as $\arg z \in (-\pi, \pi]$. The open and closed right half-planes of the complex plane are denoted by

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \quad \text{and} \quad \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\},$$

respectively. The identity map on a set B is denoted by $\operatorname{Id}_B : B \to B$ and we write $\mathbf{1}_{B_0} : B \to \{0, 1\}$ for the indicator function of a subset $B_0 \subseteq B$.

Throughout this article, $(X, \|\cdot\|_X)$ denotes a Banach space over the complex scalar field $\mathbb{C}$; the real case can be treated using complexifications. The Banach space of bounded linear operators from X to a Banach space Y with the uniform operator norm is denoted by $\mathscr{L}(X; Y)$; for $X = Y$ we set $\mathscr{L}(X) := \mathscr{L}(X; X)$. The notation $A: \mathsf{D}(A) \subseteq X \to X$ indicates that A is a (possibly unbounded) linear operator on X with domain $\mathsf{D}(A)$ and graph $\mathsf{G}(A) := \{(x, Ax) : x \in \mathsf{D}(A)\}$.

Let $(S, \mathscr{A}, \mu)$ be a measure space. For $f: S \to \mathbb{R}$ and $x \in X$, define $f \otimes x: S \to X$ by $[f \otimes x](s) := f(s)x$. A function $f: S \to X$ is said to be strongly μ-measurable if it is the μ-almost everywhere ("a.e.") limit of μ-simple functions, i.e., linear combinations of $\mathbf{1}_B \otimes x$ with $B \in \mathscr{A}$, $\mu(B) < \infty$ and $x \in X$. For $p \in [1, \infty]$, let $L^p(S; X)$ denote the Bochner space of (equivalence classes of) p-integrable functions with norm

$$\|f\|_{L^p(S;X)} := \begin{cases} (\int_S \|f(s)\|_X^p \mathrm{d}\mu(s))^{1/p}, & \text{if } p \in [1, \infty); \\ \operatorname{esssup}_{s \in S} \|f(s)\|_X, & \text{if } p = \infty. \end{cases}$$

Intervals $J \subseteq \mathbb{R}$ are equipped with the Lebesgue σ-algebra and measure. The Banach space of bounded continuous functions $u: J \to X$, endowed with the supremum norm, is denoted by $(C_b(J; X), \|\cdot\|_\infty)$.

2.2. Strongly measurable semigroups of bounded linear operators

A family $(S(t))_{t \geq 0}$ of bounded linear operators on a complex Banach space X is said to be a *semigroup* if $S(0) = \mathrm{Id}_X$ and, for all $t, s \geq 0$, we have $S(t + s) = S(t)S(s)$. It is called a *strongly measurable semigroup* if, in addition, the orbit $t \mapsto S(t)x$ of any $x \in X$ is a strongly measurable mapping from $[0, \infty)$ to X. For a strongly measurable semigroup, it holds that $t \mapsto S(t)x$ is continuous on $(0, \infty)$ for every $x \in X$, with continuity at zero if and only if $x \in \overline{\mathsf{D}(A)}$. In what follows, we exclusively consider *locally bounded* strongly measurable semigroups, which satisfy

$$\exists M_0 \in [1, \infty),\ w \in \mathbb{R} : \quad \|S(t)\|_{\mathscr{L}(X)} \leq M_0 e^{-wt}, \quad \forall t \in [0, \infty). \tag{2.1}$$

Analogously to the relation between a matrix G and its corresponding matrix exponential function $S(t) := e^{tG}$, we can more generally associate an *infinitesimal generator* $G: \mathsf{D}(G) \subseteq X \to X$ to any locally bounded strongly measurable semigroup $(S(t))_{t \geq 0}$. It is defined by the following property: For all $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > w$, the operator $\lambda \mathrm{Id}_X - G$ admits a bounded inverse, and we have

$$\langle (\lambda \mathrm{Id}_X - G)^{-1} x, x^* \rangle = \int_0^\infty e^{-\lambda t} \langle S(t)x, x^* \rangle \mathrm{d}t \quad \text{for all } x \in X \text{ and } x^* \in X^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its dual X^*. We refer the reader to [12, Appendix K] for a more detailed summary of the theory of (strongly) measurable semigroups.

The above notions will be applied to the operator A from (1.4) on X, on which we impose the following standing assumptions:

Assumption 2.1. Let $-A: \mathsf{D}(A) \subseteq X \to X$ be the infinitesimal generator of a locally bounded strongly measurable semigroup $(S(t))_{t \geq 0} \subseteq \mathscr{L}(X)$, which satisfies (2.1). More precisely, we suppose that $(S(t))_{t \geq 0}$ is either

(i) *uniformly bounded*, meaning that $w \in [0, \infty)$, or
(ii) *uniformly exponentially stable*, meaning that $w \in (0, \infty)$.

We may sometimes additionally assume that $(S(t))_{t\geq 0}$ is

(iii) *bounded analytic*, i.e., $(0, \infty) \ni t \mapsto S(t) \in \mathscr{L}(X)$ admits a bounded holomorphic extension to $\Sigma_\varphi := \{z \in \mathbb{C} : |\arg z| < \varphi\}$ for some $\varphi \in (0, \frac{1}{2}\pi)$.

2.3. Solution concepts for first-order abstract Cauchy problems

In this section we define and relate the concepts of strong, mild and L^p-solutions to the first-order Cauchy problem (1.2) on intervals of the form $J = (t_0, \infty)$, where $t_0 \in [-\infty, \infty)$. For a more detailed reference, see [12, Chapter 17].

Let $L^1_{\mathrm{loc}}(\overline{J}; X)$ denote the space of strongly measurable functions from $\overline{J}$ to X which are integrable on every compact subset of $\overline{J}$. For $b < a$ we set $\int_a^b := -\int_b^a$.

Definition 2.2 (Strong solution). A strongly measurable function $u : J \to X$ is said to be a *strong solution* of (1.2) associated with $f \in L^1_{\mathrm{loc}}(\overline{J}; X)$ if

(i) $u(t) \in \mathsf{D}(A)$ for almost all $t \in J$ and $t \mapsto Au(t) \in L^1_{\mathrm{loc}}(\overline{J}; X)$;
(ii) for almost all $t \in J$ we have

$$\begin{aligned} u(t) + \int_{t_0}^t Au(\tau)\mathrm{d}\tau &= x + \int_{t_0}^t f(\tau)\mathrm{d}\tau \qquad &&\text{if } t_0 \in \mathbb{R}; \\ u(t) + \int_0^t Au(\tau)\mathrm{d}\tau &= u(0) + \int_0^t f(\tau)\mathrm{d}\tau \qquad &&\text{if } t_0 = -\infty. \end{aligned} \tag{2.2}$$

As antiderivatives of locally integrable functions are continuous, see [11, Proposition 2.5.9], it follows that any strong solution admits a continuous representative, so that the pointwise evaluation of u in (2.2) is meaningful. In fact, identifying u with this representative, it turns out to be (classically) differentiable for almost all $t \in J$, where it holds that $u'(t) + Au(t) = f(t)$, see [12, Equation (17.3)]. This implies $u' = f - Au \in L^1_{\mathrm{loc}}(\overline{J}; X)$, hence u is weakly differentiable and its weak derivative $\partial_t u$ coincides a.e. with u', again by [11, Proposition 2.5.9].

Next, we turn to the definition of a mild solution to (1.2). In Proposition 2.4, we will see that mild solutions are weaker than strong solutions.

Definition 2.3 (Mild solution). Suppose that Assumption 2.1(ii) is satisfied and let $J := (t_0, \infty)$ for a given $t_0 \in [-\infty, \infty)$. The *mild solution* to (1.2) with $f \in L^p(J; X)$ for $p \in [1, \infty]$ is the function $u \in C_{\mathrm{b}}(\overline{J}; X)$ defined for all $t \in \overline{J}$ by

$$\begin{aligned} u(t) &:= S(t - t_0)x + \int_{t_0}^t S(t-\tau)f(\tau)\mathrm{d}\tau \qquad &&\text{if } t_0 \in \mathbb{R} \text{ and } x \in \overline{\mathsf{D}(A)}; \\ u(t) &:= \int_{-\infty}^t S(t-\tau)f(\tau)\mathrm{d}\tau &&\text{if } t_0 = -\infty. \end{aligned} \tag{2.3}$$

The continuity of the mild solution u defined by (2.3) follows from [12, Proposition K.1.5(3)] and Proposition 2.6(b). The following is a slight extension of [12, Proposition 17.1.3] for the class of time intervals considered in this work.

Proposition 2.4. *Suppose that Assumption 2.1(ii) holds. Let* $J := (t_0, \infty)$ *for a given* $t_0 \in [-\infty, \infty)$, $f \in L^p(J; X)$ *for some* $p \in [1, \infty]$ *and* $x \in \overline{\mathsf{D}(A)}$ *if* $t_0 \in \mathbb{R}$. *Then, for every* $u \in C_{\mathrm{ub}}(\overline{J}; X)$, *the following assertions are equivalent:*

(a) *u is a strong solution of (1.2) in the sense of Definition 2.2;*
(b) *u is the mild solution of (1.2) in the sense of Definition 2.3 and u is (classically) differentiable almost everywhere with* $u' \in L^1_{\mathrm{loc}}(\overline{J}; X)$;
(c) *u is the mild solution of (1.2) in the sense of Definition 2.3,* $u(t) \in \mathsf{D}(A)$ *for almost all* $t \in J$ *and* $t \mapsto Au(t) \in L^1_{\mathrm{loc}}(\overline{J}; X)$.

Proof. If $t_0 \in \mathbb{R}$, then the required modifications of the proof of [12, Proposition 17.1.3] are straightforward. We briefly comment on the case $t_0 = -\infty$. Let $\lambda \in \mathbb{C}$ be such that $\lambda \operatorname{Id}_X + A$ admits a bounded inverse.

(a) $\Longrightarrow$ (b): For u as in Definition 2.2 and $t \in \mathbb{R}$, we define $v\colon (-\infty, t] \to X$ by

$$v(\tau) := (\lambda \operatorname{Id}_X + A)^{-1} S(t-\tau) u(\tau), \quad \tau \in (-\infty, t].$$

Fixing $t' < t$, and arguing as in the original proof–except for integrating over (t', t) instead of $(0, t)$–we find

$$u(t) = S(t-t')u(t') + \int_{t'}^{t} S(t-\tau) f(\tau) \mathrm{d}\tau.$$

As $t' \to -\infty$, the first term vanishes by $u \in C_{\mathrm{b}}(\mathbb{R}; X)$ and Assumption 2.1(ii), and the second term converges to $\int_{-\infty}^{t} S(t-\tau) f(\tau) \mathrm{d}\tau$ by dominated convergence.

(b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a): Use the following analog to [12, Equation (17.4)]:

$$\int_0^t A(\lambda \operatorname{Id}_X + A)^{-1} u(\tau) \mathrm{d}\tau = -(\lambda \operatorname{Id}_X + A)^{-1} \left[u(t) - u(0) - \int_0^t f(\tau) \mathrm{d}\tau \right]. \quad \square$$

Finally, we introduce a stronger notion of solutions, which will later be used as the basis for an analogous solution concept for the fractional problem:

Definition 2.5 (L^p-solution). Let $J := (t_0, \infty)$ for some $t_0 \in [-\infty, \infty)$ and $f \in L^p(J; X)$ with $p \in [1, \infty]$. A strong solution u to (1.2) (in the sense of Definition 2.2) is said to be an *L^p-solution* if $t \mapsto Au(t) \in L^p(J; X)$.

2.4. Fractional parabolic calculus

Fractional calculus classically refers to the idea of generalizing the basic operations of calculus, namely differentiation and integration, beyond the integer-order cases (which simply consist of repeated application). For example, a fractional derivative $\partial_t^\alpha f$ of a sufficiently smooth given function $f\colon \mathbb{R} \to \mathbb{R}$ should be defined for non-integer $\alpha \in (0, \infty)$, in such a way that $\partial_t^{1/2}\partial_t^{1/2} f = \partial_t f$. To this end, a common first object of study is the *Riemann–Liouville fractional integral*, defined for suitable f by

$$I_{\mathrm{RL}}^s f(t) := \frac{1}{\Gamma(s)} \int_{-\infty}^t (t-\tau)^{s-1} f(\tau)\,\mathrm{d}\tau, \quad s \in (0,\infty), \quad t \in \mathbb{R},$$

Since this is a fractional-order generalization of the integration operator I_{RL}^1, which can itself be viewed as an inverse of the derivative ∂_t, we can interpret I_{RL}^s as ∂_t^{-s}. This leads us to define the following fractional derivatives:

$$[\partial_t^s]_{\mathrm{RL}} := \partial_t^{\lceil s \rceil} I^{\lceil s \rceil - s} \quad \text{and} \quad [\partial_t^s]_{\mathrm{C}} := I^{\lceil s \rceil - s} \partial_t^{\lceil s \rceil},$$

which are known as the Riemann–Liouville and Caputo fractional (time) derivatives, respectively. For a comprehensive overview of these and other fractional time derivatives, as well as their application in fractional differential equations, we refer to the monographs [13,25,26].

Under the assumption that $-A$ generates a semigroup $(S(t))_{t\geq 0}$ satisfying Assumption 2.1(ii), we can extend these notions to define the lesser-known fractional *parabolic* integration and differentiation operators $\mathfrak{I}^s$ and $\mathfrak{D}^s$, which are the rigorous definitions of the expressions $(\partial_t + A)^{-s}$ and $(\partial_t + A)^s$ from Sect. 1, respectively. For $s \in (0, \infty)$, let the integration kernel $k_s\colon \mathbb{R} \to \mathscr{L}(X)$ be defined by $k_s(\tau) := \frac{1}{\Gamma(s)} \tau_+^{s-1} S(\tau)$ for all $\tau \in \mathbb{R}$. Given $u\colon \mathbb{R} \to X$, its *Riemann–Liouville type fractional parabolic integral* $\mathfrak{I}^s u\colon \mathbb{R} \to X$ *of order* s is defined by

$$\mathfrak{I}^s u(t) := k_s * u(t) := \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} S(\tau) u(t-\tau)\,\mathrm{d}\tau \tag{2.4}$$

whenever this Bochner integral converges for a.e. $t \in \mathbb{R}$. For $s = 0$ we set $\mathfrak{I}^0 := \mathrm{Id}_X$. When viewed as a linear operator, the fractional parabolic integral $\mathfrak{I}^s$ turns out to map its domain $L^p(\mathbb{R}; X)$ boundedly to itself for every $s \in [0, \infty)$. In fact, we have the following properties of $(\mathfrak{I}^s)_{s\in[0,\infty)}$, which will be used throughout this work:

Proposition 2.6. *Suppose that Assumption 2.1(ii) holds. Let $p \in [1, \infty]$ and $s \in [0, \infty)$. The following assertions hold:*

(a) $\mathfrak{I}^s \in \mathscr{L}(L^p(\mathbb{R}; X))$ *with* $\|\mathfrak{I}^s\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leq \frac{M_0}{w^s}$.

(b) $\mathfrak{I}^s \in \mathscr{L}(L^p(\mathbb{R}; X); C_{\mathrm{b}}(\mathbb{R}; X))$ *if*

$$\begin{cases} p = 1 & \text{and} \quad s \in [1, \infty), \quad \text{or}, \\ p \in (1, \infty) & \text{and} \quad s \in (1/p, \infty). \end{cases} \tag{2.5}$$

(c) $\mathfrak{I}^{s_1}\mathfrak{I}^{s_2}u = \mathfrak{I}^{s_1+s_2}u$ *a.e. for all* $s_1, s_2 \in [0,\infty)$ *and* $u \in L^p(\mathbb{R}; X)$.

(d) *Given* $x \in X$, *if* $p \in [1, \frac{1}{1-s_1})$ *and* $s_1 \in (0,1)$ *or* $p \in [1,\infty]$ *and* $s_1 \in [1,\infty)$, *then we have* $k_{s_1} \otimes x \in L^p(\mathbb{R}; X)$ *and* $\mathfrak{I}^{s_2}(k_{s_1} \otimes x) = k_{s_1+s_2} \otimes x$ *for all* $s_2 \in [0,\infty)$.

Proof. Estimate (2.1) implies $\|k_s\|_{L^1(\mathbb{R};\mathscr{L}(X))} \le M_0 w^{-s}$ for all $s \in (0,\infty)$, so that Minkowski's integral inequality [27, Section A.1] yields (a).

If $p' \in [1,\infty]$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $u \in L^p(\mathbb{R}; X)$, then $k_s \in L^{p'}(\mathbb{R}; \mathscr{L}(X))$ for s as in the statement of (b), and the result follows from Hölder's inequality and the continuity of translations in $L^q(\mathbb{R}; X)$ or $L^q(\mathbb{R}; \mathscr{L}(X))$ for $q \in [1,\infty)$.

Assertions (c) and (d) follow by combining the semigroup property of $(S(t))_{t\ge 0}$, Fubini's theorem and [23, Equation (5.12.1)]. □

For $p \in [1,\infty]$, let $W^{1,p}(\mathbb{R}; X)$ denote the Bochner–Sobolev space consisting of functions $u \in L^p(\mathbb{R}; X)$ whose weak derivative $\partial_t u$ also belongs to $L^p(\mathbb{R}; X)$. Identifying $A \colon \mathsf{D}(A) \subseteq X \to X$ with $\mathcal{A} \colon L^p(\mathbb{R}; \mathsf{D}(A)) \subseteq L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)$ defined by $[\mathcal{A}u](\,\cdot\,) := Au(\,\cdot\,)$, we can view $\partial_t + A$ as an operator on $L^p(\mathbb{R}; X)$ with domain $L^p(\mathbb{R}; \mathsf{D}(A)) \cap W^{1,p}(\mathbb{R}; X)$. In conjunction with the operators $(\mathfrak{I}^s)_{s\ge 0}$ from (2.4), this leads to the definition of the *Riemann–Liouville type fractional parabolic derivative of order* $s \in [0,\infty)$:

$$\begin{aligned} \mathfrak{D}^s u &:= (\partial_t + A)^{\lceil s\rceil}\mathfrak{I}^{\lceil s\rceil - s}u, \\ u \in \mathsf{D}(\mathfrak{D}^s) &:= \{u \in L^p(\mathbb{R}; X) : \mathfrak{I}^{\lceil s\rceil - s}u \in \mathsf{D}((\partial_t + A)^{\lceil s\rceil})\}. \end{aligned} \tag{2.6}$$

Note that we do not explicitly indicate the dependence of $\mathfrak{D}^s$ and $\mathfrak{I}^s$ on $p \in [1,\infty]$ in the notation, instead leaving it to be inferred from context.

Remark 2.7. While the terminology "fractional parabolic" is inspired by the case $A = -\Delta$ acting on a function space such as $X = L^2(\mathcal{D})$, our setting is considerably more general.

Remark 2.8. Let us briefly elaborate on our choice for the Riemann–Liouville type operators $\mathfrak{D}^s$ and $\mathfrak{I}^s$ as the rigorous interpretations of $(\partial_t + A)^s$ and $(\partial_t + A)^{-s}$, respectively. Under some additional assumptions on A and X, it can be shown that the sum operator $\partial_t + \mathcal{A}$ admits an extension $\mathcal{B}$ which is *sectorial* [12, Section 16.3]. For this class of operators, fractional powers can be defined as in [12, Section 15.2]. If, for instance, $-A$ generates an exponentially bounded *strongly continuous* semigroup on a Hilbert space $X = H$, then it holds that $\mathcal{B}^{-s} = \mathfrak{I}^s$ for all $s \in [0,\infty)$, cf. [14, Proposition 3.2 and Equation (3.9)]. Then [12, Proposition 15.1.12(2)] implies

$$\mathcal{B}^s = \mathcal{B}^{\lceil s\rceil}\mathcal{B}^{s-\lceil s\rceil} \quad \text{with} \quad \mathsf{D}(\mathcal{B}^s) = \{u \in L^p(\mathbb{R}; X) : u \in \mathsf{D}(\mathcal{B}^{\lceil s\rceil})\}.$$

Hence in this situation we find that $\mathcal{B}^s$ is an extension of $\mathfrak{D}^s$. In particular, if we have $\mathcal{B} = \partial_t + \mathcal{A}$ (a property which is closely related to the maximal L^p-regularity of A (see [12, Proposition 17.3.14])), then in fact $\mathcal{B}^s = \mathfrak{D}^s$.

Thus, there is a close relation between the Riemann–Liouville fractional parabolic operators and fractional powers of sectorial extensions of $\partial_t + \mathcal{A}$. We choose the former viewpoint for the sake of simplicity, generality and consistency with the analogous definitions for $A = 0$ in fractional calculus texts such as [8,13,25,26].

2.5. Phillips functional calculus

If Assumption 2.1(i) holds and $f : \overline{\mathbb{C}}_+ \to \mathbb{C}$ can be written as the Laplace transform of a complex Borel measure μ of bounded variation on $[0, \infty)$, i.e., if we have

$$f(z) = \mathcal{L}[\mu](z) := \int_{[0,\infty)} e^{-zs} \mathrm{d}\mu(s)$$

for all $z \in \overline{\mathbb{C}}_+$, then we define the operator

$$f(A) := \left[\int_{[0,\infty)} e^{-sz} \mathrm{d}\mu(s)\right](A) := \int_{[0,\infty)} S(s)\mathrm{d}\mu(s) \in \mathscr{L}(X).$$

The map $f \mapsto f(A)$, called the *Phillips functional calculus* for A, is an algebra homomorphism from the space of Laplace transforms to $\mathscr{L}(X)$ (see [10, Remark 3.3.3]).

Note that $S(t) = \mathcal{L}[\delta_t](A) = (e^{-zt})(A)$, where δ_t denotes the Dirac measure concentrated at $t \in [0, \infty)$. Moreover, for any $\alpha, \varepsilon \in (0, \infty)$ we can define the negative fractional powers of $A + \varepsilon \operatorname{Id}_X$ by

$$(A + \varepsilon \operatorname{Id}_X)^{-\alpha} := [(z + \varepsilon)^{-\alpha}](A) = \mathcal{L}\left[\frac{s^{\alpha-1} e^{-\varepsilon s}}{\Gamma(\alpha)} \mathrm{d}s\right](A),$$

see [10, Proposition 3.3.5]. These can be used to define $(A + \varepsilon \operatorname{Id}_X)^{\alpha}$ and, in turn, $A^{\alpha} : \mathsf{D}(A^{\alpha}) \subseteq X \to X$ in a manner which is consistent with other common definitions of fractional powers, cf. [10, Propositions 3.1.9 and 3.3.2]. Under Assumption 2.1(ii), we can also allow for $\varepsilon = 0$ directly in the above to define (see [10, Corollary 3.3.6]):

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \sigma^{\alpha-1} S(\sigma)\mathrm{d}\sigma \in \mathscr{L}(X). \tag{2.7}$$

3. Fractional-order inhomogeneous abstract Cauchy problem on $\mathbb{R}$

In this section, we consider the inhomogeneous abstract Cauchy problem associated to the fractional operator $\mathfrak{D}^s$ and $f \in L^p(\mathbb{R}; X)$, where $s \in (0, \infty)$, $p \in [1, \infty]$ and $J = \mathbb{R}$:

$$\mathfrak{D}^s u(t) = f(t), \quad t \in \mathbb{R}. \tag{3.1}$$

Note that we do not impose any initial data here since the problem is posed on the entire real line. The solution concepts for (3.1) which we will define are the following fractional-order analogs to the notion of L^p-solutions and mild solutions (see Definitions 2.5 and 2.3, respectively):

Definition 3.1 (L^p-solution). Suppose that Assumption 2.1(ii) holds. Let $s \in (0, \infty)$, $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}; X)$. Then $u \in L^p(\mathbb{R}; X)$ is called an L^p-*solution* to (3.1) if $u \in \mathsf{D}(\mathfrak{D}^s)$ and the equation (3.1) holds almost everywhere on $\mathbb{R}$.

It is a consequence of Proposition 3.3(b) that the L^p-solution to (3.1) is unique if it exists. The question of existence of the L^p-solution for every $f \in L^p(\mathbb{R}; X)$ is highly nontrivial; in the case $s = 1$, it characterizes a deep property of the linear operator A called *maximal* L^p*-regularity*. Since the present article is primarily concerned with the concept of mild solutions, to be defined next, we shall not investigate this matter further, and instead refer to [12, Chapter 17] for a more detailed account of the topic of maximal L^p-regularity.

Definition 3.2 (Mild solution). Suppose that Assumption 2.1(ii) holds. Let $s \in (0, \infty)$ and $p \in [1, \infty]$ satisfy (2.5). The *mild solution* to (3.1) with $f \in L^p(\mathbb{R}; X)$ is the function $u \in C_b(\mathbb{R}; X)$ defined for all $t \in \mathbb{R}$ by

$$u(t) := \mathfrak{I}^s f(t) = \frac{1}{\Gamma(s)} \int_{-\infty}^{t} (t-\tau)^{s-1} S(t-\tau) f(\tau) \mathrm{d}\tau.$$

The mild solution exists and is unique by definition, since it is given by an explicit formula. Moreover, in view of Proposition 2.6(b), it is indeed continuous under the given assumptions on s and p.

The next proposition shows that the fractional parabolic derivative and integral are inverse to each other whenever the respective left-hand sides are well-defined. In particular, it implies that L^p-solutions are mild solutions whenever the parameters s and p are such that (2.5) holds (see Corollary 3.5).

Proposition 3.3. *Suppose that Assumption 2.1(ii) holds. Let $s \in [0, \infty)$, $p \in [1, \infty]$ and $u \in L^p(\mathbb{R}; X)$. Then the following assertions hold:*

(a) If $\mathfrak{I}^s u \in \mathsf{D}(\mathfrak{D}^s)$, then $\mathfrak{D}^s \mathfrak{I}^s u = u$ a.e.
(b) If $u \in \mathsf{D}(\mathfrak{D}^s)$, then $\mathfrak{I}^s \mathfrak{D}^s u = u$ a.e.

Proof. (a) For $s = 1$, $v := \mathfrak{I}^1 u$ is the mild solution to (1.2) with $f := u \in L^p(J; X)$. Moreover, since $v \in W^{1,p}(J; X) \cap L^p(J; X)$, the conditions of Proposition 2.4(b)-(c) are satisfied, so that v is a strong solution, which proves the base case.

Now let $k \in \mathbb{N}$ and suppose that (a) holds for $s = k$. If $\mathfrak{I}^{k+1} u \in \mathsf{D}(\mathfrak{D}^{k+1})$, then by definition we have $\mathfrak{I}^{k+1} u \in \mathsf{D}(\mathfrak{D}^k)$ and $\mathfrak{D}^k \mathfrak{I}^{k+1} u \in \mathsf{D}(\mathfrak{D}^1)$. By Proposition 2.6(c), this means that $\mathfrak{I}^k \mathfrak{I}^1 u \in \mathsf{D}(\mathfrak{D}^k)$ and $\mathfrak{D}^k \mathfrak{I}^k \mathfrak{I}^1 u \in \mathsf{D}(\mathfrak{D}^1)$. Combining the former expression with the induction hypothesis yields

$$\mathfrak{D}^k \mathfrak{I}^{k+1} u = \mathfrak{D}^k \mathfrak{I}^k \mathfrak{I}^1 u = \mathfrak{I}^1 u, \quad \text{a.e.,} \tag{3.2}$$

and thus $\mathfrak{I}^1 u \in \mathsf{D}(\mathfrak{D}^1)$ by the latter. It follows that $\mathfrak{D}^1 \mathfrak{I}^1 u = u$ a.e. by the base case $s = 1$ of (a). Using (3.2) once more, this becomes $\mathfrak{D}^1 \mathfrak{D}^k \mathfrak{I}^{k+1} u = u$ a.e., whose left-hand side equals $\mathfrak{D}^{k+1} \mathfrak{I}^{k+1} u$ a.e. by the definition of integers powers of $\mathfrak{D}$. This proves (a) for $s = k+1$, and thus for all $s \in \mathbb{N}$ by induction since $k \in \mathbb{N}$ was arbitrary.

For $s \in (0,\infty) \setminus \mathbb{N}$, the assertion follows upon combining the definition (2.6) of $\mathfrak{D}^s$ with Proposition 2.6(c) and the integer case:

$$\mathfrak{D}^s \mathfrak{I}^s u = (\partial_t + A)^{\lceil s \rceil} \mathfrak{I}^{\lceil s \rceil - s} \mathfrak{I}^s u = (\partial_t + A)^{\lceil s \rceil} \mathfrak{I}^{(\lceil s \rceil - s) + s} u = u \quad \text{a.e.}$$

(b) The case $s = 1$ follows from Proposition 2.4 (ii) $\implies$ (b) with $f := u' + Au$, and the integer case follows by induction. For fractional s, fix $u \in \mathsf{D}(\mathfrak{D}^s)$ and note

$$\mathfrak{I}^{\lceil s \rceil - s} \mathfrak{I}^s \mathfrak{D}^s u = \mathfrak{I}^{\lceil s \rceil - s} \mathfrak{I}^s (\partial_t + A)^{\lceil s \rceil} \mathfrak{I}^{\lceil s \rceil - s} u = \mathfrak{I}^{\lceil s \rceil} (\partial_t + A)^{\lceil s \rceil} \mathfrak{I}^{\lceil s \rceil - s} u = \mathfrak{I}^{\lceil s \rceil - s} u,$$

a.e. Since (a) implies that $\mathfrak{I}^{\lceil s \rceil - s}$ is injective, we conclude $\mathfrak{I}^s \mathfrak{D}^s u = u$ a.e. □

Combining Propositions 2.6(b) and 3.3(b) yields the following corollaries:

Corollary 3.4. *Suppose that Assumption 2.1(ii) holds. If $s \in [0,\infty)$ and $p \in [1,\infty]$ satisfy (2.5), then we have $\mathsf{D}(\mathfrak{D}^s) \subseteq C_b(\mathbb{R}; X)$.*

Corollary 3.5. *Suppose that Assumption 2.1(ii) is satisfied and let u be an L^p-solution to (3.1) in the sense of Definition 3.1 for some $s \in (0,\infty)$, $p \in [1,\infty]$. If s and p satisfy (2.5), then u is the mild solution in the sense of Definition 3.2.*

Proof. If u is an L^p-solution, then $u \in \mathsf{D}(\mathfrak{D}^s)$ and $\mathfrak{D}^s u = f$ holds almost everywhere. Thus, by Proposition 3.3(b), we can apply $\mathfrak{I}^s$ on both sides to obtain $u = \mathfrak{I}^s f$ a.e., and we have $u \in C_b(\mathbb{R}; X)$ by Corollary 3.4 (or by Proposition 2.6(b) directly). □

4. Dirichlet problem associated to the fractional parabolic derivative operator

In this section we turn to the main subject of the present work, namely the natural abstract Dirichlet problem associated to $\mathfrak{D}^s$, which consists in finding a function $u : \mathbb{R} \to X$ satisfying

$$\begin{cases} \mathfrak{D}^s u(t) = 0, & t \in (t_0, \infty), \\ u(t) = g(t), & t \in (-\infty, t_0], \end{cases} \tag{4.1}$$

for $s \in (0,\infty)$, $t_0 \in \mathbb{R}$ and sufficiently regular $g : (-\infty, t_0] \to X$. Recall from Sect. 2.4 that $\mathfrak{D}^s$ denotes the Riemann–Liouville type fractional parabolic differentiation operator acting on functions from $\mathbb{R}$ to X, which is our interpretation of the operator $(\partial_t + A)^s$ appearing in (1.4), as motivated by Remark 2.8.

As in the previous sections, we begin by defining the notion of an L^p-solution to (4.1) (cf. Definitions 2.5 and 3.1) and subsequently define the mild solution (cf. Definitions 2.3 and 3.2), which is the rigorous formulation of the solution formula formally given by (1.5). As before, we note that the existence and uniqueness of the mild solution are immediate from the definition; for the L^p-solution, we have uniqueness but the matter of existence is outside of the scope of this work, analogously to the discussion Definition 3.1.

Definition 4.1 (L^p-solution). Suppose that Assumption 2.1(ii) holds. Let $s \in (0, \infty)$, $p \in [1, \infty]$, $t_0 \in \mathbb{R}$ and $g \in L^p(-\infty, t_0; X)$. Then $u \in L^p(\mathbb{R}; X)$ is called an L^p-*solution* to (4.1) if $u \in \mathsf{D}(\mathfrak{D}^s)$ and both equations in (4.1) hold almost everywhere on their respective sub-intervals of $\mathbb{R}$. In particular, we have $g \in \mathsf{D}(\mathfrak{D}^s)$ on $(-\infty, t_0]$.

Definition 4.2 (Mild solution). Suppose that Assumption 2.1(i) holds. Let $s \in (0, \infty)$, $p \in [1, \infty]$, $t_0 \in \mathbb{R}$ be given and let $g \in C_b((-\infty, t_0]; X) \cap \mathsf{D}(\mathfrak{D}^{(s-1)\vee 0})$ be such that $\mathfrak{D}^{(s-1)\vee 0} g \in C_b((-\infty, t_0]; X)$. The *mild solution* to (4.1) with initial datum g is the function $u \in C_b(\mathbb{R} \setminus \{t_0\}; X)$ defined by

$$u(t) := g(t), \quad t \in (-\infty, t_0],$$

and, for $s \in (0, \infty) \setminus \mathbb{N}$,

$$\begin{aligned} u(t) &:= \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau + 1} S((t - t_0)(\tau + 1)) g(t_0 - (t - t_0)\tau) \mathrm{d}\tau \\ &\quad + \sum_{k=1}^{\lfloor s \rfloor} \frac{(t - t_0)^{\{s\}+k-1}}{\Gamma(\{s\} + k)} S(t - t_0) \mathfrak{D}^{\{s\}+k-1} g(t_0), \quad t \in (t_0, \infty), \end{aligned} \tag{4.2}$$

whereas for $s = n \in \mathbb{N}$, we set

$$u(t) := \sum_{k=0}^{n-1} \frac{(t - t_0)^k}{k!} S(t - t_0) \mathfrak{D}^k g(t_0), \quad t \in (t_0, \infty). \tag{4.3}$$

The following proposition shows that the mild solution is indeed well-defined in the sense that it possesses the continuity properties asserted in Definition 4.2. Its proof is postponed to Sect. 4.1, in which we also state and prove some additional properties of the mild solution.

Proposition 4.3. *Suppose Assumption 2.1(i) holds. Let $s \in (0, \infty)$, $p \in [1, \infty]$, $t_0 \in \mathbb{R}$ be given and let $g \in L^p(-\infty, t_0; X)$ be as in Definition 4.2. Then the mild solution u to (4.1) satisfies $u \in C_b(\mathbb{R} \setminus \{t_0\}; X)$ and, for all $t \in (t_0, \infty)$,*

$$\|u(t)\|_X \le M_0 \overline{\Gamma}(s, w(t - t_0)) \max\{\|g\|_\infty, \|\mathfrak{D}^{\{s\}} g(t_0)\|_X, \ldots, \|\mathfrak{D}^{s-1} g(t_0)\|_X\},$$

where $M_0 \in [1, \infty)$ and $w \in [0, \infty)$ are as in (2.1).

If moreover $g(t), \mathfrak{D}^{\{s\}+k} g(t_0) \in \overline{\mathsf{D}(A)}$ for all $t \in (-\infty, t_0]$ and $k \in \{0, \ldots, \lfloor s \rfloor\}$, then we in fact have $u \in C_b(\mathbb{R}; X)$.

Remark 4.4. Let us emphasize that the solution formula can fail to be continuous at t_0 even in the first-order case $u(t) = S(t - t_0)x$ if $x \notin \overline{\mathsf{D}(A)}$. As an example, we can take $X = C_b(\mathbb{R})$, $A = -\Delta$ and $x(\xi) = \sin(\xi^2)$. Then $-A$ generates the analytic heat semigroup and $\overline{\mathsf{D}(A)} = C_{ub}(\mathbb{R})$ is the space of bounded and uniformly continuous functions on $\mathbb{R}$, cf. [20, Corollary 3.1.9]. In this case, $\|u(t)\|_\infty \le 1$ for all $t \in [t_0, \infty)$, but $S(t - t_0)x$ does not converge uniformly to x as $t \downarrow t_0$.

The motivation for the solution formulae in Definition 4.2 is provided by the following theorem, which shows that any L^p-solution to (4.1) is a mild solution whenever $s \in (0, \infty)$ and $p \in [1, \infty]$ are such that (4.2)-(4.3) are meaningful.

Theorem 4.5. *Suppose that Assumption 2.1(ii) is satisfied and let u be an L^p-solution to (4.1) in the sense of Definition 4.1 for some $p \in [1, \infty]$, $s \in (0, \infty)$ and $t_0 \in \mathbb{R}$. If s and p satisfy (2.5), then u is the mild solution to (4.1) in the sense of Definition 4.2.*

The proof of Theorem 4.5 is presented in Sect. 4.2, where the integer-order and fractional-order cases are treated separately. Before proceeding to the next subsection, we consider the following important example of a situation in which we can write down an explicit mild solution formula for (1.2):

Example 4.6. (The fractional heat operator $(\partial_t - \Delta)^s$ on $L^2(\mathbb{R}^d)$). Let us consider the function space $X = L^2(\mathbb{R}^d)$ and differential operator $A = -\Delta$ for $d \in \mathbb{N}$, i.e., the negative Laplacian on the full Euclidean space $\mathbb{R}^d$ of dimension $d \in \mathbb{N}$. By classical results (see for instance [29, Section 13.6.(c)]), we know that $-A = \Delta$ generates the *heat semigroup* $(S(t))_{t \geq 0}$, which is given by the (spatial) convolution with the *Gauss–Weierstrass kernel*. That is, for all $t \in (0, \infty)$, $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have

$$[S(t)f](x) = \int_{\mathbb{R}^d} K_t(x - y) f(y) \mathrm{d}y,$$

where $K_t : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$K_t(x) := \frac{1}{(\sqrt{4\pi t})^d} \exp\left(-\frac{\|x\|^2_{\mathbb{R}^d}}{4t}\right).$$

Substituting these formulae into equation (4.2) for some sufficiently regular initial datum function $g : (-\infty, t_0] \times \mathbb{R}^d \to \mathbb{R}$, we obtain an explicit formula for the mild solution $u : \mathbb{R}^{d+1} \to \mathbb{R}$ to (4.1). For example, if $t_0 = 0$ and $s \in (0, 1)$, then we have for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$:

$$\begin{aligned} u(t, x) &= \frac{\sin(\pi s)}{\pi} \int_0^\infty \int_{\mathbb{R}^d} \frac{\tau^{-s}}{\tau + 1} K_{t(\tau+1)}(x - y) g(-t\tau, y) \mathrm{d}y \mathrm{d}\tau \\ &= \frac{\sin(\pi s)}{\pi(\sqrt{4\pi t})^d} \int_0^\infty \int_{\mathbb{R}^d} \tau^{-s} (\tau + 1)^{-\frac{d}{2}-1} \exp\left(-\frac{\|x - y\|^2_{\mathbb{R}^d}}{4t(\tau + 1)}\right) g(-t\tau, y) \mathrm{d}y \mathrm{d}\tau. \end{aligned}$$

If $s \in (1, 2)$ (still with $t_0 = 0$), then we instead find for all $t \in (0, \infty)$ and $x \in \mathbb{R}^d$:

$$\begin{aligned} u(t, x) &= \frac{\sin(\pi\{s\})}{\pi(\sqrt{4\pi t})^d} \int_0^\infty \int_{\mathbb{R}^d} \tau^{-\{s\}} (\tau + 1)^{-\frac{d}{2}-1} \exp\left(-\frac{\|x - y\|^2_{\mathbb{R}^d}}{4t(\tau + 1)}\right) g(-t\tau, y) \mathrm{d}y \mathrm{d}\tau \\ &\quad + \frac{t^{\{s\}}}{\Gamma(\{s\} + 1)} S(t) \mathfrak{D}^{\{s\}} g(0). \end{aligned} \tag{4.4}$$

The fractional parabolic derivative $\mathfrak{D}^\alpha f$ of any sufficiently regular $f\colon \mathbb{R} \to X$ (e.g., $f \in \mathsf{D}(\mathfrak{D}^1)$) admits the following Marchaud type representation for $\alpha \in (0, 1)$ and $t \in \mathbb{R}$ (cf. [21, Proposition 3.2.1] or [28, Equation (1.2)]):

$$\mathfrak{D}^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \sigma^{-\alpha-1}[S(\sigma) f(t-\sigma) - f(t)]\mathrm{d}\sigma.$$

Therefore, supposing that g is sufficiently regular, we have for all $t \in (0, \infty)$,

$$\begin{aligned}
&\frac{t^{\{s\}}}{\Gamma(\{s\}+1)} S(t)\mathfrak{D}^{\{s\}} g(0) \\
&\quad = -\frac{t^{\{s\}}\sin(\pi\{s\})}{\pi} \int_0^\infty \sigma^{-\{s\}-1}[S(t+\sigma)g(-\sigma) - S(t)g(0)]\mathrm{d}\sigma \\
&\quad = -\frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \tau^{-\{s\}-1}[S(t(\tau+1))g(-\tau t) - S(t)g(0)]\mathrm{d}\tau,
\end{aligned}$$

where we used the reflection formula for the gamma function [23, Equation (5.5.3)] to obtain the prefactor on the second line, and the change of variables $\sigma = t\tau$ to yield the third line. Substituting the heat semigroup once more, we obtain, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\frac{t^{\{s\}}}{\Gamma(\{s\}+1)}[S(t)\mathfrak{D}^{\{s\}} g(0)](x) \\
&\quad = \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \int_{\mathbb{R}^d} \tau^{-\{s\}-1}[K_t(x-y)g(0,y) - K_{t(1+\tau)}(x-y)g(-\tau t, y)]\mathrm{d}y\mathrm{d}\tau \\
&\quad = \frac{\sin(\pi\{s\})}{\pi(\sqrt{4\pi t})^d} \int_0^\infty \int_{\mathbb{R}^d} \tau^{-\{s\}-1}\Bigg[\exp\Bigg(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t}\Bigg)g(0,y) \\
&\qquad\qquad - (\tau+1)^{-\frac{d}{2}} \exp\Bigg(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t(\tau+1)}\Bigg)g(-\tau t, y)\Bigg]\mathrm{d}y\mathrm{d}\tau.
\end{aligned}$$

Substituting this expression into (4.4), we conclude that the mild solution to the Dirichlet problem associated to the fractional heat operator $(\partial_t - \Delta)^s$, with $s \in (1, 2)$ and sufficiently regular initial datum g, admits the explicit expression

$$\begin{aligned}
u(t,x) &= \frac{\sin(\pi\{s\})}{\pi(\sqrt{4\pi t})^d} \int_0^\infty \int_{\mathbb{R}^d} \Bigg[\tau^{-\{s\}-1}\exp\Bigg(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t}\Bigg)g(0,y) \\
&\quad + \Big(\tau^{-\{s\}}(\tau+1)^{-\frac{d}{2}-1} - \tau^{-\{s\}-1}(\tau+1)^{-\frac{d}{2}}\Big)\exp\Bigg(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t(\tau+1)}\Bigg)g(-t\tau, y)\Bigg]\mathrm{d}y\mathrm{d}\tau
\end{aligned}$$

for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$.

4.1. Properties of the mild solution

In this section, we further investigate the mild solution concept introduced in Definition 4.2, by establishing some of its key properties. To this end, we start with the central observation that formula (4.2) has close connections to the normalized upper incomplete gamma function, whose principal branch $\overline{\Gamma}(\alpha, \cdot): \mathbb{C}\setminus(-\infty, 0) \to \mathbb{C}$ for $\alpha \in \mathbb{C}_+$ is defined by

$$\overline{\Gamma}(\alpha, z) := \frac{1}{\Gamma(\alpha)} \int_z^\infty \zeta^{\alpha-1} e^{-\zeta} \mathrm{d}\zeta, \quad z \in \mathbb{C} \setminus (-\infty, 0),$$

integrating over any contour from z to ∞ avoiding $(-\infty, 0)$ (see [23, Chapter 8]).

The relation to (4.2) follows from the following identities. For $\alpha \in (0, 1)$ and $z \in \mathbb{C}_+ \cup \{0\}$, [23, Equations (5.5.3), (13.4.4) and (13.6.6)] yield

$$\overline{\Gamma}(\alpha, z) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{\tau^{-\alpha}}{1+\tau} e^{-z(1+\tau)} \mathrm{d}\tau. \tag{4.5}$$

In particular, for $t \in (0, \infty)$, the change of variables $\sigma := t(1+\tau)$ produces

$$\overline{\Gamma}(\alpha, tz) = \frac{t^\alpha \sin(\pi\alpha)}{\pi} \int_t^\infty (\sigma - t)^{-\alpha} \sigma^{-1} e^{-\sigma z} \mathrm{d}\sigma. \tag{4.6}$$

Moreover, for all $\alpha \in (0, \infty)$ and $n \in \mathbb{N}$, we have the following recurrence relations [23, Equations (8.8.12) and (8.4.10)]:

$$\overline{\Gamma}(\alpha, z) = \overline{\Gamma}(\{\alpha\}, z) + \sum_{k=1}^{\lfloor\alpha\rfloor} \frac{z^{k+\{\alpha\}-1}}{\Gamma(k+\{\alpha\})} e^{-z}, \quad \overline{\Gamma}(n, z) = \sum_{k=0}^{n-1} \frac{z^k}{k!} e^{-z}. \tag{4.7}$$

As a first application of these identities, we present the following proof:

Proof of Proposition 4.3. The estimate on $\|u(t)\|_X$ follows by applying the triangle inequality, (2.1) and identities (4.5)-(4.7) to (4.2). The continuity assertions rely on the strong continuity of $(S(t))_{t\geq 0}$ (see the remarks Assumption 2.1). They are immediate for the terms involving $\mathfrak{D}^{\{s\}+k} g(t_0)$. For the integral term, we note that the norm of the integrand is dominated by $\tau \mapsto M_0 \|g\|_\infty \frac{\tau^{-\{s\}}}{\tau+1}$, which is integrable in view of (4.5). Combined with the continuity of

$$t \mapsto S((t-t_0)(\tau+1)) g(t_0 - (t-t_0))$$

on (t_0, ∞) and possibly at t_0, the dominated convergence theorem yields the result. □

Next we comment on the precise way in which the fractional-order mild solution formula (4.2) of Definition 4.2 reduces to formula (4.3) for integer orders $s = n \in \mathbb{N}$. Substituting $s = n$ (i.e., $\lfloor s \rfloor = n$ and $\{s\} = 0$) in the higher-order terms of (4.2) and shifting the index of summation yields

$$\sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0)[(\partial_t + A)^k g](t_0), \quad \forall t \in (t_0, \infty), \tag{4.8}$$

as desired. Moreover, the first term in (4.2) vanishes as required, provided that the integral remains convergent for $\{s\} = 0$. This occurs under Assumption 2.1(ii), but may fail in general if only Assumption 2.1(i) is satisfied, hence in this case we cannot argue via direct substitution. Instead, we have to consider limits as $s \to n$. Let u_s denote the mild solution from Definition 4.2 of order $s \in (0,\infty)\setminus\mathbb{N}$. Then we have

$$u_{n+\varepsilon}(t) = \frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty \frac{\tau^{-\varepsilon}}{\tau+1} S((t-t_0)(\tau+1))g(t_0-(t-t_0)\tau)\mathrm{d}\tau \tag{4.9}$$

$$+\sum_{k=0}^{n-1}\frac{(t-t_0)^{k+\varepsilon}}{\Gamma(k+\varepsilon+1)}S(t-t_0)\mathfrak{D}^{k+\varepsilon}g(t_0), \tag{4.10}$$

$$u_{n-\varepsilon}(t) = \frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty \frac{\tau^{\varepsilon-1}}{\tau+1} S((t-t_0)(\tau+1))g(t_0-(t-t_0)\tau)\mathrm{d}\tau \tag{4.11}$$

$$+\sum_{k=1}^{n-1}\frac{(t-t_0)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)}S(t-t_0)\mathfrak{D}^{k-\varepsilon}g(t_0), \tag{4.12}$$

for all $\varepsilon \in (0,1)$ and $t \in (t_0,\infty)$. Substituting $\varepsilon = 0$ into the summations on lines (4.10) and (4.12), and comparing the resulting expressions with (4.8), we see that the integral terms on lines (4.9) and (4.11) should converge to zero and $S(t-t_0)g(t_0)$, respectively, as $\varepsilon \to 0$, in order to recover the integer-order case (formally, since we cannot expect the continuity of $\varepsilon \mapsto \mathfrak{D}^{k\pm\varepsilon}g(t_0)$ in general). The following proposition states when these convergences hold:

Proposition 4.7. *Let $t_0 \in \mathbb{R}$ and $g \in C_\mathrm{b}((-\infty,t_0];X)$ be given. If Assumption 2.1(i) is satisfied, then for all $t \in (t_0,\infty)$ it holds that*

$$\frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty \frac{\tau^{\varepsilon-1}}{\tau+1} S((t-t_0)(\tau+1))g(t_0-(t-t_0)\tau)\mathrm{d}\tau \to S(t-t_0)g(t_0)$$

as $\varepsilon \to 0$. If, in addition, Assumption 2.1(ii) is satisfied, then

$$\frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty \frac{\tau^{-\varepsilon}}{\tau+1} S((t-t_0)(\tau+1))g(t_0-(t-t_0)\tau)\mathrm{d}\tau \to 0 \tag{4.13}$$

as $\varepsilon \to 0$.

Proof. Fix $t_0 \in \mathbb{R}$ and $t \in (t_0,\infty)$. First we define the function $f_{t,t_0}\colon \mathbb{R} \to X$ by

$$f_{t,t_0}(r) := \begin{cases} S(t-t_0-r)g(t_0+r), & r \in (-\infty,0]; \\ S(t-t_0)g(t_0), & r \in (0,\infty), \end{cases}$$

which is bounded and continuous at $r = 0$ by the assumptions on $(S(t))_{t\ge 0}$ and g. Next, for any $\varepsilon \in (0,1)$ we define $\psi_{t,t_0,\varepsilon}\colon \mathbb{R} \to [0,\infty)$ by

$$\psi_{t,t_0,\varepsilon}(r) := \frac{(t-t_0)^{1-\varepsilon}\sin(\pi\varepsilon)}{\pi}(r+(t-t_0))_+^{-1} r_+^{\varepsilon-1}, \quad r \in \mathbb{R}.$$

Shifting the integration variable by $t-t_0$ and applying (4.6) with $s = 1-\varepsilon$ and $z = 0$, we find

$$\int_{\mathbb{R}} \psi_{t,t_0,\varepsilon}(r)\mathrm{d}r = \frac{(t-t_0)^{1-\varepsilon}\sin(\pi\varepsilon)}{\pi}\int_{t-t_0}^{\infty} r^{-1}(r-(t-t_0))^{\varepsilon-1}\mathrm{d}r = 1.$$

Moreover, we have for any $\delta > 0$:

$$\int_{\{|r|\geq\delta\}} |\psi_{t,t_0,\varepsilon}(r)|\mathrm{d}r = \frac{(t-t_0)^{1-\varepsilon}\sin(\pi\varepsilon)}{\pi}\int_{\delta}^{\infty}(r+t-t_0)^{-1}r^{\varepsilon-1}\mathrm{d}r$$
$$\leq \frac{(t-t_0)^{1-\varepsilon}\sin(\pi\varepsilon)}{\pi}\int_{\delta}^{\infty} r^{\varepsilon-2}\mathrm{d}r = \frac{(t-t_0)^{1-\varepsilon}\delta^{\varepsilon-1}\sin(\pi\varepsilon)}{\pi(1-\varepsilon)} \to \frac{(t-t_0)\cdot 0}{\pi\delta} = 0$$

as $\varepsilon \to 0$. Together, these observations show that the family $(\psi_{t,t_0,\varepsilon})_{\varepsilon\in(0,1)}$ forms an approximate identity as $\varepsilon \to 0$ in the sense of [9, Definition 1.2.15]. Since the change of variables $\sigma := (t-t_0)\tau$ yields

$$\frac{\sin(\pi\varepsilon)}{\pi}\int_0^{\infty}\frac{\tau^{\varepsilon-1}}{\tau+1}S((t-t_0)(\tau+1))g(t_0-(t-t_0)\tau)\mathrm{d}\tau = [\psi_{t,t_0,\varepsilon} * f_{t,t_0}](0),$$

the first assertion now follows by applying the obvious vector-valued generalization of [9, Theorem 1.2.19(2)], which gives

$$[\psi_{t,t_0,\varepsilon} * f_{t,t_0}](0) \to f_{t,t_0}(0) = S(t-t_0)g(t_0) \quad \text{as } \varepsilon \to 0.$$

For the second assertion, suppose that Assumption 2.1(ii) holds. By Proposition 4.3, the left-hand side of (4.13) is bounded above by $M_0\overline{\Gamma}(\varepsilon, w(t-t_0))\|g\|_\infty$ for all $t \in (t_0, \infty)$. Since $w(t-t_0) > 0$, this expression tends to zero as $\varepsilon \to 0$. □

The final corollary concerns the choice $g \equiv x \in \mathsf{D}(A^{(s-1)\vee 0})$ in Definition 4.2, in which case the solution can be expressed in terms of an operator-valued counterpart of the upper incomplete gamma function. Namely, for $\alpha \in (0,1)$ and $t \in (0,\infty)$, we use the Phillips calculus from Sect. 2.5 to define

$$\overline{\Gamma}(\alpha, tA) := [z \mapsto \overline{\Gamma}(\alpha, tz)](A) = \mathcal{L}\left[\frac{t^\alpha \sigma_+^{-1}(\sigma-t)_+^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\mathrm{d}\sigma\right](A) \in \mathscr{L}(X),$$

see (4.6). For $\alpha \in [1,\infty)$, such a Laplace transform representation is no longer available, so in this case we instead define $\overline{\Gamma}(\alpha, tA)$ by analogy with (4.7):

$$\overline{\Gamma}(\alpha, tA)x := \overline{\Gamma}(\{\alpha\}, tA)x + \sum_{k=1}^{\lfloor\alpha\rfloor}\frac{t^{k+\{\alpha\}-1}}{\Gamma(k+\{\alpha\})}A^{k+\{\alpha\}-1}S(t)x, \tag{4.14}$$

where $x \in \mathsf{D}(A^{\alpha-1})$. For $t = 0$ we set $\overline{\Gamma}(\alpha, 0A) := \mathrm{Id}_X$. Although $\overline{\Gamma}(\alpha, tA)$ is unbounded in general, under the additional Assumption 2.1(ii) we have, cf. [24, Chapter 2, Theorem 6.13(c)]. For all $\beta \in [0,\infty)$, there exists an $M_\beta \in [1,\infty)$ such that

$$\|A^\beta S(t)\|_{\mathscr{L}(X)} \leq M_\beta t^{-\beta}e^{-wt}, \quad \forall t \in (0,\infty). \tag{4.15}$$

Putting these observations together, we obtain the following formula for the solution with initial datum $g \equiv x$:

Corollary 4.8. *Suppose Assumption 2.1(i) holds. Let $s \in (0,\infty)\setminus\mathbb{N}$, $p \in [1,\infty]$ and $t_0 \in \mathbb{R}$. If $g \equiv x$ for some given $x \in \mathsf{D}(A^{(s-1)\vee 0})$ and $s \in (0,1)$ or Assumption 2.1(ii) holds, then the solution u to (4.1) from Definition 4.2 becomes*

$$u(t) = \overline{\Gamma}(s, (t-t_0)A)x, \quad \forall t \in (t_0,\infty). \tag{4.16}$$

If, in addition, $s \in (0,1)$ or Assumption 2.1(ii) is satisfied, then

$$\|u(t)\|_X \leq \left[M_0\overline{\Gamma}(\{s\}, w(t-t_0)) + e^{-w(t-t_0)} \sum_{k=1}^{\lfloor s\rfloor} \frac{M_{\{s\}+k-1}}{\Gamma(\{s\}+k)} \right] \|x\|_X, \tag{4.17}$$

where $w \in [0,\infty)$ and $M_0, M_{\{s\}}, \ldots, M_s \in [1,\infty)$ are as in equations (2.1) and (4.15).

Proof. The substitution $g \equiv x$ and the change of variables $\sigma := (t-t_0)(1+\tau)$ in the first term of (4.2) produces

$$\frac{\sin(\pi\{s\})}{\pi}(t-t_0)^{\{s\}} \int_{t-t_0}^{\infty} \sigma^{-1}(\sigma-(t-t_0))^{-\{s\}} S(\sigma) \mathrm{d}\sigma\, x,$$

which is equal to $\overline{\Gamma}(\{s\}, (t-t_0)A)x$ by (4.6).

Now suppose that $s \in (1,\infty)$ and let Assumption 2.1(ii) be satisfied. Then for any $\alpha \in [0,\infty)$ we have

$$\mathfrak{I}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} S(\tau) x \mathrm{d}\tau = A^{-\alpha}x$$

by (2.7), so that

$$\mathfrak{D}^\beta g(t) = (\partial_t + A)^{\lceil\beta\rceil} \mathfrak{I}^{\lceil\beta\rceil-\beta} g(t) = A^{\lceil\beta\rceil} A^{\beta-\lceil\beta\rceil} x = A^\beta x$$

for all $\beta \in [0, s-1]$, hence the remaining terms of (4.2) become

$$\sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{\{s\}+k-1}}{\Gamma(\{s\}+k)} S(t-t_0) A^{\{s\}+k-1} x,$$

proving (4.16) in view of (4.14). Estimate (4.17) follows from (2.1) (and (4.15)). □

4.2. Proof of the relation between mild solutions and L^p-solutions

The aim of this section is to prove Theorem 4.5. We will prove the integer-order and fractional-order cases separately in the following two subsections. Before proceeding to do so, we make a preliminary observation which applies to both cases:

If Assumption 2.1(ii) holds, and $u \in \mathsf{D}(\mathfrak{D}^s)$ is an L^p-solution to (4.1) for some $t_0 \in \mathbb{R}$, $p \in [1,\infty]$ and $s \in (0,\infty)$ satisfying (2.5), then $u \in C_{\mathrm{b}}(\mathbb{R}; X)$ by Corollary 3.4. Moreover, if $s \geq 1$, then we have $\mathfrak{D}^{s-1}u \in \mathsf{D}(\mathfrak{D}^1)$ by the definition (2.6), and thus $\mathfrak{D}^{s-1}u \in C_{\mathrm{b}}(\mathbb{R}; X)$ by applying Corollary 3.4 once more. Since $g \equiv u$ on $(-\infty, t_0]$, this shows that the continuity properties of g are as in Definition 4.2.

4.2.1. Integer-order case

Let us consider the case $s = n \in \mathbb{N}$, in which the operator $\mathfrak{D}^n = (\partial_t + A)^n$ is local in time. Note that the first-order case $s = 1$ was already treated in Sect. 2.3. The following proposition is the key ingredient in the proof of Theorem 4.5 for $s = n$.

Proposition 4.9. *Suppose that Assumption 2.1(ii) holds. Let $n \in \mathbb{N}$, $p \in [1, \infty]$, $t_0 \in \mathbb{R}$ and $u \in \mathsf{D}((\partial_t + A)^n)$. For all $t \in J := (t_0, \infty)$, we have*

$$\mathfrak{I}^n_{t_0}(\partial_t + A)^n u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t - t_0)^k}{k!} S(t - t_0)[(\partial_t + A)^k u](t_0). \tag{4.18}$$

Here, $\mathfrak{I}^s_{t_0} \in \mathscr{L}L^p(J; X)$ denotes the (Riemann–Liouville type) fractional parabolic integral, defined by

$$\mathfrak{I}^s_{t_0} u(t) := \frac{1}{\Gamma(s)} \int_{t_0}^t (t - \tau)^{s-1} S(t - \tau) u(\tau) \mathrm{d}\tau, \tag{4.19}$$

for all $u \in L^p(J; X)$ and almost every $t \in (t_0, \infty)$.

Proof. We use induction on $n \in \mathbb{N}$. For the base case $n = 1$, let us fix an arbitrary

$$u \in W^{1,p}(J; X) \cap L^p(J; \mathsf{D}(A)) \hookrightarrow C_\mathrm{b}(\overline{J}; X).$$

Since $u(t) \in \mathsf{D}(A)$ a.e., we find in particular that $u(t_0) \in \overline{\mathsf{D}(A)}$. Now the result follows by applying Proposition 2.4(a) $\Longrightarrow$ (b) with $f := u' + Au \in L^p(J; X)$.

Now suppose that the statement is true for some $n \in \mathbb{N}$. We present the argument for $t_0 = 0$, the other cases being analogous. Fix $u \in \mathsf{D}((\partial_t + A)^{n+1})$ and apply the induction hypothesis to the function $(\partial_t + A)u \in \mathsf{D}((\partial_t + A)^n)$, yielding

$$(\partial_t + A)u(t) = \mathfrak{I}^n_{t_0}(\partial_t + A)^{n+1} u(t) + \sum_{k=0}^{n-1} \frac{t^k}{k!} S(t)[(\partial_t + A)^{k+1} u](0)$$

for all $t \in \overline{J}$. Applying $\mathfrak{I}^1$ to both sides of the above equation, we can use the case $n = 1$ along with Propositions 2.6(c)–(d) to find

$$\begin{aligned} u(t) &= \mathfrak{I}^{n+1}_{t_0}(\partial_t + A)^{n+1} u(t) + \sum_{k=0}^{n-1} \frac{t^{k+1}}{(k+1)!} S(t)[(\partial_t + A)^{k+1} u](0) + S(t)u(0) \\ &= \mathfrak{I}^{n+1}_{t_0}(\partial_t + A)^{n+1} u(t) + \sum_{k=0}^{n} \frac{t^k}{k!} S(t)[(\partial_t + A)^k u](0). \end{aligned}$$

□

We can now prove the integer-order case of Theorem 4.5:

Proof of Theorem 4.5 *(for $s = n \in \mathbb{N}$).* Let u be an L^p-solution to (4.1). By Definition 4.2 and the observations in the beginning of Sect. 4.2, we have $u \in C_{\mathrm{b}}(\mathbb{R}; X)$, $u \equiv g$ on $(-\infty, t_0]$ and $(\partial_t + A)^n u = 0$ a.e. on (t_0, ∞). Applying $\mathfrak{I}^s_{t_0}$ to both sides of the latter and using Proposition 4.9 on the left-hand side, we find for all $t \in (t_0, \infty)$:

$$u(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0)[(\partial_t + A)^k u](t_0).$$

Note that the operators $(\partial_t + A)^k$ are local in time and that, in fact, we can choose to interpret ∂_t as a left derivative. Thus, since $u \equiv g$ on $(-\infty, t_0]$, we obtain (4.3). □

If $u \in \mathsf{D}((\partial_t + A)^k)$ is sufficiently regular, say if $u \in C^j(\overline{J}; \mathsf{D}(A^{k-j}))$ (j times continuously differentiable) for all $j \in \{0, \dots, k\}$, then we have the pointwise binomial expansion

$$[(\partial_t + A)^k u](t) = \sum_{j=0}^{k} \binom{k}{j} A^{k-j} u^{(j)}(t), \quad \forall t \in \overline{J},$$

where $u^{(j)}$ denotes the jth (classical) derivative of u. Substituting this into (4.18), using the definition of binomial coefficients, interchanging the order of summation and shifting the inner summation index yields

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{t^k}{k!} S(t)[(\partial_t + A)^k u](0) &= \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{t^k}{j!(k-j)!} S(t) A^{k-j} u^{(j)}(0) \\ &= \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{t^k}{j!(k-j)!} S(t) A^{k-j} u^{(j)}(0) = \sum_{j=0}^{n-1} \frac{t^j}{j!} \sum_{\ell=0}^{n-j-1} \frac{t^\ell}{\ell!} S(t) A^\ell u^{(j)}(0). \end{aligned}$$

Moreover, note that for $n \in \mathbb{N}$, $x \in \mathsf{D}(A^{n-1})$ and $t \in (0, \infty)$ we have

$$\overline{\Gamma}(n, tA)x = \sum_{k=0}^{n-1} \frac{t^k}{k!} S(t) A^k x,$$

cf. (4.7). Together, these observations imply that, in this situation, equation (4.18) takes on the following form:

$$\mathfrak{I}^n_{t_0} (\partial_t + A)^n u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} \overline{\Gamma}(n-k, (t-t_0)A) x_k. \tag{4.20}$$

4.2.2. Fractional-order case

Now we turn to the proof of Theorem 4.5 in the case that $s \in (0, \infty) \setminus \mathbb{N}$. It relies on the following result, which is the fractional-order analog to Proposition 4.9.

Theorem 4.10. *Suppose that Assumption 2.1(ii) is satisfied. Let $f \in L^p(-\infty, t_0; X)$ for some $p \in [1, \infty]$ and $t_0 \in \mathbb{R}$, and let $\widetilde{f} \in L^p(\mathbb{R}; X)$ denote its extension by zero to the whole of $\mathbb{R}$. Let $s \in (0, \infty)\setminus\mathbb{N}$ be such that (2.5) is satisfied. Then the following identity holds for all $t \in (t_0, \infty)$:*

$$\mathfrak{I}^s \widetilde{f}(t) = \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau + 1} S((t - t_0)(\tau + 1))\mathfrak{I}^s f(t_0 - (t - t_0)\tau)\mathrm{d}\tau + \sum_{k=1}^{\lfloor s \rfloor} \frac{(t - t_0)^{\{s\}+k-1}}{\Gamma(\{s\} + k)} S(t - t_0)\mathfrak{I}^{\lfloor s \rfloor - k+1} f(t_0). \tag{4.21}$$

Before proving this result, we show how it can indeed be used to finish the proof of Theorem 4.5:

Proof of Theorem 4.5 (for $s \in (0, \infty)\setminus\mathbb{N}$). Let $u \in \mathsf{D}(\mathfrak{D}^s)$ be an L^p-solution to (4.1). Applying $\mathfrak{D}^s$ to the second line of (4.1) yields $\mathfrak{D}^s u = \mathfrak{D}^s g$ a.e. on $(-\infty, t_0)$.

Combined with the first line of (4.1), i.e., $\mathfrak{D}^s u = 0$ a.e. on (t_0, ∞), we find

$$\mathfrak{D}^s u = \widetilde{f} \quad \text{a.e. on } \mathbb{R},$$

where $f := \mathfrak{D}^s g \in L^p(-\infty, t_0; X)$. Now we apply $\mathfrak{I}^s$ to both sides of this equation, and use Proposition 3.3(b) and Theorem 4.10 to the left-hand and right-hand sides, respectively. Together, this yields, for all $t \in (t_0, \infty)$,

$$u(t) = \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau + 1} S((t - t_0)(\tau + 1))\mathfrak{I}^s[\mathfrak{D}^s g](t_0 - (t - t_0)\tau)\mathrm{d}\tau + \sum_{k=1}^{\lfloor s \rfloor} \frac{(t - t_0)^{\{s\}+k-1}}{\Gamma(\{s\} + k)} S(t - t_0)\mathfrak{I}^{\lfloor s \rfloor - k+1}[\mathfrak{D}^s g](t_0).$$

In order to conclude that u satisfies equation (4.2), it remains to note that $\mathfrak{I}^s \mathfrak{D}^s g = g$ and $\mathfrak{I}^{\lfloor s \rfloor - k+1}\mathfrak{D}^s g = \mathfrak{D}^{s-(\lfloor s \rfloor - k+1)} g = \mathfrak{D}^{\{s\}+k-1} g$ for all $k \in \{1, \dots, \lfloor s \rfloor\}$. Indeed, the former follows from the natural analog of Proposition 3.3(b) for functions defined on $(-\infty, t_0]$; for the latter, let $m \in \mathbb{N}_0$ be such that $m \le \lfloor s \rfloor$, for which we have

$$\begin{aligned}\mathfrak{I}^m \mathfrak{D}^s g &= \mathfrak{I}^m \mathfrak{D}^{\lfloor s \rfloor} \mathfrak{I}^{\lfloor s \rfloor - s} g = \mathfrak{I}^m \mathfrak{D}^m \mathfrak{D}^{\lfloor s \rfloor - m} \mathfrak{I}^{\lfloor s \rfloor - s} g = \mathfrak{D}^{\lfloor s \rfloor - m} \mathfrak{I}^{\lfloor s \rfloor - m-(s-m)} g \\ &= \mathfrak{D}^{\lfloor s-m \rfloor} \mathfrak{I}^{\lfloor s-m \rfloor - (s-m)} g = \mathfrak{D}^{s-m} g.\end{aligned}$$

Here, we used the definition of $\mathfrak{D}^s$, the additivity of integer powers of $\mathfrak{D}$, and the aforementioned analog to Proposition 3.3(b). This completes the proof that u is a mild solution in the sense of Definition 4.2. □

The proof of Theorem 4.10 involves expressing an integral in terms of fractional binomial coefficients, given by [23, Equations (1.2.6), (5.2.4) and (5.2.5)]:

$$\binom{\alpha}{k} := \frac{1}{k!}\prod_{\ell=0}^{k-1}(\alpha - \ell) = \frac{\Gamma(\alpha + 1)}{k!\,\Gamma(\alpha - k + 1)}, \quad \alpha \in (0, \infty),\ k \in \mathbb{N}_0. \tag{4.22}$$

Since the author is not aware of a direct reference for the following integral identity, a proof is presented below for the sake of self-containedness.

Lemma 4.11. *For $\alpha \in (0,1)$, $a, b \in (0,\infty)$ and $n \in \mathbb{N}_0$ we have*

$$\frac{\sin(\pi\alpha)}{\pi}\int_0^{a/b}\frac{\tau^{-\alpha}(a-b\tau)^{\alpha+n-1}}{\tau+1}\mathrm{d}\tau = (a+b)^{\alpha+n-1} - \sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}a^{n-k}b^{k+\alpha-1}. \tag{4.23}$$

Proof. By the change of variables $\sigma := \frac{b}{a}\tau$ and [23, Equation (5.5.3)], the validity of the identity (4.23) is equivalent to that of

$$\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha+n-1}}{\sigma+\frac{b}{a}}\mathrm{d}\sigma = a^{1-n}b^{-\alpha}(a+b)^{\alpha+n-1} - \sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k-1}. \tag{4.24}$$

We will verify this identity using induction on $n \in \mathbb{N}_0$. The base case $n = 0$ is a consequence of [23, Equations (5.12.4) and (5.12.1)]:

$$\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha-1}}{\sigma+\frac{b}{a}}\mathrm{d}\sigma = \left(1+\tfrac{b}{a}\right)^{\alpha-1}\left(\tfrac{b}{a}\right)^{-\alpha} = ab^{-\alpha}(a+b)^{\alpha-1}.$$

Now suppose that (4.24) holds for a given $n \in \mathbb{N}_0$. In order to establish the identity for $n+1$, we write $1-\sigma = 1+\frac{b}{a}-(\sigma+\frac{b}{a})$ and apply the induction hypothesis and [23, Equation (5.12.1)], respectively, to the resulting two integrals:

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha+n}}{\sigma+\frac{b}{a}}\mathrm{d}\sigma\\ &\quad= \frac{1+\frac{b}{a}}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha+n-1}}{\sigma+\frac{b}{a}}\mathrm{d}\sigma - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\sigma^{-\alpha}(1-\sigma)^{\alpha+n-1}\mathrm{d}\sigma\\ &\quad= a^{-n}b^{-\alpha}(a+b)^{\alpha+n} - \left(1+\frac{b}{a}\right)\sum_{k=1}^{n}\binom{\alpha+n-1}{n-j}\left(\frac{b}{a}\right)^{j-1} - \frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)}.\end{aligned}$$

For the latter two terms, we have

$$\begin{aligned}\left(1+\frac{b}{a}\right)\sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k-1}&+\frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)}\\&=\sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k-1}+\sum_{k=0}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k}\\&=\sum_{k=1}^{n}\left[\binom{\alpha+n-1}{n-k}+\binom{\alpha+n-1}{n-k+1}\right]\left(\frac{b}{a}\right)^{k-1}+\left(\frac{b}{a}\right)^{n}\\&=\sum_{k=1}^{n+1}\binom{\alpha+n}{n+1-k}\left(\frac{b}{a}\right)^{k-1}.\end{aligned}$$

Indeed, to obtain the second line we note that $\frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)}=\binom{\alpha+n-1}{n}$ by (4.22), which is the term $k=0$ of the second summation on the second line; shifting its index of summation and splitting off the last term yields the first expression on the third line. The final step uses [23, Equation (1.2.7)] and the fact that $(\frac{b}{a})^n$ corresponds to the term $k=n+1$ in the desired formula. Putting the previous two displays together proves the induction step and thereby the lemma. □

Remark 4.12. An alternative way to derive (4.23) is by noting that the integral can be expressed in terms of a hypergeometric function [23, Equation (15.6.1)] to which one can apply the transformation formula [23, Equation (15.8.2)]. This results in a difference of two hypergeometric functions, whose definitions can be written out to, respectively, yield an infinite and a finite sum: The former is the fractional binomial expansion of $(a+b)^{\alpha+n-1}$ and the latter consists of its first n terms, and together this gives (4.23). In particular, we note that (4.23) is formally equal to the tail of a fractional binomial series. The proof of Lemma 4.11 is more direct and avoids the need to address the convergence of an infinite series.

Proof of Theorem 4.10. Fixing $t\in(t_0,\infty)$, the semigroup law implies

$$S(t-r)=S((t-t_0)(\tau+1))S(t_0-(t-t_0)\tau-r) \tag{4.25}$$

for all $\tau\in(0,\infty)$ and $r\in(-\infty,t_0-(t-t_0)\tau)$. This identity, followed by (2.1), Hölder's inequality and equation (4.5) yields

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^\infty\int_{-\infty}^{t_0-(t-t_0)\tau}\left\|\frac{\tau^{-\{s\}}(t_0-(t-t_0)\tau-r)^{\gamma-1}}{\tau+1}S(t-r)f(r)\right\|_X\mathrm{d}r\,\mathrm{d}\tau\\&\le M_0\int_0^\infty\frac{\tau^{-\{s\}}}{\tau+1}e^{-w(t-t_0)(\tau+1)}\int_{-\infty}^{t_0-(t-t_0)\tau}\|k_s(t_0-(t-t_0)\tau-r)f(r)\|_X\mathrm{d}r\,\mathrm{d}\tau\\&\le\frac{M_0\pi}{\sin(\pi\{s\})}\overline{\Gamma}(\{s\},w(t-t_0))\|k_s\|_{L^{p'}(0,\infty;\mathscr{L}(X))}\|f\|_{L^p(\mathbb{R};X)}<\infty.\end{aligned}$$

This justifies the use of Fubini's theorem in the following:

$$\begin{aligned}
&\int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1))\mathfrak{I}^s f(t_0-(t-t_0)\tau)\mathrm{d}\tau \\
&\quad= \frac{1}{\Gamma(s)} \int_0^\infty \int_{-\infty}^{t_0-(t-t_0)\tau} \frac{\tau^{-\{s\}}(t_0-(t-t_0)\tau-r)^{s-1}}{\tau+1} S(t-r) f(r)\mathrm{d}r\mathrm{d}\tau \\
&\quad= \frac{1}{\Gamma(s)} \int_{-\infty}^{t_0} \biggl[\int_0^{\frac{t_0-r}{t-t_0}} \frac{\tau^{-\{s\}}(t_0-r-(t-t_0)\tau)^{s-1}}{\tau+1}\mathrm{d}\tau\biggr] S(t-r) f(r)\mathrm{d}r,
\end{aligned}$$

where we used (4.25) once more. Lemma 4.11 and equation (4.22) produce

$$\begin{aligned}
&\frac{\sin(\pi\{s\})}{\pi} \int_0^{\frac{t_0-r}{t-t_0}} \frac{\tau^{-\{s\}}(t_0-r-(t-t_0)\tau)^{\{s\}+\lfloor s\rfloor-1}}{\tau+1}\mathrm{d}\tau \\
&\quad= (t-r)^{s-1} - \sum_{k=1}^{\lfloor s\rfloor} \frac{\Gamma(s)}{(\lfloor s\rfloor-k)!\,\Gamma(\{s\}+k)} (t_0-r)^{\lfloor s\rfloor-k}(t-t_0)^{k+\{s\}-1}.
\end{aligned}$$

The previous two displays and the identity $S(t-r) = S(t-t_0)S(t_0-r)$ yield

$$\begin{aligned}
&\frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1))\mathfrak{I}^s f(t_0-(t-t_0)\tau)\mathrm{d}\tau \\
&= -\sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{\{s\}+k-1}S(t-t_0)}{\Gamma(\{s\}+k)} \frac{1}{(\lfloor s\rfloor-k)!} \int_{-\infty}^{t_0} (t_0-r)^{\lfloor s\rfloor-k} S(t_0-r) f(r)\mathrm{d}r \\
&\quad+ \frac{1}{\Gamma(s)} \int_{-\infty}^{t_0} (t-r)^{s-1} S(t-r) f(r)\mathrm{d}r \\
&= -\sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{\{s\}+k-1}S(t-t_0)}{\Gamma(\{s\}+k)} \mathfrak{I}^{\lfloor s\rfloor-k+1} f(t_0) + \mathfrak{I}^s \widetilde{f}(t)
\end{aligned}$$

which is precisely (4.21). The final assertion follows from Proposition 3.3(b). □

5. Comparison to Riemann–Liouville and Caputo Cauchy problems

In this section, we compare the Dirichlet problem (4.1) to fractional-order abstract Cauchy problems of the form

$$(\partial_t + A)^s u(t) = 0, \quad t \in (t_0, \infty),$$

augmented with initial conditions which depend on the interpretation of the abstract space–time operator $(\partial_t + A)^s$, acting on functions $u\colon J \to X$ with $J = (t_0, \infty)$ for $t_0 \in \mathbb{R}$ (instead of $J = \mathbb{R}$ as in the previous sections). More precisely, we will interpret $(\partial_t + A)^s$ as a Riemann–Liouville or Caputo type fractional parabolic derivative, respectively, on $L^p(J;X)$ for $p \in [1,\infty]$, and determine the corresponding initial conditions and mild solution formulae (see Definitions 5.2 and 5.3).

For fractional time derivatives ∂_t^s, i.e., the case $A=0$, the resulting solution concepts are well-known and commonly studied (see for instance [13, Chapter 3]). In this case, the analog to (2.4) has less favorable mapping properties [13, Section 2.3] and the well-posedness of (1.5) is less clear. We will show that, as in the case $A=0$, the lowest-order term of the solution to the Riemann–Liouville type initial value problem has a singularity at t_0 in general, whereas the Caputo initial value problem yields the same solution for any two $s_1, s_2 \in (n, n+1)$ for $n \in \mathbb{N}_0$. In contrast, the solution from Definition 4.2 is continuous at t_0 under mild assumptions on g or $(S(t))_{t\geq 0}$ and changes for all choices of $s \in (0,\infty)$.

Firstly, let us recall the Riemann–Liouville type fractional parabolic integral $\mathfrak{I}_{t_0}^s$ on $L^p(J;X)$ defined by (4.19). Then, the Riemann–Liouville and Caputo type fractional parabolic *derivatives* are, respectively, defined by

$$\mathfrak{D}_{\mathrm{RL}}^s := (\partial_t + A)^{\lceil s\rceil}\mathfrak{I}_{t_0}^{\lceil s\rceil - s} \quad \text{and} \quad \mathfrak{D}_{\mathrm{C}}^s := \mathfrak{I}_{t_0}^{\lceil s\rceil - s}(\partial_t + A)^{\lceil s\rceil}$$

on their maximal domains. In order to derive mild solution formulae for L^p-solutions to the equations $\mathfrak{D}_{\mathrm{RL}}^s u = 0$ and $\mathfrak{D}_{\mathrm{C}}^s u = 0$, we proceed analogously to [13, Chapter 3] and express $\mathfrak{I}_{t_0}^s\mathfrak{D}_{\mathrm{RL}}^s u$ and $\mathfrak{I}_{t_0}^s\mathfrak{D}_{\mathrm{C}}^s u$ in terms of initial data from u (compare with Proposition 4.9 and Theorem 4.10), so that applying $\mathfrak{I}_{t_0}^s$ on both sides of the equations motivates the definitions. The integer-order case $s = n \in \mathbb{N}$, where

$$\mathfrak{D}^n = \mathfrak{D}_{\mathrm{RL}}^n = \mathfrak{D}_{\mathrm{C}}^n = (\partial_t + A)^n,$$

was treated in Sect. 4.2.1. From these results, we derive the following proposition regarding $\mathfrak{D}_{\mathrm{RL}}^s$ and $\mathfrak{D}_{\mathrm{C}}^s$ for fractional $s \in (0,\infty)\backslash\mathbb{N}$:

Proposition 5.1. *Let Assumption 2.1(i) be satisfied. If $s \in (0,\infty)\setminus\mathbb{N}$, $t_0 \in \mathbb{R}$ and $p \in [1,\infty]$ and $u \in \mathsf{D}(\mathfrak{D}_{\mathrm{RL}}^s)$, then for almost all $t \in J := (t_0,\infty)$:*

$$\begin{aligned}\mathfrak{I}_{t_0}^s\mathfrak{D}_{\mathrm{RL}}^s u(t) = u(t) &- \frac{(t-t_0)^{\{s\}-1}}{\Gamma(\{s\})}S(t-t_0)\mathfrak{I}_{\mathrm{RL}}^{1-\{s\}}u(t_0)\\ &- \sum_{k=1}^{\lfloor s\rfloor}\frac{(t-t_0)^{k+\{s\}-1}}{\Gamma(k+\{s\})}S(t-t_0)\mathfrak{D}_{\mathrm{RL}}^{k+\{s\}-1}u(t_0).\end{aligned}$$

If $u \in \mathsf{D}(\mathfrak{D}_{\mathrm{C}}^s)$ is such that $u \in C^j(\overline{J};\mathsf{D}(A^{n-1-j}))$ for all $j \in \{0,\ldots,n-1\}$, then we have for almost all $t \in J := (t_0,\infty)$:

$$\mathfrak{I}_{t_0}^s\mathfrak{D}_{\mathrm{C}}^s u(t) = u(t) - \sum_{k=0}^{\lfloor s\rfloor}\frac{(t-t_0)^k}{k!}\overline{\Gamma}(\lceil s\rceil - k, (t-t_0)A)u^{(k)}(t_0) \quad \textit{a.e.}$$

Proof. For the sake of notational convenience we only present the case $t_0 = 0$. The definition of $\mathfrak{D}_{\mathrm{RL}}^s$, along with Propositions 2.6(c)-(d) and 4.9, yields

$$
\begin{aligned}
\mathfrak{I}_0^{\lceil s\rceil-s}\mathfrak{I}_0^s\mathfrak{D}_{\mathrm{RL}}^s u &= \mathfrak{I}_0^{\lceil s\rceil}\mathfrak{D}_{\mathrm{RL}}^s u = \mathfrak{I}_0^{\lceil s\rceil}(\partial_t + A)^{\lceil s\rceil}\mathfrak{I}_0^{\lceil s\rceil-s} u \\
&= \mathfrak{I}_0^{\lceil s\rceil-s} u - \sum_{k=0}^{n-1} \frac{(\,\cdot\,)^k}{k!} S(\,\cdot\,)[(\partial_t + A)^k \mathfrak{I}_0^{\lceil s\rceil-s} u](0) \\
&= \mathfrak{I}_0^{\lceil s\rceil-s}\left[u - \sum_{k=0}^{n-1} \frac{(\,\cdot\,)^{k-\lceil s\rceil+s} S(\,\cdot\,)}{\Gamma(k+s-\lceil s\rceil+1)} [(\partial_t + A)^k \mathfrak{I}_0^{\lceil s\rceil-s} u](0) \right]
\end{aligned}
$$

for any $u \in \mathsf{D}(\mathfrak{D}_{\mathrm{RL}}^s)$. The first assertion then follows from Proposition 3.3 and the injectivity of $\mathfrak{I}_0^{\lceil s\rceil-s}$.

If $u \in \mathsf{D}(\mathfrak{D}_{\mathrm{C}}^s)$, combining the definition with Proposition 2.6(c) produces

$$
\mathfrak{I}_0^s\mathfrak{D}_{\mathrm{C}}^s u = \mathfrak{I}_0^s\mathfrak{I}_0^{\lceil s\rceil-s}(\partial_t + A)^{\lceil s\rceil} u = \mathfrak{I}_0^{\lceil s\rceil}(\partial_t + A)^{\lceil s\rceil} u,
$$

so that the result follows from Proposition 4.9 and the discussion below it, in particular equation (4.20). □

Note that $\mathfrak{I}_0^{1-\{s\}}u$ need not vanish at $t_0 = 0$. Indeed, even if it is continuous, it may not satisfy (4.19) pointwise, as evidenced by the example $u := k_{\{s\}} \otimes x$ for $p \in [1, \frac{1}{\{s\}-1})$ and $x \in \overline{\mathsf{D}(A)} \setminus \{0\}$ (see Proposition 2.6(d)).

Proposition 5.1 motivates the following definition of the Riemann–Liouville fractional abstract Cauchy type problem and its corresponding solution.

Definition 5.2. Let Assumption 2.1(i) be satisfied. For any $s \in (0, \infty)\setminus\mathbb{N}$ and $t_0 \in \mathbb{R}$, the mild solution to the Riemann–Liouville abstract Cauchy type problem

$$
\begin{cases}
\mathfrak{D}_{\mathrm{RL}}^s u(t) = 0, & t \in J := (t_0, \infty), \\
\mathfrak{I}_{t_0}^{1-\{s\}} u(t_0) = x_0 \in X, & \\
\mathfrak{D}_{\mathrm{RL}}^{k+\{s\}-1} u(t_0) = x_k \in \overline{\mathsf{D}(A)}, & k \in \{1, \ldots, \lfloor s\rfloor\},
\end{cases}
\tag{5.1}
$$

is the function $u_{\mathrm{RL}} \in C(J; X)$ defined by

$$
u_{\mathrm{RL}}(t) := \sum_{k=0}^{\lfloor s\rfloor} \frac{(t-t_0)^{k+\{s\}-1}}{\Gamma(k+\{s\})} S(t-t_0)x_k, \quad t \in J. \tag{5.2}
$$

Compared with Definition 4.2, we first note that the terms $k \in \{1, \ldots, \lfloor s\rfloor\}$ in (5.2) are almost identical to those of (4.2), up to the difference between taking Riemann–Liouville type fractional parabolic derivatives of the function u defined on J and Weyl type derivatives of g defined on $\mathbb{R} \setminus J$. The remaining term, on the other hand, differs significantly. In (5.1), we see that x_0 is the prescribed value of $\mathfrak{I}_{t_0}^{1-\{s\}}u$ at t_0 and u_{RL} is continuous at t_0 if and only if $x_0 = 0$, in view of the singularity occurring there for $x_0 \neq 0$. In contrast, the solution to (4.1) given by Definition 4.2 is bounded by Proposition 4.3, does in fact prescribe the value $u(t_0) = g(t_0)$ and is continuous on $\mathbb{R}$ under some further regularity assumptions.

The following definition of a Caputo type initial value problem and corresponding solution can also be derived from Proposition 5.1:

Definition 5.3. Let Assumption 2.1(i) be satisfied. For any $s \in (0, \infty)\setminus\mathbb{N}$ and $t_0 \in \mathbb{R}$, the mild solution to the Caputo abstract Cauchy problem

$$\begin{cases} \mathfrak{D}^s_{\mathrm{C}} u(t) = 0, & t \in J := (t_0, \infty), \\ u^{(k)}(t_0) = x_k \in \mathsf{D}(A^{\lfloor s \rfloor - k}), & k \in \{0, \ldots, \lfloor s \rfloor\}. \end{cases} \tag{5.3}$$

is the function $u_{\mathrm{C}} \in C(J; X)$ defined by

$$u_{\mathrm{C}}(t) := \sum_{k=0}^{\lfloor s \rfloor} \frac{(t - t_0)^k}{k!} \overline{\Gamma}(\lceil s \rceil - k, (t - t_0)A)x_k, \quad t \in J.$$

Note that this definition has the same form as the integer-order abstract Cauchy problem from Definition 4.2, i.e., formula (4.3). Analogously, for sufficiently regular x_k or $(S(t))_{t\geq 0}$, this solution allows for the specification of the value of $u_{\mathrm{C}}(t_0)$. However, in contrast to the solution in the sense of Definition 4.2, we observe that the form of u_{C} only changes "discretely in s," i.e., the solutions for any two $s_1, s_2 \in (n, n+1)$, $n \in \mathbb{N}_0$ are given by the same formula.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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