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Applying optimization algorithms in different types of optimization problems

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Introduction

This is the `2024-2025`-version of the book. Go to /CME4501 to view the most recent version of this book, or adapt the year in /CME4501/2024 to the year when you took the course.

This book focuses on how to apply optimization algorithms to specific optimization problems. It does so using python, scipy and pymoo. It contains the preparations for the practical sessions in CME4501 Engineering Systems Optimization. All other information about this course is available on Brightspace.

How to use this interactive book

Contents

This book focuses on how to apply optimization algorithms to specific optimization problems. It contains the preparations for the practical sessions in CME4501 Engineering Systems Optimization. All other information about this course is available on Brightspace.

Interactive features

This book includes interactive features!

- Interactive quizzes: Some page include interactive 'H5p' quizzes. This allows you to check your understanding. Any interactions you have with this system are not stored, unless you login into H5p.
- Python interactivity: Click 🚀 –> **Live Code** on the top right corner of interactive page to start up a python-kernel in your browser! Any interactions you do here are not stored. You can also download the page as a notebook to apply the content on your own computer.

Spot a mistake?

If you spot any mistakes, you can click on  –>  , login with a GitHub account and report your issue. It'll be solved soon!

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If you'd like to make this TeachBook more personal by adding (private or public) annotations I can recommend the Hypothesis extension. This is only for your own use, I won't monitor public post on this platform.

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This is the `2024-2025`-version of the book. Go to /CME4501 to view the most recent version of this book, or adapt the year in /CME4501/2024 to the year when you took the course.

Contact information

Contents

This module is taught by Maria Nogal and Tom van Woudenberg. The practical sessions are taught by Tom. Please contact Tom if you've any questions, feedback or when you've personal circumstances which we should know.

Tom van Woudenberg

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- T.R.vanWoudenberg@tudelft.nl

Python preparations

In this book you can use the interactive python functionality to run the code in your browser. However, you'll need to install Python to be able to do the analysis on your own computer. The following chapter will guide you through the installation, show you the basics of Jupyter Notebooks and Python, and show you a template of how we'll code optimization problems.

Python installation instruction

Contents

Python is a popular language for research computing, and great for general-purpose programming as well. Installing all of its research packages individually can be a bit difficult, so you're recommended to use Anaconda, an all-in-one installer. To make sure you're on the same versions you can set up an 'environment'.

Installation Anaconda

We will teach Python using the Jupyter Notebook, a programming environment that runs in a web browser (Jupyter Notebook will be installed by Anaconda). For this to work you will need a reasonably up-to-date browser. The current versions of the Chrome, Safari and Firefox browsers are all supported (some older browsers, including Internet Explorer version 9 and below, are not). Notebooks can be run after you install Python and the appropriate packages.

Installation on Windows | **Installation on MAC** | **Installation on Linux**

1. https://www.anaconda.com/products/individual#download-section with your web browser.
2. Download the Anaconda for Windows installer with Python 3. (If you are not sure which version to choose, you probably want the 64-bit Graphical Installer Anaconda3-…-Windows-x86_64.exe)
3. Install Python 3 by running the Anaconda Installer, using all of the defaults for installation except make sure to check Add Anaconda to my PATH environment variable.

Source: Carpentries [2023]

Exercise 1

Set up the environment

Now we will make sure you have an environment to get started in this course. This year we will use Python 3.11, so we are going to make sure that's installed and ready to go. Other Python packages that are not included in the base installation will be installed in your environment when we need to use them, to make sure we know exactly what is going on when working with code in this course.

The following steps will create an Anaconda environment called `optimization` and install Python version 3.11. Even if you already have Python 3.11, it is still good practice to create a dedicated Anaconda environment for each of your major projects. Please make sure you have upgraded Anaconda before proceeding with these steps:

1. Open Anaconda prompt from the start menu

2. Execute: `conda create -n optimization python=3.11 numpy matplotlib scipy pymoo` (this may take several minutes)
3. Activate: `conda activate optimization`
4. Check: you should now see `optimization` displayed somewhere in the prompt between parenthesis, like this: `(optimization)`

One important reminder: throughout the course you should be using the optimization environment every time you use Python, except when instructed otherwise. All you have to do is remember to use the command `conda activate optimization` prior to opening your files, or select the new environment in the Anaconda Navigator prior to opening any python-program.

Add other packages

If you want to install additional packages, you can do so by using `pip install <package name>`. Make sure you do this from within your activated environment.

Source: MUDE [2023]

Jupyter Notebook and Python basics

Contents

After installing Anaconda and setting up the environment, start the Navigator and select the `optimization environment`. This can be done on the top of the screen at `All application` on `base (root)`: change `base (root)` to `optimization`. Or in the left tab under `Environments`, click `optimization.` Then, launch the Jupyter Notebook from the list of applications under `home`, which will open the Jupyter Notebook dashboard in your webbrowser (starting in your Documents folder on Windows and your home directory on a Mac):

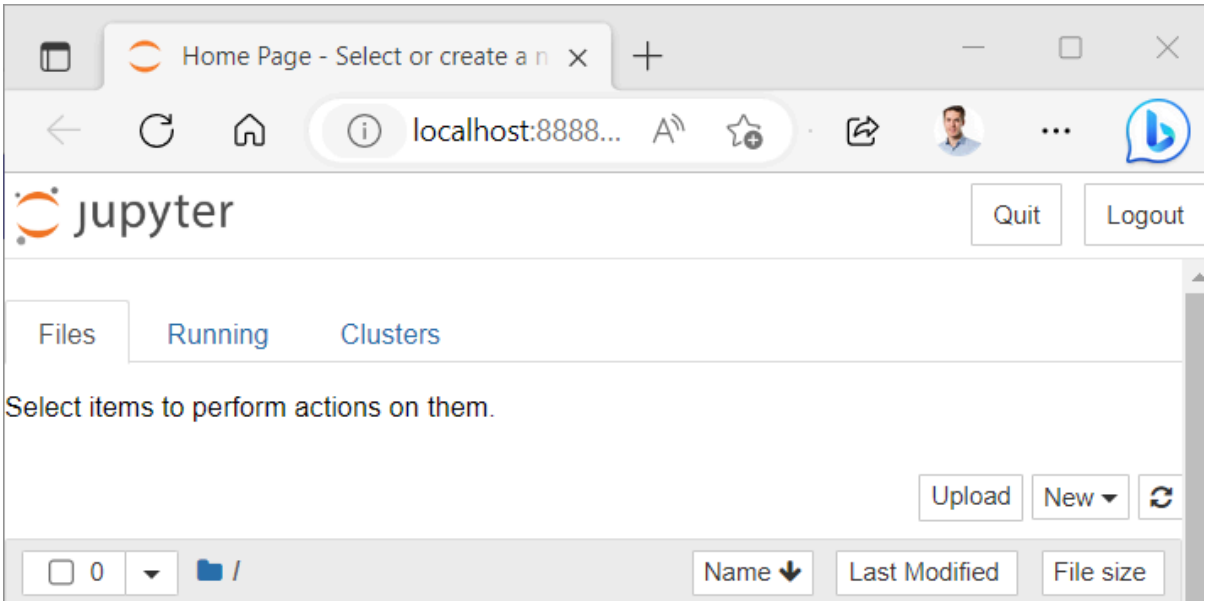


Fig. 1 Jupyter notebook start page

Use the dashboard to surf to the directory where you stored your Notebooks (you can only surf to lower directories not higher directories) and click on the one you want to open. Launch a new notebook by clicking on the “New” button and then selecting “Python 3”:

Exercise 2

Working in a Jupyter Notebook

Notebooks consist of cells which can contain code or text/figures (in markdown language). You can choose this cell type for each cell:

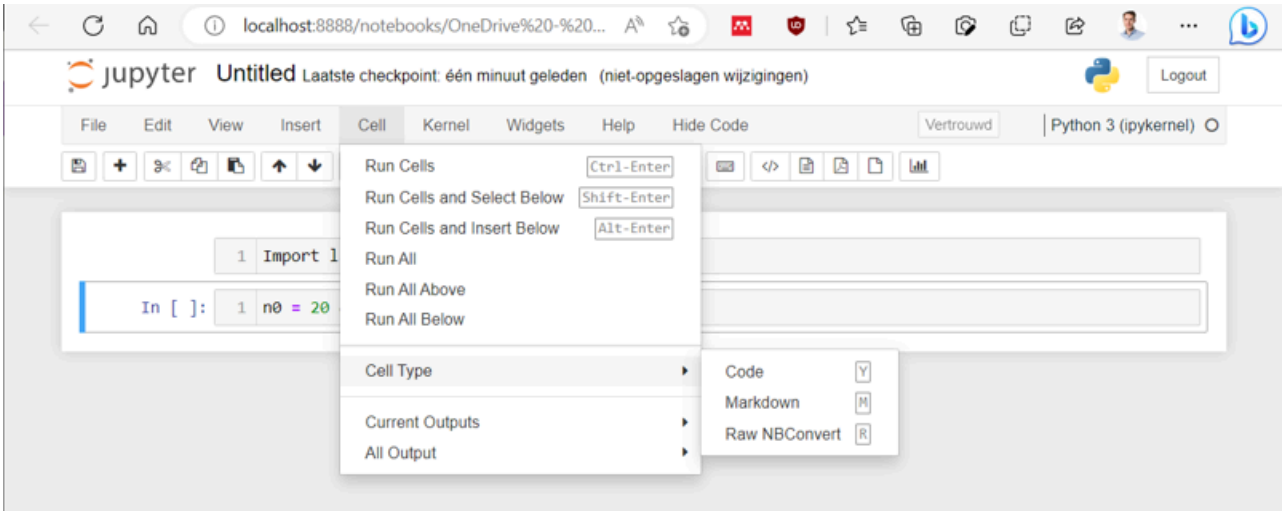


Fig. 2 Jupyter Notebook cell types

Both code and text cells can be run by clicking `execute` or pressing `shift + enter`. Code cells run, resulting in a counter between the square brackets, while text cells are just rendered:

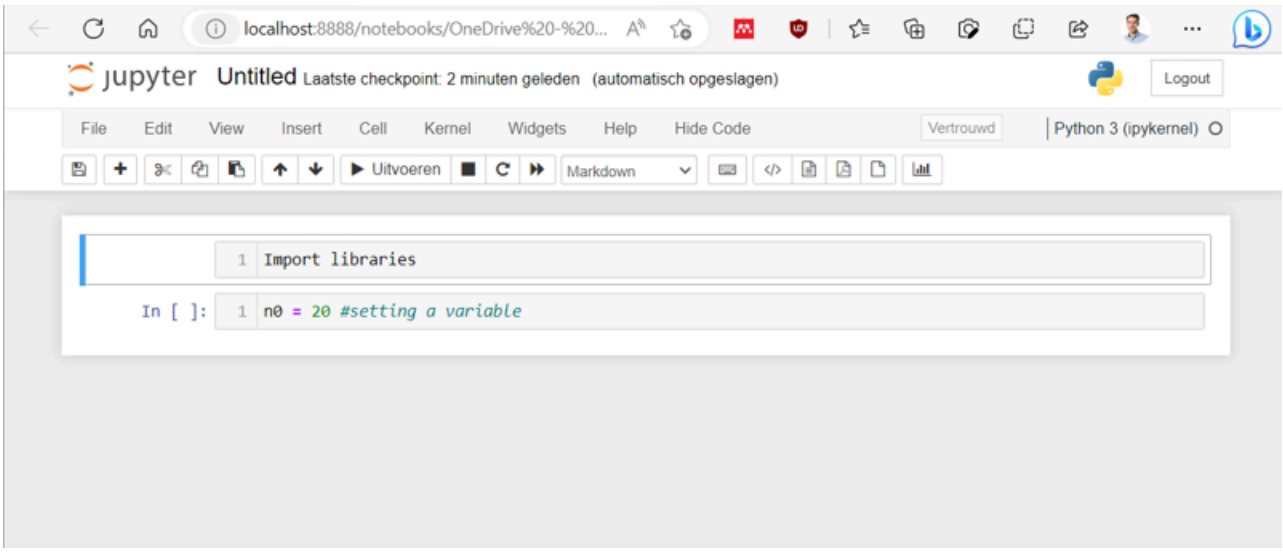


Fig. 3 Before running cells

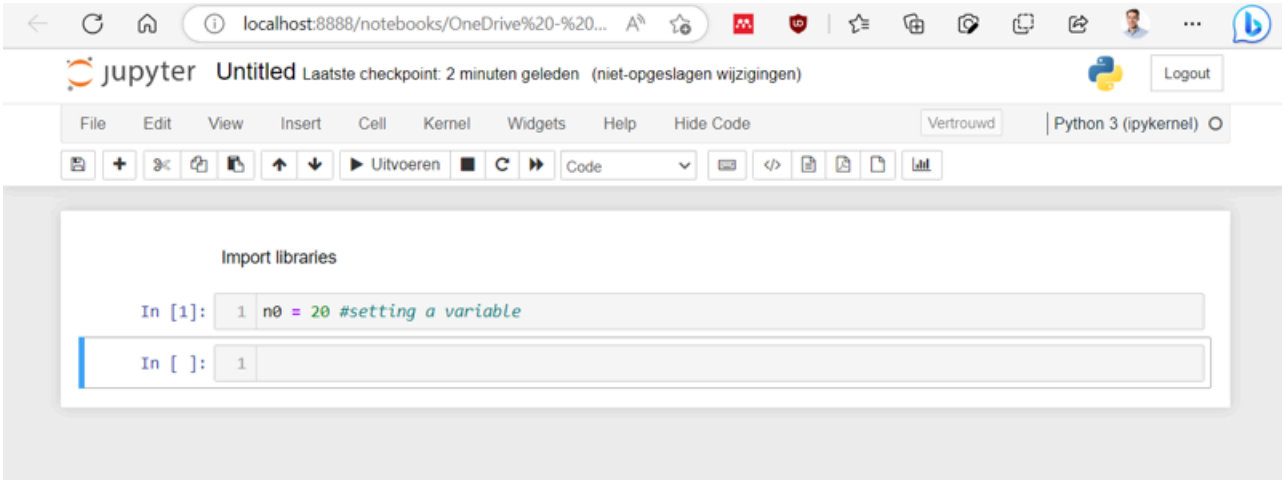


Fig. 4 After running cells

Exercise 3

Tutorial Python

If you've no experience in using Python, please work through this material on your own. Specifically, for this course you'll need:

1. Variables, operators and functions.
2. Modules, conditions, data structures and loops
3. Numpy
4. Matplotlib
5. Errors

Template for optimization problems in Jupyter Notebooks

Contents

For this course, our Jupyter Notebooks follow the same structure for all problems by defining the following sections:

- Import libraries
- Define variables
- Define objective function
- Define constrain function(s) if needed
- Solve the problem
- Postprocess results if needed

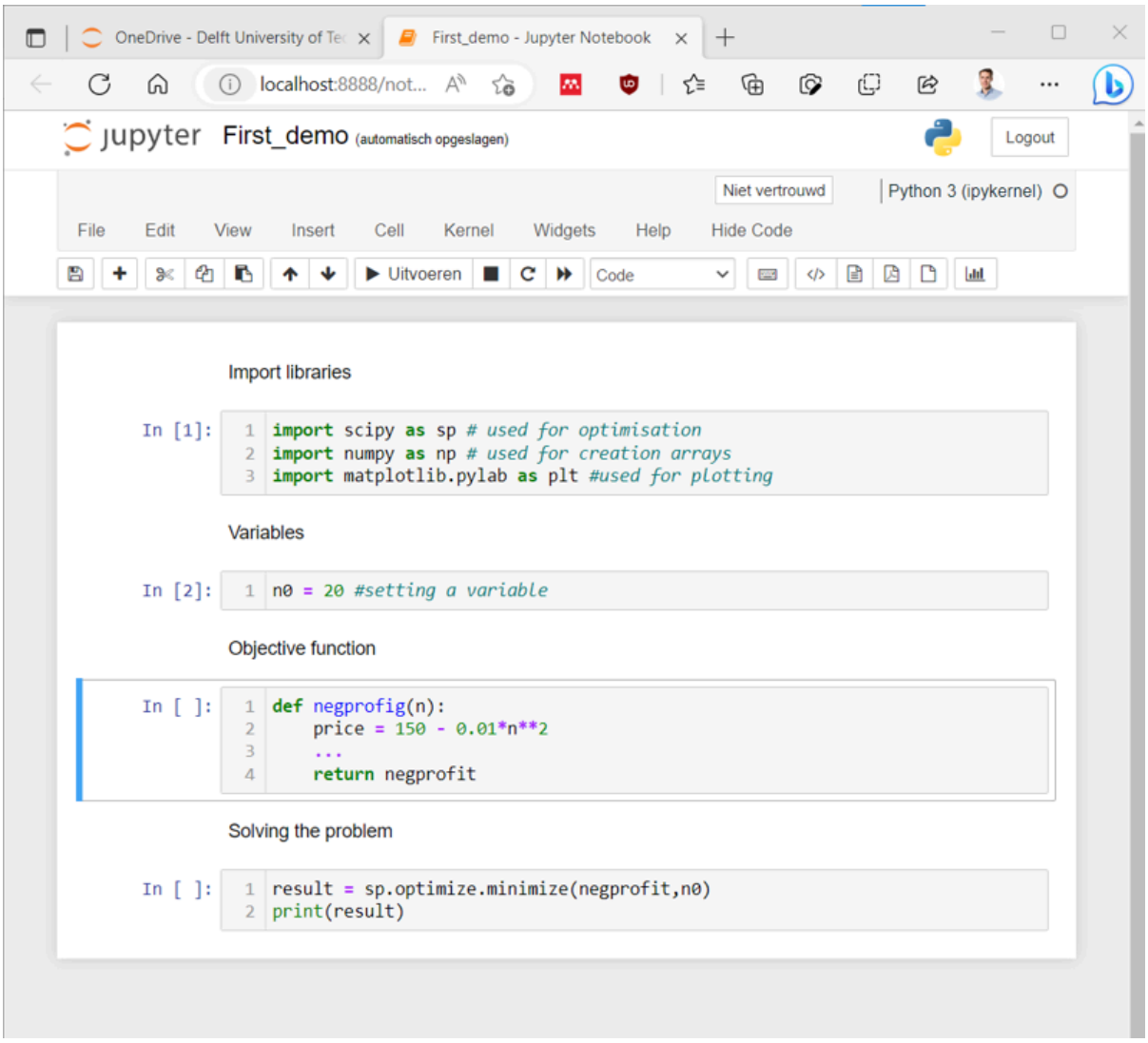


Fig. 5 General layout Jupyter Notebooks in this course

The structure is shown for an example which will be treated in-depth later.

Import libraries

Python cannot do optimization on its own, therefore we make use of separate packages which are installed with Anaconda:

- `scipy` used for optimization
- `numpy` used for matrix algebra
- `matplotlib` used for plotting

Libraries are imported with `import ... as ...` which directly abbreviates the packages for later use.

```
import scipy as sp
import numpy as np
import matplotlib.pylab as plt
```

Exercise 4

Define variables

In this part of the notebook we can store all our variables in the form of integers, floats and arrays.

```
n0 = 20
```

Exercise 5

Define objective function

In this part we define our constraint function as a `callable` with `def ...: ... return ...` These functions can contain multiple lines of calculation but should return one values which is minimized.

```
def negprofit(n):
    price = 150 - 0.01 * n**2
    revenues = price * n
    totalcost = 75 * n
    profit = revenues - totalcost
    return -profit
```

Exercise 6

Define constraint function

Similar to the objective function, the constrain function(s) can be defined, which may return multiple values

Solve the problem

Making using of the `scipy`-library, the problem is solved. Using `print`, the result is shown

```
result = sp.optimize.minimize(negprofit,n0)
print(result)
```

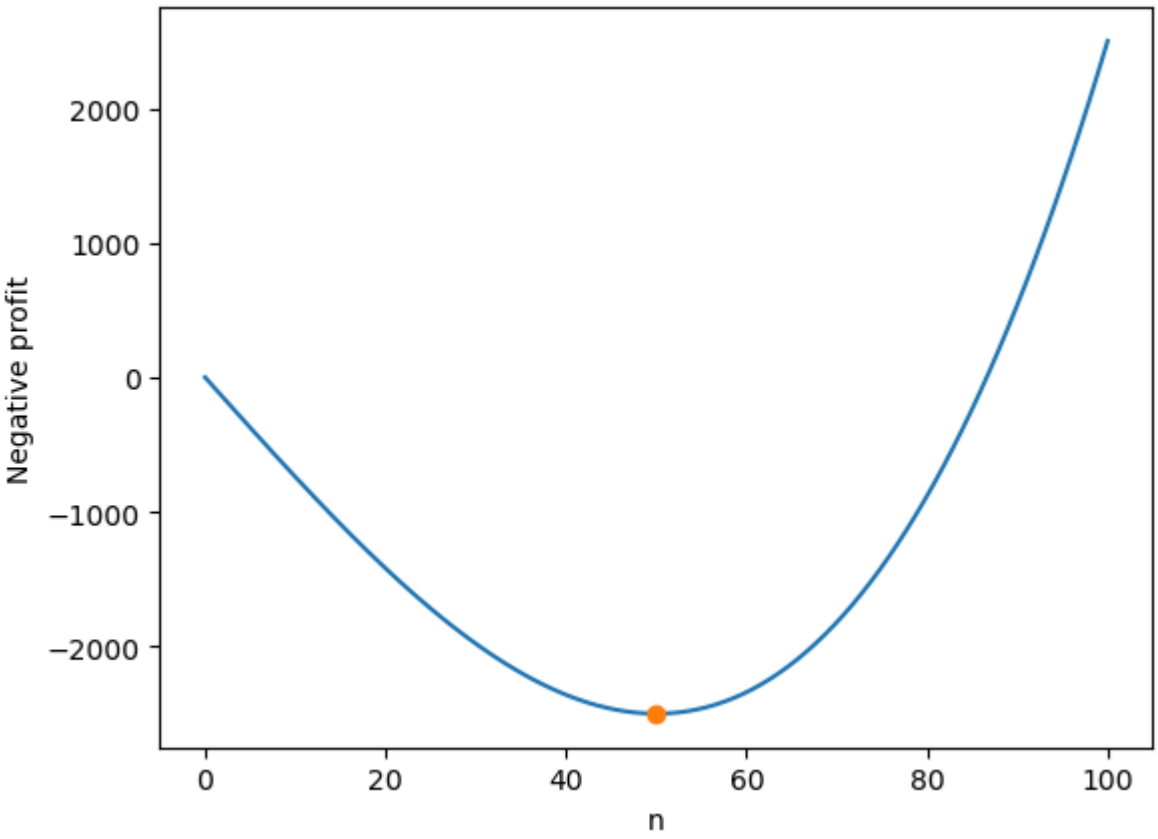
```
  message: Optimization terminated successfully.
  success: True
   status: 0
      fun: -2499.9999999998727
        x: [ 5.000e+01]
      nit: 8
      jac: [ 0.000e+00]
 hess_inv: [[ 3.503e-01]]
     nfev: 22
     njev: 11
```

Postprocess results

If needed, we can postprocess the result or analyse the problem in another way

```
# checking the results with exhaustive search

n_range = np.linspace(0,100,100)
negprofit_result = negprofit(n_range)
plt.plot(n_range,negprofit_result)
plt.plot(result.x,result.fun,'o')
plt.xlabel('n')
plt.ylabel('Negative profit');
```



Exercise 7

Unconstrained optimization using scipy

Contents

In this chapter, we'll cover how to apply `scipy.optimize.minimize` to unconstrained optimization problems. As a reminder, unconstrained optimization considers:

$$\min_x f(x) \tag{1}$$

with:

- x: the design variable of length n
- f: the objective function.

Method

In this course, we're making use of the function `scipy.optimize.minimize`. The documentation of this function is available here: https://docs.scipy.org/doc/scipy-1.15.0/reference/generated/scipy.optimize.minimize.html [The SciPy community, 2024]. In this course we'll cover only the relevant parts.

For unconstrained optimization we need to run at least `scipy.optimize.minimize(fun, x0, ...)` with:

- `fun`, the objective function $f(x)$ to be minimized. `fun` is a callable. The `scipy.optimize.minimize` function takes care of defining and inputting our design variable x.
- `x0`, the initial guess for our design variable x. It needs to be a `ndarray` with length n

The function `scipy.optimize.minimize` outputs an object `scipy.optimize.OptimizeResult`. with:

- `scipy.optimize.OptimizeResult.x` the optimized solution of the design variable x. It is a `ndarray` with length n
- `scipy.optimize.OptimizeResult.success`, a indication whether or not the optimizer was executed successfully. It is a `bool`, indicating `True` or `False`
- `scipy.optimize.OptimizeResult.message`, a message describing the cause of termination of the optimization algorithm. It is a `str`.
- `scipy.optimize.OptimizeResult.fun`, the values of the optimized objective function f. It is a `int` or `float`
- `scipy.optimize.OptimizeResult.nit`, the number of iteration performed by the optimizer. It is a `int`

Exercise 8

Homework exercise: Bathymetry maps

Contents

Click 🚀 –> Live Code to activate live coding on this page!

Problem

It is desired to determine the number of bathymetry maps n of a local area that should be produced to maximize the profit of a company. The total cost of production and distribution is €75 per unit n. The revenues are proportional to the number of units multiplied by its price: $Revenues = n \cdot Price$

The demand depends on the price ($n = \sqrt{100 \cdot (Price - 150)}$), as shown in the graph:

▶ Show code cell source

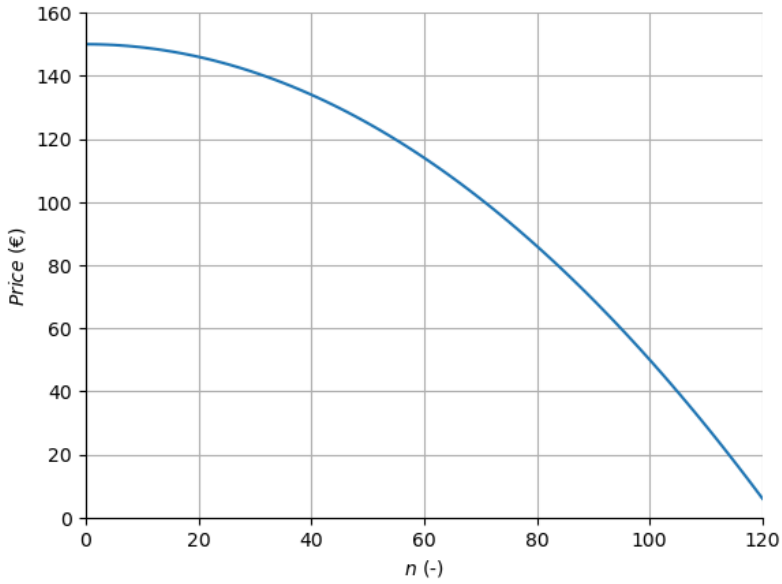


Fig. 6 $Price = 150 - 0.01n^2$

The profit can be estimated as the revenues minus the total costs.

This problem could be solved by taking the derivative and solving the root of the derivative. However, those steps might not be possible for more complex problems, in that case optimization is required.

Model

We need to define our problem in the form of unconstrained optimization (1).

The function for the profit can be found by combining the relations in the problem statement. However, this is the profit which should be maximized. To turn this into a minimization problem, the profit can be multiplied with -1. The final model of this problem results in:

$$\min_n \left(75n - \left(150 - 0.01n^2\right)n\right) \tag{2}$$

Exercise 9

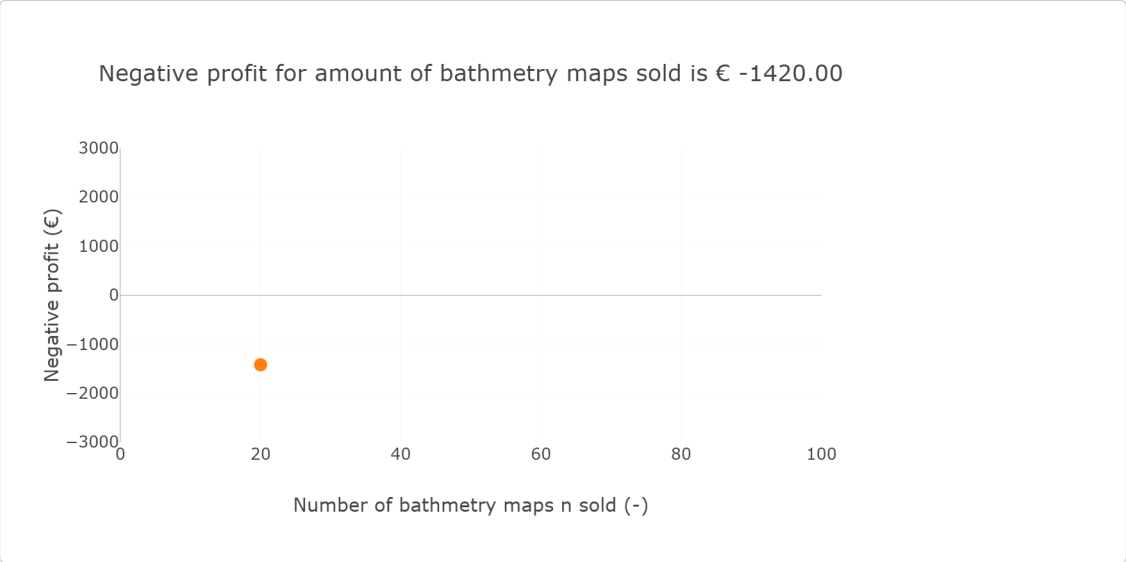
Find best solution manually

An approach to solve this problem might be to try out some values. You can do so in the applet below. The plot below shows the negative profit for some number of bathmetry maps sold.

Exercise 10

Try and adjust the values for n, the number of bathmetry maps sold. How small can you get the negative profit?

#maps n sold 20 Show objective function



As this case is only one-dimensional and the potential range of values is limited, this approach (exhaustive search) is valid: evaluating all possible values for n doesn't take a lot of computing power. The resulting values from the objective function show a clear minimum. For problem in which the objective function required more computational power or the amount of dimensions of the design variables increases, this approach quickly becomes infeasible.

Method

Now let's solve this problem using an optimization method. This model is described using `scipy.optimize.minimize` according to the standard structure in this course

Import libraries

For this problem, we'll use all three packages `scipy`, `numpy`, `matplotlib`.

```
import scipy as sp
import numpy as np
import matplotlib.pylab as plt
```

Define the variables

There are very few variables in this problem. In fact, the only variable we have to specify is the initial guess for the optimization algorithm. The objective function will be treated later. The length of n doesn't have to be specified.

```
1 n0 = 20
```

run | run all | add cell | clear

Exercise 11

Define the objective function

In the objective function, the formula derived above in (2) ean be inserted. Or, each individual step can be calculated on a seperate line. Again, note that the profit is multiplied with -1 to maximize the profit in the minimization formulation. This results in:

```
1 def negprofit(n):
2     price = 150 - 0.01 * n**2
3     revenues = price * n
4     totalcost = 75 * n
5     profit = revenues - totalcost
6     return -profit
```

run | run all | add cell | clear

Solve the problem

Now, the problem can be solved. The result is stored in the variables `result` which is printed.

```
result = sp.optimize.minimize(negprofit,n0)
print(result)
```

run | run all | add cell | clear

```
  message: Optimization terminated successfully.
  success: True
   status: 0
      fun: -2499.9999999998727
        x: [ 5.000e+01]
      nit: 8
      jac: [ 0.000e+00]
 hess_inv: [[ 3.503e-01]]
     nfev: 22
     njev: 11
```

Exercise 12

Postprocess results

As seen before, this problem is very small and can be solved by evaluating all possible values (or applying algebra). These values can be plotted and the optimum solution is clearly in the minimum.

```
n_range = np.linspace(0,100,100)
negprofit_result = negprofit(n_range)
plt.figure()
plt.plot(n_range,negprofit_result)
plt.plot(result.x,result.fun,'o');
plt.xlabel('$n$')
plt.ylabel('Negative profit');
ax = plt.gca()
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
ax.spines['bottom'].set_position('zero')
ax.spines['left'].set_position('zero')
```

run | run all | add cell | clear

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
Cell In[8], line 1
----> 1 n_range = np.linspace(0,100,100)
      2 negprofit_result = negprofit(n_range)
      3 plt.figure()

NameError: name 'np' is not defined
```

Exercise 13

Exercise 14

Click 🚀 –> Live Code and adapt the code to answer the following question.

Homework exercise: Curve-fitting

Contents

Click 🚀 –> **Live Code** to activate live coding on this page!

Problem

8 data points are collected where X_i is the explanatory variable and Y_i the response variable. It is believed that the data follows the trend $Y = aX^b + c$ where a, b and c are the parameters to be determined to fit the model.

X_i	Y_i
0.97	0.97
0	0.06
0.5	0.7
0.85	0.74
0.7	0.2
0.19	0.34
0.41	0.29
0.78	0.94

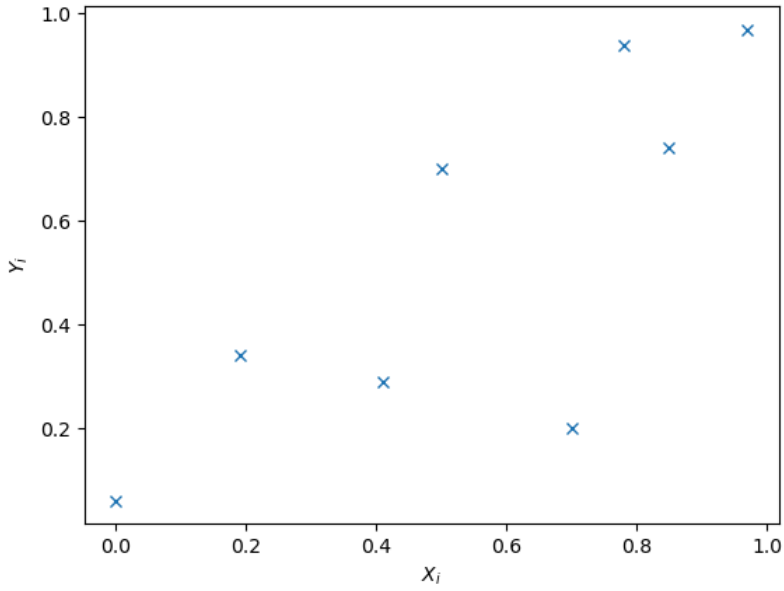


Fig. 7 8 data points

Examples of the source of this data can be:

- Stress of a pre-stressed concrete specimen based on the deformation when applied an axial force

- Permeability-coefficient for compacted sand based on the grain size
- Job performance based on the number of working hours
- etc.

Model

Although it's possible to solve this problem too with a direct mathematical approach which doens't require optimization, we'll define our problem in the form of unconstrained optimization (1).

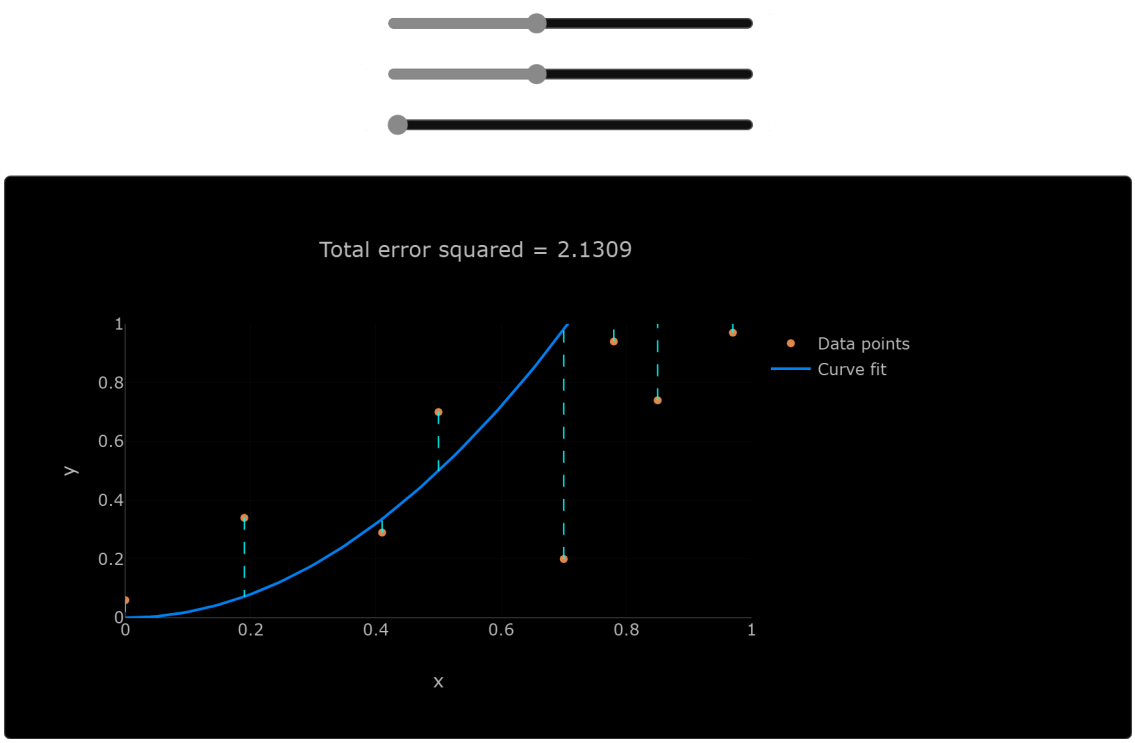
The problem is modelled as a minimization of the squared error, where error is the difference between $aX_i^b + C$ and Y_i

$$\min_{a,b,c} \sum_{i=1}^{8} \left(aX_i{}^b + c - Y_i \right)^2 \tag{3}$$

Find best solution manually

Exercise 15

Try and adjust the values for a, b and c. How small can you get the total error squared?



Method

Again, this model is described using `scipy.optimize.minimize` according to the standard structure in this course

Import libraries

```
import scipy as sp
import numpy as np
import matplotlib.pyplot as plt
```

Define the variables

In this problem, there are 3 design variables: a, b and c. In `scipy.optimize`, these variables are part of one array. You don't have to specify this variable beforehand, but the optimization algorithm will define the number of variables on your other input. Furthermore, input data is given which is required to specify explicitly. Additionally, an initial guess is required for the otimization algorithm.

```
xi = np.array([0.97, 0   , 0.5, 0.85, 0.7, 0.19, 0.41, 0.78])
yi = np.array([0.97, 0.06, 0.7, 0.74, 0.2, 0.34, 0.29, 0.94])
abc0 = np.array([4., 2., 0.5]) #initial guess
```

run | run all | add cell | clear

Define objective function

The objective function was defined in (3). Because `a`, `b`, `c` and `xi` are all numpy arrays, we can simply use `*`, `**` and `+` to deal with the multiplication, exponent and summation of those arrays. `np.sum` is used to sum up all components of the array which is the result of `(yi-y_est)**2`

```
def squarederror(abc):
    a = abc[0]
    b = abc[1]
    c = abc[2]
    y_est = a * xi ** b + c
    error = np.sum((yi-y_est)**2)
    return error
```

run | run all | add cell | clear

Solve the problem

Now we can solve the actual problem

```
result2 = sp.optimize.minimize(squarederror,abc0)
print(result2)
```

run | run all | add cell | clear

```
  message: Optimization terminated successfully.
  success: True
   status: 0
      fun: 0.33422039804883197
        x: [ 8.028e-01  1.231e+00  1.233e-01]
      nit: 22
      jac: [-8.568e-07  1.006e-07 -1.036e-06]
hess_inv: [[ 6.552e-01 -4.308e-01 -3.826e-01]
           [-4.308e-01  1.803e+01  2.737e+00]
           [-3.826e-01  2.737e+00  6.369e-01]]
     nfev: 96
     njev: 24
```

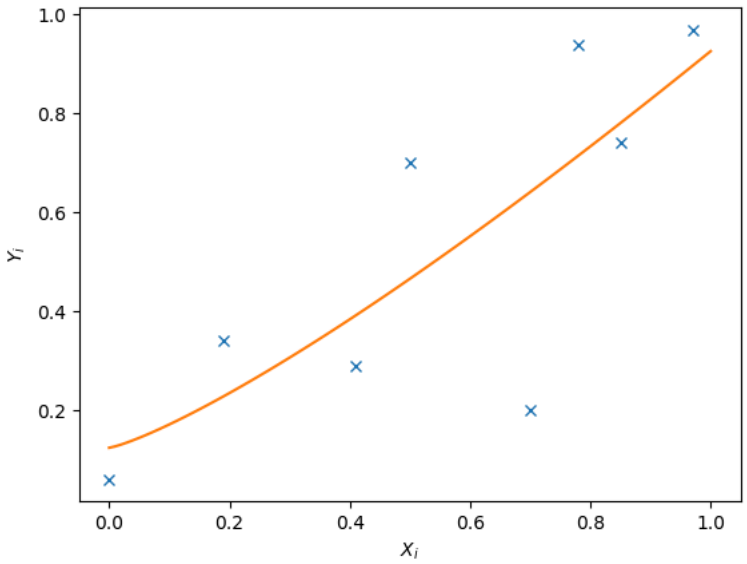
Exercise 16

Postprocess results

As this problem involves three design variables, a plot of all possible design variables with the objective function is not possible because it would require four dimensions. However, we can plot the data which the obtained solution to see how it looks like

```
x_range = np.linspace(0,1,100)
plt.plot(xi,yi,'x');
plt.plot(x_range,result2.x[0] * x_range ** result2.x[1] + result2.x[2])
plt.xlabel('$X_i$')
plt.ylabel('$Y_i$');
```

run | run all | add cell | clear



Exercise

Exercise 17

Click 🚀 –> **Live Code** and adapt the code to answer the following question.

Linear constrained optimization using scipy

Contents

In this chapter, we'll cover how to apply `scipy.optimize.linprog` to linear constrained optimization problems. As a reminder, linear constrained optimization considers:

$$\begin{aligned} &\min_x f(x) \\ &\text{such that} \quad g_j(x) \leq 0 \qquad j = 1, m \\ &\qquad\qquad\quad h_k(x) = 0 \qquad k = 1, p \\ &\qquad\qquad\quad x_i^l \leq x_i \leq x_i^u \; i = 1, n \end{aligned} \tag{4}$$

with:

- $f(x)$, the linear objective function
- x, the n design variables
- $g_j(x)$, the m linear inequality constraints
- $h_k(x)$, the p linear inequality constraints
- x_k^l and x_k^u, the n low er and upper bounds of the design variable

Because all functions are linear, this problem can be rewritten as:

$$\begin{aligned} &\min_x \; c^T x \\ &\text{such that} \quad A_{ub} x \leq b_{ub} \\ &\qquad\qquad\quad A_{eq} x = b_{eq} \\ &\qquad\qquad\quad x_i^l \leq x_i \leq x_i^u \; i = 1, n \end{aligned} \tag{5}$$

with:

- x, the n design variables

- c, the n coefficients of the linear objective function
- A_{ub}, the inequality constraint matrix of m inequality constraints
- b_{ub}, the inequality constraint vector of m inequality constraints
- A_{eq}, the equality constraint matrix of p equality constraints
- b_{eq}, the equality constraint vector of p equality constraints
- x_i^l and x_i^u, the n lower and upper bounds of the design variable

Method

For linear programs, we can use the function `scipy.optimize.linprog`. In contrast to `scipy.optimize.minimize`, this function is limited to linear functions. The documentation of this function is available here: https://docs.scipy.org/doc/scipy-1.15.0/reference/generated/scipy.optimize.linprog.html [The SciPy community, 2024]. In this course we'll cover only the relevant parts.

Again, we won't use all options, but a minimum requirement for our problem is the command `scipy.optimize.linprog(c, A_ub, b_ub, A_eq, b_eq, bounds, ...)` with:

- `c`, a onedimensional numpy array with the n coefficients of the linear objective function c.
- `A_ub`, a twodimensional numpy array with the n coefficient of the m linear inequality constraints matrix A_{ub}.
- `b_ub`, a onedimensional numpy array with the upper bound of the m linear inequality constraint vector b_{ub}.
- `A_eq`, a twodimensional numpy array with the n coefficient of the p linear equality constraints matrix A_{eq}.
- `b_eq`, a onedimensional numpy array with value of the p linear equality constraint vector b_{eq}.
- `Bounds`: A sequence of i `(min, max)` pairs for each element in x, defining the minimum x_i^l and maximum values x_i^u of that decision variable. Use `None` to indicate that there is no bound.

The function `scipy.optimize.linprog` outputs an object `scipy.optimize.OptimizeResult` similar as `scipy.optimize.minimize` explained for unconstrained optimization.

Exercise 18

Homework exercise: Book distribution

Contents

Click 🚀 –> **Live Code** to activate live coding on this page!

Problem

We'll consider planning the shipment of books from distribution centers to stores where they are needed. There are three distribution centers at different cities: Amsterdam, Den Haag and Rotterdam. They have 250, 130 and 235 books. There are four stores in Haarlem, Utrecht, Delft and Breda. They ordered 70, 230, 240 and 70 books. All stores can receive books from all distribution centers, as shown below:

There are the following costs in € of transportation per book:

From \ To	Haarlem	Utrecht	Delft	Breda
Amsterdam	7	11	16	26
Den Haag	7	13	5	20
Rotterdam	16	28	7	12

The goal is to minimize the shipping costs while meeting demand.

Model

We need to define our model in the form of a linear constrained optimization model (4) and to apply `scipy.optimize.linprog` in matrix notation (5).

We'll define the model as follows:

- Design variables: how many books go from which distribution center to which store
- Objective function: minimum shippings costs
- Inequality constraint functions: don't run out of stock in warehouses
- Equality constraint functions: meet demand of stores
- Bounds: book are only transported from distribution centers to stores.

Find best solution manually

Exercise 19

Try and adjust the book transports yourself. Can you find the optimal solution?

Amsterdam → Haarlem 0 books

Amsterdam → Utrecht 0 books

Amsterdam → Delft 0 books

Amsterdam → Breda 0 books

Den Haag → Haarlem 0 books

Den Haag → Utrecht 0 books

Den Haag → Delft 0 books

Den Haag → Breda 0 books

Rotterdam → Haarlem 0 books

Rotterdam → Utrecht 0 books

Rotterdam → Delft 0 books

Rotterdam → Breda 0 books

Total costs: €0

bankrupt @ Amsterdam

bankrupt @ Den Haag

bankrupt @ Rotterdam

didn't receive the correct amount of books @ Haarlem: 70 too few books received

didn't receive the correct amount of books @ Utrecht: 230 too few books received

didn't receive the correct amount of books @ Delft: 240 too few books received

didn't receive the correct amount of books @ Breda: 70 too few books received

Design variables

Let's start with our design variables. In this case that could be the amount of books x transported from each distribution center $i = [1\,(\text{Amsterdam}), 2\,(\text{Den Haag}), 3\,(\text{Rotterdam})]$ to each store $j = [1\,(\text{Haarlem}), 2\,(\text{Utrecht}), 3\,(\text{Delft}), 4\,(\text{Breda})]$:

$$x_{ij} = \begin{bmatrix} x_{\text{Amsterdam}\ \to\text{Haarlem}} & x_{\text{Amsterdam}\ \to\text{Utrecht}} & x_{\text{Amsterdam}\ \to\text{Delft}} & x_{\text{Amsterdam}\ \to\text{Breda}} \\ x_{\text{Den Haag}\ \to\text{Haarlem}} & x_{\text{Den Haag}\ \to\text{Utrecht}} & x_{\text{Den Haag}\ \to\text{Delft}} & x_{\text{Den Haag}\ \to\text{Breda}} \\ x_{\text{Rotterdam}\ \to\text{Haarlem}} & x_{\text{Rotterdam}\ \to\text{Utrecht}} & x_{\text{Rotterdam}\ \to\text{Delft}} & x_{\text{Rotterdam}\ \to\text{Breda}} \end{bmatrix} \tag{6}$$

$$x_{ij} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

Exercise 20

Alternatively, the design variables can be expressed as follows:

$$x = \begin{bmatrix} x_{\text{Amsterdam}} & \rightarrow \text{Haarlem} \\ x_{\text{Amsterdam}} & \rightarrow \text{Utrecht} \\ x_{\text{Amsterdam}} & \rightarrow \text{Delft} \\ x_{\text{Amsterdam}} & \rightarrow \text{Breda} \\ x_{\text{Den Haag}} & \rightarrow \text{Haarlem} \\ x_{\text{Den Haag}} & \rightarrow \text{Utrecht} \\ x_{\text{Den Haag}} & \rightarrow \text{Delft} \\ x_{\text{Den Haag}} & \rightarrow \text{Breda} \\ x_{\text{Rotterdam}} & \rightarrow \text{Haarlem} \\ x_{\text{Rotterdam}} & \rightarrow \text{Utrecht} \\ x_{\text{Rotterdam}} & \rightarrow \text{Delft} \\ x_{\text{Rotterdam}} & \rightarrow \text{Breda} \end{bmatrix} \tag{7}$$

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} & x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}^T$$

Objective function

Now we can define the costs as the sum of the amount of books transported with the cost per transport as in (4)in the form of $\min_x f(x_{ij})$:

$$\min_x f(x_{ij}) = 7x_{11} + 11x_{12} + 16x_{13} + 26x_{14} + 7x_{21} + 13x_{22} + 5x_{23} + 20x_{24} + 16x_{31} + 28x_{32} + 7x_{33} + 12x_{34} \tag{8}$$

This can be converted to matrix notation. In case of the design variables defined as in (5) in the form $\min_x c^T x$ with c:

$$c = \begin{bmatrix} 7 & 11 & 16 & 26 & 7 & 13 & 5 & 20 & 16 & 28 & 7 & 12 \end{bmatrix}^T \tag{9}$$

Exercise 21

Inequality constraints

Let's continue with the inequality constraints, which should deal with that the stock of the distribution centers should always be bigger than 0. Or, stated differently, the sum of the amount of books transported out of each distribution center should be small or equal than the stock of each distribution center. These can be defined in the form $g(x_{ij}) \leq 0$ as:

$$
\begin{aligned}
g_1\left(x_{ij}\right) &= x_{11} + x_{12} + x_{13} + x_{14} - 250 \\
g_2\left(x_{ij}\right) &= x_{21} + x_{22} + x_{23} + x_{24} - 130 \\
g_3\left(x_{ij}\right) &= x_{31} + x_{32} + x_{33} + x_{34} - 235
\end{aligned}
\tag{10}
$$

Or in matrix notation in the form $A_{ub}x \leq b_{ub}$ as:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix} x \leq \begin{bmatrix} 250 \\ 130 \\ 235 \end{bmatrix}
\tag{11}
$$

Exercise 22

Equality constraints

Finally, let's define the inequality constraints, which should deal with that the each store receives the correct amount of books. Or, stated differently, the sum amount of paper books transported to each store should be equal to the demand of that store. These can be defined in the from $h\left(x_{ij}\right) = 0$ as:

$$
\begin{aligned}
h_1\left(x_{ij}\right) &= x_{11} + x_{21} + x_{31} - 70 \\
h_2\left(x_{ij}\right) &= x_{12} + x_{22} + x_{32} - 230 \\
h_3\left(x_{ij}\right) &= x_{13} + x_{23} + x_{33} - 240 \\
h_3\left(x_{ij}\right) &= x_{14} + x_{24} + x_{34} - 70
\end{aligned}
\tag{12}
$$

Or in matrix notation in the form $A_{eq}x = b_{eq}$ as:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix} x = \begin{bmatrix} 70 \\ 230 \\ 240 \\ 70 \end{bmatrix}
\tag{13}
$$

Bounds

The number of books cannot be negative (assuming that book are only transported from distribution centers to stores). The maximum number of books transported could be the number of book available in a distribution center, but this is already defined by the constraint functions. Therefore, the bounds can be defined as:

$$
0 < x_i \; i = 1, 2, \ldots, 12
\tag{14}
$$

Method

Now let's solve this problem using an optimization method.

Import libraries

As this problem is higher-dimensional, we cannot easily plot the solution space. Therefore, we won't import `matplotlib`.

```
import scipy as sp
import numpy as np
```

Define variables

As before, the (length of) design variable itself doesn't have to be specified. So there's actually nothing be be done here.

Define objective function

The objective function was defined in (9), which gives:

```
c = np.array([7, 11, 16, 26, 7, 13, 5, 20, 16, 28, 7, 12])
```

run | run all | add cell | clear

Define constraint functions

The constraint functions were defined in (11) and (13), which can be coded as follows:

```
A_ub = np.array([[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0],
                 [0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0],
                 [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1]])
b_ub = np.array([250, 130, 235])

A_eq = np.array([[1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0],
                 [0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0],
                 [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0],
                 [0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1]])
b_eq = np.array([70, 230, 240, 70])
```

run | run all | add cell | clear

Bounds

The bounds were defined in (14) and result in:

```
bounds = np.array([[0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None],
          [0, None]])
```

run | run all | add cell | clear

Solve the problem

```
result = sp.optimize.linprog(c, A_ub, b_ub, A_eq, b_eq, bounds)
print(result)
```

run | run all | add cell | clear

```
       message: Optimization terminated successfully. (HiGHS Status 7: Optimal)
       success: True
        status: 0
           fun: 5380.0
             x: [ 2.000e+01  2.300e+02  0.000e+00  0.000e+00  5.000e+01
                  0.000e+00  8.000e+01  0.000e+00  0.000e+00  0.000e+00
                  1.600e+02  7.000e+01]
           nit: 7
         lower:  residual: [ 2.000e+01  2.300e+02  0.000e+00  0.000e+00
                             5.000e+01  0.000e+00  8.000e+01  0.000e+00
                             0.000e+00  0.000e+00  1.600e+02  7.000e+01]
                marginals: [ 0.000e+00  0.000e+00  1.100e+01  1.600e+01
                             0.000e+00  2.000e+00  0.000e+00  1.000e+01
                             7.000e+00  1.500e+01  0.000e+00  0.000e+00]
         upper:  residual: [       inf        inf        inf        inf
                                   inf        inf        inf        inf
                                   inf        inf        inf        inf]
                marginals: [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00
                             0.000e+00  0.000e+00  0.000e+00  0.000e+00
                             0.000e+00  0.000e+00  0.000e+00  0.000e+00]
         eqlin:  residual: [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00]
                marginals: [ 9.000e+00  1.300e+01  7.000e+00  1.200e+01]
       ineqlin:  residual: [ 0.000e+00  0.000e+00  5.000e+00]
                marginals: [-2.000e+00 -2.000e+00 -0.000e+00]
mip_node_count: 0
mip_dual_bound: 0.0
       mip_gap: 0.0
```

Exercise 23

Click 🚀 –> Live Code to activate live coding on this page.

Class exercise: Furniture manufacturing

> ***Added in version v2024.1.0:*** After class February 19th
>
> Solutions in text and downloads

A furniture manufacturing company wishes to determine how many tables, chairs, desks and bookcases it should make to optimize its profit.

These products utilize two different types of lumber, the company has on hand:

- 1500 m of type 1
- 1000 m of type 2.

There are 800 labour-hours available for the job.

Sales forecasts plus back orders require the company to make at least:

- 40 tables
- 130 chair
- 30 desk
- no more than 10 bookcases.

Each table, chair, desk, and bookcase requires:

- of lumber type 1: 5, 1, 9, and 12 m, respectively
- of lumber type 2: 2, 3, 4, and 1 m, respectively.

A table requires 3 labour-hours to make, a chair 2 hours, a desk 5 hours, and a bookcase 3 hours.

The company makes a total of profit:

- €12 on a table
- €5 on a chair
- €15 on a desk

- €10 on a bookcase.

Exercise 24 (Model)

Write out the linear programming model of this problem.

Solution to Exercise 24 (Model)

$$\min_x \quad [-12 \quad -5 \quad -15 \quad -10]x$$

with $$x = \begin{bmatrix} x_{table} \\ x_{chair} \\ x_{desk} \\ x_{bookcases} \end{bmatrix}$$

such that $$\begin{bmatrix} 5 & 1 & 9 & 12 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 5 & 3 \end{bmatrix} x \leq \begin{bmatrix} 1500 \\ 1000 \\ 800 \end{bmatrix}$$

$$[1 \quad -4 \quad 0 \quad 0]x = 0$$

$$\begin{bmatrix} 40 \\ 160 \\ 30 \\ 0 \end{bmatrix} \leq x \leq \begin{bmatrix} \infty \\ \infty \\ \infty \\ 10 \end{bmatrix}$$

Exercise 25 (Method)

Now let's solve this problem using an optimization method.

Click 🚀 –> Live Code to activate live coding on this page!

```
import scipy as sp
import numpy as np
```

```
#YOUR CODE HERE
```

Solution to Exercise 25 (Method)

```
import scipy as sp
import numpy as np
```

```
c = np.array([-12, -5, -15, -10])
```

```
A_ub = np.array([[5, 1, 9, 12],
                 [2, 3, 4, 1 ],
                 [3, 2, 5, 3]])
b_ub = np.array([1500, 1000, 800])
```

```
bounds = np.array([[40, None],
          [130, None],
          [30, None],
          [0, 10]])
```

```
result = sp.optimize.linprog(c, A_ub=A_ub, b_ub=b_ub, A_eq=None, b_eq=None, bour
print(result.x)
print(result.fun)
```

```
[130. 130.  30.   0.]
-2660.0
```

The optimal solution is:

- 130 tables
- 130 chairs
- 30 desk
- 0 bookcases

With a profit of €2660

Exercise 26 (Additional exercise)

What happens if you impose that four chairs should be made for every table?

```
#YOUR CODE HERE
```

Solution to Exercise 26 (Additional exercise)

```
A_eq = np.array([[4,-1,0,0]])
b_eq = np.array([0])
```

```
result = sp.optimize.linprog(c, A_ub=A_ub, b_ub=b_ub, A_eq=A_eq, b_eq=b_eq, bour
print(result.x)
print(result.fun)
```

```
[ 40. 160.  66.  10.]
-2370.0
```

A different solution is found with a lower profit:

- 40 tables
- 160 chairs
- 66 desk
- 10 bookcases

With a profit of €2370

Nonlinear constrained optimization using scipy

Contents

In this chapter, we'll cover how to apply `scipy.optimize.minimize` to nonlinear constrained optimization problems. As a reminder, nonlinear constrained optimization considers:

$$\begin{aligned} &\min_x f(x) \\ &\text{such that} \quad g_j(x) \leq 0 \qquad j = 1, m \\ &\qquad\qquad\quad h_k(x) = 0 \qquad k = 1, p \\ &\qquad\qquad\quad x_i^l \leq x_i \leq x_i^u \; i = 1, n \end{aligned} \tag{15}$$

with:

- $f(x)$, the linear or nonlinear objective function.
- x, the n design variables
- $g_j(x)$, the m linear or nonlinear inequality constraints
- $h_k(x)$, the p linear or nonlinear inequality constraints
- x_k^l and x_k^u, the n low er and upper bounds of the design variable

Method

For linear programs, we can use the function `scipy.optimize.minimize` again. The documentation of this function is available here: https://docs.scipy.org/doc/scipy-1.15.0/reference/generated/scipy.optimize.minimize.html [The SciPy community, 2024]. In this course we'll cover only the relevant parts.

For unconstrained optimization we need to run at least `scipy.optimize.minimize(fun, x0, bounds, constraints, ...)` with:

- `fun`, the function representing the objective function $f(x)$ to be minimized. `fun` is a callable. The `scipy.optimize.minimize` function takes care of defining and inputting our design variable x.
- `x0`, the initial guess for our design variable x. It needs to be a `ndarray` with length n
- `Bounds`: A sequence of i `(min, max)` pairs for each element in x, defining the minimum x_i^l and maximum values x_i^u of that decision variable. `None` is used to specify no bound.
- `constraints`, a single or a list of constraint objective either being:
  - `scipy.optimize.LinearConstraint`
  - `scipy.optimize.NonlinearConstraint`

Exercise 27

As you can see, the constraints are stored in an object `scipy.optimize.LinearConstraint` and/or `scipy.optimize.NonlinearConstraint`. These function have the following input, for `scipy.optimize.NonlinearConstraint(fun, lb, ub, ...)`:

- `fun`, the function representing the constraint function $g(x)$ or $h(x)$ to be minimized. Again, `fun` is a callable. The `scipy.optimize.minimize` function takes care of defining and inputting our design variable x.
- `lb` and `ub`, two arrays containing the lower- and upper bounds for each of the constraint functions $g(x)$. This lower bound can be `-np.inf` or `np.inf` to represent one-sides constraints. If the lower- and upper bound are set to the same value, an equality function is modelled.

For linear constraints, the constraint function is stored in a matrix again: `scipy.optimize.NonlinearConstraint(A, lb, ub, ...)`:

- `A`, a twodimensional numpy array with the n coefficient of the m linear inequality constraints matrix A_{ub}.
- `lb` and `ub` as for `scipy.optimize.NonlinearConstraint`

Please note that unlike with linear constraints optimization, the right-hand-side of the constraints are not stored in an upper bound vector, but defined with `lb` and `ub`.

Exercise 28

The function `scipy.optimize.linprog` outputs an object `scipy.optimize.OptimizeResult` similar as `scipy.optimize.minimize` explained for unconstrained optimization.

Homework exercise: Breakwater blocks

Contents

Click 🚀 –> Live Code to activate live coding on this page!

Problem

It is desired to design the cheapest breakwater block (box-shaped). The price of the brick will depend only on the amount of used material. It is required that each face has a minimum surface of 0.8 m^2. Also, for stability reasons, the block weight has to be larger than 3000 kg. Let's assume concrete density of 2500 kg$/m^2$.

Fig. 8 Breakwater blocks

Model

We need to define our model in the form of a nonlinear constrained optimization model (15) to apply `scipy.optimize.minimize`.

We'll define the model as follows:

- Design variables: width, height and depth of a block
- Objective function: minimum volume of the block
- Inequality constraint functions: minimum surface area of each face of 0.8 m^2 and maximum weight of 3000 kg
- Equality constraint functions: none
- Bounds: positive dimensions

Design variables

Let's start with our design variables. In this case a logical choice could be the width, height and depth of our block

$$x = \begin{bmatrix} x_{width} \\ x_{depth} \\ x_{height} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{16}$$

Objective function

Now we can define the objective function as the product of the dimension to represent $\min_x f(x)$ in (15):

$$\min_x f(x) = x_1 \cdot x_2 \cdot x_3 \tag{17}$$

Inequality constraints

Let's continue with the inequality constraints, which should deal with the required positive dimensions, minimum surface area of each face of 0.8 m^2 and maximum weight of 3000 kg. These can be defined in the form $g(x_{ij}) \leq 0$ as:

$$\begin{aligned} g_1(x) &= -x_1 \cdot x_2 + 0.8 \\ g_2(x) &= -x_2 \cdot x_3 + 0.8 \\ g_3(x) &= -x_1 \cdot x_3 + 0.8 \\ g_4(x) &= -x_1 \cdot x_2 \cdot x_3 \cdot 2500 + 3000 \end{aligned} \tag{18}$$

Exercise 29

Bounds

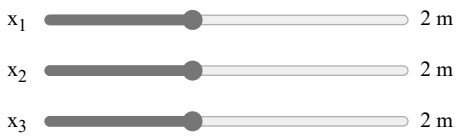
The dimensions of the block cannot be negative. Therefore, the bounds can be defined as:

$$0 < x_i \, i = 1, 2, 3 \tag{19}$$

Find best solution manually

Exercise 30

Try and adjust the values for x_1, x_2 and x_3. Can you find the optimal solution?



Objective function: 8.000 m^3

- Constraint function 1: -3.20 m^2
- Constraint function 2: -3.20 m^2
- Constraint function 3: -3.20 m^2
- Constraint function 4: -17000 kg

Method

Now let's solve this problem using an optimization method.

Import libraries

```
import scipy as sp
import numpy as np
```

Define variables

As before, we don't need to specify our variable x itself as defined in (16). However, this optimization method requires an initial guess. An arbitrary value is chosen here:

```
x0 = np.array([5,0,1])
```

run | run all | add cell | clear

Define objective function

The objective function was defined in (17), which gives:

```
def func(x):
    vol = x[0]*x[1]*x[2]
    return vol
```

run | run all | add cell | clear

Define constrain functions

The constraint functions were defined in (18). We had no equality constraints. Unlike before with the method to solve linear constrained problem, we need an object which defines the upper and lower bounds. As this problem has only an upper bound of 0, the lower bound is set to ∞ which is `np.inf` in python. Note that a single constraint object can include multiple constraints.

```
def nonlinconfun(x):
    c1 = 0.8 - x[0]*x[1]
    c2 = 0.8 - x[0]*x[2]
    c3 = 0.8 - x[1]*x[2]
    c4 = 3000 - 2500 * x[0] * x[1] * x[2]
    return np.array([c1,c2,c3,c4])

cons = sp.optimize.NonlinearConstraint(nonlinconfun, np.array([-np.inf,-np.inf,-np.inf,-np.inf]), np.
```

run run all add cell clear

Define bounds

The bounds were defined in (19) and result in:

```
bounds = [[0, None],
          [0, None],
          [0, None]]
```

run run all add cell clear

Solve the problem

Now let's solve the problem. The `cons` object can be added directly, in the case of equality constraints as well you can define a list of constrainer objects as an input.

```
result = sp.optimize.minimize(fun = func,x0 = x0,bounds = bounds,constraints=cons)
print(result)
```

run run all add cell clear

Exercise 31

Click 🚀 –> Live Code to activate live coding on this page.

Class exercise: Suspended beams

> **Added in version v2024.2.0:** After class February 26th
>
> Solutions in text and downloads

3 beams are suspended on ropes A, B, C, D, E and G:

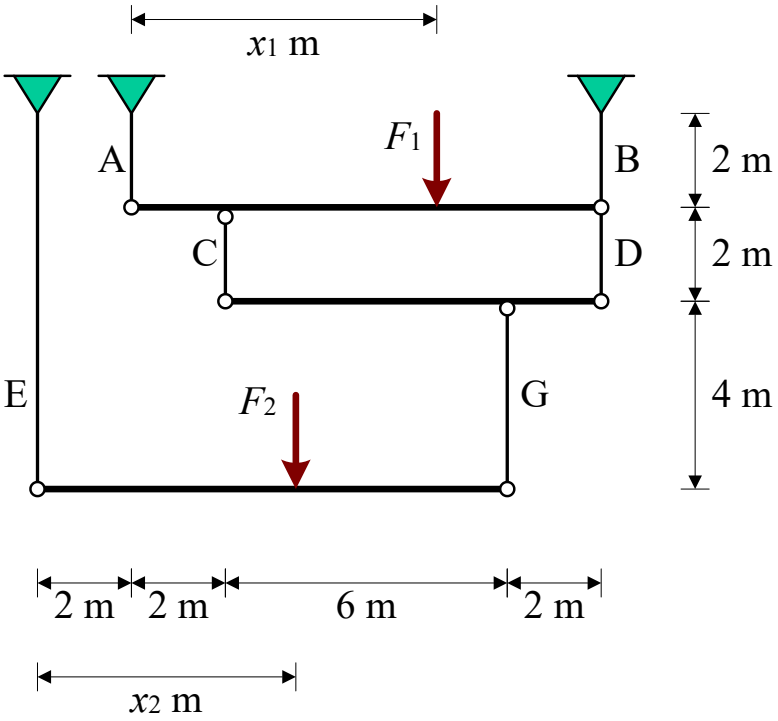


Fig. 9 Suspended beams

Find the permissible total load which doesn't exceed the maximum tension N in rope A and B, C and D, and E and G of respectively 300 N, 200 N and 100 N. What is the tension in the ropes for the permissible total load?

> **Exercise 32 (Model)**
>
> Write out the nonlinear programming model of this problem.

Solution to Exercise 32 (Model)

From mechanics it follows that:

$$
\begin{aligned}
\sum F_{\mathrm{v}}^{\mathrm{EG}} &= 0 \rightarrow N_E + N_G - F_2 = 0 \\
\sum T_{\mathrm{E}}^{\mathrm{EG}} &= 0 \rightarrow 10N_G - x_2 F_2 = 0 \\
\sum F_{\mathrm{v}}^{\mathrm{CD}} &= 0 \rightarrow N_C + N_D - N_G = 0 \\
\sum T_{\mathrm{C}}^{\mathrm{CD}} &= 0 \rightarrow 8N_D - 6N_G = 0 \\
\sum F_{\mathrm{v}}^{\mathrm{AB}} &= 0 \rightarrow N_A + N_B - N_C - N_D - F_1 = 0 \\
\sum T_{\mathrm{A}}^{\mathrm{AB}} &= 0 \rightarrow 10N_B - 10N_D - F_1 x_1 - 2N_C = 0
\end{aligned}
$$

The problem can be stated as follows:

$$
\min_x \left(-F_1 - F_2\right)
$$

with

$$
x = \begin{bmatrix} F_1 \\ F_2 \\ x_1 \\ x_2 \end{bmatrix}
$$

such that

$$
\begin{aligned}
N_A\left(F_1, F_2, x_1, x_2\right) &\leq 300 \\
N_B\left(F_1, F_2, x_1, x_2\right) &\leq 300 \\
N_C\left(F_1, F_2, x_1, x_2\right) &\leq 200 \\
N_D\left(F_1, F_2, x_1, x_2\right) &\leq 200 \\
N_E\left(F_1, F_2, x_1, x_2\right) &\leq 100 \\
N_G\left(F_1, F_2, x_1, x_2\right) &\leq 100 \\
0 &\leq F_1 \\
0 &\leq F_2 \\
0 \leq x_1 &\leq 10 \\
0 \leq x_2 &\leq 10
\end{aligned}
$$

This could be solved by solving the systems of equations when evaluation the constraints, or solving the systems of equation symbolically on beforehand. In the second case, we can rephrase the model as:

$$\min_x \left(-F_1 - F_2\right)$$

with $$x = \begin{bmatrix} F_1 \\ F_2 \\ x_1 \\ x_2 \end{bmatrix}$$

such that

$$-\frac{F_1 x_1}{10} + F_1 + \frac{F_2 x_2}{50} \leq 300$$

$$\frac{F_1 x_1}{10} + \frac{2F_2 x_2}{25} \leq 300$$

$$\frac{F_2 x_2}{40} \leq 200$$

$$\frac{3F_2 x_2}{40} \leq 200$$

$$-\frac{F_2 x_2}{10} + F_2 \leq 100$$

$$\frac{F_2 x_2}{10} \leq 100$$

$$0 \leq F_1$$

$$0 \leq F_2$$

$$0 \leq x_1 \leq 10$$

$$0 \leq x_2 \leq 10$$

Exercise 33 (Method)

Now let's solve this problem using an optimization method.

Click 🚀 –> **Live Code** to activate live coding on this page!

```
import scipy as sp
import numpy as np
```

```
#YOUR CODE HERE
```

Solution to Exercise 33 (Method)

```
import scipy as sp
import numpy as np
```

```
x0 = np.array([500, 100, 2, 9])
```

```
def func(x):
    return -x[0] - x[1]
```

```
# solving the systems of equations during every iteration
def nonlinconfun(x):
    F_1 = x[0]
    F_2 = x[1]
    x_1 = x[2]
    x_2 = x[3]
    A = np.array([[0,0,0,0,1,1],
                  [0,0,0,0,0,10],
                  [0,0,1,1,0,-1],
                  [0,0,0,8,0,-6],
                  [1,1,-1,-1,0,0],
                  [0,10,-2,-10,0,0]])
    b = np.array([F_2, x_2*F_2, 0, 0, F_1, F_1*x_1])
    sol = np.linalg.solve(A,b)
    return sol

cons = sp.optimize.NonlinearConstraint(nonlinconfun, np.array([-np.inf,-np.inf,-
```

```
bounds = [[0, None],
          [0, None],
          [0, 10],
          [0, 10]]
```

```
result = sp.optimize.minimize(fun = func, x0 = x0, bounds = bounds,constraints=c
print(result.fun)
print(result.x)
print(nonlinconfun(result.x))
```

```
-700.0000000000525
[549.95615294 150.04384706   4.7270118    3.33528152]
[300.         300.          12.51096176  37.53288529 100.
  50.04384706]
```

```
x0 = np.array([1000,1000,10,10])
```

```
result = sp.optimize.minimize(fun = func, x0 = x0, bounds = bounds,constraints=c
print(result.fun)
print(result.x)
print(nonlinconfun(result.x))
```

```
-700.0
[500.  200.    4.4   5. ]
[300. 300.  25.  75. 100. 100.]
```

We get different results for different initial guesses

- With initial guess: $F_1 = 500$, $F_2 = 100$, $x_1 = 2$, $x_2 = 9$:
  - Total permissible load ≈ 700
  - $F_1 \approx 550$, $F_2 = 150$, $x_1 = 4.73$, $x_2 = 3.34$
  - $N_A = 300$, $N_B = 300$, $N_C \approx 12.5$, $N_D \approx 37.5$, $N_E \approx 100$, $N_G \approx 50.0$
- With initial guess: $F_1 = 1000$, $F_2 = 1000$, $x_1 = 10$, $x_2 = 10$:
  - Total permissible load $= 700$
  - $F_1 = 500$, $F_2 = 200$, $x_1 = 4.4$, $x_2 = 5$
  - $N_A = 300$, $N_B = 300$, $N_C \approx 25$, $N_D \approx 75$, $N_E \approx 100$, $N_G \approx 100.0$

Multi-objective optimization using scipy

Contents

In this chapter, we'll cover how to solve multi-objective optimization problem using `scipy`. As a reminder, nonlinear constrained optimization considers:

$$\min_x f_1(x), f_2(x) \tag{20}$$

with:

- $f_1(x)$ and $f_2(x)$, the linear or nonlinear objective functions.
- x, the n design variables
- Constraints and bounds as for single-objective optimization problems.

Model

Three different ways of solving multi-objective optimization problems were introduced, which all effectively convert the problem to a single-objective optimization problem. All of this is assuming minimization problems:

1. Weighted objective function: setting pre-determined weight on the two objectives. In general this requires the two objectives to have a comparable unit:

$$\min_x \left(\delta_{1,\text{predefined}} \cdot f_1(x) + \delta_{2\text{predefined}} \cdot f_2(x)\right) \tag{21}$$

2. Goal attainment, minimizing the maximum difference with respect to two goal values for the objectives. Again, this requires the two objectives to have a comparable unit:

$$\min_x \left(\max \left(f_1\left(x\right) - f_{1,\text{goal}} \ , f_2\left(x\right) - f_{2,\text{goal}} \ \right)\right) \tag{22}$$

3. Pareto front: finding many possible optimal solution for a large set weights. It's good practise to normalize the objective functions.

$$\begin{gathered}\min_x \ \left(\delta_i \cdot f_{1,\text{normalized}} \quad (x) + \delta_j \cdot f_{2,\text{normalized}} \quad (x)\right) \\ \text{with } 0 < \delta_i < 1 \text{ and } \delta_j = 1 - \delta_i\end{gathered} \tag{23}$$

All of these methods could also be applied to problems which include more than two goals.

Exercise 34

Normalize objective functions

Normalizing the objectives functions can be done by setting the domain of every goal f between 0 and 1 by finding (or estimating) the lower and upper bounds for these objective functions within the domain:

$$\begin{gathered}f_{\text{normalized}} \quad (x) = \frac{f\left(x\right) - \min_x \ \left(f\left(x\right)\right)}{\max_x \ \left(f\left(x\right)\right) - \min_x \ \left(f\left(x\right)\right)} \\ \text{with } \min_x \ \left(f\left(x\right)\right), \max_x \ \left(f\left(x\right)\right) \text{ for } x_i^l \leq x_i \leq x_i^u \text{ with } i = 1, n\end{gathered} \tag{24}$$

Note that here are various ways to normalize functions / data. This is just an example.

Exercise 35

Homework exercise: engine power vs emissions

Contents

Click 🚀 –> Live Code to activate live coding on this page!

Problem

In this problem we're trying to optimize a diesel engine for maximum power and minimum CO_2 emissions. Depending on the optimal engine speed, the power and CO_2 emissions change.

The following data is available:

Rotations per minute	CO_2 Emissions	Power
800	708	161.141
1000	696.889	263.243
1200	688.247	330.51
1400	682.897	381.561
1700	684.955	391.17
1800	697.3	370

This data is interpolated to obtain a continuous relation:

▶ Show code cell source

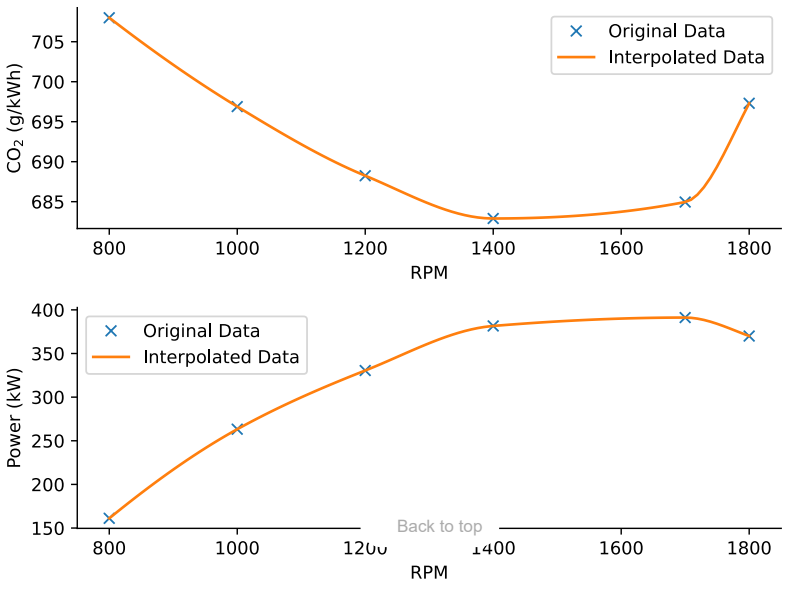


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Fig. 10 Interpolation of CO_2 emissions and Power

Model

We'll define the model as follows:

- Design variables: scalar value representing the rotations per minute (RPM)
- Objective function: weighted sum of the power and CO_2 emissions, max of difference with goals or pareto front
- Bounds: only interpolation so between 800 and 1800 RPM

Design variables

Let's start with our design variables. In this case a logical choice could be the width, height and depth of our block

$$x = \text{rotations per minute} \tag{25}$$

Objective function

Let's explore all three possible objective functions as defined in (21), (22) and (23):

Weighted objective function

We can define the weighted objective function as defined by (21) with some predefined weights. For example a weight of $\frac{1}{3}$ for the power value and $\frac{2}{3}$ for the CO_2 emissions.

$$\min_x \left(-\frac{1}{3} \cdot f_{\text{power}} \quad (x) + \frac{2}{3} \cdot f_{\text{CO }_2 \text{ emissions}} \quad (x) \right) \tag{26}$$

Please note that we need a minus for the power objective because it's an maximization objective and we now apply the weights to the non-normalized functions while they have different units. Therefore, it might be wise to apply normalization as defined by (24).

Goal attainment

If we define two fails for the power and CO_2 emissions, we can apply goal attainment. Those goals could be 460 kW for the generated power and 640 g/kWh for the CO_2 emissions. Now the objective function can as in (22) as:

$$\min_x \left(\max \left(460 - f_{\text{power}}\left(x\right), f_{\text{CO}_2\text{emissions}}\left(x\right) - 640 \right) \right) \tag{27}$$

Pareto front

Finally, we can find the pareto front by defining the problem as in (23):

$$\min_x \left(-\delta_i \cdot f_{\text{power} \quad \text{,normalized}} \quad (x) + \delta_j \cdot f_{\text{CO }_2 \text{ emissions} \quad \text{,normalized}} \quad (x) \right) \tag{28}$$

For normalization, we could take the value from our plot, but for more real-life complex problems the dimensions are higher and you cannot find the maximum and minimum of each single objective function. Therefore, let's estimate the maxima and minima as:

	CO_2 Emissions	**Power**
Minimum value	680	150
Maximum value	710	400

Another approach would be to optimize for both CO_2 Emissions and Power separately, and use the respective objective function values as minimum and maximum values.

With a linear normalization, this gives:

$$
\begin{aligned}
f_{\text{power, normalized}}(x) &= \frac{f_{\text{power}}(x) - 150}{400 - 150} \\
f_{\text{CO}_2\text{ emissions, normalized}}(x) &= \frac{f_{\text{CO}_2\text{ emissions}}(x) - 680}{710 - 680}
\end{aligned}
\tag{29}
$$

To find the full pareto front this optimization model has to be solved for a large set of δ_i and δ_j

Bounds

Let's limit our solution to within our interpolation domain. Therefore, the bound can be defined as:

$$
800 < x < 1800 \tag{30}
$$

Find best solution manually

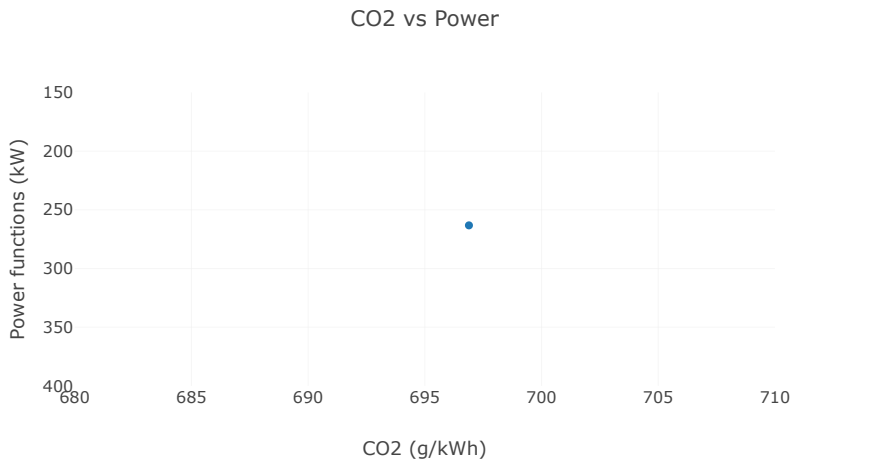
Exercise 36

Try and adjust the values for x. Can you find the optimal solution? How does it change for different models?

RPM 1000

Weighted Objective: = - 1/3 * POWfunc(1000) + 2/3 * CO2func(1000) = - 1/3 * 263.24 + 2/3 * 696.89 = 376.85

Goal Attainment: max(460 - POWfunc(1000), CO2func(1000) - 640) = max(460 - 263.24, 696.89 - 640) = 196.76



Method

Now let's solve this problem using an optimization method. The interpolated data is stored in `CO2func` and `POWfunc`.

Import libraries

```
import scipy as sp
import numpy as np
import matplotlib.pyplot as plt
```

Define variables

As before, we don't need to specify our variable x itself as defined in (16). However, this optimization method requires an initial guess. An arbitrary value is chosen here:

```
1 x0 = np.array(1200)
```

run run all add cell clear

Define objective function

Let's define the objective function for each of the three models

Weighted objective function

The objective function was defined in (26), which gives:

```
1 def weighted_obj(x):
2     delta_p = 1/3
3     delta_c = 1 - delta_p
4     return -delta_p * POWfunc(x) + delta_c * CO2func(x)
```

run run all add cell clear

Goal attainment

The objective function was defined in (27), which gives:

```
def goal_attainment(x):
    Pt = 460
    Ct = 640
    return max(Pt - POWfunc(x),CO2func(x)-Ct)
```

run run all add cell clear

Pareto front

For the pareto front, the objective functions needed to be normalized as defiend by (29):

```
def POWfunc_normalized(x):
    return (POWfunc(x) - 150)/(400 - 150)

def CO2func_normalized(x):
    return (CO2func(x) - 680)/(710 - 680)
```

run run all add cell clear

The objective function was defined in (28), which gives:

```
def weighted_obj_pareto(x):
    return -delta_p * POWfunc_normalized(x) + delta_c * CO2func_normalized(x)
```

run run all add cell clear

in which `delta_p` and `delta_c` are defined when solving this problem

Define bounds

The one single bound was defined in (30) and results in:

```
bounds = [[800,1800]]
```

run | run all | add cell | clear

Solve the problem

Now let's solve the problem for each of the three models. We're gonna time them to see how long the analysis takes. Please note that when running this in the browser, the `CPU times` gives invalid values.

Weighted objective function

```
%%time
result = sp.optimize.minimize(fun = weighted_obj, x0 = x0, bounds = bounds)
print(result)
```

run | run all | add cell | clear

Goal attainment

```
%%time
result2 = sp.optimize.minimize(fun = goal_attainment, x0 = x0, bounds=bounds)
print(result2)
```

run run all add cell clear

Pareto front

For the pareto front we need to solve this problem for a collection of weights. The results are stored in a list

```
%%time
x_pareto_opt =[]
delta_p_list = np.linspace(0,1,101)
delta_c_list = 1 - delta_p_list
for i in range(101):
    delta_p = delta_p_list[i]
    delta_c = delta_c_list[i]
    result_i = sp.optimize.minimize(fun = weighted_obj_pareto,x0=x0,bounds=bounds)
    x_pareto_opt.append(result_i.x[0])
```

run run all add cell clear

Now the pareto front can be plotted by evaluating the objectives functions for our collection of optimum values for x:

```
P_pareto_opt = POWfunc(x_pareto_opt)
C_pareto_opt = CO2func(x_pareto_opt)

plt.figure()
plt.plot(C_pareto_opt,P_pareto_opt,'x')
for i in range(101):
    if i%10 == 0:
        plt.annotate(f"($\delta_i,\delta_j$)=({round(delta_p_list[i],1)},{round(delta_c_list[i],1)})
plt.ylabel('Power (kW)')
plt.xlabel('CO$_2$ (g/kWh)')
plt.title('Pareto Front')
ax = plt.gca()
ax.invert_yaxis()
ax.spines['right'].set_color('none')
```

run run all add cell clear

Exercise 37

Exercise 38

Click 🚀 –> Live Code and answer the following question:

Class exercise: Ice cream cone

> ***Added in version v2024.3.0:*** After class March 10th
>
> Solutions in text and downloads

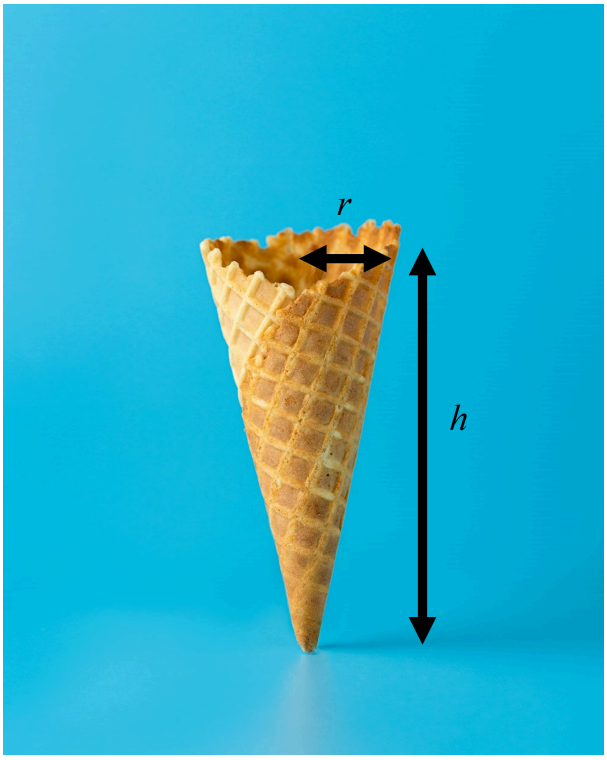


Fig. 11 ice cone

We want to minimize the surface area of a cone, while not making the gap on the top too big (in terms of area). Let's define this as a multi-objective optimization problem with

1. the first objective to minimize the ice cream cone area $\pi r\sqrt{r^2+h^2}$
2. the second objective to minimize the ice cream cone area + the area of the gap $\pi r\sqrt{r^2+h^2}+\pi r^2$.

The total volume inside the cone must be at least 200.

Find the pareto front using normalized objective functions

Exercise 39 (Model)

Write out the model of this problem.

Solution to Exercise 39 (Model)

$$\min_{r,h}\left(\delta_{Area}\cdot\frac{Area\,(r,h)-\min_{r,h}Area\,(r,h)}{\max_{r,h}Area\,(r,h)-\min_{r,h}Area\,(r,h)}+\delta_{Total}\cdot\frac{Total\,(r,}{\max_{r,h}Tota}\right.$$

$$\text{with } Area\,(r,h)=\pi r\sqrt{r^2+h^2}$$

$$\text{and } Total\,(r,h)=Area\,(r,h)+\frac{4}{3}\pi r^2$$

$$\text{such that}\frac{1}{3}\pi r^2 h$$

The minimum and maximum $Area$ and Gap could be find by solving two single-objective problems:

$$\min_{r,h}\pi r\sqrt{r^2+h^2}$$

$$\text{such that } \frac{1}{3}\pi r^2 h\geq 200$$

```
import scipy as sp
import numpy as np
```

```
def Area(x):
    return np.pi* x[0] * np.sqrt(x[0]**2 + x[1]**2)
def Total(x):
    return np.pi* x[0] * np.sqrt(x[0]**2 + x[1]**2) + 4/3 * np.pi*x[0]**2

x0 = np.array([5,10])
def nonlinconfun(x):
    c = np.pi / 3 * x[0]**2 * x[1]
    return c

cons = sp.optimize.NonlinearConstraint(nonlinconfun, 200, np.inf)

bounds = [[0, np.inf],
          [0, np.inf]]
```

```
result = sp.optimize.minimize(fun = Area, x0 = x0, bounds = bounds, constraints
print(result)
print(Area(result.x))
print(Total(result.x))
```

```
 message: Optimization terminated successfully
 success: True
  status: 0
     fun: 143.23025086897755
       x: [ 5.131e+00  7.256e+00]
     nit: 6
     jac: [ 3.722e+01  1.316e+01]
    nfev: 18
    njev: 6
143.23025086897755
253.48896806009228
```

$$\min_{r,h} \frac{4}{3}\pi r^3$$

$$\text{such that } \frac{1}{3}\pi r^2 h \geq 200$$

```
result = sp.optimize.minimize(fun = Total, x0 = x0, bounds = bounds, constraints
print(result)
print(Area(result.x))
print(Total(result.x))
```

```
 message: Optimization terminated successfully
 success: True
  status: 0
     fun: 224.73872628150949
       x: [ 3.840e+00  1.296e+01]
     nit: 10
     jac: [ 7.804e+01  1.157e+01]
    nfev: 27
    njev: 9
162.98691601464716
224.73872628150949
```

Leading to:

- $\min_{r,h} Area\,(r,h) \approx 143$
- $\max_{r,h} Area\,(r,h) \approx 162$
- $\min_{r,h} Total\,(r,h) \approx 224$
- $\max_{r,h} Total\,(r,h) \approx 253$

Exercise 40 (Method)

Now let's solve this problem using an optimization method.

Click 🚀 –> **Live Code** to activate live coding on this page!

```
import scipy as sp
import numpy as np
```

```
#YOUR CODE HERE
```

Solution to Exercise 40 (Method)

```
def weighted_obj_pareto(x):
    return delta_Area * ( Area(x) -143) / ( 162 - 143) + delta_Total * ( Total(x
```

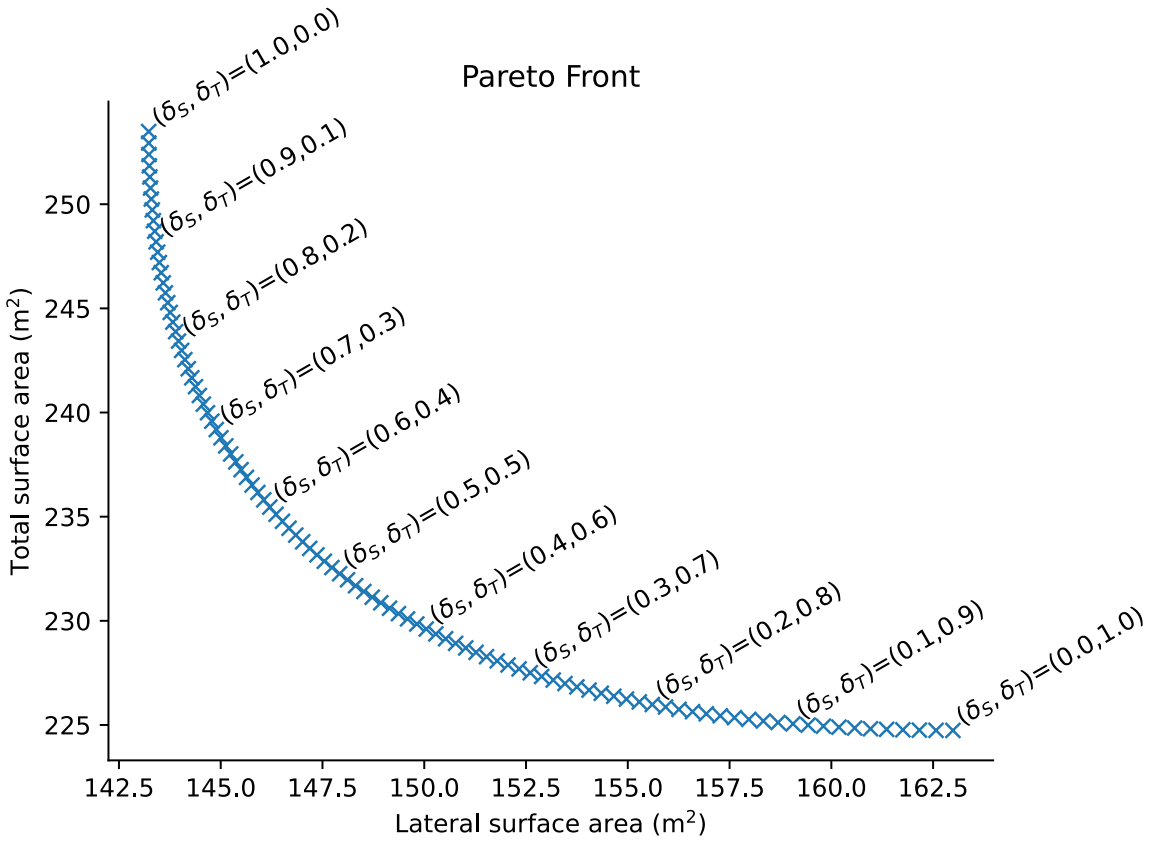
```
%%time
x_pareto_opt =[]
delta_Area_list = np.linspace(0,1,101)
delta_Area_opt = []
delta_Total_list = 1 - delta_Area_list
delta_Total_opt = []
for i in range(101):
    delta_Area = delta_Area_list[i]
    delta_Total = delta_Total_list[i]
    result_i = sp.optimize.minimize(fun = weighted_obj_pareto,x0=x0,bounds=bound
    if result_i.success:
        x_pareto_opt.append(result_i.x)
        delta_Area_opt.append(delta_Area)
        delta_Total_opt.append(delta_Total)
    else:
        print(result_i.message)
```

```
CPU times: total: 234 ms
Wall time: 227 ms
```

```
import matplotlib.pyplot as plt
%config InlineBackend.figure_formats = ['svg']
```

```
Area_pareto_opt = []
Total_pareto_opt = []
for i in range(len(x_pareto_opt)):
    Area_pareto_opt.append(Area(x_pareto_opt[i]))
    Total_pareto_opt.append(Total(x_pareto_opt[i]))

plt.figure()
plt.plot(Area_pareto_opt,Total_pareto_opt,'x')
for i in range(len(x_pareto_opt)):
    if i%10 == 0:
        plt.annotate(f"($\delta_S,\delta_T$)=({round(delta_Area_opt[i],1)},{rou
plt.xlabel('Lateral surface area (m$^2$)')
plt.ylabel('Total surface area (m$^2$)')
plt.title('Pareto Front')
ax = plt.gca()
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
```


Pareto Front
Total surface area (m^2)
Lateral surface area (m^2)
250
245
240
235
230
225
142.5
145.0
147.5
150.0
152.5
155.0
157.5
160.0
162.5
(δ_S, δ_T)=(1.0,0.0)
(δ_S, δ_T)=(0.9,0.1)
(δ_S, δ_T)=(0.8,0.2)
(δ_S, δ_T)=(0.7,0.3)
(δ_S, δ_T)=(0.6,0.4)
(δ_S, δ_T)=(0.5,0.5)
(δ_S, δ_T)=(0.4,0.6)
(δ_S, δ_T)=(0.3,0.7)
(δ_S, δ_T)=(0.2,0.8)
(δ_S, δ_T)=(0.1,0.9)
(δ_S, δ_T)=(0.0,1.0)

Metaheuristics using scipy and pymoo

Contents

In this chapter, we'll cover how to solve all optimization problem using metaheuristics, which are methods. Depending on the actual metaheuristic different models can be solved, but in general metaheuristics are more versatile than the gradient-based methods.

Model

The two method covered in this chapter will be applied on our most general nonlinear constrained optimization problem as defined before in (15).

Method

Two different methods will be treated: `scipy.optimize.differential_evolution` and the genetic algorithm from `pymoo.optimize`. There are many different methods available in different programming languagues, these two methods are only an example of what you could use.

Scipy

`scipy` has implemented, among others, the differential evolution method [Storn and Price, 1997]. The documentation of this function is available here: https://docs.scipy.org/doc/scipy-1.15.0/reference/generated/scipy.optimize.differential_evolution.html [The SciPy community, 2024]. In this course we'll cover only the relevant parts.

We need to run at least `scipy.optimize.differential_evolution(fun, x0, bounds, constraints, integrality, strategy, popsize, mutation, recombination, ...)` with:

- `fun`, the function representing the objective function $f(x)$ to be minimized. `fun` is a callable. The `scipy.optimize.minimize` function takes care of defining and inputting our design variable x.
- `x0`, the initial guess for our design variable x. It needs to be a `ndarray` with length n
- `Bounds`: A sequence of i `(min, max)` pairs for each element in x, defining the minimum x_i^l and maximum values x_i^u of that decision variable. For this function `None` cannot be used to specify no bound. The values `min` and `max` need to be finite.
- `constraints`, a single or a list of constraint objective defined in the same way as in nonlinear constrained optimization with:
  - `scipy.optimize.LinearConstraint`
  - `scipy.optimize.NonlinearConstraint`
- `integrality`, an `ndarray` specifying whether part of the n design variables is constrained to integer values.
- `strategy`, optional `string` argument which allows you to select another differential evolution strategy. The default is `best1bin`
- `popsize`, an optional `integer` argument which allows you to set the total population size.
- `mutation`, an optional `float` or `tuple(float,float)` argument which allows you to set the mutation constant.
- `recombination`, an optional `float` argument which allows you to set the recombinatino constant.

Please note that are even more options to adapt the optimization algorithm.

The function `scipy.optimize.differential_evolution` outputs an object `scipy.optimize.OptimizeResult` similar as `scipy.optimize.minimize` explained for unconstrained optimization.

Exercise 41

pymoo

`pymoo` has implemented, among others, the genetic algorithm. The documentation of this function, although very limited, is available here: https://pymoo.org/ [Blank and Deb, 2020]. Again, we'll cover only the relevant parts. Make sure you install pymoo as explained here in the section Add other packages and in the documentation of pymoo.

The main function we need is `pymoo.minimize(problem, algorithm, ...)` with:

- `problem`, pymoo object containing the problem
- `algorithm`, pymoo object containing the method

This results in an object `pymoo.core.result.Result` with:

- `Result.x`, the solution found
- `Result.F`, value of the objective function for the solution
- `Result.G`, value of the constraint functions for the solution
- `Result.time`, the time required to run the algorithm

The problem needs to be defined in an object, therefore we'll use `pymoo.problems.functional(n_var, objs, constr_ieq=[], constr_eq=[], xl, xu, ...)` with:

- `n_var`, the number of design variables n, this needs to be an `int`.

- `objs`, the objective function f to be minimized, this needs to be a `callable`.
- `constr_ieq`, list of m inquality constraint functions g, this needs to be a list of `callable`
- `constr_eq`, list of n equality constraint functions h, this needs to be a list of `callable`.
- `xl`, `Float` or `ndarray` of length n representing the lower bounds of the design variables.
- `xu`, `Float` or `ndarray` of length n representing the upper bounds of the design variables.

As a method, we'll use the genetic algorithm. This is stored in the object `pymoo.algorithms.soo.nonconvex.ga(pop_size=100, sampling=<pymoo.operators.sampling.rnd.FloatRandomSampling object>, selection=<pymoo.operators.selection.tournament.TournamentSelection object>, crossover=<pymoo.operators.crossover.sbx.SBX object>, mutation=<pymoo.operators.mutation.pm.PM object>, survival=<pymoo.algorithms.soo.nonconvex.ga.FitnessSurvival object>, ...)` with:

- `pop_size`, `int` defining size of the population
- `sampling`, pymoo object defining how sampling should happen. If you want to solve integer problems, input must be `pymoo.operators.sampling.rnd.IntegerRandomSampling()`
- `selection`, pymoo object defining how selection should happen
- `crossover`, pymoo object defining how crossover should happen. If you want to solve integer problems, input must be `pymoo.operators.crossover.sbx.SBX(repair=pymoo.operators.repair.rounding.RoundingRepair())`
- `mutation`, pymoo object defining how mutation should happen. If you want to solve integer problems, input must be `pymoo.operators.mutation.pm.PM(repair=pymoo.operators.repair.rounding.RoundingRepair())`
- `survival`, pymoo object defining how survival should happen

Error

Note that pymoo doesn't work in the interactive coding functionality of this website. Please install pymoo on your computer to use it locally.

Exercise 42

Homework exercise: Construction site

Contents

Click 🚀 –> **Live Code** to activate live coding on this page! Unfortunately, the `pymoo` package doesn't work yet in this book. Download this page to run the cells on your own computer.

Problem

As managers of a construction site, we have to decide the required equipment to move a given volume of excavated soil. In order to achieve the deadline for this working unit, we need to guarantee an averaged efficiency of $2700 \ \frac{\mathrm{m}^3}{h}$ during one month.

We have our own equipment, and also we can sub-contract another (just one) company. The efficiency of each equipment and the cost are given in the table:

Own equip.		Company 1		Company 2	
Eff. $\left(\frac{\mathrm{m}^3}{h}\right)$	Cost $\left(\frac{€}{h}\right)$	Eff. $\left(\frac{\mathrm{m}^3}{h}\right)$	Cost $\left(\frac{€}{h}\right)$	Eff. $\left(\frac{\mathrm{m}^3}{h}\right)$	Cost $\left(\frac{€}{h}\right)$
200	500	470	4.000	640	5.400
240	800	700	5.700	730	5.500
265	1.000	800	6.500	775	6.800
330	1.500				

Then, we can use part or all of our own equipment with some of the equipment options provided by another company.

What is the optimal equipment combination that minimizes the cost?

Model

We'll define the model as follows:

- Design variables: Whether or not to use each type of equipment
- Objective function: sum of the costs of the selected equipment
- Constraint functions: make use of one sub-contractor and create an efficiency of at least $2700\ \frac{\mathrm{m}^3}{h}$

Design variables

Let's start with our design variables. let's define them as a list of integer values either being 0 or 1, which selects the equipment type.

$$x = \begin{bmatrix} x_{\text{First item of own equipment}} \\ x_{\text{Second item of own equipment}} \\ x_{\text{Third item of own equipment}} \\ x_{\text{Fourth item of own equipment}} \\ x_{\text{First item of company 1}} \\ x_{\text{Second item of company 1}} \\ x_{\text{Third item of company 1}} \\ x_{\text{First item of company 2}} \\ x_{\text{Second item of company 2}} \\ x_{\text{Third item of company 2}} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}$$

The design variables either being 0 or 1 can be defined mathemetically:

$$x_i \in \{0, 1\}\, i = 1, 2, \ldots, 10 \tag{32}$$

Objective function

Now we can define the objective function as the product of the dimension to represent $\min_x f(x)$ in (4):

$$f(x) = 500 \cdot x_1 + 800 \cdot x_2 + 1000 \cdot x_3 + 1500 \cdot x_4 + 4000 \cdot x_5 + 5700 \cdot x_6 + 650 \tag{33}$$

As this is a linear relation, this can be converted to matrix notation. In case of the design variables defined as in (5) in the form $\min_x c^T x$ with c:

$$c = \begin{bmatrix} 500 & 800 & 1000 & 1500 & 4000 & 5700 & 6500 & 5400 & 5500 & 6800 \end{bmatrix}^T \tag{34}$$

Constraint function

We need to define two constraint functions, let's start with the inequality constraint functions as $g(x) \leq 0$ as defined in (4):

$$g(x) = -200 \cdot x_1 - 240 \cdot x_2 - 265 \cdot x_3 - 330 \cdot x_4 - 470 \cdot x_5 - 700 \cdot x_6 - 800 \cdot x \tag{35}$$

As this is a linear relation, this can be converted to matrix notation as defined in (5) in the form $A_{ub}x \leq b_{ub}$:

$$\begin{bmatrix}-200 & -240 & -265 & -330 & -470 & -700 & -800 & -640 & -730 & -775\end{bmatrix} x \tag{36}$$

Let's add the equality constraint function. As soon as we hire one piece of equipment this should be violated. It is defined as $h\left(x\right) \leq 0$ as in (15):

$$h\left(x\right) = \max\left(x_5, x_6, x_7\right) \cdot \max\left(x_8, x_9, x_{10}\right) \tag{37}$$

Clearly, this function is not linear, so we cannot define it in matrix formulation.

Method

Now let's solve this problem using both `scipy` and `pymoo`.

Import libraries

```
import numpy as np
```

SciPy

For SciPy this is easy:

```
import scipy as sp
```

pymoo

For pymoo we need to import a lot more and be more specific:

```
from pymoo.problems.functional import FunctionalProblem as FunctionalProblem
from pymoo.algorithms.soo.nonconvex.ga import GA
from pymoo.optimize import minimize
from pymoo.operators.sampling.rnd import IntegerRandomSampling
from pymoo.operators.crossover.sbx import SBX
from pymoo.operators.mutation.pm import PM
from pymoo.operators.repair.rounding import RoundingRepair
```

Error

Note that pymoo doesn't work in the interactive coding functionality of this website. Please download this notebook to use it locally.

Define variables

This is done differently in SciPy and pymoo.

SciPy

As before, we don't need to specify our variable x itself as defined in (31). However, we do need to specify that we have integers (specifically only 0 and 1 which will be covered by the bounds) In SciPy we use an array of booleans to specify which design variables are integers:

```
integers=np.array([True,True,True,True,True,True,True,True,True,True])
```

pymoo

In pymoo we don't have to specify that have integers, but we have to adapt all of our functions later on so that the outputs are integers. Furthermore, we must specify explicitly how many design variables we have:

```
n_var = 10
```

Define bounds

Now let's continue with the bounds, specified by (31) too:

SciPy

The bounds are defined as before:

```

```

```
bounds = [[0,1],
          [0,1],
          [0,1],
          [0,1],
          [0,1],
          [0,1],
          [0,1],
          [0,1],
          [0,1],
          [0,1]]
```

pymoo

In pymoo, the bounds are defined as separate arrays:

```
xl = np.array([0,0,0,0,0,0,0,0,0,0])
xu = np.array([1,1,1,1,1,1,1,1,1,1])
```

Define objective function

Let's define the objective function as defined in (34).

```
def obj(x):
    return np.array([500, 800, 1000, 1500, 4000, 5700, 6500, 5400, 5500, 6800])@x
```

Define constraint function

Let's define the constraint function as defined in (36) and (37)

```
g = np.array([-200, -240, -265, -330, -470, -700, -800, -640, -730, -775])

def nonlinconfun(x):
    return max(x[4:7]) * max(x[7:])
```

SciPy

In SciPy we need to store the functions in a scipy-object including the bounds

```
lincon = sp.optimize.LinearConstraint(g, lb=-np.inf, ub=-2700)
nonlincon = sp.optimize.NonlinearConstraint(nonlinconfun, 0, 0)
```

pymoo

In pymoo the functions can be inserted directly in the problem object later on. However, this requires a function for the linear constraints instead of just the array

```
def linconfun(x):
    return g@x + 2700
```

Solve the problem

Now let's solve the problem using both SciPy and pymoo.

SciPy

```
result_scipy = sp.optimize.differential_evolution(func = obj,bounds = bounds,constraints=
print(result_scipy)
```

```
             message: Optimization terminated successfully.
             success: True
                 fun: 19000.0
                   x: [ 1.000e+00  1.000e+00  0.000e+00  1.000e+00
                        1.000e+00  1.000e+00  1.000e+00  0.000e+00
                        0.000e+00  0.000e+00]
                 nit: 34
                nfev: 1226
          population: [[ 1.000e+00  1.000e+00 ...  0.000e+00  0.000e+00]
                       [ 1.000e+00  1.000e+00 ...  0.000e+00  0.000e+00]
                       ...
                       [ 1.000e+00  1.000e+00 ...  0.000e+00  0.000e+00]
                       [ 1.000e+00  1.000e+00 ...  0.000e+00  0.000e+00]]
 population_energies: [ 1.900e+04  1.900e+04 ...  1.900e+04  1.900e+04]
              constr: [array([ 0.000e+00]), array([ 0.000e+00])]
    constr_violation: 0.0
               maxcv: 0.0
```

Pymoo

In pymoo we have to define the problem and algorithm as objects, and call them from the `pymoo.minimize` function. For the `GA` functions, we repair some stuff so that we only deal with integer values

```
```

```
problem = FunctionalProblem(n_var=n_var, objs=obj, constr_ieq=[linconfun],constr_eq = [no
algorithm = GA(sampling=IntegerRandomSampling(),crossover=SBX(repair=RoundingRepair()),mu
```

Now we can solve the problem

```
result_pymoo = minimize(problem, algorithm)
print(result_pymoo.X)
print(result_pymoo.F)
```

```
[0 1 1 1 1 1 1 0 0 0]
[19500.]
```

Exercise 43

Class exercise: choose yourself!

Solve one of the previous class exercises using metaheuristics!

- Class exercise: Furniture manufacturing
- Class exercise: Suspended beams
- Class exercise: Ice cream cone

Exercise 44 (Method)

Let's solve this problem using a metaheuristic!

Click 🚀 –> **Live Code** to activate live coding on this page!

```
import scipy as sp
import numpy as np
```

```
#YOUR CODE HERE
```

Stochastic workshop

Deterministic

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import differential_evolution
import scipy as sp

def reliability(beta0_b, d_b, L_b, Thoriz):
    # --- Reliability estimation
    # beta_bt: reliability curve over time for bridge b
    # beta0_b: Initial reliability of bridge b at t=0
    # d_b, L_b: when the degradation starts and reaches beta=0 for each bridge
    # Thoriz: maximum time studied
    nb = len(beta0_b)
    beta_bt = np.zeros((nb, Thoriz))

    for b in range(nb):
        #horziontal branch
        timesteps = int(np.floor(d_b[b]))
        beta_bt[b, :timesteps] = beta0_b[b]

        #degradation till L_b
        servicelife = int(np.floor(L_b[b]))
        t = np.arange(timesteps + 1, servicelife + 1)
        beta_bt[b, timesteps:servicelife] = beta0_b[b] / (L_b[b] - d_b[b]) * (L_b[b] - t)

    return beta_bt

def MaintenanceApplication(beta_bt, X_bt, Thoriz):
    # --- Given a maintenance strategy given by X_bt, where 1 means that bridge
    # b undergoes maintenance at time t, this function gives the updated
    # reliability profile
    updatedbeta_bt = beta_bt.copy()
    InterventionTimes = np.where(np.sum(X_bt, axis=0) > 0)[0]

    for t in InterventionTimes:
        intervbrid = np.where(X_bt[:, t] == 1)[0]
        timecells = Thoriz - t - 1
        updatedbeta_bt[intervbrid, t+1:Thoriz] = beta_bt[intervbrid, :timecells]

    return updatedbeta_bt

def Frequency2Schedule(y_b, nb, nt, Thoriz):
    # Transforms a vector indicating the frequency to a schedule matrix [0/1]
    freq_b = y_b.reshape((nt, nb)).T
    cumFreq_b = cumulateFreq(freq_b)
    X_bt = np.zeros((nb, Thoriz))

    for b in range(nb):
        idx = cumFreq_b[b, cumFreq_b[b, :] != 0].astype(int)
        idx = idx[idx <= Thoriz]
        if len(idx) > 0:
            X_bt[b, idx - 1] = 1

    return X_bt

def cumulateFreq(freq_b):
    # creates the cumulative frequency
    nb, nt = freq_b.shape
    cumFreq_b = np.zeros((nb, nt))

    for b in range(nb):
        f_b = freq_b[b, freq_b[b, :] != 0]
        cumFreq_b[b, :len(f_b)] = np.cumsum(f_b)
```

```
    return cumFreq_b

def CountNinterv(y_b, nt, nb):
    # from vector y_b, it computes the number of interventions per bridge
    y_b01 = y_b != 0
    freq_b = y_b01.reshape((nt, nb)).T
    ninterv_b = np.sum(freq_b, axis=1)
    return ninterv_b
```

```
Thoriz = 25
# for each bridge b:  beta0 d(years) L(years) C(Meuros)
data = np.array([
    [1, 5, 10, 0.6],
    [1, 6, 11, 0.8],
    [1, 7, 15, 1.0],
    [1, 10, 20, 0.7]
])
beta0_b, d_b, L_b, C_b = data[:, 0], data[:, 1], data[:, 2], data[:, 3]

# Annual resources (M euro)
R = 1.4

# Initial reliability estimation
nb = len(beta0_b)
beta_bt0 = reliability(beta0_b, d_b, L_b, Thoriz)

plt.figure()
col = ['r', 'b', 'g', 'm']
for b in range(nb):
    plt.plot(range(1, Thoriz + 1), beta_bt0[b, :], f'-{col[b]}', linewidth=2)
plt.grid()
plt.xlim([1, Thoriz])
plt.ylim([0, 1])
plt.xlabel('Time, t (years)')
plt.ylabel('Reliability index, β')
plt.title('Reliability Index Over Time')
plt.show()

# Deterministic Optimization
# Variables ->  e.g., freq=[10 2 0 0 ; 5, 5 0 0; 0 0 0 0];

maxInter_b = np.ceil(Thoriz / L_b).astype(int) + 1 #max number of interventions expected
nt = max(maxInter_b) #to dimension the matrix, max interventions experienced by the worst
nvars = nb * nt #number of design variables
integers = np.array([True] * nvars) #all variables are integers

#  Range of definition: [0,max number of years with no interventions==L_b]

lb = np.zeros(nvars)
ub = np.zeros(nvars)
pos = 0
for b in range(nb):
    # This guarantees interventions freq. within the service life (if conducted)
    ub[pos:pos + maxInter_b[b]] = L_b[b] - 1
    # this forces zeros after the max number of interventions expected
    if maxInter_b[b] < nb:
        ub[pos + maxInter_b[b]:pos + nb - 1] = 0
    pos += nt

def nonlcon(y_b):
    #  From frequency to a schedule matrix [0/1]
    X_bt = Frequency2Schedule(y_b, nb, nt, Thoriz)

    # Constraint 1 >>> the last intervention is within Thoriz --> LastInterv<Thoriz --> L
    lastInterv_b = np.max(cumulateFreq(y_b.reshape((nt, nb)).T), axis=1)
    c1 = lastInterv_b - Thoriz - 1

    # Constraint 2 >>> C_b*X_bt <= R   for all t
    c2 = np.sum(C_b[:, None] * X_bt, axis=0) - R

    # Constraint 3 >>> beta_bt>0  --> sum(sum(beta_bt))<=0
    beta_bt = MaintenanceApplication(beta_bt0, X_bt, Thoriz)
```

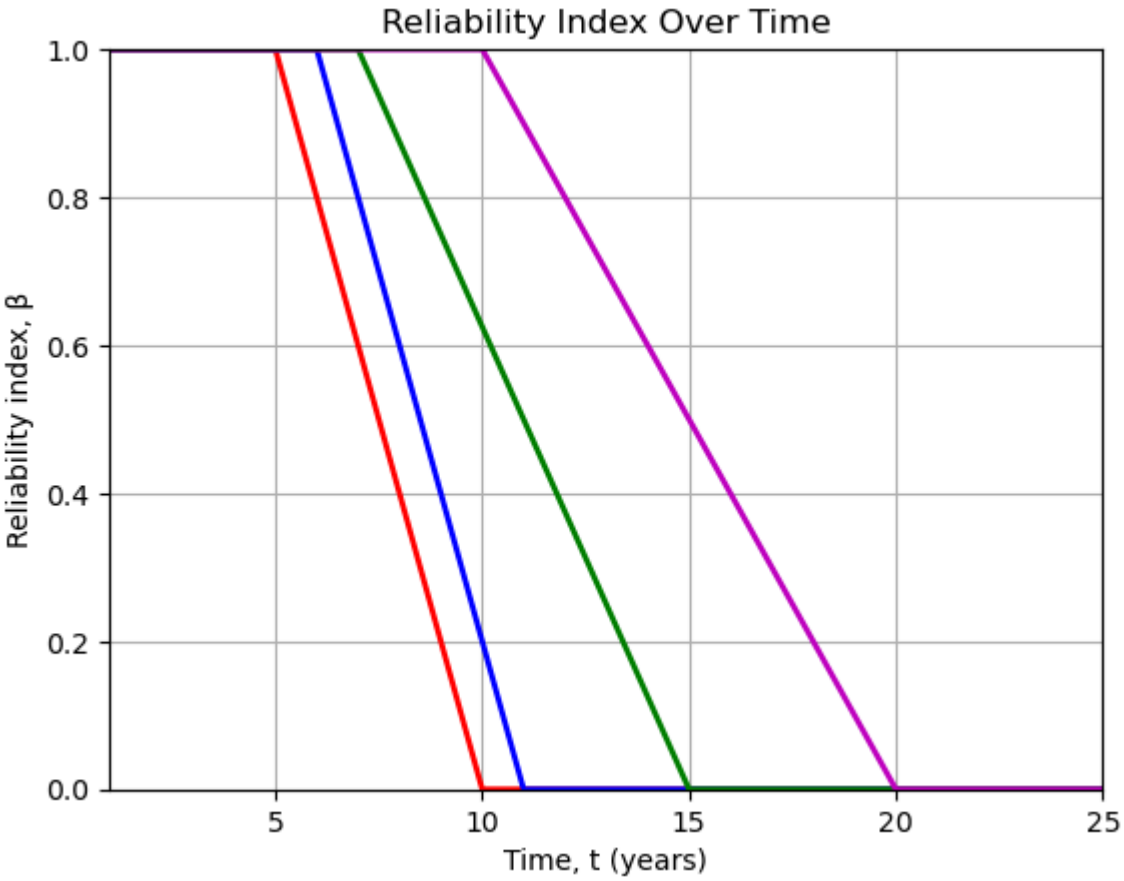
```
    c3 = np.array([np.sum(beta_bt == 0)])

    c = np.concatenate([c1, c2, c3])
    return c

cons = sp.optimize.NonlinearConstraint(nonlcon, np.array([-np.inf] * (nb + Thoriz + 1)),

def fun(y_b):
    #  minimize cost trying to go for the largest time between interventions (== max y_b)
    ninterv_b = CountNinterv(y_b, nt, nb)
    return np.dot(C_b, ninterv_b)

bounds = [[lb[i], ub[i]] for i in range(nvars)]
```




```
result = differential_evolution(fun, bounds, constraints=cons, integrality=integers, disp
print(result)
```

```
differential_evolution step 1: f(x)= 11.0
differential_evolution step 2: f(x)= 11.0
differential_evolution step 3: f(x)= 8.200000000000001
differential_evolution step 4: f(x)= 8.200000000000001
differential_evolution step 5: f(x)= 8.200000000000001
differential_evolution step 6: f(x)= 8.200000000000001
differential_evolution step 7: f(x)= 8.200000000000001
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             message: Optimization terminated successfully.
             success: True
                 fun: 4.5
                   x: [ 0.000e+00  8.000e+00 ...  1.400e+01  0.000e+00]
                 nit: 517
                nfev: 17373
          population: [[ 0.000e+00  8.000e+00 ...  1.400e+01  0.000e+00]
                       [ 0.000e+00  7.000e+00 ...  1.400e+01  0.000e+00]
                       ...
                       [ 0.000e+00  8.000e+00 ...  1.300e+01  0.000e+00]
                       [ 0.000e+00  9.000e+00 ...  1.400e+01  0.000e+00]]
 population_energies: [ 4.500e+00  4.500e+00 ...  4.500e+00  4.500e+00]
              constr: [array([ 0.000e+00,  0.000e+00, ...,  0.000e+00,
                               0.000e+00])]
    constr_violation: 0.0
                maxcv: 0.0
```
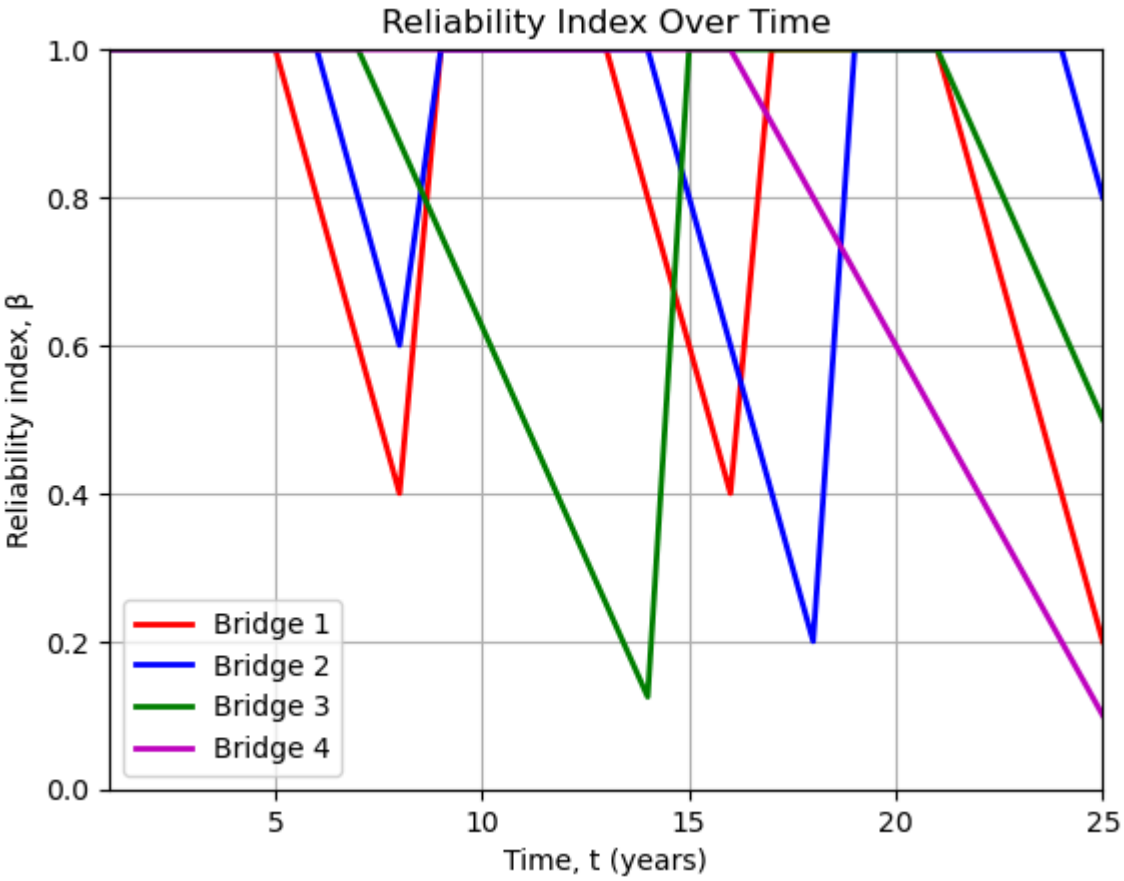

```python
Yopt = result.x
OptInterv = cumulateFreq(Yopt.reshape((nt, nb)).T)
X_bt = Frequency2Schedule(Yopt, nb, nt, Thoriz)
beta_bt = MaintenanceApplication(beta_bt0, X_bt, Thoriz)

print(f'Min Cost = {result.fun:.2f}')
ninterv = np.sum(CountNinterv(Yopt, nt, nb))
print(f'Number of interventions = {ninterv}')
c = nonlcon(Yopt)
BudgetIssues = np.sum(c[-Thoriz - 1:-1] > 0)
print(f'Number of time intervals with exceeding budget = {BudgetIssues}')
reliabIssues = c[-1]
print(f'Number of time intervals with reliability issues = {reliabIssues}')

plt.figure()
h = []
leg = []
for b in range(nb):
    h.append(plt.plot(range(1, Thoriz + 1), beta_bt[b, :], f'-{col[b]}', linewidth=2))
    leg.append(f'Bridge {b + 1}')
plt.legend(leg, loc='best')
plt.grid()
plt.xlim([1, Thoriz])
plt.ylim([0, 1])
plt.xlabel('Time, t (years)')
plt.ylabel('Reliability index, β')
plt.title('Reliability Index Over Time')
plt.show()
```

```
Min Cost = 4.50
Number of interventions = 6
Number of time intervals with exceeding budget = 0
Number of time intervals with reliability issues = 0.0
```


Reliability Index Over Time
Reliability index, β
Time, t (years)
1.0
0.8
0.6
0.4
0.2
0.0
5
10
15
20
25
Bridge 1
Bridge 2
Bridge 3
Bridge 4

Robust

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import differential_evolution
import scipy as sp

def reliability(beta0_b, d_b, L_b, Thoriz):
    # --- Reliability estimation
    # beta_bt: reliability curve over time for bridge b
    # beta0_b: Initial reliability of bridge b at t=0
    # d_b, L_b: when the degradation starts and reaches beta=0 for each bridge
    # Thoriz: maximum time studied
    nb = len(beta0_b)
    beta_bt = np.zeros((nb, Thoriz))

    for b in range(nb):
        #horziontal branch
        timesteps = int(np.floor(d_b[b]))
        beta_bt[b, :timesteps] = beta0_b[b]

        #degradation till L_b
        servicelife = int(np.floor(L_b[b]))
        t = np.arange(timesteps + 1, servicelife + 1)
        beta_bt[b, timesteps:servicelife] = beta0_b[b] / (L_b[b] - d_b[b]) * (L_b[b] - t)

    return beta_bt

def MaintenanceApplication(beta_bt, X_bt, Thoriz):
    # --- Given a maintenance strategy given by X_bt, where 1 means that bridge
    # b undergoes maintenance at time t, this function gives the updated
    # reliability profile
    updatedbeta_bt = beta_bt.copy()
    InterventionTimes = np.where(np.sum(X_bt, axis=0) > 0)[0]

    for t in InterventionTimes:
        intervbrid = np.where(X_bt[:, t] == 1)[0]
        timecells = Thoriz - t - 1
        updatedbeta_bt[intervbrid, t+1:Thoriz] = beta_bt[intervbrid, :timecells]

    return updatedbeta_bt

def Frequency2Schedule(y_b, nb, nt, Thoriz):
    # Transforms a vector indicating the frequency to a schedule matrix [0/1]
    freq_b = y_b.reshape((nt, nb)).T
    cumFreq_b = cumulateFreq(freq_b)
    X_bt = np.zeros((nb, Thoriz))

    for b in range(nb):
        idx = cumFreq_b[b, cumFreq_b[b, :] != 0].astype(int)
        idx = idx[idx <= Thoriz]
        if len(idx) > 0:
            X_bt[b, idx - 1] = 1

    return X_bt

def cumulateFreq(freq_b):
    # creates the cumulative frequency
    nb, nt = freq_b.shape
    cumFreq_b = np.zeros((nb, nt))

    for b in range(nb):
        f_b = freq_b[b, freq_b[b, :] != 0]
        cumFreq_b[b, :len(f_b)] = np.cumsum(f_b)
```

```
    return cumFreq_b

def CountNinterv(y_b, nt, nb):
    # from vector y_b, it computes the number of interventions per bridge
    y_b01 = y_b != 0
    freq_b = y_b01.reshape((nt, nb)).T
    ninterv_b = np.sum(freq_b, axis=1)
    return ninterv_b
```

```python
Thoriz = 25
# for each bridge b:  beta0 d1(years) d2(years) L1(years) L2(years) C(Meuros)
data = np.array([
    [1, 3, 7, 9, 11, 0.6],
    [1, 5, 7, 9, 13, 0.8],
    [1, 6, 8, 10, 20, 1.0],
    [1, 9, 11, 19, 21, 0.7]
])
beta0_b, d1_b, d2_b, L1_b, L2_b, C_b = data[:, 0], data[:, 1], data[:, 2], data[:, 3], da

# Annual resources (M euro)
R = 1.4

# Initial reliability estimation
nb = len(beta0_b)
beta_bt01 = reliability(beta0_b, d1_b, L1_b, Thoriz)
beta_bt02 = reliability(beta0_b, d2_b, L2_b, Thoriz)

plt.figure()
col = ['r', 'b', 'g', 'm']
for b in range(nb):
    plt.plot(range(1, Thoriz + 1), beta_bt01[b, :], f'-{col[b]}', linewidth=2)
    plt.plot(range(1, Thoriz + 1), beta_bt02[b, :], f'-{col[b]}', linewidth=2)
plt.grid()
plt.xlim([1, Thoriz])
plt.ylim([0, 1])
plt.xlabel('Time, t (years)')
plt.ylabel('Reliability index, β')
plt.title('Reliability Index Over Time')
plt.show()

# Deterministic Optimization
# Variables ->  e.g., freq=[10 2 0 0 ; 5, 5 0 0; 0 0 0 0];

maxInter_b = np.ceil(Thoriz / L1_b).astype(int) + 1 #max number of interventions expected
nt = max(maxInter_b) #to dimension the matrix, max interventions experienced by the worst
nvars = nb * nt #number of design variables
integers = np.array([True] * nvars) #all variables are integers

#  Range of definition: [0,max number of years with no interventions==L_b]

lb = np.zeros(nvars)
ub = np.zeros(nvars)
pos = 0
for b in range(nb):
    # This guarantees interventions freq. within the service life (if conducted)
    ub[pos:pos + maxInter_b[b]] = L2_b[b] - 1
    # this forces zeros after the max number of interventions expected
    if maxInter_b[b] < nb:
        ub[pos + maxInter_b[b]:pos + nb - 1] = 0
    pos += nt

def nonlcon(y_b):
    #  From frequency to a schedule matrix [0/1]
    X_bt = Frequency2Schedule(y_b, nb, nt, Thoriz)

    # Constraint 1 >>> the last intervention is within Thoriz --> LastInterv<Thoriz --> L
    lastInterv_b = np.max(cumulateFreq(y_b.reshape((nt, nb)).T), axis=1)
    c1 = lastInterv_b - Thoriz - 1

    # Constraint 2 >>> C_b*X_bt <= R   for all t
    c2 = np.sum(C_b[:, None] * X_bt, axis=0) - R
```

```
    # Constraint 3 >>> beta_bt>0 for the lowest reliability
    beta_bt = MaintenanceApplication(beta_bt01, X_bt, Thoriz)
    c3 = np.array([np.sum(beta_bt == 0)])

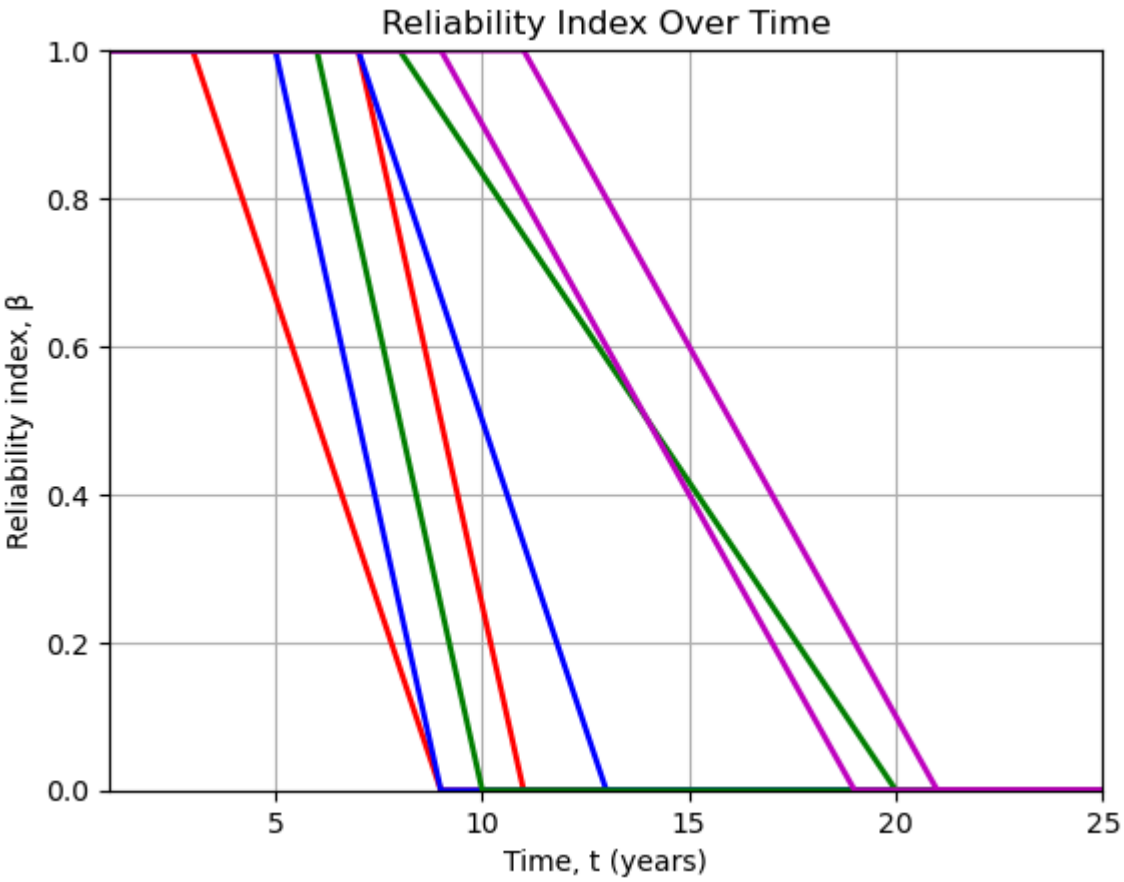
    # Constraint 4 >>> beta_bt>0 for the highest reliability
    beta_bt = MaintenanceApplication(beta_bt02, X_bt, Thoriz)
    c4 = np.array([np.sum(beta_bt == 0)])

    c = np.concatenate([c1, c2, c3, c4])
    return c

cons = sp.optimize.NonlinearConstraint(nonlcon, np.array([-np.inf] * (nb + Thoriz + 2)),

def fun(y_b):
    #  minimize cost trying to go for the largest time between interventions (== max y_b)
    ninterv_b = CountNinterv(y_b, nt, nb)
    return np.dot(C_b, ninterv_b)

bounds = [[lb[i], ub[i]] for i in range(nvars)]
```




```
result = differential_evolution(fun, bounds, constraints=cons, integrality=integers, disp
print(result)
```

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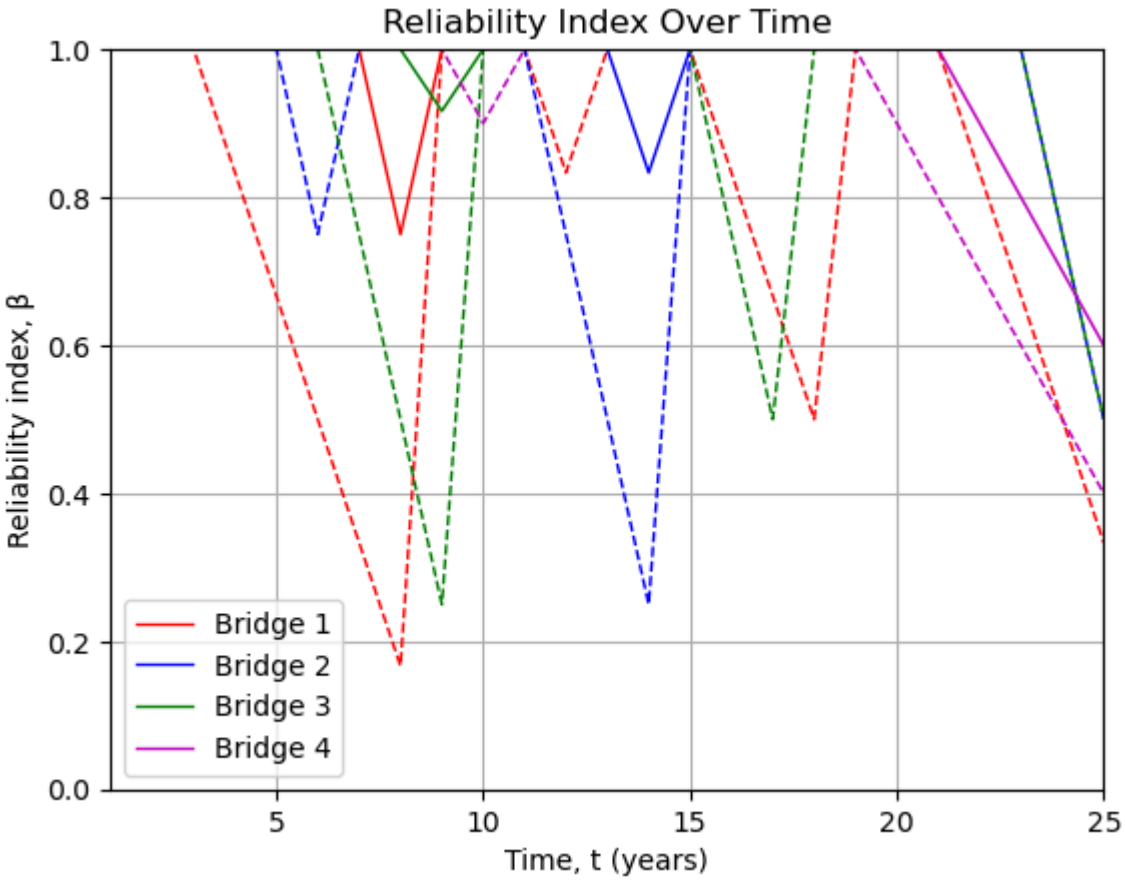
differential_evolution step 1034: f(x)= 6.9
             message: Optimization terminated successfully.
             success: True
                 fun: 6.9
                   x: [ 8.000e+00  6.000e+00 ...  8.000e+00  0.000e+00]
                 nit: 1034
                nfev: 11133
          population: [[ 8.000e+00  6.000e+00 ...  8.000e+00  0.000e+00]
                       [ 7.000e+00  7.000e+00 ...  7.000e+00  0.000e+00]
                       ...
                       [ 8.000e+00  7.000e+00 ...  8.000e+00  0.000e+00]
                       [ 8.000e+00  7.000e+00 ...  9.000e+00  0.000e+00]]
 population_energies: [ 6.900e+00  6.900e+00 ...  6.900e+00  6.900e+00]
              constr: [array([ 0.000e+00,  0.000e+00, ...,  0.000e+00,
                               0.000e+00])]
    constr_violation: 0.0
                maxcv: 0.0
```

```
Yopt = result.x
OptInterv = cumulateFreq(Yopt.reshape((nt, nb)).T)
X_bt = Frequency2Schedule(Yopt, nb, nt, Thoriz)
beta1_bt = MaintenanceApplication(beta_bt01, X_bt, Thoriz)
beta2_bt = MaintenanceApplication(beta_bt02, X_bt, Thoriz)

print(f'Min Cost = {result.fun:.2f}')
ninterv = np.sum(CountNinterv(Yopt, nt, nb))
print(f'Number of interventions = {ninterv}')
c = nonlcon(Yopt)
BudgetIssues = np.sum(c[-Thoriz - 1:-1] > 0)
print(f'Number of time intervals with exceeding budget = {BudgetIssues}')
reliabIssues = c[-1]
print(f'Number of time intervals with reliability issues = {reliabIssues}')

plt.figure()
h = []
leg = []
for b in range(nb):
    h.append(plt.plot(range(1, Thoriz + 1), beta1_bt[b, :], f'--{col[b]}', linewidth=1))
    plt.plot(range(1, Thoriz + 1), beta2_bt[b, :], f'-{col[b]}', linewidth=1, label = f'B
plt.legend(loc='best')
plt.grid()
plt.xlim([1, Thoriz])
plt.ylim([0, 1])
plt.xlabel('Time, t (years)')
plt.ylabel('Reliability index, β')
plt.title('Reliability Index Over Time')
plt.show()
```

```
Min Cost = 6.90
Number of interventions = 9
Number of time intervals with exceeding budget = 0
Number of time intervals with reliability issues = 0.0
```



Stochastic

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import differential_evolution
import scipy as sp

def reliability(beta0_b, d_b, L_b, Thoriz):
    # --- Reliability estimation
    # beta_bt: reliability curve over time for bridge b
    # beta0_b: Initial reliability of bridge b at t=0
    # d_b, L_b: when the degradation starts and reaches beta=0 for each bridge
    # Thoriz: maximum time studied
    nb = len(beta0_b)
    beta_bt = np.zeros((nb, Thoriz))

    for b in range(nb):
        #horziontal branch
        timesteps = int(np.floor(d_b[b]))
        beta_bt[b, :timesteps] = beta0_b[b]

        #degradation till L_b
        servicelife = int(np.floor(L_b[b]))
        t = np.arange(timesteps + 1, servicelife + 1)
        beta_bt[b, timesteps:servicelife] = beta0_b[b] / (L_b[b] - d_b[b]) * (L_b[b] - t)

    return beta_bt

def MaintenanceApplication(beta_bt, X_bt, Thoriz):
    # --- Given a maintenance strategy given by X_bt, where 1 means that bridge
    # b undergoes maintenance at time t, this function gives the updated
    # reliability profile
    updatedbeta_bt = beta_bt.copy()
    InterventionTimes = np.where(np.sum(X_bt, axis=0) > 0)[0]

    for t in InterventionTimes:
        intervbrid = np.where(X_bt[:, t] == 1)[0]
        timecells = Thoriz - t - 1
        updatedbeta_bt[intervbrid, t+1:Thoriz] = beta_bt[intervbrid, :timecells]

    return updatedbeta_bt

def Frequency2Schedule(y_b, nb, nt, Thoriz):
    # Transforms a vector indicating the frequency to a schedule matrix [0/1]
    freq_b = y_b.reshape((nt, nb)).T
    cumFreq_b = cumulateFreq(freq_b)
    X_bt = np.zeros((nb, Thoriz))

    for b in range(nb):
        idx = cumFreq_b[b, cumFreq_b[b, :] != 0].astype(int)
        idx = idx[idx <= Thoriz]
        if len(idx) > 0:
            X_bt[b, idx - 1] = 1

    return X_bt

def cumulateFreq(freq_b):
    # creates the cumulative frequency
    nb, nt = freq_b.shape
    cumFreq_b = np.zeros((nb, nt))

    for b in range(nb):
        f_b = freq_b[b, freq_b[b, :] != 0]
        cumFreq_b[b, :len(f_b)] = np.cumsum(f_b)
```

```
    return cumFreq_b

def CountNinterv(y_b, nt, nb):
    # from vector y_b, it computes the number of interventions per bridge
    y_b01 = y_b != 0
    freq_b = y_b01.reshape((nt, nb)).T
    ninterv_b = np.sum(freq_b, axis=1)
    return ninterv_b

def ProbFulfilment(beta_bti0, X_bt, Thoriz):
    nb, _, nMCS = beta_bti0.shape

    # --- MCS
    Achivement_bi = np.zeros((nb, nMCS))
    for i in range(nMCS):
        beta_bt = MaintenanceApplication(beta_bti0[:, :, i], X_bt, Thoriz)
        failures_b = np.sum(beta_bt == 0, axis=1)
        Achivement_bi[:, i] = (failures_b == 0)

    ProbRel_b = np.sum(Achivement_bi, axis=1) / nMCS
    return ProbRel_b
```

```
Thoriz = 25
# for each bridge b:  beta0 mu_d(years) sigma_d(years) mu_L(years) sigma_L(years) C(Meuro
data = np.array([
    [1, 5, 2 / 1.96, 10, 1 / 1.96, 0.6],
    [1, 6, 1 / 1.96, 11, 2 / 1.96, 0.8],
    [1, 7, 1 / 1.96, 15, 5 / 1.96, 1.0],
    [1, 10, 1 / 1.96, 20, 1 / 1.96, 0.7]
])
beta0_b, mud_b, sigd_b, muL_b, sigL_b, C_b = data[:, 0], data[:, 1], data[:, 2], data[:,

# Annual resources (M euro)
R = 1.4

# Sample size of the probabilistic analysis
nMCS=10

# --- Initial reliability estimation
nb = len(beta0_b)  # number of bridges

# nMCS cases of d and L
d_bi, L_bi = np.zeros((nb, nMCS)), np.zeros((nb, nMCS))
for b in range(nb):
    d_bi[b, :] = np.random.normal(mud_b[b], sigd_b[b], nMCS)
    L_bi[b, :] = np.random.normal(muL_b[b], sigL_b[b], nMCS)

beta_bti0 = np.zeros((nb, Thoriz, nMCS))
for i in range(nMCS):
    beta_bti0[:, :, i] = reliability(beta0_b, d_bi[:, i], L_bi[:, i], Thoriz)  # reliabil

plt.figure()
col = ['r', 'b', 'g', 'm']
for i in range(nMCS):
    for b in range(nb):
        plt.plot(range(1, Thoriz + 1), beta_bti0[b, :, i], f'-{col[b]}', linewidth=0.5)
for b in range(nb):
    minbeta_t0 = reliability([beta0_b[b]], [np.min(d_bi[b, :])], [np.min(L_bi[b, :])], Th
    maxbeta_t0 = reliability([beta0_b[b]], [np.max(d_bi[b, :])], [np.max(L_bi[b, :])], Th
    plt.plot(range(1, Thoriz + 1), minbeta_t0, f'-{col[b]}', linewidth=3)
    plt.plot(range(1, Thoriz + 1), maxbeta_t0, f'-{col[b]}', linewidth=3)
plt.grid()
plt.xlim([1, Thoriz])
plt.ylim([0, 1])
plt.xlabel('Time, t (years)')
plt.ylabel('Reliability index, β')
plt.title('Reliability Index Over Time')
plt.show()

# Deterministic Optimization
# Variables ->  e.g., freq=[10 2 0 0 ; 5, 5 0 0; 0 0 0 0];

maxInter_b = np.ceil(Thoriz / np.min(L_bi, axis=1)).astype(int) + 1  # max number of inte
nt = max(maxInter_b)  # to dimension the matrix, max interventions experienced by the wor
nvars = nb * nt #number of design variables
integers = np.array([True] * nvars) #all variables are integers

#  Range of definition: [0,max number of years with no interventions==L_b]

lb = np.zeros((nvars, 1))
# upper bound
ub = lb.copy()
pos = 0
for b in range(nb):
    # This guarantees interventions freq. within the service life (if conducted)
```

```
    ub[pos:pos + maxInter_b[b]] = np.max(L_bi[b, :], axis=0) - 1  # the largest service l
    # this forces zeros after the max number of interventions expected
    if maxInter_b[b] < nb:
        ub[pos + maxInter_b[b]:pos + nb - 1] = 0
    pos += nt
ub = np.ceil(ub)

def nonlcon(y_b):
    #  From frequency to a schedule matrix [0/1]
    X_bt = Frequency2Schedule(y_b, nb, nt, Thoriz)

    # Constraint 1 >>> the last intervention is within Thoriz --> LastInterv<Thoriz --> L
    lastInterv_b = np.max(cumulateFreq(y_b.reshape((nt, nb)).T), axis=1)
    c1 = lastInterv_b - Thoriz - 1

    # Constraint 2 >>> C_b*X_bt <= R   for all t
    c2 = np.sum(C_b[:, None] * X_bt, axis=0) - R

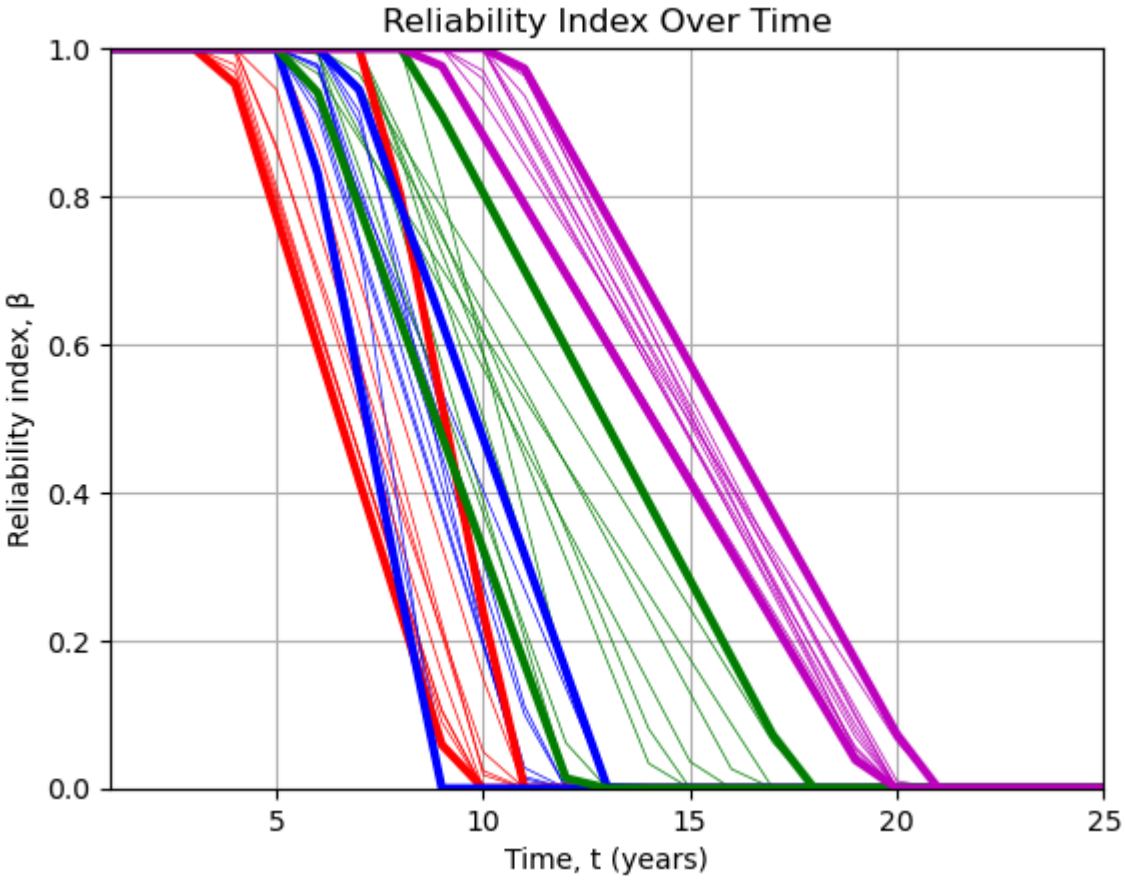
    # Constraint 3 >>> Prob(beta_bt>0)>=0.90  --> sum(sum(beta_bt))<=0 for all b
    ProbRel_b = ProbFulfilment(beta_bti0, X_bt, Thoriz)
    c3 = 0.90 - ProbRel_b

    c = np.concatenate([c1, c2, c3])
    return c

cons = sp.optimize.NonlinearConstraint(nonlcon, np.array([-np.inf] * (nb + Thoriz + nb)),

def fun(y_b):
    #  minimize cost trying to go for the largest time between interventions (== max y_b)
    ninterv_b = CountNinterv(y_b, nt, nb)
    return np.dot(C_b, ninterv_b)

bounds = [[lb[i][0], ub[i][0]] for i in range(nvars)]
```



```
result = differential_evolution(fun, bounds, constraints=cons, integrality=integers, disp
print(result.x)
```

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differential_evolution step 1: f(x)= 11.7
differential_evolution step 2: f(x)= 11.7
differential_evolution step 3: f(x)= 11.7
differential_evolution step 4: f(x)= 9.3
differential_evolution step 5: f(x)= 9.3
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differential_evolution step 7: f(x)= 9.3
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differential_evolution step 638: f(x)= 6.9
differential_evolution step 639: f(x)= 6.9
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differential_evolution step 640: f(x)= 6.9
differential_evolution step 641: f(x)= 6.9
differential_evolution step 642: f(x)= 6.9
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differential_evolution step 692: f(x)= 6.1
differential_evolution step 693: f(x)= 6.1
differential_evolution step 694: f(x)= 6.1
differential_evolution step 695: f(x)= 6.1
differential_evolution step 696: f(x)= 6.1
differential_evolution step 697: f(x)= 6.1
differential_evolution step 698: f(x)= 6.1
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differential_evolution step 700: f(x)= 6.1
differential_evolution step 701: f(x)= 6.1
differential_evolution step 702: f(x)= 6.1
differential_evolution step 703: f(x)= 6.1
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differential_evolution step 704: f(x)= 6.1
differential_evolution step 705: f(x)= 6.1
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differential_evolution step 707: f(x)= 6.1
differential_evolution step 708: f(x)= 6.1
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differential_evolution step 711: f(x)= 6.1
differential_evolution step 712: f(x)= 6.1
differential_evolution step 713: f(x)= 6.1
differential_evolution step 714: f(x)= 6.1
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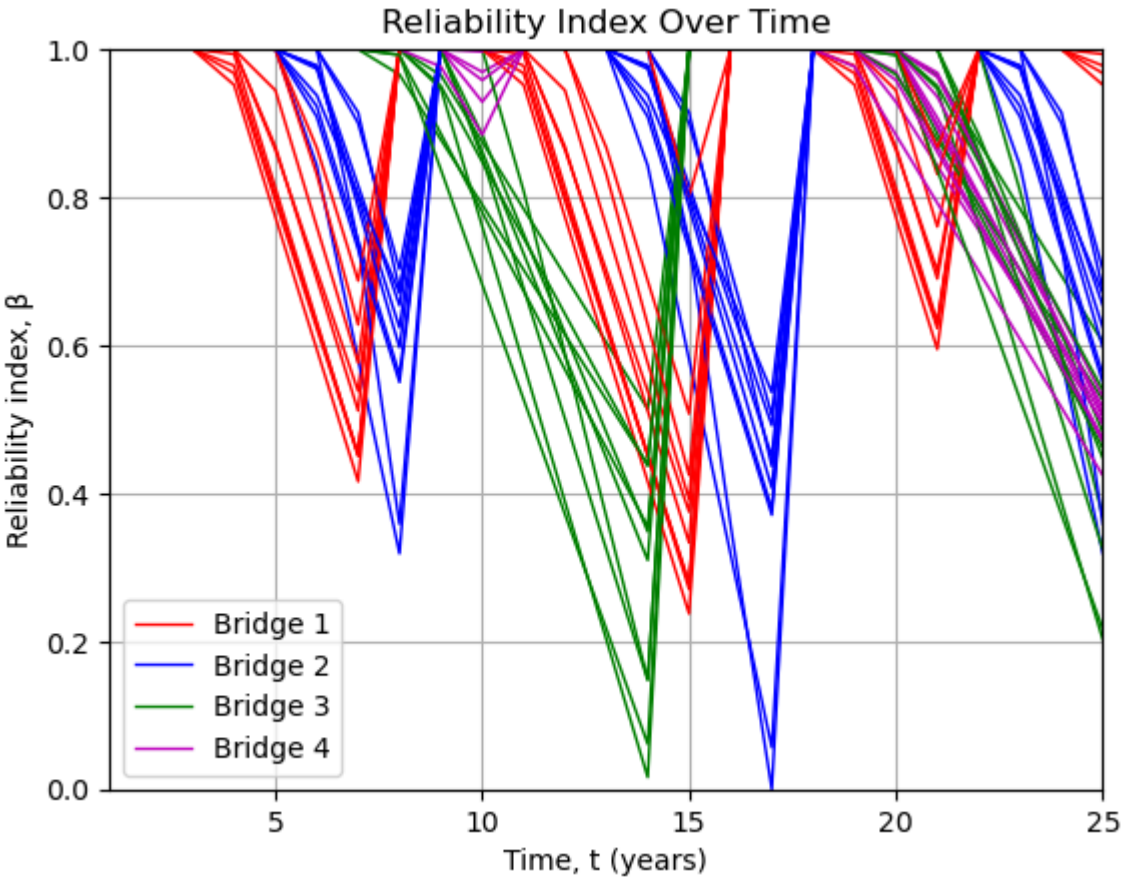
differential_evolution step 727: f(x)= 6.1
differential_evolution step 728: f(x)= 6.1
differential_evolution step 729: f(x)= 6.1
differential_evolution step 730: f(x)= 6.1
differential_evolution step 731: f(x)= 6.1
differential_evolution step 732: f(x)= 6.1
differential_evolution step 733: f(x)= 6.1
differential_evolution step 734: f(x)= 6.1
differential_evolution step 735: f(x)= 6.1
differential_evolution step 736: f(x)= 6.1
differential_evolution step 737: f(x)= 6.1
differential_evolution step 738: f(x)= 6.1
differential_evolution step 739: f(x)= 6.1
differential_evolution step 740: f(x)= 6.1
[ 7.  0.  2.  0.  8.  8.  0.  0.  6.  9. 12. 10.  0.  0.  0.  0.]
```

```python
Yopt = result.x
OptInterv = cumulateFreq(Yopt.reshape((nt, nb)).T)
X_bt = Frequency2Schedule(Yopt, nb, nt, Thoriz)

print(f'Min Cost = {result.fun:.2f}')
ninterv = np.sum(CountNinterv(Yopt, nt, nb))
print(f'Number of interventions = {ninterv}')
c = nonlcon(Yopt)
BudgetIssues = np.sum(c[-Thoriz - 1:-1] > 0)
print(f'Number of time intervals with exceeding budget = {BudgetIssues}')
for b in range(1, nb + 1):
    probfulfilment = 0.90 - c[-nb + b - 1]
    print(f'Prob(reliability>0) for bridge {b} = {probfulfilment:.4f}')

plt.figure()
h = []
leg = []
for i in range(nMCS):
    beta_bt = MaintenanceApplication(beta_bti0[:, :, i], X_bt, Thoriz)  # optimal mainten
    for b in range(nb):
        h.append(plt.plot(range(1, Thoriz + 1), beta_bt[b, :], f'-{col[b]}', linewidth=1)
        if i == 0:
            leg.append(f'Bridge {b + 1}')
plt.legend(leg, loc='best')
plt.grid()
plt.xlim([1, Thoriz])
plt.ylim([0, 1])
plt.xlabel('Time, t (years)')
plt.ylabel('Reliability index, β')
plt.title('Reliability Index Over Time')
plt.show()
```

```
Min Cost = 6.10
Number of interventions = 8
Number of time intervals with exceeding budget = 0
Prob(reliability>0) for bridge 1 = 1.0000
Prob(reliability>0) for bridge 2 = 0.9000
Prob(reliability>0) for bridge 3 = 1.0000
Prob(reliability>0) for bridge 4 = 1.0000
```


Reliability Index Over Time
Reliability index, β
Time, t (years)
1.0
0.8
0.6
0.4
0.2
0.0
5
10
15
20
25
Bridge 1
Bridge 2
Bridge 3
Bridge 4

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Contents

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How the book is made

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Changelog

Contents

This changelog will include all changes, except for minor adjustments like typos.

v2024.3.0 After class February 26th

- Added solutions to text and downloads Coding of optimization problems - Multi-objective optimization using `scipy` - Class exercise: Ice cream cone
- See full changelog here

v2024.2.0 After class February 26th

- Added solutions to text and downloads Coding of optimization problems - Nonlinear constrained optimization using `scipy` - Class exercise: Suspended beams
- See full changelog here

v2024.1.0 After class February 19th

- Added solutions to text and downloads Coding of optimization problems - Linear constrained optimization using `scipy` - Class exercise: Book distribution
- See full changelog here

v2024.0.0 Start course

- Restructured book with part on introduction, coding of optimization problems and miscellaneous
- Added class exercises to book
- Various improvements of student-experience
- See full changelog here