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Finite volume discretization of the incompressible Navier-Stokes equations in general coordinates on staggered grids

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Abstract

Using standard tensor notation, a coordinate invariant finite volume discretization of the incompressible Navier-Stokes equations is derived on staggered grids in general coordinates. No effort is made to avoid Christoffel symbols. The metric quantities are approximated in a specific way, satisfying a certain geometric identity. It is shown that instead of the contravariant velocity components U^α the quantities $V^\alpha = \sqrt{g}U^\alpha$ should be used as unknowns, where $\sqrt{g}$ is the Jacobian of the mapping. Satisfactory accuracy is obtained on reasonably smooth grids.

1 Introduction

As part of an information system for the solution of the Navier-Stokes equations (ISNaS-project, [1]) we are developing an incompressible Navier-Stokes solution method. It has been decided to use finite volume discretization on structured grids. Finite volume (and difference) methods for the incompressible Navier-Stokes equations may be divided in two classes: those using a staggered arrangement of the unknowns in the grid (marker-and-cell placement, [2]) and those where all unknowns reside in the same grid points (collocated approach). Both classes differ markedly in the ease of implementation in general coordinates.

In the staggered approach the angle between coordinate lines and velocity components plays a role. Hence transformation of velocity unknowns seems unavoidable, if restrictions on the coordinate transformation are to be avoided. As a consequence many metric coefficients (notably the Christoffel symbols (for brevity: CS)) enter the formulation. As a consequence the derivation and programming of the discretization becomes a tedious affair; computing time and memory requirements are increased; and inaccurate discretizations may result, unless certain precautions are taken, which will be the main point of discussion in this paper. In the collocated approach these troubles do not occur, since the Cartesian velocity components can be maintained as unknowns. The collocated and the staggered approach in general coordinates are discussed in [3], where further references may be found.

Here staggered grids are used. The discretization of the equations of fluid dynamics in general coordinates is discussed in [4] using (Gibbs) vector notation; this notation is also used in [5], [6], for incompressible Navier-Stokes on staggered grids. In conjuction with the finite volume method this leads naturally to a certain discretization in which the CS do not occur explicitly. However, certain quantities appear that approximate the CS. The vector formulation leads naturally to a certain discrete implementation of the CS. We will use tensor-notation, and allow the occurrence of CS. The resulting stencils are found to be somewhat smaller than with the vector formulation. The vector formulation can be obtained as a special case of the tensor formulation in the manner indicated in section 6.

2 Boundary-fitted coordinate mapping

Concepts from tensor analysis will be used; for an introduction see [7]. Let $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) : G \to \Omega$ be a one-to-one boundary-fitted coordinate mapping in 2D from a rectangle G to a physical domain Ω. In G a uniform grid is defined by

$$G_h = \{\boldsymbol{\xi} : \xi^1_{i-1/2} = i\Delta\xi^1,\ \ \xi^2_{j-1/2} = j\Delta\xi^2,\ \ i = 0,1,2,\cdots,I,\ \ j = 0,1,2,\cdots,J\} \tag{1}$$

It is assumed that the only information available about the mapping are the values $\mathbf{x}_{ij} = \mathbf{x}(\xi^1_i, \xi^2_j)$. The quadrilaterals with these points as vertices (in Ω or in G) are called the (primary) cells of the grid. The covariant basevectors $\mathbf{a}_{(\alpha)} = \partial\mathbf{x}/\partial\xi^\alpha$ are computed in a staggered arrangement in the centers of the cell faces according to

$$a^\alpha_{(1)}(i, j+1/2) = (x^\alpha_{i+1/2,j+1/2} - x^\alpha_{i-1/2,j+1/2})/\Delta\xi^1 \tag{2}$$

and similarly for $a^\alpha_{(2)}(i+1/2, j)$. In other points (cell centers or vertices) $\mathbf{a}_{(\alpha)}$ is obtained from the values in face centers by averaging in the obvious way. The contravariant basevectors $\mathbf{a}^{(\alpha)} = grad\ \xi^\alpha$ are obtained by orthogonality:

$$\mathbf{a}^{(1)}(i+1/2, j) = \{(a^2_{(2)}, -a^1_{(2)})/\sqrt{g}\}|_{i+1/2,j},\quad \mathbf{a}^{(2)}(i, j+1/2) = \{(-a^2_{(1)}, a^1_{(1)})/\sqrt{g}\}|_{i,j+1/2} \tag{3}$$

with $\sqrt{g} = a^1_{(1)}a^2_{(2)} - a^2_{(1)}a^1_{(2)}$ the Jacobian. The metric tensor is obtained by $g_{\alpha\beta} = \mathbf{a}_{(\alpha)} \cdot \mathbf{a}_{(\beta)}, g^{\alpha\beta} = \mathbf{a}^{(\alpha)} \cdot \mathbf{a}^{(\beta)}$. The CS are defined by

$$\left\{ {\alpha \atop \gamma\beta} \right\} = \mathbf{a}^{(\alpha)} \cdot \frac{\partial \mathbf{a}_{(\gamma)}}{\partial \xi^\beta} = \left\{ {\alpha \atop \beta\gamma} \right\} \tag{4}$$

These are computed by central differences after computing $\mathbf{a}^{(\alpha)}$ and $\mathbf{a}_{(\gamma)}$ in the required points. This completes our description of how the metric quantities are handled.

3 The divergence theorem

We have $\int\limits_{\Omega} div\mathbf{u}d\Omega = \oint\limits_{\Gamma} \mathbf{u}\cdot \boldsymbol{d\Gamma}$. We apply this to a primary cell with staggered representation of $\mathbf{u} : U^1 (U^\alpha = \mathbf{a}^{(\alpha)} \cdot \mathbf{u},\;\; \mathbf{u} = U^\alpha \mathbf{a}_{(\alpha)})$ is given in $(i+1/2, j)$, U^2 in $(i, j+1/2)$. The faces of the cells are defined to be straight in Ω. Then the face with center at $(i+1/2, j)$ is given by $(x^2, -x^1)|_{i+1/2,j-1/2}^{i+1/2,j+1/2} = (\sqrt{g}\mathbf{a}^{(1)}\Delta\xi^2)(i+1/2,j)$. Handling the other faces similarly one obtains

$$\int\limits_{\Omega} div\mathbf{u}d\Omega \cong V^2|_{i,j-1/2}^{i,j+1/2}\Delta\xi^1 + V^1|_{i-1/2,j}^{i+1/2,j}\Delta\xi^2 \tag{5}$$

with $V^\alpha = \sqrt{g}U^\alpha$. Now suppose $\mathbf{u} =$ constant. We have

$$\int\limits_{\Omega} div\mathbf{u}d\Omega \cong (-a_{(1)}^2 u_1 + a_{(1)}^1 u_2)|_{i,j-1/2}^{i,j+1/2}\Delta\xi^1 + (a_{(2)}^2 u_1 - a_{(2)}^1 u_2)|_{i-1/2,j}^{i+1/2,j}\Delta\xi^2 = 0 \tag{6}$$

exactly because of (2). This is an important advantage of approximating the metric quantities in the manner described in the preceding section. Equation (6) means that we satisfy exactly a discrete version of the following geometric identity

$$\oint\limits_{\Gamma} \boldsymbol{d\Gamma} = 0 \text{ or } \oint\limits_{\Gamma} a_\beta^{(\alpha)} d\Gamma_\alpha = \oint\limits_{\Gamma} a_{(\alpha)}^\beta d\Gamma^\alpha = 0 \tag{7}$$

with $d\Gamma_\alpha = d\Gamma^\alpha$ the Cartesian components of $\boldsymbol{d\Gamma}$.

4 Choice of unkowns

Some publications using a staggered arrangement of coordinate-invariant (hence, non-Cartesian) unknowns in general coordinates are [8], [9], [10], [11], [5], [6], [12]. Like [5], [6] we will use $V^\alpha = \sqrt{g}U^\alpha$ as velocity unknowns. This preference is based on the following novel argument. Let $\mathbf{u}$ be a constant vector field. Compute its contravariant representation V^α on the staggered grid. From V^α, compute the corresponding Cartesian components $\tilde{u}^\alpha$ in cell-centers, using averaging to obtain V^α and $\mathbf{a}_{(\alpha)}$ in the cell-centers. Then one finds $\tilde{\mathbf{u}} = \mathbf{u}$ exactly. This is not the case if U^α is used. Practical experience also learns that V^α is to be preferred, as will be seen in section 8.

5 Tensor formulation

Coordinate-invariant formulations of physical laws can be given in (Gibbs) vector notation, or in tensor notation. Vector notation is used in [5], [6] and [4]; the last reference provides valuable general guidelines for discretization in general coordinates. The difference between vector and tensor notation is not immaterial; we will use tensor notation and obtain a somewhat smaller discretization stencil than [5], [6]. The underlying cause of differences is the way the CS are treated. CS do not occur explicitly when vector notation is used, but are implicitly approximated in a specific way. Tensor formulation seems more flexible and general, but requires more care to obtain accurate discretizations. The vector formulation can be obtained as a special case in a way that we will indicate.

Let us first consider the following conservation law in tensor notation:

$$T^{\alpha\beta}_{,\beta} = 0, \quad T^{\alpha\beta}_{,\beta} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} T^{\alpha\beta}}{\partial \xi^{\beta}} + \{ {}^{\ \alpha}_{\gamma\beta} \} T^{\gamma\beta} \tag{8}$$

Approximation of (8) implies approximation of $\{ {}^{\ \alpha}_{\gamma\beta} \}$ with (4) which involves second derivatives of the mapping and hence may cause inaccuracy when the mapping is non-smooth. However, our implementation of $\{ {}^{\ \alpha}_{\gamma\beta} \}$ gives good results on reasonably smooth grids. The occurrence of CS may be avoided by contracting (8) with constant vector fields $\mathbf{w}^{(\gamma)}$:

$$W^{(\gamma)}_{\alpha} T^{\alpha\beta}_{,\beta} = (W^{(\gamma)}_{\alpha} T^{\alpha\beta})_{,\beta} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^{\beta}} (\sqrt{g} W^{(\gamma)}_{\alpha} T^{\alpha\beta}) \tag{9}$$

Work and storage required for $W^{(\gamma)}_{\alpha}$ are equivalent to that for $\{ {}^{\ \alpha}_{\gamma\beta} \}$, and the resulting stencils are somewhat larger than with (8). For a certain choice of $W^{(\gamma)}_{\alpha}$ the tensor-equivalent of the vector notation discretizations of [5], [6] and [4] is obtained; because of lack of space we do not go into more detail.

For the Navier-Stokes equations we have

$$T^{\alpha\beta} = U^{\alpha} U^{\beta} + g^{\alpha\beta} p - \nu (g^{\alpha\gamma} U^{\beta}_{,\gamma} + g^{\gamma\beta} U^{\alpha}_{,\gamma}) \tag{10}$$

6 Finite volume discretization

The continuity equation is discretized according to (5). For (8) shifted control volumes are used, centered at V^1-points $(i + 1/2, j)$ for $\alpha = 1$ and V^2-points $(i, j + 1/2)$ for $\alpha = 2$, as is usual for staggered discretization. Taking $\alpha = 1$ for example we have, using (8):

$$\begin{aligned} \int_{\Omega} T^{1\beta}_{,\beta} d\Omega &= \iint \left[\frac{\partial \sqrt{g} T^{1} \beta}{\partial \xi \beta} + \sqrt{g} \{ {}^{\ \alpha}_{\gamma\beta} \} T^{\gamma\beta} \right] d\xi^1 d\xi^2 \\ &= (\sqrt{g} T^{11})|^{i+1,j}_{ij} + (\sqrt{g} T^{12})|^{i+1/2,j+1/2}_{i+1/2,j-1/2} + [\sqrt{g} \{ {}^{\ 1}_{\gamma\beta} \} T^{\gamma\beta}]|_{i+1/2,j} \end{aligned} \tag{11}$$

where we have taken $\Delta\xi^1 = \Delta\xi^2 = 1$. Equation (10) is substituted, U^{α} is replaced by $V^{\alpha}/\sqrt{g}$, and the viscous terms are discretized in straightforward fashion. Some terms can be eliminated using (5). The resulting 19-point stencil is depicted in Fig. 1. Taking into account the fact that mixed derivatives occur these 19 points would seem the minimum required.

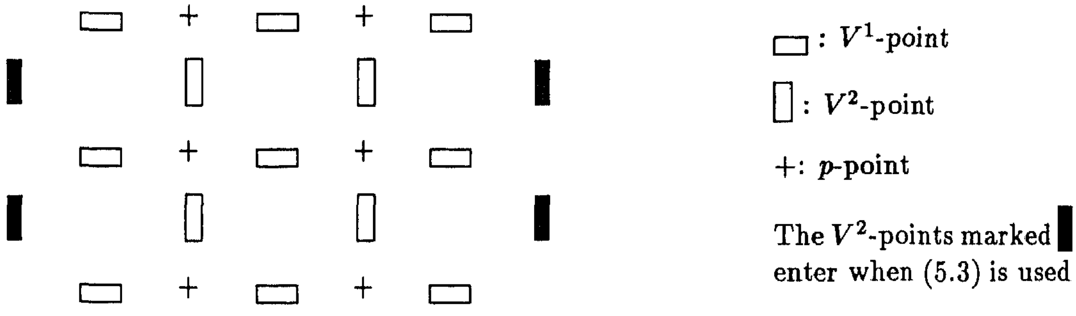


Figure 1: Stencil for V^1-momentum equation.

With equation (9), equivalent to [5], [6] and [4], 23 points are needed. The molecule for the V^2-momentum equation is obtained by rotation. Because of space limitations we do not discuss the boundary conditions. Suffice it to say that these are implemented in the usual way by using virtual points and/or one-sided differences. Note that we have central differencing for the convection terms.

7 Time discretization and solution method

We solve the time-dependent incompressible Navier-Stokes equation, which may be denoted formally as

$$\frac{\partial U^\alpha}{\partial t} + T^{\alpha\beta}_{,\beta} = 0, \quad U^\alpha_{,\alpha} = 0 \tag{12}$$

After spatial discretization this results in a system of ordinary differential equations that may be written as

$$\begin{aligned} \frac{d\mathbf{V}_m}{dt} &= \mathbf{f}(\mathbf{V}_m, \mathbf{V}_b) + G\mathbf{p} \\ D_m \mathbf{V}_m + D_b \mathbf{V}_b &= 0 \end{aligned} \tag{13}$$

where the algebraic vector $\mathbf{V}_m$ contains all non-prescribed V^α values in grid points, $\mathbf{f}$ is a nonlinear algebraic function defined by the discretization, $\mathbf{V}_b$ contains V^α values prescribed by the boundary conditions, G, D_m and D_b are matrices and $\mathbf{p}$ contains the unknown p values. We use a standard time-accurate pressure correction method as introduced in [13] in the second-order time-accurate version described in [14]. Crank-Nicolson time discretization of (13) gives

$$(\mathbf{V}^{n+1}_m - \mathbf{V}^n_m)/\Delta t = \theta \mathbf{f}(\mathbf{V}^{n+1}_m, \mathbf{V}^{n+1}_b) + (1-\theta)\mathbf{f}(\mathbf{V}^n_m, \mathbf{V}^n_b) + \theta G\mathbf{p}^{n+1} + (1-\theta)G\mathbf{p}^n \tag{14}$$

$$D_m \mathbf{V}^{n+1}_m + D_b \mathbf{V}^{n+1}_b = 0 \tag{15}$$

With $\theta = 1/2$ one expects $O(\Delta t^2)$ accuracy, but until now computations have only been performed with $\theta = 1$. With the pressure correction method (14) is replaced by

$$(\mathbf{V}^*_m - \mathbf{V}^n_m)/\Delta t = \theta \mathbf{f}(\mathbf{V}^*_m, \mathbf{V}^{n+1}_b) + (1-\theta)\mathbf{f}(\mathbf{V}^n_m, \mathbf{V}^n_b) + G\mathbf{p}^n \tag{16}$$

Subtracting (16) from (14) gives

$$(V_m^{n+1} - \mathbf{V}_m^*)/\Delta t = \theta \mathbf{f}(\mathbf{V}_m^{n+1}, \mathbf{V}_b^{n+1}) - \theta f(\mathbf{V}_m^*, \mathbf{V}_b^{n+1}) + \theta G(\mathbf{p}^{n+1} - \mathbf{p}^n) \tag{17}$$

In [14] it is shown that without loss of order of accuracy this can be approximated by

$$(\mathbf{V}_m^{n+1} - \mathbf{V}_m^*)/\Delta t = \theta G(\mathbf{p}^{n+1} - \mathbf{p}^n) \tag{18}$$

It follows that

$$D_m(\mathbf{V}_m^{n+1} - V_m^*)/\Delta t = \theta D_m G(\mathbf{p}^{n+1} - \mathbf{p}^n) \tag{19}$$

from which follows the following equation for $\mathbf{p}^{n+1}$:

$$\theta D_m G(\mathbf{p}^{n+1} - \mathbf{p}^n) = -(D_b \mathbf{V}_b^{n+1} + D_m V_m^*)/\Delta t \tag{20}$$

One easily verifies that the matrix $\theta D_m G$ corresponds to a nine-point stencil; unfortunately, the matrix is non-symmetric for a non-orthogonal grid. Equation (20) is solved by a GMRES iterative method ([15]) further developed for the present problem in [16]. The nonlinear system (16) is Newton-linearized without loosing order of accuracy; for example, a term like $V_{ij}^* \, V_{i+1,j}^*$ is approximated by $V_{ij}^n V_{i+1,j}^* + V_{ij}^* V_{i+1,j}^n - V_{ij}^n V_{i+1,j}^n$.

8 Some results

Two cases will be described. The first case is the flow in the parallellogram-shaped domain of Fig. 2, with boundary conditions and right-hand-side chosen such that the exact solution in Cartesian variable is given by

$$u^1 = \sin t \sin x \sin y, \quad u^2 = \sin t \cos x \cos y, \quad p = \sin t(\sin x \cos y). \tag{21}$$

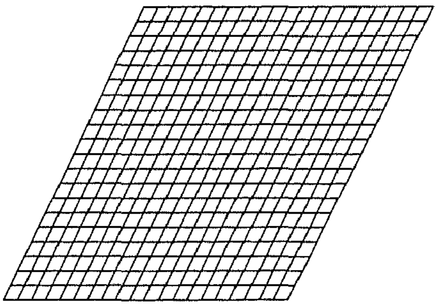


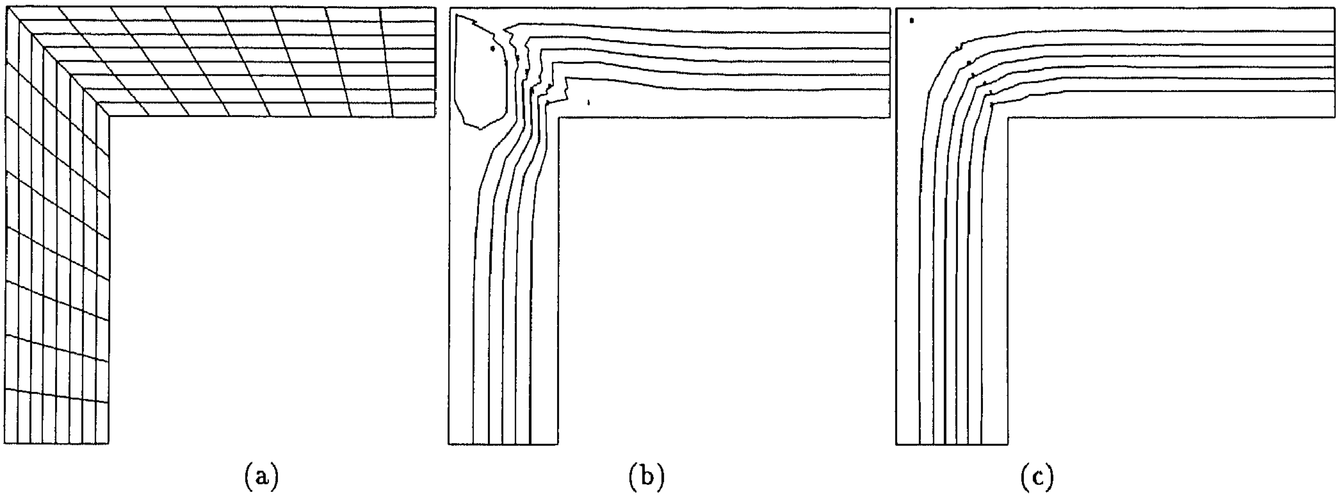
Figure 2: Grid for case 1.

The solution is computed with V^α unknows; the result is transformed to Cartesian components in the cell vertices. The l_2-norm of the error is listed in table 1. We expect $\Delta p = O(h)$, $\Delta u, \Delta v = O(h^2)$. The results are more or less expected. Time-accuracy is found to be $O(\Delta t)$.

Grid	20*20	40*40	80*80
Δu	.023	.0081	.0029
Δv	.013	.0046	.0016
Δp	.043	.016	.010

Table 1: Results for case 1. $Re = 20, t = 1.55, \Delta t = .003125$.

The next case is the flow through an L-shaped duct. At the inflow boundary a parabolic velocity profile is prescribed and the tangential velocity component is set zero, at the outflow boundary the normal stress and the tangential velocity component are set zero. Fig. 3 gives some results.



(a) (b) (c)

Figure 3: Results for case 2: $Re = 10$. (a): grid; (b): streamlines with U^α unknowns; (c) streamlines with V^α unknowns.

The results of Fig. 3(c) agree very well with results obtained with a standard finite element code; the same is true for the pressure (not shown). Fig. 3(b) shows that with U^α unknowns erroneous results are obtained.

9 Conclusions

We have shown that with a certain implementation of the metric quantities finite volume discretization in terms of a standard tensor formulation of the incompressible Navier-Stokes equations results of satisfactory accuracy are obtained, provided $V^\alpha = \sqrt{g}U^\alpha$ are used as contravariant velocity unknowns.

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