Delft University of Technology
Faculty of Electrical Engineering, Mathematics & Computer Science
Delft Institute of Applied Mathematics

On the positive stabilizability and controllability of linear discrete-time systems

(Dutch title: Over de positieve stabiliseerbaarheid en bestuurbaarheid van lineaire discrete-tijd systemen)

Thesis submitted to the
Delft Institute of Applied Mathematics
as partial fulfillment of the requirements

for the degree of

BACHELOR OF SCIENCE
in
APPLIED MATHEMATICS

by

S.A.C. Enschedé

Delft, The Netherlands
July 2020

BSc thesis APPLIED MATHEMATICS

"On the positive stabilizability and controllability of linear discrete-time systems"

"Dutch title: Over de positieve stabiliseerbaarheid en bestuurbaarheid van lineaire discrete-tijd systemen"

S.A.C. Enschedé

Delft University of Technology

Defended publicly on July 2nd 2020

An electronic version of this thesis is available at
`https://repository.tudelft.nl/`.

Supervisor

Dr. J. W. van der Woude

Other committee members

Dr. Ir. G. F. Nane   Dr. Y. van Gennip

July, 2020   Delft

Abstract

This paper is a research in positive controllability and stabilizability for discrete-time linear systems. Several conditions are given and explained for both single input as multiple input systems. Also the connection between controllability and reachability will be mentioned. Furthermore, a method for multiple input positive controllability is analysed.

Keywords: discrete-time linear systems, positive control, stabilizing control

Preface

This report is written in order to obtain the degree Bachelor of Science in Applied Mathematics at Delft University of Technology, The Netherlands. The research was supervised by Jacob van der Woude, and has been done within the Mathematical Physics department of the faculty EEMCS.

For 10 weeks I have been doing research in positive controllability and stabilizability for discrete-time linear systems. Since I am interested in the Mathematical Physics and mostly system theory, this project seemed to be in line with my interests. With limited background knowledge in system theory, I started this project in April. From then I worked full-time. The goal of this project became clarifying what is already known, and why that holds. With this project I have gained a lot of knowledge and experience.

I want to thank my supervisor Jacob van der Woude for his help and support during my project. Furthermore I would like to thank Yves van Gennip and Tina Nane for their participation in my thesis commitee. Lastly, I want to thank my fellow students and family for their moral support.

List of symbols

Symbol	Meaning
$x(t)$	state in continuous-time systems at time t
$\dot{x}$	derivative of the state
$x(k)$	state in discrete-time systems at timestep k
u	input
A	$(n \times n)$ matrix
B	$(n \times m)$ matrix
b	$(n \times 1)$ matrix, for single input systems
$\mathbb{R}^n_+$	positive part of the coordinate system
$\mathbb{C}$	the set of complex numbers
I_n	$n \times n$ identity matrix

Table 1: List of symbols

Contents

1

Introduction to system theory

For the reader to understand this report, some basic knowledge in mathematical system theory is required. In this chapter basic system theory, stabilizability and controllability will be explained. In the last section positive controllability is introduced.

1.1 Systems

To start with the definition of a system. As stated on the website (*Cambridge Dictionary*), a system is

> "A set of connected things or devices that operate together"

A system is dependent on its surroundings. The influence of the surroundings on the system, is called the input. The other way around, the influence of the system on its surroundings, is the output. The system creates a relationship between its input and output. (Olsder et al., 2011)

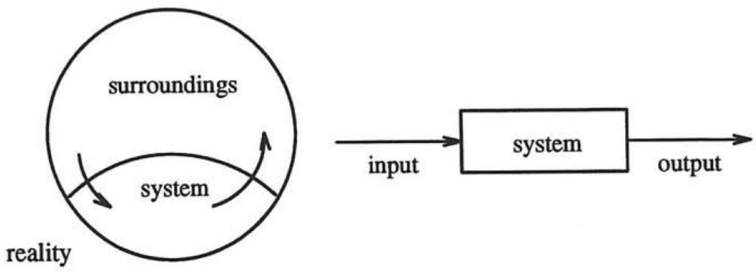

Figure 1.1: System, source: (Olsder et al., 2011)

In this report, we only look at linear systems, which are systems of the form

$$\dot{x} = Ax(t) + Bu(t) \qquad t \geq 0 \tag{1.1}$$

In the basis, most systems are not linear. That means that the equations the system consists of, are not linear. When these equations are linearized, the system can be written in the matrix form (1.1). (Olsder et al., 2011)

A system can be a continuous-time system, or a discrete-time system. In this report, we only look at the linear discrete-time systems. These type of systems contain difference equations and are of the form

$$x(k+1) = Ax(k) + Bu(k) \qquad k = 0, 1, 2, \ldots \tag{1.2}$$

Any further mentions of a system in this report will be discrete-time systems.

The focus of this report is positive stabilizability and controllability, hence we first consider stabilizability and controllability in general.

1.2 Stabilizability

Stabilizability is the ability to steer the system (1.2) to zero, with a control input that can be defined over an infinite amount of time.

(Olsder et al., 2011) give the following equivalence with discrete-time system stabilizability:

> Consider the pair (A, B), where A is a real $n \times n$ matrix and B a real $n \times m$ matrix of system (1.2). Then the following statements are equivalent:
>
> 1. The pair (A, B) is stabilizable
> 2. $\text{rank}(zI - A, B) = n$ for all $z \in \mathbb{C}$ with $|z| \geq 1$
> 3. $\text{rank}(\lambda I - A, B) = n$ for all eigenvalues λ of matrix A with $|\lambda| \geq 1$

A discrete-time system is stable if the eigenvalues of the matrix A all lie in the open unit disk.

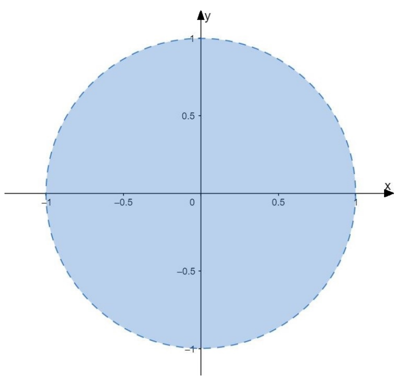

Figure 1.2: The open unit disk

1.3 Controllability

Controllability of a system is the ability to steer the system from any state to any other state in finite time (Olsder et al., 2011). We define the following notation:
$x(k, x(0), u)$ is the state x after k timesteps taken, started at $x(0) = x_0$ under control input u.
Then we find for controllability:

> For all $x_0, x_1 \in \mathbb{R}^n$ there exists a time $k > 0$ and a sequence $u := u(0), u(1), \ldots$ such that $x(k, x_0, u) = x_1$.

To find out if a system is controllable, one can look at the rank of the controllability matrix. This matrix is defined as

$$\begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \tag{1.3}$$

and will be used frequently in the rest of the report. Whenever the rank of matrix (1.3) is equal to n, the system is controllable.
An equivalence of statements as given in the section of stabilizability, is given for controllability:

> Consider the pair (A, B) of system (1.2). Then the following statements are equivalent
>
> 1. The pair (A, B) is controllable
> 2. $\text{rank}(sI - A, B) = n$ for all $s \in \mathbb{C}$
> 3. $\text{rank}(\lambda I - A, B) = n$ for all eigenvalues λ of matrix A

(Olsder et al., 2011)

Whenever a system is controllable, there exists a control input that steers the system to stability, independent of the general stability of the system. That means with a certain input sequence u, the state variables can be placed to an equilibrium. (*Researchgate, How stabilizability and controllability interconnect*).

1.4 Positive control

This report will cover positive controllability and stabilizability. As the name suggests, it puts a restriction on the control input $u(k)$. For general controllability we have $u(k) \in \mathbb{R}^m$. This changes for positive controllability to $u(k) \in \mathbb{R}^m_+$, which in practice means that every entry of $u(k)$ belongs to the set $[0, \infty)$.

2

Single input

In this chapter a single input system is discussed. That is, the dimension of matrix B is $n \times 1$ where with multiple input, the dimension is $n \times m$ for a positive integer $m > 1$. For convenience, the notation of matrix B changes to b. In this chapter we will cover the system

$$\begin{aligned} x(k+1) &= Ax(k) + bu(k) \\ x(k) &\in \mathbb{R}^n, u(k) \in \mathbb{R}_+ \end{aligned} \tag{2.1}$$

Any further mentions of a system in this chapter will refer system (2.1), unless otherwise specified.

When we write out the solutions of $x(k+1)$, we find

$$\begin{aligned} x(1) &= Ax(0) + bu(0) \\ x(2) &= Ax(1) + bu(1) \\ &= A(Ax(0) + bu(0)) + bu(1) \\ &= A^2x(0) + Abu(0) + bu(1) \\ &\vdots \end{aligned} \tag{2.2}$$

Therefore we find the general solution of system (2.1)

$$x(k) = A^k x(0) + \sum_{l=0}^{k-1} A^l bu(k-1-l) \tag{2.3}$$

2.1 Evans and Murthy

Evans and Murthy (1977) did research in single input positive controllability. For their main results, we start with two lemma's with proofs.

Lemma 1 System (2.1) is completely controllable with $u(k) \in [0, \infty), k = 0, 1, \dots$ if and only if it is completely reachable with $u(k) \in [0, \infty), k = 0, 1, \dots$.

Note that controllability is the ability to go from any starting state to another state in finite time, and reachability is the ability to reach every arbitrary state starting from the origin in finite time.

Lemma 2 If A, b satisfy the relationship

$$\sum_{i=0}^{r} a_i A^i b = 0 \qquad \text{with } a_i > 0 \quad i = 0, 1, \dots, r$$

then any vector x in the subspace $V \subseteq \mathbb{R}^n$ spanned by $A^i b$, $i = 0, 1, \dots, r$ can be expressed as a linear combination of $A^i b$, $i = 0, 1, \dots, r$ with positive coefficients. That is, for any $x \in V$ there exist (nonunique) $c_i > 0, i = 0, 1, \dots, r$ such that

$$x = \sum_{i=0}^{r} c_i A^i b$$

Where r is the sum of the algebraic multiplicities of the real eigenvalues of A. For example, if A has only real eigenvalues $\{1, 2, 2, 3\}$, then $r = 4$. This is analogue to the sum of dimensions of the J_i matrices in figure 2.1 which will be explained later.

Lemma 1 follows from the definitions of complete controllability and complete reachability. Reachability is the ability to reach every state when started in the origin (Olsder et al., 2011).

For lemma 2 we find all $A^i b$ are dependent with positive coefficients, since for $a_i > 0$, $\sum_{i=0}^{r} a_i A^i b = 0$, $i = 0, 1, \dots, r$. Therefore, any negative coefficients can be turned positive with the use of equation (2.4).

$$A^k b = -\frac{1}{a_k}(a_1 Ab + \dots + a_{k-1} A^{k-1} b + a_{k+1} A^{k+1} b + \dots + a_r A^r b) \tag{2.4}$$

Which means that for any x spanned by $A^i b$, x can be expressed in a linear combination of $A^i b$ with only positive coefficients. For example, let $x = -Ab + A^2 b$. Then x has a negative coefficient in it's representation. In the assumption that $a_1 Ab + \dots + a_r A^r b = 0$, we write $Ab = -\frac{a_2}{a_1} A^2 b - \dots - \frac{a_r}{a_1} A^r b$. All the coefficients are now negative. Therefore we can write

$$\begin{aligned} x &= -(-\frac{a_2}{a_1} A^2 b - \dots - \frac{a_r}{a_1} A^r b) + A^2 b \\ &= (\frac{a_2}{a_1} A^2 b + \dots + \frac{a_r}{a_1} A^r b) + A^2 b \end{aligned}$$

All coefficients are now positive.

With the knowledge of the two lemmas, the main result of (Evans and Murthy, 1977) is given in a theorem.

Theorem The system (2.1) with $u(k) \in [0, \infty), k = 0, 1, \ldots$ is completely controllable on $\mathbb{R}^n$ if and only if

1. rank$[b\ Ab\ \ldots A^{n-1}b] = n$
2. A has no real eigenvalues $\lambda \geq 0$

The proof for this theorem is given in the article. The question considered in this report is more about why this needs to hold. The first point is trivial. For positive controllability to hold, we need at least controllability for $u \in \mathbb{R}$. For controllability, we need the rank of the controllability matrix to be equal to n. The proof of that is given in (Olsder et al., 2011).
Therefore, we take a look at the second point. We seperate this statement into two parts: one where A is diaganolizable and one where A is not diagonalizable.
Whenever A is diagonalizable, we can write A in a new form such that on the diagonal the eigenvalues are shown. See figure 2.1. We can divide this matrix into 3 smaller

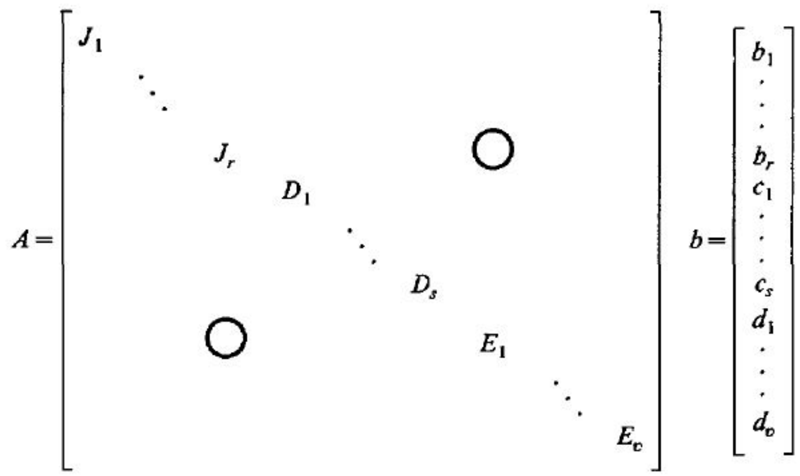

Figure 2.1: Diagonal matrix form of A, source: (Evans and Murthy, 1977)

matrices. A part with real eigenvalues, a part with complex eigenvalues with negative real parts and a part with complex eigenvalues with positive real parts.
Every J_i, D_i and E_i has only one eigenvalue repeating on it's diagonal. The matrices are of forms given in figure 2.2 where $G_i > 0$ and $\mu_i \geq 0$.

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & \bigcirc & \\ & & & \ddots & \\ & \bigcirc & \ddots & & 1 \\ & & & & \lambda_i \end{bmatrix} \quad D_i = \begin{bmatrix} e_i & n & & & \\ & e_i & n & \bigcirc & \\ & & & \ddots & \\ & \bigcirc & \ddots & & n \\ & & & & e_i \end{bmatrix} \quad E_i = \begin{bmatrix} f_i & n & & & \\ & f_i & n & \bigcirc & \\ & & & \ddots & \\ & \bigcirc & \ddots & & n \\ & & & & f_i \end{bmatrix}$$

$$e_i = \begin{bmatrix} -G_i & w_i \\ -w_i & -G_i \end{bmatrix} \quad f_i = \begin{bmatrix} \mu_i & \theta_i \\ -\theta_i & \mu_i \end{bmatrix} \quad n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Figure 2.2: Matrices J_i, D_i and E_i, source: (Evans and Murthy, 1977)

Since the theorem only talks about $\lambda \geq 0$, we consider the block with J_i matrices only. Then the solution of the system is

$$x(k) = A^k x(0) + \sum_{l=0}^{k-1} A^l b u(k-1-l)$$

$$\text{With } A = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix}$$

Consider the sign of the last element of A. This is a real eigenvalue. Whenever we take this eigenvalue positive, the sign of the last element of $A^l b$ is the same as the sign of the last element of b_p, see figure 2.3. Therefore, with a positive control input, the sign of the control system is the same as the sign of the last element of b_p. Hence, we find with the solution that the sign of the state cannot be arbitrarily assigned with positive control input.

$$(J_p)^i b_p = \begin{bmatrix} (\lambda_p)^i & \cdots & \\ & \ddots & \\ \bigcirc & & (\lambda_p)^i \end{bmatrix} \begin{bmatrix} b_{p1} \\ \vdots \\ b_{pj} \end{bmatrix}.$$

Figure 2.3: Consider the last element of $J_p^i b_p$, source: (Evans and Murthy, 1977)

Whenever A is not diagonalizable, we can still write A in it's Jordan canonical form. This gives a matrix in the form shown in figure 2.4. Then we can use the same argument as given above.

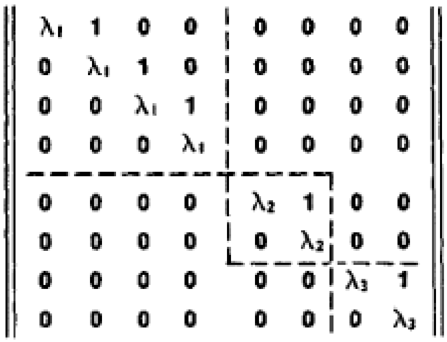

Figure 2.4: Jordan canonical form, source: (*Jordan matrix*)

2.2 Leenheer and Nešić

In the article of (Leenheer and Nešić, 1998), positive stabilizability is the main focus. It states:

Theorem System (2.1) is positively stabilizable if and only if

1. The pair (A, b) is stabilizable
2. A has no eigenvalues $\lambda \geq 1$

The eigenvalues of A are in this theorem restricted to the open unit disk and the real negative values. This is different from the eigenvalue restriction of (Evans and Murthy, 1977). In the next section we will look at this difference.

2.2.1 Eigenvalue restrictions

Evans and Murthy	**Leenheer and Nešić**
Positive controllability	Positive stabilizability
$\text{Rank}(\lambda I - A, b) = n$ for all eigenvalues λ of A	$\text{Rank}(\lambda I - A, b) = n$ for all eigenvalues λ of A with $\lambda \geq 1$
A has no real eigenvalues $\lambda \geq 0$	A has no real eigenvalues $\lambda \geq 1$

We separate the problem into a few parts.

Firstly, we look at the eigenvalues that lie in the open unit disk. These eigenvalues are for discrete-time systems stable eigenvalues (Olsder et al., 2011). Hence all eigenvalues that are already in the open unit disk, are stable and therefore don't need to be controllable to go to zero. The system will stabilise itself in time.

Secondly, the unstable negative eigenvalues. That means the real eigenvalues that are smaller or equal to 1 ($\lambda \in (-\infty, -1]$). In the theorem of (Evans and Murthy, 1977), we find that whenever A has a negative eigenvalue, it can still be positive controllable. Therefore, the unstable negative eigenvalues are positively controllable and thus can be controlled to stability. Hence, (Leenheer and Nešić, 1998) allows the eigenvalues to be negative.

Thirdly, the eigenvalues that are greater or equal to 1. These eigenvalues are not stable. Therefore we want these eigenvalues to be controllable. However, they do not satisfy the theorem of (Evans and Murthy, 1977) and therefore cannot be positive controllable or positive stabilizable.

Lastly, the complex eigenvalues. This part can be divided again: eigenvalues with negative real parts, purely imaginary eigenvalues and eigenvalues with positive real parts. The idea centers around the characteristic equation. These three cases are all explained in (Evans and Murthy, 1977) and will be left out of this report.

2.3 Nguyen

Another paper on controllability under restricted input is that of Nguyen (1986). It's main focus is on the set of states that are controllable to the origin in finite steps. The system considered differs from (2.1). Here, we look at

$$x(k+1) = Ax(k) + u(k) \quad x(k) \in \mathbb{R}^n \quad u(k) \in \Gamma \subset \mathbb{R}^n \tag{2.5}$$

Where Γ is a convex subset of $\mathbb{R}^n$ containing zero.

Since (Nguyen, 1986) not only looks at $u(k) \in \mathbb{R}^n_+$, there are some other theorems in restricted control theory.

> **Theorem** The system (A, Γ), with convex control set Γ satisfying $0 \in \Gamma$, is locally controllable if and only if the transpose matrix A^T has neither real eigenvectors, with positive eigenvalue supporting to Γ, nor complex eigenvectors, with nonzero complex eigenvalue orthogonal to Γ.

To simplify, we list it as follows:
The system (A, Γ) is locally controllable if and only if

1. A^T has no real eigenvector with positive eigenvalues supporting to Γ
2. A^T has no complex eigenvector with nonzero complex eigenvalue orthogonal to Γ

For the definition of local controllability and the proof of the theorem above, we define four sets.

$$
\begin{aligned}
F_k &= \sum_{i=0}^{k-1} A^i \Gamma \\
F &= \bigcup_{k-1}^{\infty} F_k \\
S_k &= \{x \in \mathbb{R}^n : -A^k x \in F_k\} \\
S &= \bigcup_{k=1}^{\infty} S_k
\end{aligned}
\tag{2.6}
$$

Local controllability has the property that 0 belongs to the interior of S from (2.6). A system is locally reachable if 0 belongs to the interior of F, where the interior of F will be denoted as: int F.

The proof is divided into two cases. A case where zero is assumed to be an eigenvalue, and a case where zero is assumed *not* to be an eigenvalue. The first case, where zero is an eigenvector, is explained in detail in the article (Nguyen, 1986). The second case is explained further below.

We assume that zero is not an eigenvalue of A^T. Then A^k is a nonsingular matrix for all $k = 1, 2, \ldots$. The proof then follows from a few equivalences.
Using the definition, the system is locally controllable if and only if there exists a neighbourhood $V(0)$ of the origin and an integer p, such that $A^p V(0) \subset F_p$. Since A^p is nonsingular, $A^p V(0) \subset F_p$ is equivalent to $0 \in \text{int } F_p$. This is the property of local reachability. For which there is another theorem:

> **Theorem** The system (A, Γ), with convex set Γ satisfying $0 \in \Gamma$, is locally reachable if and only if the transpose matrix A^T has neither real eigenvectors, with nonnegative eigenvalue supporting Γ, nor complex eigenvectors orthogonal to Γ.

Hence we find that the theorem of local reachability, implies the local controllability. In short that is:

Local controllability

$\Updownarrow$

$\exists V(0)$ and p such that $A^p V(0) \subset F_p$

$\Updownarrow$

$0 \in \text{int } F_p$

$\Updownarrow$

Local reachability

$\Updownarrow$

A^T has neither real eigenvectors with nonnegative eigenvalue supporting Γ nor complex eigenvectors orthogonal to Γ

3

Multiple input

The following chapter discusses multiple input systems. Where in the previous chapter the dimension of the matrix B was restricted to $n \times 1$, in this chapter the dimension will be $n \times m$, for n and m in $\mathbb{N}$. The system covered in this chapter is:

$$\begin{gathered} x(k+1) = Ax(k) + Bu(k) \\ x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m_+ \end{gathered} \tag{3.1}$$

In this chapter a description of the method of (Yoshida et al., 1994) to show positive controllability will be explained.

3.1 Yoshida

The goal in the paper of (Yoshida et al., 1994) is to show a new method to proof positive controllability. This method is about decomposing a system into smaller subsystems. The test for positive controllability can be simplified, since the sizes of the subsystems are smaller than that of the system given in (3.1). We can decompose the system (3.1) into two subsystems

$$x_0(k+1) = A_0 x_0(k) + B_0 u(k) \tag{3.2}$$

$$x_r(k+1) = A_r x_r(k) + B_r u(k) \tag{3.3}$$

When both these systems are positive controllable, the system (3.1) is positive controllable. That is since positive controllability is invariant under nonsingular state transformations (Yoshida et al., 1994).

A_0 is a nilpotent $(n_0 \times n_0)$ matrix. That means A_0 has only 0 as eigenvalue and $A_0^p = 0$ for some integer p. A_r holds the remaining, nonzero, eigenvalues and is therefore an invertible $(n_r \times n_r)$ matrix. Note that $n = n_0 + n_r$. Since A_0 is nilpotent, system

(3.1) is positive controllable if and only if (3.3) is positive controllable. Proof of this statement can be found in (Yoshida et al., 1994).

To determine whether a system is positive controllable or not, the following theorem is stated:

> **Theorem** System (3.3) is positive controllable if and only if there exist positive integers $\{N, h, K_1, K_2, \dots, K_h\}$ and positive scalars $\{c_{K_1}, c_{K_2}, \dots, c_{K_h}\}$ which satisfy the following three conditions
>
> 1. $c_{K_1}e_{K_1} + c_{K_2}e_{K_2} + \cdots + c_{K_h}e_{K_h} = 0$
> 2. $1 \leq K_1 < K_2 < \cdots < K_h \leq Nm$
> 3. $\text{rank}[e_{K_1}, e_{K_2}, \dots, e_{K_h}] = n_r < h \leq Nm$

The proof of the theorem is given in the article. This is the first step in the method of (Yoshida et al., 1994). The system (3.3) can be decomposed in smaller systems as well.

$$x_t(k+1) = A_t x_t(k) + B_t u(k) \tag{3.4}$$

$$x_q(k+1) = A_q x_q(k) + B_q u(k) \tag{3.5}$$

Where the eigenvalues of A_t are strictly negative or the imaginary part of the eigenvalue is nonzero, and the eigenvalues of A_q are strictly positive. With this decomposition, we find system (3.4) is positive controllable with similar reasoning as shown in section (2.2.1). Therefore, (Yoshida et al., 1994) concludes the system (3.3) is positive controllable if and only if

1. The system (3.5) is positive controllable
2. The rank of the controllability matrix of A_r is equal to n_r

Thus the positive controllability of (3.3) depends on that of (3.5). To simplify even further, we decompose (3.5) into more subsystems. Whenever (3.5) has Q distinct eigenvalues, we decompose the system into Q subsystems using jordan blocks. These systems are of the form

$$x_i(k+1) = A_i x_i(k) + B_i u(k) \quad i = 1, 2, \dots, Q \tag{3.6}$$

$$A_q = \begin{bmatrix} A_1 & 0 & \dots & \dots \\ 0 & A_2 & 0 & \dots \\ \dots & \dots & \ddots & \dots \\ \dots & \dots & 0 & A_Q \end{bmatrix}$$

$$A_i = \begin{bmatrix} A_{i1} & 0 & \dots & \dots \\ 0 & A_{i2} & 0 & \dots \\ \dots & \dots & \ddots & \dots \\ \dots & \dots & 0 & A_{ir(i)} \end{bmatrix}$$

$$A_{ij} = \text{Jordan form for } \lambda_i \text{ of order } n_{ij}$$

$$
\begin{aligned}
B_q &= \left[B_1^T, \quad B_2^T, \quad \ldots, \quad B_Q^T\right]^T \\
B_i &= \left[B_{i1}^T, \quad B_{i2}^T, \quad \ldots, \quad B_{ir(i)}^T\right]^T \\
B_{ij} &= \left[b_{n_{ij}ij}^T, \quad \ldots, \quad b_{2ij}^T, \quad b_{1ij}^T\right]^T
\end{aligned}
$$

$$
\begin{aligned}
i &= 1, 2, \ldots, Q \\
j &= 1, 2, \ldots, r(i)
\end{aligned}
$$

Now, (3.5) is positive controllable if and only if (3.6) is positive controllable for all $i = 1, 2, \ldots, Q$.

With these decompositions, we get to the main result of (Yoshida et al., 1994). Define the following system for any positive scalar λ and any positive integer F:

$$
\begin{aligned}
x^*(k+1) &= A^* x^*(k) + B^* u(k) \\
A^* &= \lambda I_F \\
B^* &= \left[\xi_1, \quad \xi_2, \quad \ldots, \quad \xi_m\right]
\end{aligned}
\tag{3.7}
$$

Then the main results are

Theorem The system (3.7) is positive controllable if and only if there exists positive integers $\{h, K_1, K_2, \ldots, K_h\}$ and positive scalars $\{c_{K_1}, c_{K_2}, \ldots, c_{K_h}\}$ which satisfies the following three conditions:

1. $c_{K_1}\xi_{K_1} + c_{K_2}\xi_{K_2} + \cdots + c_{K_h}\xi_{K_h} = 0$
2. $1 \leq K_1 < K_2 < \cdots < K_h \leq m$
3. $\text{rank}[\xi_{K_1}, \xi_{K_2}, \ldots, \xi_{K_h}] = F < h \leq m$

Theorem The system (3.5) is positive controllable if and only if the system described by

$$
S_i^* : x_i^*(k+1) = A_i^* x_i^*(k) + B_i^* u(k) \tag{3.8}
$$

where

$$
A_i^* = \lambda_i I_{r(i)} \tag{3.9}
$$

$$
B_i^* = \begin{bmatrix} b_{1i1} \\ b_{1i2} \\ \cdots \\ b_{1ir(i)} \end{bmatrix} \tag{3.10}
$$

is positive controllable for each $i = 1, 2, \ldots, Q$.

4
Example

In this chapter, we will show positive controllability of the example of (Yoshida et al., 1994) with the theorem of (Heemels and Stoorvogel, 1998).

The article of (Yoshida et al., 1994) contains an example with the matrices A and B defined as

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad B := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & 4 \\ -1 & 0 & 2 \end{bmatrix} \tag{4.1}$$

The multiple-input discrete-time systems described with the matrices of (4.1), is a positive controllable system. The steps using the method of (Yoshida et al., 1994) can be found in the article itself. Here, we will apply another theorem stated in an article of (Heemels and Stoorvogel, 1998).

(A, B) is positive controllable if and only if

1. (A, B) is controllable
2. All real eigenvectors v of A^T corresponding to a positive eigenvalue of A^T have the property that $B^T v$ has at least one strictly positive component

Note that eigenvectors are not unique. For every eigenvector v, $-v$ is also an eigenvector for the same eigenvalue. Therefore, $B^T v$ must also have at least one strictly negative component.

First, we consider the controllability of (A, b) using the rank of the controllability matrix $\begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix}$. In this case, the controllability matrix is equal to:

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & -1 & -2 & -3 & 1 & 2 & 3 & -1 & -2 & -3 & 1 & 2 & 3 \\ 1 & -2 & 1 & 3 & -6 & 3 & 9 & -18 & 9 & 27 & -54 & 27 & 81 & -162 & 81 \\ 0 & 3 & 4 & -1 & 9 & 14 & -6 & 27 & 48 & -27 & 81 & 162 & -108 & 243 & 540 \\ -1 & 0 & 2 & -3 & 0 & 6 & -9 & 0 & 18 & -27 & 0 & 54 & -81 & 0 & 162 \end{bmatrix}$$

Which can be converted to it's row reduced echelon form:

$$\begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 & 3.0 & -1.1 & -2.0 & 6.4 & -2.7 & -3.9 & 23.0 & -10.0 & -14.0 & 68.0 \\ 0 & 1.0 & 0 & 0 & 0 & -6.1 & 2.2 & 3.9 & -13.0 & 5.5 & 7.9 & -46.0 & 21.0 & 28.0 & -140.0 \\ 0 & 0 & 1.0 & 0 & 0 & 3.1 & -0.34 & 0.083 & 10.0 & -2.2 & 0.17 & 33.0 & -9.9 & 0.59 & 110.0 \\ 0 & 0 & 0 & 1.0 & 0 & -0.93 & 3.1 & 0.71 & -1.3 & 8.5 & 1.4 & -3.7 & 24.0 & 5.0 & -6.3 \\ 0 & 0 & 0 & 0 & 1.0 & 2.1 & -0.91 & 1.7 & 4.9 & -2.9 & 6.5 & 18.0 & -12.0 & 18.0 & 58.0 \end{bmatrix}$$

From here we easily find that the rank is equal to 5, which implies that the system (A, B) is controllable.

Now, the eigenvectors of A^T corresponding to the positive eigenvalues of A^T are

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.2}$$

For every eigenvector in (4.2), we check $B^T v$:

$$B^T v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad B^T v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \tag{4.3}$$

Both have a strict positive and strict negative component, which satisfies the theorem. Hence we can conclude the system described by (4.1) is positive controllable using the theorem of (Heemels and Stoorvogel, 1998).

5
Conclusion

The aim of this report was to find out what is now known about positive controllability and stabilizability and to look at the reason why this holds. This research focused only on discrete-time linear systems.

In chapter 2 single input systems were discussed. For this, mainly three articles were used: (Evans and Murthy, 1977), (Leenheer and Nešić, 1998) and (Nguyen, 1986).
The eigenvalues of the matrix A became very important. For positive controllability, A cannot have real eigenvalues greater or equal to zero. However, for positive stabilizability, the eigenvalues are allowed to be in the open unit disk. Therefore A cannot have eigenvalues greater or equal to one. The article (Nguyen, 1986) connected local positive controllability to local reachability.

In chapter 3 a multiple input system was discussed. A lot less is known about the positive controllability and stabilizability of these systems, and that is why it has been limited to only (Yoshida et al., 1994).
The method of (Yoshida et al., 1994) depends on subsystems. Positive controllability is invariant under nonsingular state transformations. The dimension of a subsystem is smaller than that of the given system, therefore the test for positive controllability can be simplified.

6

Future research

This research has provided an overview of positive controllability and stabilizability. However, there is a lot more to research.

To start, this research has only been done for discrete-time linear systems. It can be extended with continuous-time systems. The difference in location of stable eigenvalues will change the statements made in this report.

Next, the conclusion of (Evans and Murthy, 1977) ends with the following:

> For multi-input systems it is conjectured that the result is similar to that in (Brammer, 1972) for continuous time, but the development is not straightforward.

If we introduce Farkas' lemma (Aardal, Iersel, and Janssen, 2019):

> Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two assertions is true:
>
> 1. There exists an $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.
> 2. There exists a $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$.

There can be a link between the problems described in this report and Farkas' lemma, together with the article (Brammer, 1972). Unfortunately, it has not been possible to specify this research due to time limitations.

Bibliography

[1] Cambridge University Press 2020. *Cambridge Dictionary.* URL: https://dictionary.cambridge.org/dictionary/english/system. (accessed: 01.06.2020).

[2] K. Aardal, L. van Iersel, and R. Janssen. *Lecture Notes AM2020, Optimization.* 2019.

[3] R. F. Brammer. "Controllability in linear autonomous systems with positive controllers". In: *SIAM Journal on Control* 10.2 (1972), pp. 339–353.

[4] M. E. Evans and D. N. P. Murthy. "Controllability of Discrete-Time Systems with Positive Controls". In: *IEEE Transactions on automatic control* 22.6 (1977), pp. 942–945. DOI: 10.1109/TAC.1977.1101648.

[5] A. Gambier. *Researchgate, How stabilizability and controllability interconnect.* URL: https://www.researchgate.net/post/How_do_stabilizability_and_controllability_interconnect. (accessed: 08.06.2020).

[6] W. P. M. H. Heemels and A. A. Stoorvogel. "Positive stabilizability of a linear continuous-time system". In: *Measurement and control systems : internal report* 98I.01 (1998).

[7] *Jordan matrix.* URL: https://encyclopedia2.thefreedictionary.com/Jordan+block. (accessed: 05.06.2020).

[8] P. de Leenheer and D. Nešić. "A note on stabilizing and time-optimal controllers for discrete-time linear systems with positive controls". In: *IFAC Nonlinear control systems design* 31.17 (1998), pp. 657–662. DOI: https://doi.org/10.1016/S1474-6670(17)40412-5.

[9] K. S. Nguyen. "On the null-controllability of linear discrete-time systems with restrained controls". In: *Journal of optimization theory and applications* 50.2 (1986), 313–329.

[10] G. J. Olsder et al. *Mathematical Systems Theory.* VSSD, 2011.

[11] H. Yoshida et al. "Controllability of multiple input discrete-time linear systems with positive controls". In: *Trans. of the society of instrument and control engineers* 30.3 (1994), pp. 243–252. DOI: https://doi.org/10.9746/sicetr1965.30.243.