PHASE PORTRAITS OF QUADRATIC SYSTEMS

HIGHER ORDER SINGULARITIES AND SEPARATRIX CYCLES

PHASE PORTRAITS OF QUADRATIC SYSTEMS

HIGHER ORDER SINGULARITIES AND SEPARATRIX CYCLES

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus,
Prof.drs. P.A. Schenck,
in het openbaar te verdedigen ten overstaan
van een commissie door het College van Dekanen
daartoe aangewezen
op maandag 29 mei 1989 te 14.00 uur

door

PAUL DE JAGER,
geboren te Ridderkerk,
wiskundig ingenieur

Aan mijn moeder

CONTENTS

SAMENVATTING

In dit proefschrift worden fasebeelden van kwadratische stelsels differentiaalvergelijkingen bestudeerd.

Ten eerste wordt een classificatie gegeven van alle fasebeelden met een hogere orde singulier punt met twee eigenwaarden gelijk aan nul. Voor kwadratische stelsels blijkt dat er slechts vier types hogere orde singuliere punten met twee eigenwaarden gelijk aan nul voorkomen. Deze punten zijn de vierde orde zadelknoop, het derde orde punt met elliptische en hyperbolische sector, het derde orde zadelpunt en de tweede orde snavel.

Vervolgens wordt een classificatie gegeven van alle fasebeelden met een derde of vierde orde singulier punt met één eigenwaarde gelijk aan nul. In dit geval is het een vierde orde zadelknoop, een derde orde zadelpunt of een derde orde knoop.

In alle kwadratische stelsels, met een hogere orde singulier punt, die bekeken worden in dit proefschrift kan ten hoogste één grenskringloop voorkomen.

In het tweede deel van dit proefschrift wordt een overzicht gegeven van alle types begrensde separatrixlussen, die voorkomen in kwadratische stelsels. Voor elk van de vijf types separatrixlussen, worden de volgende drie vragen bekeken:

1. Hoeveel grenskringlopen kunnen worden omsloten door een separatrixlus?
2. Welke typen spiralen kunnen binnen een separatrixlus liggen?
3. Onder welke voorwaarden is een separatrixlus stabiel/instabiel?

CHAPTER 1

PHASE PORTRAITS FOR QUADRATIC SYSTEMS WITH A HIGHER ORDER SINGULARITY WITH TWO ZERO EIGENVALUES

1. INTRODUCTION

In a survey paper [5] on general properties of quadratic systems of differential equations in the plane Coppel states that what remains to be done is to determine all possible phase portraits of such systems, this being of great practical value. The present paper aims at giving a contribution in this direction. By a quadratic system is meant the system

$$\begin{aligned} \dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \equiv P(x,y), \\ \dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \equiv Q(x,y), \end{aligned} \tag{1}$$

for the functions $x = x(t)$, $y = y(t)$, where $\cdot = \frac{d}{dt}$ and $a_{ij}, b_{ij} \in \mathbb{R}$.

Since quadratic systems without singular points in the finite part of the plane have been classified by Gasull, Sheng Li-Ren and Llibre [8], we may assume that (1) has at least one singular point in the finite part of the plane and we may shift the origin into this point: $a_{00} = b_{00} = 0$. Two limiting cases are then also classified: the linear case ($a_{20} = a_{11} = a_{02} = b_{20} = b_{11} = b_{02} = 0$) and the homogeneous case ($a_{10} = a_{01} = b_{10} = b_{01} = 0$) [6], [8]. If both linear and quadratic terms are present a lot of work remains to be done. If one or both eigenvalues in the singularity are zero, yet at least one linear term remains after transformation to the origin, a higher order singularity or multiple equilibrium point exists.

In this paper systems with a higher order singular with two zero eigenvalues are considered. Such higher order points were classified by Berlinskii [2], who makes use of other papers in Russian, which are not easily accessible. At present it is more convenient to use the classification of multiple equilibrium points for analytic systems as given by Andronov *et al.* in Chapter 9 of their book [1], of which an English translation is available. Also, the notion of order of a singular point (or multiplicity of an equilibrium point) as used in [2] may be improved somewhat by using that, implicitly present in the analysis given in [1]. The results of a renewed classification [10], however, agree with those given by Berlinskii (see also [4]).

If the higher order singular point is in the origin, the quadratic system may be brought into a normal form by an affine transformation. If both eigenvalues are zero this normal form reads [1, p. 346].

$$\begin{aligned} \dot{x} &= y + ax^2 + bxy + cy^2 \equiv P(x,y), \\ \dot{y} &= \quad\;\; dx^2 + exy + fy^2 \equiv Q(x,y). \end{aligned} \tag{2}$$

The order of the singular point in the origin may be defined as the maximum number of common zeros near the origin in the unfoldings of the functions $P(x,y)$ and $Q(x,y)$. This is the same as the maximum number of zeros of the unfolding of $\psi(x) \equiv Q\{x,\phi(x)\}$, where $\phi(x)$ is defined by $P\{x,\phi(x)\} = 0$, satisfying $\phi(0) = \phi'(0) = 0$; thus if $\psi(x) = a_n x^n + \ldots$, with $a_n \neq 0$, n is the order of the singular point. On the basis of Theorems 66 and 67, Chapter 9 of [1] there are for (2) the following cases:

(i) a fourth order saddle node, if $a \neq 0$, $d = e = 0$, $f \neq 0$; index 0,

(ii) a third order point, having an elliptic and a hyperbolic sector, if $ae > 0$, $d = 0$; index 1,

(iii) a third order saddle point, if $ae < 0$, $d = 0$; index -1,

(iv) a cusp point; the phase portrait is the union of two hyperbolic sectors and two separatrices, both tangent to the x-axis, if $d \neq 0$, order 2; index 0.

In all cases div $\{P(0,0),Q(0,0)\} = 0$.

In the present paper the possible phase portraits are given for all quadratic systems with a higher order singularity having two zero eigenvalues.

It should be noted that the classification in this paper is not complete in the sense that some global problems, such as the number of limit cycles and the relative position of separatrices, are not completely solved in all cases. The phase portraits are characterized in the usual way: by the number, position and character of the singular points; by the number and position of the periodic solutions; by the position of the separatrices and by the behaviour at infinity. Standard arguments will be used, such as local linearisation in

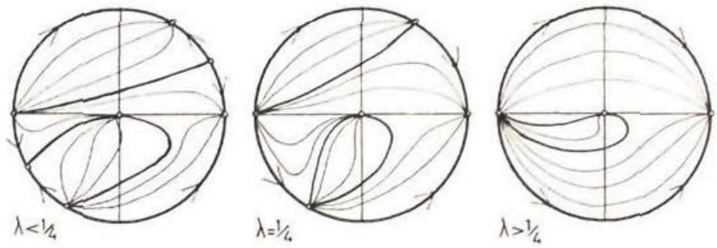


Figure 2: Phase portraits with a fourth order saddle node.

3. QUADRATIC SYSTEMS WITH A THIRD ORDER SINGULAR POINT WITH AN ELLIPTIC AND A HYPERBOLIC SECTOR

If (0,0) is a third order singular point with an elliptic and hyperbolic sector, (2), with $d = 0$, $ae > 0$, may be written, if necessary by applying an affine transformation and/or a scaling of t, in the form

$$\begin{aligned} \dot{x} &= y + \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2 \equiv P(x,y), \\ \dot{y} &= xy \qquad\qquad\qquad\qquad\quad \equiv Q(x,y), \end{aligned} \tag{5}$$

with $\lambda_1 > 0$, $\lambda_2 \in \{0,1\}$, $\lambda_3 \in \mathbb{R}$.

As may be seen in Figure 3, the elliptic sector occurs in the lower half plane and the hyperbolic sector in the upper half plane. Obviously $y \equiv 0$ consists of orbits of (5), and it seems that the horizontal axis separates the hyperbolic sector from the other sector, as sketched in Figure 3a. It appears, however, that the location of the separatrix needs a more detailed analysis and that a situation as in Figure 3b also must be taken into consideration.

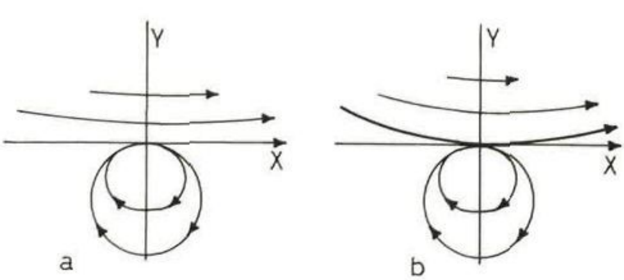


Figure 3 : Third order point with elliptic sector.

Blowing up the singularity in (0,0) by means of the transformation $y = w_1 x$, $x = x$, yields, from (5)

$$\begin{aligned} \dot{x} &= \lambda_1 x^2 + x w_1 + \lambda_2 x^2 w_1 + \lambda_3 x^2 w_1^2, \\ \dot{w}_1 &= (1 - \lambda_1)\, x w_1 - w_1^2 - \lambda_2 x w_1^2 - \lambda_3 x w_1^3. \end{aligned} \tag{6}$$

The behaviour of the integral curves of (5) in the first quadrant of the x,y-plane is reflected in that of the curves of (6) in the first quadrant of the x,w_1-plane, of which a qualitative sketch is given in Figure 4. It is not difficult to show, that except along the x- and w_1-axis, also in the direction $w_1 = \frac{1}{2}(1 - 2\lambda_1)\, x$ integral path(s) are approching the singular point. Apart from a hyperbolic sector, a parabolic sector therefore exists in

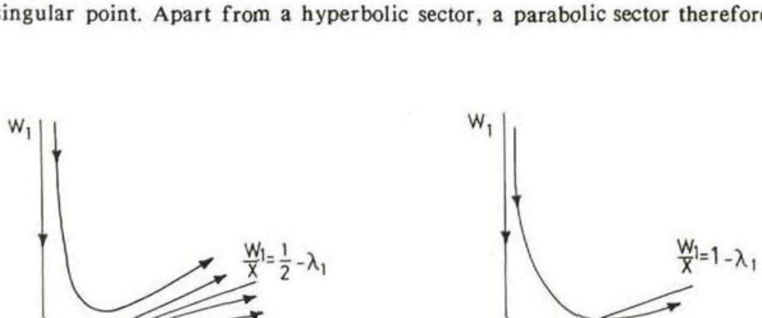


Figure 4 : x,w_1-plane in the neighbourhood of (0,0) of equation (6).

the first quadrant of the x,w_1-plane and the x,y-plane if $0 < \lambda_1 < \frac{1}{2}$ and the separatrix configuration is then as in Figure 3b. The 0-isocline $[\frac{dw_1}{dx} = 0]$ approaches the origin along the line $w_1 = (1 - \lambda_1)\,x$, as a result of which it is not certain, for $\frac{1}{2} \le \lambda_1 < 1$, whether or not there is a parabolic sector in the first quadrant tangent to the x-axis. In order to answer this question a further 'blow up' is needed. By the transformation $w_1 = wx$, $x = x$, (6), becomes

$$\begin{aligned} \dot{x} &= \lambda_1 x^2 + x^2 w + \lambda_2 x^3 w + \lambda_3 x^4 w^2, \\ \dot{w} &= (1 - 2\lambda_1)\, xw - 2xw^2 - 2\lambda_2 x^2 w^2 - 2\lambda_3 x^3 w^3, \end{aligned} \tag{7}$$

or

$$\frac{dw}{dx} = \frac{(1 - 2\lambda_1)\, w - 2w^2 - 2\lambda_2 x w^2 - 2\lambda_3 x^2 w^2}{\lambda_1 x + xw + \lambda_2 x^2 w + \lambda_3 x^3 w^2}, \tag{8}$$

what shows that for $\lambda_1 \ge \frac{1}{2}$, in the first quadrant there is $\frac{dw}{dx} < 0$ near the origin, as a result of which there will be no parabolic sector in the first quadrant of the x,w-plane, the x,w_1-plane and the x,y-plane for $\lambda_1 \ge \frac{1}{2}$, and the separatrix configuration is as in Figure 3a.

System (5) has no limit cycles. In fact, it can be shown that for $\lambda_2 = 1$ the system has no periodic solutions, since $B(x,y) = y^{-2\lambda_1 - 1}$ is a Dulac function yielding

$$\frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) = \lambda_2 y^{-2\lambda_1},$$

which is of constant sign in a half plane $y \gtrless 0$. A periodic solution must then cross the x-axis, which is not possible since it consists of integral curves. For $\lambda_2 = 0$ the system is symmetric around the y-axis, which excludes limit cycles, since all possible singular points are on the y-axis and a limit cycle has to have a singular point in its interior. Another argumentation would be to use div $(BP,BQ) \equiv 0$, when applying Green's theorem to an annulus with a limit cycle as one of the boundary curves [11]. For $\lambda_2 = 0$, the

Dulac function acts as an integrating factor to yield the solutions

$$\text{for } \lambda_1 \neq \tfrac{1}{2},1:\quad y \equiv 0 \text{ and } \frac{2}{2\lambda_1 - 1}\, y + x^2 + \frac{\lambda_3}{\lambda_1 - 1}\, y^2 + \lambda_4 |y|^{2\lambda_1} = 0,\ \lambda_4 \in \mathbb{R},$$

$$\text{for } \lambda_1 = \tfrac{1}{2}:\quad y \equiv 0 \text{ and } -2y \ln|y| + x^2 - 2\lambda_3 y^2 + \lambda_4 y = 0 \qquad ,\ \lambda_4 \in \mathbb{R},$$

$$\text{for } \lambda_1 = 1:\quad y \equiv 0 \text{ and } 2y + x^2 - 2\lambda_3 y^2 \ln|y| + \lambda_4 y^2 = 0 \qquad ,\ \lambda_4 \in \mathbb{R},$$

which also include the periodic solutions of (5) for $\lambda_3 < 0$ (see also [12]).
The character of the other finite singular point P_1: $(0, -\lambda_3^{-1})$, (for $\lambda_3 \neq 0$) may be analyzed by local linearization and using the symmetry for $\lambda_2 = 0$.
In order to determine the phase portraits of (5) we consider the cases $\lambda_2 = 0$ and $\lambda_2 \neq 0$ separately.

Case $\lambda_2 = 0$. From (3) follows that the singular points at infinity are given by

$$C_2(\theta) \equiv \left[(1 - \lambda_1)\cos^2\theta - \lambda_3 \sin^2\theta\right] \sin\theta.$$

For $\lambda_1 = 1$, $\lambda_3 = 0$, all points on $\rho \equiv 1$ are singular, or, if the factor $(1 - \rho)$ is divided out in (4), ordinary points. In the x,y-plane the integral curves are conics through P_0.
For $(\lambda_1, \lambda_3) \neq (1,0)$, $C_2(\theta) = 0$ shows that there is a singular point P_2 at $\theta = 0(\pi)$, and for $\lambda \equiv \lambda_3 (1 - \lambda_1)^{-1} > 0$, a point P_4 at $\theta = \operatorname{arccot} -\sqrt{\lambda}$, and a point P_5 at $\theta = \operatorname{arccot} \sqrt{\lambda}$; these points coincide for $\lambda = 0$ in point P_3 at $\theta = \frac{\pi}{2}$ $\left(\frac{3\pi}{2}\right)$.
The character of these points may be found by local linearization and using Theorems 65, 66 and 67 of Chapter 9 of [1]. The character of the singular points is listed in Table I and the phase portraits are indicated in the λ_1,λ_3-parameter plane in Figure 5 and given in Figure 6. It may be seen, that for $\lambda_3 < 0$, equation (5) has periodic solutions and the phase portraits may also be found in the classification of quadratic systems with a centre as given by Vulpe [12].

TABLE I.

Portrait	$\lambda_1 - \frac{1}{2}$	$\lambda_1 - 1$	λ	λ_3	P_0	P_1	P_2	P_3	P_4	P_5
1	-		+	+	Pwes*	S	S		N	N
2	0,+	-	+	+	Pwes*	S	S		N	N
3		0		+	Pwes*	S	N*			
4		+	-	+	Pwes*	S	N			
5	-		0	0	Pwes*		S	Pwes*		
6	0,+	-	0	0	Pwes*		S	Pwes*		
7		0		0	Pwes*					
8		+	0	0	Pwes*		N	S*		
9	-		-	-	Pwes*	C	S			
10	0,+	-	-	-	Pwes*	C	S			
11		0		-	Pwes*	C	S*			
12		+	+	-	Pwes*	C	N		S	S

* higher order point

C = center point

Pwes = point with an elliptic and hyperbolic sector

S = saddle point

N = node

SN = saddle node

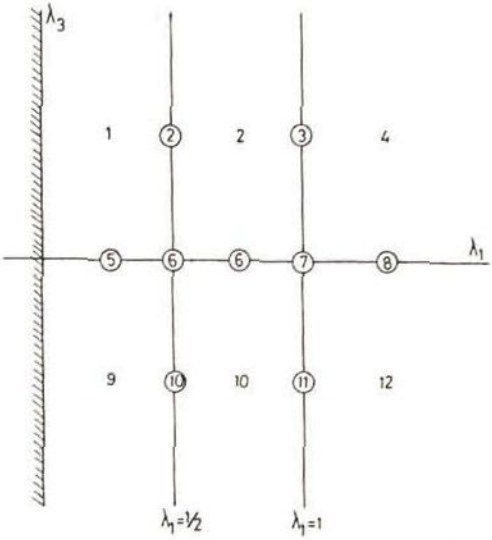


Figure 5: The λ_1,λ_3 parameter plane for the case $\lambda_2 = 0$.

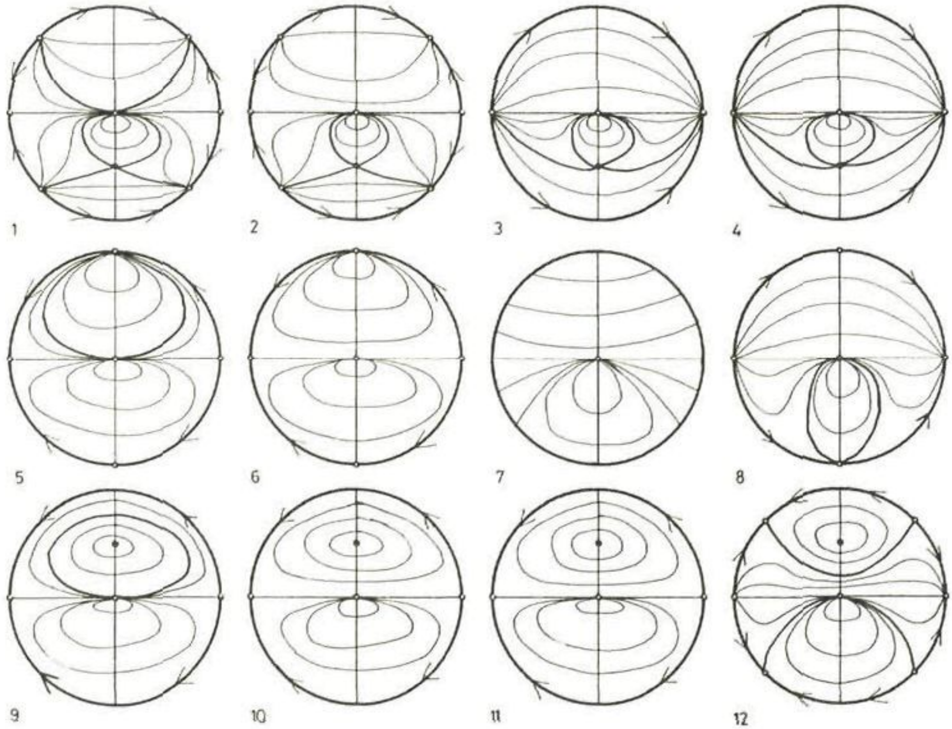


Figure 6 : Phase portraits for the case $\lambda_2 = 0$.

Case $\lambda_2 = 1$. From (3) follows, that the singular points at infinity are given by

$$C_2(\theta) = \left[(1 - \lambda_1)\cos^2\theta - \sin\theta\cos\theta - \lambda_3 \sin^2\theta\right] \sin\theta = 0,$$

so that there is a point P_2 at $\theta = 0(\pi)$, and, for $\lambda \equiv 1 + 4\lambda_3 (1 - \lambda_1) > 0$, a point P_4 at $\theta = \operatorname{arccot} - \frac{1}{2}(1 + \sqrt{\lambda})(\lambda_1 - 1)^{-1}$ and a point P_5 at $\theta = \operatorname{arccot} - \frac{1}{2}(1 - \sqrt{\lambda})(\lambda_1 - 1)^{-1}$; these two points coincide for $\lambda = 0$ at point P_3 at $\theta = \operatorname{arccot} - \frac{1}{2}(\lambda_1 - 1)^{-1}$.

Moreover, for $\lambda_1 = 1$, the point P_4 coincides with P_2, and point P_5 is situated in $\theta = \operatorname{arccot} - \lambda_3$. The character of these points may be found by local linearization and using Theorem 65 of Chapter 9 of [1]. The character of the singular points is listed in Table II and the phase portraits are indicated in the λ_1,λ_3-parameter plane in Figure 7 and given in Figure 8.

4. QUADRATIC SYSTEMS WITH A THIRD ORDER SADDLE POINT AND TWO ZERO EIGENVALUES

If (0,0) is a third order saddle point with two zero eigenvalues, (2), with $d = 0$, $ae < 0$, may be written, if necessary by applying an affine transformation and/or a scaling of t, in the form

$$\begin{aligned} \dot{x} &= y + \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2 \equiv P(x,y)\,, \\ \dot{y} &= xy \equiv Q(x,y), \end{aligned} \tag{9}$$

with $\lambda_1 < 0$, $\lambda_2 \in \{0,1\}$, $\lambda_3 \in \mathbb{R}$.

As illustrated in Figure 9, the saddle point in P_0: (0,0) consists of four hyperbolic sectors separated by four separatrices: the positive x-axis and a curved separatrix tangent to it in P_0, and the negative x-axis and a curved separatrix tangent to it in P_0; both curved separatrices lying in the upper half plane.

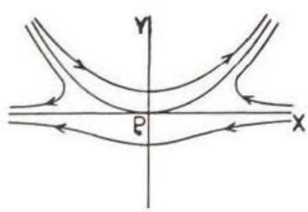


Figure 9 : Third order saddle point.

As for the previous case it can be shown that (9) has no limit cycles by using the same Dulac function $B(x,y) \equiv y^{-2\lambda_1 - 1}$. For $\lambda_2 = 0$, this Dulac function acts as an integrating factor to yield the solutions

$$y \equiv 0 \text{ and } \frac{2}{2\lambda_1 - 1}\, y + x^2 + \frac{\lambda_3}{\lambda_1 - 1}\, y^2 + \lambda_4 |y|^{2\lambda_1} = 0,\ \lambda_4 \in \mathbb{R},$$

which includes also the periodic solution of (9).

The other finite singular point P_1: $(0,-\lambda_3^{-1})$, (for $\lambda_3 \neq 0$) may be analyzed by local linearization and using the symmetry for $\lambda_2 = 0$.

From (3) follows, that the singular points at infinity are given by

$$C_2(\theta) \equiv \left[(1 - \lambda_1)\cos^2\theta - \lambda_2 \sin\theta\cos\theta - \lambda_3 \sin^2\theta\right] \sin\theta = 0,$$

so that there is a point P_2 at $\theta = 0(\pi)$, and, for $\lambda \equiv \lambda_2^2 + 4\lambda_3(1 - \lambda_1) > 0$, a point P_4 at $\theta = \operatorname{arccot} - \frac{1}{2}(\lambda_2 + \sqrt{\lambda})(\lambda_1 - 1)^{-1}$, and a point P_5 at $\theta = \operatorname{arccot} - \frac{1}{2}(\lambda_2 - \sqrt{\lambda})(\lambda_1 - 1)^{-1}$; these two points coincide for $\lambda = 0$ in the point P_3 at $\operatorname{arccot} - \frac{1}{2}\lambda_2(\lambda_1 - 1)^{-1}$. The character of these points may be found by local linearization and using Theorems 65, 66 and 67 of Chapter 9 of [1]. The character of the singular points is listed in Table III and the phase portraits are given in Figure 10. Use should be made of the argument that $\theta' < 0$ on P_0P_4 and P_0P_5 (and thereby on P_0P_3).

It may be noted that a singularity with an elliptic and a hyperbolic sector occurs at infinity in phase portrait 2 of Figure 10. In the same way as in section 3 one may show, by blowing up the singularity, that the separatrices of the singular point with elliptic sector coincide with the equator of the Poincaré sphere.

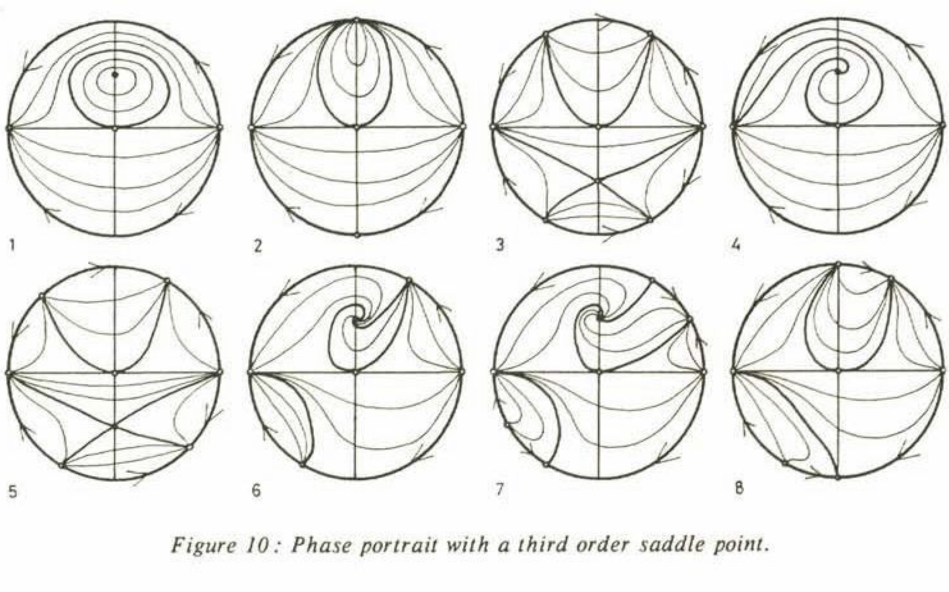


Figure 10 : Phase portrait with a third order saddle point.

TABLE III.

λ	λ_2	λ_3	P_0	P_1	P_2	P_3	P_4	P_5
-	0	-	S*	C	N			
0	0	0	S*		N	Pwes*		
+	0	+	S*	S	N		N	N
+	1	+	S*	S	N		N	N
+	1	0	S*		N		N	SN*
+	1	-	S*	N,f	N		N	S
0	1	-	S*	N,f	N	SN*		
-	1	-	S*	N,f	N			

* higher order point

5. QUADRATIC SYSTEMS WITH A CUSP POINT

If (0,0) is a second order cusp point, system (2), with $d \neq 0$, may be written, if necessary by applying an affine transformation and/or scaling of t, in the form:

$$\begin{aligned} \dot{\bar{x}} &= \bar{y} + a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + a_{02}\bar{y}^2 \equiv P(\bar{x},\bar{y}), \\ \dot{\bar{y}} &= \bar{x}^2 + b_{11}\bar{x}\bar{y} + b_{02}\bar{y}^2 \equiv Q(\bar{x},\bar{y}), \end{aligned} \tag{10}$$

with $a_{20}, a_{11}, a_{02}, b_{11}, b_{02} \in \mathbb{R}$.

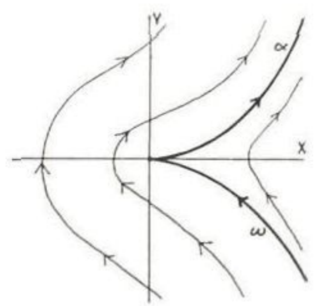


Figure 11: Second order cusp point.

As illustrated in Figure 11 the cusp point $P_0(0,0)$ consists of two hyperbolic sectors, separated by two separatrices. These two separatrices are both tangent to the positive $\bar{x}$-axis in P_0. In a neighbourhood of P_0 the α-separatrix of P_0 lies in the first quadrant and the ω-separatrix lies in the fourth quadrant.

In order to determine the phase portraits of system (10), the cases $\Delta = b_{11}^2 - 4b_{02} < 0$, $\Delta = 0$ and

$\Delta > 0$ will be considered separately. If $\Delta < 0$ the curve $Q(\bar{x},\bar{y}) = 0$ only consists of the point (0,0) and system (10) has no finite singular points besides P_0. If $\Delta = 0$ the curve $Q(\bar{x},\bar{y}) = 0$ consists of two coinciding straight lines system (10) may have one second order singular point besides P_0. In the last case the curve $Q(\bar{x},\bar{y}) = 0$ consists of two straight lines and system (10) may have two first order singular points besides the point P_0.

THE CASE $\Delta < 0$

From (3) follows that the singular points at infinity for system (10) are given by

$$C_2(\theta) = \cos^3\theta + (b_{11} - a_{20}) \cos^2\theta \sin\theta + (b_{02} - a_{11}) \cos\theta \sin^2\theta - a_{02} \sin^3\theta = 0 \quad (11)$$

or equivalently $\cot^3\theta + (b_{11} - a_{20}) \cot^2\theta + (b_{02} - a_{11}) \cot\theta - a_{02} = 0.$

Now three different cases may be distinguished. Case a: there is just one real solution $\cot\theta$ of equation (11). Case b: there are two different real solutions of (11); one of them is simple and the other one is double. Case c: equation (11) has three different real solutions.

If the affine transformation $\bar{x} = x + \alpha y$, $\bar{y} = y$, where α is a properly chosen constant, is applied to (10), there may be obtained

$$\begin{aligned} \dot{x} &= y + \lambda_1 x^2 + \lambda_2 xy &&\equiv P(x,y), \\ \dot{y} &= x^2 + \lambda_3 xy + \lambda_4 y^2 &&\equiv Q(x,y), \end{aligned} \qquad (12)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in R$ and $\lambda_3^2 - 4\lambda_4 < 0$.

In fact we take the constant $\alpha = \cot\theta$, where α is the unique solution of (11) in case a, α is the double solution in case b and the middle one of the three solutions in case c. From

(11) then follows that the singular points at infinity of (12) are given by

$$C_2(\theta) = \left[\cos^2\theta + (\lambda_3 - \lambda_1)\cos\theta \sin\theta + (\lambda_4 - \lambda_2)\sin^2\theta\right] \cos\theta = 0. \tag{13}$$

For $\Delta < 0$ system (12) has just one singular point in the finite part of the plane: the second order cusp point P_0 in (0,0). Since there are no other singular points in the finite part of the plane, P_0 is the only possible candidate to be situated inside a limit cycle. However, in a quadratic system a limit cycle has to surround a focus [5]. Thus for $\Delta < 0$ there is no limit cycle. In order to investigate the singular points at infinity the cases a, b and c will be considered separately.

Case a: Since there is one singular point at infinity ($\theta = \frac{1}{2}\pi$) there should be satisfied either $\lambda_3 - \lambda_1 = \lambda_4 - \lambda_2 = 0$ or $(\lambda_3 - \lambda_1)^2 - 4(\lambda_4 - \lambda_2) < 0$ as follows from (13).

Case b: Since the double singular point at infinity is at $\theta = \frac{1}{2}\pi$, it follows from (13) that $\lambda_2 = \lambda_4$. Since there is a second singular point at infinity at $\theta = \operatorname{arccot}(\lambda_1 - \lambda_3)$ we know that $\lambda_1 - \lambda_3 \neq 0$ and without loss of generality we may assume that $\lambda_1 - \lambda_3 < 0$.

Case c: Since the middle one of the three singular points at infinity is at $\theta = \frac{1}{2}\pi$, there follows from (13) that $\lambda_2 - \lambda_4 > 0$. Without loss of generality we may assume that $\lambda_1 \geq 0$. Besides the singular point at $\theta = \frac{1}{2}\pi$ the system has two other singular points at infinity at $\theta = \operatorname{arccot}\left[\frac{1}{2}(\lambda_1 - \lambda_3 - \sqrt{\lambda})\right]$ and at $\theta = \operatorname{arccot}\left[\frac{1}{2}(\lambda_1 - \lambda_3 + \sqrt{\lambda})\right]$, where $\lambda = (\lambda_3 - \lambda_1)^2 - 4(\lambda_4 - \lambda_2)$.

The character of the singular points at infinity may be found by local linearisation and using Theorems 65, 66 and 67 of [1]. The results of this analysis are listed in Table IV and the phase portraits are given in Figure 12. For the cases b and c use should be made of the argument that, as $\lambda_2 > 0$, $\dot{x} = \lambda_1\lambda_2^{-2}$ on the line $x = -\lambda_2^{-1}$.

TABLE IV: CASE $\Delta < 0$

Case	$\lambda_4 - \lambda_2$	P_0	P_3	P_4	P_5
a	+	CP^*	N		
a	0	CP^*	N^*		
b		CP^*	SN^*	N	
c		CP^*	S	N	N

CP = cusp point

$*$ higher order point

$P_3: \theta = \frac{1}{2}\pi$

$P_4: \frac{1}{2}\pi < \theta < \pi$

$P_5: 0 < \theta < \frac{1}{2}\pi$

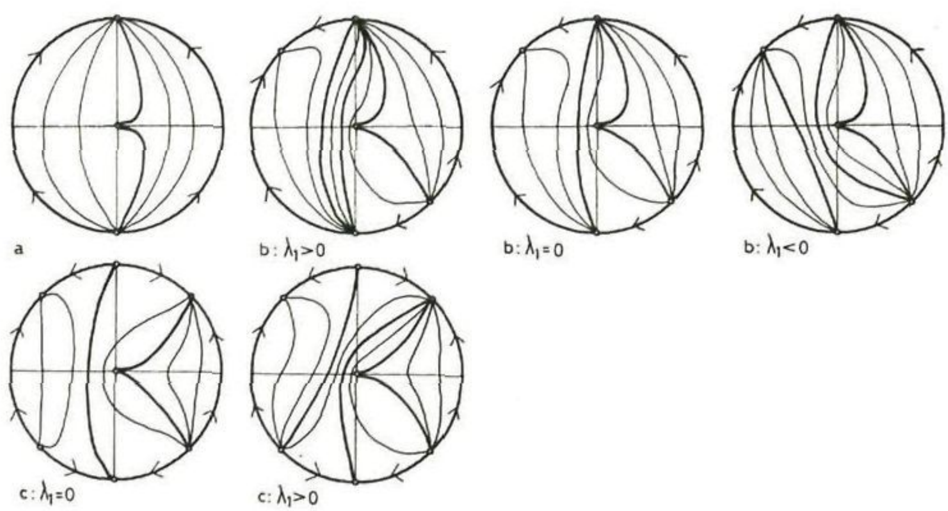


Figure 12: CASE $\Delta < 0$.

THE CASE $\Delta = 0$

The singular points at infinity of system (10) are given by (11). Just as in the previous section we may apply the affine transformation $\bar{x} = x + \alpha y$, $\tilde{y} = y$ where α is a properly chosen constant. System (10) then becomes

$$\begin{aligned} \dot{x} &= y + \lambda_1 x^2 + \lambda_2 xy &&\equiv P(x,y), \\ \dot{y} &= \quad x^2 + \lambda_3 xy + \lambda_4 y^2 &&\equiv Q(x,y), \end{aligned} \tag{14}$$

where $\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in R$ and $\lambda_3^2 - 4\lambda_4 = 0$.

From (11) follows that the singular points at infinity of (14) are given by

$$C_2(\theta) = \left[\cos^2\theta + (\lambda_3 - \lambda_1)\cos\theta\sin\theta + (\lambda_4 - \lambda_2)\sin^2\theta\right]\cos\theta = 0. \tag{15}$$

Now the cases $\lambda_3 = 0$ and $\lambda_3 \neq 0$ will be considered separately.

CASE I: $\lambda_3 = 0$

In this case holds $\lambda_3 = \lambda_4 = 0$. We may achieve, by applying an affine transformation and scaling of t that $\lambda_1 \geq 0$.

System (14) has just one finite singular point: the cusp point P_0. Since a limit cycle has to surround a focus, it is obvious that the system has no limit cycles.

Now the cases where system (14) has respectively one, two or three singular points at infinity, will be considered separately in Ia, Ib and Ic.

<u>*Case Ia*</u>*:* Since there is one singular point at infinity at $\theta = \frac{1}{2}\pi$ there should be satisfied either $\lambda_1 = \lambda_2 = 0$ or $\lambda_1^2 + 4\lambda_2 < 0$ as follows from (15).

<u>*Case Ib*</u>*:* Since the double singular point at infinity is at $\theta = \frac{1}{2}\pi$, it follows from (15) that $\lambda_2 = 0$. Because there is a second singular point at infinite at $\theta = \operatorname{arccot}(\lambda_1)$ it is clear that $\lambda_1 > 0$.

Case Ic: Because the middle one of the three singular points at infinity is at $\theta = \frac{1}{2}\pi$, there follows from (15) that $\lambda_2 > 0$. The two other singular points are at $\theta = \operatorname{arccot}(\frac{1}{2}\lambda_1 + \frac{1}{2}\sqrt{\lambda})$ and $\theta = \operatorname{arccot}(\frac{1}{2}\lambda_1 - \frac{1}{2}\sqrt{\lambda})$ where $\lambda = \lambda_1^2 + 4\lambda_2$.

The character of the singular points at infinity may be found by local linearisation and using Theorems 65, 66 and 67 of [1]. The results of this analysis are listed in Table V and the phase portraits are given in Figure 13.
For case Ib we notice that the separatrix of P_3 at $\theta = \frac{3}{2}\pi$ intersects the negative x-axis in a point A. It is easy to show that the separatrix remains below the straight line from point A to the singular point at infinity in the first quadrant and above the α- separatrix of P_0, and thus will tend to the singular point at infinity in the first quadrant.
For case Ic we notice that as $\lambda_2 > 0$, along the line $x = -\lambda_2^{-1}$ holds $\dot{x} = \lambda_1\lambda_2^{-1}$.
Now it can easily been seen that the phase portraits are as given in Figure 13.

TABLE V: CASE $\Delta = 0$: Ia,b,c

Case		P_0	P_3	P_4	P_5
Ia	$\lambda_1 = \lambda_2 = 0$	CP	N*		
Ia	$\lambda_1^2 + 4\lambda_2 < 0$	CP	N*		
Ib	$\lambda_2 = 0, \lambda_1 > 0$	CP	SN*	N	
Ic	$\lambda_2 > 0$	CP	S*	N	N

* higher order point

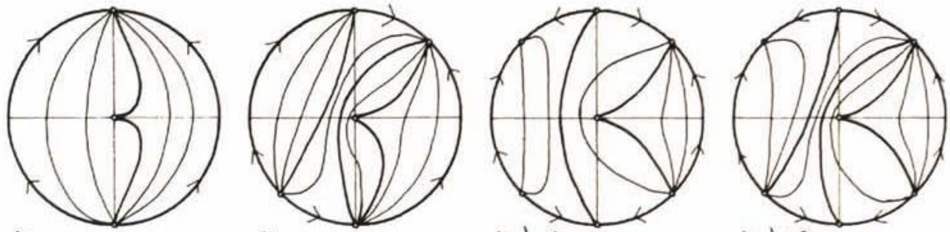


Figure 13 : CASE $\Delta = 0$: Ia,b,c.

CASE II : $\lambda_3 \neq 0$

By a scaling of y and t system (10) may be put into the form

$$\begin{aligned} \dot{x} &= y + \lambda_1 x^2 + \lambda_2 xy \equiv P(x,y), \\ \dot{y} &= x^2 - 2xy + y^2 \equiv Q(x,y). \end{aligned} \tag{16}$$

Besides the cusp point P_0 system (16) has a second finite singular point P_1 at $((-\lambda_1 - \lambda_2)^{-1},(-\lambda_1 - \lambda_2)^{-1})$ for $\lambda_1 + \lambda_2 \neq 0$ and no other finite singular point if $\lambda_1 + \lambda_2 = 0$. The character of this point may be analysed by using Theorems 65, 66 and 67 of [1]. P_1 is a cusp point for $2\lambda_1 + \lambda_2 = 0$ and a second order saddle node for $2\lambda_1 + \lambda_2 \neq 0$. The nodal part of this saddle node is stable for $(2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2) > 0$ and unstable for $(2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2) < 0$.

Since a limit cycle has to surround a focus it is clear that system (16) has no limit cycles.

Now the cases where system (16) has respectively one, two or three singular points at infinity, will be considered separately in IIa, IIb and IIc.

Case IIa: Since there is one singular point P_3 at infinity there should be satisfied either $\lambda_1 + 2 = \lambda_2 - 1 = 0$ or $(\lambda_1 + 2)^2 + 4(\lambda_2 - 1) < 0$ as follows from (15). The character of the singular points and the phase portraits are given in Table VI and Figure 14.

We note that in Figure 14, phase portrait 2 the relative position of the α-separatrix of P_0 and the ω-separatrix of P_1 can be found in the following way: For small positive values of λ_1, the position of the α-separatrix of P_0 in phase portrait 2 is close to that for $\lambda_1 = 0$, as given in phase portrait 1. Therefore it should tend to the nodal part of P_1.

For small negative values of $2\lambda_1 + \lambda_2$ this position is close to that for $2\lambda_1 + \lambda_2 = 0$ as given in phase portrait 3 and therefore the α-separatrix should tend to P_3. From the continuity of the vector field it may be concluded that the case in which the α-separatrix of P_0 and the ω-separatrix of P_1 coincide, also occurs. So all three relative positions of the separatrices, suggested in portrait 2 of Figure 14 occur.

TABLE VI: CASE $\Delta = 0$: IIa

Portrait		λ_1	$2\lambda_1+\lambda_2$	P_0	P_1	P_3
1	$\lambda_1 + 2 = \lambda_2 - 1 = 0$	-	-	CP*	SN*	N*
1	$(\lambda_1+2)^2 + 4(\lambda_2-1) < 0$	-	-	CP*	SN*	N
1	$(\lambda_1+2)^2 + 4(\lambda_2-1) < 0$	0	-	CP*	SN*	N
2	$(\lambda_1+2)^2 + 4(\lambda_2-1) < 0$	+	-	CP*	SN*	N
3	$(\lambda_1+2)^2 + 4(\lambda_2-1) < 0$	+	0	CP*	CP*	N
4	$(\lambda_1+2)^2 + 4(\lambda_2-1) < 0$	+	+	CP*	SN*	N

* higher order point.

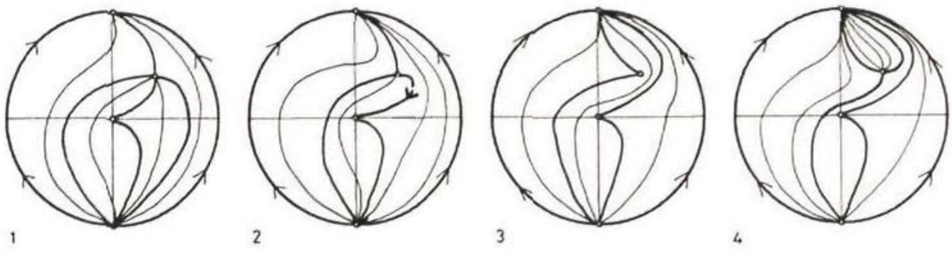


Figure 14 : CASE Δ = 0 : IIa.

<u>Case IIb</u>: Since the double singular point at infinity P_3 is at $\theta = \frac{1}{2}\pi$, it follows from *(15) that* $\lambda_2 = 1$. *Since there is a second singular point* P_4 *at* $\theta = \operatorname{arccot}(\lambda_1 + 2)$ *we know* that $\lambda_1 \neq -2$. The character of the singular points and all phase portraits are given in Table VII and Figure 15.

We notice that the relative position of the α-separatrix of P_3 at $\theta = 1\frac{1}{2}\pi$ and the right ω-separatrix of P_1 in phase portrait 6 cannot be found with standard arguments. Comparing phase portrait 6 with phase portrait 5 and 7, in the same way as in Case IIa, we may conclude that all three relative positions of the separatrices, suggested in phase portraits 6, occur.

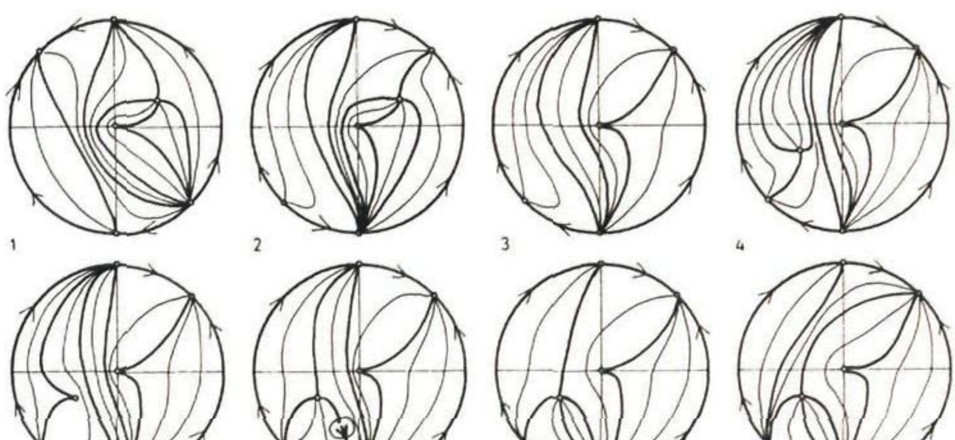


Figure 15 : CASE Δ = 0 : IIb.

TABLE VII: CASE $\Delta = 0$: IIb

Portrait	λ_1	P_0	P_1	P_3	P_4
1	$\lambda_1 < -2$	CP^*	SN^*	SN^*	N
2	$-2 < \lambda_1 < -1$	CP^*	SN^*	SN^*	N
3	$\lambda_1 = -1$	CP^*		SN^*	N^*
4	$-1 < \lambda_1 < -\frac{1}{2}$	CP^*	SN^*	SN^*	N
5	$\lambda_1 = -\frac{1}{2}$	CP^*	CP^*	SN^*	N
6	$-\frac{1}{2} < \lambda_1 < 0$	CP^*	SN^*	SN^*	N
7	$\lambda_1 = 0$	CP^*	SN^*	SN^*	N
8	$0 < \lambda_1$	CP^*	SN^*	SN^*	N

* higher order point

Case IIc: Since there are three different singular points at infinity and P_3 at $\theta = \frac{1}{2}\pi$ is the middle one, it follows from (15) that $\lambda_2 > 1$. Besides the point P_3 there are two singular points P_4 and P_5 at $\theta = \text{arccot}\,(\frac{1}{2}\lambda_1 + 1 + \frac{1}{2}\sqrt{\lambda})$ and at $\theta = \text{arccot}\,(\frac{1}{2}\lambda_1 + 1 + - \frac{1}{2}\sqrt{\lambda})$ where $\lambda = (\lambda_1 + 2)^2 + 4(\lambda_2 - 1)$. The character of the singular points and the phase portraits are given in Table VIII and Figure 16.

Comparing phase portrait 5 with phase portraits 4 and 6 we notice that all three relative positions of the α-separatrix of P_3 and the ω-separatrix of P_1, suggested in phase portrait 5 can occur.

TABLE VIII: CASE $\Delta = 0$: IIc

Portrait	λ_1	$\lambda_1 + \lambda_2$	$2\lambda_1 + \lambda_2$	P_0	P_1	P_3	P_4	P_5
1	-	-	-	CP*	SN*	S	N	N
2	-	0	-	CP*		S	N*	N
3	-	+	-	CP*	SN*	S	N	N
4	-	+	0	CP*	CP*	S	N	N
5	-	+	+	CP*	SN*	S	N	N
6	0	+	+	CP*	SN*	S	N	N
7	+	+	+	CP*	SN*	S	N	N

* higher order point

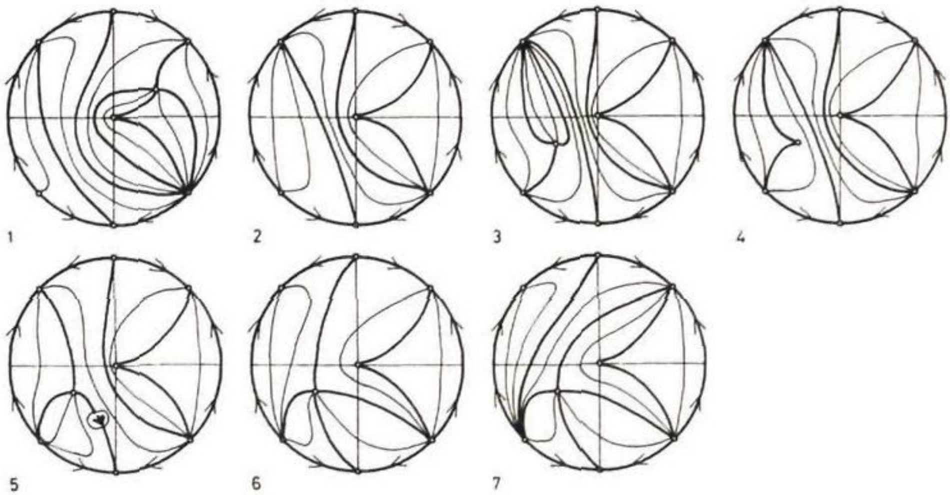


Figure 16 : CASE $\Delta = 0$: IIc.

THE CASE $\Delta > 0$

The singular points at infinity of system (10) are given by (11). Just as in the last two sections we may apply the affine transformation $\bar{x} = x + \alpha y$, $\bar{y} = y$ where α is a properly chosen constant. Then by scaling of x,y and t system (4) may be put into the form

$$\begin{aligned} \dot{x} &= y + \lambda_1 x^2 + \lambda_2 xy , \\ \dot{y} &= \quad x^2 + 2\lambda_3 xy + (\lambda_3^2 - 1)y^2, \end{aligned} \tag{17}$$

where $\lambda_1 \in R_0^+$ and $\lambda_2,\lambda_3 \in R$.
From (11) follows that the singular points at infinity of (17) are given by:

$$C_2(\theta) = \left[\cos^2\theta + (2\lambda_3 - \lambda_1)\cos\theta\sin\theta + (\lambda_3^2 - 1 - \lambda_2)\sin^2\theta\right]\cos\theta = 0. \tag{18}$$

Besides the cusp point P_0 system (17) can have up till two other finite singular points. Let P_1 be $((1 - \lambda_3)\,\mu_1^{-1},\mu_1^{-1})$ for $\mu_1 \neq 0$ and let P_2 be $((-1 - \lambda_3)\,\mu_2^{-1},\mu_2^{-1})$ for $\mu_2 \neq 0$, where $\mu_1 = (1 - \lambda_3)(\lambda_1(\lambda_3 - 1) - \lambda_2)$ and $\mu_2 = (-1 - \lambda_3)(\lambda_1(\lambda_3 + 1) - \lambda_2)$. The character of these points may be found by local linearisation as long as they are not a center in linear approximation. However, P_1 is a center in linear approximation for $\mu_3^2 - 4\mu_1 < 0$ and $\mu_3 = 0$ where $\mu_3 = ((1 - \lambda_3)(2\lambda_1 - 2) + \lambda_2)\,\mu_1^{-1}$. For further investigation one may use the four focal values as given in [3]. It then follows that P_1 is a center for $\lambda_1 + \lambda_3 = 0$, a stable first order fine focus for $\lambda_1 + \lambda_3 > 0$ and an unstable first order fine focus for $\lambda_1 + \lambda_3 < 0$.
If point P_1 is a node or a focus it is stable for $\mu_3 < 0$ and unstable for $\mu_3 > 0$. If P_2 is a node it is stable for $\lambda_3 > -1$ and unstable for $\lambda_3 < -1$. The character of these points is listed in Table IX.

TABLE IX

Point P_1	$\mu_3^2 - 4\mu_1$	μ_3	μ_1	$\lambda_1 + \lambda_3$
S	+		-	
N	+,0		+	
F	-	+,-		
C	-	0		0
FF	-	0		+,-

Point P_2	μ_2
S	+
N	-

FF = fine focus

According to [5] P_1 *is the only candidate to be situated inside a limit cycle since this* point is a (fine) focus for certain values of the parameters. It is known that exactly one limit cycle is generated out of P_1 if it is a stable (unstable) fine focus and the parameters are varied in such a way that P_1 becomes an unstable (stable) focus. Then, if the parameters are varied further in such a way that P_1 becomes a node, the limit cycle must have been disappeared. In [6] Coppel proofs that a quadratic system with a cusp point has at most one simple limit cycle. This implies that the unique limit cycle generated out of P_1 remains the only one as long as it exists, and that the system has no limit cycles if P_1 is a fine focus. The only way this limit cycle may disappear is by vanishing in a separatrix cycle. Numerical calculations also indicate that the limit cycle disappears in a separatrix cycle, either formed by the separatrices of the saddle point P_2 (Case $\Delta > 0$: a,b,c) or by the separatrices of two singular points at infinity (Case $\Delta > 0$: b,c).

Now the case where system (17) has respectively one, two or three singular points at infinity, will be considered separately in a, b and c.

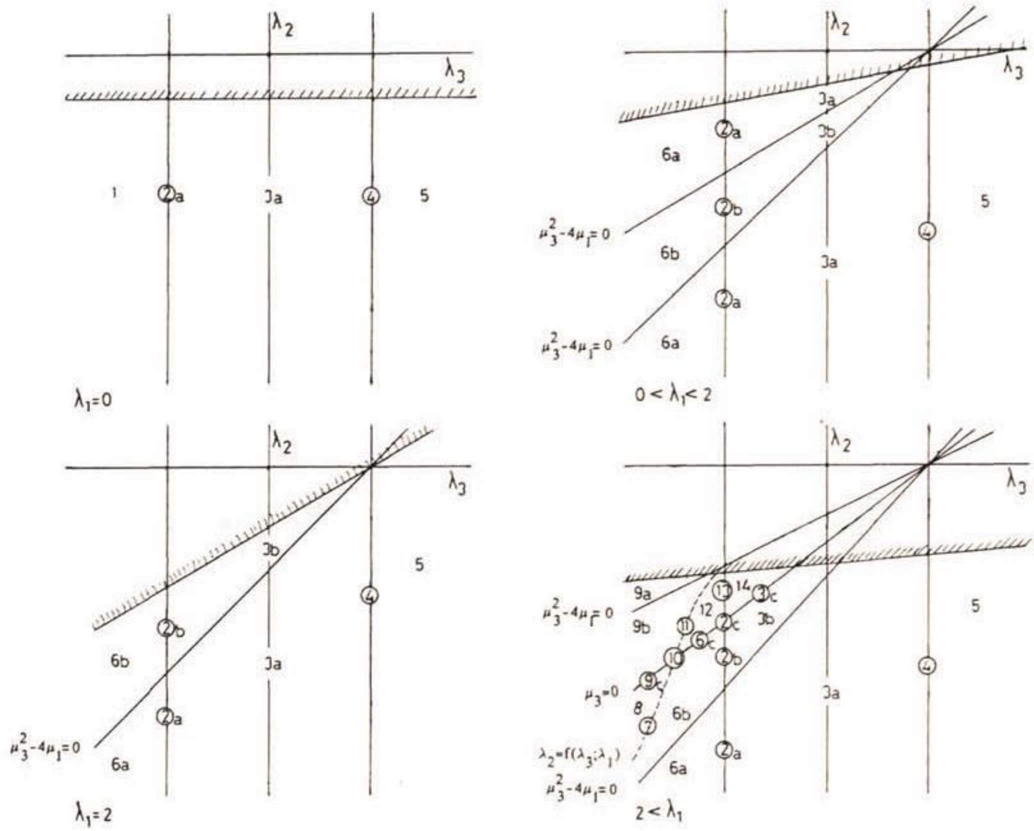
Case a: Since there is one singular point P_3 at infinity at $\theta = \frac{1}{2}\pi$ there should be satisfied either $\lambda_1^2 - 4\lambda_3\lambda_1 + 4\lambda_2 + 4 < 0$ or $2\lambda_3 - \lambda_1 = \lambda_3^2 - \lambda_2 - 1 = 0$. Although the phase portraits in the last case have one singular point at infinity, it will be treated in case b because the properties of system (17) are almost similar to the properties of system

(17) in case b. The character of singular point P_3 at infinity is listed in Table X. For four different choices of λ_1 the (λ_3,λ_2)-parameter plane is given in Figure 17. The phase portraits are indicated in Figure 17, listed in Table XI and drawn in Figure 18.

TABLE X: CASE $\Delta > 0$: a

Point P_3	$\lambda_3^2 - 1$
S	-
SN*	0
N	+

We note that in the portraits 2c, 3c, 6c and 9c, P_1 is a first order fine focus, which is stable in the portraits 2c, 3c and 6c and unstable in portrait 9c. If the parameters vary in such a way that P_1 becomes a focus and that the stability changes, exactly one limit cycle is generated out of P_1 (portraits 8, 12, 13 and 14). This limit cycle is stable in the portraits 12, 13 and 14 and is unstable in portrait 8. If the parameters are continued to vary in the same way, the limit cycle of portraits 8 and 12 seems to disappear in a separatrix cycle (portraits 7 and 11), which is indicated in Figure 17 by the dashed curve $\lambda_2 = f(\lambda_3;\lambda_1)$. If the parameters are continued to vary in the same way in portraits 13 and 14 the limit cycle will also blow up. However, before the limit cycle disappear in a separatrix cycle, one or two singular points at infinity appear. So, the portraits with these limit cycles may be found in Figures 20 and 22.



$$\mu_1 = (1 - \lambda_3)(\lambda_1(\lambda_3 - 1) - \lambda_2)$$
$$\mu_2 = (-1 - \lambda_3)(\lambda_1(\lambda_3 + 1) - \lambda_2)$$
$$\mu_3 = ((1 - \lambda_3)(2\lambda_1 - 2) + \lambda_2)\cdot\mu_1^{-1}$$

Figure 17 : CASE $\Delta > 0$: a.

TABLE XI: CASE $\Delta > 0$: a

Portrait	P_0	P_1	P_2	P_3				
1	CP	N	S	N	$\lambda_3<-1$	$\lambda_1=0,\ \lambda_3<-1$		
2a	CP	N		SN	$\lambda_3=-1$	$\mu_3^2-4\mu_1\geq 0$		
2b	CP	F		SN	$\lambda_3=-1$	$\mu_3^2-4\mu_1<0$	$\mu_3<0$	
2c	CP	FF		SN	$\lambda_3=-1$	$\mu_3^2-4\mu_1<0$	$\mu_3=0$	
3a	CP	N	N	S	$-1<\lambda_3<1$	$\mu_3^2-4\mu_1\geq 0$		
3b	CP	F	N	S	$-1<\lambda_3<1$	$\mu_3^2-4\mu_1<0$	$\mu_3<0$	
3c	CP	FF	N	S	$-1<\lambda_3<1$	$\mu_3^2-4\mu_1<0$	$\mu_3=0$	
4	CP		N	SN	$\lambda_3=1$			
5	CP	S	N	N	$\lambda_3>1$			
6a	CP	N	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1\geq 0$		
6b	CP	F	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3<0$	$\lambda_2<f(\lambda_3;\lambda_1)$
6c	CP	FF	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3=0$	$\lambda_1+\lambda_3>0$
7	CP	F	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3<0$	$\lambda_2=f(\lambda_3;\lambda_1)$
8	CP	F	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3<0$	$\lambda_2>f(\lambda_3;\lambda_1)$
9a	CP	N	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1\geq 0$		
9b	CP	F	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3>0$	$\lambda_2>f(\lambda_3;\lambda_1)$
9c	CP	FF	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3=0$	$\lambda_1+\lambda_3<0$
10	CP	C	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3=0$	$\lambda_1+\lambda_3=0$
11	CP	F	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3>0$	$\lambda_2=f(\lambda_3;\lambda_1)$
12	CP	F	S	N	$\lambda_3<-1$	$\mu_3^2-4\mu_1<0$	$\mu_3>0$	$\lambda_2<f(\lambda_3;\lambda_1)$
13	CP	F		SN	$\lambda_3=-1$	$\mu_3^2-4\mu_1<0$	$\mu_3>0$	
14	CP	F	N	S	$-1<\lambda_3<1$	$\mu_3^2-4\mu_1<0$	$\mu_3>0$	

$$\mu_1 = (1 - \lambda_3)(\lambda_1(\lambda_3 - 1) - \lambda_2)$$

$$\mu_2 = (-1 - \lambda_3)(\lambda_1(\lambda_3 + 1) - \lambda_2)$$

$$\mu_3 = ((1 - \lambda_3)(2\lambda_1 - 2) + \lambda_2)\cdot\mu_1^{-1}$$

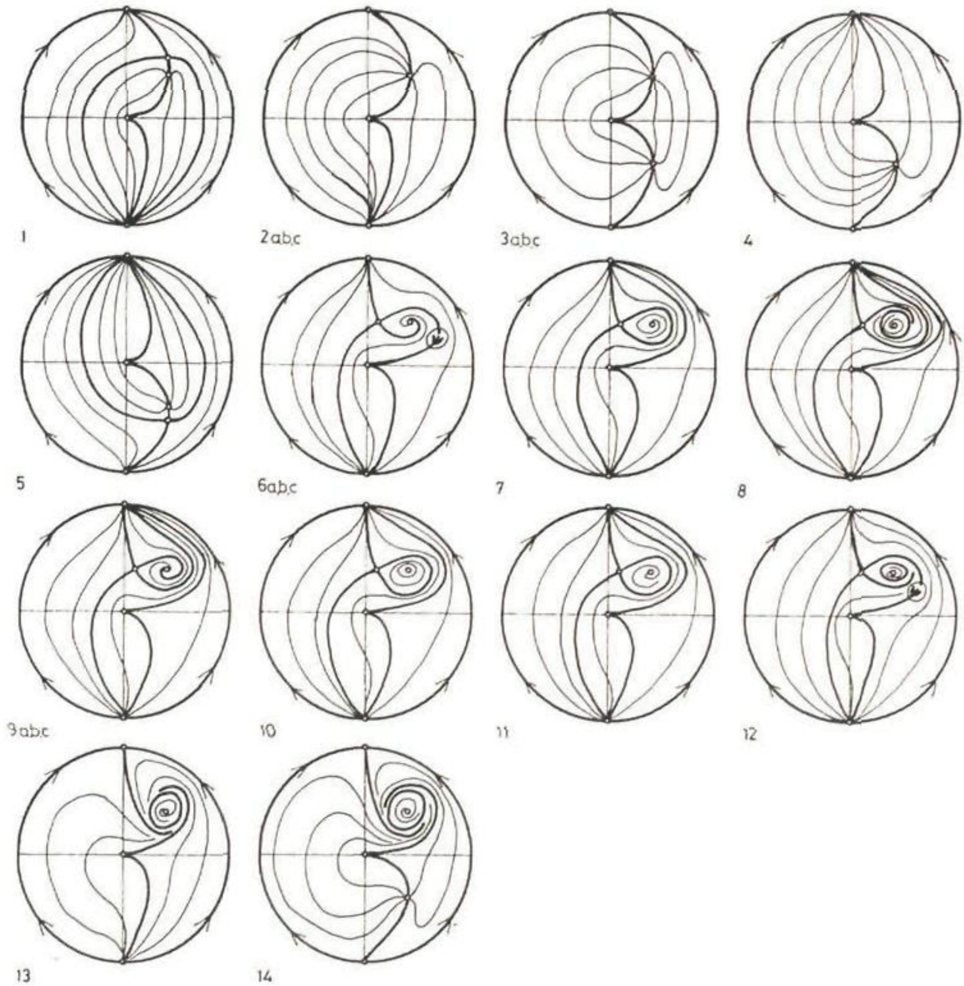


Figure 18 : CASE $\Delta > 0$: a.

According to [6] the limit cycle in the portraits 8, 12, 13 and 14 is unique and numerical calculations strongly indicate that in all other phase portraits of Figure 18 the system has no limit cycles. Comparing phase portrait 6 with phase portraits 1 and 7, we notice that all three relative positions of the α-separatrix of P_0 and the ω-separatrix of P_2, suggested in portrait 6, occur. Comparing phase portrait 12 with phase portraits 11 and 13 we also may conclude that all three relative positions of the separatrices occur.

<u>*Case b:*</u> Since the double singular point at infinity P_3 is at $\theta = \frac{1}{2}\pi$, it follows from (18) that $\lambda_2 = \lambda_3^2 - 1$. Besides the singular point P_3 there is a second singular point P_4 at $\theta = \text{arccot}\,(\lambda_1 - 2\lambda_3)$. It may be noticed that for $\lambda_1 - 2\lambda_3 = 0$ P_3 and P_4 coincide. It is the second part of case a which should be treated in section b.

The character of the singular points P_3 and P_4 is listed in Table XII and the (λ_1,λ_3)-parameter plane is given in Figure 19. The phase portraits are indicated in Figure 19 and given in Figure 20, and their portraits are listed in Table XIII.

It should be noticed that in the portraits 4a, 8a and 12a, P_1 is a stable first order fine focus. If the parameters vary in such a way that P_1 becomes an unstable focus (portraits 3, 7 and 11), exactly one stable limit cycle is generated out of P_1. Numerical analysis strongly indicate that if the parameters are continued to vary in the same way, the limit cycle disappears in a separatrix cycle (portraits 2, 6 and 10), which is indicated in Figure 19 by the dashed curve $\lambda_1 = f(\lambda_3)$.

According to [6] the limit cycle in the portraits 3, 7 and 11 is unique. Numerical calculations strongly indicate that in all other phase portraits of Figure 20 the system has no limit cycles. Comparing phase portraits 1b and 1c with phase portraits 1a and 2, we may conclude that all three relative positions of the α-separatrix of P_2 and the ω- separatrix P_3, suggested in portrait 1b,c, occur.

In the phase portraits 19, 22, 27, 28 and 29 P_3 is a higher order singular point with an elliptic sector. Applying the Poincaré transformation $x = \frac{u}{z}$, $y = \frac{1}{z}$ and the 'blow-up' transformation $u = u$, $z = wu$ or $u = u$, $z = wu^2$, it is easy to show that in all portraits, except 28, the separatrix coincide with the circle $\rho = 1$.

TABLE XII: CASE $\Delta > 0$: b

Point P_3	λ_3^2-1	$\lambda_1-2\lambda_3$	λ_3	Order
N	+	0		3
SN	+	-,+		2
SN	0	0		4
S	0	+	+	3
Pwes	0	-	+	3
Pwes	0		-	3
S	-	0		3
SN	-	-,+		2

Point P_4	$(\lambda_1-\lambda_3-1)(\lambda_1-\lambda_3+1)$
N	+
SN*	0
S	-

* higher order point

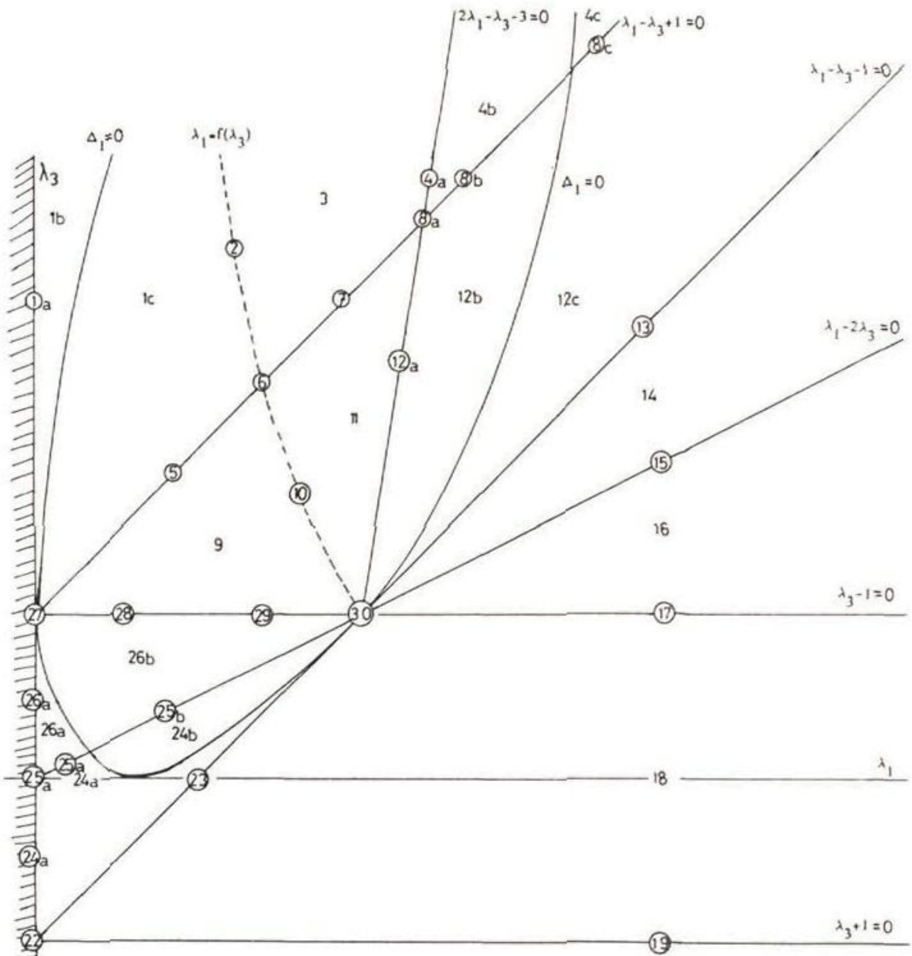


Figure 19 : CASE $\Delta > 0$: b.

TABLE XIII: CASE $\Delta > 0$: b

Por-trait	P_0	P_1	P_2	P_3	P_4	λ_3-1	λ_3+1	Δ_1	$2\lambda_1-\lambda_3-3$	$\lambda_1-2\lambda_3$	$\lambda_1-\lambda_3+1$	$\lambda_1-\lambda_3-1$	λ_1	$\lambda_1-f(\lambda_3)$
1a	CP													
1b	CP	N	S	SN	N	+		+	-				0	
1c	CP	N	S	SN	N	+		+0	-		-		+	-
2	CP	F	S	SN	N	+		-	-		-		+	-
3	CP	F	S	SN	N	+		-	-		-			0
4a	CP	F	S	SN	N	+		-	-		-			+
4b	CP	FF	S	SN	N	+		-	0		-			
4c	CP	F	S	SN	N	+		-	+		-			
5	CP	N	S	SN	N	+		+0	+		-			
6	CP	F		SN	SN	+		-	-		0			-
7	CP	F		SN	SN	+		-	-		0			0
8a	CP	F		SN	SN	+		-	-		0			+
8b	CP	FF		SN	SN	+		-	0		0			
8c	CP	F		SN	SN	+		-	+		0			
9	CP	N		SN	SN	+		+0	+		0			
10	CP	F	N	SN	S	+		-	-		+			-
11	CP	F	N	SN	S	+		-	-		+			0
12a	CP	F	N	SN	S	+		-	-		+			+
12b	CP	FF	N	SN	S	+		-	0		+			
12c	CP	F	N	SN	S	+		-	+		+			
13	CP	N	N	SN	S	+		+0	+		+	-		
14	CP		N	SN	SN	+						0		
15	CP	S	N	SN	N	+				-		+		
16	CP	S	N	N		+				0				
17	CP	S	N	SN	N	+				+				
18	CP		N	S	N	0						+		
19	CP	S	N	SN	N	-	+					+		
20	CP	S		Pwes	N		0					+		
21	CP	S	N	SN	N		-					+	+	
22	CP	S	N	SN	N		-					+	0	
23	CP			Pwes	SN		0					0	0	
24a	CP		N	SN	SN	-	+					0	+	
24b	CP	N	N	SN	S			+0		+		-		
25a	CP	F	N	SN	S			-		+		-		
25b	CP	N	N	S		-		+0		0				
26a	CP	F	N	S		-		-		0				
26b	CP	N	N	SN	S	-		+0		-				
27	CP	F	N	SN	S	-		-		-				
28	CP			Pwes	SN	0							0	
29	CP		N	Pwes	S	0		-					≤1	
30	CP		N	Pwes	S	0		-					>1	
			N	SN		0		0					+	

$\Delta_1 = (2\lambda_1 - \lambda_3 - 1)^2 - 4\lambda_3$

1a

1b,c

2

3

4a,b,c

5

6

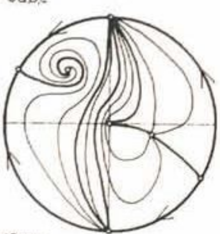
7

8a,b,c

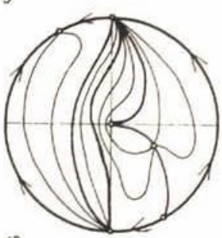
9

10

11



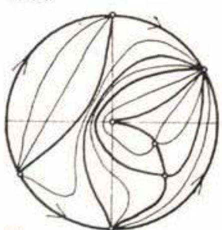
12a,b,c



13

14

15



16

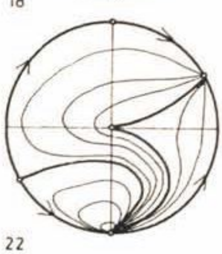
17

18

19

20

21



22

23

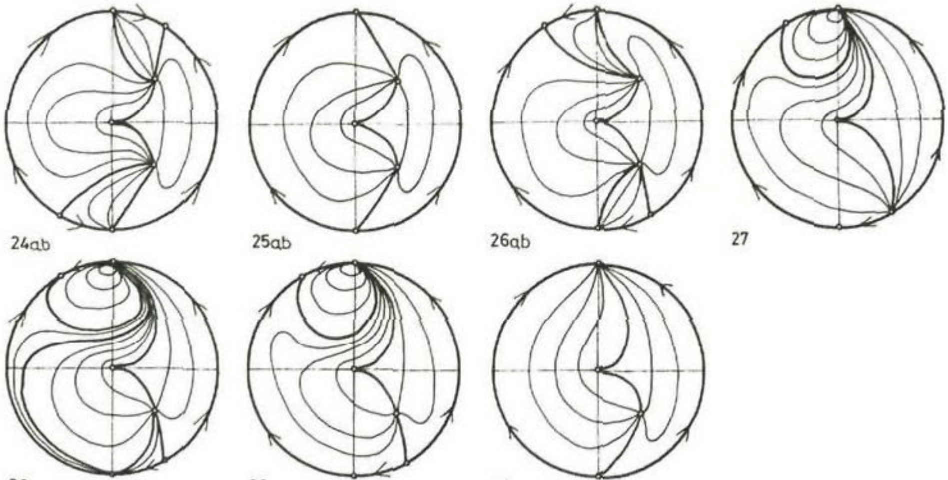


Figure 20 : CASE Δ > 0 : b.

Case c: There are three different singular points at infinity if $\lambda_1^2 - 4\lambda_1\lambda_3 + 4\lambda_2 + 4 > 0$. Apart from P_3 at $\theta = \frac{1}{2}\pi$, (18) has a positive and a negative root if $\lambda_2 - \lambda_3^2 + 1 > 0$. Besides the point P_3 there are two singular points P_4 and P_5 at $\theta = \text{arccot}\,(\frac{1}{2}\lambda_1 - \lambda_3 + \frac{1}{2}\sqrt{\lambda})$ and at $\theta = \text{arccot}\,(\frac{1}{2}\lambda_1 - \lambda_3 - \frac{1}{2}\sqrt{\lambda})$ where $\lambda = \lambda_1^2 - 4\lambda_1\lambda_3 + 4\lambda_2 + 4$. The character of these points is listed in Table XIV.

TABLE XIV: CASE Δ > 0: c

Point P_3	$\lambda_3^2 - 1$
S	+
SN*	0
N	-

Point P_4	
N	$\lambda_2 > 0 \vee \lambda_2 > \lambda_1(\lambda_3 - 1)$
SN*	$\lambda_3 < 1 \wedge \lambda_2 = \lambda_1(\lambda_3 - 1)$
S	$\lambda_3 < 1 \wedge \lambda_2 < \lambda_1(\lambda_3 - 1)$

Point P_5	$\lambda_3 - 1$	$\lambda_2 - \lambda_1(\lambda_3 + 1)$	$\lambda_2 - \lambda_1(\lambda_3 - 1)$
N		+	+
SN*		0	+
S		-	+
SN*	+	-	0
S	-	-	0
N	+	-	-
S	-	-	-

For four different values of λ_1 the (λ_3,λ_2)-parameter plane is given in Figure 21. The phase portraits are listed in Table XV and in Figure 22.

It should be noticed that in the portraits 15a, 21a and 27a, P_1 is a stable first order fine focus. If the parameters vary in such a way that P_1 becomes an unstable focus (portraits 14, 20 and 26), exactly one stable limit cycle is generated out of P_1. Numerical analysis strongly indicate that if the parameters are continued to vary in the same way, the limit cycle disappears in a separatrix cycle (portraits 13, 19 and 25), which is indicated in Figure 21 by the dashed curve $\lambda_3 = f(\lambda_2;\lambda_1)$.

According to [6] the limit cycle in the portraits 14, 20 and 26 is unique. Numerical calculations strongly indicate that in all other phase portraits of Figure 22 the system has no limit cycles.

Comparing phase portrait 12 with phase portraits 11 and 13, we notice that all three relative positions of the separatrices, suggested in phase portrait 12, occur.

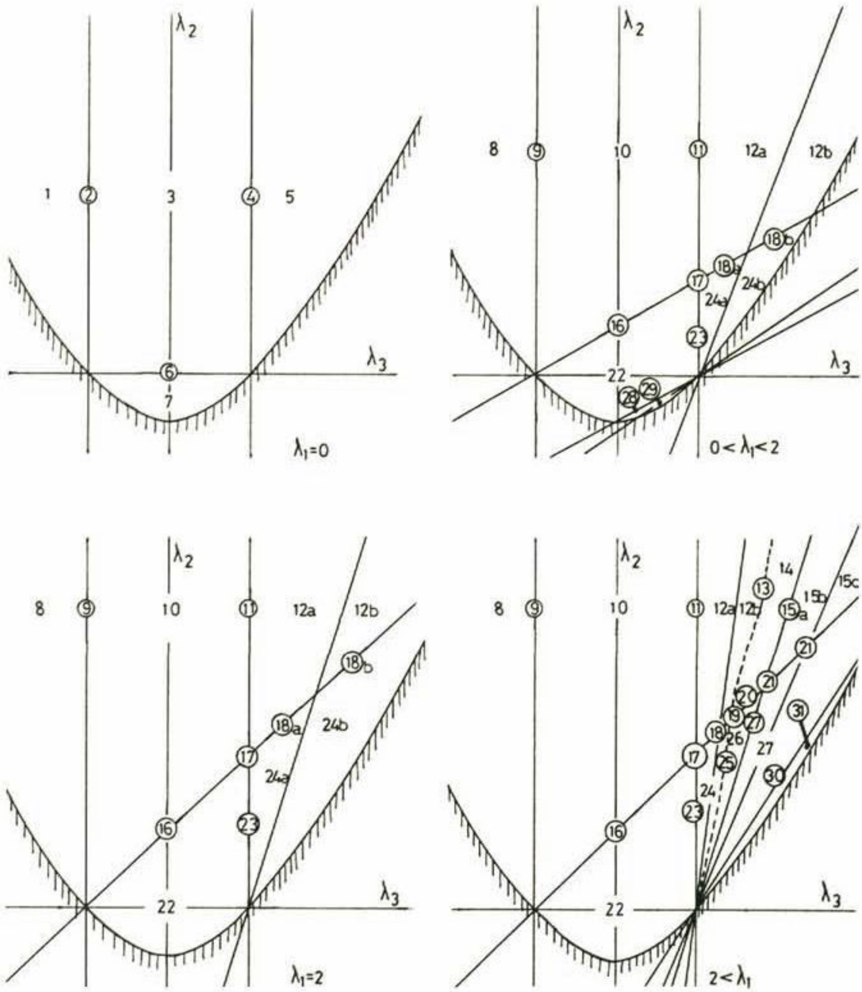
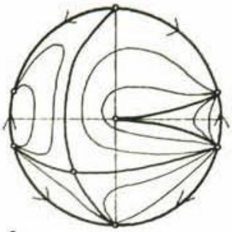


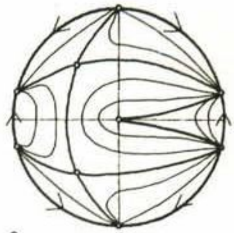
Figure 21 : CASE Δ > 0 : c.

TABLE XV: CASE $\Delta > 0$: c

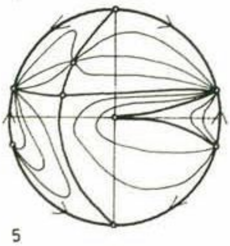
Por-trait	P_0	P_1	P_2	P_3	P_4	P_5	λ_1	λ_3-1	λ_3+1	λ_2	$\mu_3^2-4\mu_1$	μ_3	$\lambda_2-\lambda_1(\lambda_3+1)$	$\lambda_2-\lambda_1(\lambda_3-1)$	$\lambda_3-f(\lambda_2,\lambda_1)$
1	CP	S	N	S	N	N	0		-	+					
2	CP	S		SN	N	N	0		0	+					
3	CP	S	S	N	N	N	0	-	+	+					
4	CP		S	SN	N	N	0	0		+					
5	CP	N	S	S	N	N	0	+		+					
6	CP			N	SN	SN	0			0					
7	CP	N	N	N	S	S	0			-					
8	CP	S	N	S	N	N	+		-						
9	CP	S		SN	N	N	+		0						
10	CP	S	S	N	N	N	+	-	+				+		
11	CP		S	SN	N	N	+	0					+		
12a	CP	N	S	S	N	N	+	+			+0	+	+		
12b	CP	F	S	S	N	N	+	+			-	+	+		
13	CP	F	S	S	N	N	+	+			-	+	+		-
14	CP	F	S	S	N	N	+	+			-	+	+		0
15a	CP	FF	S	S	N	N	+	+			-	0	+		+
15b	CP	F	S	S	N	N	+	+			-	-	+		
15c	CP	N	S	S	N	N	+	+			+0	-	+		
16	CP	S		N	N	SN	+	-	+				0		
17	CP			SN	N	SN	+	0					0		
18a	CP	N		S	N	SN	+	+			+0	+	0		
18b	CP	F		S	N	SN	+	+			-	+	0		
19	CP	F		S	N	SN	+	+			-	+	0		-
20	CP	F		S	N	SN	+	+			-	+	0		0
21a	CP	FF		S	N	SN	+	+			-	0	0		+
21b	CP	F		S	N	SN	+	+			-	-	0		
21c	CP	N		S	N	SN	+	+			+0	-	0		
22	CP	S	N	N	N	S	+	-	+				-	+	
23	CP		N	SN	N	S	+	0					-		
24a	CP	N	N	S	N	S	+	+			+0	+	-		
24b	CP	F	N	S	N	S	+	+			-	+	-		
25	CP	F	N	S	N	S	+	+			-	+	-		-
26	CP	F	N	S	N	S	+	+			-	+	-		0
27a	CP	FF	N	S	N	S	+	+			-	0	-		+
27b	CP	F	N	S	N	S	+	+			-	-	-		
27c	CP	N	N	S	N	S	+	+			+0	-	-		
28	CP		N	N	SN	S	+	-	+					0	
29a	CP	N	N	N	S	S	+	-	+		+0			-	
29b	CP	F	N	N	S	S	+	-	+		-			-	
30	CP		N	S	N	SN	+	+						0	
31	CP	S	N	S	N	N	+	+						-	

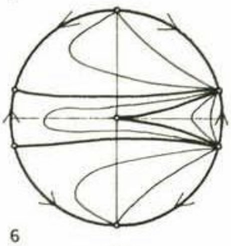

1

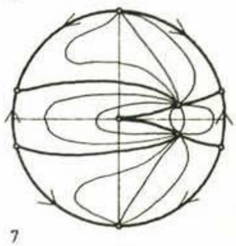

2


3


4


5

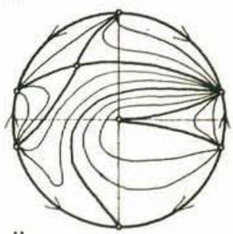

6


7

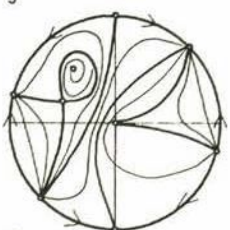

8


9


10


11

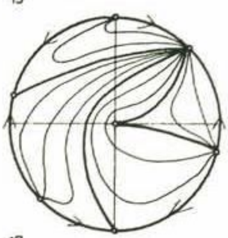

12a,b

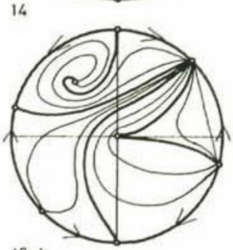

13


14


15a,b,c


16


17

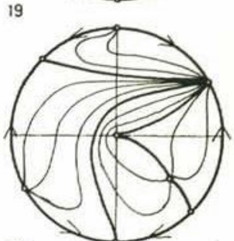

18a,b

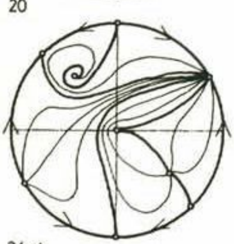

19


20


21a,b,c


22


23


24a,b

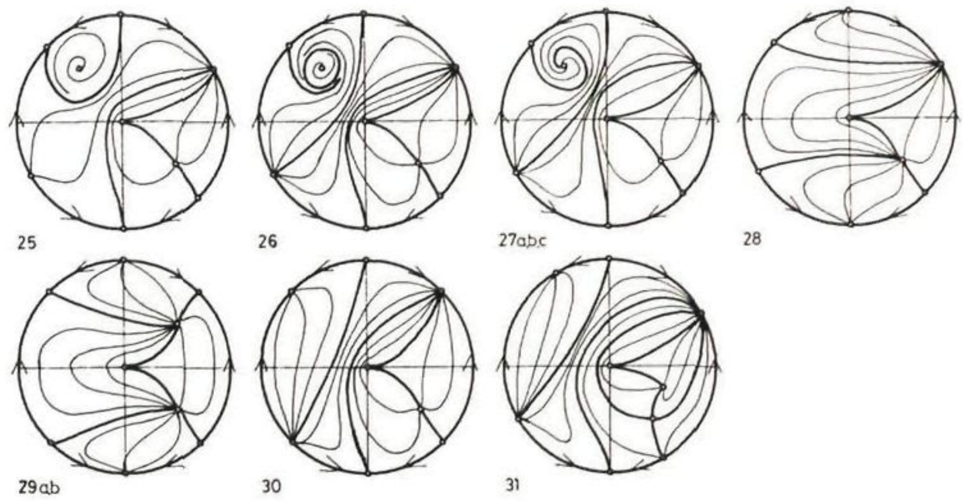


Figure 22: CASE $\Delta > 0$: c

REFERENCES

1. A.A. Andronov, E.A. Leontovich, J.J. Gordon and A.G. Maier, "Qualitative theory of second-order dynamic systems", Israel Program for Scientific Translation, Jeruzalem, Wiley, New York, 1973.

2. A.N. Berlinskii, On the behaviour of integral curves of a differential equation, *Izv. Vyss. Ucebn. Zaved. Matematika*, No. 2, **15** (1960) 3-18, (Russian), translated by the National Lending Library for Science and Technology, Boston Spa, Yorkshire, England, Russian Translating Programme RIS 5158, June 1969.

3. T.R. Blows and N.G. Lloyd, "The number of limit cycles of certain polynomial differential equations", Proceedings of the Royal Society of Edinburgh 98A, pp. 215-239, 1984.

4. C. Chicone and D.S. Shafer, Separatrix and limit cycles of quadratic systems and Dulac's theorem, *Trans. of the American Mathematical Society*, No. 2, **278** (1983).

5. W.A. Coppel, A survey of quadratic systems, *Journ. of Diff. Eq.* **2** (1966), 293-304.

6. W.A. Coppel, Quadratic systems with a degenerate critical point, *Bull. Austral. Math. Soc.* **38** (1988), 1-10.

7. Date, Tsumotu, Classification and analysis of two-dimensional real homogeneous quadratic differential equation systems, *Journ. of Diff. Eq.* **32** (1979), 311-334.

8. Gasull, Armengol, Sheng Li-Ren, and Jaume Llibre, "Chordal quadratic systems", Universitat Autonoma de Barcelona, Bellaterra, Barcelona.

9. T.A. Newton, Two dimensional homogeneous quadratic differential systems, *SIAM Review* **20** (1978), 120-138.

10. Private communication, W.T. van Horssen.

11. J.W. Reyn, Phase portraits of a quadratic system of differential equations occurring freqently in applications, *Nieuw Archief voor Wiskunde*, Series 4, Deel 5, No. 1, March 1987.

12. N.I. Vulpe, Affine-invariant conditions for the topological discrimination of quadratic systems with a center, *Diff. Uravn.*, No. 3, **19** (1983), 371-379.

CHAPTER 2

PHASE PORTRAITS FOR QUADRATIC SYSTEMS WITH A HIGHER ORDER SINGULARITY

THIRD AND FOURTH ORDER POINTS WITH ONE ZERO EIGENVALUE

1. INTRODUCTION

In a survey paper [3] on general properties of quadratic systems of differential equations in the plane Coppel states that what remains to be done is to determine all possible phase portraits of such systems, this being of great practical value. The present paper aims to give a contribution in this direction. By a quadratic system is meant the system

$$\begin{aligned} \overset{\circ}{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \overset{\circ}{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned} \tag{1}$$

for the functions $x = x(t)$, $y = y(t)$, where $\circ = \frac{d}{dt}$ and $a_{ij}, b_{ij} \in R$.

Since quadratic systems without singular points in the finite part of the plane have been classified by Gasull, Sheng Li-Ren and Llibre [5], we may assume that system (1) has at least one singular point in the finite part of the plane. Without loss of generality we may assume that one of the finite singular points is situated in the origin: $a_{00} = b_{00} = 0$. Two limiting cases are classified: the linear case ($a_{20} = a_{11} = a_{02} = b_{20} = b_{11} = b_{02} = 0$) and the homogeneous case ($a_{10} = a_{01} = b_{10} = b_{01} = 0$) [4], [9]. If both linear and quadratic terms are present a lot of work remains to be done. If one or both eigenvalues of the linear part of system (1) are zero the singular point in the origin is a higher order singular point. All phase portraits for quadratic systems with a singular point, having two zero eigenvalues, have been classified in [6], [7] and [8]. In the present report the possible phase portraits are given for quadratic systems with a third order or fourth order singular point, having one zero eigenvalue. They include the fourth order saddle node, the third order saddle point and the third order node (see [1], page 337 and [8]).

If a singular point with one zero eigenvalue is situated in the origin, the quadratic system may be brought into a normal form by an affine transformation. According to [8] this normal form reads:

$$\begin{aligned} \overset{\circ}{x} &= \phantom{y + {}} a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \overset{\circ}{y} &= y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned} \tag{2}$$

where the singular point in the origin is a

(i) fourth order saddle node, if $a_{20} = a_{11} = 0$ and $a_{02} \cdot b_{20} \neq 0$; index 0,

(ii) third order saddle point, if $a_{20} = 0$ and $a_{11} \cdot b_{20} > 0$; index -1,

(iii) third order node, if $a_{20} = 0$ and $a_{11} \cdot b_{20} < 0$; index 1.

It should be noted that the classification in this paper is not complete in the sense that some global problems, such as the number of limit cycles and the relative position of separatrices, are not completely solved in all cases. The phase portraits are characterized in the usual way: by the number, position and character of the singular points; by the number and position of the periodic solutions; by the position of the separatrices and by the behaviour at infinity. Standard arguments will be used, such as local linearisation in singular points, if possible Dulac functions, integrating factors, continuity and index arguments. For the investigation of singular points at infinity a slightly different transformation then given by Poincaré will be used, by putting $x = \rho\,(1 - \rho)^{-1} \cos\theta$, $y = \rho\,(1 - \rho)^{-1} \sin\theta$ where $0 \leq \rho < 1$ and $0 \leq \theta < 2\pi$. System (2) then becomes, with $\rho\,(1 - \rho)\dfrac{d}{dt} = \dfrac{d}{d\tau} = \circ$

$$\begin{aligned} \overset{\circ}{\rho} &= \rho^2\,(1 - \rho)^2\,B_1(\theta) + \rho^3\,(1 - \rho)\,C_1(\theta), \\ \overset{\circ}{\theta} &= \rho\,(1 - \rho)\,B_2(\theta) + \rho^2\,C_2(\theta) \qquad , \end{aligned} \tag{3}$$

where

$$B_1(\theta) = \sin\theta\cos\theta, \quad B_2(\theta) = -\sin^2\theta,$$

$$C_1(\theta) = a_{20}\cos^3\theta + (a_{11} + b_{20})\sin\theta\cos^2\theta + (a_{02} + b_{11})\sin^2\theta\cos\theta + b_{02}\sin^3\theta,$$

$$C_2(\theta) = b_{20}\cos^3\theta + (b_{11} - a_{20})\sin\theta\cos^2\theta + (b_{02} - a_{11})\sin^2\theta\cos\theta - a_{02}\sin^3\theta.$$

Singular points at infinity are then represented on $\rho \equiv 1$ and appear in diametrically opposed pairs and are indicated in the present report by the value of θ in the interval $[0,\pi>$. The singular point in the interval $[0,\pi>$ is indicated by P_i and the diametrically opposed point by P_i'.

If $C_2(\theta) \not\equiv 0$, $\rho \equiv 1$ consists of integral curves and possibly of singular points. In order to include such a curve into considerations using index theory, an extension of the usual Poincaré index of a planar vector field will be adopted by regarding $\rho \equiv 1$ as the limiting positions of a closed curve near it [10].

If $C_2(\theta) \not\equiv 0$, it can then be deduced that the sum of the indices of the singular points on $\rho \leq 1$ is equal to 1, where for the index of a singular point on $\rho = 1$ only the vector field for $\rho \leq 1$ is considered.

2. QUADRATIC SYSTEMS WITH A FOURTH ORDER SADDLE NODE

If (0,0) is a fourth order saddle node system (2), may be written, if necessary by applying on affine transformation and/or scaling of t, in the form

$$\begin{aligned} \overset{\circ}{\bar{x}} &= \bar{y}^2, \\ \overset{\circ}{\bar{y}} &= \bar{y} + \bar{x}^2 + b_{11}\bar{x}\bar{y} + b_{02}\bar{y}^2, \end{aligned} \tag{4}$$

with $b_{11}, b_{02} \in R$.

From (3) follows that the singular points at infinity for system (4) are given by

$$C_2(\theta) = \cos^3\theta + b_{11}\cos^2\theta\sin\theta + b_{02}\cos\theta\sin^2\theta - \sin^3\theta = 0, \tag{5}$$

or equivalenty by $\cot^3\theta + b_{11}\cot^2\theta + b_{02}\cot\theta - 1 = 0$.

Now three different cases may be distinguished. Case 2a: there is just one real solution $\cot\theta$ of equation (5). Case 2b: there are two different real solutions of (5); one of them is simple and the other one is double. Case 2c: equation (5) has three different real solutions. If the affine transformation $\bar{x} = x + \alpha y$, $\bar{y} = y$, where α is a properly chosen constant, is applied to (4), there may be obtained,

$$\begin{aligned} \overset{\circ}{x} &= \lambda_2(y + x^2 + \lambda_1 xy) \quad, \\ \overset{\circ}{y} &= y + x^2 + \lambda_1 xy - \lambda_2^{-1}y^2, \end{aligned} \tag{6}$$

where $\lambda_1 \in R$, $\lambda_2 \in R\backslash\{0\}$ and take values in the λ_1,λ_2-plane (Figure 2) depending on the choice of α.

In fact we take the constant $\alpha = \cot\theta$, where α is the unique solution of (5) in case 2a, α is the double solution in case 2b and the middle one of the three solutions in case 2c. From (3) then follows that the singular points at infinity of (6) are given by

$$C_2(\theta) = \cot^3\theta + (\lambda_1 - \lambda_2) \cot^2\theta + (-\lambda_2^{-1} - \lambda_1\lambda_2) \cot\theta = 0. \tag{7}$$

As illustrated in Figure 1, the fourth order saddle node P_0 consists of two hyperbolic sectors and a parabolic sector.

The two separatrices which separate the parabolic sector from the hyperbolic sectors are tangent to the line $x - \lambda_2 y = 0$ in P_0. The third separatrix and the integral curves in the parabolic sector are tangent to the x-axis in P_0.

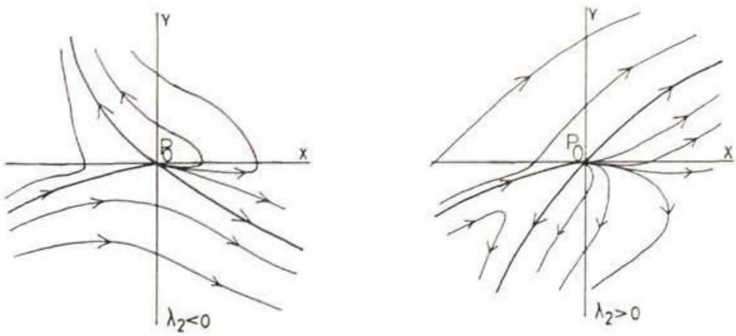


Figure 1: Fourth order saddle node.

System (6) has just one singular point in the finite part of the plane: the fourth order saddle node P_0 in (0,0). Since there are no other singular points in the finite part of the plane, P_0 is the only possible candidate to be situated inside a limit cycle. However, in a quadratic system a limit cycle has to surround a focus [3]. Thus there is no limit cycle. In order to investigate the singular points at infinity the cases 2a, 2b and 2c will be considered separately.

Case 2a: Since there is one singular point at infinity ($\theta = \frac{1}{2}\pi$) there should be satisfied either $\lambda_1 - \lambda_2 = \lambda_2^{-1} + \lambda_1\lambda_2 = 0$ (or equivalently $\lambda_1 = \lambda_2 = -1$) or $(\lambda_1 + \lambda_2)^2 + 4\lambda_2^{-1} < 0$ as follows from (7).

<u>*Case 2b:*</u> Since the double singular point at infinity is at $\theta = \frac{1}{2}\pi$, it follows from (7) that $\lambda_2^{-1} + \lambda_1\lambda_2 = 0$. Since there is a second singular point at infinity at $\theta = \operatorname{arccot}(\lambda_2 - \lambda_1)$ we know that $\lambda_2 - \lambda_1 \neq 0$.

<u>*Case 2c:*</u> Since the middle one of the three singular points at infinity is at $\theta = \frac{1}{2}\pi$ there follows from (7) that $\lambda_2^{-1} + \lambda_1\lambda_2 > 0$.

Besides the singular point at $\theta = \frac{1}{2}\pi$ the system has two other singular points at infinity at $\theta = \operatorname{arccot}(\frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_1 - \frac{1}{2}\sqrt{\lambda})$ and $\theta = \operatorname{arccot}(\frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_1 + \frac{1}{2}\sqrt{\lambda})$ where $\lambda = (\lambda_1 + \lambda_2)^2 + 4\lambda_2^{-1}$.

The character of the singular points at infinity may be found by local linearisation and using Theorems 65, 66 and 67 of [1]. The results of this analysis are listed in Table 1 and the phase portraits are indicated in the λ_1,λ_2-parameter plane in Figure 2 and given in Figure 3.

TABLE 1: Case 2

Case	Portrait	λ	$\lambda_2^{-1}+\lambda_1\lambda_2$	$\lambda_1-\lambda_2$	λ_2	P_0	P_1	P_2	P_3
2a	1a		0	0	-1	SN*	N*		
2a	1b	-	-			SN*	N		
2b	2		0	+	-	SN*	SN*	N	
2b	3		0	-	-	SN*	SN*		N
2b	4		0	-	+	SN*	SN*		N
2c	5		+		-	SN*	S	N	N
2c	6		+		+	SN*	N	S	N

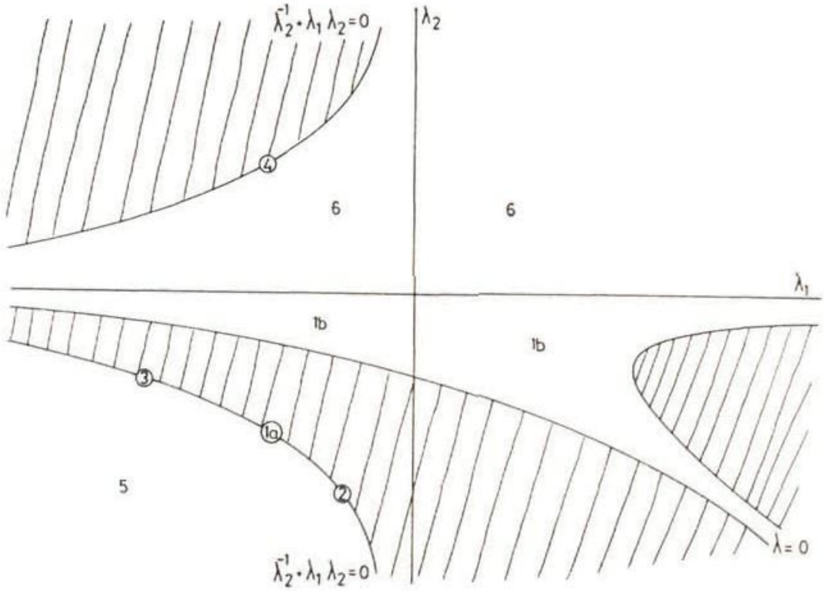
* higher order point $\qquad \lambda = (\lambda_1 + \lambda_2)^2 + 4\lambda_2^{-1}$

S = saddle point
N = node
SN = saddle node

P_1: $\theta = \frac{1}{2}\pi$
P_2: $\frac{1}{2}\pi < \theta < \pi$
P_3: $0 < \theta < \frac{1}{2}\pi$

P_1, P_2 and P_3 are the singular points at infinity.

For portrait 2 in Figure 3 use should be made of the argument that in the x,y-plane a straight line in the direction of P_2 may be drawn such that all integral curves in the parabolic sector of P_0, remain under this straight line.



$\lambda = (\lambda_1 + \lambda_2)^2 + 4\,\lambda_2^{-1}$

Figure 2: Case 2.

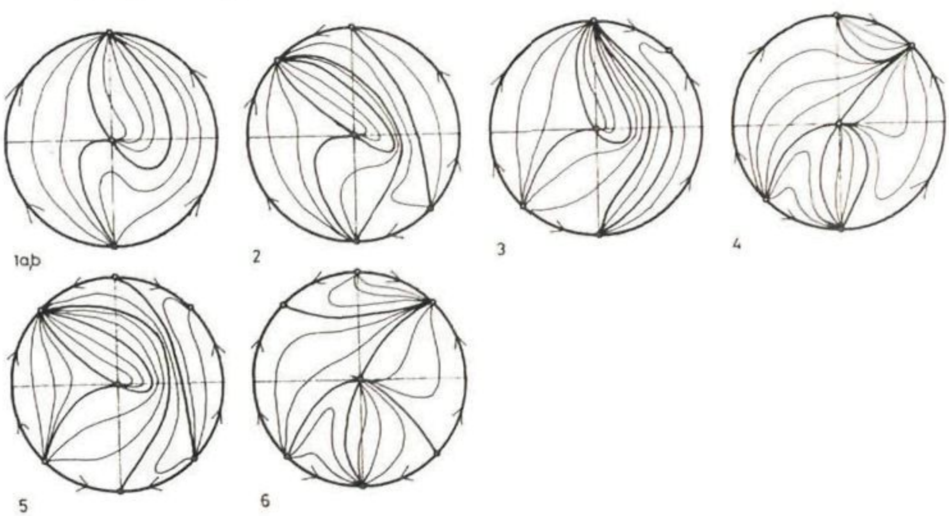


Figure 3: Case 2.

3. QUADRATIC SYSTEMS WITH A THIRD ORDER SADDLE POINT

If (0,0) is a third order saddle point system (2), may be written, if necessary by applying an affine transformation and/or a scaling of t, in the form

$$\begin{aligned} \mathring{\bar{x}} &= \bar{x}\bar{y} + a_{02}\bar{y}^2, \\ \mathring{\bar{y}} &= \bar{y} + \bar{x}^2 + b_{11}\bar{x}\bar{y} + b_{02}\bar{y}^2, \end{aligned} \tag{8}$$

where $a_{02}, b_{02} \in R$ and $b_{11} \in R_0^+$.

The cases $a_{02} = 0$ and $a_{02} \neq 0$ will be considered separately.

<u>Case 3I</u>: $a_{02} = 0$.

In this case system (8) may be rewritten in the form

$$\begin{aligned} \mathring{x} &= xy, \\ \mathring{y} &= y + x^2 + \lambda_1 xy + \lambda_2 y^2, \end{aligned} \tag{9}$$

where $\lambda_1 \in R_0^+$ and $\lambda_2 \in R$.

As illustrated in Figure 4, the third order saddle point $P_0(0,0)$ consists of four hyperbolic sectors, which are separated by four separatrices. Two of the separatrices are tangent to the x-axis in P_0 and the two others coincide with the y-axis.

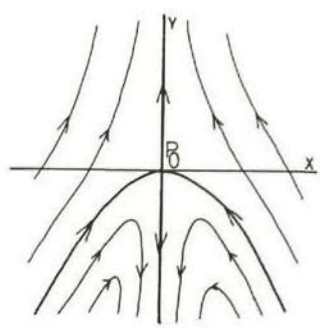


Figure 4 : Third order saddle point.

Apart from P_0 there is a second finite singular point P_1 in $(0,-\lambda_2^{-1})$ if $\lambda_2 \neq 0$. By local linearisation one finds that P_1 is a stable node for $\lambda_2 > 0$ and a saddle point for $\lambda_2 < 0$. It is clear that system (9) has no limit cycles since a limit cycle has to surround a (fine) focus [3].

From (3) follows that the singular points at infinity for system (9) are given by

$$C_2(\theta) = \cos^3\theta + \lambda_1 \cos^2\theta \sin\theta + (\lambda_2 - 1)\cos\theta \sin^2\theta = 0, \tag{10}$$

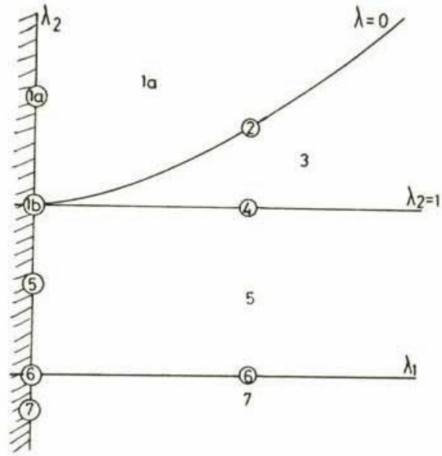
so that there is exactly one singular point P_2 at $\theta = \frac{1}{2}\pi$ for $\lambda_1 = \lambda_2 - 1 = 0$ or $\lambda < 0$ where $\lambda \equiv \lambda_1^2 - 4(\lambda_2 - 1)$. System (9) has two singular points if either $\lambda = 0$ and $\lambda_1 > 0$ or $\lambda_2 = 1$ and $\lambda_1 > 0$: P_2 at $\theta = \frac{1}{2}\pi$ and P_3 at $\theta = \operatorname{arccot}(-\frac{1}{2}\lambda_1)$. System (9) has three singular points if $\lambda > 0$ and $\lambda_2 \neq 1$: P_2 at $\theta = \frac{1}{2}\pi$, P_3 at $\theta = \operatorname{arccot}(-\frac{1}{2}\lambda_1 - \frac{1}{2}\sqrt{\lambda})$ and P_4 at $\theta = \operatorname{arccot}(-\frac{1}{2}\lambda_1 + \frac{1}{2}\sqrt{\lambda})$.

The character of these singular points is listed in Table 2 and the phase portraits are indicated in the λ_1,λ_2-parameter plane in Figure 5 and drawn in Figure 6.

TABLE 2: Case 3I: $a_{02} = 0$.

Portrait	λ	λ_1	λ_2	λ_2-1	P_0	P_1	P_2	P_3	P_4
1a	-	0,+			S*	N	N		
1b	0	0			S*	N	N*		
2	0	+			S*	N	N	SN*	
3	+			+	S*	N	N	N	S
4	+			0	S*	N	SN*	N	
5	+	0,+	+	-	S*	N	S	N	N
6	+	0,+	0		S*		SN*	N	N
7	+	0,+	-		S*	S	N	N	N

* higher order point $\qquad \lambda \equiv \lambda_1^2 - 4(\lambda_2 - 1)$



$$\lambda = \lambda_1^2 - 4(\lambda_2 - 1)$$

Figure 5 : Case 3I : $a_{02} = 0$.

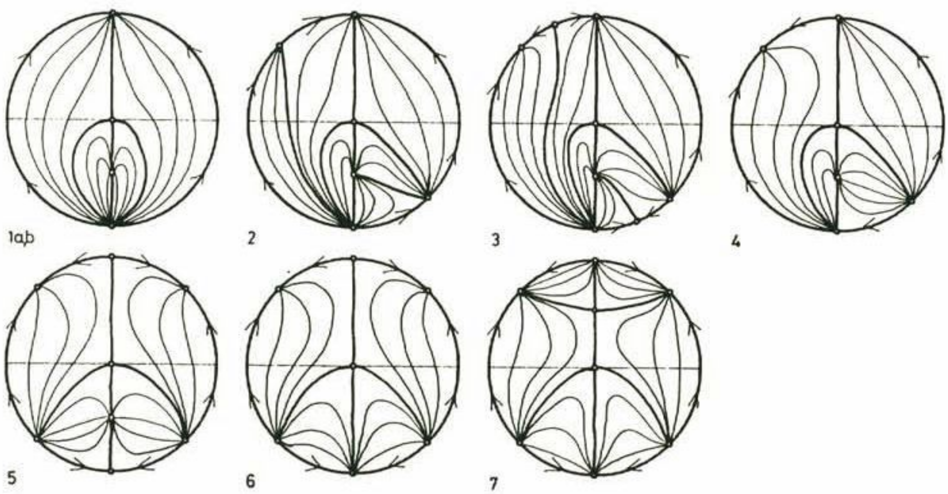


Figure 6 : Case 3I : $a_{02} = 0$.

Case 3II: $a_{02} \neq 0$.

From (3) follows that the singular points at infinity for system (8) are given by

$$C_2(\theta) = \cos^2\theta + b_{11}\cos^2\theta\sin\theta + (b_{02} - 1)\cos\theta\sin^2\theta - a_{02}\sin^3\theta = 0.$$

or equivalently $$\cot^3\theta + b_{11}\cot^2\theta + (b_{02} - 1)\cot\theta - a_{02} = 0. \qquad (11)$$

Now three different cases may be distinguished. Case 3IIa: there is just one real solution $\cot\theta$ of equation (11). Case 3IIb: there are two different real solutions of (11); one of them is simple and the other one is double. Case 3IIc: equation (11) has three different real solutions.

If the affine transformation $\bar{x} = \alpha x + \alpha y$, $\bar{y} = y$, where α is a properly chosen constant, is applied to (8), there may be obtained, if necessary by applying a scaling of x and y

$$\begin{aligned} \overset{\circ}{x} &= x + \lambda_1 x^2 + \lambda_2 xy + y^2, \\ \overset{\circ}{y} &= x \qquad\quad + \lambda_3 xy + y^2, \end{aligned} \qquad (12)$$

where λ_1, λ_2 and $\lambda_3 \in R$ and $\lambda_2 < \lambda_3$.

Restrictions of the values of λ_1, λ_2 and λ_3 will follow depending on the choice of α.

In fact we take the constant $\alpha = \cot\theta$, where α is the unique solution of (11) in case 3IIa, α is the double solution in case 3IIb and the middle one of the three solutions in case 3IIc.

From (3) follows that the singular points at infinity of (12) are given by

$$C_2(\theta) = [\sin^2\theta + (\lambda_2 - 1)\sin\theta\cos\theta + (\lambda_1 - \lambda_3)\cos^2\theta]\sin\theta = 0. \qquad (13)$$

As illustrated in Figure 7, the third order saddle point P_0 of system (12) consists of four hyperbolic sectors, which are separated by four separatrices. Two of the separatrices are tangent to the y-axis in P_0 and the other two are tangent to the line $y = x$ in P_0.

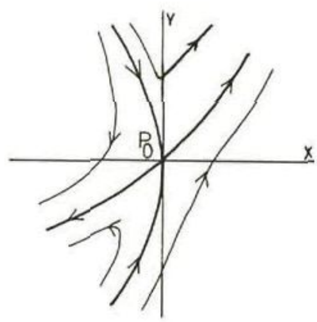


Figure 7 : Third order saddle point.

Apart from the third order saddle point P_0 system (12) has one other finite singular point P_1: $(-(\lambda_2 - \lambda_3)^2 \lambda_1^{-1}\lambda_4^{-1}, (\lambda_2 - \lambda_3)\lambda_1^{-1}\lambda_4^{-1})$ if $\lambda_1 \cdot \lambda_4 \neq 0$, where $\lambda_4 = \lambda_1 + \lambda_3(\lambda_3 - \lambda_2)$. The character of this point may be found by local linearisation as long as it is not a center in linear approximation. However, P_1 is a center in linear approximation for $\lambda_6^2 - 4\lambda_5 < 0$ and $\lambda_6 = 0$ where $\lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1}\lambda_4^{-1}$ and

$\lambda_6 = [\lambda_1^2 - \lambda_1(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_2 + 2) - \lambda_3(\lambda_3 - \lambda_2)^2]\lambda_1^{-1}\lambda_4^{-1}$.

For further investigation one may use the four local values as given in [2].

If then follows that P_1 is an unstable first order fine focus.

If point P_1 is a node or a focus it is stable for $\lambda_6 < 0$ and unstable for $\lambda_6 > 0$.

The character of P_1 is listed in Table 3.

TABLE 3.

Point P_1	$\lambda_6^2-4\lambda_5$	λ_5	λ_6
S	+	-	
N	+,0	+	
F	-	+	+,-
FF	-	+	0

P_1 is the only candidate to be situated inside a limit cycle since this point is a (fine) focus for certain values of the parameters. It is known that exactly one unstable limit cycle is *generated out of* P_1 *if the parameters are varied in such a way that* P_1 *becomes a stable* focus. Then, if the parameters are varied further in such a way that P_1 becomes a node, the limit cycle must have disappeared. The limit cycle may have disappeared in two different ways: It is possible that another limit cycle surrounds the limit cycle that was generated out of P_1 and that the two limit cycles together formed a semi-stable limit cycle which disappeared before P_1 became a node. Another possibility is that the limit cycle was blown up until it disappeared in a separatrix cycle.

Numerical calculations strongly indicate that the last possibility occurs. The limit cycle blows up until it disappears in a separatrix cycle, formed by the separatrices of the third order saddle point P_0. It also seems that this limit cycle is unique. Now the cases where system (12) has respectively one, two or three singular points at infinity, will be considered separately in 3IIa, 3IIb and 3IIc.

Case 3IIa: Since there is one singular point at infinity, at $\theta = 0$ and indicated by P_2 there should be satisfied either $(\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_3) < 0$ or $\lambda_2 - 1 = \lambda_1 - \lambda_3 = 0$.

Although the phase portraits in the last case have one singular point at infinity, it will be treated in case 3IIb because the properties of system (12) for $\lambda_2 - 1 = \lambda_1 - \lambda_3 = 0$ are similar to the properties of system (12) in case 3IIb.

For all values of the parameters P_2 is a node. The character of the finite singular point P_1 follows from Table 3.

For two fixed values of λ_3 the (λ_1,λ_2)-parameter plane is given in Figure 8 and the phase portraits are indicated in this figure, listed in Table 4 and drawn in Figure 9.

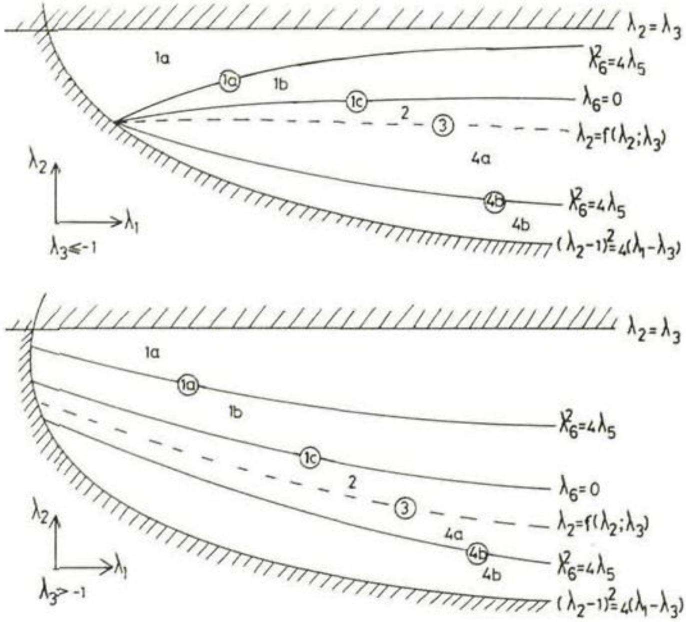


Figure 8 : (λ_1,λ_2)-parameter plane for constant λ_3 in case 3IIa.

We note that in portrait 1c P_1 is an unstable first order fine focus. If the parameters vary in such a way that P_1 becomes a stable focus exactly one unstable limit cycle is generated out of P_1 (portrait 2). If the parameters are continued to vary in the same way, this limit cycle seems to disappear in a separatrix cycle (portrait 3), which is indicated in Figure 8 by the dashed curve $\lambda_2 = f(\lambda_1;\lambda_3)$.

Numerical analysis strongly indicates that the limit cycle in portrait 2 is unique and that in all other phase portraits of Figure 9 the system has no limit cycles.

TABLE 4: Case 3IIa: $(\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_3) < 0$.

Portrait	$\lambda_6^2-4\lambda_5$	λ_6	$\lambda_2-f(\lambda_1;\lambda_3)$	P_0	P_1	P_2
1a	+,0	+		S*	N	N
1b	-	+		S*	F	N
1c	-	0		S*	FF	N
2	-	-	+	S*	F	N
3	-	-	0	S*	F	N
4a	-	-	-	S*	F	N
4b	+,0	-		S*	N	N

$$\lambda_4 = \lambda_1 + \lambda_3(\lambda_3 - \lambda_2) \qquad \lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1}\lambda_4^{-1}$$
$$\lambda_6 = [\lambda_1^2 - \lambda_1(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_2 + 2) - \lambda_3(\lambda_3 - \lambda_2)^2]\lambda_1^{-1}\lambda_4^{-1}.$$

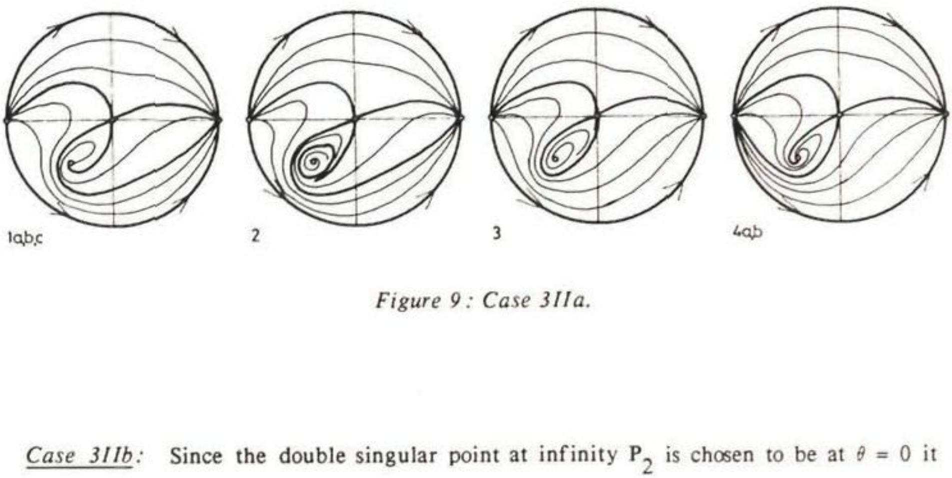


Figure 9 : Case 3IIa.

Case 3IIb: Since the double singular point at infinity P_2 is chosen to be at $\theta = 0$ it follows from (13) that $\lambda_1 = \lambda_3$. Apart from the singular point P_2 there is a second singular point at infinity P_3 at $\theta = \operatorname{arccot}(1 - \lambda_2)$. We also allow $1 - \lambda_2 = 0$, then P_2 and P_3 coincide. It is the second case of case 3IIa which was postponed to the present case.

The character of the singular points P_2 and P_3 is listed in Table 5 and the (μ_1,μ_2)-parameter plane is given in Figure 10 where $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_1 - \lambda_2$.

The phase portraits are listed in Table 6 and are drawn in Figure 11.

It should be noticed that in the portraits 6a, 10a and 14a, P_1 is an unstable first order fine focus. If the parameter vary in such a way that P_1 becomes a stable focus (portraits 5, 9 and 13), exactly one unstable limit cycle is generated out of P_1. Numerical analysis strongly indicate that if the parameters are continued to vary in the same way, the limit cycle disappears in a separatrix cycle (portraits 4, 8 and 12), what is indicated in Figure 10 by the dashed curve $\mu_2 = f(\mu_1)$.

Numerical analysis also indicates that the limit cycle in the portraits 5, 9 and 13 is unique and that the quadratic system has no limit cycles in all other portraits of Figure 11. Comparing phase portraits 11 with phase portraits 7 and 12, we may conclude that all three relative position of the α-separatrix of P_2 and the ω-separatrix P_0, suggested in portrait 11, occur.

In phase portrait 2 P_2 is a higher order singular point with an elliptic sector. Applying the Poincaré transformation $u = \frac{y}{x}$, $z = \frac{1}{x}$ and the 'blow-up' transformation u = u, z = wu it is easy to show that the separatrix coincides with the circle $\rho = 1$ (see Appendix 1, also [7]).

TABLE 5: Case 3IIb

Point P_2	μ_1	$\mu_1-\mu_2-1$
Pwes*	0	
N*		0
SN*	+,-	+,-

Point P_3	$\mu_1-\mu_2-1$
N	+,-
N*	0

* higher order point

Pwes = Point with elliptic sector

TABLE 6: Case 3IIb: $\lambda_1 = \lambda_3$

Portrait	P_0	P_1	P_2	P_3	μ_1	$\mu_1-\mu_2-1$	$\lambda_6^2-4\lambda_5$	λ_6	$\mu_2-f(\mu_1)$
1	S*	N	SN*	N	-	-	+	+	
2	S*		Pwes*	N	0	-			
3a	S*	N	SN*	N	+	-	+,0		
3b	S*	F	SN*	N	+	-	-	-	-
4	S*	F	SN*	N	+	-	-	-	0
5	S*	F	SN*	N	+	-	-	-	+
6a	S*	FF	SN*	N	+	-	-	0	
6b	S*	F	SN*	N	+	-	-	+	
6c	S*	N	SN*	N	+	-	+,0	+	
7	S*	F	N*		+	0	-	-	-
8	S*	F	N*		+	0	-	-	0
9	S*	F	N*		+	0	-	-	+
10a	S*	FF	N*		+	0	-	0	
10b	S*	F	N*		+	0	-	+	
10c	S*	N	N*		+	0	+,0	+	
11	S*	F	SN*	N	+	+	-	-	-
12	S*	F	SN*	N	+	+	-	-	0
13	S*	F	SN*	N	+	+	-	-	+
14a	S*	FF	SN*	N	+	+	-	0	
14b	S*	F	SN*	N	+	+	-	+	
14c	S*	N	SN*	N	+	+	+,0	+	

* higher order point

$\lambda_4 = \lambda_1 + \lambda_3 (\lambda_3 - \lambda_2)$ $\qquad \lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1} \lambda_4^{-1}$ $\qquad \mu_1 = \lambda_1$

$\lambda_6 = [\lambda_1^2 - \lambda_1 (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_2 + 2) - \lambda_3 (\lambda_3 - \lambda_2)^2] \lambda_1^{-1} \lambda_4^{-1}$ $\qquad \mu_2 = \lambda_1 - \lambda_2$

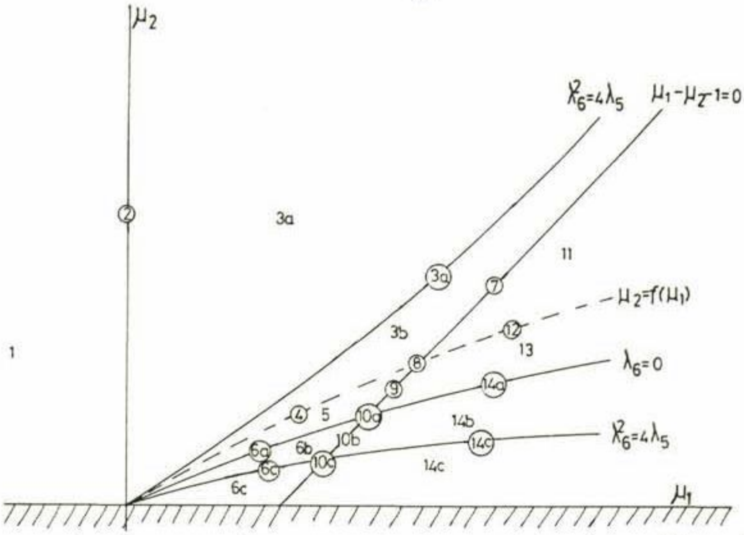


Figure 10 : Case 3IIb.

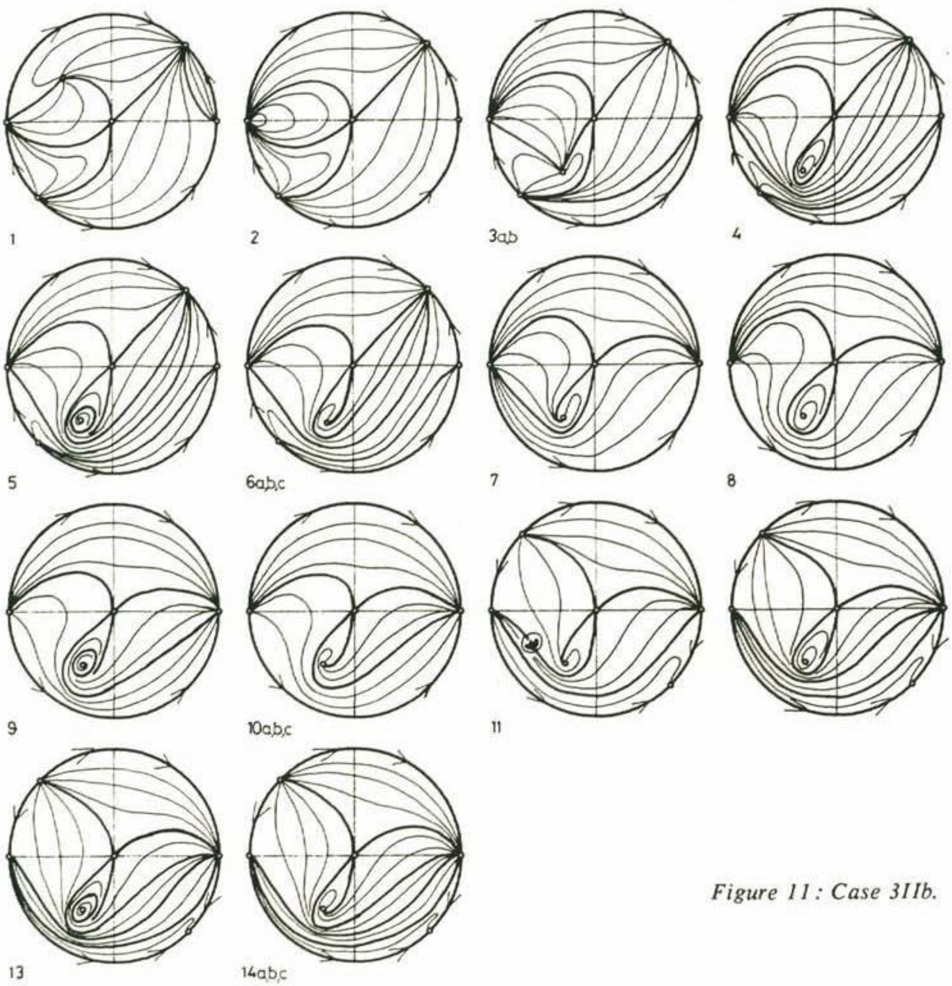


Figure 11 : Case 3IIb.

Case 3IIc: According to (13) there are three different singular points at infinity for $(\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_3) > 0$ and $\lambda_1 - \lambda_3 \neq 0$. In order that P_2 at $\theta = 0$ is intermediate between a positive root P_3 and a negative root P_4 of equation (13), there should be satisfied $\lambda_1 - \lambda_3 < 0$. P_3 and P_4 are situated at $\theta = \operatorname{arccot}(-\frac{1}{2} - \frac{1}{2}\lambda_2 + \frac{1}{2}\sqrt{\lambda})$ and at $\theta = \operatorname{arccot}(-\frac{1}{2} - \frac{1}{2}\lambda_2 - \frac{1}{2}\sqrt{\lambda})$ respectively, where $\lambda = (\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_2)$.
The character of these points is listed in Table 7 and for $\lambda_3 \leq 0$ and $\lambda_3 > 0$ the (λ_1, λ_2)-parameter plane is given in Figure 12. The phase portraits are listed in Table 8 and are drawn in Figure 13.

TABLE 7: Case 3IIc

Point P_2	λ_1
N	-
SN	0
S	+

Point P_3
N

Point P_4	λ_1
N	-
SN	0
S	+

$$\lambda_4 = \lambda_1 + \lambda_3(\lambda_3 - \lambda_2)$$

It should be noticed that in portrait 8a P_1 is an unstable first order fine focus. If the parameters vary in such a way that P_1 becomes a stable focus (portrait 7), exactly one unstable limit cycle is generated out of P_1.
Numerical analysis strongly indicates that if the parameters are continued to vary in the same way, the limit cycle disappears in a separatrix cycle (portrait 6), which is indicated in Figure 12 by the dashed curve $\lambda_2 = f(\lambda_1; \lambda_3)$.
Numerical calculations also indicate that the limit cycle in portrait 7 is unique and that the quadratic system has no other limit cycles in all other portraits of Figure 13. Comparing phase portrait 5 with phase portraits 4 and 6, we may conclude that all three relative positions of the separatrices, suggested in portrait 5, occur.

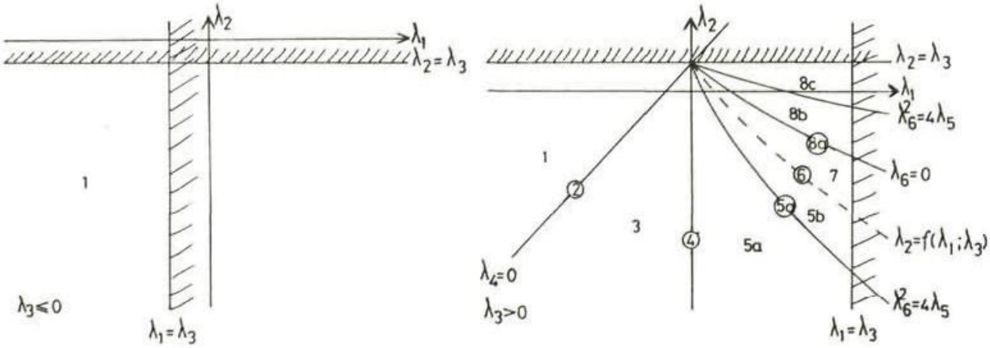


Figure 12: Case 3IIc.

TABLE 8: Case 3IIc: $\lambda_1 - \lambda_3 < 0$

Portrait	P_0	P_1	P_2	P_3	P_4	λ_1	λ_4	$\lambda_6^2-4\lambda_5$	λ_6	$\lambda_2-f(\lambda_1,\lambda_3)$
1	S*	N	N	N	S	-	-			
2	S*		N	N	SN*	-	0			
3	S*	S	N	N	N	-	+			
4	S*		SN*	N	N	0				
5a	S*	N	S	N	N	+		+,0	-	
5b	S*	F	S	N	N	+		-	-	-
6	S*	F	S	N	N	+		-	-	0
7	S*	F	S	N	N	+		-	-	+
8a	S*	FF	S	N	N	+		-	0	
8b	S*	F	S	N	N	+		-	+	
8c	S*	N	S	N	N	+		+,0	+	

$\lambda_4 = \lambda_1 + \lambda_3 (\lambda_3 - \lambda_2)$ $\qquad\qquad$ $\lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1}\lambda_4^{-1}$

$\lambda_6 = [\lambda_1^2 - \lambda_1 (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_2 + 2) - \lambda_3 (\lambda_3 - \lambda_2)^2] \lambda_1^{-1}\lambda_4^{-1}$

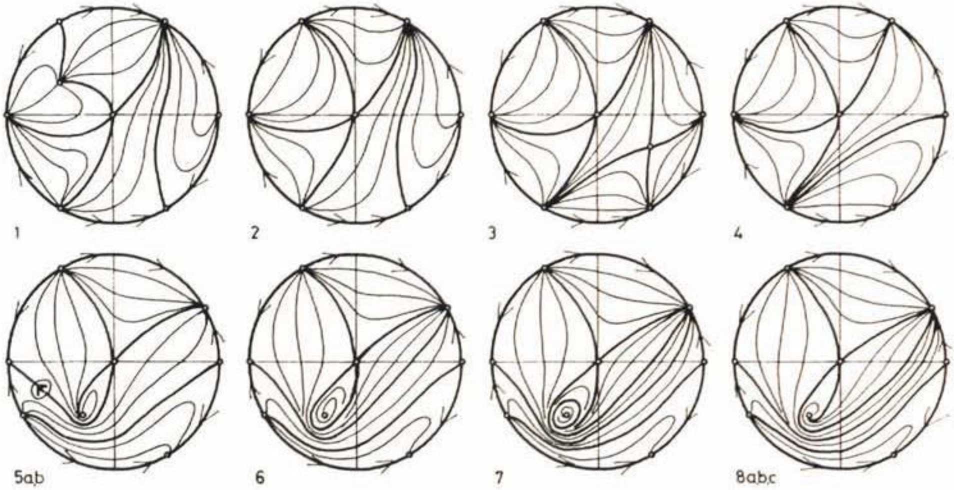


Figure 13: Case 3IIc.

4. QUADRATIC SYSTEMS WITH A THIRD ORDER NODE

If (0,0) is a third order node system (2) may be written, if necessary by applying an affine transformation and/or a scaling of t, in the form

$$\begin{aligned}\overset{\circ}{\bar{x}} &= -\bar{x}\bar{y} + a_{02}\bar{y}^2,\\ \overset{\circ}{\bar{y}} &= \bar{y} + \bar{x}^2 + b_{11}\bar{x}\bar{y} + b_{02}\bar{y}^2,\end{aligned} \tag{14}$$

where $a_{02}, b_{02} \in R$ and $b_{11} \in R_0^+$.

The cases $a_{02} = 0$ and $a_{02} \neq 0$ will be considered separately.

Case 4I: $a_{02} = 0$.

In this case system (14) may be rewritten in the form

$$\begin{aligned}\overset{\circ}{x} &= -xy,\\ \overset{\circ}{y} &= y + x^2 + \lambda_1 xy + \lambda_2 y^2,\end{aligned} \tag{15}$$

where $\lambda_1 \in R_0^+$ and $\lambda_2 \in R$.

As illustrated in Figure 14, the third order node $P_0(0,0)$ is unstable. All trajectories that tend to P_0 are tangent to the x-axis in P_0, or coincide with the negative or positive point of the y-axis.

Besides P_0 there is a second finite singular point P_1 in $(0,-\lambda_2^{-1})$ if $\lambda_2 \neq 0$.

By local linearisation one finds that P_1 is a stable node for $\lambda_2 < 0$ and a saddle point for $\lambda_2 > 0$. It is clear that system (15) has no limit cycles since a limit cycle has to surround a (fine) focus.

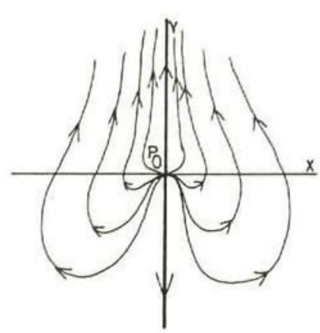


Figure 14: Third order node.

From (3) follows that the singular points at infinity for system (15) are given by

$$C_2(\theta) = \cos^3\theta + \lambda_1 \cos^2\theta \sin\theta + (\lambda_2 + 1) \cos\theta \sin^2\theta = 0, \tag{16}$$

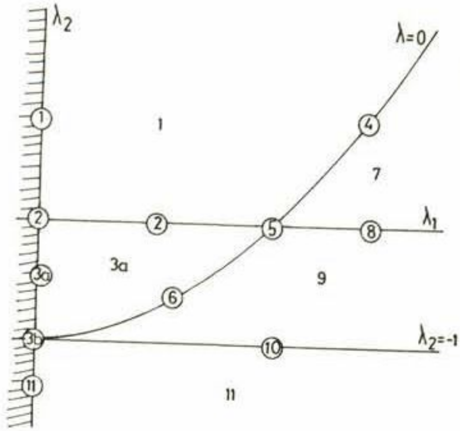
so that there is exactly one singular point P_2 at $\theta = \frac{1}{2}\pi$ if $\lambda_1 = \lambda_2 + 1 = 0$ or $\lambda < 0$ where $\lambda \equiv \lambda_1^2 - 4(\lambda_2 + 1)$. System (15) has two singular points if either $\lambda = 0$ and $\lambda_1 > 0$ or $\lambda_2 = -1$ and $\lambda_1 > 0$: P_2 at $\theta = \frac{1}{2}\pi$ and P_3 at $\theta = \operatorname{arccot}(-\frac{1}{2}\lambda_1)$. System (15) has three singular points if $\lambda > 0$ and $\lambda_2 \neq -1$: P_2 at $\theta = \frac{1}{2}\pi$, P_3 at $\theta = \operatorname{arccot}(-\frac{1}{2}\lambda_1 - \frac{1}{2}\sqrt{\lambda})$ and P_4 at $\theta = \operatorname{arccot}(-\frac{1}{2}\lambda_1 + \frac{1}{2}\sqrt{\lambda})$.

The character of these singular points is listed in Table 9 and the phase portraits are indicated in the λ_1,λ_2-parameter plane in Figure 15 and drawn in Figure 16.

TABLE 9: Case 4I: $a_{02} = 0$

Portrait	λ	λ_1	λ_2	λ_2+1	P_0	P_1	P_2	P_3	P_4
1	-		+		N*	S	N		
2	-		0		N*		SN*		
3a	-		-		N*	N	S		
3b	0	0			N*	N	S*		
4	0		+		N*	S	N	SN*	
5	0		0		N*		SN*	SN*	
6	0		-		N*	N	S	SN*	
7	+		+		N*	S	N	S	N
8	+		0		N*		SN*	S	N
9	+		-	+	N*	N	S	S	N
10	+			0	N*	N	SN*	S	
11	+			-	N*	N	N	S	S

* higher order point $\qquad \lambda = \lambda_1^2 - 4(\lambda_2 + 1)$



$\lambda = \lambda_1^2 - 4(\lambda_2 + 1).$

Figure 15 : Case 4I : $a_{02} = 0$.

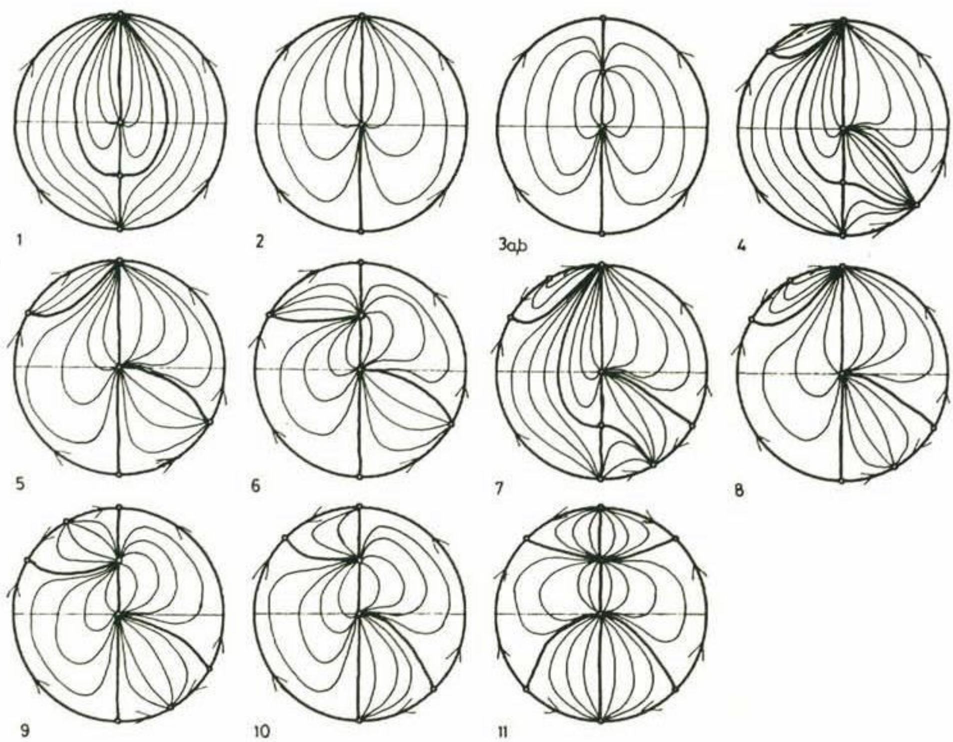


Figure 16 : Case 4I : $a_{02} = 0$.

Case 4II: $a_{02} \neq 0$.

From (3) follows that the singular points at infinity for system (14) are given by

$$
\begin{aligned}
&C_2(\theta) = \cos^3\theta + b_{11}\cos^2\theta \sin\theta + (b_{02} + 1)\cos\theta \sin^2\theta - a_{02}\sin^3\theta = 0 \\
&\text{or equivalently } \cot^3\theta + b_{11}\cot^2\theta + (b_{02} + 1)\cot\theta - a_{02} = 0.
\end{aligned}
\tag{17}
$$

Now three different cases may be distinguished. Case 4IIa: there is just one real solution $\cot\theta$ of equation (17). Case 4IIb: there are two different real solutions of (17), one of them is simple and the other one is double. Case 4IIc: equation (17) has three different real solutions.

If the affine transformation $\bar{x} = \alpha x + \alpha y$, $\bar{y} = y$, where α is a properly chosen constant, is applied to (14), there may be obtained, if necessary by applying a scaling x and y

$$
\begin{aligned}
\overset{\circ}{x} &= x + \lambda_1 x^2 + \lambda_2 xy + y^2, \\
\overset{\circ}{y} &= x \qquad\quad + \lambda_3 xy + y^2,
\end{aligned}
\tag{18}
$$

where λ_1, λ_2 and $\lambda_3 \in R$ and $\lambda_2 > \lambda_3$, and further restrictions on $\lambda_1, \lambda_2, \lambda_3$ will follow depending on the choice of α.

In fact we take the constant $\alpha = \cot\theta$, where α is the unique solution of (17) in case 4IIa, α is the double solution in case 4IIb and the middle one of the three solutions in case 4IIc.

From (3) then follows that the singular points at infinity of (18) are given by

$$
C_2(\theta) = [\sin^2\theta + (\lambda_2 - 1)\sin\theta\cos\theta + (\lambda_1 - \lambda_3)\cos^2\theta]\sin\theta = 0. \tag{19}
$$

As illustrated in Figure 17, the third order node $P_0(0,0)$ of system (18), is unstable. All trajectories that tend to P_0 are tangent to the y-axis except two which are tangent to the line $y = x$.

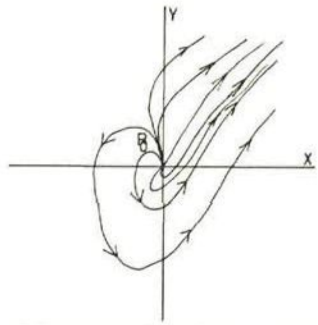


Figure 17 : Third order node.

Apart from the third order node P_0, if $\lambda_1 \cdot \lambda_4 \neq 0$ system (18) has one other finite singular point: P_1 in $(-(\lambda_2 - \lambda_3)^2 \lambda_1^{-1} \lambda_4^{-1}, (\lambda_2 - \lambda_3) \lambda_4^{-1})$ where $\lambda_4 = \lambda_1 + \lambda_3 (\lambda_3 - \lambda_2)$.

The character of this point may be found by local linearisation as long as it is not a center in linear approximation. However, P_1 is a center in linear approximation for $\lambda_6^2 - 4 \lambda_5 < 0$ and $\lambda_6 = 0$ where $\lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1} \lambda_4^{-1}$ and
$\lambda_6 = [\lambda_1^2 - \lambda_1 (\lambda_3 - \lambda_2) (\lambda_3 - \lambda_2 + 2) - \lambda_3 (\lambda_3 - \lambda_2)^2] \lambda_1^{-1} \lambda_4^{-1}$.

For further investigation one may use the four focal values as given in [2]. It then follows that P_1 is a stable first order fine focus.

If point P_1 is a node or a focus it is stable for $\lambda_6 < 0$ and unstable for $\lambda_6 > 0$. The character of P_1 is listed in Table 10.

TABLE 10

Point P_1	$\lambda_6^2 - 4\lambda_5$	λ_5	λ_6
S	+	-	
N	+,0	+	
F	-	+	+,-
FF	-	+	0

P_1 is the only candidate to be situated inside a limit cycle since this point is a (fine) focus for certain values of the parameters. It is known that exactly one stable limit cycle is generated out of P_1 if P_1 is a stable fine focus and the parameters are varied in such a way that P_1 becomes an unstable focus. Then, if the parameters are varied further in such a way that P_1 becomes a node, the limit cycle must have disappeared. The limit cycle may have disappeared in two different ways: It is possible that another limit cycle surrounds the limit cycle that was generated out of P_1 and that the two limit cycles together formed a semi-stable limit cycle which disappeared before P_1 became a node. Another possibility is that the limit cycle was blown up until it disappeared in a separatrix cycle.

Numerical calculations strongly indicate that the last possibility occurs. The limit cycle blows up until it disappears in a separatrix cycle, formed by the separatrices of two singular points at infinity. It also seems that this limit cycle is unique.

Now the case where system (18) has respectively one, two or three singular points at infinity, will be considered separately in 4IIa, 4IIb and 4IIc.

Case 4IIa: Since there is one singular point P_2 at infinity at $\theta = 0$ there should be satisfied either $(\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_3) < 0$ or $\lambda_2 - 1 = \lambda_1 - \lambda_3 = 0$.

Although the phase portraits in the last case have one singular point at infinity, it will be treated in case 4IIb because the properties of system (18) for $\lambda_2 - 1 = \lambda_1 - \lambda_3 = 0$ are similar to the properties of system (18) in case 4IIb.

The character of the singular points is given in Table 11. For a fixed value of λ_3 the (λ_1,λ_2)-parameter plane is given in Figure 18 and the phase portraits are indicated in this figure, listed in Table 11 and drawn in Figure 19.

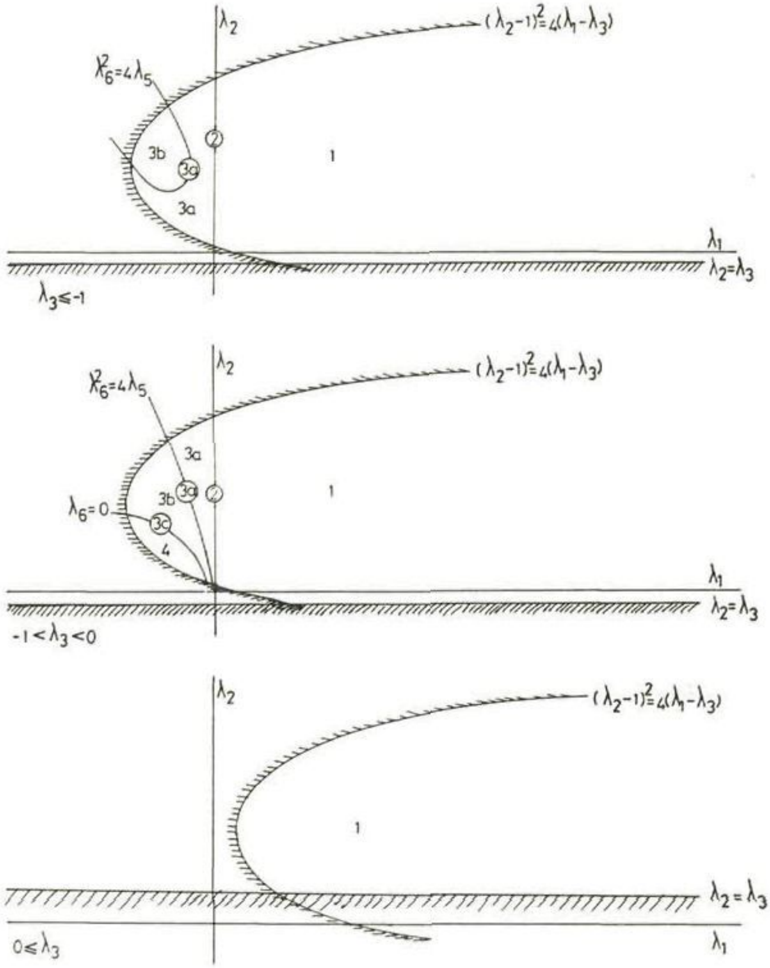


Figure 18 : (λ_1,λ_2)-parameter plane for constant λ_3 in case 4IIa.

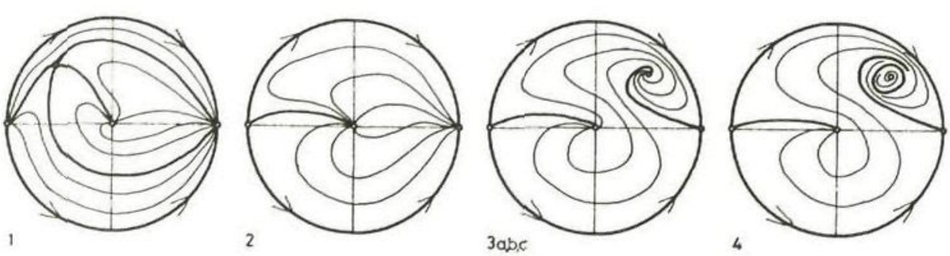


Figure 19 : Case 4IIa.

TABLE 11: Case 4IIa: $(\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_3) < 0$

Portrait	λ_1	$\lambda_6^2-4\lambda_5$	λ_6	P_0	P_1	P_2
1	+			N*	S	N
2	0			N*		SN*
3a	-	+,0		N*	N	S
3b	-	-	-	N*	F	S
3c	-	-	0	N*	FF	S
4	-	-	+	N*	F	S

* higher order point

$$\lambda_4 = \lambda_1 + \lambda_3(\lambda_3 - \lambda_2) \qquad \lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1}\lambda_4^{-1}$$
$$\lambda_6 = [\lambda_1^2 - \lambda_1(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_2 + 2) - \lambda_3(\lambda_3 - \lambda_2)^2]\lambda_1^{-1}\lambda_4^{-1}$$

We note that in portrait 3c P_1 is a stable first order fine focus. If the parameters vary in such a way that P_1 becomes an unstable focus exactly one stable limit cycle is generated out of P_1 (portrait 4). If the parameters are continued to vary in the same way, this limit cycle blows up. However, before the limit cycle disappears in a separatrix cycle, one or two singular points at infinity appear. So, the phase portraits with these separatrix cycles may be found in Figures 21 and 23. Numerical analysis strongly indicates that the limit cycle in portrait 4 is unique and that in all other portraits of Figure 19 the system has no limit cycles.

Case 4IIb: Since the double singular point at infinity P_2 is at $\theta = 0$, it follows from (19) that $\lambda_1 = \lambda_3$. Apart from the singular point P_2 there is a second singular point P_3 at $\theta = \text{arccot}(1 - \lambda_2)$ which coincides with P_3 if $\lambda_2 = 1$. It is the second part of case 4IIa which would be treated here. The character of the singular points P_2 and P_3 is listed in Table 12 and the (μ_1,μ_2)-parameter plane is given in Figure 20 where $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_1 - \lambda_2$. The phase portraits are listed in Table 13 and are drawn in Figure 21.

TABLE 12: Case 4IIb

Point P_2	μ_1	$\mu_1-\mu_2-1$
S*	0	-
SN*	0	0
Pwes*	0	+
N*	+	0
S*	-	0
SN*	+,-	+,-

Point P_3	μ_2+1
S	-
SN*	0
N	+

* higher order point

It should be noticed that in portrait 5c P_1 is a stable first order fine focus. If the parameters vary in such a way that P_1 becomes an unstable focus (portrait 6) exactly one stable limit cycle is generated out of P_1.

Numerical analysis strongly indicate that if the parameters are continued to vary in the same way, the limit cycle disappears in a separatrix cycle (portrait 7), which is indicated in Figure 20 by the dashed curve $\mu_2 = f(\mu_1)$.

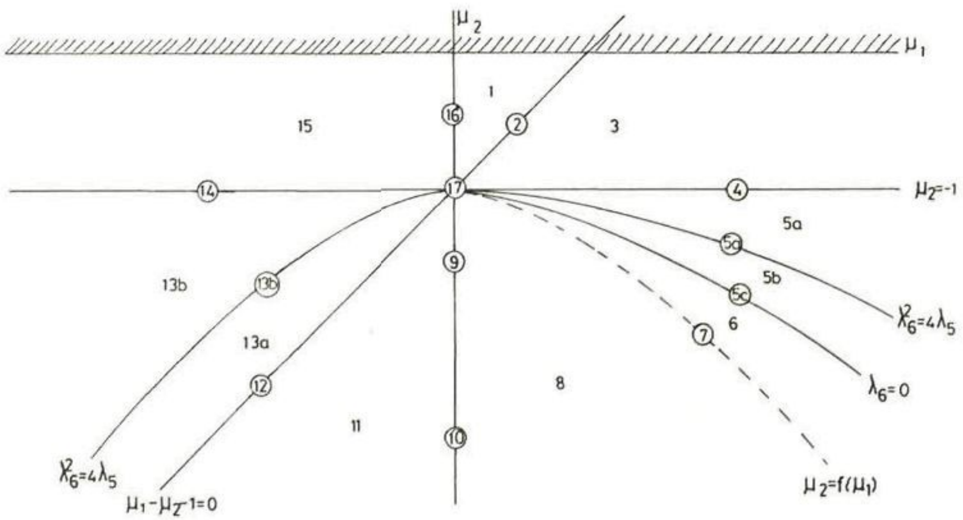


Figure 20 : The (μ_1,μ_2)-parameter plane in case 4IIb.

Numerical analysis also indicates that the limit cycle in portrait 6 is unique and that the quadratic system has no limit cycles in all other portraits of Figure 21.

In phase portraits 9 and 10 P_2 is a higher order singular point with an elliptic sector. Applying the Poincaré transformation $u = \frac{y}{x}$, $z = \frac{1}{x}$ and the 'blow-up' transformation $u = u$, $z = wu$ or $u = u$, $z = wu^2$, it is easy to show that only for $\mu_2 \le -2$ (phase portrait 10) the separatrices coincide with the circle $\rho = 1$ (see Appendix 2 and also [7]).

TABLE 13: Case 4IIB: $\lambda_1 = \lambda_3$

Portrait	P_0	P_1	P_2	P_3	μ_1	μ_2+1	$\mu_1-\mu_2-1$	λ_6	$\lambda_6^2-4\lambda_5$	
1	N*	S	SN*	N	+	+	-			
2	N*	S	N*		+	+	0			
3	N*	S	SN*	N	+	+	+			
4	N*		SN*	SN*	+	0	+			
5a	N*	N	SN*	S	+	-	+	-	+,0	
5b	N*	F	SN*	S	+	-	+	-	-	
5c	N*	FF	SN*	S	+	-	+	0	-	
6	N*	F	SN*	S	+	-	+	+	-	$\mu_2>f(\mu_1)$
7	N*	F	SN*	S	+	-	+	+	-	$\mu_2=f(\mu_1)$
8	N*	F	SN*	S	+	-	+	+	-	$\mu_2<f(\mu_1)$
9	N*		Pwes*	S	0	-	+			$-2<\mu_2<-1$
10	N*		Pwes*	S	0	-	+			$\mu_2\le-2$
11	N*	F	SN*	S	-	-	+	-	-	
12	N*	F	S*		-	-	0	-	-	
13a	N*	F	SN*	S	-	-	-	-	-	
13b	N*	N	SN*	S	-	-	-	-	+,0	
14	N*		SN*	SN*	-	0	-			
15	N*	S	SN*	N	-	+	-			
16	N*		S*	N	0	+	-			
17	N*		SN*		0	0	0			

* higher order point

$\lambda_4 = \lambda_1 + \lambda_3(\lambda_3 - \lambda_2)$ $\qquad \lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1} \lambda_4^{-1}$ $\qquad \mu_1 = \lambda_1$

$\lambda_6 = [\lambda_1^2 - \lambda_1(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_2 + 2) - \lambda_3(\lambda_3 - \lambda_2)^2]\lambda_1^{-1}\lambda_4^{-1}$ $\qquad \mu_2 = \lambda_1 - \lambda_2$

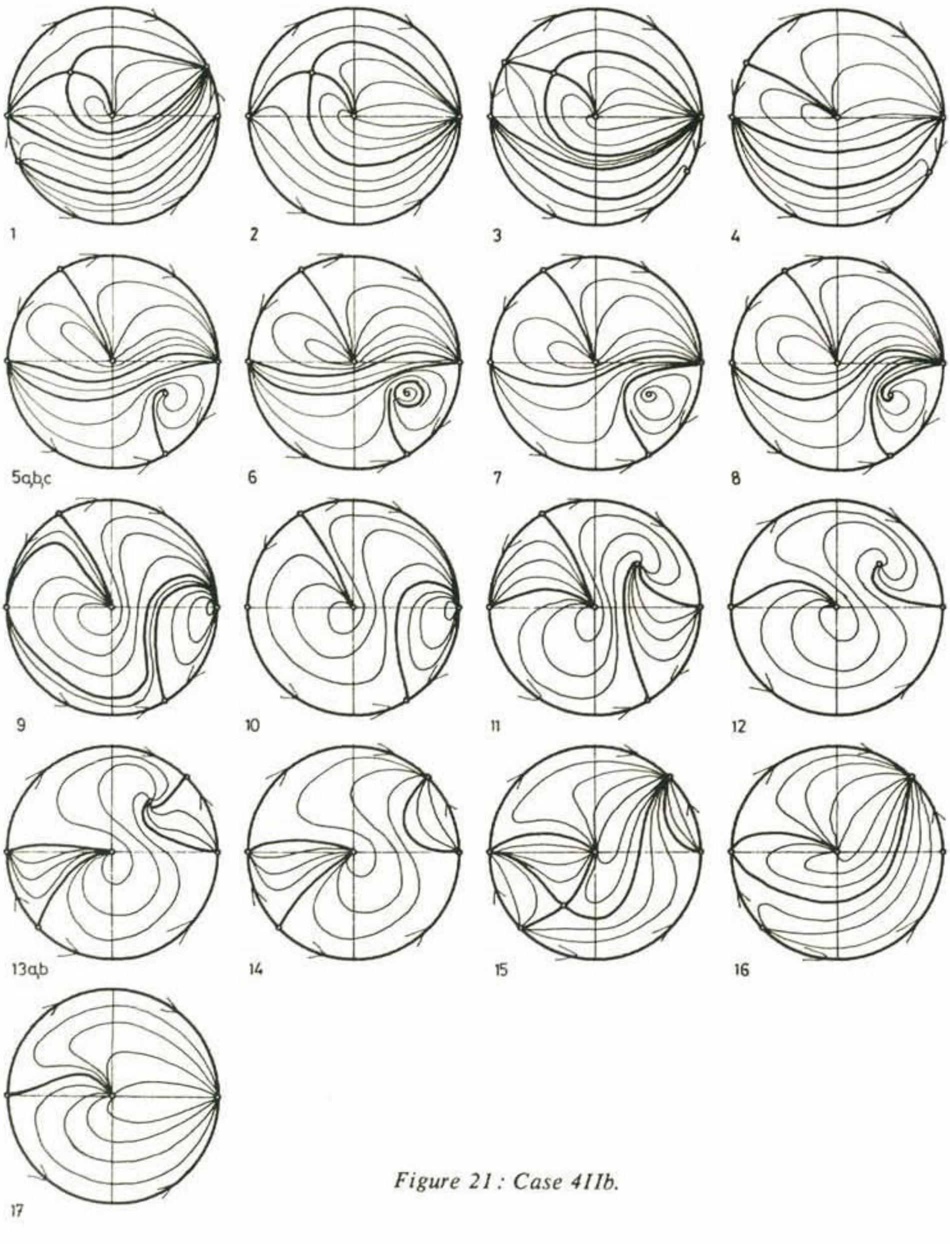


Figure 21 : Case 4IIb.

Case 4IIc: According to (19) there are three different singular points at infinity for $(\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_3) > 0$ and $\lambda_1 - \lambda_3 \neq 0$. In order that P_2 at $\theta = 0$ is intermediate between a positive root P_3 and a negative root P_4 of equation (13), there should be satisfied $\lambda_1 - \lambda_3 < 0$. P_3 and P_4 are situated at $\theta = \text{arccot}\,(-\frac{1}{2} - \frac{1}{2}\lambda_2 + \frac{1}{2}\sqrt{\lambda})$ and at $\theta = \text{arccot}\,(-\frac{1}{2} - \frac{1}{2}\lambda_2 - \frac{1}{2}\sqrt{\lambda})$, respectively, where $\lambda \equiv (\lambda_2 - 1)^2 - 4(\lambda_1 - \lambda_3)$. The character of these points is listed in Table 14 and for $\lambda_3 < 0$, $\lambda_3 = 0$ and $\lambda_3 > 0$ the (λ_1,λ_2)-parameter plane is given in Figure 22.

TABLE 14: Case 4IIc

Point P_2	λ_1
N	-
SN*	0
S	+

Point P_3	λ_1	λ_4
N	+,0	
N	-	-
SN*	-	0
S	-	+

Point P_4	λ_1	λ_4
S	-,0	
S	+	-
SN*	+	0
N	+	+

* higher order point

$\lambda_4 = \lambda_1 + \lambda_3(\lambda_3 - \lambda_2)$

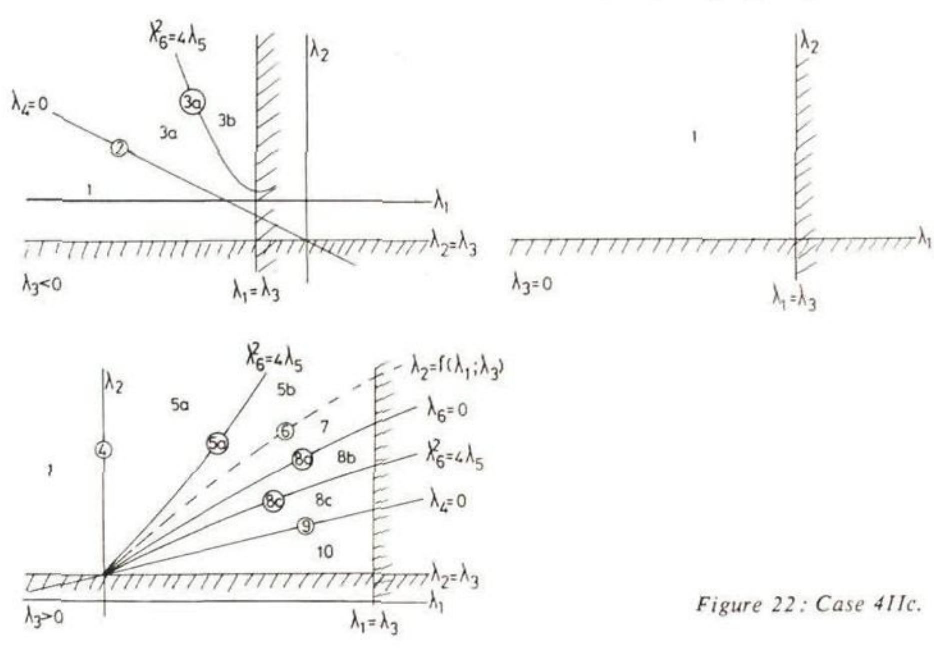


Figure 22: Case 4IIc.

The phase portraits are listed in Table 15 and are drawn in Figure 23.

It should be noticed that in phase portrait 8a P_1 is a stable first order fine focus. If the parameters vary in such a way that P_1 becomes an unstable focus (portrait 7), exactly one stable limit cycle is generated out of P_1.

Numerical calculations strongly indicate that if the parameters are continued to vary in the same way, the limit cycle disappears in a separatrix cycle (portrait 6), which is indicated in Figure 22 by the dashed curve $\lambda_2 = f(\lambda_1;\lambda_3)$.

Numerical calculations also indicate that the limit cycle in portrait 7 is unique and that *the quadratic system has no limit cycles in all other portraits of Figure 23.*

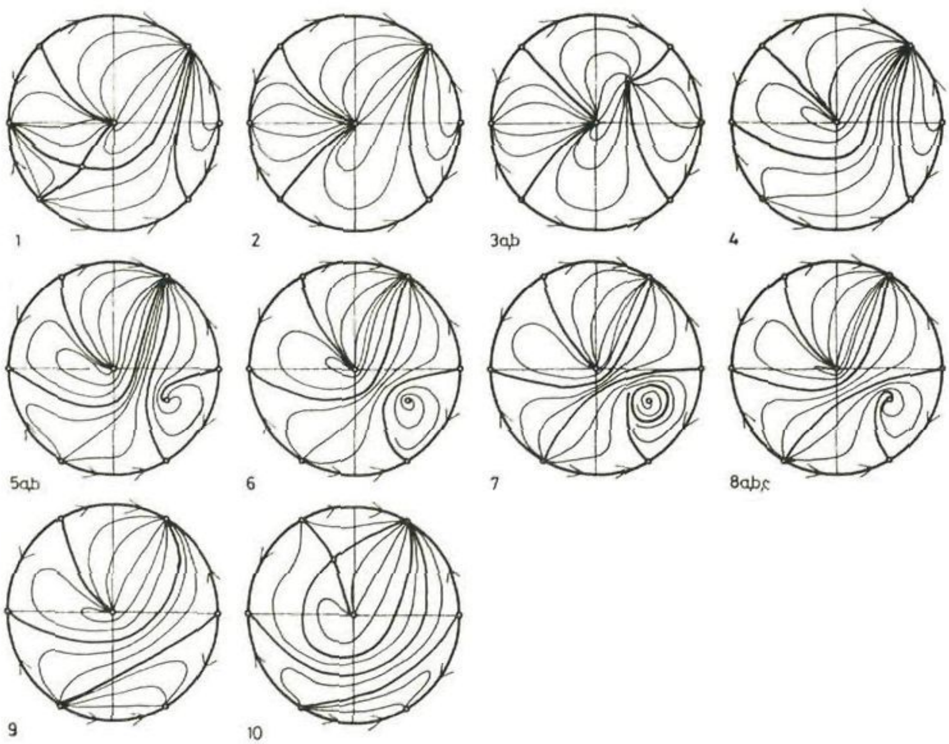


Figure 23: Case 4IIc.

TABLE 15: Case 4IIc: $\lambda_1 - \lambda_3 < 0$

Portrait	P_0	P_1	P_2	P_3	P_4	λ_1	λ_4	$\lambda_6^2-4\lambda_5$	λ_6	$\lambda_2-f(\lambda_1;\lambda_3)$
1	N*	S	N	N	S	-	-			
2	N*		N	SN	S	-	0			
3a	N*	N	N	S	S	-	+	+,0	-	
3b	N*	F	N	S	S	-	+	-	-	
4	N*		SN*	N	S	0				
5a	N*	N	S	N	S	+	-	+,0	+	
5b	N*	F	S	N	S	+	-	-	+	+
6	N*	F	S	N	S	+	-	-	+	0
7	N*	F	S	N	S	+	-	-	+	-
8a	N*	FF	S	N	S	+	-	-	0	
8b	N*	F	S	N	S	+	-	-	-	
8c	N*	N	S	N	S	+	-	+,0	-	
9	N*		S	N	SN*	+	0			
10	N*	S	S	N	N	+	+			

$\lambda_4 = \lambda_1 + \lambda_3 (\lambda_3 - \lambda_2)$ $\qquad\qquad$ $\lambda_5 = (\lambda_3 - \lambda_2)^3 \lambda_1^{-1} \lambda_4^{-1}$

$\lambda_6 = [\lambda_1^2 - \lambda_1 (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_2 + 2) - \lambda_3 (\lambda_3 - \lambda_2)^2] \lambda_1^{-1} \lambda_4^{-1}$

REFERENCES

1. Andronov, A.A., E.A. Leontovich, J.J. Gordon, and A.G. Maier, *Qualitative theory of second-order dynamic systems*, Israel Program for Scientific Translation, Jeruzalem, Wiley, New York (1973).
2. Blows, T.R. and N.G. Lloyd, The number of limit cycles of certain polynomial differential equations, Proceedings of the Royal Society of Edinburgh 98A (1984) pp. 215-239.
3. Coppel, W.A., A survey of quadratic systems, *Journ. of Diff. Eq.*, 2 (1966) pp. 293-304.
4. Date, Tsumotu, Classification and analysis of two-dimensional real homogeneous quadratic differential equation systems, *Journ. of Diff. Eq.*, 32 (1979) pp. 311-334.
5. Gasull, Armengol, Sheng Li-Ren, and Jaume Llibre, Chordal quadratic systems, Universitat Autonoma de Barcelona, Bellaterra, Barcelona.
6. De Jager, P., Phase portraits for quadratic systems with a higher order singularity. II. A second order point with two zero eigenvalues. Report 88-32 of the Faculty of Mathematics and Informatics, Delft University of Technology, 1988.
7. De Jager, P. and J.W. Reyn, Addendum to Phase portraits for quadratic systems with a higher order singularity. I. Third and fourth points with two zero eigenvalues, Delft Progress Report, 12 (1988) pp. 33-37.
8. De Jager, P. and J.W. Reyn, Phase portraits for quadratic systems with a higher order singularity. I. Third and fourth order points with two zero eigenvalues, Proc. First Int. Conf. on Ind. and Appl. Math. ICIAM87, contributions from the Netherlands, Paris, 1987, CWI-Tract 36, Amderdam, pp. 85-97.
9. Newton, T.A., Two dimensional homogeneous quadratic differtential systems, *SIAM Review*, 20 (1978) pp. 120-138.
10. Reyn, J.W., Phase portraits of a quadratic system of differential equations occurring frequently in applications, *Nieuw Archief voor Wiskunde*, Serie 4, Deel 5, Nr. 2 (1987) pp. 107-155.

APPENDIX 1

For system (12) in case 3IIb and the phase portrait for $\lambda_1 = \mu_1 = 0$, $\lambda_1 - \lambda_2 = \mu_2 > 0$ there should be determined whether the separatrices of the singular point with elliptic sector P_2 at infinity coincide with the equator of the Poincaré sphere (Figure 24a) or not (Figure 24b).

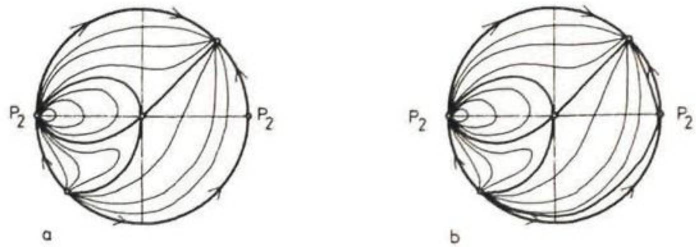


Figure 24.

To find out which one of the Figures 24a and 24b is the correct phase portrait a more detailed analysis of the singular point P_2 is needed. One may apply the Poincaré transformation $u = \frac{y}{x}$, $z = \frac{1}{x}$. System (12) then becomes with $\lambda_1 = 0$,

$$\begin{aligned} \overset{\circ}{u} &= z^{-1}\,(z + (1 - \lambda_2)\,u^2 - uz - u^3), \\ \overset{\circ}{z} &= z^{-1}\,(- \lambda_2 uz - z^2 - u^2 z), \end{aligned} \tag{A1.1}$$

or

$$\frac{dz}{du} = \frac{- \lambda_2 uz - z^2 - u^2 z}{z + (1 - \lambda_2)\,u^2 - uz - u^3}, \tag{A1.2}$$

where $-\lambda_2 = \mu_2 > 0$.

The singular point P_2 of system (12) is transformed to the origin in system (A1.2). The question which one of the phase portraits in Figure 24 is correct, is equal to the question whether the separatrices of the point with elliptic sector of system (A1.2), coincide with the positive and negative u-axis (Figure 25a) or not (Figure 25b).

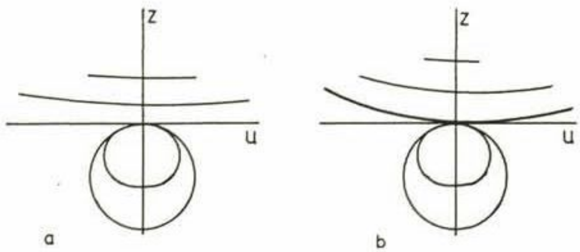


Figure 25.

Blowing up the singularity in the origin by means of the transformation u = u, z = wu, yields, from (A1.2),

$$\frac{dw}{du} = \frac{-uw - w^2}{(1 - \lambda_2)\, u^2 + uw - u^3 - u^2 w} . \qquad \text{(A1.3)}$$

The behaviour of the curves of (A1.2) in the first quadrant near the origin of the u,z plane is reflected in that of the curves of (A1.3) in the first quadrant of the u,w plane which is sketched in Figure 26. Because $\frac{dw}{du}$ is negative in the first quadrant near the origin, the singularity in (0,0) of system (A1.2) will not have a parabolic sector in the first quadrant as in Figure 25b. For the second quadrant of Figure 25b the same

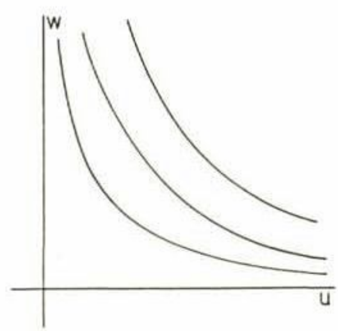


Figure 26.

argument may be used to exclude a parabolic sector.

From the fact that the separatrices of the singularity in the origin of system (A1.2) coincide with the positive and negative u-axis (Figure 25a), one may conclude that the correct phase portraits of the original system (12) is given in Figure 24a.

APPENDIX 2

For system (18) in case 4IIb the phase portrait for $\lambda_1 = \mu_1 = 0$, $\lambda_1 - \lambda_2 = \mu_2 < -1$ cannot be found with standard arguments. It is not clear whether the separatrices of the singular point with elliptic sector P_2 at infinity coincide with the equator of the Poincaré sphere (Figure 27a) or not (Figure 27b).

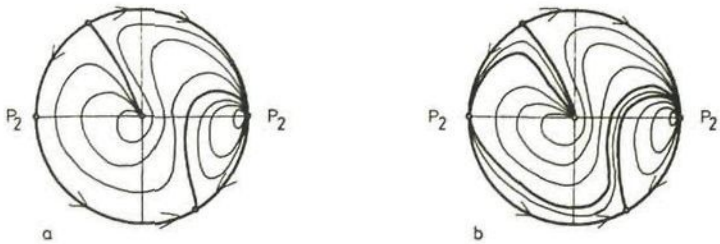


Figure 27.

To make a more detailed analysis of the singular point P_2 one may use the Poincaré transformation $u = \frac{y}{x}$, $z = \frac{1}{x}$. System (18) then becomes with $\lambda_1 = 0$

$$\begin{aligned} \overset{\circ}{u} &= z^{-1}\,(z + (1 - \lambda_2)\, u^2 - uz - u^3), \\ \overset{\circ}{z} &= z^{-1}\,(\quad - \lambda_2 uz - z^2 - u^2 z \quad), \end{aligned} \tag{A2.1}$$

or

$$\frac{dz}{du} = \frac{-\lambda_2 uz - z^2 - u^2 z}{z + (1 - \lambda_2)\, u^2 - uz - u^3}\,, \tag{A2.2}$$

where $-\lambda_2 = \mu_2 < -1$.

The singular point P_2 of system (18) is transformed to the origin in system (A2.2). Just as in Appendix 1 the question that remains is: do the separatrices of the point with elliptic sector in the origin of sysem (A2.2) coincide with the positive and negative u-axis (Figure 28a) or not (Figure 28b).

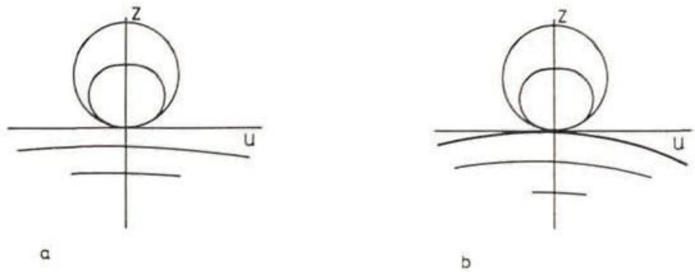


Figure 28.

Blowing up the singularity in the origin by means of the transformation $u = u$, $z = wu$, yields, from (A2.2)

$$\frac{dw}{du} = \frac{-uw - w^2}{(1 - \lambda_2)\, u^2 + uw - u^3 - u^2 w}. \tag{A2.3}$$

The behaviour of the integral curves of (A2.2) in the fourth quadrant near the origin of the u,z-plane is reflected in that of the curves of (A2.3) in the fourth quadrant of the u,w-plane of which a qualitative sketch is given in Figure 29.

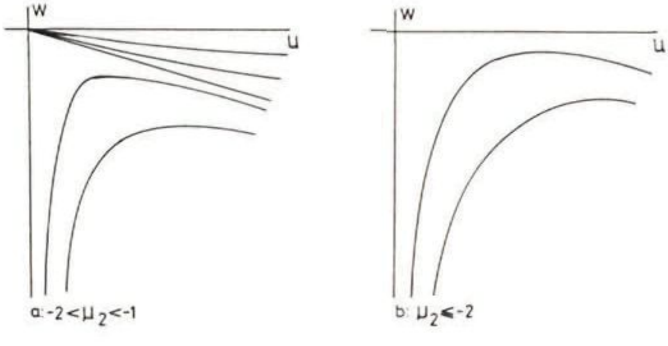


Figure 29.

It is not difficult to show for system (A2.3) that except along the u-axis and the w-axis, also in the direction $w = (-1 - \frac{1}{2}\mu_2)\,u$ integral path(s) are approaching the origin. Therefore, apart from the hyperbolic sector, a parabolic sector exists in the fourth quadrant of the u,w-plane and the u,z-plane if $-2 < \mu_2 < -1$. In the same way one can show that there is a parabolic sector in the third quadrant if $-2 < \mu_2 < -1$ and thus the correct phase portrait for $-2 < \mu_2 < -1$ is Figure 27b.

The O-isodine $[\frac{dw}{du} = 0]$ approaches the origin along the line $u + w = 0$ as a result of which it is not a priori certain, for $\mu_2 \le -2$, whether or not there is a parabolic sector in the fourth quadrant tangent to the u-axis (Figure 29b). In order to answer this question a further 'blow up' is needed. By the transformation $w = w_1 u$, $u = u$ (A2.3) becomes

$$\frac{dw_1}{du} = \frac{(\lambda_2 - 2)\,w_1 + uw_1 - 2w_1^2 + uw_1^2}{(1 - \lambda_2)\,u - u^2 + uw_1 - u^2 w_1}\,. \tag{A2.4}$$

Then it is easy to show that for $\mu_2 \le -2$ $\frac{dw_1}{du}$ is positive in the fourth quadrant near the origin as a result of which there will be no parabolic sector in the fourth quadrant of the u,w-plane (Figure 29b) or in the u,z-plane (Figure 28a) for $\mu_2 \le -2$. In the same way one can show that there is no parabolic sector in the third quadrant of the u,z-plane for $\mu_2 \le -2$ and thus that Figure 27a gives the correct phase portrait if $\mu_2 \le -2$.

CHAPTER 3

BOUNDED SEPARATRIX CYCLES IN QUADRATIC SYSTEMS

I. INTRODUCTION

In this report bounded separatrix cycles in quadratic systems of differential equations

$$\begin{aligned}\dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \equiv P(x,y),\\ \dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \equiv Q(x,y),\end{aligned} \tag{1}$$

where $\cdot = \frac{d}{dt}$ and $a_{ij}, b_{ij} \in \mathbb{R}$, will be studied. In a survey paper [8] on general properties of quadratic systems Coppel states that what remains to be done is to determine all possible phase portraits of such systems. The present report aims to give a contribution by giving several properties of bounded separatrix cycles in quadratic systems.

First it will be proved that just five different types of bounded separatrix cycles can occur in a quadratic system. For (almost) each of these types conditions will be deduced to determine the stability of the separatrix cycle.

Another question that will be answered is what types of singular points may be surrounded by each type of separatrix cycle.

The most difficult problem that will be treated is the question whether a separatrix cycle may surround one or more limit cycles. This question is related to the second part of Hilbert's sixteenth problem [13]: find the maximum number of limit cylces in a polynomial system of differential equations of degree N. The question whether for a given polynomial system the number of limit cycles is finite, was taken up by Dulac [12], but some doubt has risen with respect to the validity of all of his proofs.

Recently Bamón [4] gave a proof that the number of limit cycles in a given quadratic system is finite. The essence of the problem is to show that an accumulation of limit cycles towards a separatrix cycle is not possible. The first step was set by Chicone and Shafer [7], who showed that a bounded separatrix cycle in a quadratic system surrounds at most finitely many limit cycles. This raises the question, also mentioned by

Chicone and Shafer [7], whether a separatrix cycle in a quadratic system at all can surround one or more limit cycles. An affirmate answer was given by Bamón [3] by giving an example of a class of quadratic systems, with a separatrix cycle with at least one limit cycle in its interior. In a paper by Drachman, van Gils and Zhang Zhi-fen [11] an example is given of a class of quadratic systems with exactly one limit cycle surrounded by a separatrix cycle.

In the present report an example will be given of a separatrix cycle which surrounds at least two limit cycles.

The presentation of this report is as follows. In section 2 some definitions and lemmas will be given that are needed in the following sections. In section 3 a survey will be given of all types of bounded separatrix cycles that possibly may occur in quadratic systems. In the sections 4, 5, 6 and 7 examples will be given of the five possible types of bounded separatrix cycles and for each type the questions mentioned in the beginning of this section will be treated. In the last section also an example is given of an unbounded separatrix cycle which surrounds at least two limit cylces.

2. SOME DEFINITIONS AND LEMMAS

As a preparation for the lemmas given in this section it is necessary to give some definitions.

According to [1] one may divide the neighbourhood of a singular point of a quadratic system in several regions each being of one of the following three types: parabolic, elliptic or hyperbolic.

Definition 1. A separatrix is a trajectory, entering or leaving a singular point, which forms the boundary in the neighbourhood of the singular point of two regions of which one is hyperbolic.

Definition 2. A separatrix cycle is a simple closed curve composed of singular points and separatrices, which is the limit continuum for all trajectories in the neighbourhood of the separatrix cycle in the region surrounded by the separatrix cycle.

The fact that a separatrix cycle is defined here as a limit continuum for trajectories inside the separatrix cycle excludes the possibility that a separatrix cycle can surround a center. The cycles surrounding a center can be found in Vulpe's classification of quadratic systems with a center [22].

Definition 3. A bounded separatrix cycle is a separatrix cycle situated in the finite part of the plane.

It should be emphasized that in this report the word 'limit cycle' stands for a simple limit cyle.

Lemma 1 (Tung's lemma [8]).

For a quadratic system the total number of singular points and contact points on a straight line L is at most two.

Proof. Let L be the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ where $\lambda \in R$ and consider a quadratic system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$. If one substitutes $x = a_1 + \lambda b_1$ and $y = a_2 + \lambda b_2$ in the equation $b_2 \cdot P(x,y) - b_1 \cdot Q(x,y) = 0$ to find the singular points and the contact points, the result is that an equation in λ of at most second degree is obtained. So the total number of singular points and contact points is at most two.

Lemma 2.

A bounded separatrix cycle of a quadratic system contains either one, two or three singular points.

Proof. Since P(x,y) and Q(x,y) of equation (1) are conics, a quadratic system has at most four finite singular points. If a quadratic system has a bounded separatrix cycle, exactly one singular point is situated inside this separatrix cycle (see [7]). So the separatrix cycle may contain at most three singular points.

Lemma 3.

In a quadratic system both the interior of a limit cycle and of a bounded separatrix cycle are convex regions.

Proof. If the region R surrounded by the limit cycle or the bounded separatrix cycle is not convex, there are two points in R such that the line segment between them contains a point outside R. This implies that the line through these three points has at least three points of contact or singular points. That is in contradiction with lemma 1.

Lemma 4.

If a bounded separatrix cycle of a quadratic system contains two singular points P_1 and P_2 then the line segment between P_1 and P_2 is part of the separatrix cycle.

Proof. According to lemma 3 the interior of a bounded separatrix cycle is convex. Hence, the line segment between P_1 and P_2 is either a part of the separatrix cycle or lies in its interior. However, the line segment between P_1 and P_2 cannot be part of the interior of the separatrix cycle since that would imply that some trajectories inside the separatrix cycle intersect the line segment twice and that there is a point of contact on P_1P_2. That is in contradiction with lemma 1. Thus the line segment P_1P_2 must be part of the separatrix cycle.

Lemma 5.

If a quadratic system has finitely many finite singular points, the sum of the orders of these singular points is at most four.

Proof. Trivial, since $P(x,y) = 0$ and $Q(x,y) = 0$ in equation (1) are conics.

Lemma 6.

A singular point on a bounded separatrix cycle of a quadratic system can only be one of the following types:

(1) saddle point, order 1;

(2) saddle node, order 2, one eigenvalue zero;

(3) saddle point, order 3, one eigenvalue zero.

Proof. According to definitions 1 and 2 a singular point on a bounded separatrix cycle must have at least one hyperbolic sector bounded by two separatrices. It is clear that a first order saddle point is the only first order point to satisfy this condition.
A classification of higher order singular points in quadratic systems yields the following types of points [17]:

If the divergence in the singular point is zero:

(i) saddle point, order 4;

(ii) saddle point, order 3;

(iii) point with elliptic and hyperbolic sector, order 3;

(iv) cusp point, order 2.

If the divergence in the singular point is unequal to zero:

(i) saddle node, order 4;

(ii) saddle, order 3;

(iii) node, order 3;

(iv) saddle node, order 2.

Fourth order saddle nodes are excluded as singular points on a bounded separatrix cycle, since they are, according to lemma 5, the only singular point in a quadratic system and a separatrix cycle should contain another singular point in its interior.
According to the classification of quadratic systems with a third order singular point with divergence zero, [17], [18], a bounded separatrix cycle will never contain a third order saddle point or a point with elliptic sector.
The cusp point is also excluded because in its neighbourhood the behaviour of the separatrices is such that they cannot be part of a separatrix cycle since this needs to be convex (see [15]). Obviously the third order node will never be part of a separatrix cycle since it has no hyperbolic region.

Lemma 7.

Suppose a quadratic system has a bounded separatrix cycle. Suppose the parameters of the system can be varied in such a way that in every nonsingular point of the separatrix cycle the vector field either does not rotate or rotates in either clockwise direction all along the cycle or anti-clockwise. Then the parameters of the system can be changed such that the separatrix cycle disappears and that at least one limit cycle is generated out of the separatrix cycle.

Proof. Suppose a quadratic system has a bounded separatrix cycle S. Without loss of generality one may assume that S is stable and that the direction of the vector field is as given in Figure 1. Although the separatrix cycle S in Figure 1 contains exactly one singular point, which is a saddle point, it is not relevant for the proof, that holds for every bounded separatrix cycle.

Since S is stable it is clear that a closed curve Γ_1 may be drawn inside S, such that all trajectories that intersect Γ_1 do so from the inside of Γ_1 to the outside.

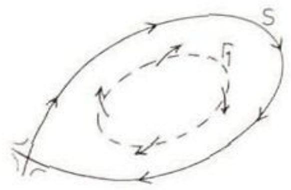


Figure 1.

Now suppose that the parameters of the quadratic system are changed in such a way that the vector field on the non-singular points of S either does not rotate or rotates to the right. Then the nonsingular points that formed S in the original system, form a curve Γ_2 such that all trajectories that intersect Γ_2 enter the region surrounded by Γ_2 (Figure 2).

The separatrix cycle S has disappeared and, if the change of the parameters is small enough, the curve Γ_1 is still a curve without contact with flow from the inside towards the outside.

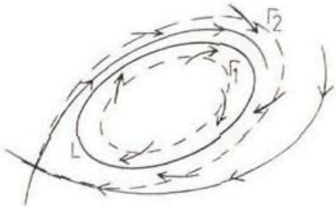


Figure 2.

Now according to the Poincaré-Bendixon theorem one may conclude that the region between Γ_1 and Γ_2 contains at least one limit cycle L.

Lemma 8.

Let P be a saddle point in a quadratic system and let L and M be the tangents to the separatrices in P. Then L and M are either a trajectory or a line without contact (i.e. all trajectories intersect the line either from left to right or from right to left).

Proof. P may be taken in the origin, then the quadratic system can be brought into the normal form ([1], p. 134)

$$\dot{x} = \lambda_1 x + a_{20}x^2 + a_{11}xy + a_{02}y^2 = P(x,y),$$
$$\dot{y} = \lambda_2 y + b_{20}x^2 + b_{11}xy + b_{02}y^2 = Q(x,y),$$

where the x-axis and the y-axis are the tangents to the separatrices in P.

Along the y-axis holds $P(0,y) = a_{02}y^2$. So if $a_{02} = 0$ the y-axis is a trajectory and if $a_{02} \neq 0$ the y-axis is a line without contact. Similarly it can be shown that the x-axis is a trajectory if $b_{20} = 0$ and a line without contact if $b_{20} \neq 0$.

Lemma 9.

For a quadratic system the following proposition holds. Consider a line segment PR and let Q be a point on PR. If the segment PR contains no singular points and the trajectories intersect segment PR in P and Q in the same direction but in R in opposite direction then all trajectories intersect segment PQ in the same direction (Figure 3).

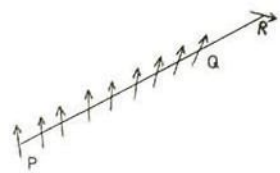


Figure 3.

Proof. Suppose there is at least one point S on the segment PQ where the trajectory intersects segment PR in the same direction as in R. Because the vector field of the quadratic system varies continuously along segment PR, there must be at least one contact point on each of the segments PS, SQ and QR. However, this is in contradiction with lemma 1, so all trajectories intersect PQ in the same direction.

Conjecture 1.

In a quadratic system a singular point may be surrounded by at most three limit cycles.

Although a proof of this conjecture is not known it seems highly improbable that a quadratic system may have a nest of four or more limit cycles. In several papers this conjecture has been made (see [9], [24]).

A more detailed form of this conjecture states that in a quadratic system a third order fine focus cannot be surrounded by limit cycles, a second order fine focus by at most one limit cycle, a first order fine focus by at most two limit cycles and a strong focus by at most three limit cycles. However, until now the only thing that has been proved completely is that a third order fine focus is never surrounded by a limit cycle [19].

3. THE FIVE TYPES OF BOUNDED SEPARATRIX CYCLES

Theorem 1.

There are five types of bounded separatrix cycles which may occur in a quadratic system:

S_1: a separatrix cycle with one singular point: a first order saddle point;

S_2: a separatrix cycle with one singular point: a second order saddle node;

S_3: a separatrix cycle with one singular point: a third order saddle with nonzero divergence;

S_4: a separatrix cycle with two singular points: two first order saddle points;

S_5: a separatrix cycle with two singular points: a first order saddle point and a second order saddle node.

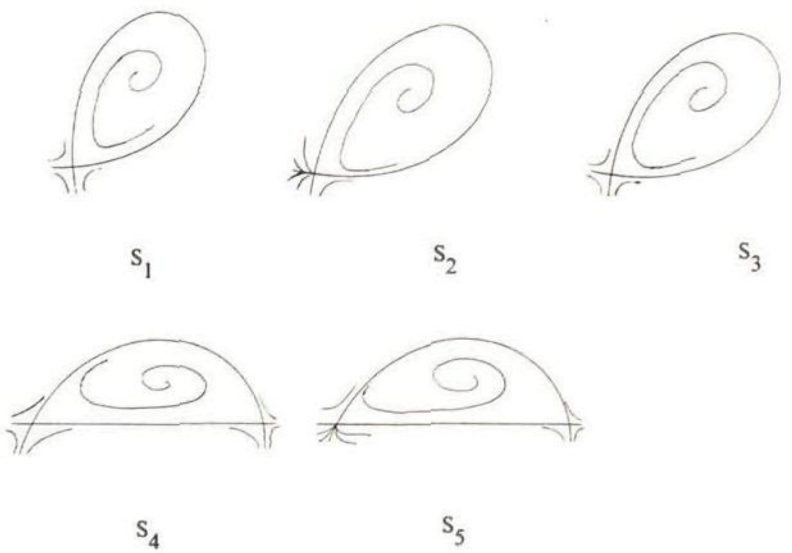


Figure 4.

Proof. According to lemma 2 a bounded separatrix cycle contains either one, two or three singular points. In [10] Dong Jinzhu showed that a bounded separatrix cycle, according to the definition given in section 2, which contains three singular points does not occur in quadratic systems. So a bounded separatrix cycle contains either one or two singular points.

If it contains just one singular point then, according to lemma 6, it can be only a first order saddle, a second order saddle node or a third order saddle.

Since a separatrix cycle surrounds one first order singular point (see [7]) it is clear *from the lemmas 5 and 6 that a separatrix cycle which contains two singular points,* must contain either two first order saddle points or a first order saddle point and a second order saddle node. According to lemma 4 the line segment between the two singular points on the separatrix cycle is part of the separatrix cycle.

That all types can be realized will be shown in the following sections by giving examples of quadratic systems with separatrix cycles of each type.

4. BOUNDED SEPARATRIX CYCLES WITH ONE SINGULAR POINT: A FIRST ORDER SADDLE POINT

Theorem 2.

A quadratic system can have a bounded separatrix cycle with a first order saddle point (S_1).

- (i) This S_1 surrounds either a strong focus, a first order fine focus or a second order fine focus.
- (ii) The region surrounded by the S_1 can contain at least uptil two limit cycles.
- (iii) A S_1 is stable (unstable) of the divergence of the saddle point on the S_1 is negative (positive).
- (iv) On the condition that conjecture 1 is true, a S_1 cannot surround more than two limit cycles.

Proof. First part (i) and part (ii) will be proved by giving examples of such quadratic systems. These quadratic systems are given in lemma 10.

Lemma 10.

Consider the system

$$\begin{aligned} \dot{x} &= \lambda_1 x - y + (1 + \lambda_2)\, x^2 + bxy + y^2 \equiv P(x,y), \\ \dot{y} &= x + \frac{1}{2}\, x^2 + (4b - 2)\, xy \equiv Q(x,y). \end{aligned} \tag{2}$$

For all λ_1 and λ_2, with $|\lambda_1| << 1$ and $|\lambda_2| << 1$, there exist a value of $b \in [2\frac{1}{2}, 10]$ such that system (2) has a S_1.

This lemma will be proved using the following five properties.

Property 1.

System (2) has two singular points: $P_1(0,1)$ and $P_2(0,0)$. P_1 is a saddle point and $\text{div}\,(P(0,1),Q(0,1)) > 0$. P_2 is a (fine) focus wherein the four focal values are given by

$$\text{div}\,(P(0,0),Q(0,0)) = \lambda_1$$

$$V_1 = -\lambda_2(1 - b)$$

$$V_2 = \frac{-7\frac{1}{2}\, b + 2\frac{1}{8} + \lambda_2(\ldots\ldots)}{(\frac{1}{4} + (2 + \lambda_2)^2)^2} \cdot \frac{8\frac{1}{2}\, b + \lambda_2(\ldots\ldots)}{\frac{1}{4} + (2 + \lambda_2)^2} \cdot \left[\frac{8\frac{1}{2}\, b + \lambda_2(\ldots\ldots)}{\frac{1}{4} + (2 + \lambda_2)^2} - 5 \right]$$

$$V_3 > 0 \text{ for } b = 2\frac{1}{2} \text{ and } \lambda_1 = \lambda_2 = 0.$$

Proof. It is easy to show that there are just two singular points and to find the topological structure of these points. With the expressions given in [5] the four focal values in P_2 may be calculated.

Property 2.

If $b \in <2\frac{1}{2}, 10>$, $\lambda_1 < 0$, $\lambda_2 > 0$ and $|\lambda_1| << |\lambda_2| << 1$ system (2) has at least two small limit cycles, surrounding P_2.

Proof. If b, λ_1 and λ_2 satisfy the conditions mentioned above, then, according to property 1, div $(P(0,0),Q(0,0)) < 0$, $V_1 > 0$ and $V_2 < 0$. These conditions are sufficient to have at least two small limit cycles surrounding P_2 (see [5]).

Property 3.

Let L_1 and L_2 be the tangents to the α-separatrix and the ω-separatrix in P_1, respectively. Let L_3 be the line $y = \frac{1}{2 - 4b}$. Let P_3 be the point of intersection of L_2 and L_3 and let P_4 be the point of intersection of L_1 and L_3. Then the α-separatrix intersects segment P_2P_3 in point A and the ω-separatrix intersects the segment P_2P_3 in point B. Both points A and B lie within the triangle $P_1P_3P_4$ (Figure 5).

Proof. It is easy to show that L_1 and L_2 are not trajectories. From lemma 8 it follows that L_1 and L_2 are lines without contact.

Along the horizontal line L_3 holds $Q(x, \frac{1}{2 - 4b}) = \frac{1}{2} x^2 \geq 0$. It may then easily be seen that the α-separatrix starting at P_1 into the triangle $P_1P_3P_4$ must cross the y-axis on the interval $<\frac{1}{2 - 4b}, 0>$ and subsequently a point of segment P_2P_3. Similarly it can be shown that the ω-separatrix intersects P_2P_3.

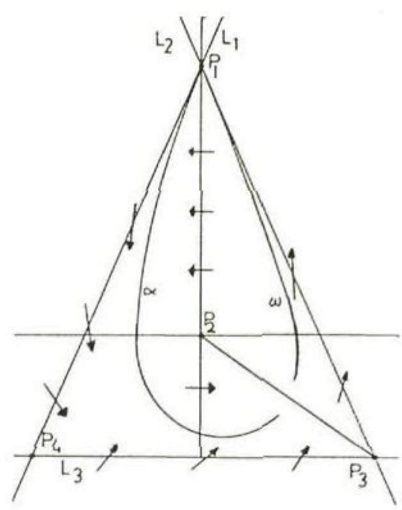


Figure 5.

Property 4.

If $b = 2\frac{1}{2}$ and $\lambda_1 = \lambda_2 = 0$ then $P_2A > P_2B$ (Figure 6).

Proof. Instead of system (2) we will use the equivalent system

$$\begin{aligned} \dot{x} &= -10^8 y + 10x^2 + 25xy + 10y^2 \equiv P(x,y), \\ \dot{y} &= 10^8 x + 5x^2 + 80xy \equiv Q(x,y). \end{aligned} \tag{3}$$

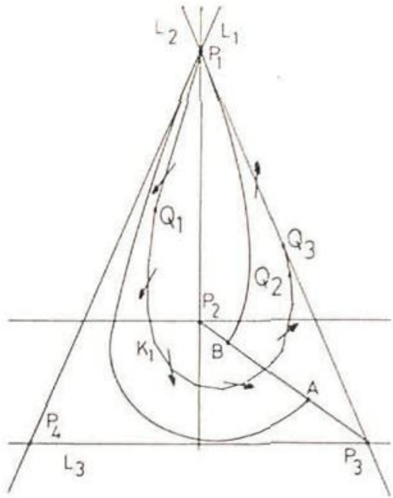


Figure 6.

It is possible to construct a closed curve K_1: $P_1Q_1Q_2Q_3P_1$ in such a way that all trajectories that intersect K_1 move out of the region surrounded by K_1 for increasing values of t.

Part 1: P_1Q_1. In system (3) P_1 is in $(0,10^7)$ and Q_1 is taken in $(-10^5,9790000)$. The line segment P_1Q_1 is given by $x = -10^5\mu$, $y = 10^7 - 210000\,\mu$ with $\mu \in \langle 0,1]$ and the vector field on the segment P_1Q_1 by

$$\begin{aligned} P &= 10^9\,(1066\,\mu^2 - 46000\,\mu), \\ Q &= 10^9\,(1730\,\mu^2 - 90000\,\mu). \end{aligned}$$

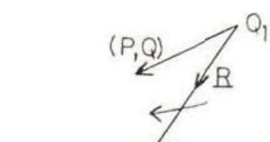


Figure 7.

It is easy to show that all trajectories intersect line segment P_1Q_1 from right to left.

Part 2: Q_1Q_2. Q_1Q_2 is a polygonal arc with 233 points where Q_1 is the first point and Q_2 (1233854, 4200646) is the last point. This polygonal arc will be constructed using the following algorithm:

Let $\underline{R}$ be the vector in Q_1 which is obtained by rotating the vector (P,Q) in Q_1 a small angle in counter-clockwise direction. Let L be the line $\underline{x} = \underline{x}_{Q_1} + \mu\underline{R}$ with $\mu \in R$. Then find a μ_1 and a μ_2 (with $0 < \mu_1 < \mu_2$) such that the trajectories intersecting L in the points where $\mu = 0$ and $\mu = \mu_1$ do so in the same direction, and in the point where $\mu = \mu_2$ in opposite direction (Figure 7). The point on L where $\mu = \mu_1$, is the second point of the polygonal arc and, according to lemma 9, all trajectories intersecting the segment of L corresponding to $0 \le \mu \le \mu_1$ do so in the same direction. This procedure is then repeated starting in the point where $\mu = \mu_1$ and continued till point Q_2 near L_2 is reached. For this calculation computer program 1 (appendix 1) is used which executes this algorithm in such a way, that all calculations are made with integers, so that rounding off errors will not occur. Thus in 232 steps a polygonal arc with 233 vertices was constructed, such that all trajectories intersecting the polygonal arc do so in the same direction.

Part 3: Q_2Q_3. Let L_4 be the line through Q_2 given by $x = 1233854 - 109\,\mu$, $y = 4200646 + 543\,\mu$ with $\mu \in R$. Let Q_3 be the point of intersection of L_4 with L_2 (for Q_3: $\mu = 164674/35$). Along L_4 the vector field is given by

$$P = -98811778151580 - \mu \cdot 6067978460 + \mu^2 \cdot 1587625,$$
$$Q = 545636088041300 - \mu \cdot 4724083780 - \mu^2 \cdot 4675555.$$

It is easy to show that all trajectories intersect segment Q_2Q_3 from left to right.

Part 4: Q_3P_1. Segment Q_3P_1 is a part of line L_2. In property 3 it has already been shown that all trajectories intersect L_2 from left to right.

With the help of property 3 it can easily be shown that the ω-separatrix lies within the region surrounded by K_1 and that the α-separatrix lies outside this region, thus $P_2A > P_2B$.

Property 5.

If $b = 10$ and $\lambda_1 = \lambda_2 = 0$ then $P_2A < P_2B$ (Figure 8).

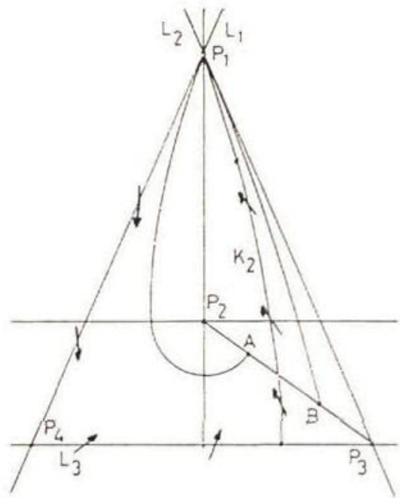


Figure 8.

Proof. Instead of system (2) we will use the equivalent system

$$\dot{x} = -10^8 y + 10x^2 + 100xy + 10y^2 \equiv P(x,y),$$
$$\dot{y} = 10^8 x + 5x^2 + 380xy \equiv Q(x,y). \qquad (4)$$

It is possible to construct a closed curve K_2: $P_1Q_1Q_2P_4P_1$ in such a way that all trajectories that intersect K_2 move into the region surrounded by K_2, for increasing values of t.

Part 1: P_1Q_1. In system (4) P_1 is in $(0,10^7)$ and Q_1 is taken in (50000, 9349500). The line segment P_1Q_1 is given by $x = 50000\,\mu$, $y = 10^7 - 650500\,\mu$ with $\mu \in \langle 0,1]$ and the vector field on the segment P_1Q_1 by

$$P = 10^5 \quad (10040025\,\mu^2 - 150500000\,\mu),$$
$$Q = 10^5\,(-123470000\,\mu^2 + 1950000000\,\mu).$$

It is easy to show that all trajectories intersect line segment P_1Q_1 from right to left.

Part 2: Q_1Q_2. Q_1Q_2 is a polygonal arc with 118 points. Just as is mentioned in property 4 a similar polygonal arc, starting in Q_1, is constructed. In 117 steps a polygonal arc was constructed, such that the last vertex lies below the line L_3. The point Q_2 will be taken in the point of intersection of this polygonal arc and L_3. For the calculation of the polygonal arc Q_1Q_2 computer program 2 is used (appendix 1). It is obvious that all trajectories intersect Q_1Q_2 from right to left.

Part 3: Q_2P_4 and P_4P_1. These two line segments are parts of the lines L_3 and L_1. In property 3 it has already been shown that all trajectories that intersect the segments Q_2P_4 and P_4P_1, move into the triangle $P_1P_4P_3$ for increasing values of t.

With the help of property 3 it can easily be shown that the α-separatrix lies within the region surrounded by K_2 and that the ω-separatrix lies outside this region, thus $P_2A < P_2B$.

Conclusion. If the parameters b, λ_1 and λ_2 vary then A and B shift continuously along the segment P_2P_3. From properties 4 and 5 it follows that if b increases from $2\frac{1}{2}$ to 10 (and $\lambda_1 = \lambda_2 = 0$ for $b = 2\frac{1}{2}$ and $b = 10$) there must be at least one $b \in \langle 2\frac{1}{2}, 10\rangle$ where $P_2A = P_2B$ and thus system (2) has a S_1.

Now it is easy to proof part (i) of theorem 2. If $\lambda_1 = \lambda_2 = 0$ one may increase b from $b = 2\frac{1}{2}$ to $b = 10$. For at least one value of b system (2) has a S_1. During this variation of b the singular point P_2 is a second order fine focus (according to property 1) and so an example is given of a S_1 which surrounds a second order fine focus. In the same way one may show that if b increases from $b = 2\frac{1}{2}$ to $b = 10$ and $\lambda_2 = 0$, $\lambda_1 < 0$ and $|\lambda_1| << 1$ that for at least one value of b a S_1 appears which surrounds a first order fine focus. Finally an example of a S_1 which surrounds a strong focus may be given increasing b from $b = 2\frac{1}{2}$ to $b = 10$, while $\lambda_1 < 0$, $\lambda_2 > 0$, $|\lambda_1| << |\lambda_1| << 1$.

To proof part (ii) of theorem 2 system (2) in lemma 10 may also be used since the example above of a S_1 which surrounds a strong focus satisfies the conditions mentioned in property 2. So at the same time it is an example of a quadratic system with a S_1 which surrounds (at least) two limit cycles.

It may be noted that examples of systems with a S_1 which surrounds exactly one or no limit cycles are already known (see [3], [10]).

The proof of part (iii) of theorem 2 is simple since it is known (p. 304 in [2]) that, if the separatrices of a singular point form a separatrix cycle, it is stable (unstable) if the divergence in the singular point is negative (positive).

A complete proof of part (iv) is not known. However, the following arguments indicate strongly that part (iv) is true.
Suppose that a quadratic system has a S_1 which surrounds three or more limit cycles. Then the parameters of the system may be changed, in such a way that the vector field rotates in every nonsingular point in the same direction. One way to achieve such a bifurcation is to take quadratic system (1) and to consider the system

$$\begin{aligned} \dot{x} &= \cos\alpha \cdot P(x,y) + \sin\alpha \cdot Q(x,y), \\ \dot{y} &= \cos\alpha \cdot Q(x,y) - \sin\alpha \cdot P(x,y). \end{aligned} \tag{5}$$

It is easy to check that for $\alpha = 0$ system (5) equals system (1) and that a change of the parameter α gives a rotation of the complete vector field. According to lemma 7 the bifurcation of α may be done in such a way that the S_1 disappears and that at least one limit cycle appears. Since a small change of α does not change the number of limit cycles already surrounded by the S_1, it results in a nest of at least four limit cycles. This however is in contradiction with conjecture 1.

Now one could ask the question whether a S_1 in a quadratic system can surround a third order fine focus. Although a complete proof is not known, it seems to be impossible. First it may be noted that for a special class of quadratic systems the answer is no. Liang Zhao Jun proved in [20] that a quadratic system, with a third order fine focus, transformed into the normal form given in [23], p. 350, has no separatrix cycle surrounding the fine focus, for $n = 0$.
Another indication is that a third order fine focus is never surrounded by a limit cycle, as has been proved independently by Cerkas [6] and Li [19]. Since a separatrix

cycle surrounding a third order fine focus can be seen as a limiting case of a quadratic system with a third order fine focus surrounded by a limit cycle, it seems to be impossible.

A third argument that a S_1 would never surround a third order fine focus would be the following one. Suppose a quadratic system has a S_1 which surrounds a third order fine focus. Then one can change the parameters such that the singular point remains a third order fine focus. Within this class of parameter changes it seems to be possible that simultaneously the S_1 disappears generating at least one limit cycle (see lemma 7). However, the result would be a quadratic system with a third order fine focus, surrounded by at least one limit cycle. This is in contradiction with the proofs of Cerkas [6] and Li [19].

5. BOUNDED SEPARATRIX CYCLES WITH ONE SINGULAR POINT: A SECOND ORDER SADDLE NODE

Theorem 3.

A quadratic system can have a bounded separatrix cycle with one singular point: a second order saddle node (S_2).

(i) A S_2 can surround one limit cycle and it is highly improbable that it can surround more than one limit cycle.

(ii) A S_2 can surround a strong focus or a first order fine focus. It seems impossible that a S_2 surrounds a second order fine focus and it is impossible that it surrounds a third order fine focus.

(iii) A S_2 is stable (unstable) if the divergence of the saddle node on the S_2 is negative (positive).

Proof. In order to proof part (i) of theorem 3 one may consider the following quadratic system

$$\dot{x} = \lambda_1 x - y + \lambda_2 x^2 + 4xy + y^2,$$
$$\dot{y} = x \quad - \quad x^2 - xy \quad , \tag{6}$$

where $0 \le \lambda_1 << 1$ and $-\frac{3}{4} \le \lambda_2 \le -\frac{1}{2}$.

System (6) has the following properties. There are three singular points in the finite part of the x,y-plane: a second order saddle node in $P_0(0,1)$, a stable focus in $P_1((-\lambda_1 - 3)(\lambda_2 - 3)^{-1}, (\lambda_1 + \lambda_2)(\lambda_2 - 3)^{-1})$ and a (fine) focus P_2 in the origin. One may use the focal values as given in [5] to show that P_2 is an unstable focus if $\lambda_1 > 0$, a stable first order fine focus if $\lambda_1 = 0$ and $\lambda_2 < -\frac{1}{2}$ and an unstable second order fine focus if $\lambda_1 = 0$ and $\lambda_2 = -\frac{1}{2}$.

At infinity system (6) has just one singular point: a saddle point which is situated in the second and fourth quadrant.

Now the proof of theorem 3 will be completed by using the lemmas 11 and 12.

Lemma 11.

For $\lambda_1 = 0$ and $\lambda_2 = -\frac{1}{2}$ the phase portrait of system (6) is as given in Figure 9. It should be noted that it is not decided whether there are limit cycles surrounding P_1 and/or P_2 or not.

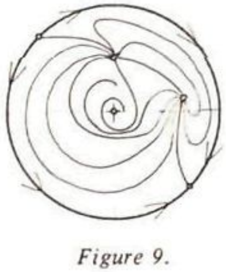


Figure 9.

Proof. Using the mentioned properties of system (6) and the general properties of quadratic systems one may draw the phase portrait in Figure 9 apart from the fact that it is still necessary to show that the left α-separatrix of P_0 has the indicated position relative to the ω-separatrix of P_0.

It is also not known whether there are limit cycles surrounding P_0 and/or P_1.

The first problem may be solved by constructing a polygon without contact. This polygon which begins on P_0P_2 and ends at $y = 1 - 4x$ consists of 47 vertices is constructed in such a way that all trajectories which intersect the polygon do so in the same way (see Figure 10). Because the left α-separatrix of P_0 can not intersect the polygon, it must intersect the line without contact $y = 1 - 4x$ and then tend in the direction of P_1. The list of the 47 vertices of the polygon is given in Table 1. One may note that it is a lot of work to check by hand that all trajectories intersect the polygon as suggested in Figure 10. However, one may check the polygon easily by using a Computer Algebra System like for instance Macsyma.

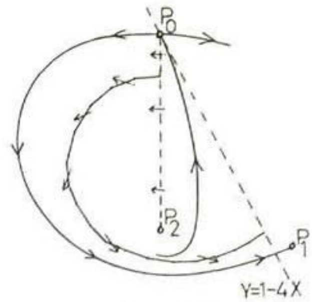


Figure 10.

The second problem is to show that the phase portrait in Figure 9 has no limit cycles. Although a proof for the nonexistence of limit cycles cannot be given, there are several indications that there are no limit cycles.

There is one numerical argument. Computer calculations strongly indicate that the left α-separatrix of P_0 tends to P_1 and that the ω-separatrix tends to P_2 instead of tending to a limit cycle which surrounds P_2.

An analytic argument that there are no limit cylces surrounding P_1 is the following one. Suppose there are limit cycles surrounding P_1. Since the left α-separatrix of P_0 tends in the direction of P_1 and P_1 is a stable focus the number of limit cycles is even. Thus there are at least two limit cycles surrounding P_1. Because P_2 is a second order fine focus one may bifurcate the parameters of the quadratic system (6) such that P_2 becomes a focus and generates two limit cycles. This would be an example of a quadratic system with a (2,2) limit cycle configuration (or even more limit cycles). But a quadratic system with a (2,2) limit cycle configuration seems highly improbable. In [24] an attempt has been made to proof the impossibility of the configuration. However, this proof is not complete (yet).

One may use almost the same arguments for the nonexistence of limit cycles

TABLE 1.

x	y	x	y
0	.8	-.7499	-.8765
-.2149	.7763	-.5529	-.9481
-.4919	.7148	-.3486	-.9981
-.7252	.6356	-.1672	-1.0221
-.9087	.5502	-.0266	-1.0253
-1.0487	.4655	.0731	-1.0168
-1.154	.3849	.1446	-1.003
-1.233	.3096	.1964	-.9871
-1.2923	.2395	.235	-.9706
-1.3367	.1738	.2645	-.9541
-1.3696	.1117	.2882	-.9375
-1.393	.053	.3079	-.9205
-1.4088	-.0036	.3246	-.9028
-1.4181	-.059	.3389	-.8842
-1.4214	-.1138	.3513	-.864
-1.4189	-.1686	.3621	-.8416
-1.4105	-.2241	.3713	-.8162
-1.3955	-.2816	.3792	-.7851
-1.3727	-.3419	.3849	-.7459
-1.3406	-.4054	.3871	-.6911
-1.2966	-.4733	.3813	-.5999
-1.2374	-.5462	.3407	-.3628
-1.1587	-.6243		
-1.0547	-.7073		
-.9193	-.7932		

surrounding P_2. Suppose there are limit cycles surrounding P_2. Since the ω-separatrix of P_0 tends in the direction of P_2 and P_2 is an unstable second order fine focus the number of limit cycles is even. This would be an example of a quadratic system with a second order fine focus which is surrounded by at least two limit cycles. Then one could change the parameters of the quadratic system such that the second order fine focus becomes a focus and generates two limit cycles. That would give a nest of at least four limit cycles. That is in contradiction with conjecture 1.

Lemma 12.

For $\lambda_1 = 0$ and $\lambda_2 = -\frac{3}{4}$ the phase portrait of system (6) is as given in Figure 11. The ω-separatrix of P_0 coincides with the straight line $y = 1 - 4x$ for $x > 0$ and the system has no limit cycles.

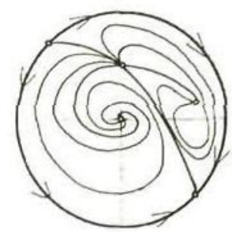


Figure 11.

Proof. Using the properties of system (6) and the straight line solution $y = 1 - 4x$ it is easy to draw the phase portrait in Figure 11. From [23] it is known that a quadratic system with a first order fine focus and a straight line solution has no limit cycles.

Using the lemmas 11 and 12 it is clear that if the parameters of system (6) are varied from $\lambda_1 = 0$ and $\lambda_2 = -\frac{1}{2}$ to $\lambda_1 \approx 0$ and $\lambda_2 = -\frac{3}{4}$ for at least one value of the parameters during this variation the left α-separatrix of P_0 and the ω-separatrix of P_0 coincide and form a S_1 since they shift continuously during the variation. The variation of the parameters may be performed in such a way that λ_1 becomes positive just before the separatrices of P_0 from the S_1. That implies that P_2 becomes an unstable focus and generates exactly one (small) (stable) limit cycle. Then, if the parameters are varied further, the separatrices of P_0 form a S_1 which surrounds the small limit cycle. So, for specific values of the parameters system (6) has a S_1 which contains a second order saddle node and surrounds at least one limit cycle (see Figure 12).

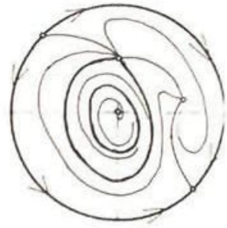


Figure 12.

It may be noted that if λ_1 equals zero during the parameter variation from $\lambda_1 = 0$, $\lambda_2 = -\frac{1}{2}$ to $\lambda_1 = 0$, $\lambda_2 = -\frac{3}{4}$, P_2 is a first order fine focus and so an example is given of a S_2 which surrounds a first order fine focus.

The question remains whether a S_2 may surround more than one limit cycle. Although a proof is not known it seems to be impossible. It is clear that this question is closely related to the question whether a S_2 may surround a second order fine focus. That however, seems to be impossible, according to part (ii) of theorem 3, which will be discussed now.

The first part of the proof of part (ii) of theorem 3 is easy since examples of quadratic systems with a S_2 which surrounds a strong focus or a first order fine focus have already been given in the proof of part (i).
The second part is harder to show. First it will be shown that a S_2 cannot surround a third order fine focus and then an indication will be given that it is most unlikely that a S_2 can surround a second order fine focus.

Suppose a quadratic system has a S_2 which surrounds a third order fine focus. Without loss of generality one may assume that the saddle node is situated in the origin and that the quadratic system has the following form (see [17])

$$\begin{aligned} \dot{x} &= \quad\; a_{20}x^2 + a_{11}xy + a_{02}y^2 = P(x,y), \\ \dot{y} &= y + b_{20}x^2 + b_{11}xy + b_{02}y^2 = Q(x,y), \end{aligned} \tag{7}$$

where $a_{20} \in \mathbb{R}/\{0\}$ and $a_{11}, a_{02}, b_{20}, b_{11}, b_{02} \in \mathbb{R}$. *Without loss of generality we may* further suppose that the third order fine focus is situated in (1,1). The parameters have to satisfy the following conditions

$$\begin{aligned} P(1,1) &= \quad a_{20} + a_{11} + a_{02} &&= 0; \\ Q(1,1) &= 1 + b_{20} + b_{11} + b_{02} &&= 0; \\ \text{Divergence}\,(1,1) &= \quad 2a_{20} + a_{11} + 1 + b_{11} + 2b_{02} &&= 0. \end{aligned} \tag{8}$$

Since the singular point in (1,1) is a fine focus it is possible to transform system (7) into the normal form, given in [5]. One has to apply the transformation $x = 1 + \bar{y}$, $y = 1 + \lambda\bar{x} + \bar{y}$, where λ is a positive parameter which satisfies the condition

$$\lambda^2(2a_{20} + a_{11}) = -1 \tag{9}$$

to yield this normal form. Then with a scaling of the variables and using the conditions (8), the system can be put into the form

$$\begin{aligned} \dot{\bar{x}} &= -\bar{y} + (\lambda(b_{02} - a_{02}))\,\bar{x}^2 - \bar{x}\bar{y} - \frac{1}{\lambda}\bar{y}^2, \\ \dot{\bar{y}} &= \bar{x} \qquad\qquad \lambda^2 a_{02}\bar{x}^2 + \frac{1}{\lambda}\bar{x}\bar{y} \quad . \end{aligned} \tag{10}$$

It may be noted that there is always a value for the parameter λ, satisfying condition (9), because it is known that the transformation to obtain the normal form can always be applied if the system has a fine focus (see [5]). Since the singular point in the origin of system (10) was supposed to be a third order fine focus, the first and second focal values have to be zero and the third one not. According to [5] these conditions give us the following relations between the parameters of system (10)

$$\begin{aligned} -1 &= 5\lambda^2 a_{02} \qquad ; \\ 2 &= \lambda^2(b_{02} - a_{02}). \end{aligned} \tag{11}$$

Now from (11) it is clear that the parameter a_{02} in (7) must be negative. On the other hand it is known that the S_2 is convex (as illustrated in Figure 13). Because the separatrices of the saddle node that are on S_2 are tangent to the x-axis and y-axis in the origin, the trajectories intersecting the positive y-axis do so as illustrated in Figure 13. That implies that $P(0,y) = a_{02}y^2 > 0$. This is in contradiction with the fact that a_{02} must be negative, so the conjecture that a S_2 may surround a third order fine focus is false.

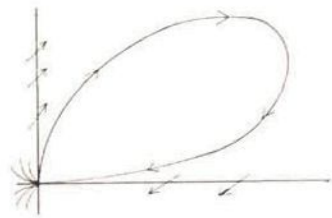


Figure 13.

The last part of the proof of theorem 3, part (ii) is the hardest. Suppose a quadratic system has a second order fine focus surrounded by a S_2. Without loss of generality one may assume that the saddle node is situated in the origin, that the quadratic system has the normal form given in (7) and that the second order fine focus is situated in (1,1).Then the parameters have to satisfy the conditions given in (8).

Since the singular part in (1,1) is a fine focus it is possible to transform system (7) into the normal form given in [5], by the transformation $x = 1 + \bar{y}$, $y = 1 + \lambda\bar{x} + \bar{y}$, where λ is a positive parameter which satisfies the condition

$$\begin{aligned} \mu_2 - \mu_1 &= \lambda^2\mu_1 \quad ; \\ \mu_1 &= 2a_{20} + a_{11} ; \\ \mu_2 &= 2b_{20} + b_{11} . \end{aligned} \tag{12}$$

Then with a scaling of the variables and using the conditions (8) and (12) the system can be put into the form (10).

It may be noted that there is always a value for λ, satisfying condition (12), because it is known that the transformation to obtain the normal form (10) can always be applied if the system has a fine focus (see [5]).

Since the singular point in the origin of system (10) was supposed to be a second order fine focus, the first focal value has to be zero and the second one does not equal zero. According to [5] these conditions give us the following necessary condition

$$1 - \lambda^2(b_{02} - a_{02}) = \lambda^4 \cdot a_{02}(a_{11} + 2b_{02}). \tag{13}$$

Since the saddle node is situated in the origin and the fine focus in (1,1) in system (7), the convex separatrix cycle is completely situated in the first quadrant of the x,y-plane, as illustrated in Figure 14. Because the separatrices of the saddle node that form the S_2 are tangent to the axis in the origin, the trajectories intersecting the y-axis do so as drawn in Figure 14. That implies that $P(0,y) = a_{02}y^2 \geq 0$.

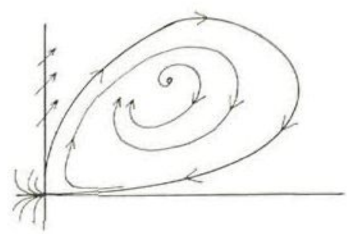


Figure 14.

Now one may find the stability of the second order fine focus by calculating the focal values as given in [5]. By using the conditions (11), (12) and (13) one may compute that the second focal value L(2) equals

$$L(2) = \frac{-2\lambda^5 a_{02}^3}{((\lambda(b_{02} - a_{02}) - \frac{1}{\lambda})^2 + (\lambda^2 a_{02})^2)^2} \cdot \frac{a_{20}}{2(a_{20} - \mu_1) - \mu_1} \cdot \frac{\mu_1}{a_{02}} \cdot \left(\frac{\mu_1}{a_{02}} - 5\right) > 0.$$

Thus the second order fine focus is unstable as illustrated in Figure 14.

Using part (iii) of theorem 3 it is clear that the S_2 in unstable and according to the *Poincaré-Bendixon Theorem there is at least one limit cycle inside the* S_2. Altogether it means that if a quadratic system has a S_2 which surrounds a second order fine focus, it simultaneously surrounds at least one limit cycle.Now it is easy to show that if a quadratic system has a second order fine focus surrounded by a S_2, it can be bifurcated such that a nest of at least four limit cycles appears. This bifurcation can be made in the following way. Suppose the quadratic system is put in the normal form given in [5].

First one can bifurcate the system such that the saddle node disappears. Of course the S_2 also disappears, generating one limit cycle. This bifurcation can be done in such a way that the second order fine focus remains a second order fine focus.

Then one can change the parameters such that the second order fine focus becomes a strong focus generating exactly two small limit cycles. Together with the limit cycle(s) which was in the original system and the limit cycle which was generated out of the S_2 the system has a nest of at least four limit cycles.

However this is in contradiction with conjecture 1, so it seems unlikely that a S_2 can surround a second order fine focus.

The proof of part (iii) of theorem 3 is simple since the argument can be used (p. 304 in [2]) that, if the separatrices of a singular point form a separatrix cycle, it is stable (unstable) if the divergence in the singular point is negative (positive).

6. BOUNDED SEPARATRIX CYCLES WITH ONE SINGULAR POINT: A THIRD ORDER SADDLE POINT WITH NONZERO DIVERGENCE

Theorem 4.

A quadratic system may have a bounded separatrix cycle which contains a third order saddle point with one zero eigenvalue (S_3).

(i) The region surrounded by a S_3 cannot contain a limit cycle.

(ii) A S_3 is stable (unstable) if the divergence in the saddle point is negative (positive).

(iii) A S_3 surrounds a strong focus.

Proof. It is no problem to give an example of a quadratic system with a S_3. In the classification of quadratic systems with a third order saddle point [16] several phase portraits are given with a S_3.

The first part of theorem 4 may be proved as follows. Suppose a quadratic system has a S_3 which surrounds at least one limit cycle. Then one can rotate the vector field in the same way as in (5). According to lemma 7 this rotation can be performed in such a way that the S_3 disappears and that at least one limit cycle is generated. It is clear that, under the condition that the change of the parameters is small enough, the limit cycles which were inside the S_3 in the original system, do not disappear. It can also be shown that the third order saddle does not disappear as a result of the bifurcation. Thus the bifurcated system has at least two limit cycles and a third order saddle point. However, this is in contradiction with the fact that a quadratic system with a third order saddle point has at most one limit cycle, as has been proved in [9].

The proof of part (ii) of theorem 4 is simple using the argument (p. 304 in [2]) that, if the separatrices of a singular point form a separatrix cycle, it is stable (unstable) if the divergence in the singular point is negative (positive).

Part (iii) of theorem 4 may be proved as follows. According to [8] a separatrix cycle surrounds either a strong focus or a fine focus. In [16] it was shown that a quadratic system with a third order saddle point with one eigenvalue zero may have a fine focus of at most order one. Now it will be shown that such a quadratic system with a first order fine focus cannot have a S_3 around it.

Suppose a quadratic system has a S_3 which surrounds a first order fine focus. One may put the system in the normal form as given in [5]. Without loss of generality one may assume that the third order saddle point is situated in (0,1). Since the system has no limit cycles (part (i) of theorem 4) one may assume that the fine focus is stable and that the S_3 is unstable. Then the system has the following form

$$\begin{aligned} \dot{x} &= -y + \lambda_1 x^2 + \lambda_2 xy + y^2 \equiv P(x,y), \\ \dot{y} &= x - \lambda_2 x^2 - xy \equiv Q(x,y), \end{aligned} \tag{14}$$

where $\lambda_1 < 0$ and $\lambda_2 > 0$.

Now one may consider the system

$$\begin{aligned} \dot{x} &= \cos\alpha \cdot P(x,y) + \sin\alpha \cdot Q(x,y), \\ \dot{y} &= \cos\alpha \cdot Q(x,y) - \sin\alpha \cdot P(x,y), \end{aligned} \tag{15}$$

where $P(x,y)$ and $Q(x,y)$ are given in equation (14).

It is clear that for $\alpha = 0$ system (14) equals system (15). Now system (15) may be bifurcated by making α positive. It is easy to check that this bifurcation causes an anti-clockwise rotation in all nonsingular points of the vector field. According to lemma 7 the S_3 disappears and at least one limit cycle is generated.

The stable first order fine focus becomes an unstable focus (Höpf bifurcation) and generates one limit cycle. It also can be shown that the third order saddle does not disappear as a result of the bifurcation. So the bifurcated system has a third order saddle point and more than one limit cycle. That is in contradiction with [9].

7. BOUNDED SEPARATRIX CYCLES WITH TWO SINGULAR POINTS

Theorem 5.

A quadratic system can have a bounded separatrix cycle with either two first order saddle points (S_4) or a first order saddle point and a second order saddle node (S_5).

(i) Neither a S_4 nor a S_5 can surround a limit cycle.

(ii) Both a S_4 and a S_5 surround a strong focus.

(iii) A S_4 is stable (unstable) if $\Sigma = 1 - \dfrac{\mu_1\mu_2}{\lambda_1\lambda_2}$ is negative (positive), where μ_1 and μ_2 are the positive eigenvalues of the saddle points on the S_4 and λ_1 and λ_2 the negative ones. The case $\Sigma = 0$ cannot occur.

(iv) A S_5 is stable (unstable) if the divergence in the saddle node is negative (positive).

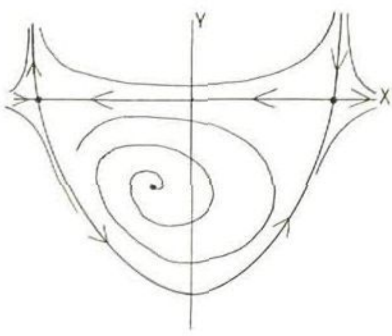
Proof. First an example of a S_4 and a S_5 will be given to show that quadratic systems can have such separatrix cycles. Consider the system

$$\begin{aligned} \dot{x} &= 1 + 2y - x^2 + \lambda xy , \\ \dot{y} &= 2\lambda y + 2xy + 2\lambda y^2, \end{aligned} \tag{15}$$

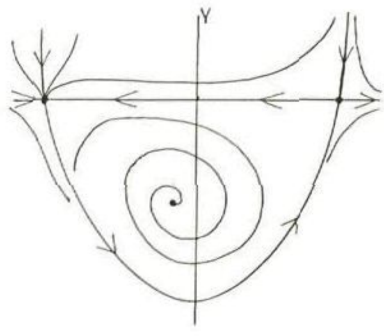
where $\lambda \in \{\frac{1}{2}, 1\}$.

Both for $\lambda = \frac{1}{2}$ and $\lambda = 1$ system (15) has the following properties. The x-axis and the parabola $y = x^2 - 1$ are trajectories of (15) with the exception of the points (1,0) and (-1,0) which are singular points. The singular point (1,0) is a saddle point and (-1,0) is a saddle point for $\lambda = \frac{1}{2}$ and a saddle node for $\lambda = 1$.

There is a singular point $(-\frac{1}{2}\lambda, -\frac{1}{2})$ situated in the region bounded by the parts of the x-axis and the parabola between (-1,0) and (1,0). This singular point is a stable focus. Now it is clear that the parts of the x-axis and the parabola between (-1,0) and (1,0) form a S_4 for $\lambda = \frac{1}{2}$ and a S_5 for $\lambda = 1$ (see Figures 15a and 15b).



a: $\lambda = \frac{1}{2}$



b: $\lambda = 1$

Figure 15.

Part (i) of theorem 5 may be proved as follows. Suppose a quadratic system has a S_4 or a S_5. Without loss of generality one may assume that the saddle point of the S_4 or S_5 is situated in the origin, the other singular point in (0,1) and that the separatrix cycle is stable and situated left of the y-axis. Then from lemma 4 it is known that the y-axis must be a trajectory of the system since the line segment between (0,0) and (0,1) is part of the separatrix cycle (see Figure 16).

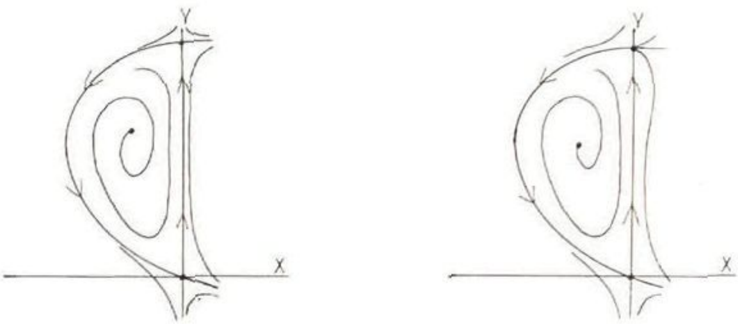


Figure 16.

After a scaling of t the system may be put in the following form

$$\begin{aligned} \dot{x} &= a_{10}x \qquad + a_{20}x^2 + a_{11}xy \qquad \equiv P(x,y), \\ \dot{y} &= b_{10}x + y + b_{20}x^2 + b_{11}xy - y^2 \equiv Q(x,y), \end{aligned} \tag{16}$$

where $a_{11} + a_{10} > 0$ if (0,1) is a saddle point and where $a_{11} + a_{10} = 0$ and $\frac{-a_{20}}{a_{11}} \neq b_{10} + b_{11}$ if (0,1) is a saddle node and where $a_{10} \in R^-$, $a_{11} \in R/\{0\}$ and $a_{20}, b_{10}, b_{11}, b_{20} \in \mathbb{R}$.

Now consider a small change of the parameter b_{10}, b_{20} and b_{11} then system (16) may be written as

$$\begin{aligned} \dot{x} &= a_{10}x \qquad + a_{20}x^2 + a_{11}xy \qquad \equiv \overline{P}(x,y), \\ \dot{y} &= \bar{b}_{10}x + y + \bar{b}_{20}x^2 + \bar{b}_{11}xy - y^2 \equiv \overline{Q}(x,y), \end{aligned} \tag{17}$$

and suppose that the condition that there is a $k \in \mathbb{R}/\{0\}$ such that $a_{10} = k(b_{10} - \bar{b}_{10})$, $a_{20} = k(b_{20} - \bar{b}_{20})$ and $a_{11} = k(b_{11} - \bar{b}_{11})$ is satisfied. To find the direction of the rotation of the vector field from (16) to (17) in each nonsingular point of the x,y-plane one may take the original system (16) and the bifurcated system (17) and calculate the innerproduct $W(x,y) = \overline{P}(x,y)Q(x,y) - P(x,y)\overline{Q}(x,y)$. After some calculations one may find that

$$W(x,y) = (a_{10}x + a_{20}x^2 + a_{11}xy)^2 \cdot k^{-1}. \tag{18}$$

It is clear that, with the exception of the ∞-isocline $P(x,y) = 0$ where the vector field does not rotate, the vector field rotates in the same direction as a result of the bifurcation. In which direction the vector field rotates, only depends on the sign of the constant k. It may be noted that since the vector field does not rotate on the ∞-isocline $P(x,y) = 0$, the y-axis remains a trajectory of the bifurcated system.

Now suppose that the separatrix cycle in system (16) surrounds one or more limit cycles. Obviously a small change of the parameters of the quadratic system will not cause the disappearance of these limit cycles.

If the sign of k is choosen properly then, according to lemma 7, the separatrix cycle disappears and at least one limit cycle is generated.

The result of this bifurcation is a quadratic system with a straight line solution (the y-axis) and at least two limit cycles. However, from [23] it is known that a quadratic system with a straight line solution has at most one limit cycle so a S_4 or a S_5 in a quadratic system will never surround a limit cycle.

Part (ii) of theorem 5 may be proved as follows. According to [8] a separatrix cycle surrounds either a strong focus or a fine focus. If it should be a fine focus it must be a first order fine focus since a quadratic system with a straight line solution (see lemma 4) may only have a fine focus of order one (see [23]). Now it will be proved that a S_4 or a S_5 never surround a first order fine focus.

Suppose a quadratic system has a S_4 or a S_5 which surrounds a first order fine focus. Without loss of generality one may assume that (a) the two singular points on the separatrix cycle are situated in (0,1) and (0,-1), (b) the separatrix cycle is stable, (c) the first order fine focus inside the separatrix cycle is situated in (1,0). As a result of these assumptions one knows that the y-axis is a trajectory (lemma 4) and that the first order fine focus is unstable (theorem 5(i)), as illustrated in Figure (17).

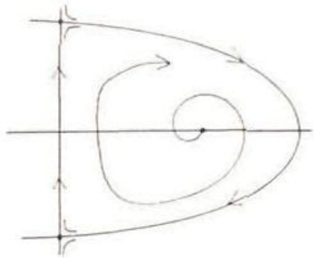


Figure 17.

A quadratic system satisfying these conditions must have the following form

$$\begin{aligned} \dot{x} &= \qquad a_{10}x - \qquad a_{10}x^2 + a_{11}xy \qquad \equiv P(x,y), \\ \dot{y} &= b_{00} + b_{10}x - (b_{00} + b_{10})\,x^2 + b_{11}xy - b_{00}y^2 \equiv Q(x,y), \end{aligned} \tag{19}$$

where $a_{10} - b_{11} = 0$ and $a_{10}, a_{11}, b_{10}, b_{11} \in R$ and $b_{00} \in R^+$.

Now consider a small change of the parameters b_{10} and b_{11}. Then system (19) may be written as

$$\begin{aligned} \dot{x} &= \qquad a_{10}x - \qquad a_{10}x^2 + a_{11}xy \qquad \equiv \overline{P}(x,y), \\ \dot{y} &= b_{00} + \bar{b}_{10}x - (b_{00} + \bar{b}_{10})\,x^2 + \bar{b}_{11}xy - b_{00}y^2 \equiv \overline{Q}(x,y), \end{aligned} \tag{20}$$

and let the condition be satisfied that there is a $k \in R/\{0\}$ such that $a_{10} = k\,(\bar{b}_{10} - b_{10})$ and $a_{11} = k\,(\bar{b}_{11} - b_{11})$.

Just as in the proof of part (i) of theorem 5 one may calculate the inner product $W(x,y) = \overline{P}(x,y)Q(x,y) - P(x,y)\overline{Q}(x,y)$, to show that for a properly choosen k the vector field rotates clockwise with the exception of the ∞-isocline $P(x,y) = 0$ where it does not rotate at all. According to lemma 7, the separatrix cycle in system (19) will disappear as a result of the change of the parameters, and at least one limit cycle will be generated out of it. Using the direction of the vector field in a neighbourhood of the point (1,0) one may show that the unstable first order fine focus in system (19) becomes a stable strong focus in system (20), generating exactly one limit cycle (Höpf bifurcation).

The total result of this is a quadratic system with a straight line solution and at least two limit cycles. However, from [23] it is known that a quadratic system with a straight line solution has at most one limit cycle so a S_4 or a S_5 will never surround a fine focus (see also [24]).

The proof of part (iii) of theorem 5 may be given by using [21]. In that paper it has been proved that a S_4 is stable (unstable) if $\Sigma = 1 - \frac{\mu_1\mu_2}{\lambda_1\lambda_2}$ is negative (positive), where λ_1 and λ_2 are the positive eigenvalues of the two saddle points on the S_4 and μ_1 and μ_2 the negative ones. The only case in which the stability of the S_4 is not determined, is when $\Sigma = 0$. However it will be shown that if $\Sigma = 0$, the quadratic system would not have a S_4.

Suppose a quadratic system has a S_4 for which holds that $\Sigma = 0$. Without loss of generality one may assume that the two saddle points on the S_4 are situated in (0,1) and (0,-1) and that the singular point inside the S_4 is situated in (1,0). According to lemma 4 the y-axis is a trajectory. A quadratic system satisfying these conditions has the following form

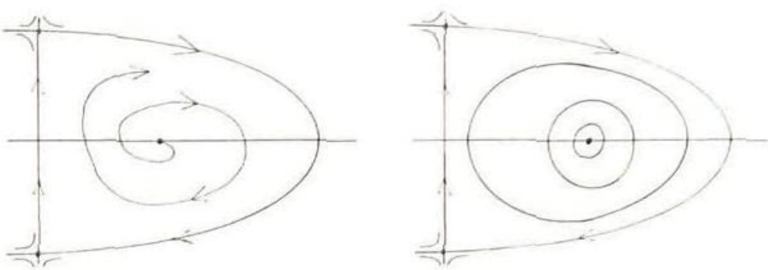
$$
\begin{aligned}
\dot{x} &= a_{11}xy \equiv P(x,y),\\
\dot{y} &= b_{00} + b_{10}x - (b_{00} + b_{10})\, x^2 + b_{11}xy - b_{00}y^2 \equiv Q(x,y),
\end{aligned}
\tag{21}
$$

where $b_{10}, b_{11} \in R$ and $a_{11}, b_{00} \in R^+$.

Now the question remains whether system (21) has a S_4 for certain values of the parameters.

Suppose system (21) has a S_4 for the parameter set $\bar{a}_{11}, \bar{b}_{00}, \bar{b}_{10}, \bar{b}_{11}$ (Figure 18a). Now one could change the parameter b_{11} from $b_{11} = \bar{b}_{11}$ to $b_{11} = 0$. It is easy to show that this variation of the parameter b_{11} results in a rotation of the vector field, with the exception of the x-axis and the y-axis where the field does not rotate. For $b_{11} = 0$ system (21) has a center in (1,0) and the separatrices of the saddle points in (0,1) and (0,-1) form a closed curve surrounding the center (Figure 18b).

It is clear that if system (21) has a S_4, the rotation of the vector field will never yield a system with a closed curve formed by the separatrices of the two saddle points, as given in Figure 18a. So the assumption that system (21) has a S_4 is false.



a b

Figure 18.

The last part of theorem 5 may be proved in the following way. Suppose a quadratic system has a S_5. Without loss of generality one may assume that the saddle node is situated in the origin and the saddle point in (0,1) and that the system has the form

$$\begin{aligned} \dot{x} &= \lambda_2 x + x^2 + a_{11}xy \equiv P(x,y), \\ \dot{y} &= y + (b_{20} + \lambda_1)\, x^2 + (b_{11} + a_{11}\lambda_1)\, xy - y^2 \equiv Q(x,y), \end{aligned} \tag{22}$$

where $\lambda_1 = \lambda_2 = 0$, $a_{11} \in R^+$, $b_{20} \in R^-$ and $b_{11} \in R$ (see Figure 19a).

In this form the divergence of the system is one in the saddle node. Now a small positive change of the parameter λ_1 results in a counter-clockwise rotation of the vector field with the exception of the points on the ∞-isocline $P(x,y) = 0$ where the field does not rotate. The result is that the S_5 disappears and that the separatrices are situated as drawn in Figure 19b. Analogously a small negative change of λ_1 results in the disappearance of the S_5 as drawn in Figure 19c.

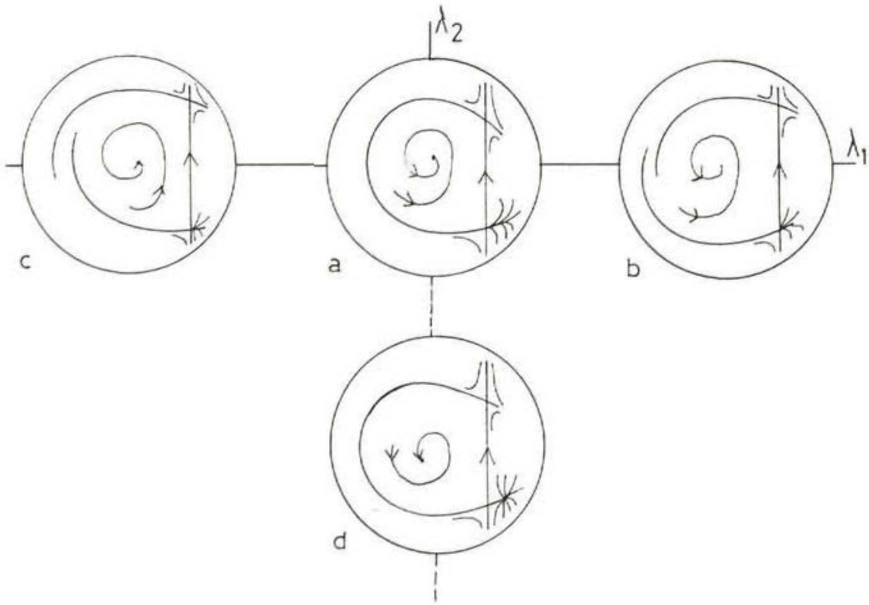


Figure 19.

If the parameter λ_2 becomes negative the singular point (0,0) becomes a saddle point and a node appears in a neighbourhood of the origin in the half plane $x > 0$.

Taking into account the relative position of the separatrices of the singular points (0,0) and (0,1) in the half plane $x < 0$ it may be clear that if the parameters are varied along a curve in the λ_1,λ_2-plane from a value corresponding to Figure 19b to one of Figure 19c, there must be at least one point such that the two separatrices coincide (independent of the path in the parameter plane).

Then if $\lambda_2 < 0$ and $|\lambda_2|$ sufficiently small, a S_4 is formed by the saddle points in

(0,0) and (0,1) as illustrated in Figure 19d for a point on a curve in the λ_1,λ_2-parameter plane, which is shown as a dashed curve. Thus a quadratic system with a S_4 may be bifurcated from a S_5.

Obviously the stability of the separatrix cycle does not change during such a bifurcation. So the stability of the S_5 may be found by calculating the stability of the S_4, according to theorem 5, part (iii). In system (22) one may calculate

$$\Sigma = 1 - \frac{\lambda_2 \cdot (-1)}{1 \cdot (\lambda_2 + a_{11})} > 0$$

if $\lambda_2 < 0$ and $|\lambda_2|$ is small enough. So the S_4 is unstable Thus one may conclude that a S_5 is unstable if the divergence in the saddle node on the S_5 is positive. Similarly it may be shown that a S_5 is stable if the divergence in the saddle node is negative.

8. AN EXAMPLE OF A QUADRATIC SYSTEM WITH AN UNBOUNDED SEPARATRIX CYCLE SURROUNDING AT LEAST TWO LIMIT CYCLES

In the previous sections a survey is given of all possible types of bounded separatrix cycles. For each type an example is given of a quadratic system having such a separatrix cycle. For some types of separatrix cycles a proof has been given how many limit cycles could be surrounded but for some other types just an indication of the maximum number of limit cycles could be given.

Now the question raises whether the same questions may be answered for unbounded separatrix cycles in quadratic systems. Bamón has already given a survey of all possible types of unbounded separatrix cycles [4], although he did not give a concrete example of a quadratic system for each type. The question how many limit cycles may be surrounded by each type of unbounded separatrix cycle has not been considered at all. The only known example of a quadratic system with an unbounded separatrix cycle which surrounds a limit cycle is given by Holmes. In [14] she investigated quadratic systems with an unbounded separatrix cycle which consists of a straight line path and the arc of the equator joining the ends of this line. According to [23] a quadratic system with a straight line solution has at most one limit cycle. So a separatrix cycle as investigated by Holmes may surround at most one limit cycle. She, however, also gave exact algebraic conditions for the existence and nonexistence of a limit cycle within the region surrounded by such a separatrix cycle.

The question remains whether a quadratic system may have an unbounded separatrix cycle which surrounds two or more limit cycles. In this section an example of a quadratic system with at least two limit cycles, surrounded by an unbounded separatrix cycle, will be presented.

Theorem 6.

Consider the quadratic system

$$\begin{aligned} \dot{x} &= \lambda x - y + ax^2 + bxy + y^2 \equiv P(x,y), \\ \dot{y} &= x + dx^2 + exy \equiv Q(x,y), \end{aligned} \tag{23}$$

where $|\lambda|$, $|a+4|$, $|b-5|$, $|d-1|$ and $|e+7|$ are sufficiently small. There exist values of these five parameters such that system (23) has an unbounded separatrix cycle with at least two limit cycles in its interior.

Proof. This theorem will be proved using the following four properties of system (23).

Property 1.

System (23) has two finite singular points: $P_1(0,1)$ and $P_2(0,0)$. P_1 is an unstable node and P_2 is a (fine) focus wherein the four focal values are given by

$$\mathrm{div}\,(P(0,0),Q(0,0)) = \lambda$$

$$L(1) = -\lambda_5$$

$$L(2) = \lambda_2\lambda_4(\lambda_4 - 5)$$

$$L(3) = -\lambda_2\lambda_4(\lambda_2^2 + (\lambda_3 - 1)(\lambda_3 - 2)).$$

$$\lambda_2 = \Delta^{-2}\,[-(e + 1)(a + 1)^2\, d + b(a + 1)\, d^2 + d^3]$$

$$\lambda_3 = \Delta^{-2}\,[a(a + 1)^3 + (d + b)(a + 1)^2\, d + (e + 1)(a + 1)\, d^2]$$

$$\lambda_4 = \Delta^{-1}\,[(a + 1)(e + 2a) + bd]$$

$$\lambda_5 = \Delta^{-1}\,[-(a + 1)\, b + (e + 2a)\, d]$$

where $\Delta = (a + 1)^2 + d^2$.

Proof. It is easy to show that system (23) has just two finite singular points and to find the topological structure of these points. With the expressions given in [5] the four focal values in P_2 may be calculated.

Property 2.

System (23) has three singular points at infinity: two saddle points P_3 and P_4 and a node P_5.

Proof. To determine the number of singular points at infinity one may use the Poincaré transformation [1]. Then it is easy to show that the third degree equation which must be solved to find the position of the singular points at infinity, has exactly three different solutions. The character of these three singular points may be found with the usual methods.

Property 3.

For $\lambda = a + 4 = b - 4{,}5 = d - 0{,}9 = e + 7 = 0$ the phase portrait of system (23) is as given in Figure 20.

Proof. Using property 1 it is easy to find that P_2 is an unstable third order fine focus. From [19] it is known that a third order fine focus in a quadratic system is never surrounded by a limit cycle. There are also no limit cycles surrounding P_1 since it is an unstable node (see [8]). With the help of property 2 and the general properties of quadratic system one may show that the phase portrait is as given in Figure 20. It may be noted that both the separatrix of P_3' and the separatrix of P_4' intersect the y-axis between P_1 and P_2 as shown in Figure 20.

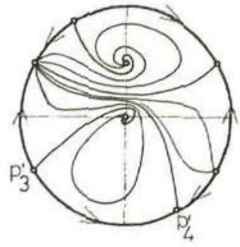


Figure 20.

Property 4.

For $\lambda = a + 4 = b - 5{,}5 = d - 1{,}1 = e + 7 = 0$ the phase portrait of system (23) is as given in figure 21.

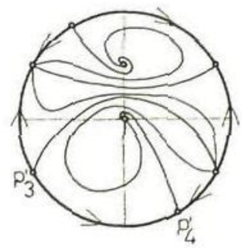


Figure 21.

Proof. If the same way as in property 3 one may conclude that P_1 is an unstable node, P_2 is a stable third order fine focus and that the system has no limit cycles. With the help of the usual methods one may draw the phase portrait. It may be noted that both the separatrices of P_3' and of P_4' intersect the y-axis between P_1 and P_2 as shown in Figure 21.

Conclusion. Using the properties 1, 2, 3 and 4 it is clear that if the parameters are varied from $\lambda = a + 4 = b - 4{,}5 = d - 0{,}9 = e + 7 = 0$ to $\lambda = a + 4 = b - 5{,}5 = d - 1{,}1 = e + 7 = 0$ there must be at least one set of parameter values such that

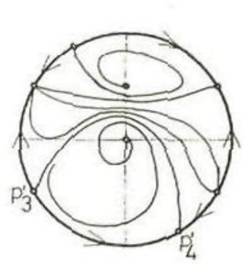


Figure 22.

the separatrices of P_3' and P_4' intersect the y-axis between P_1 and P_2 in the same point and this form an unbounded separatrix cycle. This variation of the parameters can be made in such a way that during the variation holds

$$\lambda = 0, \quad (a + 1)\, b = (e + 2a)\, d, \quad b - 5d < 0. \tag{24}$$

According to property 1 this implies that during the whole parameter variation P_2 is a stable second order fine focus (see Figure 22). What is happening is that if the parameters are varied, starting from the conditions mentioned in property 3 the unstable third order fine focus P_2 becomes a stable second order fine focus and exactly one limit cycle is generated out of it. During the parameter variation this limit cycle blows up until it disappears in the separatrix cycle (Figure 22). In this way an example is given of a quadratic system with an unbounded separatrix cycle surrounding a second order focus.

To give an example of a quadratic system with an unbounded separatrix cycle surrounding at least two limit cycles, one may also use the parameter variation from $\lambda = a + 4 = b - 4{,}5 = d - 0{,}9 = e + 7 = 0$ to $\lambda = a + 4 = b - 5{,}5 = d - 1{,}1 = e + 7 = 0$ in system (23). Again one may conclude that there is at least one set of parameter values during the parameter variation such that the separatrices of P_3' and P_4' in system (23) form an unbounded separatrix cycle. Instead of the conditions mentioned in (24) the parameter variation can be made under the conditions

$$\begin{gathered} \lambda < 0, \quad (e + 2a)\, d - b(a + 1) < 0, \quad b - 5d < 0, \\ |\lambda| << |L(1)| << |L(2)|, \quad \lambda \cdot L(1) < 0, \quad L(1) \cdot L(2) < 0, \end{gathered} \tag{25}$$

where L(1) and L(2) are defined in property 1. According to property 1 and [5], these conditions imply that during the parameter variation two small limit cycles surround

P_2 (besides the limit cycle which blows up *until it disappears in the separatrix cycle*). So, under the conditions mentioned in (25) there are sets of values of the parameters such that system (23) has an unbounded separatrix cycle surrounding at least two limit cycles (see Figure 23).

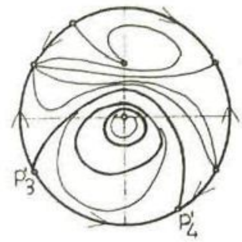


Figure 23.

REFERENCES

1. Andronov, A.A., E.A. Leontovich, J.J. Gordon, and A.G. Maier, *Qualitative theory of second-order dynamic systems*, Israel Program for Scientific Translation, Jeruzalem, Wiley, New York, 1973.
2. Andronov, A.A., J.J. Gordon, and A.G. Maier, *Theory of bifurcation of dynamical systems on a plane*, Israel Program for Scientific Translation, Jeruzalem, Wiley, New York, 197 .
3. Bamón, R., A class of planar quadratic vector fields with a limit cycle surrounded by a saddle loop, Proceedings of the American Mathematical Society, Vol. 88, No. 4 (1983), pp. 719-724.
4. Bamón, R., Solution of Dulac's problem for qudratic vector fields, Preprint, 1985, 46 pages.
5. Blows, T.R. and N.G. Lloyd, The number of limit cycles of certain polynomial differential equations, Proceedings of the Royal Society of Edinburgh 98A, 1984, pp. 215-239.
6. Cerkas, L.A., On the absence of limit cycles surrounding a third order fine focus in a planar quadratic system, *Journ. of Differential Equations*, Vol. 22, No. 11, pp. 2015- 2017 (in Russian).

7. Chicone, C. and D. Shafer, Separatrix and limit cycles of quadratic systems and Dulac's theorem, *Transactions of the American Mathematical Society*, Vol. 278, No. 2 (1983), pp. 585-612.

8. Coppel, W.A., A survey of quadratic systems, *Journ. of Differential Equations*, Vol. 2 (1966), pp. 293-304.

9. Coppel, W.A., The limit cycle configurations of quadratic systems, Research Report No. 32-1986, Mathematical Sciences Research Centre of the Australian National University, 1986.

10. Dong Jinzhu, The structure of the separatrix cycles of the system $\dot{x} = \sum_{0\leq i+k\leq 2} \cdot a_{ik}x^i y^k$, $\dot{y} = \sum_{0\leq i+k\leq 2} b_{ij}x^i y^k$, *Acta Math. Sinica*, Vol. 12 (1962), pp. 251-257 (Chinese), translated as *Chin. Math.*, Vol. 3 (1963), pp. 277-284.

11. Drachman, B., S. van Gils, and Zhang Zhi-fen, Abelian integrals for quadratic vector fields, Preprint, 1986, 17 pages.

12. Dulac, M.H., Sur les cycles limites, *Bull. Soc. Math. France*, Vol. 51 (1923), pp. 45-188.

13. Hilbert, D., *Mathematische Probleme*, Gottingen Nachrichten, 1900.

14. Holmes, C.A., Some quadratic systems with a separatrix cycle surrounding a limit cycle, Research Report CMA-K36-86, Centre for Mathematical Analysis of the Australian National University, 1986.

15. De Jager, P., Phase portraits for quadratic systems with a higher order singularity - II. A second order point with two zero eigenvalues, Report 88-32 of the Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1988.

16. De Jager, P., Phase portraits for quadratic systems with a higher order singularity - III. Third and fourth order points with zero eigenvalues, Report 88-79 of the Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1988.

17. De Jager, P. and J.W. Reyn, Phase portraits for quadratic systems with a higher order singularity - I. Third and fourth order points with two zero eigenvalues, Proc. First Intern. Conf. on Ind. and Appl. Math., ICIAM87, contributions from the Netherlands, Paris, 1987, CWI-Tract 36, Amsterdam, pp. 85-97.

18. De Jager, P. and J.W. Reyn, Addendum to Phase portraits for quadratic systems with a higher order singularity - I. Third and fourth order points with two zero eigenvalues, *Delft Progress Report*, Vol. 12 (1988), pp. 33-37.

19. Li, C., Non-existence of limit cycle around a weak focus of order three for any quadratic system, *Chinese Ann. Math.*, Vol. 78 (1986), pp. 174-190.

20. Liang, Z.J., On the critical cycles of a class of quadratic systems with a third order focus, *Journ. of Wuhan University*, Vol. 6, No. 4 (1986), pp. 411-417 (in Chinese).

21. Reyn, J.W., A stability criterion for separatrix polygons in the phase plane, *Nieuw Archief voor de Wiskunde* (3), XXVII (1979), pp. 238-254.

22. Vulpe, N.I., Affine-invariant conditions for the topological discrimination of quadratic systems with a center, *Diff. Uravn.*, Vol. 19, No. 3 (1983), pp. 371-379.

23. Ye Yanqian, Theory of limit cycles, Translations of mathematical monographs, Vol. 66, American Mathematical Society, Providence, Rhode Island, 1984.

24. Ye Yanqian, On the impossibility of (2,2) distribution of limit cycles of any real quadratic system, *Journ. of Nanjing Univ.*, 1985, pp. 161-182.

APPENDIX 1

program 1
```
10 SCREEN 2 : CLS : X0#=-100000! : Y0#=9790000!
20 FOR Z=1 TO 232 : X#=X0# : Y#=Y0# : GOSUB 160
30 DDX#=DX#-18*INT(DY#/1000) : DDY#=DY#+18*INT(DX#/1000)
40 DDX#=INT(DDX#/10) : DDY#=INT(DDY#/10)
50 IF ABS(DDX#)>999 OR ABS(DDY#)>999 THEN GOTO 40
60 G#=1
70 G#=2#*G# : GOSUB 180 : IF P#>0 THEN GOTO 70
80 G1#=1 : G2#=G#+1
90 G#=INT((G1#+G2#)/2) : GOSUB 180
100 IF P#>0 THEN G1#=G#
110 IF P#<0 THEN G2#=G#
120 IF G2#-G1#>2 THEN GOTO 90
130 X0#=X0#+G1#*DDX# : Y0#=Y0#+G1#*DDY#
140 PRINT X0#,Y0#,G1#,G2#,Z
150 NEXT Z : STOP
160 DX#=-100000000#*Y#+10#*X#*X#+25#*X#*Y#+10#*Y#*Y#
170 DY#=100000000#*X#+5#*X#*X#+80#*X#*Y# : RETURN
180 X#=X0#+G#*DDX# : Y#=Y0#+G#*DDY# : GOSUB 160
190 DX#=INT(DX#/100) : DY#=INT(DY#/100)
200 A#=DX#*DDY# : B#=DY#*DDX# : P#=A#-B# : RETURN
```

program 2
```
10 SCREEN 2 : CLS : X0#=50000! : Y0#=9349500!
20 FOR Z=1 TO 117 : X#=X0# : Y#=Y0# : GOSUB 160
30 DDX#=DX#+15*INT(DY#/1000) : DDY#=DY#-15*INT(DX#/1000)
40 DDX#=INT(DDX#/10) : DDY#=INT(DDY#/10)
50 IF ABS(DDX#)>999 OR ABS(DDY#)>999 THEN GOTO 40
60 G#=1
70 G#=2#*G# : GOSUB 180 : IF P#>0 THEN GOTO 70
80 G1#=1 : G2#=G#+1
90 G#=INT((G1#+G2#)/2) : GOSUB 180
100 IF P#>0 THEN G1#=G#
110 IF P#<0 THEN G2#=G#
120 IF G2#-G1#>2 THEN GOTO 90
130 X0#=X0#-G1#*DDX# : Y0#=Y0#-G1#*DDY#
140 PRINT X0#,Y0#,G1#,G2#,Z
150 NEXT Z : STOP
160 DX#=-100000000#*Y#+10#*X#*X#+100#*X#*Y#+10#*Y#*Y#
170 DY#=100000000#*X#+5#*X#*X#+380#*X#*Y# : RETURN
180 X#=X0#-G#*DDX# : Y#=Y0#-G#*DDY# : GOSUB 160
190 DX#=INT(DX#/100) : DY#=INT(DY#/100)
200 A#=DX#*DDY# : B#=DY#*DDX# : P#=B#-A# : RETURN
```

ACKNOWLEDGEMENT

The author of this thesis wishes to thank Prof.dr.ir. J.W. Reyn for his stimulating interest and his valuable suggestions.
This work has been supported by the Delftse Hogeschool Fonds through a stipendium awarded to the author.

CURRICULUM VITAE

De auteur van dit proefschrift werd op 11 mei 1961 geboren te Ridderkerk. Na in 1979 zijn V.W.O.-opleiding aan de Openbare Scholen Gemeenschap Walburg te Zwijndrecht te hebben voltooid, begon hij aan de studie wiskunde aan de Technische Hogeschool te Delft (thans genoemd Technische Universiteit Delft). Onder begeleiding van Prof.dr.ir. J.W. Reyn werd in de afstudeerfase onderzoek gedaan naar bifurcaties van separatrixlussen in kwadratische stelsels. In januari 1985 werd het diploma wiskundig ingenieur behaald. Op 1 februari 1985 kon hij onder begeleiding van Prof.dr.ir. J.W. Reyn beginnen aan zijn onderzoek, dankzij een stipendium toegekend door het Delfts Hogeschoolschool Fonds. Vanaf juni 1986 was hij verbonden aan de Technische Universiteit Delft waar hij het promotie-onderzoek afrondde. Tijdens zijn onderzoekperiode gaf hij enkele colleges differentiaalvergelijkingen, een practicum computer algebra en enkele voordrachten in binnen- en buitenland.