

**ON THE STRUCTURE
OF LATTICE ORDERED GROUPS**

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PROF. DR. F. LOONSTRA.**

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INTRODUCTION

In this chapter we briefly give the most important definitions and properties which are necessary for the understanding of the other chapter.

§ 1 Relations

Let A be a non empty set. The Cartesian product of a natural number n of copies of A will be denoted by A^n .

A relation ϱ on A is a subset of A^2 . We will call ϱ

<i>reflexive</i>	if $(a,a) \in \varrho$ for all $a \in A$,
<i>symmetric</i>	if $(a,b) \in \varrho$ implies $(b,a) \in \varrho$,
<i>antisymmetric</i>	if $(a,b) \in \varrho$ and $(b,a) \in \varrho$ imply $a = b$,
<i>transitive</i>	if $(a,b) \in \varrho$ and $(b,c) \in \varrho$ imply $(a,c) \in \varrho$.

A relation on A which is reflexive and transitive is called a *preorder* on A . A symmetric preorder is called an *equivalence relation* and an antisymmetric preorder is called a *partial order*.

In case ϱ is a partial order we write $a \leqslant b(\varrho)$ (a is smaller than b under the partial order ϱ) instead of $(a,b) \in \varrho$, and if no misunderstanding is possible we remove the sign (ϱ) and write $a \leqslant b$. If $a \leqslant b$ but not $a = b$, then we write $a < b$ (a is strictly smaller than b). Moreover we use the following notations: $b \geqslant a$ (resp. $b > a$) for $a \leqslant b$ (resp. $a < b$), $a \not\leqslant b$ for not $a \leqslant b$ etc. and finally $a \parallel b$ for $a \not\leqslant b$ and $b \not\leqslant a$. A set endowed with a preorder (resp. a partial order) is called a *preordered* (resp. *partially ordered*) set. A subset S of a partially ordered set A is called *convex* (with respect to the given partial order ϱ) if $a \leqslant b \leqslant c(\varrho)$ and $a, c \in S$ imply $b \in S$.

For a relation ϱ on A we define $\varrho^{-1} = \{(a,b) : (b,a) \in \varrho\}$. If ϱ is a partial order on A such that $\varrho \cup \varrho^{-1} = A^2$ we will say that ϱ is a *full order* on A and A is called a *fully ordered set* or a *chain*. A chain is called *well ordered* if every non-void subset B of A contains a smallest element i.e. B contains an element d such that $d \leqslant b$ for all $b \in B$. A finite chain is well ordered.

Let ϱ be an equivalence relation on A . As is known, ϱ induces a division of A into disjoint classes (of mutually equivalent elements) which is also called a *partition* of A . The set of these classes is denoted by A/ϱ . The class of the element a of A in this set is denoted by a^ϱ i.e. $a^\varrho = \{b \in A : (a,b) \in \varrho\}$. Conversely, a partition of A induces an equivalence relation on A . The correspondence between the equivalence relations on A and its partitions is one to one.

If π is a preorder on A , then $\varepsilon = \pi \cap \pi^{-1}$ is an equivalence relation on A and the

classes of ε are called the *classes of the preorder* π . The set A/ε can be endowed with a partial order in the following natural way: $a^\varepsilon \leq b^\varepsilon$ if and only if $(a,b) \in \pi$. In the sequel this process of passing from a preordered set A to the partially ordered set A/ε of the classes of the preorder will be used several times.

We say that a relation ϱ on A is smaller than a relation σ on A if $\varrho \subset \sigma$. This defines a partial order on the set of relations on a set A (the so called inclusion order). If the preorder π_1 on A is smaller than the preorder π_2 on A , then the equivalence relation $\varepsilon_1 = \pi_1 \cap \pi_1^{-1}$ is smaller than the equivalence relation $\varepsilon_2 = \pi_2 \cap \pi_2^{-1}$. The equivalence relation ϱ on A is smaller than the equivalence relation σ on A if and only if $a^\varrho \subset a^\sigma$ holds good for all $a \in A$. In this case we will say that the classes of ϱ are contained in the classes of σ . As a consequence we have: if the preorder π_1 is smaller than the preorder π_2 (both on the same set A), then the classes of π_1 are contained in the classes of π_2 . Partitions on a set A are ordered in correspondence with the ordering of the induced equivalence relations.

§ 2 Lattices

A *lattice* L is a non-empty set together with two binary operations $\vee$ and $\wedge$ satisfying:

- L1. $a \vee a = a$ and $a \wedge a = a$,
- L2. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$,
- L3. $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
- L4. $(a \vee b) \wedge a = a$ and $(a \wedge b) \vee a = a$.

L is called *distributive* if the following law holds.

- L5. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

In a distributive lattice we also have

- L5'. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

L5 and L5' are equivalent properties of a lattice.

The relation λ defined by $\lambda = \{(a,b) \in L^2 : a \wedge b = a\}$ is a partial order on L . Unless otherwise stated the symbol $\leq$ used in a lattice will always refer to this partial ordering. Then $a \vee b$ (resp. $a \wedge b$) turns out to be the *l.u.b.* or *join* (resp. the *g.l.b.* or *meet*) of a and b . If every non-empty subset of a lattice L has a g.l.b. and a l.u.b., then L is called a *complete* lattice; if every bounded subset has a g.l.b. and a l.u.b. we say that

L is *conditionally complete*. The partially ordered set of partitions on a set A is a complete lattice.

If L is a lattice with a minimal element 0 and a maximal element 1 and a unary operation $a \rightarrow a'$ such that

$$L6. \quad a \wedge a' = 0 \quad \text{and} \quad a \vee a' = 1,$$

then the lattice is called *complemented*; the element a' is called the *complement* of a .

The subset $[a, b] = \{x \in L : a \leq x \leq b\}$ of L is called a *closed interval* in L . It is a sublattice of L with a minimal element a and a maximal element b . A *relatively complemented* lattice is a lattice in which every closed interval is complemented. In such a lattice there exists for every element $c \in [a, b]$ an element c' such that $c \wedge c' = a$ and $c \vee c' = b$; c' is called the *complement* of c in $[a, b]$. A lattice is *sectionally complemented* if it has a minimal element 0 and every interval $[0, a]$ is complemented.

Of course a relatively complemented lattice with minimal element 0 is sectionally complemented and in a distributive lattice with minimal element 0 the converse is also true. A *Boolean algebra* is a distributive and a complemented lattice. A Boolean algebra is relatively complemented.

§ 3 Lattice ordered groups

A *partially ordered group* (*p.o.group*) is an additively written group G and at the same time a partially ordered set such that the *monotony law* is satisfied i.e. $a \leq b$ implies $a + c \leq b + c$ and $c + a \leq c + b$ for $a, b, c \in G$. The *positive cone* $P(G)$ (if no misunderstanding is possible, we write shortly P) of a *p.o.group* G consists of the elements $g \geq 0$ in G .

P has the following properties:

$$P1. \quad P \text{ is a semigroup with } 0,$$

$$P2. \quad P \cap -P = 0 \quad \text{with} \quad -P = \{x \in G : -x \in P\},$$

$$P3. \quad P \text{ is a normal subset of } G \text{ i.e. } g \in P \text{ implies } -x + g + x \in P \text{ for } x \in G.$$

For the rest of this paragraph G will stand for the *p.o.group* G . G is called *directed* if every two elements of G have an upperbound and a lowerbound in G . We have

$$P4. \quad G \text{ is directed, if and only if } P \text{ generates } G.$$

If G is a lattice and its partial order is the partial order λ of the preceding paragraph, then G is called a *lattice ordered group* (*l.group*).

In an l -group the monotony law is equivalent to

$$a + (b \vee c) + d = (a + b + d) \vee (a + c + d) \quad \text{for } a, b, c, d \in G.$$

An l -subgroup of an l -group G is a subgroup which is a sublattice of G . It should be observed that a subgroup of G which is a lattice under the induced order need not be an l -subgroup of G .

For the positive cone of an l -group we have

P5. G is an l -group, if and only if P generates G and P is a lattice under the induced order.

The properties of the positive cone decide whether G is fully ordered or not. This is seen from

P6. G is fully ordered, if and only if P generates G and P is fully ordered under the induced order.

G is called *Archimedean* if $a, b \in G$ and $na < b$ for all integers n implies $a = 0$. This means that $\{0\}$ is the only subgroup of an Archimedean l -group G , having an upper bound in G . For l -groups we have

P7. An l -group is Archimedean if and only if P is Archimedean in the following sense: $a, b \in P$ and $na < b$ for all $n \in \mathbb{N}$ implies $a = 0$.*

For later use we will give the following properties of an l -group G . For the proofs of A to H see e.g. Fuchs [4].

A. G is a distributive lattice.

B. $na \geq 0$ for $a \in G, n \in \mathbb{N}$, if and only if $a \geq 0$ in G .

C. $a \vee b = a - (a \wedge b) + b$ for $a, b \in G$. This implies: $a \vee b \leq a + b$, if and only if $a, b \in P$.

D. Two positive elements a and b of G are called *orthogonal* if $a \wedge b = 0$. From C one sees immediately that $a \vee b = a + b$, if and only if $a \wedge b = 0$ and also that orthogonal elements commute. Moreover, $a \wedge b = 0$ implies $ma \wedge nb = 0$ for $m, n \in \mathbb{N}$.

E. We define the *absolute value* $|a|$ of $a \in G$ as $|a| = a \vee -a$. Then we have the following properties:

* $\mathbb{N}$ is the set of natural numbers.

$$\begin{aligned}
|a| &> 0 \text{ for } a \neq 0, \text{ and } |0| = 0; \\
|-a| &= |a| \text{ and } |na| = n|a| \text{ for } n \in N; \\
|a-b| &= (a \vee b) - (a \wedge b).
\end{aligned}$$

If $|a|$ and $|b|$ are orthogonal then a and b commute.

The set of elements $x \in G$ such that $|x| \wedge a = 0$ for some fixed $a \in P$ is a convex l -subgroup of G .

F. The following identities hold in G .

If $\vee_\alpha a_\alpha$ (if $\wedge_\alpha a_\alpha$) exists in G (here $\vee$ and $\wedge$ may denote the l.u.b. resp. g.l.b. of an infinite set of elements) then:

1. $b + (\vee_\alpha a_\alpha) + c = \vee_\alpha (b + a_\alpha + c)$ and $b + (\wedge_\alpha a_\alpha) + c = \wedge_\alpha (b + a_\alpha + c)$
for $b, c \in G$,
2. $-(\vee_\alpha a_\alpha) = \wedge_\alpha (-a_\alpha)$ and $-(\wedge_\alpha a_\alpha) = \vee_\alpha (-a_\alpha)$.

Moreover the *infinite distributive laws* apply.

3. $b \wedge (\vee_\alpha a_\alpha) = \vee_\alpha (b \wedge a_\alpha)$ and $b \vee (\wedge_\alpha a_\alpha) = \wedge_\alpha (b \vee a_\alpha)$ for $b \in G$.

G. If $a, b_1, \dots, b_n$ are positive elements of G such that

$$a \leq b_1 + b_2 + \dots + b_n$$

then G contains positive elements $a_1, \dots, a_n$ satisfying

$$a = a_1 + a_2 + \dots + a_n$$

with $a_i \leq b_i$ ($i = 1, \dots, n$).

H. An Archimedean l -group is commutative. (We will give an original proof of this well-known property in the next chapter, § 7.) An Archimedean fully ordered group is isomorphic to a subgroup of the additive group of the real numbers with the usual ordering.

The following properties are new, it appears.

I. For $a, b \in P$ we have

$$na \wedge mb \leq (n+m-1)(a \wedge b) \leq (n+m)(a \wedge b) \text{ for } n, m \in N.$$

Proof. The last inequality is trivial and is mentioned for the sake of convenience. The proof of the first one follows from

$$(n+m-1)(a \wedge b) = (n+m-1)a \wedge (\dots) \wedge \dots \wedge (n+m-1)b.$$

Every factor $(\dots)$ in the right member contains either at least n times the element a or at least m times the element b . This means that such a factor is greater than na or mb and consequently greater than $na \wedge mb$. $\square$

COROLLARY. For $a, b \in P$ and $n \in N$ we have $nb \wedge a = n(a \wedge b) \wedge a$.

Proof. Since $a \wedge b \leq b$ we see that $n(a \wedge b) \wedge a \leq nb \wedge a$. We just proved $nb \wedge a \leq n(a \wedge b)$, hence $nb \wedge a \leq n(a \wedge b) \wedge a$. $\square$

We will also need the well-known formula $nb \wedge 0 = n(b \wedge 0)$ for all $b \in G, n \in N$.

- J. Further reflection on the infinite distributive laws under F yields the following consequences which play an essential role in the development of the theory of the $\aleph_0$ -classes in the next chapter.

Let $\{a_\alpha\}$ and $\{a_\beta\}$ be sets of elements of an l -group G . If:

1. $\vee_\beta(a_\beta \wedge a_\alpha)$ exist for all α and $\vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\}$ exists, then

$$\vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\} = \vee_{\alpha,\beta}(a_\alpha \wedge a_\beta),$$
2. $\vee_\beta(a_\beta \wedge a_\alpha) = a_\alpha$ for all α and $\vee_\alpha a_\alpha$ exists, then

$$\vee_\alpha a_\alpha = \vee_\beta\{a_\beta \wedge (\vee_\alpha a_\alpha)\} = \vee_{\alpha,\beta}(a_\alpha \wedge a_\beta),$$
3. $\vee_\beta a_\beta$ and $\vee_\alpha a_\alpha$ exist, then

$$(\vee_\alpha a_\alpha) \wedge (\vee_\beta a_\beta) = \vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\} = \vee_\beta\{\vee_\alpha(a_\beta \wedge a_\alpha)\} = \vee_{\alpha,\beta}(a_\alpha \wedge a_\beta).$$

Proof. 1. Of course $\vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\} \geq a_\alpha \wedge a_\beta$ for all α, β and if $d \geq a_\alpha \wedge a_\beta$ for all α, β , then $d \geq \vee_\beta(a_\beta \wedge a_\alpha)$ for all α . Hence $d \geq \vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\}$. Hence $\vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\}$ is the l.u.b. for the set of elements $\{a_\beta \wedge a_\alpha\}$.

2. $\vee_\beta(a_\beta \wedge a_\alpha)$ exists ($= a_\alpha$) for all α and $\vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\}$ exists ($= \vee_\alpha a_\alpha$). By property 1 this implies $\vee_{\alpha,\beta}(a_\alpha \wedge a_\beta) = \vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\} = \vee_\alpha a_\alpha$. In order to prove the remaining equality we observe that $\vee_\alpha a_\alpha \geq a_\beta \wedge (\vee_\alpha a_\alpha)$ for all β . If $d \geq a_\beta \wedge (\vee_\alpha a_\alpha)$ for all β , then $d \geq \vee_\beta(a_\beta \wedge a_\alpha) = a_\alpha$ for all α . Hence $d \geq \vee_\alpha a_\alpha$. This proves that $\vee_\alpha a_\alpha = \vee_\beta\{a_\beta \wedge (\vee_\alpha a_\alpha)\}$.

3. Since $\vee_\beta a_\beta$ exists, we have that $\vee_\beta(a_\beta \wedge a_\alpha) = a_\alpha \wedge (\vee_\beta a_\beta)$ (property F3 of this paragraph). Moreover, the existence of $\vee_\alpha a_\alpha$ implies (using the same property) $\vee_\alpha\{\vee_\beta(a_\beta \wedge a_\alpha)\} = \vee_\alpha\{a_\alpha \wedge (\vee_\beta a_\beta)\} = (\vee_\alpha a_\alpha) \wedge (\vee_\beta a_\beta)$. This proves the first equality of the above stated property. Because the conditions of the property are symmetric with respect to interchanging α and β , the second equality of the property follows immediately, while the third one is a consequence of property 1. $\square$

- K. Let $\{a_\alpha\}$, $\{b_\alpha\}$ and $\{c_\alpha\}$ be sets of elements of an l -group G such that $\vee_\alpha a_\alpha$ and $\vee_\alpha c_\alpha$ exist, then $\vee_\alpha a_\alpha = \vee_\alpha c_\alpha$ and $a_\alpha \leq b_\alpha \leq c_\alpha$ for all α imply $\vee_\alpha b_\alpha = \vee_\alpha a_\alpha$.
Proof. We have $b_\alpha \leq c_\alpha \leq \vee_\alpha c_\alpha$ for all α . Hence $\vee_\alpha c_\alpha$ is an upperbound for the set $\{b_\alpha\}$. Suppose d is an upperbound for the set $\{b_\alpha\}$ i.e. $d \geq b_\alpha$ for all α , then $d \geq a_\alpha$ for all α and this implies $d \geq \vee_\alpha a_\alpha = \vee_\alpha c_\alpha$. $\square$

§ 4 Homomorphisms

Let L and L' be lattices and φ a function from L onto L' . φ is said to be *isotone* if $x \leq y$ in L implies $\varphi(x) \leq \varphi(y)$. We use the following terms

join homomorphism if $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$,

meet homomorphism if $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ and

lattice homomorphism if φ is both a join and a meet homomorphism.

A join homomorphism as well as a meet homomorphism is an isotone function. If φ is a function from L onto L' such that $\varphi(\vee_\alpha a_\alpha) = \vee_\alpha \{\varphi(a_\alpha)\}$ whenever $\vee_\alpha a_\alpha$ exists in L , then φ is called a *suprema preserving homomorphism*. Clearly, a suprema preserving homomorphism is a join homomorphism. If G and G' are groups (semi-groups) and φ is a function from G onto G' such that $\varphi(x+y) = \varphi(x) + \varphi(y)$ then φ is called a *group homomorphism* (*semigroup homomorphism*).

In any of the preceding cases we use the following self-evident notations: $\varphi: x \rightarrow x'$ or $x' = \varphi(x)$ and $L' = \varphi(L)$ resp. $G' = \varphi(G)$; L' is called a join homomorphic image of L in case φ is a join homomorphism, etc. The word homomorphism is replaced by the word *isomorphism* if the function φ is one to one. If the function φ is one to one and L (resp. G) coincides with L' (resp. G') we replace the word homomorphism by the word *automorphism*.

We observe that if G and G' are l -groups, then the property of φ being a lattice (or join or meet) homomorphism from G onto G' need not imply that φ is a group (or semigroup) homomorphism from G onto G' . The converse need not be true either. We say that two homomorphisms φ and ψ of the l -group G onto the l -groups $\varphi(G)$ resp. $\psi(G)$ are of the same type if they are both join homomorphisms (or both meet homomorphisms etc.), even if the images $\varphi(G)$ and $\psi(G)$ do not coincide.

Let G be an l -group and let S be a class of all homomorphisms of the same type defined on G . If $\varphi \in S$, then $\varrho_\varphi = \{(x, y) \in G^2: \varphi(x) = \varphi(y)\}$ is an equivalence relation on G . We define the *classes of the homomorphism* φ as the classes of the corresponding equivalence relation ϱ_φ . We define the order relation for homomorphisms φ and ψ of the same type as follows: $\varphi \leq \psi$, if and only if $\varrho_\varphi \subset \varrho_\psi$ for the corresponding equivalence relations. This partial ordering on S has the following properties:

1. $\varphi \leq \psi$ for two homomorphisms φ and ψ of G (both of the same type) if and only if $\varphi(x) = \varphi(y)$ for $x, y \in G$ implies $\psi(x) = \psi(y)$ or, if and only if the classes of φ are contained in the classes of ψ .
2. $\varphi = \psi$, if and only if the mapping $\varphi(x) \rightarrow \psi(x)$ for $x \in G$ is an isomorphism from $\varphi(G)$ onto $\psi(G)$. In the special case of $\varphi(G)$ and $\psi(G)$ coinciding, we have $\varphi = \psi$, if and only if $\varphi(x) \rightarrow \psi(x)$ for $x \in G$ is an automorphism of $G' = \varphi(G)$.

If φ is a lattice homomorphism from a lattice L onto a lattice L' and L' has a minimal element $0'$, then the set of elements $x \in L$ such that $\varphi(x) = 0'$ is called the *kernel* of φ

and is denoted by $K(\varphi)$. In general φ is not determined by its kernel. However, if L is a sectionally complemented lattice, then the kernel does determine φ . This means that if φ and ψ are two lattice homomorphisms of a sectionally complemented lattice L onto the lattices $\varphi(L)$ and $\psi(L)$ and φ and ψ have the same kernel, then $\varphi = \psi$ in the sense of 2. above.

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§ 1 Archimedean classes

In this paragraph we will approach the notion of Archimedean classes of a lattice ordered group in another way and we will derive some new properties.

For an l .group G we define: $a \leq Nb$ for $a, b \in G$, if and only if there exists a natural number n such that $a \leq nb$. Otherwise stated: $a \leq Nb$ means that a finite multiple of b is greater than a .

It is easy to see that $\pi_N = \{(a, b) \in P^2 : a \leq Nb\}$ is a reflexive and transitive relation on the positive cone P of an l .group G . So π_N is a preorder on P . The classes of π_N (Intr., 1) are called the *Archimedean classes* of the l .group G , a^- denotes the Archimedean class of the element a . Trevisan [14] proved that for any l .group G the partially ordered set P/π_N of the Archimedean classes is a distributive lattice.*

A study of the Archimedean classes yields

THEOREM 1.1. *An Archimedean class is a subsemigroup and a convex sublattice of P .*

Proof. Let a_1 and a_2 ($a_1, a_2 \in P$) belong to the same Archimedean class a^- . This means $a_1 \leq n_1 a$ and $a_2 \leq n_2 a$, $a \leq m_1 a_1$ and $a \leq m_2 a_2$ for some $n_1, n_2, m_1, m_2 \in N$. Then we have

$$\begin{aligned} a &\leq m_1 a_1 \wedge m_2 a_2 \leq (m_1 + m_2) (a_1 \wedge a_2) & (\text{Intr., 3, I}) \\ &\leq (m_1 + m_2) (a_1 \vee a_2) \leq (m_1 + m_2) (a_1 + a_2), \end{aligned}$$

and also

$$a_1 \wedge a_2 \leq a_1 \vee a_2 \leq a_1 + a_2 \leq (n_1 + n_2) a.$$

This implies that both $a_1 \wedge a_2$ and $a_1 \vee a_2$ and $a_1 + a_2$ belong to a^- whenever we have $a_1, a_2 \in a^-$. So a^- is a subsemigroup and a sublattice of P . Suppose $a_1 \leq b \leq a_2$, then $a \leq m_1 a_1 \leq m_1 b$ and $b \leq a_2 \leq n_2 a$ which proves $b \in a^-$. Consequently a^- is convex. $\square$

The following theorem characterises the partition of P into Archimedean classes among the other partitions of P .

* The Archimedean classes were introduced by HAHN [6] for commutative fully ordered groups and by LOONSTRA [9] for commutative lattice ordered groups. Both paid more attention to the structure of the set of Archimedean classes than to the Archimedean classes themselves.

THEOREM 1.2. *The partition of P into Archimedean classes is the minimal partition of P into convex subsemigroups.*

Proof. The set of partitions of P into convex subsemigroups is not empty (e.g. P itself is an element of this set). Hence this set has a g.l.b. in the partially ordered set of all partitions (Intr., 2). This g.l.b. is a partition of P into convex subsemigroups since the intersection of any number of partitions of P into convex subsemigroups is a partition of P into convex subsemigroups. Let S be the class of the element a in this minimal partition. We will prove that $a^- \subset S$. Suppose $b \in a^-$ i.e. $b \in P$ such that $a \leq nb$ and $b \leq ma$ for some $n, m \in N$. Then we have $a \leq nb \leq nma$. Because S is a convex subsemigroup we have the following implications $a \in S \rightarrow nma \in S \rightarrow nb \in S$. If $b \notin S$, then there exists a class T such that $b \in T$ implying $nb \in T$ and therefore $S \cap T \neq \emptyset$ contradicting that S and T are classes of a partition. Consequently, we have $b \in S$. Hence $a^- \subset S$. By Theorem 1.1 the partition of P into Archimedean classes is a partition into convex subsemigroups and we just proved that this partition is smaller than the minimal partition under discussion. But it cannot be strictly smaller and this proves the theorem. $\square$

We continue with some theorems concerning the distributive lattice of Archimedean classes.

THEOREM 1.3. *The mapping $\varphi: a \rightarrow a^-$ from the positive cone P of an l.group G onto the distributive lattice of the Archimedean classes of G is a lattice homomorphism with kernel $K(\varphi) = \{0\}$, satisfying*

$$\varphi(a+b) = \varphi(a \vee b).$$

φ may be characterized as the minimal join homomorphism φ' of P which satisfies $\varphi'(a+b) = \varphi'(a \vee b)$.

Proof. The first part of the theorem is proved by showing that $(a \wedge b)^-$ and $(a \vee b)^-$ are the g.l.b. and the l.u.b. respectively of a^- and b^- . It is obvious that $(a \wedge b)^- \leq a^-$ and b^- . Assume that $c^- \leq a^-, b^-$, then we have (c, a) and $(c, b) \in \pi_N$ i.e. $c \leq n_1 a$ and $c \leq n_2 b$ for some $n_1, n_2 \in N$. If $m = \max(n_1, n_2)$, then $c \leq ma \wedge mb \leq 2m(a \wedge b)$ (Intr., 3, I). Therefore $c^- \leq (a \wedge b)^-$, establishing the first assertion.

In the second case $a^-, b^- \leq (a \vee b)^-$ is trivial and if $d^- \geq a^-, b^-$ then $k_1 d \geq a$, $k_2 d \geq b$ for some $k_1, k_2 \in N$, whence $ld \geq a \vee b$ for $l = \max(k_1, k_2)$. Consequently $d^- \geq (a \vee b)^-$.

It is immediately seen that $K(\varphi)$ is $\{0\}$ and we have $\varphi(a+b) = \varphi(a \vee b)$ because $a \vee b \leq a+b \leq 2(a \vee b)$ for $a, b \in P$ (Intr., 3, C).

To prove the last statement of the theorem, let φ' be any join homomorphism of P satisfying $\varphi'(a+b) = \varphi'(a \vee b)$. Then φ' is an isotone function which satisfies $\varphi'(na) = \varphi'(a)$ for $a \in P$. If $b \in a^-$, then we have $a \leq Nb$ and $b \leq Na$. Hence, $\varphi'(a) \leq$

$\leq \varphi'(b)$ and $\varphi'(b) \leq \varphi'(a)$. So $\varphi'(a) = \varphi'(b)$. This means that the Archimedean classes are contained in the classes of φ' or otherwise stated $\varphi \leq \varphi'$ (Intr., 4). But φ itself is a join homomorphism satisfying $\varphi(a+b) = \varphi(a \vee b)$. So it must be the minimal one. $\square$

THEOREM 1.4. *The lattice of Archimedean classes of an l.group G is a chain if and only if G is a fully ordered group.*

Proof. From Theorem 1.3 and the fact that the lattice isomorphic image of a fully ordered set (i.e. P) is a chain, follows the “if” part of the theorem. Suppose that the lattice of Archimedean classes contains two incomparable elements, say the Archimedean classes a^- and b^- , then it is clear that a and b themselves must be incomparable and so P cannot be fully ordered. An application of Intr., 3, P5 and P6 completes the proof. $\square$

A *strong unit* u of an l.group G is an element $u \in G$ such that for each $a \in G$ we have $a \leq Nu$ (Freudenthal [3]).

THEOREM 1.5. *The lattice of Archimedean classes of an l.group G has a maximal element, if and only if G has a strong unit; this maximal Archimedean class is the set of strong units of G .*

Proof. If u is a strong unit of G , then u is a positive element of G (Intr., 3, B). But then u^- is the maximal Archimedean class. Conversely, if $(u')^-$ is the maximal Archimedean class, then u' is a positive element of G such that for any $a \in P$ some $nu' < a$. But then for any $b \in G$ we have $b \leq b \vee 0 \leq Nu'$. Hence u' is a strong unit. $\square$

We finish this paragraph with a lemma which clarifies the use of words in the next paragraph, where we will introduce the term “infinite multiple of an element”.

LEMMA 1.1. *In an l.group G we have: $a \leq Nb$ for $a, b \in P$, if and only if $a = \bigvee_{n=1}^k (nb \wedge a)$ for some $k \in N$.*

Proof. Because a and b are positive, we have $b \wedge a \leq 2b \wedge a \leq \dots \leq kb \wedge a$. Hence $\bigvee_{n=1}^k (nb \wedge a) = kb \wedge a$. But then $a = \bigvee_{n=1}^k (nb \wedge a)$ implies $a = kb \wedge a$ i.e. $a \leq kb$ or $a \leq Nb$. This proves the “if” part of the lemma. Reading the proof in opposite direction yields the “only if” part of this lemma. $\square$

§ 2 $\aleph_0$ -classes

In this paragraph we introduce a partition of P which shows a remarkable correspondence with the division of P into Archimedean classes.

For an l .group G we define: $a \leq \aleph_o b$ for $a, b \in G$, if and only if $a = \bigvee_{n=1}^{\infty} (nb \wedge a)$.

Remarks. We emphasize that the expression “ $\aleph_o b$ ” has no meaning in itself, but that only the meaning of the statement “ $a \leq \aleph_o b$ ” is defined. Loosely speaking one might say that $a \leq \aleph_o b$ means that “a countable infinite multiple of b is greater than a ” (c.f. also Lemma 1.1).

We start with a sequence of lemmas which give the rules for the calculus on the new notion $a \leq \aleph_o b$.

LEMMA 2.1. *If in an l .group $a \leq Nb$ for $a, b \in P$, then $a \leq \aleph_o b$.*

Proof. Let $a \leq Nb$ for some $a, b \in P$ i.e. $a \leq kb$ for some $k \in N$. This implies $a \leq (k+p)b$ for all $p \in N$, and so $a = \bigvee_{n=1}^{k+p} (nb \wedge a)$ for all $p \in N$ (Lemma 1.1). Otherwise written $a = \bigvee_{n=1}^{\infty} (nb \wedge a)$ or $a \leq \aleph_o b$. $\square$

COROLLARY 2.1. *For any positive element a of an l .group we have $ka \leq \aleph_o a$ for all $k \in N$.*

The following example shows that the converse of this lemma is not true. Hence $a \leq \aleph_o b$ makes sense.

EXAMPLE 2.1. G is the additive group of all real valued continuous functions on $[0, 1]$. P is the set of functions $f \in G$ with $f(x) \geq 0$. Then G is an l .group. Let f be the function $f(x) = |x - \frac{1}{2}|$ on $[0, 1]$ and let g be the function $g(x) = 1$ on $[0, 1]$. We certainly have not $g \leq Nf$ (no finite multiple of f is greater than g) but we do have $g = \bigvee_{n=1}^{\infty} (nf \wedge g)$, for g is the least continuous function on $[0, 1]$ that is greater than all the functions $nf \wedge g$, $n \in N$. $\square$

The negation of $a \leq \aleph_o b$ will be denoted by $a \not\leq \aleph_o b$. It has a consequence which is used several times in this paragraph.

LEMMA 2.2. *If in an l .group G $a \not\leq \aleph_o b$ for $a, b \in P$, then there is an element $d \in G$ such that $nb \wedge a \leq d < a$ for all $n \in N$.*

Proof. We show that if we assume that there is no such d , then we would have $a = \bigvee_{n=1}^{\infty} (nb \wedge a)$. Indeed, if c is an upperbound for the elements $nb \wedge a$ ($n \in N$), then $a \wedge c$ is also an upperbound. Our assumption implies $a \wedge c \not\leq a$, but then $a \wedge c = a$ i.e. $c \geq a$. Since a is an upperbound for the elements $nb \wedge a$, a must be the least upperbound of these elements. $\square$

If in an l .group $a \leq Nc$ and $b \leq Nc$ for $a, b, c \in P$, then $a + b \leq Nc$. The following lemma concerns “infinite multiples” and is analogous to the preceding inference.

LEMMA 2.3. *If in an l.group $a \leq \aleph_o c$ and $b \leq \aleph_o c$ for $a, b, c \in P$ then $a+b \leq \aleph_o c$.*

Proof. Suppose that $a \leq \aleph_o c$, $b \leq \aleph_o c$ and $a+b \not\leq \aleph_o c$ for $a, b, c \in P$. Then, by Lemma 2.2, there exists an element d such that $kc \wedge (a+b) \leq d < a+b$ for all $k \in N$.

This implies

$$\begin{aligned} (nc \wedge a) + (mc \wedge b) &= (n+m)c \wedge (a+mc) \wedge (nc+b) \wedge (a+b) \\ &\leq (n+m)c \wedge (a+b) \leq d \end{aligned} \quad \text{for all } n, m \in N.$$

So we find

$$\begin{aligned} (nc \wedge a) + b &= (nc \wedge a) + \bigvee_{m=1}^{\infty} (mc \wedge b) \quad (\text{Intr., 3, F1}) \\ &= \bigvee_{m=1}^{\infty} \{(nc \wedge a) + (mc \wedge b)\} \leq d \quad \text{for all } n \in N. \end{aligned}$$

Hence

$$d \geq \left\{ \bigvee_{n=1}^{\infty} (nc \wedge a) \right\} + b = a+b \quad (a \leq \aleph_o c).$$

This indeed contradicts $d < a+b$. The conclusion is that $a \leq \aleph_o c$ and $b \leq \aleph_o c$ imply $a+b \leq \aleph_o c$. $\square$

COROLLARY 2.2. *If in an l.group $a \leq \aleph_o b$ for $a, b \in P$, then $na \leq \aleph_o b$ for all $n \in N$.*

Proof. This follows by induction, for, according to Lemma 2.3, $(n-1)a \leq \aleph_o b$ and $a \leq \aleph_o b$ imply $na \leq \aleph_o b$. $\square$

Another property on “finite multiples” which occurs analogously in the case of “infinite multiples” is: if in an l.group $a \leq Nb$ and $b \leq Nc$ for $a, b, c \in P$ then $a \leq Nc$. We prove

LEMMA 2.4. *If in an l.group $a \leq \aleph_o b$ and $b \leq \aleph_o c$ for $a, b, c \in P$, then $a \leq \aleph_o c$.*

Proof. From $b \leq \aleph_o c$ and Corollary 2.2 we know $\bigvee_{m=1}^{\infty} (mc \wedge nb) = nb$.
Then

$$\bigvee_{m=1}^{\infty} \{mc \wedge (nb \wedge a)\} = \bigvee_{m=1}^{\infty} \{(mc \wedge nb) \wedge a\} = nb \wedge a \quad \text{for all } n \in N \quad (\text{Intr., 3, F3})$$

We now apply Intr., 3, J2 substituting for the set of elements $\{a_\alpha\}$ the set $\{nb \wedge a, n \in N\}$ and for the set $\{a_\beta\}$ the set $\{mc, m \in N\}$. Because $\bigvee_{n=1}^{\infty} (nb \wedge a)$ exists ($=a$), and since we just proved $\bigvee_{m=1}^{\infty} \{mc \wedge (nb \wedge a)\} = nb \wedge a$ for all $n \in N$ we have

$$\begin{aligned}
a &= \bigvee_{n=1}^{\infty} (nb \wedge a) \\
&= \bigvee_{m=1}^{\infty} [mc \wedge \{\bigvee_{n=1}^{\infty} (nb \wedge a)\}] \\
&= \bigvee_{m=1}^{\infty} (mc \wedge a).
\end{aligned}$$

Hence $a \leq \aleph_o c$. $\square$

COROLLARY 2.3. *If in an l.group $a \leq b$ and $b \leq \aleph_o c$ for $a, b, c \in P$, then $a \leq \aleph_o c$.*

COROLLARY 2.4. *If in an l.group $a \leq \aleph_o b$ and $b \leq c$ for $a, b, c \in P$, then $a \leq \aleph_o c$.*

Proofs. Both corollaries are proved in the same way. We prove the first one. $a \leq b$ implies $a \leq \aleph_o b$ (Lemma 2.1) and an application of the foregoing lemma gives the desired result. $\square$

LEMMA 2.5. *If in an l.group $a \leq \aleph_o b$ and $a \leq \aleph_o c$ for $a, b, c \in P$, then $a \leq \aleph_o (b \wedge c)$.*

Proof. The assumptions of the lemma mean $a = \bigvee_{n=1}^{\infty} (nb \wedge a)$ and $a = \bigvee_{m=1}^{\infty} (mc \wedge a)$ for $a, b, c \in P$. From $a = a \wedge a = \{\bigvee_{n=1}^{\infty} (nb \wedge a)\} \vee \{\bigvee_{m=1}^{\infty} (mc \wedge a)\}$ and Intr., 3, J3 we conclude

$$a = \bigvee_{n,m=1}^{\infty} (nb \wedge mc \wedge a).$$

From this and (Intr., 3, I)

$$nb \wedge mc \wedge a \leq \{(n+m)(b \wedge c)\} \wedge a \leq a$$

it follows that (Intr., 3, K)

$$a = \bigvee_{n,m=1}^{\infty} \{(n+m)(b \wedge c)\} \wedge a = \bigvee_{k=1}^{\infty} \{k(b \wedge c) \wedge a\}$$

i.e. $a \leq \aleph_o (b \wedge c)$. $\square$

COROLLARY 2.5. *If in an l.group $a \leq \aleph_o b$ and $a \leq \aleph_o c$ for $a, b, c \in P$, then $a \leq \aleph_o (b \vee c)$ and $a \leq \aleph_o (b+c)$.*

Proof. Since $b \wedge c \leq b \vee c$, we have (Lemma 2.1) $b \wedge c \leq \aleph_o (b \vee c)$. From Lemma 2.5 we know that $a \leq \aleph_o (b \wedge c)$ and Lemma 2.4 leads to the conclusion $a \leq \aleph_o (b \vee c)$. The proof of $a \leq \aleph_o (b+c)$ is analogous. $\square$

THEOREM 2.1. *The relation $\pi_{\aleph_o} = \{(a,b) \in \mathcal{Q}^2 : a \leq \aleph_o b\}$ is a preorder on the set of positive elements of an l.group.*

Proof. From Corollary 2.1 we know that $a \leq \aleph_o a$ for all $a \in P$. Hence the relation $\pi_{\aleph_o}$ is reflexive. From Lemma 2.4 we know that the relation $\pi_{\aleph_o}$ is transitive. $\square$

The classes of $\pi_{\aleph_o}$ (Intr., 1) will be called the $\aleph_o$ -classes of the l.group G ; the class of the element $a \in P$ will be denoted by a° .

Our first aim will be to prove the analogs of the Theorems 1.1 and 1.2 on Archimedean classes. In order to get an adequate description we will introduce a closure operation in a lattice (i.e. a unary operation $S \rightarrow [S]$ on the set of subsets of the lattice such that $S \subset [S]$, $[S] = [[S]]$ and $S \subset T$ implies $[S] \subset [T]$). The concept is due to Riesz [12]. It concerns the correspondence

$$S \rightarrow [S] = \{b : b \in S \text{ or } b = \vee_\alpha a_\alpha \text{ with } a_\alpha \in S\}$$

for subsets of a lattice. A subset S is called *closed* if $S = [S]$. We formulate

THEOREM 2.2. *An $\aleph_o$ -class is a subsemigroup and a closed convex sublattice of P .*

Proof. Suppose $a^\circ = b^\circ$ for $a, b \in P$. This means $a \leq \aleph_o b$ and $b \leq \aleph_o a$. From $a \wedge b \leq a$ we conclude $a \wedge b \leq \aleph_o a$ (Lemma 2.1). From $a \leq \aleph_o a$ and $b \leq \aleph_o a$ we conclude $a + b \leq \aleph_o a$ (Lemma 2.3) and since $a \vee b \leq a + b$ (Intr., 3, C), we have $a \vee b \leq \aleph_o a$ (Corollary 2.3). Moreover, $a \leq \aleph_o a$ (Corollary 2.1) and $a \leq \aleph_o b$ imply $a \leq \aleph_o(a \wedge b)$ (Lemma 2.5), $a \leq \aleph_o(a \vee b)$ and $a \leq \aleph_o(a + b)$. Together with the first part of the proof these show that $a^\circ = b^\circ$ leads to $(a \wedge b)^\circ = (a \vee b)^\circ = (a + b)^\circ = a^\circ$. If we have $a \leq c \leq b$ and $a^\circ = b^\circ$, then $a^\circ \leq c^\circ$ (since $a \leq \aleph_o c$ i.e. $(a, c) \in \pi_{\aleph_o}$) and $c^\circ \leq b^\circ$. From this it is obvious that $a^\circ = c^\circ$, which proves the convexity of an $\aleph_o$ -class.

It remains to prove that an $\aleph_o$ -class of an element is closed. Suppose that $\vee_\alpha a_\alpha$ exists for a set of elements $\{a_\alpha\}$, such that $(a_\alpha)^\circ = a^\circ$ for all α . From $a \leq \aleph_o a_\alpha$ we conclude $a \leq \aleph_o(\vee_\alpha a_\alpha)$ (Corollary 2.4). On the other hand if, in property J2 of Intr., 3, we replace the set of elements $\{a_\beta\}$ by the set $\{na, n \in N\}$, then $\bigvee_{n=1}^\infty (na \wedge a_\alpha) = a_\alpha$ for all α and $\vee_\alpha a_\alpha$ exists. Hence $\vee_\alpha a_\alpha = \bigvee_{n=1}^\infty \{na \wedge (\vee_\alpha a_\alpha)\}$. This means $\vee_\alpha a_\alpha \leq \aleph_o a$. Consequently $\vee_\alpha a_\alpha \in a^\circ$. $\square$

The analogue of Theorem 1.2 is

THEOREM 2.3. *The partition of P into $\aleph_o$ -classes is the minimal partition of P into closed convex subsemigroups.*

Proof. P can be considered a closed convex subsemigroup of P . Therefore, the set of partitions of P into closed convex subsemigroups is not empty and consequently it has a g.l.b. in the p.o. set of all partitions of P . (Intr., 2). This g.l.b. is a partition of P into closed convex subsemigroups, because the intersection of any number of partitions of P into closed convex subsemigroups is a partition of the same type.

Let S be the class of the element $a \in P$ in this minimal partition. We will show that the $\aleph_o$ -class a° is contained in S . Let $b \in a^\circ$, then $a \leq \aleph_o b$ and $b \leq \aleph_o a$ i.e. $a = \bigvee_{n=1}^{\infty} (nb \wedge a)$ and $b = \bigvee_{n=1}^{\infty} (na \wedge b)$. From Intr., 3, I we know

$$a \wedge b \leq nb \wedge a \leq n(a \wedge b) \quad \text{and} \quad a \wedge b \leq na \wedge b \leq n(a \wedge b).$$

Consequently, the classes of the partition under discussion being convex subsemigroups, we see that $na \wedge b$ and $nb \wedge a$ belong to the same class for all $n \in N$. From the fact that this class is closed we see that: $\bigvee_{n=1}^{\infty} (na \wedge b) = b$ and $\bigvee_{n=1}^{\infty} (nb \wedge a) = a$ belong to the same class. So indeed $a^\circ \subset S$. The partition of P into $\aleph_o$ -classes is a partition of P into closed convex subsemigroups (Theorem 2.2) and as we just proved it is contained in the minimal partition into closed convex subsemigroups. But then it is the minimal partition. $\square$

From Intr., 1 we know that the set of $\aleph_o$ -classes, being the set of classes of a pre-order, can be partially ordered in a natural way. We intend to show that $(a \vee b)^\circ$ and $(a \wedge b)^\circ$ are the l.u.b. and g.l.b. respectively of a° and b° with respect to this partial order. Clearly $(a \vee b)^\circ \geq a^\circ$ and b° . If $c^\circ \geq a^\circ$ and b° , then $a \vee b \leq a + b \leq \aleph_o c$ (Intr., 3, C; Lemma 2.3 and Corollary 2.3), and consequently $(a \vee b)^\circ \leq c^\circ$. This proves that $(a \vee b)^\circ = a^\circ \vee b^\circ$. In the second case $(a \wedge b)^\circ \leq a^\circ, b^\circ$ is trivial. If $c^\circ \leq a^\circ, b^\circ$, we have $c \leq \aleph_o a$ and $c \leq \aleph_o b$ and thus $c \leq \aleph_o(a \wedge b)$ (Lemma 2.5) or $c^\circ \leq (a \wedge b)^\circ$. Consequently $(a \wedge b)^\circ = a^\circ \wedge b^\circ$. This proves that the partially ordered set of $\aleph_o$ -classes is a lattice. Now we are able to formulate

THEOREM 2.4. *The mapping $\alpha: a^- \rightarrow a^\circ$ from the lattice of Archimedean classes of an l.group G onto the lattice of the $\aleph_o$ -classes of G is a lattice homomorphism with kernel $K(\alpha) = \{0^-\}$.*

Proof. First of all we must show that α maps one Archimedean class upon one $\aleph_o$ -class i.e. α is a function. For that purpose suppose $a^- \leq b^-$ i.e. $a \leq Nb$. This implies (Lemma 2.1) $a \leq \aleph_o b$ or $a^\circ \leq b^\circ$. Because $a^- = b^-$ can be interpreted as $a^- \leq b^-$ and $b^- \leq a^-$, this statement implies $a^\circ \leq b^\circ$ and $b^\circ \leq a^\circ$ i.e. $a^\circ = b^\circ$. Hence α is a function from the lattice of Archimedean classes to the lattice of $\aleph_o$ -classes of G . That α is onto is trivial.

Above we derived that $(a \vee b)^\circ = a^\circ \vee b^\circ$. This means that α maps $a^- \vee b^- = (a \vee b)^-$ upon $(a \vee b)^\circ = a^\circ \vee b^\circ$ or in words, α is a join homomorphism. In the same way we

can demonstrate that α is a meet homomorphism. But then α is a lattice homomorphism.

It is immediately seen that the kernel $K(\alpha)$ of α is $\{0^-\}$. $\square$

COROLLARY 2.6. *The partially ordered set of the $\aleph_0$ -classes of an l.group is a distributive lattice.*

Proof. The lattice of Archimedean classes is distributive (introductory remarks of the second chapter, § 1) and a lattice homomorphic image of such a lattice is also distributive. $\square$

The mapping $\psi: a \rightarrow a^\circ$ from the positive cone of an l.group G onto the lattice of $\aleph_0$ -classes of G can be considered as the result of first applying the homomorphism φ of Theorem 1.3 on P and next the homomorphism α of Theorem 2.4 on the image (i.e. the lattice of Archimedean classes of G). In this sense ψ might be called the product of φ and α i.e. $\psi = \alpha \cdot \varphi$. It is seen from Theorem 1.3 that $\psi(a+b) = \alpha \cdot \varphi(a+b) = \alpha \cdot \varphi(a \vee b) = \psi(a \vee b)$. Moreover, the kernel of φ is $\{0\}$ and the kernel of α is $\{0^-\}$. Hence the kernel of ψ is $\{0\}$. This proves the first part of

THEOREM 2.5. *The mapping $\psi: a \rightarrow a^\circ$ from the positive cone P of an l.group G onto the distributive lattice of the $\aleph_0$ -classes is a lattice homomorphism with kernel $K(\psi) = \{0\}$, satisfying*

$$\psi(a+b) = \psi(a \vee b).$$

ψ can be characterized as the minimal suprema preserving homomorphism ψ' of P which satisfies $\psi'(a+b) = \psi'(a \vee b)$.

Proof. Let $a = \vee_\alpha a_\alpha$ for elements $a_\alpha \in P$. We prove that $\psi(a) = \vee_\alpha \psi(a_\alpha)$ or otherwise stated that a° is the l.u.b. for the elements $(a_\alpha)^\circ$ in the lattice of $\aleph_0$ -classes. It is clear that a° is an upperbound for the $(a_\alpha)^\circ$. Let b° be any upperbound for the $(a_\alpha)^\circ$, then $a_\alpha \leq \aleph_0 b$ or $a_\alpha = \bigvee_{n=1}^\infty (nb \wedge a_\alpha)$ for all α . This implies

$$a = \vee_\alpha a_\alpha = \bigvee_{n=1}^\infty \{nb \wedge (\vee_\alpha a_\alpha)\} = \bigvee_{n=1}^\infty (nb \wedge a)$$

(by Intr., 3, J2,) the set $\{a_\alpha\}$ being replaced by the set $\{nb, n \in N\}$) or $a^\circ \leq b^\circ$. So indeed, a° is the least upperbound for the $(a_\alpha)^\circ$ i.e. ψ is suprema preserving.

Suppose ψ' is a suprema preserving homomorphism of P which satisfies $\psi'(a+b) = \psi'(a \vee b)$. Then ψ' is also a join homomorphism, satisfying $\psi'(a+b) = \psi'(a \vee b)$. Consequently the homomorphism φ of Theorem 1.3 is smaller than ψ' .

For $a, b \in P$ we have $\varphi(a \wedge b) = \varphi(na \wedge b)$ for all $n \in N$. Hence $\psi'(a \wedge b) = \psi'(na \wedge b)$ for all $n \in N$. If $b = \bigvee_{n=1}^\infty (na \wedge b)$, then the suprema preserving property of ψ' yields $\psi'(b) = \psi'(a \wedge b)$. In the same way we derive from $a = \bigvee_{n=1}^\infty (nb \wedge a)$ that $\psi'(a) =$

$= \psi'(a \wedge b)$. This means that $\psi(a) = \psi(b)$ (or $a^\circ = b^\circ$) implies $\psi'(a) = \psi'(b)$. Thus ψ is smaller than ψ' . $\square$

COROLLARY 2.7. *The Archimedean classes of an l.group G are contained in the $\aleph_0$ -classes of G .*

Proof. The proof of this corollary is trivial, since the homomorphism φ of Theorem 1.3 is smaller than the homomorphism ψ of the foregoing theorem. $\square$

There is an important case in which the Archimedean classes coincide with the $\aleph_0$ -classes. Then, in other words, the mapping α of Theorem 2.4 is a lattice isomorphism. This is seen from

THEOREM 2.6. *In a fully ordered group the $\aleph_0$ -classes are the Archimedean classes. Otherwise stated: If in a fully ordered group $a \leq \aleph_0 b$ for $a, b \in P$, then $a \leq Nb$ (c.f. also Lemma 2.1).*

Proof. By Corollary 2.7, it suffices to show that the $\aleph_0$ -class a° of a positive element a of a fully ordered group G is contained in the Archimedean class a^- of a .

We already know that $\{0^-\} = \{0^\circ\}$ (Theorem 2.4). Suppose $a^\circ = b^\circ$ for elements $a, b > 0$ ($a > 0$ combined with $b = 0$, or $a = 0$ combined with $b > 0$ are impossible after Theorem 2.5). If $a = b$, then $a^- = b^-$. So we assume $a \neq b$. Then without loss of generality we may take $a < b$. If it is true that $na < b$ for all $n \in N$, then we also have $na < b - a$ for all $n \in N$. Hence $b = \bigvee_{n=1}^{\infty} (na \wedge b) \leq (b - a) \wedge b = b - a$. This implies $a \leq 0$, contradicting $a > 0$. Consequently from $a^\circ = b^\circ$ it must be concluded that $na < b$ is not valid for all $n \in N$. In a fully ordered group this means that for some $n \in N$ we have $b \leq na$. As a consequence $a^\circ = b^\circ$ implies $a^- = b^-$. This means that the $\aleph_0$ -class a° is contained in the Archimedean class a^- . $\square$

Let $L_\alpha (\alpha \in A)$ be a set of fully ordered groups. The set of all the elements $a = \langle \dots, a_\alpha, \dots \rangle$ of the complete direct sum of the L_α such that nearly all a_α vanish, is an l.subgroup of the complete direct sum. The positive elements in this subgroup are the elements for which all non-vanishing components a_α are strictly positive in L_α . The l.group thus obtained is called the *restricted cardinal sum* of the L_α . Theorem 2.6 can be extended to cases in which G is an l.group of this type.

THEOREM 2.6A. *In a restricted cardinal sum of fully ordered groups the $\aleph_0$ -classes are the Archimedean classes.*

Proof. The introductory remarks of the proof of Theorem 2.6 apply here too. So we confine ourselves to the case $a^\circ = b^\circ$ for elements $0 < a < b$. Let $a = \langle \dots, a_\alpha, \dots \rangle$ and $b = \langle \dots, b_\alpha, \dots \rangle$. Then $0 < a < b$ means $0 \leq a_\alpha \leq b_\alpha$ for all $\alpha \in A$. Suppose

$(a_{\alpha_0})^\circ < (b_{\alpha_0})^\circ$ in L_{α_0} for some $\alpha_0 \in A$. We define $a_\alpha = \langle \dots, a_\beta, \dots \rangle$ with $a_\beta = 0$ if $\beta \neq \alpha_0$, and $a_\beta = a_{\alpha_0}$ if $\beta = \alpha_0$. Then both b and $b - a_\alpha$ are upperbounds for the elements $na \wedge b$, $n \in N$. But b is the l.u.b. for these elements ($b^\circ = a^\circ$); hence $a_\alpha \leq 0$. This implies $a_{\alpha_0} = 0$. If $b_{\alpha_0} \neq 0$, then b cannot be the l.u.b. for the elements $na \wedge b$ ($n \in N$), because if we replace the α_0 -th component b_{α_0} of b by 0, we get a strictly smaller upperbound for the same elements. So $b = \bigvee_{n=1}^{\infty} (na \wedge b)$ implies $b_{\alpha_0} = 0$. This contradicts $(a_{\alpha_0})^\circ < (b_{\alpha_0})^\circ$. The conclusion is $(a_\alpha)^\circ = (b_\alpha)^\circ$ in L_α for all $\alpha \in A$.

For at most a finite number of α 's, say α_i ($i = 1, \dots, n$), we have $a_{\alpha_i} > 0$. For each such a_{α_i} there exists a natural number m_i such that $m_i a_{\alpha_i} > b_{\alpha_i}$ (Theorem 2.6). Let $m = \max(m_1, \dots, m_n)$, then it is clear that $ma > b$. Since we have $a < b$, we conclude $a^- = b^-$. $\square$

If G is the complete direct sum of fully ordered groups L_α , with the componentwise ordering (the so called *cardinal sum* of the L_α) then the $\aleph_0$ -classes need not be the Archimedean classes. The following example will show this.

EXAMPLE 2.2. Let A be the closed interval $[0, 1]$ and the L_α are the fully ordered groups of the reals for all $\alpha \in A$. Then the cardinal sum of the L_α is the lattice ordered group of all real-valued functions on $[0, 1]$. Let f be the function $f(x) = 1$ on $(0, 1]$ and $f(0) = 0$, and let g be the function $g(x) = x$ on $[0, 1]$. Then $f^\circ = g^\circ$ but $f^- > g^-$. $\square$

As in the case of the Archimedean classes of an l.group G we have

THEOREM 2.7. *The lattice of $\aleph_0$ -classes of an l.group G is a chain, if and only if G is a fully ordered group.*

The proof of this theorem is completely analogous to the proof of Theorem 1.4. We omit it.

A *pseudo strong unit* of an l.group G is defined as an element $v \in G$ such that for each $a \in G$ we have $a \leq \aleph_0 v$.

THEOREM 2.8. *The lattice of $\aleph_0$ -classes of an l.group G has a maximal element, if and only if G has a pseudo strong unit. This maximal $\aleph_0$ -class is the set of pseudo strong units of G .*

Proof. Let v be a pseudo strong unit of G , then $0 \leq \aleph_0 v$ i.e. $0 = \bigvee_{n=1}^{\infty} (nv \wedge 0)$. From $nv \wedge 0 = n(v \wedge 0)$ (Intr., 3, I) we see that $0 = \bigvee_{n=1}^{\infty} n(v \wedge 0) = v \wedge 0$. This means $v \geq 0$, hence v° exists. It is clear that v° is the maximal $\aleph_0$ -class and so: $(v')^\circ = v^\circ$ if v' is any other pseudo strong unit.

It is trivial that the maximal $\aleph_0$ -class consists of all the pseudo strong units in G . $\square$

COROLLARY 2.8. *Every strong unit is a pseudo strong unit.*

The proof of this corollary is a straightforward application of Theorem 1.5, Theorem 2.4 and Theorem 2.8 successively. Not all pseudo strong units are strong units as can be seen from Example 2.1. There f is a pseudo strong unit but not a strong unit.

§ 3 Carriers

A third important preorder on the positive cone P of an l .group G can be defined as follows:

$$\pi_c = \{(a, b) \in P^2 : x \wedge b = 0 \text{ for } x \in G \text{ implies } x \wedge a = 0\}.$$

It is immediately seen that π_c is a reflexive and a transitive relation on the positive elements of G . Hence π_c is a preorder on P . The classes of this preorder (Intr., 1) are called the *carriers* of G . The carrier of the element $a \in P$ is denoted by $a^\wedge$. The notion was introduced by Jaffard [7], [8] in a slightly different way. He discovered several theorems on this subject which are similar to our theorems on Archimedean classes and $\aleph_o$ -classes.

We mention without proof

THEOREM 3.1 (Jaffard [7]). *A carrier is a subsemigroup and a convex sublattice of P .*

THEOREM 3.2. (Jaffard [8]). *The partially ordered set of the carriers of an l .group G is a distributive lattice. The mapping $\chi: a \rightarrow a^\wedge$ from the positive cone P of G onto the lattice of the carriers of G is a lattice homomorphism with kernel $K(\chi) = \{0\}$, satisfying*

$$\chi(a+b) = \chi(a \vee b).$$

To Theorem 3.1 we add

THEOREM 3.1a. *A carrier is closed.*

Proof. Let $a^\wedge$ be a carrier and let $\vee_\alpha a_\alpha$ exist for elements $a_\alpha \in a^\wedge$. Then $x \wedge a = 0$ for $x \in G$ implies $x \wedge a_\alpha = 0$ for all α , and from Intr., 3, F3 it then follows that $x \wedge (\vee_\alpha a_\alpha) = \vee_\alpha (x \wedge a_\alpha) = 0$. Conversely, let $x \wedge (\vee_\alpha a_\alpha) = 0$, then clearly $x \wedge a_\alpha = 0$ for any α and, because $a_\alpha \in a^\wedge$, this implies $x \wedge a = 0$. This proves $\vee_\alpha a_\alpha \in a^\wedge$ hence $a^\wedge$ is closed. $\square$

A theorem of Pierce [11] reads: The mapping χ (see Theorem 3.2) can be characterized as the maximal lattice homomorphism of P with kernel $\{0\}$. This result can be extended to

THEOREM 3.3. *The mapping χ of Theorem 3.2 can be characterized as the maximal meet homomorphism χ' of P with kernel $K(\chi') = \{0\}$.*

Proof. Let χ' be any meet homomorphism of P whose kernel is $\{0\}$ and suppose $\chi'(a) = \chi'(b)$ for $a, b \in P$. Then $x \wedge (a \wedge b) = 0$ for $x \in G$ implies $\chi'(x \wedge (a \wedge b)) = = \chi'(x) \wedge \chi'(a) \wedge \chi'(b) = \chi'(x \wedge a) = \chi'(0)$

The kernel of χ' being $\{0\}$ we find that $x \wedge (a \wedge b) = 0$ and $\chi'(a) = \chi'(b)$ imply $x \wedge a = 0$. But then $a^\wedge = (a \wedge b)^\wedge$. The interchanging of a and b shows that $\chi'(a) = = \chi'(b)$ also yields $b^\wedge = (a \wedge b)^\wedge$, and thus $a^\wedge = b^\wedge$.

Hence, the classes of the meet homomorphism χ' (Intr., 4) are contained in the carriers of G . The mapping $\chi: a \rightarrow a^\wedge$ itself is a meet homomorphism with kernel $\{0\}$ (Theorem 3.2). This completes the proof. $\square$

COROLLARY 3.1. *The maximal meet homomorphism of P with kernel $\{0\}$ is a lattice homomorphism of P .*

The connection between the lattice of the $\aleph_o$ -classes and the lattice of the carriers of G follows from

THEOREM 3.4. *The mapping $\beta: a^\circ \rightarrow a^\wedge$ from the lattice of the $\aleph_o$ -classes of an l .group G onto the lattice of the carriers of G is a lattice homomorphism with kernel $\{0^\circ\}$.*

Proof. Suppose $a^\circ \leq b^\circ$ for some $a, b \in P$. Then $a \leq \aleph_o b$ i.e. $a = \bigvee_{n=1}^{\infty} (nb \wedge a)$. If $x \wedge b = 0$, then $x \wedge nb = 0$ for all $n \in N$ (Intr., 3, D); hence $x \wedge nb \wedge a = 0$ for all $n \in N$. Consequently, $x \wedge a = x \wedge \{\bigvee_{n=1}^{\infty} (nb \wedge a)\} = \bigvee_{n=1}^{\infty} (x \wedge nb \wedge a) = 0$ (Intr., 3, F). So we showed that $a^\circ \leq b^\circ$ implies $(a, b) \in \pi_c$ i.e. $a^\wedge \leq b^\wedge$. But then $a^\circ = b^\circ$ implies $a^\wedge = b^\wedge$ which proves that β is a function. Since we know $(a \wedge b)^\circ = a^\circ \wedge b^\circ$ and $(a \vee b)^\circ = a^\circ \vee b^\circ$ from the introductory remarks to Theorem 2.4 and $(a \wedge b)^\wedge = = a^\wedge \wedge b^\wedge$ and $(a \vee b)^\wedge = a^\wedge \vee b^\wedge$ from Theorem 3.2, it is proved that β is a lattice homomorphism. Obviously the kernel of β is $\{0^\circ\}$. $\square$

COROLLARY 3.2. *The $\aleph_o$ -classes of an l .group G are contained in the carriers of G .*

Goffman [5] has proved that in an Archimedean l .group the mapping $\chi: a \rightarrow a^\wedge$ of Theorem 3.2 is the unique suprema preserving homomorphism χ' of P with kernel $K(\chi') = \{0\}$, and which satisfies $\chi'(a+b) = \chi'(a \vee b)$. From Theorem 2.5 we know that $\psi: a \rightarrow a^\circ$ is a suprema preserving lattice homomorphism with kernel $\{0\}$ and $\psi(a+b) = \psi(a \vee b)$. Hence in an Archimedean l .group we have $\chi = \psi$. Otherwise stated we have proved (Intr., 4)

THEOREM 3.5. *In an Archimedean l .group the carriers are the $\aleph_o$ -classes.*

A weak unit w of an l .group G is defined as an element $w \in G$ such that $x \wedge w = 0$ for $x \in G$, if and only if $x = 0$ (Birkhoff [1]). It is clear that w is a positive element of G . The carrier $w^\wedge$ of a weak unit w is the maximal element in the lattice of the carriers of

an l .group G (Fuchs [4]) and, conversely if the lattice of carriers of an l .group G has a maximal element, then the elements of this carrier (and only these) are the weak units in G .

COROLLARY 3.3. *Every pseudo strong unit is a weak unit.*

This is a direct consequence of Theorem 2.8, of Theorem 3.4 and of the preceding observations, applied successively. Not all weak units are pseudo strong units. For example in a non-Archimedean fully ordered group all strictly positive elements are weak units but not all these elements are pseudo strong units.

However, from the foregoing theorem we know that in an Archimedean l .group the weak units are pseudo strong units. This is an extension of a result of Fuchs [4]. He proved this under the stronger hypothesis that G is a complete l .group.

§ 4 The lattice of $\aleph_0$ -classes

So far we have deduced the structure of the $\aleph_0$ -classes and of the lattice which they form, from what is known about l .groups. In this paragraph we will investigate the consequences regarding the l .group G if the lattice of its $\aleph_0$ -classes is relatively complemented.

Let G be an l .group. G is called a *full l .group*, if $\bigvee_{n=1}^{\infty} (na \wedge b)$ exists for all $a, b \in P$.

THEOREM 4.1. *If G is a full l .group, then the lattice of the $\aleph_0$ -classes of G is relatively complemented.*

Proof. Because the lattice of $\aleph_0$ -classes is a distributive lattice with minimal element 0° (Corollary 2.6), it suffices to prove that the lattice of $\aleph_0$ -classes sectionally complemented (Intr., 2). Let a° and b° be $\aleph_0$ -classes of G such that $a^\circ \leq b^\circ$, and let $c = \bigvee_{n=1}^{\infty} (na \wedge b)$. The $\aleph_0$ -classes are closed (Theorem 2.2) and $na \wedge b \in a^\circ$ for all $n \in \mathbb{N}$. So we have $c \in a^\circ$ i.e. $c^\circ = a^\circ$. It is clear that $b - c \geq 0$; hence $(b - c)^\circ$ exists. We conclude $(b - c)^\circ \vee a^\circ = (b - c)^\circ \vee c^\circ = \{(b - c) + c\}^\circ = b^\circ$ (c.f. Theorem 2.5 for the second step).

From Intr., 3, F1 we see that $a + c = \bigvee_{n=1}^{\infty} \{na \wedge (a + b)\}$ and Intr., 2, F3 shows $b \wedge (a + c) = \bigvee_{n=1}^{\infty} \{b \wedge na \wedge (a + b)\} = \bigvee_{n=1}^{\infty} (na \wedge b) = c$. This implies $(b - c) \wedge a = 0$ and, consequently, $(b - c)^\circ \wedge a^\circ = 0^\circ$. This proves that $(b - c)^\circ$ is the complement of a° in $[0^\circ, b^\circ]$. $\square$

On the other hand we have

THEOREM 5.2. *If the lattice of the $\aleph_0$ -classes of an l .group G is relatively complemented, then G is Archimedean.*

Proof. Suppose G is not Archimedean. According to Intr., 3, P7 there exists a and b ($a, b > 0$) such that $na < b$ for all $n \in \mathbb{N}$. Then we have $a^\circ \leq b^\circ$.

The condition of the theorem yields an $\aleph_0$ -class c° such that $c^\circ \wedge a^\circ = 0^\circ$ and $c^\circ \vee a^\circ = b^\circ$. From $c^\circ \wedge a^\circ = (c \wedge a)^\circ$ and Theorem 2.5 we conclude $c \wedge a = 0$. From $c^\circ \vee a^\circ = (c \vee a)^\circ = (c + a)^\circ$ and $(2b)^\circ = b^\circ$ (both consequences of Theorem 2.5) we derive $2b = \bigvee_{n=1}^{\infty} \{n(a+c) \wedge 2b\}$. Because orthogonal elements commute (Intr., 3, D) we have $n(a+c) = na + nc$, and so $n(a+c) \wedge 2b = (na + nc) \wedge 2b \leq (b + nc) \wedge 2b = b + (nc \wedge b) \leq 2b$.

From Intr., 3, K and F1 we see that $2b = \bigvee_{n=1}^{\infty} (b + (nc \wedge b)) = b + \bigvee_{n=1}^{\infty} (nc \wedge b)$; hence $b = \bigvee_{n=1}^{\infty} (nc \wedge b)$ i.e. $b^\circ \leq c^\circ$.

Since $c^\circ \vee a^\circ = b^\circ$ it follows $b^\circ = c^\circ$; hence $b^\circ \wedge a^\circ = 0^\circ$.

This last conclusion and $a^\circ \leq b^\circ$ imply $a^\circ = 0^\circ$ i.e. $a = 0$. This contradicts $a > 0$ and, consequently, the lattice of $\aleph_0$ -classes of G cannot be relatively complemented, if G is not Archimedean. $\square$

In order to formulate the next theorem we use the following abbreviations:

“Full” means: G is a full l .group.
“ Γ rel. compl.” means: the lattice Γ of $\aleph_0$ -classes of G is relatively complemented.
“Arch.” means: G is an Archimedean l .group.
“ $a^\circ = a^\wedge$ ” means: the carriers of G are the $\aleph_0$ -classes of G .
“ Λ rel. compl.” means: the lattice Λ of carriers of G is relatively complemented.
Moreover: $p \rightarrow q$ means p implies q and $\bigwedge_q \{p\} \rightarrow r$ means that whenever p and q hold simultaneously we have r ; p, q and r here stand for properties of the l .group G . The first type of implication ($p \rightarrow q$) will be called a simple implication. Then we have the

FIRST INCLUSION THEOREM. *In any l .group G we have the following implications:*

$$Full \rightarrow \Gamma \text{ rel. compl. } \left\{ \begin{array}{l} \rightarrow Arch. \rightarrow a^\circ = a^\wedge \\ \rightarrow \Lambda \text{ rel. compl.} \end{array} \right\} \rightarrow \Gamma \text{ rel. compl.}$$

The converse of the simple implications does not hold. Consequently, neither of the properties “Arch” and “ $a^\circ = a^\wedge$ ” depends on “ Λ rel. compl.”.

Proof. “Full $\rightarrow \Gamma$ rel. compl.” is proved in Theorem 4.1.

“ Γ rel. compl. $\rightarrow$ Arch.” is proved in Theorem 4.2.

“Arch. $\rightarrow a^\circ = a^\wedge$ ” is proved in Theorem 3.5.

“ Γ rel. compl. $\rightarrow \Lambda$ rel. compl.” is a consequence of the fact we just mentioned that

“ Γ rel. compl.” implies “ $a^\circ = a^\wedge$ ”,

and finally that “ Λ rel. compl.” in combination with “ $a^\circ = a^\wedge$ ” imply Γ rel. compl. is trivial.

The second part of the theorem is proved by a list of counterexamples.

1. An l -group with a relatively complemented lattice of $\aleph^\circ$ -classes which is not a full l -group. Let G be the additive l -group of all real continuous functions on $[0, 1]$ with P as the set of all functions $f(x) \geq 0$ in G (the “pointwise ordering”). The lattice of $\aleph^\circ$ -classes of G (which is the same as the lattice of carriers of G because G is Archimedean; Theorem 3.5) is relatively complemented. Let f be the function $f(x) = \max(x - \frac{1}{2}, 0)$ on $[0, 1]$ and let g be the function $g(x) = 1$ on $[0, 1]$. Then $f, g \in P$ and $\bigvee_{n=1}^{\infty} (nf \wedge g)$ does not exist in G . Hence G is not a full l -group.
2. An Archimedean l -group of which the lattice of $\aleph^\circ$ -classes (which is the lattice of carriers; Theorem 3.5) is not relatively complemented. Let G be the additive l -group of all real continuous functions on $[-1, 1]$ such that each function $f \in G$ is constant in some interval $(0, \varepsilon_f)$ with $\varepsilon_f > 0$, G being pointwise ordered. G is an Archimedean l -group. Let f be the function $f(x) = \max(-x, 0)$ on $[-1, 1]$ and let g be the function $g(x) = 1$ on $[-1, 1]$. Then $f, g \in P$ and $f^\circ \leq g^\circ$ but there exists no complement of f° in the interval $[0^\circ, g^\circ]$. Consequently, the lattice of the $\aleph_\circ$ -classes of G is not relatively complemented.
3. Any non-Archimedean fully ordered group is an l -group of which the lattice of carriers is relatively complemented (in fact it consists only of two elements) but of which the lattice of $\aleph_\circ$ -classes (a chain with more than two elements) is not relatively complemented.
4. A non Archimedean l -group in which the carriers are the $\aleph_\circ$ -classes. Let G be the additive group of real valued functions on $[0, 1]$ such that $f(x) \neq 0$ for at most finitely many x in $[0, 1]$ and let P be the set of all functions f in G such that $f(0) > 0$ or $f(x) \geq 0$. G is an l -group and it is easily seen that the carriers are the $\aleph_\circ$ -classes. If f is the function $f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$, and if g is the function $g(0) = 1$ and $g(x) = 0$ for $0 < x \leq 1$, then $f, g \in P$ and $nf < g$ for all $n \in N$. This shows that G is non-Archimedean. $\square$

§ 5 The lattice of Archimedean classes

We investigate the structure of an l -group whose lattice of Archimedean classes is relatively complemented.

THEOREM 5.1. *If the lattice of Archimedean classes of an l -group G is relatively complemented, then the Archimedean classes of G are the carriers of G .*

Proof. The mapping $\gamma: a^- \rightarrow a^+$ from the lattice of Archimedean classes onto the lattice of carriers of G is a lattice homomorphism with kernel $K(\gamma) = \{0^-\}$. In fact γ can be considered as the product of the lattice homomorphism α of Theorem 2.4 and the lattice homomorphism β of Theorem 3.4. Since a lattice homomorphism of

a relatively complemented lattice is determined by its kernel (Intr., 4) the mapping γ must be a lattice isomorphism. This proves the theorem. $\square$

The following theorem gives a necessary and sufficient condition for the l -group G for its lattice of Archimedean classes to be relatively complemented.

THEOREM 5.2. *The lattice of Archimedean classes of an l -group G is relatively complemented, if and only if for all $a, b \in P$ we have $\bigvee_{n=1}^{\infty} (na \wedge b) = \bigvee_{n=1}^k (na \wedge b)$ for some $k \in N$.*

Remark. The fulfilment of this formula for all $a, b \in P$ requires firstly that $\bigvee_{n=1}^{\infty} (na \wedge b)$ exists for all $a, b \in P$ (i.e. G is a full group cf. § 4 of this chapter) and secondly that this l.u.b. of infinitely many elements can be written as a join of a finite number of these elements.

Proof. Let the lattice of Archimedean classes of G be relatively complemented and let $a, b \in P$. Let c^- be the complement of $(a \wedge b)^-$ in $[0^-, b^-]$, i.e. $c^- \wedge (a \wedge b)^- = 0^-$ and $c^- \vee (a \wedge b)^- = b^-$. The first equality implies $c \wedge (a \wedge b) = 0$, so c and $a \wedge b$ commute (Intr., 3, D). The second equality yields $b^- = \{c + (a \wedge b)\}^-$ (Theorem 1.3) and as a consequence $b \leq k(c + (a \wedge b)) = kc + k(a \wedge b)$ for some $k \in N$. Otherwise written $b = b \wedge \{kc + k(a \wedge b)\} = b \wedge \{kc \vee k(a \wedge b)\}$ (Intr., 3, D). Hence if $j \geq k$, we have

$$\begin{aligned} ja \wedge b &= j(a \wedge b) \wedge b && \text{(Intr., 3, I)} \\ &= j(a \wedge b) \wedge b \wedge \{kc \vee k(a \wedge b)\} && \text{(Intr., 3, A)} \\ &= b \wedge [\{j(a \wedge b) \wedge kc\} \vee \{j(a \wedge b) \wedge k(a \wedge b)\}] && ((a \wedge b) \wedge c = 0 \text{ and} \\ &= b \vee \{0 \vee k(a \wedge b)\} && \text{Intr., 3, D)} \\ &= b \wedge k(a \wedge b) = ka \wedge b. \end{aligned}$$

But then $\bigvee_{n=1}^k (na \wedge b) = \bigvee_{n=1}^j (na \wedge b)$ for all $j \geq k$. This means $\bigvee_{n=1}^k (na \wedge b) = \bigvee_{n=1}^{\infty} (na \wedge b)$.

For the proof of the converse suppose $a^- < b^-$ in the lattice of Archimedean classes of G . If $c = \bigvee_{n=1}^{\infty} (na \wedge b) = \bigvee_{n=1}^k (na \wedge b) = ka \wedge b$ we prove that $(b-c)^-$ ($b-c$ is clearly positive) is the complement of a^- in $[0^-, b^-]$. For the proof of $(b-c) \wedge a = 0$ we refer to the last part of the proof of Theorem 4.1. Since $ka \geq c$, we have $b^- \leq (b-c+ka)^- = (b-c)^- \vee (ka)^- = (b-c)^- \vee a^-$. But $(b-c)^-, a^- \leq b^-$ so $(b-c)^- \vee a^- = b^-$. $\square$

For the following theorem we use the abbreviations and notations of the First inclusion theorem (§ 4 of this chapter). Moreover we use the following abbreviations:

" Δ rel. compl." means: the lattice Δ of Archimedean classes of G is relatively complemented.

" $a^- = a^\wedge$ " means: the Archimedean classes of G are the carriers of G .

" $a^- = a^\circ$ " means: the Archimedean classes of G are the $\aleph_0$ -classes of G .

Then we have the

SECOND INCLUSION THEOREM. *In an l .group G we have the following implications:*

$$\Delta \text{ rel. compl. } \left\{ \begin{array}{l} \rightarrow a^- = a^\wedge \rightarrow a^- = a^\circ \\ \rightarrow \text{full} \rightarrow \Gamma \text{ rel. compl.} \end{array} \right\} \rightarrow \Delta \text{ rel. compl.}$$

The converse of the simple implications does not hold. Consequently, each of the properties " $a^- = a^\wedge$ " and " $a^- = a^\circ$ " on the one hand are independent from each of the properties "full" and " Γ rel. compl." on the other hand.

Proof. " Δ rel. compl. $\rightarrow a^- = a^\wedge$ " is proved in Theorem 5.1.

" $a^- = a^\wedge \rightarrow a^- = a^\circ$ " is a direct consequence of Corollary 2.7 and of Corollary 3.2.

" Δ rel. compl. $\rightarrow$ full" is a part of Theorem 5.2.

"Full $\rightarrow \Gamma$ rel. compl." is proved in Theorem 4.1.

Finally the fact that " $a^- = a^\circ$ " combined with " Γ rel. compl." imply " Δ rel. compl.", is trivial.

Just as for the First inclusion theorem we give a list of counterexamples in order to prove the second part of the theorem.

1. Example 4 of the first inclusion theorem shows an l .group in which the Archimedean classes are the carriers but whose lattice of Archimedean classes is not relatively complemented.
2. In a fully ordered non-Archimedean group the Archimedean classes are the $\aleph_0$ -classes (Theorem 2.6) but they are not the carriers (except $\{0^-\} = \{0^0\} = \{0^\wedge\}$).
3. A full l .group of which the lattice of Archimedean classes is not relatively complemented. Let G be the group of all real valued functions on $[0, 1]$ with the pointwise ordering. This is a conditionally complete l .group and so it cannot be but a full l .group. If f and g are as in Example 2.1, then $f^- < g^-$, but there is no complement of f^- in the interval $[0^-, g^-]$.
4. That " Γ rel. compl." does not imply "full" for an l .group G has already been proved in the First inclusion theorem. $\square$

The complete direct sum $\sum_{\alpha \in A} L_\alpha$ of fully ordered groups L_α is an l .group and is called the *cardinal sum* of the L_α . A subdirect sum of the L_α is called a *lattice ordered vector group*; this means that a lattice ordered vector group G is an l .subgroup of a cardinal sum of fully ordered groups L_α such that for any $a_\alpha \in L_\alpha$ there exists an element $a \in G$ with α -th component a_α . (The restricted cardinal sum of fully ordered

groups (p. 24) is a lattice ordered vector group, but not conversely). In the sequel we will use the convention of making no distinction between an l -group G and a lattice and group isomorphic image of G . Then an important theorem of A. H. Clifford says that a commutative l -group is a lattice ordered vector group. Using this result we can prove

THEOREM 5.3. *If the lattice of Archimedean classes of an l -group G is relatively complemented, then G is an l -group of real valued functions on a set A with the pointwise ordering.*

Proof. If the lattice of Archimedean classes of G is relatively complemented, then G is a full group (Second inclusion theorem), hence Archimedean (First inclusion theorem) hence commutative (Intr., 3, H). Thus G is a subdirect sum of fully ordered groups L_α , $\alpha \in A$ (see above). We prove that the L_α are Archimedean. Suppose a_α and b_α are positive elements of L_α such that $na_\alpha < b_\alpha$ for all $n \in N$. Let a resp. b be elements of G with components a_α and b_α in L_α . Then $(na \wedge b)_\alpha = na_\alpha \wedge b_\alpha = na_\alpha$ for all $n \in N$. From the proof of the first part of Theorem 5.2 we know that there exists a $k \in N$ such that $ka \wedge b = ja \wedge b$ for all $j \geq k$. And so $ka \wedge b = (k+1)a \wedge b$. This implies $ka_\alpha = (ka \wedge b)_\alpha = \{(k+1)a \wedge b\}_\alpha = (k+1)a_\alpha$ hence $a_\alpha = 0$. Now Intr., 3, P7 implies that L is Archimedean and by Intr., 3, H L is a subgroup of the reals. This proves the theorem. $\square$

If the lattice of Archimedean classes of an l -group G is a Boolean algebra, then this lattice is relatively complemented (Intr., 2) and it also has a maximal element. So by the foregoing theorem, G is a group of real valued functions on a set A , and by Theorem 1.5 G has a strong unit u . Obviously since u is a positive element of G , all components u_α of u are positive and we assert that all u_α are strictly positive. Indeed, let $u_\alpha = 0$ for some $\alpha \in A$ and let a_α be a strictly positive element of L . Now, there is an element $a \in G$ with α -th component a_α . For all $n \in N$ we have $(nu)_\alpha = nu_\alpha = 0 < a_\alpha$. This implies $a \not\leq nu$ for all $n \in N$, contradicting that u is a strong unit. Then the mapping $a = \langle \dots, a_\alpha, \dots \rangle \rightarrow a' = \langle \dots, a_\alpha/u_\alpha, \dots \rangle$ is a lattice and group isomorphism from G onto an l -group G' of real valued functions on A with the pointwise ordering, such that $u \rightarrow u' = \langle \dots, 1, \dots \rangle$ (all components of u' are 1). Apparently, u' is called the *unit function* on A . This proves

THEOREM 5.3a. *If the lattice of Archimedean classes of an l -group G is a Boolean algebra, then G is an l -group of real valued functions on a set A , with the pointwise ordering, and such that the unit functions belongs to G .*

Let V be a vector space over the field of the real numbers R . Moreover, let V also be a lattice ordered group in which $a \geq 0$ in V and $\lambda \geq 0$ in R imply $\lambda a \geq 0$ in V . Then V is called a *vector lattice*.

A *function lattice* is a vector lattice of real valued functions on a set A , with the

pointwise ordering. A *simple function* on a set A is a real valued function on A that has only a finite number of values. (Obviously if A is a finite set, then all real valued functions on A are simple functions). If f and g are simple functions on A , then λf ($\lambda \in \mathbb{R}$), $f+g$, $f \vee g = \max(f, g)$ and $f \wedge g = \min(f, g)$ are also simple functions on A . Hence we can speak of a function lattice of simple functions on a set. In a function lattice on a set A the unit function ($f(\alpha) = 1$ for all $\alpha \in A$) will be denoted by f_1 ; the *zero function* (i.e. the vanishing function on A) will be denoted by f_0 .

The next theorem characterizes vector lattices with a Boolean algebra of Archimedean classes.

THEOREM 5.4. *A vector lattice V has a Boolean algebra of Archimedean classes, if and only if V is a function lattice of simple functions on a set A such that V contains the unit function on A .*

Proof. Any vector lattice is a subdirect sum of fully ordered vector lattices (Birkhoff [1]). The vector lattice V is also an l -group, hence the fully ordered vector lattices meant in the preceding sentence are Archimedean (c.f. the proof of Theorem 5.3). But then they are the fully ordered groups of the real numbers with the usual ordering. Hence V is a function lattice on a set A . V also contains the unit function f_1 on A (Theorem 5.3a), and $(f_1)^-$ is the maximal element of the Boolean algebra of Archimedean classes. If f is any element of V then $|f| < Nf_1$. This implies that all elements of V are bounded functions on A .

Suppose $g \in V$ and that g is not a simple function on A . Then g has an infinite number of values. Because g is a bounded function, the set of values must have an accumulation point λ . Let $f = |g - \lambda f_1|$, then $f \in V$, $f > 0$ and 0 is an accumulation point of the values of f . Now, for any function $h \in V$ such that $h \wedge f = f_0$, we then have that 0 is an accumulation point for the values of $h + f = h \vee f$. This implies that we never have $n(h \vee f) \geq f_1$ for $n \in \mathbb{N}$. Consequently, f^- has no complement in the Boolean algebra of Archimedean classes. This is a contradiction. The conclusion is that V is a function lattice of simple functions on A and such that $f_1 \in V$.

Conversely let V be such a function lattice. The lattice of Archimedean classes of V is a distributive lattice with minimal element $(f_0)^-$ (§ 1 of this chapter). Clearly f_1 is a strong unit of V hence the lattice of Archimedean classes has a maximal element $(f_1)^-$ (Theorem 1.5).

Let f be any positive element of V i.e. f is a positive simple function on A . Then the finite set of strictly positive values of f contains a minimum $\mu > 0$. Let $g = (f_1 - \mu^{-1}f) \vee f_0$, then $g \in V$. We will prove that g^- is the complement of f^- in the lattice of Archimedean classes of V . Let $\alpha \in A$, then clearly either $f(\alpha) = 0$ or $f(\alpha) \geq \mu$ i.e. $g(\alpha) = 0$. Hence $f \wedge g = f_0$ and thus $f^- \wedge g^- = (f_0)^-$. Moreover, this implies $\mu^{-1}f \wedge g = f_0$; hence $\mu^{-1}f \vee g = \mu^{-1}f + g$. Then $f^- \vee g^- = (\mu^{-1}f)^- \vee g^- = (\mu^{-1}f \vee g)^- = (\mu^{-1}f + g)^- = (f_1 \vee \mu^{-1}f)^- = (f_1)^-$ (using Theorem 1.3). Consequently, the lattice of Archimedean classes is complemented. $\square$

§ 6 l -ideals and normalizers

For some (fixed) element x of an l -group G we define $a_x = -x + a + x$ i.e. a_x is the image of a under the inner group automorphism induced by x . We know that $a_x \geq 0$, if and only if $a \geq 0$ (Intr., 3, P3). So for any element $a \in P$, it makes sense to speak of the Archimedean class, the $\aleph_0$ -class and the carrier of both a and a_x . The group automorphism referred to above induces several lattice automorphisms.

THEOREM 6.1. *Let x be any (fixed) element of an l -group G , then:*

- (i) *the mapping $a \rightarrow a_x$ from G onto itself is a group and lattice automorphism,*
- (ii) *the mapping $a^- \rightarrow (a_x)^-$ from the lattice of Archimedean classes onto itself is a lattice automorphism,*
- (iii) *the mapping $a^\circ \rightarrow (a_x)^\circ$ from the lattice of $\aleph_0$ -classes onto itself is a lattice automorphism,*
- (iv) *the mapping $a^\wedge \rightarrow (a_x)^\wedge$ from the lattice of carriers onto itself is a lattice automorphism.*

Proof. (i) We know that $a \rightarrow a_x$ is a group automorphism. That it is also a lattice automorphism, follows from

$$(a \vee b)_x = -x + (a \vee b) + x = (-x + a + x) \vee (-x + b + x) = a_x \vee b_x$$

and $(a \wedge b)_x = a_x \wedge b_x$.

(ii) First prove that the mapping $a^- \rightarrow (a_x)^-$ is one to one i.e. if $b \in a^-$, then $b_x \in (a_x)^-$ and conversely. Since $a \leq nb$ implies $a_x = -x + a + x \leq -x + nb + x = n(-x + b + x) = nb_x$, we see that $(a, b) \in \pi_N$ implies $(a_x, b_x) \in \pi_N$. Therefore, $b \in a^-$ (this means $(a, b) \in \pi_N$ and $(b, a) \in \pi_N$) implies $b_x \in (a_x)^-$. Conversely, if $b_x \in (a_x)^-$, then from the foregoing sentence it follows that $b = (b_x)_{-x} \in \{(a_x)_{-x}\}^- = a^-$.

Next we have to prove that $(a_x)^- \vee (b_x)^- = \{(a \vee b)_x\}^-$ and $(a_x)^- \wedge (b_x)^- = \{(a \wedge b)_x\}^-$. Since both formulae are proved in the same way we restrict ourselves to the proof of the first. We have $(a_x)^- \vee (b_x)^- = (a_x \vee b_x)^-$ (Theorem 1.3) and $a_x \vee b_x = (a \vee b)_x$ (c.f. part 1 of this proof). Hence $(a_x)^- \vee (b_x)^- = \{(a \vee b)_x\}^-$.

(iii) Let $(a, b) \in \pi_{\aleph_0}$ i.e. $a = \bigvee_{n=1}^{\infty} (nb \wedge a)$, then $a_x = -x + a + x = -x + \{\bigvee_{n=1}^{\infty} (nb \wedge a)\} + x = \bigvee_{n=1}^{\infty} \{(-x + nb + x) \wedge (-x + a + x)\} = \bigvee_{n=1}^{\infty} (nb_x \wedge a_x)$. This means $(a_x, b_x) \in \pi_{\aleph_0}$. Therefore, $b \in a^\circ$ (i.e. $(a, b) \in \pi_{\aleph_0}$ and $(b, a) \in \pi_{\aleph_0}$) implies $b_x \in (a_x)^\circ$. The proofs of: $b_x \in (a_x)^\circ$ implies $b \in a^\circ$ and the proofs of $(a_x)^\circ \vee (b_x)^\circ = \{(a \vee b)_x\}^\circ$ and $(a_x)^\circ \wedge (b_x)^\circ = \{(a \wedge b)_x\}^\circ$ are analogous to the proofs of the corresponding data for Archimedean classes in part (ii).

(iv) As in part (iii) the only statement that has to be proved individually is: $(a, b) \in \pi_c$ implies $(a_x, b_x) \in \pi_c$. So let $y \wedge b = 0$ imply $y \wedge a = 0$. Then $z \wedge b_x = 0$ implies $(x + z - x) \wedge b = 0$ and thus $(x + z - x) \wedge a = 0$. Consequently, $z \wedge a_x = 0$.

This completes the proof. $\square$

COROLLARY 6.1. *If the chain of Archimedean classes of a fully ordered group G (Theorem 1.4) is well ordered, then every Archimedean class is a normal subset of G . (c.f. also Fuchs [4] p. 82 Corollary 14).*

Proof. Suppose there exists an element $x \in G$ such that the set of Archimedean classes a^- with $a^- \neq (a_x)^-$ is not empty. Let b^- be the minimal element of this set. Then $b^- < (b_x)^-$, hence $(b_{-x})^- < b^-$. The definition of b^- implies $(b_{-x})^- = \{(b_{-x})_x\}^- = b^-$, and this contradicts $(b_{-x})^- < b^-$. Hence $a^- = (a_x)^-$ for all $x \in G$ and all $a \in P$ i.e. the Archimedean classes are normal. $\square$

An example of a fully ordered group in which the Archimedean classes are not normal is given by Chehata [2].

The mapping $a \rightarrow a_x$ from G onto itself is the identity automorphism for all $x \in G$, if and only if G is an abelian group. One might ask what happens if the mapping $a^- \rightarrow (a_x)^-$ from the lattice of Archimedean classes onto itself is the identity automorphism for all $x \in G$. One can put the same questions about the corresponding cases $a^\circ \rightarrow (a_x)^\circ$ and $a^\wedge \rightarrow (a_x)^\wedge$. These questions will be answered in the sequel.

An l -ideal of an l -group G is a normal and convex l -subgroup of G . Every l -ideal of G is the kernel of a lattice and group homomorphism from G onto an l -group G' , and if two lattice and group homomorphisms of G have the same kernel, then they are equal (in the sense of Intr., 4).

THEOREM 6.2. *The following three properties are equivalent in any l -group G :*

- (i) *The mapping $a^- \rightarrow (a_x)^-$ from the lattice of Archimedean classes onto itself is the identity automorphism for all $x \in G$,*
- (ii) *The Archimedean classes of G are normal subsets of G ,*
- (iii) *Every convex and directed subgroup is an l -ideal of G .*

Proof. (i) Since $a^- = (a_x)^-$ for all $x \in G$ means $a_x \in a^-$ for all $x \in G$ we have that (i) and (ii) are equivalent.

(ii) $\rightarrow$ (iii) Let G be an l -group with normal Archimedean classes and suppose S is a convex and directed subgroup of G . Obviously, S contains together with the positive element a the Archimedean class a^- of a . Since $a_x \in a^-$ for all $x \in G$, we have $a_x \in S$ for all $x \in G$. This implies that the positive cone of S is a normal subset of G . But S is generated by its positive cone (Intr., 3, P4). Hence S is normal in G . $P(S)$ is a sublattice of G . Viz. if $P(S)$ contains a and b , it also contains $a + b$, and because $a \leq a \vee b \leq a + b$ (Intr., 3, C) the convexity of S implies $a \vee b \in P(S)$. Trivially $a \wedge b \in P(S)$ (from $0 \leq a \wedge b \leq a$). Using again that $P(S)$ generates S , we may conclude from Intr., 3, P5 that S is a sublattice of G .

(iii) $\rightarrow$ (ii) Suppose that the Archimedean class a^- is not normal i.e. there exists $x \in G$ such that $(a_x)^- \neq a^-$. Then we have two possibilities for $(a_x)^-$ and a^- : either $(a_x)^- \not\leq a^-$ or $(a_x)^- < a^-$. In the first case the convex and directed subgroup consisting of those elements $b \in G$ with $|b| \leq Na$ cannot be normal, for if this subgroup were normal then we would have $|a_x| = a_x \leq Na$ i.e. $(a_x)^- \leq a^-$, contradicting $(a_x)^- \not\leq a^-$.

In the second case the convex and directed subgroup consisting of those $b \in G$ with $|b| \leq Na_x$ is not normal, because this would imply $|(a_x)^-| = a \leq Na_x$ i.e. $a^- \leq (a_x)^-$, contradicting $(a_x)^- < a^-$. This proves that in either of the possibilities not all convex and directed subgroups are l -ideals of G . $\square$

COROLLARY 6.2. *If the chain of Archimedean classes of a fully ordered group G is well ordered, then every convex subgroup is an l -ideal of G .*

Proof. Obviously, in a fully ordered group every subgroup is directed. So this corollary is a direct consequence of Corollary 6.1 and Theorem 6.2. $\square$

The closed l -ideals of an l -group G play the same role with respect to the suprema preserving group homomorphisms of G as the l -ideals do with respect to the lattice and group homomorphisms. This is seen from

THEOREM 6.3. *The kernel of a suprema preserving group homomorphism φ of an l -group G onto an l -group G' is a closed l -ideal of G . Conversely, a closed l -ideal of G is the kernel of a suprema preserving group homomorphism of G onto an l -group G' .*

Proof. If φ is a suprema preserving group homomorphism, then φ is a join homomorphism (Intr., 4) and a meet homomorphism (from $\varphi(a \wedge b) = \varphi(-(-a \vee -b)) = -(-\varphi(a) \vee -\varphi(b)) = \varphi(a) \wedge \varphi(b)$). Hence φ is a lattice and group homomorphism from G onto G' . So the kernel $K(\varphi)$ of φ is an l -ideal of G . Let $\vee_a a_\alpha$ exist for elements $a_\alpha \in K(\varphi)$, then the suprema preserving property of φ ensures that $\varphi(\vee_a a_\alpha) = \vee_a \varphi(a_\alpha)$, hence $\vee_a a_\alpha \in K(\varphi)$. Consequently, $K(\varphi)$ is closed.

Let K be a closed l -ideal of G , then K is the kernel of a lattice and group homomorphism φ from G onto an l -group G' . Suppose $\vee_a a_\alpha$ exists in G , then we will show that $\varphi(\vee_a a_\alpha) = \vee_a \varphi(a_\alpha)$ i.e. we will show that $\varphi(\vee_a a_\alpha)$ is the l.u.b. in G' for the elements $\varphi(a_\alpha)$ of G' . Clearly, $\varphi(\vee_a a_\alpha)$ is an upperbound for the elements $\varphi(a_\alpha)$. Suppose $b' \in G'$ is any upperbound for the elements $\varphi(a_\alpha)$. There exists $b \in G$ such that $b' = \varphi(b)$. Then we have $\varphi\{(a_\alpha \vee b) - b\} = \varphi(a_\alpha \vee b) - \varphi(b) = \varphi(a_\alpha) \vee \varphi(b) - \varphi(b) = b' - b' = 0'$, hence $(a_\alpha \vee b) - b \in K$. Because $\vee_a \{(a_\alpha \vee b) - b\} = \{(\vee_a a_\alpha) \vee b\} - b$ (Intr., 3, F) exists and because K is closed we have $\vee_a \{(a_\alpha \vee b) - b\} \in K$. This implies $\varphi[\vee_a \{(a_\alpha \vee b) - b\}] = \varphi\{(\vee_a a_\alpha) \vee b\} - b' = \varphi(\vee_a a_\alpha) \vee b' - b' = 0'$. Consequently, $\varphi(\vee_a a_\alpha) \vee b' = b'$ or otherwise written $\varphi(\vee_a a_\alpha) \leq b'$. This proves that $\varphi(\vee_a a_\alpha)$ is the l.u.b. for the elements $\varphi(a_\alpha)$, hence φ is suprema preserving. $\square$

The correspondence of the l -ideals and the closed l -ideals is also established by the following analog of Theorem 6.2.

THEOREM 6.4. *The following three properties are equivalent in any l -group G :*

- (i) *The mapping $a^\circ \rightarrow (a_x)^\circ$ from the lattice of $\aleph_o$ -classes onto itself is the identity automorphism for all $x \in G$,*
- (ii) *The $\aleph_o$ -classes of G are normal subsets of G ,*
- (iii) *Every closed convex directed subgroup is a closed l -ideal of G .*

Proof. It is clear that (i) and (ii) are equivalent.

(ii) $\rightarrow$ (iii) This is proved by replacing the word Archimedean class by $\aleph_o$ -class, the word convex and directed subgroup by convex and directed closed subgroup and a^- by a° in the proof of (ii) $\rightarrow$ (iii) of Theorem 6.2.

(iii) $\rightarrow$ (ii) The last part of Theorem 6.2 yields the result by using the substitutions of the preceding sentence and by simultaneously setting $\aleph_o a$ for Na . $\square$

The analog of Theorem 6.2 for carriers instead of Archimedean classes is proposition 13 on p. 116 of Fuchs [4].

We know that the normalizer of any subset of a group is a subgroup of that group. Of course, this remains valid in an l -group G . But then naturally the question arises whether the normalizer of a subset of G is not only a subgroup, but also a sublattice (hence an l -subgroup) of G . The next theorem gives a number of subsets of G , for which this is the case.

THEOREM 6.5. *The normalizer $N(S)$ of a subset S of an l -group G is an l -subgroup of G in case S is an element, an Archimedean class, an $\aleph_o$ -class or a carrier of G .*

Proof. The foregoing remarks make it sufficient to prove that the normalizers mentioned in the theorem are sublattices of G . Let a be any element of G and assume $x, y \in N(a)$ i.e. $a_x = a_y = a$. Then

$$\begin{aligned}
 a_{x \vee y} &= -(x \vee y) + a + (x \vee y) && \text{(Intr., 3, monotony law)} \\
 &= \{(-x + a + x) \wedge (-y + a + x)\} \vee \{(-x + a + y) \wedge (-y + a + y)\} && (a_x = a_y = a) \\
 &= a \wedge \{(-y + a + x) \vee (-x + a + y)\} \\
 &= a \wedge \{(-y + x + a) \vee (-x + y + a)\} \\
 &= a \wedge \{|-y + x| + a\} = a.
 \end{aligned}$$

This proves $x \vee y \in N(a)$.

In case the subset S of the theorem is either an Archimedean class or an $\aleph_o$ -class or a carrier of G , then the assertions: $x \in N(S), y \in N(S)$ imply $x \vee y \in N(S)$ are proved

analogously. We will give the proof in case S is an Archimedean class. Let a^- be any Archimedean class of G and assume $(a_x)^- = (a_y)^- = a^-$. Then

$$\begin{aligned}
(a_{x \vee y})^- &= \{-(x \vee y) + a + (x \vee y)\}^- & (a_{x \vee y} \geq 0) \\
&= [\{-(x \vee y) + a + (x \vee y)\} \vee 0]^- \\
&= [\{(-x + a + x) \wedge (-y + a + x)\} \vee \{(-x + a + y) \wedge (-y + a + y)\} \vee 0]^- \\
&\quad \text{(Intr. 3A)} \\
&= \left\{ [(-x + a + x) \wedge \{(-y + a + x) \vee 0\}] \vee [\{(-x + a + y) \vee 0\} \wedge (-y + a + y)] \right\}^- \\
&\quad \text{(Theorem 1.3)} \\
&= [(-x + a + x)^- \wedge \{(-y + a + x) \vee 0\}^-] \vee [\{(-x + a + y) \vee 0\}^- \wedge (-y + a + y)^-] \\
&= [(a_x)^- \wedge \{(-y + a + x) \vee 0\}^-] \vee [\{(-x + a + y) \vee 0\}^- \wedge (a_y)^-] \\
&\quad ((a_x)^- = (a_y)^- = a^-) \\
&= a^- \wedge \{(-y + a + x) \vee (-x + a + y) \vee 0\}^-.
\end{aligned}$$

Since we have:

$$(-y + a + x), (-x + a + y) \leq (-y + a + x) \vee (-x + a + y)$$

it follows

$$0 \leq -y + 2a + y = -y + a + x - x + a + y \leq 2 \{(-y + a + x) \vee (-x + a + y)\}$$

hence

$$(-y + a + x) \vee (-x + a + y) \geq 0 \quad \text{(Intr., 3, B)}$$

We may continue as follows:

$$\begin{aligned}
\{(-y + a + x) \vee (-x + a + y) \vee 0\}^- &= \{(-y + a + x) \vee (-x + a + y)\}^- & \text{(Theorem 1.3)} \\
&= [2\{(-y + a + x) \vee (-x + a + y)\}]^- & \text{(Theorem 1.3)} \\
&\geq (-y + 2a + y)^- \\
&= \{2(-y + a + y)\}^- \\
&= (-y + a + y)^- = a^-
\end{aligned}$$

Since we found above $(a_{x \vee y})^- = a^- \wedge \{(-y + a + x) \vee (-x + a + y) \vee 0\}^-$, this implies $(a_{x \vee y})^- = a^-$.

In all four cases for S (S is an element, an Archimedean class etc.) we know that $N(S)$ is a subgroup and that $x \in N(S)$, $y \in N(S)$ imply $x \vee y \in N(S)$. From $x \wedge y = x - (x \vee y) + y$ (Intr., 3, C), we conclude that $x \in N(S)$, $y \in N(S)$ imply also $x \wedge y \in N(S)$. This proves the theorem. $\square$

Remark. Because $a \in a^- \subset a^\circ \subset a^\wedge$ we have $N(a) \subset N(a^-) \subset N(a^\circ) \subset N(a^\wedge)$.

It is possible to determine certain subgroups of G that are contained in the normalizers $N(S)$ of the foregoing theorem. This means, for example, that if a^- is an Archimedean class of G we can indicate a subgroup H of G such that $H \subset N(a^-)$. Of course, for every element $x \in H$ the mapping $a^- \rightarrow (a_x)^-$ of Theorem 6.1 is the identity automorphism. The last part of this paragraph will be devoted to this problem.

We start with the following observations: Let a be a positive element of G , then all elements x with $x \wedge a = 0$ commute with a (Intr., 3, D). So, if we denote the set of these elements x by a^* (i.e. $a^* = \{x \in G : x \wedge a = 0 \text{ for } a \in P\}$), we conclude that $a^* \subset N(a) \subset N(a^-) \subset N(a^\circ) \subset N(a^\wedge)$.

Next suppose that $y \in a^-$, then $(y \vee a)^- = a^-$. Consequently, $(-y + a + y)^- = \{-y + (a \vee y) + y\}^- = [\{-y + (a \vee y)\} + y]^- = [y + \{-y + (a \vee y)\}]^- = (a \vee y)^- = a^-$ (the first equality follows from Theorem 6.1, and the third from Theorem 1.3). This proves $a^- \subset N(a^-)$. In the same way it is proved that $a^\circ \subset N(a^\circ)$ and $a^\wedge \subset N(a^\wedge)$.

The intersection of any number of l -subgroups of an l -group G is an l -subgroup of G . So we can speak of the l -subgroup generated by a subset S of G . This l -subgroup will be denoted by (S) . The l -subgroup generated by the set theoretic union $S \cup T$ of two subsets S and T of G is denoted by $(S+T)$.

The foregoing remarks and the fact that the normalizers $N(a)$, $N(a^-)$, $N(a^\circ)$ and $N(a^\wedge)$ are l -subgroups of G (Theorem 6.5) prove

THEOREM 6.6. *In any l -group G we have for $a \geq 0$*

- (i) $(a^*) \subset N(a)$
- (ii) $(a^* + a^-) \subset N(a^-)$
- (iii) $(a^* + a^\circ) \subset N(a^\circ)$
- (iv) $(a^* + a^\wedge) \subset N(a^\wedge)$.

Remarks. 1. It can be proved that the l -subgroup $(a^* + a^-)$ is the direct sum of the l -subgroups (a^*) and (a^-) i.e. $(a^* + a^-) = (a^*) \oplus (a^-)$. This follows immediately from $(a^*) \cap (a^-) = (0)$. The same is true for the l -subgroups $(a^* + a^\circ)$ and $(a^* + a^\wedge)$ appearing in (iii) and (iv) of Theorem 6.6.

- 2. All four subgroups (a^*) , (a^-) , (a°) and $(a^\wedge)$ are convex. This implies that the subgroups $(a^* + a^-)$, $(a^* + a^\circ)$ and $(a^* + a^\wedge)$ are convex. The proof needs Intr., 3, G.
- 3. The set of elements which are orthogonal to all elements of a subset S of positive elements of G is denoted by S^* , and $(S^*)^*$ is denoted by S^{**} . From this definition and the definition of $a^\wedge$ it is clear that $a^\wedge \subset a^{**}$ for $a \in P$. Hence $(a^\wedge) \subset (a^{**})$. We prove that $(a^{**}) \subset (a^\wedge)$. Let $b \in a^{**}$ and suppose $x \wedge a = 0$, then $x \in a^*$. Hence $x \wedge b = 0$. This means $b^\wedge \leq a^\wedge$. Since $(a^\wedge)$ is convex this implies that $b \in (a^\wedge)$. Consequently, the positive cone of (a^{**}) is contained in the positive cone of $(a^\wedge)$, and therefore $(a^{**}) \subset (a^\wedge)$. But then (by remark 1) the last part of Theorem 6.6 can be read as $(a^* + a^{**}) = (a^*) \oplus (a^{**}) \subset N(a^\wedge)$.

§ 7 Archimedean lattice ordered groups

We start with two important theorems on lattice ordered groups.

1. An Archimedean l -group can be embedded lattice and group isomorphically in a complete l -group.
2. A complete l -group is commutative.

Both theorems are based upon contributions of several mathematicians (c.f. Fuchs [4] p. 136–140 and p. 146–149). An immediate consequence of these theorems is the fact that an Archimedean l -group is commutative. We will give a fresh proof of this result without using the embedding of an Archimedean l -group in a complete l -group. The proof is based upon quite elementary theorems of Chehata [2] and Sik [13] and on lemma 7.1 below.

The theorem of Chehata says:

In a fully ordered group we have for $a, b \in P$: $n|\langle a, b \rangle| < a \vee b$ for all $n \in N$, where $\langle a, b \rangle$ denotes the commutator of a and b .

The theorem of Sik that will be used is:

If the carriers of an l -group are normal, then the l -group is a lattice ordered vector group. (For an easy proof of this theorem c.f. Fuchs [4] p. 124)

LEMMA 7.1. *An Archimedean l -group is an l -group with normal carriers.*

Proof. Suppose the carriers are not normal then there exist $a, x \in G$ $a > 0$ and $x \neq 0$ such that; $(-x + a + x) \wedge a = 0$ (Fuchs [4] p. 107). But then

$$\begin{aligned}
 (-|x| + a + |x|) \wedge a &= \{(-x \wedge x) + a + (x \vee -x)\} \wedge a \\
 &= [\{(-x + a + x) \wedge (x + a + x)\} \vee \{(-x + a - x) \wedge (x + a - x)\}] \wedge a \\
 &= \{0 \wedge (x + a + x)\} \vee \{(-x + a - x) \wedge 0\} \\
 &= 0 \wedge \{(x + a + x) \vee (-x + a - x)\} = 0.
 \end{aligned}$$

This last equality is valid because

$$(x + a + x) \vee (-x + a - x) \geq (x + a + x) \vee (-x - a - x) = |x + a + x| \geq 0.$$

So we find $(-|x| + a + |x|) \wedge a = 0$ and this implies (Intr., 3, D):

$$(-|x| + ma + |x|) \wedge ma = 0.$$

It follows that $(ma + |x|) \wedge (|x| + ma) = |x|$, which implies

$$ma = ma \wedge (ma + |x|) \wedge (|x| + ma) = ma \wedge |x|.$$

Hence $ma \leq |x|$ for all $m \in N$ and $a, |x| > 0$.

This contradicts the Archimedean property. $\square$

It is not difficult to extend the theorem of Chehata referred to above to a much larger class of l .groups. In fact we have

THEOREM 7.1. *In a lattice ordered vector group we have for $a, b \in P$: $n|\langle a, b \rangle| < a \vee b$ for all $n \in N$.*

Proof. Let G be an l .group which satisfies the conditions of the theorem (c.f. p. 32). Let for any element $c \in G$ c_α denote the component of c in L_α and suppose $a, b \in P$. Then we have $(n|\langle a, b \rangle|)_\alpha = n|\langle a_\alpha, b_\alpha \rangle|$. By Chehata's theorem we have $n|\langle a_\alpha, b_\alpha \rangle| < a_\alpha \vee b_\alpha = (a \vee b)_\alpha$ for all $n \in N$. Since this holds for all components we find: $n|\langle a, b \rangle| < a \vee b$ for all $n \in N$. $\square$

Remark. An l .group in which this theorem is not valid can be found in Birkhoff [1] p. 291.

A consequence of the foregoing lemma and theorem is

COROLLARY 7.1. *An Archimedean l .group is commutative.*

Proof. Let G be an Archimedean l .group. We indicate the following steps:

1. G is an l .group with normal carriers (Lemma 7.1).
2. G is a lattice ordered vector group (theorem of Sik).
3. In G we have for $a, b \in P$: $n|\langle a, b \rangle| < a \vee b$ for all $n \in N$. (Theorem 7.1).
4. Since G is Archimedean this leads to $\langle a, b \rangle = 0$ for $a, b \in P$ (Intr., 3, P7 and E).
So the positive elements of G commute.
5. G is commutative, since G is generated (as a group) by its positive elements (Intr., 3, P5). $\square$

APPENDIX

After the completion of the manuscript of this thesis our attention was drawn to the preprint of a forthcoming book by W. A. J. Luxemburg and A. C. Zaanen [10] on Riesz spaces. Some findings of the book are closely related to theorems of this thesis. This appendix compares the relevant notions and results. The book by Luxemburg and Zaanen will be referred to as L & Z.

A *Riesz space* is a vector lattice (c.f. p. 33 of this thesis). Otherwise stated: a Riesz space V is a commutative l -group which is closed with respect to multiplication with the reals and such that: $\lambda(a+b) = \lambda a + \lambda b$ and $\lambda(a \vee b) = \lambda a \vee \lambda b$ for λ real and $a, b \in V$.

We copy the following definitions from L & Z.

A subset of S of V is called *solid* if $a \in S$ and $|b| \leq |a|$ imply $b \in S$. A solid linear subspace A of V is called an *ideal* in V .

The ideal A in V is called a σ -*ideal* in V whenever it follows from $a_n \in A$ ($n = 1, 2, \dots$) and $a = \bigvee_{n=1}^{\infty} a_n$ in V that $a \in A$.

The ideal A in V is called a *band* in V whenever it follows from $a_\alpha \in A$ for all α in the arbitrary index set $\{\alpha\}$ and $a = \bigvee_\alpha a_\alpha$ that $a \in A$.

The smallest ideal (σ -ideal, band) in V containing the element $a \in V$ is called *the principal ideal (principal σ -ideal, principal band)* generated by a .

If S is a subset of V , the set S^d is defined as follows

$$S^d = \{b \in V : |b| \wedge |a| = 0 \text{ for all } a \in S\}.$$

S^d is called the *disjoint complement* of S . The set $(S^d)^d$ will be denoted by S^{dd} .

In the sequel we will make these definitions fit to an l -group G and we will show the connections with the notions of Archimedean class, $\aleph_0$ -class and carrier.

Let G be an l -group. The meaning of the notions solid subset and disjoint complement of a subset of G will be the same as for a Riesz space.

A. The smallest solid subgroup, containing the element $a \in G$, is called the *principal subgroup* generated by a . The Archimedean class $|a|^-$ is contained in the principal subgroup generated by a and consequently, $(|a|^-)$ is contained in this subgroup.

But $(|a|^-)$ is a solid subgroup (in fact $(|a|^-) = \{b \in G : |b|^- \leq |a|^- \}$) in which the elements of $|a|^-$ are the strong units. We conclude that the correspondence

$$H(a) \rightarrow |a|^- = \{\text{the set of strong units in } H(a)\}$$

between the principal subgroups of G and the Archimedean classes of G is one to one.

The inverse correspondence is

$$|a|^- \rightarrow H(a) = (|a|^-).$$

B. A solid subgroup H is called σ -closed whenever it follows from $a_n \in H$ ($n = 1, 2, 3, \dots$) and $a = \bigvee_{n=1}^{\infty} a_n$ in G that $a \in H$. The smallest σ -closed subgroup in G , containing an element $a \in G$, is called the *principal σ -closed subgroup* generated by a . The $\aleph_0$ -class $|a|^\circ$ is contained in the principle σ -closed subgroup generated by a and, consequently $(|a|^\circ)$ is contained in this subgroup. But $(|a|^\circ)$ is a σ -closed subgroup (in fact $(|a|^\circ) = \{b \in G : |b|^\circ \leq |a|^\circ\}$) in which the elements of $|a|^\circ$ are the pseudo strong units. We conclude that the correspondence

$$H_\sigma(a) \rightarrow |a|^\circ = \{\text{the set of pseudo strong units in } H_\sigma(a)\}$$

between the principal σ -closed subgroups of G and the $\aleph_0$ -classes of G is one to one.

The inverse correspondence is

$$|a|^\circ \rightarrow H_\sigma(a) = (|a|^\circ).$$

C. We recall that a solid subgroup H in G is *closed* whenever it follows from $a_\alpha \in H$ for all α of the arbitrary index set $\{\alpha\}$ and $a = \bigvee_\alpha a_\alpha$ that $a \in H$. The smallest closed subgroup in G that contains the element $a \in G$ is called the *principal closed subgroup* generated by a . Theorem 2.2 proves that an $\aleph_0$ -class is closed and consequently the subgroup $(|a|^\circ) = H_\sigma(a)$ is closed. Thus in an l -group the notions of principal σ -closed subgroup and principal closed subgroup coincide. This is the analog in l -groups of the following theorem of L & Z: In a Riesz space the notions of principal σ -ideal and principal band coincide.

D. The set a^{dd} for $a \in G$ is a solid subgroup of G . The carrier $|a|^\wedge$ is contained in this subgroup and, consequently, $(|a|^\wedge)$ is contained in the solid subgroup a^{dd} . But $(|a|^\wedge)$ contains the solid subgroup a^{dd} (this follows from $(|a|^\wedge) = \{b \in G : |b|^\wedge \leq |a|^\wedge\}$) in which the elements of $|a|^\wedge$ are the weak units. Consequently, the correspondence

$$a^{dd} \rightarrow |a|^\wedge = \{\text{the set of weak units of } a^{dd}\}$$

between the solid subgroups a^{dd} for $a \in G$ and the carriers of G is one to one.

The inverse correspondence is

$$|a|^\wedge \rightarrow a^{dd} = (|a|^\wedge).$$

Remark. The set $|a|^{**}$ defined in § 6 of the second chapter is the positive cone of a^{dd} .

E. Theorem 3.5 proves that in an Archimedean l -group the carriers are the $\aleph_o$ -classes while the converse is not true (example 4 in the proof of the First inclusion theorem). By B, C and D this yields: If an l -group G is Archimedean, then the principal closed subgroup generated by a coincides with a^{dd} for all $a \in G$, but not conversely. This theorem shows a correspondence with the following theorem on Riesz spaces proved by L & Z. It says: A Riesz space V is Archimedean, if and only if $A = A^{dd}$ for every band A in V .

From Theorem 2.6 and A, B and C above we derive: In a fully ordered Riesz space the principal ideals coincide with the principal bands.

L & Z define in connection with a Riesz space the notion “*principal projection property*” and prove: A Riesz space V has the principal projection property, if and only if $\bigvee_{n=1}^{\infty} (na \wedge b)$ exists for all positive elements a and b of V . For l -groups this is the defining property of a full l -group (c.f. p. 28).

L & Z proved that in a Riesz space the principal projection property implies the Archimedean property but not conversely. We proved that if an l -group G is a full l -group, then the lattice of $\aleph_o$ -classes of G is relatively complemented, and this, in its turn implies the Archimedean property. Moreover, no one of these implications holds conversely (First inclusion theorem).

If we apply this to the special case that the l -group G is a Riesz space V and if we use the terminology of L & Z we conclude the following:

For Riesz spaces V the property “*The distributive lattice of principal bands of V is relatively complemented*” is intermediate between “ *V has the principal projection property*” and “ *V is Archimedean*”. That the first property does not coincide with the second is seen from the first example in the proof of the First inclusion theorem; that the first property does not coincide with the third follows from the second example in the same proof.

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SUMMARY

In this summary G denotes a lattice ordered group, P is the positive cone of G and a stands for any element of P . Findings by others are put in accolades. All paragraphs referred to are from the second chapter.

In § 2 the notion of an $\aleph_o$ -class of G is introduced. The §§ 1 to 3 deal with the following: The Archimedean class a^- of a is a subset of the $\aleph_o$ -class a° of a , and a° is contained in the carrier $a^\wedge$ of a . The structure of the Archimedean classes and the $\aleph_o$ -classes of G is investigated. In a fully ordered group (and also in a restricted cardinal sum of fully ordered groups) the Archimedean classes coincide with the $\aleph_o$ -classes. In an Archimedean lattice ordered group the $\aleph_o$ -classes coincide with the carriers. The partially ordered set of the $\aleph_o$ -classes {resp. of the Archimedean classes and of the carriers} is a distributive lattice. The mappings $\varphi: a \rightarrow a^-$ from P onto the lattice of Archimedean classes and $\psi: a \rightarrow a^\circ$ from P onto the lattice of $\aleph_o$ -classes are examined. {The corresponding mapping from P onto the lattice of carriers is studied by P. Jaffard a.o.}

In § 4 the structure of G having a relatively complemented lattice of $\aleph_o$ -classes is treated. The First inclusion theorem (p. 29) shows the connection between this and other properties of G .

In § 5 the structure of G having a relatively complemented lattice of Archimedean classes is dealt with. The Second inclusion theorem (p. 32) correlates this and other properties of G . If G is a vectorlattice, then it is proved that G is a function lattice of simple functions such that G contains the unit function, if and only if G has a Boolean algebra of Archimedean classes (Theorem 5.4).

In § 6 we start a study on normalizers in a lattice ordered group. The connection is shown between properties of the lattices of the Archimedean classes and the $\aleph_o$ -classes on the one hand, and the l -ideals and closed l -ideals on the other hand. The normalizers of the subsets a , a^- , a° and $a^\wedge$ turn out to be sublattices of G .

§ 7 describes an original and straightforward proof of the well-known property of the commutativity of an Archimedean lattice ordered group.

SAMENVATTING

In het onderstaande stelt G een traliegroep voor, P is de positieve kegel van G en a is een willekeurig element van P . Reeds bekende resultaten worden in deze samenvatting tussen akkoladen vermeld. Alle hieronder genoemde paragrafen zijn uit het tweede hoofdstuk.

In § 2 wordt het begrip $\aleph_o$ -klasse van G ingevoerd. De §§ 1 t.m. 3 gaan over het volgende: De archimediese klasse a^- van a is een deelverzameling van de $\aleph_o$ -klasse a° van a en a° is bevat in de zgn. „carrier” $a^\wedge$ van a . In een volledig geordende groep (en ook in een beperkte kardinale som van volledig geordende groepen) vallen de archimediese klassen samen met de $\aleph_o$ -klassen; in een archimediese traliegroep vallen de $\aleph_o$ -klassen samen met de carriers. De structuur van de archimediese klassen en de $\aleph_o$ -klassen wordt nagegaan. De partieel geordende verzameling der $\aleph_o$ -klassen {resp. van de archimediese klassen en van de carriers} is een distributief tralie. De afbeeldingen $\varphi: a \rightarrow a^-$ van P op het tralie der archimediese klassen en $\psi: a \rightarrow a^\circ$ van P op het tralie der $\aleph_o$ -klassen worden onderzocht {de overeenkomstige afbeelding van P op het tralie der carriers is onderzocht door P. Jaffard e.a.}.

§ 4 heeft betrekking op de structuur van G indien het tralie der $\aleph_o$ -klassen relatief gekomplementeerd is. De eerste insluitstelling (blz. 29) geeft de samenhang tussen de genoemde eigenschap en andere eigenschappen van G .

§ 5 heeft betrekking op de structuur van G indien het tralie der archimediese klassen relatief gekomplementeerd is. De tweede insluitstelling (blz. 32) legt het verband tussen deze en andere eigenschappen van G . Indien G een vektortralie is wordt bewezen: G is een funktietralie van enkelvoudige funkties dat de eenheidsfunctie bevat, dan en slechts dan als het tralie der archimediese klassen van G een algebra van Boole is (stelling 5.4).

In § 6 wordt een begin gemaakt met de studie van de normalisatoren in een traliegroep. Het verband wordt gelegd tussen eigenschappen der tralies van de archimediese klassen en de $\aleph_o$ -klassen enerzijds en de normalisatoren van a^- en a° anderzijds. Voor de deelverzamelingen a , a^- , a° en $a^\wedge$ blijkt de normalisator een deeltralie van G te zijn.

In § 7 wordt een nieuw en meer rechtstreeks bewijs gegeven voor de stelling dat een archimediese traliegroep kommutatief is.

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BIOGRAPHY

The author of this thesis was born in the Hague in 1938. In 1955 he finished the high-school and started a three years job on the mathematical department of a life insurance company. He entered the Leiden University in 1958, after he obtained the "M.O.-A" certificate in mathematics. In 1959/60 R.C.N. (Reactor Centre of the Netherlands) enabled him to do a half years research at the atomic institute in Kjeller (Norway). In the course of his study he also was a teacher in mathematics at a high-school during four years. In 1964 he graduated in mathematics and physics at the Leiden University and he was appointed as a member of the staff in the Department of Mathematics of the University of Technology, Delft.

STELLINGEN

I

a

De bewering van Birkhoff¹ dat het tralie der „carriers” van een kommutatieve traliegroep een algebra van Boole is, is onjuist. Zelfs voor een archimediese traliegroep met een zwakke eenheid geldt de bewering niet.²

b

De bewering van Birkhoff³ dat het tralie der „carriers” van een niet-kommutatieve traliegroep niet noodzakelijk een algebra van Boole is, is juist. Het gegeven voorbeeld is fout.⁴

¹ G. Birkhoff, *Lattice Theory*, 3rd edition (New York 1967), blz. 310.

² De eerste insluitstelling van dit proefschrift, blz. 29.

³ I.c., blz. 311.

⁴ P. Jaffard, *Contributions à l'étude des groupes ordonnés*. J. Math. Pures Appl. (1950) 203–280.

II

Luxemburg en Zaanen¹ bewijzen dat in een vektortralie V geldt:

$$2(|a| \vee |b|) = |a+b| + |a-b| \text{ voor } a, b \in V \text{ en } 2(|a| \wedge |b|) = ||a+b| - |a-b|| \text{ voor } a, b \in V.$$

¹ W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces*, preprint (1967).

Een generalisatie van deze stelling luidt:

voor een traliegroep G zijn de volgende eigenschappen gelijkwaardig:

- (i) G is kommutatief,
- (ii) $2(|a| \vee |b|) = |a+b| + |a-b|$ voor $a, b \in G$,
- (iii) $2(|a| \wedge |b|) = ||a+b| - |a-b||$ voor $a, b \in G$.

III

Stelt G een traliegroep voor, dan geldt voor $a, b \in G$:

$$|a| \wedge |b| = 0 \text{ dan en slechts dan als } |a+b| = |a-b| = |b+a| = |-b+a|$$

Voor het bijzondere geval dat G een vektortralie is, is deze stelling een direkt gevolg van de onder II genoemde stelling van Luxemburg en Zaanen. Zie ook: Wiskundige opgaven 1964, opgave 192.

IV

Iedere rij $\{c_n\}$, waarvan de elementen geschreven kunnen worden als $c_n = na \wedge b$ voor positieve elementen a en b uit een traliegroep G , heet een *basisrij* in G . Zij $\{c_n\}$ een basisrij in G en $\{m(n)\}$ een monotoon stijgende rij van natuurlijke getallen, dan heet de rij $\{d_n\}$, met $d_n = c_{m(n)} - c_n$, een *verschilrij* in G . Voor de begrippen *nulrij* en *fundamenteaalrij* wordt verwezen naar: L. Fuchs, Teilweise geordnete algebraische Strukturen, (Budapest 1966), blz. 150.

G is een archimediese traliegroep dan en slechts dan als iedere verschilrij in G een nulrij is.

Deze stelling doet vermoeden dat in een archimediese traliegroep iedere basisrij een fundamenteaalrij is.

V

Ten onrechte beweert van Veen¹ dat de door hem genoemde voorwaarde voor de lineaire afhankelijkheid van een stelsel functies voldoende is.

¹ Dr. L. Kuipers & Dr. R. Timman, Handboek der Wiskunde, 2e druk (Delft 1966). Hoofdstuk VIII: Dr. S. C. van Veen, Gewone differentiaalvergelijkingen (par. 5.4).

VI

De homologische benadering van de theorie der abelse groepen verdient ruimere belangstelling; zij biedt niet alleen methodologische voordelen, maar zij verschaft ook een beter inzicht in de structuur van belangrijke klassen van abelse groepen.

VII

Een van de paradoxen van Jones¹ luidt: „The more human-like a computer becomes, the less work it does”. De snelle ontwikkeling op het gebied van informatiebewerkende automaten heeft het paradoxale karakter van deze uitspraak achterhaald. Het is een onjuiste uitspraak.

¹ P. D. Jones, Thirteen programming paradoxes. Datamation (1966).

VIII

Het is gewenst dat – in beginsel – alle leden van de wetenschappelijke corpora van universiteiten en hogescholen, voor zover ze bij het onderwijs zijn betrokken, jaarlijks optreden als gekommitteerden/deskundigen bij de eindexamens van scholen voor het voorbereidend wetenschappelijk onderwijs. Deze werkzaamheid zou tot hun normale taak moeten behoren.

IX

De samenstelling van de zogenaamde „groep”, genoemd in het rapport-van Os,¹ is te beperkt; er zouden ook studenten toe moeten behoren.

¹ „Structuur van het wetenschappelijk corps”. Rapport van de kommissie ad hoc van de Academische Raad, herzien (1968).

X

Het gesprek over de medezeggenschap van de werknemers in de onderneming komt in Nederland niet boven een akademies niveau uit. De vakvereniging, als meest belanghebbende, kan hierin verbetering brengen door een aantal ondernemingen in eigendom te verwerven en deze in te richten als experimenteerbedrijven c.q. modelbedrijven.

XI

Het minste dat de senaat der Technische Hogeschool Delft kan doen – en dus moet doen – om de kosten van het promoveren te verlagen, is duidelijk en openbaar te verklaren dat zij algehele vrijstelling geeft van de verplichting om het proefschrift te laten drukken. Iedere wijze van vermenigvuldigen die goed leesbare kopieën van het proefschrift oplevert, dient te worden goedgekeurd.

