MSc Thesis

VERIFICATION AND NUMERICAL IMPLEMENTATION OF A 3D LIQUEFACTION MODEL

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Preface

This report is aimed at verification of a 3D liquefaction model for DIANA software. The model has originally been developed by Nishimura (2002). In the first chapter of this report, a detailed study of the model itself will be made. To this end, the expressions involved in the model will be derived and their agreement with those given in the original model will be checked. Further elaboration of the model will also be made to have a better understanding of the model. The derivations and elaborations helped to identify some aspects of the model which need further improvement before being made available for customers.

A Fortran program which is based on this model is also already available. In the second chapter, this subroutine will be studied in detail and its consistency with the expressions given in the earlier chapter will be checked. First, each variable involved in the program will be defined according to those given in chapter one. The function of each subroutine involved in the program will be explained with further elaborations when necessary. The flow of analysis of the whole program will be reviewed in detail by recalling the expressions given in chapter one.

The third chapter will be about the input data file which will be used for the liquefaction analysis. This data file contains the material and state parameters which will be used in the program discussed in chapter two. Each component of the data file will be discussed. In this chapter, a guide for the determination of the material parameters from laboratory tests will also be given.

Analytical verification of the program for simple boundary conditions will be the main task in chapter four. For this, a simple shear model will be constructed and analytical calculations will be made for computing the resulting deviatoric stresses from a given strain. A liquefaction analysis will be made by DIANA software for a similar model and the results will be compared with the analytical ones.

The verification process will be extended further in chapter five by comparing DIANA software results with laboratory observation and other numerical simulations. Liquefaction analyses will be made for different types of loading and drainage conditions. From the result by DIANA software, important graphs will be plotted and compared with those from laboratory observations. Depending on the results of the comparisons, explanations will be given.

Depending on the discussions in the previous chapters, the last chapter gives conclusions and recommendations.

Acknowledgement

Thanks to God for helping and finally letting me express my gratefulness to those peoples who helped me to complete this thesis. My first appreciation goes to Prof. Molenkamp, whose continuous support and suggestions has been helpful to go forward. This thesis was done in a company which produces state-of-the-art civil engineering softwares, which wouldn't be possible without the allowance of Dr. Schreppers. I would like to acknowledge him for letting me do the thesis at the company. In addition , his supports and comments has been instrumental. I would also like to extend my gratitude for Dr. Hendriks who was also giving helpful ideas and suggestions.

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Asaye Dilbo

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Summary

With the advancement of solution techniques and solving computers, 3D analysis of civil engineering problems has increasingly become more interesting. The multiple spring model is one of the tools to give good solutions to 3D liquefaction analyses. In this model, the deviatoric stress is determined in a finite number of springs distributed over virtual planes in the soil element for which liquefaction analysis is to be undergone. Among the several options for the distribution of the virtual planes in the soil element, it was previously found that an icosahedral distribution results in an isotropic model.

For the displacement based analysis which is going to be discussed in this report, the global strains will be decomposed into one-dimensional strains in each spring through transformation matrices. Then the Masing rule after several modifications will be used to obtain stress ratio from those transformed strains. The product of the stress ratio and the mean effective stress gives the shear stress in each spring. The global shear stress of the soil mass is calculated from the shear stress in each spring through transformation matrices.

The model also uses stress-dilatancy relationships to calculate volumetric strain due to dilatancy which enables to calculate the volumetric strain due to consolidation. Expressions for a curve of isotropic compression or swelling help to calculate the mean effective stress. Along with the stress ratio, it is this mean effective stress that will be used to calculate the shear stress in each spring.

In this report, it is discovered that the icosahedral distribution of planes results in an isotropic behavior. However, the way the springs are oriented on those plane as described in the original model by Nishimura (2002) will not result in an isotropic behavior. At the end of the report, suggestions will be given to overcome this anisotropy. It will also be seen that the volumetric strain due to dilatancy is overestimated in the model. The source for the overestimation is discovered and will be forwarded for further improvement. Suggestions for the modification of the hysteresis loop when the stress ratio in the past is exceeded will also be given.

List of symbols

Scalar quantities

x,y,z – Cartesian coordinates of the original coordinate system
$x'''\ y'''\ z'''$ - Cartesian coordinates of the rotated coordinate system
θ – the rotation angle around z axis at the first stage of transformation of coordinated
ϕ – the rotation angle around y' axis at the second stage of transformation of coordinated
ζ – the rotation angle around z" axis at the third stage of transformation of coordinated
l_i- the direction cosine between the global x-axis and the rotated coordinate system
m_i- the direction cosine between the global x-axis and the rotated coordinate system
n_i- the direction cosine between the global x-axis and the rotated coordinate system
e_i – vector containing the unit vectors in the original coordinate system
$e_{i'''}$ – vector containing the unit vectors in the rotated coordinate system
ε_{ij} – strain quantity in each direction
ε_v – volumetric strain
p – mean effective stress
n- the number of springs in the icosahedral distribution
$G_{\tan}^{(i)}$ - tangential shear modulus of each spring
B- bulk modulus
$\gamma^{(i)}$ - shear strain of each spring
$R_{\max}^{*}$ - maximum stress ratio at infinite strain
$G_{\max}$ -shear modulus at small strain
$G_{\max,o}$ -shear modulus at small strain at initial mean effective stress
$k_{\max}$ -spring stiffness
$k_{\max,o}$ -spring stiffness at initial mean effective stress
γ_r - reference shear strain
$\gamma_{r,o}$ - reference shear strain at initial mean effective stress
$R^{(i)}$ - stress ratio of each spring
$R_{rev}^{(i)}$ - stress ratio for reversal point of each spring
$\gamma_{rev}^{(i)}$ - shear strain at reversal point of each spring
$\gamma_a^{(i)}$ - shear strain amplitude of each spring
$R_a^{(i)}$ - maximum shear stress ratio
η – reduction factor of the hysteresis loop
$\gamma_{\max}^{(i)}$ - maximum shear strain of each spring
$C^{(i)}$ – scaling factor for $R_{\max}$
ΔW- total strain energy.
W – elastic work done.
h– damping ratio.
R' – stress ratio after hardening is considered.

H_p – factor to consider hardening effect
$G_{eq}^{(i)}$ - equivalent elastic tangent shear stiffness
$d\varepsilon_v^d$ - increment of volumetric strain due to dilatancy
$d\varepsilon_v^c$ - increment of volumetric strain due to consolidation
$d\gamma^{p,(i)}$ - increment of plastic shear strain
$d\gamma^{e,(i)}$ - increment of elastic shear strain
N_d – slope of stress ratio versus stress dilatancy curve.
Cc – compression index
p_y' - yield stress
$d\varepsilon_{v,y}^c$ - increment of volumetric strain at yield stress
B_o – bulk stiffness at initial mean effective stress.

Matrix quantities

[M] – transformation matrix containing the direction cosines.
[ε] – strain tensor
{ ε }- strain vector in the original coordinate system
{ ε''' }- strain vector in the rotated coordinate system
[T_ε] – overall transformation matrix for engineering strains
[$T_{\varepsilon,\theta}$] –transformation matrix for engineering strains around global z axis
[$T_{\varepsilon,\phi}$] –transformation matrix for engineering strains around rotated y' axis
[$T_{\varepsilon,\zeta}$] –transformation matrix for engineering strains around rotated z'' axis
[σ] – stress tensor
{ σ }- stress vector in the original coordinate system
{ σ''' }- stress vector in the rotated coordinate system
[T_σ] – overall transformation matrix for stresses
{ I }- isotropic unit vector
[N] – selector matrix
{ τ }- deviatoric stress vector in the original coordinate system
{ $\varepsilon^{(i)}$}- strain vector of the rotated coordinate system
{ $\gamma^{(i)}$}- strain vector of each spring
[G]- tangential shear stiffness matrix
[B]- bulk stiffness matrix
[K]- tangential stiffness matrix
[D] – stiffness matrix for dilatancy effect
[C] – compliance matrix
[K'] – overall tangential stiffness matrix with dilatancy effect

CHAPTER ONE

DEFINITION OF THE MULTIPLE SPRING MODEL

1.1. Background of the multi-spring model

The multiple inelastic spring model is shown in fig 1 below. The model consists of an infinite number of non-linear springs and can take into account the effects of principal stress axes rotation. Rotation of principal axes is the phenomenon by which the principal stress axes rotates during progressive shearing of the soil element as it often occurs during cyclic liquefaction processes. When an external force is applied at the center of the model, the surrounding springs deform and the center point moves. By assuming that the external force represents the shear stress and that the displacement of the center point stands for the shear strain under plane strain conditions, shear distortion of the soil can be predicted.

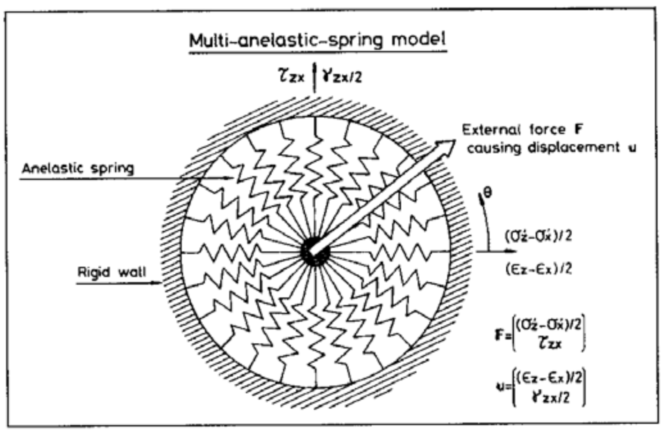


Fig. 1.1. Multiple spring model

1.2. General structure of the model

The essence of multiple shear mechanism is to express two- or three-dimensional shear stress-strain relationships as a summation of one-dimensional stress-strain relationships mobilized in virtual planes, which are oriented to various directions inside a soil element as shown in Fig. 1.2. Each plane contains two-dimensional shear stress and shear strain and hence two degrees of freedoms on each plane. To reduce the degree of freedom to one, the shear in the virtual plane is further broken down into several one-dimensional shear mechanisms.

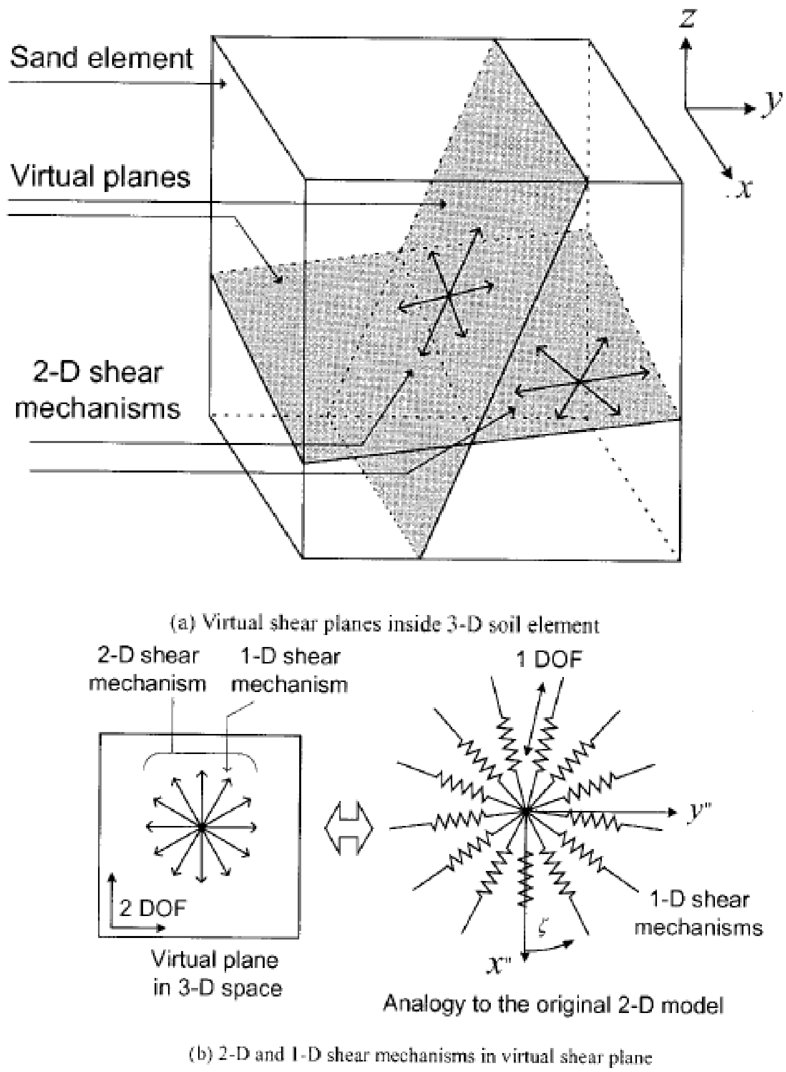


Figure 1.2 . a) Virtual planes inside the 3-D soil element b) Decomposition of 3-D shear into 2-D and 1-D shear mechanisms.

The distribution of the prepared virtual planes determines characteristics of modeled shear behaviors. If the orientation of the planes which contain the springs aligned with constant intervals is regular and omni-directional, the model will be isotropic; otherwise it will become anisotropic. The strain is related to stresses in such a manner that, first, the shear strains of one-dimensional springs are calculated from strain components in an overall system. Then, the corresponding shear stresses are obtained based on spring characteristics and summed to become the shear stress in the overall system.

1.3. Rotation of coordinate systems

The shear strain in each one-dimensional shear mechanism is obtained by coordinate transformation from the global shear strain. The coordinate transformation between the global xyz and the coordinate of a particular spring $x'''y'''z'''$ follows the following steps as

shown in Figs. 1.3 through 1.6 below. The aim of the axes rotation is to have one of the cartesian axes of the new coordinate system aligned with each of the springs.
First, a new system $x'y'z'$ will be formed by rotating the original xyz coordinate system by angle θ around z axis.

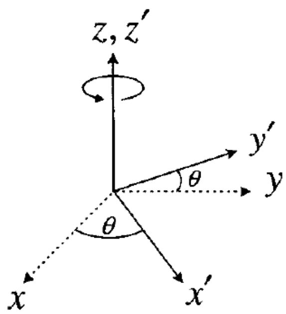


Figure 1.3 . Rotation of the coordinate *xyz* system by an angle θ around *z* axis.

Further rotation of the $x'y'z'$ coordinate system by angle ϕ around y' axis will result in a new $x''y''z''$ system.

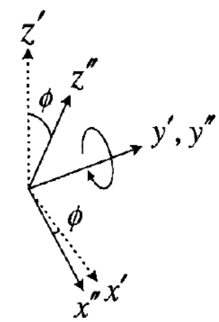


Figure 1.4. Rotation of the coordinate $x'y'z'$ system by an angle ϕ around *y'* axis.

The plane $x''y''$ which is shown in fig. 1.5 below is assumed to correspond to one of the virtual shear planes inside the 3-D soil element shown in fig. 1.2.

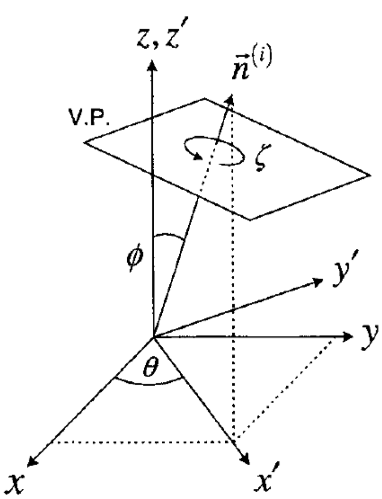


Figure 1.5. The $x''y''$ plane containing the springs

Finally the $x''y''z''$ will be rotated step-by-step around z'' axis by an angle ζ so that the y''' axis of the new $x'''y'''z'''$ system will lie on each of the one-dimensional shear mechanisms.

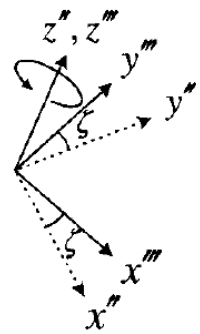


Figure 1.6. Rotation of the coordinate $x''y''z''$ system by an angle ζ around z'' axis.

Hence, the full transformation between the original xyz and the new $x'''y'''z'''$ axis involves coordinate rotation with regard to three angles θ, ϕ and ζ. The angles between a vector and the positive axes of the original coordinate system are termed as direction angles. The cosines of these direction angles are termed as direction cosines. The direction cosines between the original and the new coordinate system are given as l_i, m_i, and n_i in table 1.1.

Table 1.1 . Direction cosines between the original and the rotated coordinate system

	x	y	z
x'''	l_1	m_1	n_1
y'''	l_2	m_2	n_2
z'''	l_3	m_3	n_3

1.4. Stress and strain transformation

The stresses and strains given in original xyz coordinate system can be transformed into $x'''y'''z'''$ coordinate system. If the unit vectors along the x, y and z axes of the original coordinate system be $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ respectively. And the corresponding unit vectors in the transformed coordinate system are $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$. Let $e_i = \{\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \mathbf{e}_3\}^T$ and $e_{i'''} = \{\hat{\mathbf{e}}_1 \;\; \hat{\mathbf{e}}_2 \;\; \hat{\mathbf{e}}_3\}^T$. Then following relationship can be developed between $\{e_i\}$ and $\{e_{i'''}\}$.

$$e_{i'''} = [M]_{i'''i}\, e_i \qquad\qquad e_i = [M]^T_{ii'''}\, e_{i'''} \tag{1.1}$$

In which the coordinate transformation matrix containing the direction cosines given in table 1 is given by,

$$[M]_{i'''i} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \quad and \quad [M]^T_{ii'''} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{1.2}$$

It can be proved that the matrix $[M]_{i'''i}$ is an orthogonal matrix i.e. $[M]^T_{ii'''} = [M]^{-1}_{ii'''}$

A. Strain transformation

The strain tensor is given by the expression:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \varepsilon_{ij} e_i \otimes e_j \tag{1.3}$$

The same strain tensor can be expressed in the $x'''y'''z'''$ coordinate system as:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{x'''x'''} & \varepsilon_{x'''y'''} & \varepsilon_{x'''z'''} \\ \varepsilon_{y'''x'''} & \varepsilon_{y'''y'''} & \varepsilon_{y'''z'''} \\ \varepsilon_{z'''x'''} & \varepsilon_{z'''y'''} & \varepsilon_{z'''z'''} \end{bmatrix} = \varepsilon_{i'''j'''} e_{i'''} \otimes e_{j'''} \tag{1.4}$$

Depending on this, the following relation can be developed,

$$[\varepsilon] = \varepsilon_{i'''j'''} e_{i'''} \otimes e_{j'''} = \varepsilon_{ij} e_i \otimes e_j$$
$$\Rightarrow \varepsilon_{i'''j'''} e_{i'''} \otimes e_{j'''} = \varepsilon_{ij} [M]^T_{ii'''} e_{i'''} \otimes [M_{jj'''}]^T e_{j'''} = \varepsilon_{ij} [M]^T_{ii'''} [M]^T_{jj'''} e_{i'''} \otimes e_{j'''}$$
$$= [M]_{i'''i} \varepsilon_{ij} [M]^T_{jj'''} e_{i'''} \otimes e_{j'''}$$

$$\Rightarrow \varepsilon_{i'''j'''} = [M]_{i'''i} \varepsilon_{ij} [M]^T_{jj'''} \tag{1.5}$$

Therefore,

$$[\varepsilon_{i'''j'''}] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{1.6}$$

$$= \begin{bmatrix} l_1\varepsilon_{xx} + m_1\varepsilon_{yx} + n_1\varepsilon_{zx} & l_1\varepsilon_{xy} + m_1\varepsilon_{yy} + n_1\varepsilon_{zy} & l_1\varepsilon_{xz} + m_1\varepsilon_{yz} + n_1\varepsilon_{zz} \\ l_2\varepsilon_{xx} + m_2\varepsilon_{yx} + n_2\varepsilon_{zx} & l_2\varepsilon_{xy} + m_2\varepsilon_{yy} + n_2\varepsilon_{zy} & l_2\varepsilon_{xz} + m_2\varepsilon_{yz} + n_2\varepsilon_{zz} \\ l_3\varepsilon_{xx} + m_3\varepsilon_{yx} + n_3\varepsilon_{zx} & l_3\varepsilon_{xy} + m_3\varepsilon_{yy} + n_3\varepsilon_{zy} & l_3\varepsilon_{xz} + m_3\varepsilon_{yz} + n_3\varepsilon_{zz} \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{1.7}$$

Considering the symmetry of strain tensor, the matrix product leads to the following:

$$\varepsilon_{x'''x'''} = l_1^2\varepsilon_{xx} + m_1^2\varepsilon_{yy} + n_1^2\varepsilon_{zz} + 2l_1m_1\varepsilon_{xy} + 2m_1n_1\varepsilon_{yz} + 2l_1n_1\varepsilon_{zc}$$
$$\varepsilon_{x'''y'''} = \varepsilon_{y'''x'''} = l_1l_2\varepsilon_{xx} + m_1m_2\varepsilon_{yy} + n_1n_2\varepsilon_{zz} + (l_1m_2 + l_2m_1)\varepsilon_{xy} + (m_1n_2 + m_2n_1)\varepsilon_{yz} + (l_3n_2 + l_2n_3)\varepsilon_{zx}$$
$$\varepsilon_{x'''z'''} = \varepsilon_{z'''x'''} = l_1l_3\varepsilon_{xx} + m_1m_3\varepsilon_{yy} + n_1n_3\varepsilon_{zz} + (l_1m_3 + l_3m_1)\varepsilon_{xy} + (m_1n_3 + m_3n_1)\varepsilon_{yz} + (l_1n_3 + l_3n_1)\varepsilon_{zx}$$
$$\varepsilon_{y'''y'''} = l_2^2\varepsilon_{xx} + m_2^2\varepsilon_{yy} + n_2^2\varepsilon_{zz} + 2l_2m_2\varepsilon_{xy} + 2m_2n_2\varepsilon_{yz} + 2l_2n_2\varepsilon_{zx}$$
$$\varepsilon_{y'''z'''} = \varepsilon_{z'''y'''} = l_2l_3\varepsilon_{xx} + m_2m_3\varepsilon_{yy} + n_2n_3\varepsilon_{zz} + (l_2m_3 + l_3m_2)\varepsilon_{xy} + (m_2n_3 + m_3n_2)\varepsilon_{yz} + (l_2n_1 + l_1n_2)\varepsilon_{zx}$$
$$\varepsilon_{z'''z'''} = l_3^2\varepsilon_{xx} + m_3^2\varepsilon_{yy} + n_3^2\varepsilon_{zz} + 2l_3m_3\varepsilon_{xy} + 2m_3n_3\varepsilon_{yz} + 2l_3n_3\varepsilon_{zx}$$

These equations can be put in the matrix format as follows:

$$\begin{Bmatrix} \varepsilon_{x'''x'''} \\ \varepsilon_{y'''y'''} \\ \varepsilon_{z'''z'''} \\ \varepsilon_{x'''y'''} \\ \varepsilon_{y'''z'''} \\ \varepsilon_{z'''x'''} \end{Bmatrix} = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & 2l_1m_1 & 2m_1n_1 & 2l_1n_1 \\ l_2^2 & m_2^2 & n_2^2 & 2l_2m_2 & 2m_2n_2 & 2l_2n_2 \\ l_3^2 & m_3^2 & n_3^2 & 2l_3m_3 & 2m_3n_3 & 2l_3n_3 \\ l_1l_2 & m_1m_2 & n_1n_2 & l_1m_2 + l_2m_1 & m_1n_2 + m_2n_1 & l_3n_2 + l_2n_3 \\ l_2l_3 & m_2m_3 & n_2n_3 & l_2m_3 + l_3m_2 & m_2n_3 + m_3n_2 & l_2n_1 + l_1n_2 \\ l_1l_3 & m_1m_3 & n_1n_3 & l_3m_1 + l_1m_3 & m_3n_1 + m_1n_3 & l_1n_3 + l_3n_1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} \quad (1.8)$$

This matrix equation can also be written as in the following form to give the engineering shear strains.

$$\begin{Bmatrix} \varepsilon_{x'''x'''} \\ \varepsilon_{y'''y'''} \\ \varepsilon_{z'''z'''} \\ 2\varepsilon_{x'''y'''} \\ 2\varepsilon_{y'''z'''} \\ 2\varepsilon_{z'''x'''} \end{Bmatrix} = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & l_1m_1 & m_1n_1 & l_1n_1 \\ l_2^2 & m_2^2 & n_2^2 & l_2m_2 & m_2n_2 & l_2n_2 \\ l_3^2 & m_3^2 & n_3^2 & l_3m_3 & m_3n_3 & l_3n_3 \\ 2l_1l_2 & 2m_1m_2 & 2n_1n_2 & l_1m_2 + l_2m_1 & m_1n_2 + m_2n_1 & l_3n_2 + l_2n_3 \\ 2l_2l_3 & 2m_2m_3 & 2n_2n_3 & l_2m_3 + l_3m_2 & m_2n_3 + m_3n_2 & l_2n_1 + l_1n_2 \\ 2l_1l_3 & 2m_1m_3 & 2n_1n_3 & l_3m_1 + l_1m_3 & m_3n_1 + m_1n_3 & l_1n_3 + l_3n_1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} \quad (1.9)$$

The strain vectors in the original and rotated coordinate systems can then be written as:

$$\{\varepsilon\} = \{\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}\}^T \quad (1.10)$$

$$\{\varepsilon'''\} = \{\varepsilon_{x'''} \quad \varepsilon_{y'''} \quad \varepsilon_{z'''} \quad \gamma_{x'''y'''} \quad \gamma_{y'''z'''} \quad \gamma_{z'''x'''}\}^T \quad (1.11)$$

Therefore, eq. (1.9) can be written as:

$$\{\varepsilon'''\} = [T_\varepsilon]\{\varepsilon\} \quad (1.12)$$

Where $[T_\varepsilon]$ is the transformation matrix of engineering strains from the xyz coordinate system to $x'''y'''z'''$.The transformation matrix $[T_\varepsilon]$ will have the format as in (1.9), namely

$$[T_\varepsilon]=\begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & l_1m_1 & m_1n_1 & l_1n_1 \\ l_2^2 & m_2^2 & n_2^2 & l_2\,m_2 & m_2n_2 & l_2n_2 \\ l_3^2 & m_3^2 & n_3^2 & l_3m_3 & m_3n_3 & l_3n_3 \\ 2l_1l_2 & 2m_1m_2 & 2n_1n_2 & l_1m_2+l_2m_1 & m_1n_2+m_2n_1 & l_3n_2+l_2n_3 \\ 2l_2l_3 & 2m_2m_3 & 2n_2n_3 & l_2m_3+l_3m_2 & m_2n_3+m_3n_2 & l_2n_1+l_1n_2 \\ 2l_1l_3 & 2m_1m_3 & 2n_1n_3 & l_3m_1+l_1m_3 & m_3n_1+m_1n_3 & l_1n_3+l_3n_1 \end{bmatrix} \quad (1.13)$$

As mentioned earlier, the rotations in this model are aimed to align the z''' axis with the respective spring. Hence the strains in the rotated coordinate system belong to that of the springs. In the subsequent parts, the strain vector $\{\varepsilon'''\}$ will be written as $\{\varepsilon^{(i)}\}$. A superscript (i) will always indicate that the quantity being considered belongs to a spring.

The overall transformation matrix $[T_\varepsilon]$ can also be obtained as a product of the three matrices $[T_{\varepsilon,\theta}]$,$[T_{\varepsilon,\phi}]$ and $[T_{\varepsilon,\zeta}]$ which respectively represent the coordinate transformation processes given in figs. (1.3), (1.4) and (1.6) respectively.

When a new system $x'y'z'$ will be formed by rotating the original xyz coordinate system by angle θ around z axis, the transformation matrix to the new coordinate system $[T_{\varepsilon,\theta}]$ can be obtained by substituting the correct values of the direction cosines in eq. (1.13).

Referring to fig 1.3, the direction cosines for the first rotation around z axis will be:

$$\begin{aligned} &l_1=\cos\theta && l_2=\cos(\frac{\pi}{2}+\theta)=-\sin\theta && l_3=\cos\frac{\pi}{2}=0 \\ &m_1=\cos(\frac{\pi}{2}-\theta)=\sin\theta && m_2=\cos\theta && m_3=\cos\frac{\pi}{2}=0 \\ &n_1=\cos(\frac{\pi}{2})=0 && n_2=\cos(\frac{\pi}{2})=0 && n_3=\cos 0=1 \end{aligned} \quad (1.14)$$

Substitution of these values in eq. (1.13) will result in,

$$[T_{\varepsilon,\theta}]=\begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & \frac{1}{2}\sin 2\theta & 0 & 0 \\ \sin^2\theta & \cos^2\theta & 0 & -\frac{1}{2}\sin 2\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin 2\theta & \sin 2\theta & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \quad (1.15)$$

Referring to fig 1.4, the direction cosines for the second rotation around y' axis will be:

$$
\begin{array}{lll}
l_1 = \cos\phi & l_2 = \cos\frac{\pi}{2} = 0 & l_3 = \cos(\frac{\pi}{2} - \phi) = \sin\phi \\
m_1 = \cos\frac{\pi}{2} = 0 & m_2 = \cos 0 = 1 & m_3 = \cos\frac{\pi}{2} = 0 \\
n_1 = \cos(\frac{\pi}{2} + \phi) = -\sin\phi & n_2 = \cos(\frac{\pi}{2}) = 0 & n_3 = \cos\phi
\end{array}
\tag{1.16}
$$

Substitution of these values in eq. (1.13) will result in,

$$
[T_{\varepsilon,\phi}] = \begin{bmatrix}
\cos^2\phi & 0 & \sin^2\phi & 0 & 0 & -\frac{1}{2}\sin 2\phi \\
0 & 1 & 0 & 0 & 0 & 0 \\
\sin^2\phi & 0 & \cos^2\phi & 0 & 0 & \frac{1}{2}\sin 2\phi \\
0 & 0 & 0 & \cos\phi & -\sin\phi & 0 \\
0 & 0 & 0 & \sin\phi & \cos\phi & 0 \\
\sin 2\phi & 0 & -\sin 2\phi & 0 & 0 & \cos 2\phi
\end{bmatrix}
\tag{1.17}
$$

Referring to fig 1.6, the direction cosines for the third rotation around z*"* will be:

$$
\begin{array}{lll}
l_1 = \cos\zeta & l_2 = \cos(\frac{\pi}{2} + \zeta) = -\sin\zeta & l_3 = \cos\frac{\pi}{2} = 0 \\
m_1 = \cos(\frac{\pi}{2} - \zeta) = \sin\zeta & m_2 = \cos\zeta & m_3 = \cos\frac{\pi}{2} = 0 \\
n_1 = \cos(\frac{\pi}{2}) = 0 & n_2 = \cos(\frac{\pi}{2}) = 0 & n_3 = \cos 0 = 1
\end{array}
\tag{1.18}
$$

Substitution of these values in eq. (1.13) will result in,

$$
[T_{\varepsilon,\zeta}] = \begin{bmatrix}
\cos^2\zeta & \sin^2\zeta & 0 & \frac{1}{2}\sin 2\zeta & 0 & 0 \\
\sin^2\zeta & \cos^2\zeta & 0 & -\frac{1}{2}\sin 2\zeta & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\sin 2\zeta & \sin 2\zeta & 0 & \cos 2\zeta & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\zeta & -\sin\zeta \\
0 & 0 & 0 & 0 & \sin\zeta & \cos\zeta
\end{bmatrix}
\tag{1.19}
$$

The overall transformation matrix will be obtained as:

$$
[T_\varepsilon] = [T_{\varepsilon,\zeta}][T_{\varepsilon,\phi}][T_{\varepsilon,\theta}] \tag{1.20}
$$

B. Stress transformation

The stress transformation follows the same procedure as the strain. The stress tensor can be given in the original and rotated coordinate system as:

$$[\sigma]=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}=\sigma_{ij}e_i\otimes e_j=\begin{bmatrix}\sigma_{x'''x'''} & \sigma_{x'''y'''} & \sigma_{x'''z'''}\\ \sigma_{y'''x'''} & \sigma_{y'''y'''} & \sigma_{y'''z'''}\\ \sigma_{z'''x'''} & \sigma_{z'''y'''} & \sigma_{z'''z'''}\end{bmatrix}=\sigma_{i'''j'''}e_{i'''}\otimes e_{j'''} \tag{1.21}$$

Depending on this, the following relation can be developed as it was done for the strain transformation,

$$\sigma_{i'''j'''}=[M]_{i'''i}\,\sigma_{ij}\,[M]^T_{jj'''} \tag{1.22}$$

Following the same procedure as for the strain transformation matrix, the stresses in the original and rotated coordinate systems can be related as:

$$\begin{Bmatrix}\sigma_{x'''x'''}\\ \sigma_{y'''y'''}\\ \sigma_{z'''z'''}\\ \sigma_{x'''y'''}\\ \sigma_{y'''z'''}\\ \sigma_{z'''x'''}\end{Bmatrix}=\begin{bmatrix}l_1^2 & m_1^2 & n_1^2 & 2l_1m_1 & 2m_1n_1 & 2l_1n_1\\ l_2^2 & m_2^2 & n_2^2 & 2l_2m_2 & 2m_2n_2 & 2l_2n_2\\ l_3^2 & m_3^2 & n_3^2 & 2l_3m_3 & 2m_3n_3 & 2l_3n_3\\ l_1l_2 & m_1m_2 & n_1n_2 & l_1m_2+l_2m_1 & m_1n_2+m_2n_1 & l_1n_2+l_2n_1\\ l_2l_3 & m_2m_3 & n_2n_3 & l_2m_3+l_3m_2 & m_2n_3+m_3n_2 & l_2n_3+l_3n_1\\ l_1l_3 & m_1m_3 & n_1n_3 & l_3m_1+l_1m_3 & m_3n_1+m_1n_3 & l_1n_3+l_3n_1\end{bmatrix}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix} \tag{1.23}$$

The stress vectors in the original and rotated coordinate systems can then be written as:

$$\{\sigma\}=\{\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}\}^T \tag{1.24}$$

$$\{\sigma'''\}=\{\sigma_{x'''} \quad \sigma_{y'''} \quad \sigma_{z'''} \quad \tau_{x'''y'''} \quad \tau_{y'''z'''} \quad \tau_{z'''x'''}\}^T \tag{1.25}$$

Hence, eq. (1.23) can be written as:

$$\{\sigma'''\}=[T_\sigma]\{\sigma\} \tag{1.26}$$

Where $[T_\sigma]$ is the transformation matrix of stresses from the xyz coordinate system to $x'''y'''z'''$.The transformation matrix $[T_\sigma]$ will have the following format identical to eq. (1.8), namely

$$[T_\sigma]=\begin{bmatrix} l_1^{\,2} & m_1^{\,2} & n_1^{\,2} & 2\,l_1 m_1 & 2m_1 n_1 & 2l_1 n_1 \\ l_2^{\,2} & m_2^{\,2} & n_2^{\,2} & 2\,l_2\,m_2 & 2m_2 n_2 & 2l_2 n_2 \\ l_3^{\,2} & m_3^{\,2} & n_3^{\,2} & 2\,l_3 m_3 & 2m_3 n_3 & 2l_3 n_3 \\ l_1 l_2 & m_1 m_2 & n_1 n_2 & l_1 m_2 + l_2 m_1 & m_1 n_2 + m_2 n_1 & l_1 n_2 + l_2 n_1 \\ l_2 l_3 & m_2 m_3 & n_2 n_3 & l_2 m_3 + l_3 m_2 & m_2 n_3 + m_3 n_2 & l_2 n_3 + l_3 n_2 \\ l_1 l_3 & m_1 m_3 & n_1 n_3 & l_3 m_1 + l_1 m_3 & m_3 n_1 + m_1 n_3 & l_1 n_3 + l_3 n_1 \end{bmatrix} \tag{1.27}$$

The inverse the stress transformation matrix can be derived from eq. (1.22) as:

$$\sigma_{i''''j''''} = [M]_{i''''i}\,\sigma_{ij}\,[M]^T_{jj''''} \quad \Rightarrow \quad \sigma_{ij} = [M]^T_{ii''''}\,\sigma_{i''''j''''}\,[M]_{j''''j} \tag{1.28}$$

Therefore,

$$[\sigma_{ij}] = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{x'''x'''} & \tau_{x'''y'''} & \tau_{x'''z'''} \\ \tau_{y'''x'''} & \sigma_{y'''y'''} & \tau_{y'''z'''} \\ \tau_{z'''x'''} & \tau_{z'''y'''} & \sigma_{z'''z'''} \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{1.29}$$

$$= \begin{bmatrix} l_1\sigma_{x'''x'''} + l_2\tau_{y'''x'''} + l_3\tau_{z'''x'''} & l_1\tau_{x'''y'''} + l_2\sigma_{y'''y'''} + l_3\tau_{z'''y'''} & l_1\tau_{x'''z'''} + l_2\tau_{y'''z'''} + l_3\sigma_{z'''z'''} \\ m_1\sigma_{x'''x'''} + m_2\tau_{y'''x'''} + m_3\tau_{z'''x'''} & m_1\tau_{x'''y'''} + m_2\sigma_{y'''y'''} + m_3\tau_{z'''y'''} & m_1\tau_{x'''z'''} + m_2\tau_{y'''z'''} + m_3\sigma_{z'''z'''} \\ n_1\sigma_{x'''x'''} + n_2\tau_{y'''x'''} + n_3\tau_{z'''x'''} & n_1\tau_{x'''y'''} + n_2\sigma_{y'''y'''} + n_3\tau_{z'''y'''} & n_1\tau_{x'''z'''} + n_2\tau_{y'''z'''} + n_3\sigma_{z'''z'''} \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{1.30}$$

Considering symmetry of the stress tensor, the matrix product leads to the following:

$$\begin{aligned}
\sigma_{xx} &= l_1^{\,2}\sigma_{x'''x'''} + l_2^{\,2}\sigma_{y'''y'''} + l_3^{\,2}\sigma_{z'''z'''} + 2l_1 l_2\tau_{x'''y'''} + 2l_2 l_3\tau_{y'''z'''} + 2l_1 l_3\tau_{z'''x'''} \\
\tau_{xy} &= l_1 m_1\sigma_{x'''x'''} + l_2 m_2\sigma_{y'''y'''} + l_3 m_3\sigma_{z'''z'''} + (l_1 m_2 + l_2 m_1)\tau_{x'''y'''} + (m_1 n_2 + m_2 n_1)\tau_{y'''z'''} + (l_3 n_2 + l_2 n_3)\tau_{z'''x'''} \\
\tau_{xz} &= l_1 n_1\sigma_{x'''x'''} + l_2 n_2\sigma_{y'''y'''} + l_2 n_2\sigma_{z'''z'''} + (l_1 m_3 + l_3 m_1)\tau_{x'''y'''} + (m_1 n_3 + m_3 n_1)\tau_{y'''z'''} + (l_1 n_3 + l_3 n_1)\tau_{z'''x'''} \\
\sigma_{yy} &= m_1^{\,2}\sigma_{x'''x'''} + m_2^{\,2}\sigma_{y'''y'''} + m_3^{\,2}\sigma_{z'''z'''} + 2m_1\,m_2\tau_{x'''y'''} + 2m_2 m_3\tau_{y'''z'''} + 2m_1 m_3\tau_{z'''x'''} \\
\tau_{yz} &= m_1 n_1\sigma_{x'''x'''} + m_2 n_2\sigma_{y'''y'''} + m_3 n_3\sigma_{z'''z'''} + (l_2 m_3 + l_3 m_2)\tau_{x'''y'''} + (m_2 n_3 + m_3 n_2)\tau_{y'''z'''} + (l_2 n_1 + l_1 n_2)\tau_{z'''x'''} \\
\sigma_{zz} &= n_1^{\,2}\sigma_{x'''x'''} + n_2^{\,2}\sigma_{y'''y'''} + n_3^{\,2}\sigma_{z'''z'''} + 2l_3 m_3\tau_{x'''y'''} + 2m_3 n_3\tau_{y'''z'''} + 2l_3 n_3\tau_{z'''x'''}
\end{aligned}$$

This leads to the following relationship:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} l_1^2 & l_2^2 & l_3^2 & 2l_1l_2 & 2l_2l_3 & 2l_1l_3 \\ m_1^2 & m_2^2 & m_3^2 & 2m_1m_2 & 2m_2m_3 & 2m_1m_3 \\ n_1^2 & n_2^2 & n_3^2 & 2n_1n_2 & 2n_2n_3 & 2n_1n_3 \\ l_1m_1 & l_2m_2 & l_3m_3 & l_2m_1+l_1m_2 & l_3m_2+l_2m_3 & l_3m_1+l_1m_3 \\ m_1n_1 & m_2n_2 & m_3n_3 & m_2n_1+m_1n_2 & m_3n_2+m_2n_3 & m_3n_1+m_1n_3 \\ l_1n_1 & l_2n_2 & l_3n_3 & l_2n_1+l_1n_2 & l_3n_2+l_2n_3 & l_3n_1+l_1n_3 \end{bmatrix} \begin{Bmatrix} \sigma_{x'''x'''} \\ \sigma_{y'''y'''} \\ \sigma_{z'''z'''} \\ \sigma_{x'''y'''} \\ \sigma_{y'''z'''} \\ \sigma_{z'''x'''} \end{Bmatrix} \quad (1.31)$$

In short, eq. (1.31) can be written as:

$$\{\sigma\} = [T_\sigma]^{-1}\{\sigma'''\} \quad (1.32)$$

In which the inverse of the stress transformation matrix $[T_\sigma]^{-1}$ is given by:

$$[T_\sigma]^{-1} = \begin{bmatrix} l_1^2 & l_2^2 & l_3^2 & 2l_1l_2 & 2l_2l_3 & 2l_1l_3 \\ m_1^2 & m_2^2 & m_3^2 & 2m_1m_2 & 2m_2m_3 & 2m_1m_3 \\ n_1^2 & n_2^2 & n_3^2 & 2n_1n_2 & 2n_2n_3 & 2n_1n_3 \\ l_1m_1 & l_2m_2 & l_3m_3 & l_2m_1+l_1m_2 & l_3m_2+l_2m_3 & l_3m_1+l_1m_3 \\ m_1n_1 & m_2n_2 & m_3n_3 & m_2n_1+m_1n_2 & m_3n_2+m_2n_3 & m_3n_1+m_1n_3 \\ l_1n_1 & l_2n_2 & l_3n_3 & l_2n_1+l_1n_2 & l_3n_2+l_2n_3 & l_3n_1+l_1n_3 \end{bmatrix} \quad (1.33)$$

Comparing eqs. (1.13) and (1.33), the following relationship can be deduced.

$$[T_\sigma]^{-1} = [T_\varepsilon]^T \quad (1.34)$$

It follows that the following relationships will also be valid between the transformation matrices of stress and strain after each rotation.

$$\begin{aligned} [T_{\sigma,\theta}]^{-1} &= [T_{\varepsilon,\theta}]^T \\ [T_{\sigma,\phi}]^{-1} &= [T_{\varepsilon,\phi}]^T \\ [T_{\sigma,\zeta}]^{-1} &= [T_{\varepsilon,\zeta}]^T \\ [T_\sigma]^{-1} &= [T_{\sigma,\theta}]^{-1}[T_{\sigma,\phi}]^{-1}[T_{\sigma,\zeta}]^{-1} \end{aligned} \quad (1.35)$$

1.5. Shear strain in the springs

1.5.1. Decomposition into isotropic and deviatoric components

The multiple mechanism model decomposes strain components in three-dimensional space into numerous one-dimensional strain components by means of coordinate transformation.

The strain vector $\{\varepsilon\}$ which is given by eq. (1.10) can be decomposed into isotropic and deviatoric components.

$$\{\varepsilon\} = \frac{\varepsilon_v}{3}\{I\} + \{\gamma\} \tag{1.36}$$

In which,

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{1.37}$$

$$\{I\} = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\}$$

$$\{\gamma\} = \left\{\varepsilon_{xx} - \frac{\varepsilon_{vol}}{3} \quad \varepsilon_{yy} - \frac{\varepsilon_{vol}}{3} \quad \varepsilon_{zz} - \frac{\varepsilon_{vol}}{3} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}\right\} \tag{1.38}$$

The stress component given by eq. (1.24) can also be decomposed in a similar way into isotropic component and deviatoric component.

$$\{\sigma\} = p\{I\} + \{\tau\} \tag{1.39}$$

In which, the hydrostatic pressure p is given by,

$$p = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \tag{1.40}$$

And the deviatoric stress component $\{\tau\}$ is written as:

$$\{\tau\} = \{\sigma_x - p \quad \sigma_y - p \quad \sigma_z - p \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}\}^T \tag{1.41}$$

1.5.2. Decomposition of shear mechanisms

Multiplication of the strain vector in eq. (1.10) with the transformation matrix between the i^{th} inelastic spring and the global coordinate system of the soil element for strain $[T_\varepsilon^{(i)}]$ gives the strain vector of each spring, $\{\varepsilon^{(i)}\}$.

$$\{\varepsilon^{(i)}\} = [T_\varepsilon^{(i)}]\{\varepsilon\} = \{\varepsilon_x^{(i)} \quad \varepsilon_y^{(i)} \quad \varepsilon_z^{(i)} \quad \gamma_{xy}^{(i)} \quad \gamma_{yz}^{(i)} \quad \gamma_{zx}^{(i)}\}^T \tag{1.42}$$

Thus shear strain of a particular shear mechanism (namely, the shear strain in y''' direction) is the extracted from the strain vector of each spring as:

$$\{\gamma^{(i)}\} = [N]\{\varepsilon^{(i)}\} = [N][T_\varepsilon^{(i)}]\{\varepsilon\} = \{0 \quad 0 \quad 0 \quad 0 \quad \gamma_{yz}^{(i)} \quad 0\}^T \tag{1.43}$$

The matrix [*N*] is given by:

$$[N]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.44}$$

Alternatively, the value of the shear strain of each spring can be derived as:

$$\gamma^{(i)} = \{n\}^T \{\varepsilon^{(i)}\} \tag{1.45}$$

where $\{n\}$ is a vector given by:

$$\{n\} = \{0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0\}^{\mathrm{T}} \tag{1.46}$$

With the repetition of the procedure above for all springs, three dimensional strains in the original xyz coordinate system is decomposed into n one-dimensional shear strains.

1.6. Aggregate shear stress

The shear stress of the same mechanism, $\{\tau^{(i)}\}$, is then obtained from $\{\gamma^{(i)}\}$ via a one-dimensional shear stress-strain relationship. The obtained shear stress in the mechanism is transformed into stress in the original xyz coordinate system as follows:

$$\begin{aligned} &\{\tau\}^{(i)}_{xyz} = [T^{(i)}_{\sigma}]^{-1}\{\tau^{(i)}\} \\ &\textit{where} \quad \{\tau^{(i)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad \tau^{(i)} \quad 0\}^T \end{aligned} \tag{1.47}$$

The total strain increment is distributed for each spring. Hence, whenever shear stresses or strains are calculated back for the soil element, the average of the contribution from each spring should be taken. Thus, the total shear stress imposed to the soil is calculated by taking the summation (actually an average) of $\{\tau^{(i)}\}$ for n number of springs which will be considered in eq. (1.50) as

$$\{\tau\} = \frac{1}{n}\sum_{i=1}^{n}\{\tau^{(i)}\}_{xyz} = \frac{1}{n}\sum_{i=1}^{n}[T^{(i)}_{\sigma}]^{-1}\{\tau^{(i)}\} \tag{1.48}$$

Finally, the stress vector $\{\sigma\}$ is obtained by adding the mean principal stress, p, to the first three rows of $\{\tau\}$ in accordance with eq. (1.39).The derivation of p will follow later.

1.7. Formation of the tangent stiffness matrix

The basic tangent stiffness matrix can be developed from the equations developed so far. If the tangent shear modulus of the ith shear spring is given as $G^{(i)}_{\tan}$, the relationship between shear stress increment and shear strain increment is

$$\left\{d\tau^{(i)}\right\}=G_{\tan}^{(i)}\left\{d\gamma^{(i)}\right\} \tag{1.49}$$

For the total soil mass, substitution of eq. (1.48) for global deviatoric stress increment will lead to

$$\{d\tau\}=\frac{1}{n}\sum_{i=1}^{n}G_{\tan}^{(i)}[T_{\sigma}^{(i)}]^{-1}\{d\gamma^{(i)}\} \tag{1.50}$$

Inserting eq. (1.43) in this equation and using the relationship in eq. (1.34) in this equation yields,

$$\begin{aligned}\{d\tau\}&=\frac{1}{n}\sum_{i=1}^{n}G_{\tan}^{(i)}[T_{\sigma}^{(i)}]^{-1}[N][T_{\varepsilon}^{(i)}]\{d\varepsilon\}\\&=\frac{1}{n}\sum_{i=1}^{n}G_{\tan}^{(i)}[T_{\varepsilon}^{(i)}]^{T}[N][T_{\varepsilon}^{(i)}]\{d\varepsilon\}\end{aligned} \tag{1.51}$$

Since the volumetric strain $\{d\varepsilon_v\}$ doesn't change with coordinate transformation and since the product $[N]\{d\varepsilon_v\}$ becomes a zero matrix, combination of eq. (1.36) and eq. (1.51) gives:

$$\begin{aligned}\{d\tau\}&=\frac{1}{n}\sum_{i=1}^{n}G_{\tan}^{(i)}[T_{\varepsilon}^{(i)}]^{T}[N][T_{\varepsilon}^{(i)}]\left(\{d\gamma\}+\{d\varepsilon_v\}\right)\\&=\frac{1}{n}\sum_{i=1}^{n}G_{\tan}^{(i)}[T_{\varepsilon}^{(i)}]^{T}\left([N][T_{\varepsilon}^{(i)}]\{d\gamma\}+[N]\{d\varepsilon_v\}\right)\\&=\frac{1}{n}\sum_{i=1}^{n}G_{\tan}^{(i)}[T_{\varepsilon}^{(i)}]^{T}[N][T_{\varepsilon}^{(i)}]\{d\gamma\}\end{aligned} \tag{1.52}$$

The shear stress increment and the shear strain increment vectors can be related to eachother through the overall tangent shear stiffness matrix [G] as:

$$\{d\tau\}=[G]\{d\gamma\} \tag{1.53}$$

Referring to eqs (1.52) and (1.53), the overall tangent shear stiffness matrix can be given as:

$$[G]=\frac{1}{n}\sum_{i=1}^{n}G_{\tan}^{(i)}[T_{\varepsilon}^{(i)}]^{T}[N][T_{\varepsilon}^{(i)}] \tag{1.54}$$

For isotropic elastic material the shear stiffness matrix is given by:

$$[G]=\begin{bmatrix} \frac{4}{3}G & -\frac{2}{3}G & -\frac{2}{3}G & 0 & 0 & 0 \\ -\frac{2}{3}G & \frac{4}{3}G & -\frac{2}{3}G & 0 & 0 & 0 \\ -\frac{2}{3}G & -\frac{2}{3}G & \frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{1.55}$$

The incremental isotropic effective stress-strain relation is described by:

$$dp'\{I\}=[B]\frac{d\varepsilon_v}{3}\{I\} \tag{1.56}$$

in which the isotropic vector {I}, the volumetric strain ε_v and the mean effective stress *p'* have been defined by (1.37) and (1.40). Elaboration of eq. (1.56) in matrix form gives:

$$\begin{Bmatrix} dp' \\ dp' \\ dp' \\ 0 \\ 0 \\ 0 \end{Bmatrix}=\begin{bmatrix} B & B & B & 0 & 0 & 0 \\ B & B & B & 0 & 0 & 0 \\ B & B & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \frac{d\varepsilon_v}{3} \\ \frac{d\varepsilon_v}{3} \\ \frac{d\varepsilon_v}{3} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.57}$$

Where B is the tangent bulk modulus for compression (or swelling) and is given in terms of incremental values of effective stress and volumetric strain as:

$$B=\frac{dp'}{d\varepsilon_v} \tag{1.58}$$

Next, substituting eqs.(1.53) and (1.56) in eq.(1.39) gives:

$$\{d\sigma'\}=dp'\{I\}+\{d\tau\}=[G]\{d\gamma\}+[B]\frac{d\varepsilon_v}{3}\{I\}=[K]\{d\varepsilon\} \tag{1.59}$$

Hence the tangential stiffness matrix [K] relates the incremental effective stress vector $\{d\sigma'\}$ with incremental strain $\{d\varepsilon\}$.The overall tangent stiffness matrix [K] is then given by the sum of the shear stiffness matrix [G] and the overall tangent compression (or swelling) stiffness matrix [B].

$$[K] = [G] + [B] \tag{1.60}$$

1.8. Distribution of constituent springs

The multi-spring model presented here deals with only isotropic behavior. Thus, the distribution of the springs on the virtual planes and the distribution of the planes themselves should also be isotropic. Regular orientation of springs on each virtual plane can be achieved by simply distributing them evenly with a constant angle θ between them.

For even distribution of the virtual planes, their orientation is determined with the aid of an icosahedron which consists of twenty facets and twelve apices. Vectors directed from the center of an icosahedron to its apices coincide with the normal vectors of the planes ($\overline{n}^{(i)}$ in Figure 1.). In this model, the total number of the virtual planes is increased by using the center of gravity of the planes together with the apices. Thus, 32 planes, 12 based on the apices and 20 on the additional points will be used. By distributing 6 single-degree-of-freedoms shear mechanisms on each of them, 192 one-dimensional shear mechanisms will be employed. Note that the actual calculation is required for only half of them, considering the symmetry of the icosahedron with regard to the xy plane.

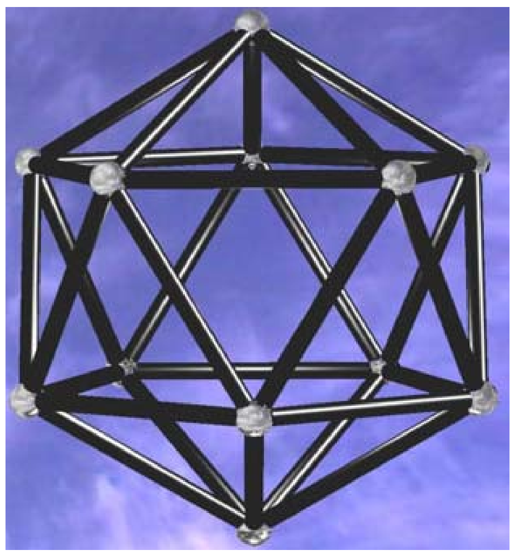


Fig 1.7. An icosahedron

If prepared planes and shear mechanisms on each plane are numbered from 1 … j…. n_p and 1 …. k …… n_s respectively, constituent springs are to be numbered 1 … i …. n, then the following formula should be met.

$$i = (j-1)n_s + k \tag{1.61}$$

1.9. One dimensional stress-strain relationship

The three-dimensional stresses and strains are decomposed into stresses and strains of single-degree of freedom with the help of the multiple mechanism model. Hence the behavior of each one-dimensional shear mechanism is a crucial part of the behavior in three-dimensional model.

1.9.1. Masing's rule

A shear stress-strain relationship of the constituent one-dimensional shear mechanism is formulated based on the extended Masing's rule along with several modifications and a hyperbolic skeleton curve.

The extended Masing's rule outlines the following four basic main points:

1. For initial loading, a stress-strain relationship is prescribed by a skeleton curve (backbone curve) (Masing ,1926)
2. When reloading or unloading occurs from the initial loading, the stress-strain relationship forms a loop which is obtained by enlarging the skeleton curve by variable factors in size.(Pyke, 1977)
3. If the previous maximum shear strain is exceeded, a stress-strain relationship follows a skeleton curve again. (Finn et. al.,1977, Jennings, 1977)
4. If a hysteresis loop intersects a previous loading or unloading curve, a stress-strain relationship follows that previous curve (Finn et. al. , 1977)

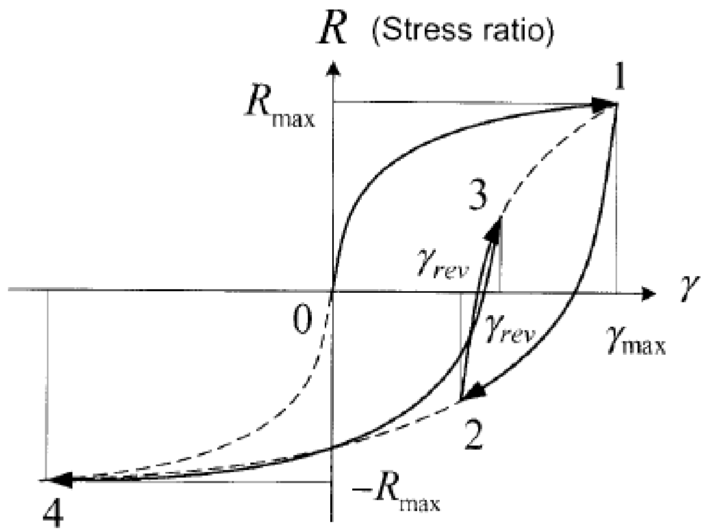


Figure 1.8. Stress ratio-strain relationship using the modified Masing's rule

The information about the recent reversal point is necessary in the extended Masing's rule for choosing the hysteresis loop to be followed when the strain amplitude is subsequently increased. Hence this rule requires memorization of all loading reversal points when cyclic strain amplitude is diminished with the number of loading cycles. However, this requires immense amount of computer memory in three-dimensions rendering this method to be impractical.

The proposed model simplifies the problem of huge memory requirement by creating hysteresis loops connecting only two points: the point of the most recent reversal point and the point of the maximum ever-experienced shear stress ratio.

The skeleton curve in this model which is shown by curve 0-1 in Fig. 1.8 is given by the hyperbolic relationship of Hardin and Drenevich (1972b). This hyperbolic relationship is for each spring is given by:

$$\tau^{(i)} = \frac{\gamma^{(i)}}{\dfrac{1}{G_{\max}^{(i)}} + \dfrac{\gamma^{(i)}}{\tau_{\max}^{(i)}}} \tag{1.62}$$

Dividing both sides of this equation by the mean effective stress *p'* leads to:

$$\frac{\tau^{(i)}}{p'} = \frac{\gamma^{(i)}}{\dfrac{p'}{G_{\max}^{(i)}} + \dfrac{p'\gamma^{(i)}}{\tau_{\max}^{(i)}}} \tag{1.63}$$

The ratio of shear stress to mean effective stress gives the stress ratio R.

$$R^{(i)} = \frac{\tau^{(i)}}{p'} \qquad and \qquad R_{\max}^{*} = \frac{\tau_{\max}^{(i)}}{p'} \tag{1.64}$$

Hence eq. (1.63) can also be written as:

$$R^{(i)} = \frac{\dfrac{G_{\max}^{(i)}}{p'}\gamma^{(i)}}{1 + \dfrac{\left|\gamma^{(i)}\right|}{\tau_{\max}^{(i)} / G_{\max}^{(i)}}} \tag{1.65}$$

Note that the absolute value of shear strain is taken in the denominator in eq. (1.65) to keep the positive sign in front of it valid for all ranges of shear strain.

For each constituent spring, the non-dimensional stiffness parameter k_{max} and the reference shear strain γ_r can be defined as:

$$k_{\max} = \frac{G_{\max}^{(i)}}{p'} \quad and \qquad \gamma_r = \frac{\tau_{\max}^{(i)}}{G_{\max}^{(i)}} \tag{1.66}$$

Substituting the expression for the mean stress *p'* from eq. (1.64) in this equation, the relationship between these two parameters can be established.

$$k_{\max} = \frac{G_{\max}^{(i)}}{\tau_{\max}^{(i)} / R_{\max}^{(i)}} = \frac{R_{\max}^{*}}{\gamma_r} \tag{1.67}$$

Hence, the equation of the skeleton curve for each spring becomes:

$$R^{(i)} = \frac{k_{\max}\gamma^{(i)}}{1 + \dfrac{\left|\gamma^{(i)}\right|}{\gamma_r}} \tag{1.68}$$

in which the superscript i indicates that the quantities belong to constituent springs. The graph of $R^{(i)}$ versus $\gamma^{(i)}$ can be plotted as shown in Fig. 1.9. From the graph, it can be seen that at $\gamma = 0$, the slope of the tangent curve is k_{max}. At the point where this tangent line intersects the horizontal line at $R = R_{max}$, the value of γ will be equal to γ_r.

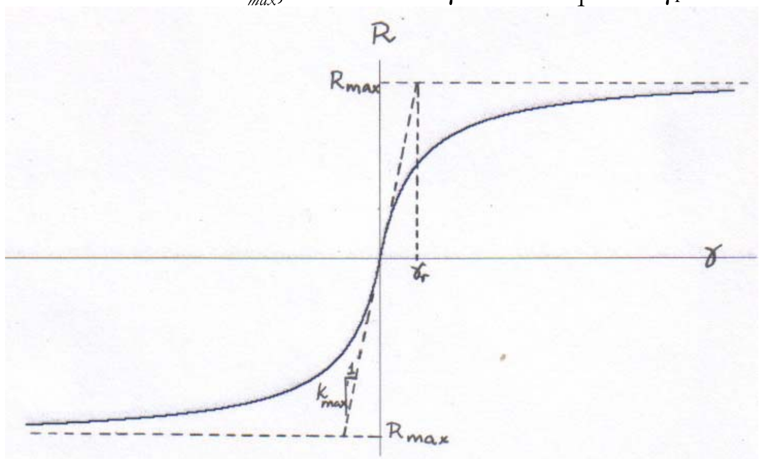


Figure 1.9 The skeleton curve on R versus γ space.

Normalizing both sides of eq. (1.68) by R_{max} gives :

$$\frac{R^{(i)}}{R_{\max}} = \frac{k_{\max}\gamma^{(i)}}{R_{\max}\left(1+\dfrac{\left|\gamma^{(i)}\right|}{\gamma_r}\right)} = \frac{\dfrac{R_{\max}}{\gamma_r}\gamma^{(i)}}{R_{\max}\left(1+\dfrac{\left|\gamma^{(i)}\right|}{\gamma_r}\right)} = \frac{\dfrac{\gamma^{(i)}}{\gamma_r}}{1+\dfrac{\left|\gamma^{(i)}\right|}{\gamma_r}} \tag{1.69}$$

From eq. (1.69), it can be observed that the stress ratio R will be equal to half of the maximum value R_{max} when the shear strain is equal to γ_r. This is the equation of the skeleton curve under normalized space. The skeleton curve looks like:

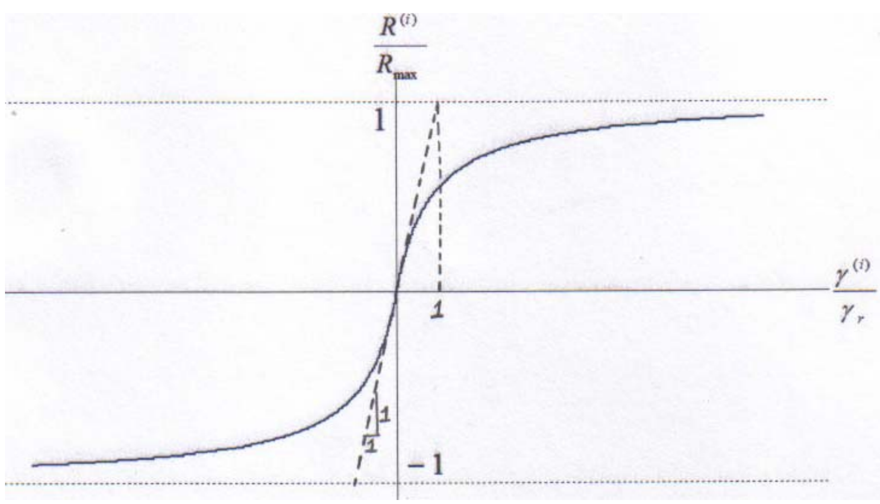


Figure 1.10. The skeleton curve on the normalized space.

The slope of the skeleton curve at any point can be obtained by taking the derivative of eq. (1.69)

$$\frac{\partial\left(\dfrac{R^{(i)}}{R_{\max}}\right)}{\partial\left(\dfrac{\gamma^{(i)}}{\gamma_r}\right)}=\frac{1}{\left(1+\dfrac{\gamma^{(i)}}{\gamma_r}\right)^2} \tag{1.70}$$

The slope of the skeleton curve at different levels of shear strain can be assessed. When the shear strain is zero, the slope of the tangent skeleton curve is equal to one. This tangent line also intersects the horizontal line at $R = R_{max}$, when the value of γ is equal to γ_r. As the level of shear strain goes to infinity, the slope of the curve will be zero.

1.9.2. Determination of spring parameters:

Defining the spring parameters k_{max}, $G^{(i)}_{\max}$ and $\tau^{(i)}_{\max}$ and identifying their relationship with global quantities at this stage is important. In this subsection, these issues will be addressed. The stiffness parameter k_{max} can be given as a function of the overall maximum shear modulus of an element, $G_{\max}$ and mean effective stress p'. Combination of eqs. (1.54) and (1.66) gives:

$$[G_{\max}]=\frac{1}{n}\sum_{i=1}^{n}p'k_{\max}[T_\varepsilon^{(i)}]^T[N][T_\varepsilon^{(i)}] \tag{1.71}$$

For isotropy of the model, the spring stiffness k_{max} should have the same value for all the springs. Hence the above equation becomes:

$$[G_{\max}]=k_{\max}p'\left(\frac{1}{n}\sum_{i=1}^{n}[T_\varepsilon^{(i)}]^T[N][T_\varepsilon^{(i)}]\right) \tag{1.72}$$

For icosahedron distribution of springs, the matrix in the bracket can be computed as:

$$\frac{1}{192}\sum_{i=1}^{192}[T_\varepsilon^{(i)}]^T[N][T_\varepsilon^{(i)}]=\begin{bmatrix}\frac{4}{15} & -\frac{2}{15} & -\frac{2}{15} & 0 & 0 & 0\\ -\frac{2}{15} & \frac{4}{15} & -\frac{2}{15} & 0 & 0 & 0\\ -\frac{2}{15} & -\frac{2}{15} & \frac{4}{15} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{5} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5}\end{bmatrix} \tag{1.73}$$

The shear stiffness matrix [G] was given by eq. (1.55). Substitution of the expressions in eqs.(1.54) and (1.73) in eq. (1.72) and using eq. (1.66), the following relationships between the overall tangent shear modulus of the soil G_{max}, the effective stress and the stiffness of each spring k_{max} can be established:

$$k_{\max} = \frac{5G_{\max}}{p'} \tag{1.74}$$

In addition, for icosahedral distribution of planes, comparison of eqs. (1.54),(1.55) and (1.72) along with the calculated matrix expression in eq. (1.73) leads to the relationship between the maximum shear stiffness of the soil element $G_{\max}$ and that of the individual springs $G_{\max}^{(i)}$ as:

$$G_{\max} = \frac{1}{5} G_{\max}^{(i)} \tag{1.75}$$

With the help of eq. (1.48), the shear strengths of the soil element in torsion shear test and that of the springs in icosahedral manner can be related.

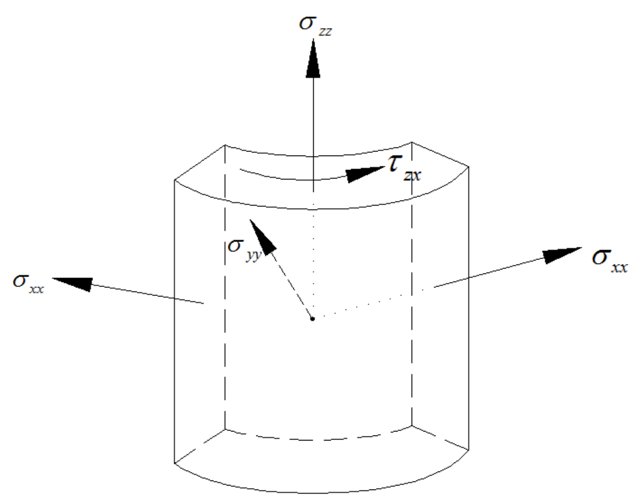


Figure 1. 11. Stress directions in isotropic torsion shear test.

The deviatoric stress vector in torsion shear test for the shear component applied in xy, yz and zx directions respectively are given by:

$$\{\tau\} = \{\sigma_x - p \quad \sigma_y - p \quad \sigma_z - p \quad \tau_{xy} \quad 0 \quad 0\}^T \tag{1.76a}$$

$$\{\tau\} = \{\sigma_x - p \quad \sigma_y - p \quad \sigma_z - p \quad 0 \quad \tau_{yz} \quad 0\}^T \tag{1.76b}$$

$$\{\tau\} = \{\sigma_x - p \quad \sigma_y - p \quad \sigma_z - p \quad 0 \quad 0 \quad \tau_{zx}\}^T \tag{1.76c}$$

The expression of the shear stress level in the springs $\tau^{(i)}$ in terms of the maximum attainable shear stress in the springs $\tau_{\max}^{(i)}$ can be derived utilizing eqs. (1.63), (1.64) and (1.66). The derivation leads to the following relationship.

$$\tau^{(i)} = \frac{\tau_{\max}^{(i)} \gamma^{(i)}}{\gamma_r + \left|\gamma^{(i)}\right|} \tag{1.77}$$

The deviatoric stresses in the springs and that of the soil element in the isotropic torsion shear test can be related using eq. (1.48). Substituting the expression of $\tau^{(i)}$ from eq. (1.77) in eq. (1.48), an equation relating the maximum deviatoric stresses in the isotropic torsion shear test and in the springs can be obtained as:

$$\{\tau\} = \frac{1}{n}\sum_{i=1}^{192}[T_\sigma^{(i)}]^{-1}\frac{\{\tau_{\max}^{(i)}\}\gamma^{(i)}}{\gamma_r + \left|\gamma^{(i)}\right|} \tag{1.78}$$

Where,

$$\{\tau_{\max}^{(i)}\} = \tau_{\max}^{(i)}\{0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0\}^T \tag{1.79}$$

If it is assumed that the ultimate stress of the overall element is mobilized when all of the springs take their own ultimate stresses, the term $\{\tau_{\max}^{(i)}\}$ can be taken out of the summation. The expression for the shear strains in the springs ($\gamma^{(i)}$) was given by eq. (1.43). Substituting this expression in eq. (1.78),

$$\{\tau\} = \left(\frac{1}{n}\sum_{i=1}^{192}[T_\sigma^{(i)}]^{-1}\frac{\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}}{\gamma_r + \left|\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}\right|}\right)\{\tau_{\max}^{(i)}\} \tag{1.80}$$

By applying a global shear strain in one direction, a simple shear test for each of the directions xy, yz and zx is simulated. Computing the term in the bracket of eq. (1.80) for all springs, the relationship between the shear strength of in torsion shear test and icosahedral distribution can be established. Depending on the level of shear strain applied in different directions in the torsion shear test, the ratio of the shear stress in the torsion shear test $\tau_{\max}$ and that of the springs $\tau_{\max}^{(i)}$ vary in the manner shown in the following graph.

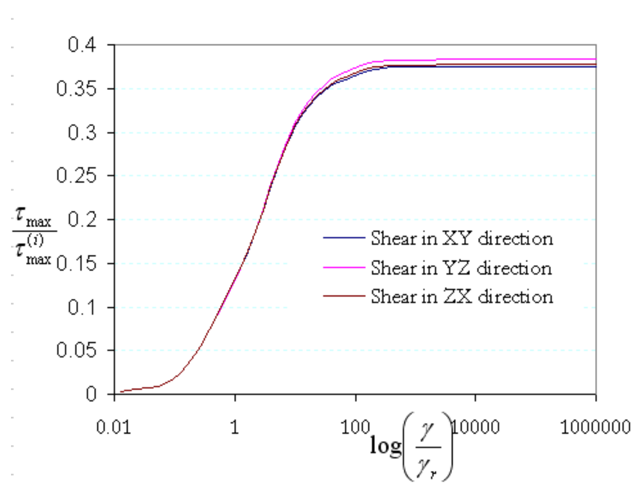


Fig. 1.12. Variation of $\frac{\tau_{\max}}{\tau_{\max}^{(i)}}$ along with the shear strain level in isotropic torsion shear test.

The graph shows that the ratio $\frac{\tau_{\max}}{\tau_{\max}^{(i)}}$ converges to a certain value as the strain level increases. This happens when the shear strain level in the springs $\{n\}^T [T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}$ is large as compared to the reference shear strain γ_r. In that case, the relationship in eq. (1.80), can be approximated as:

$$\{\tau\} = \left(\frac{1}{n} \sum_{i=1}^{192} [T_\sigma^{(i)}]^{-1} \frac{\{n\}^T [T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}}{\left|\{n\}^T [T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}\right|} \right) \{\tau_{\max}^{(i)}\} \tag{1.81}$$

Substituting the values of the stress vectors in each direction given by eq. (1.76) from torsion shear test and that of the spring from eq. (1.79) in eq. (1.81), the shear strengths of the soil element and that of the springs can be related.

Torsional shear applied in xy direction

For shear strain applied in xy direction, eq. (1.81) can be more elaborated as:

$$\begin{Bmatrix} \sigma_x - p \\ \sigma_x - p \\ \sigma_x - p \\ \tau_{xy/\max} \\ 0 \\ 0 \end{Bmatrix} = \left(\frac{1}{n} \sum_{i=1}^{n} [T_\sigma^{(i)}]^{-1} \frac{\{n\}^T [T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}}{\left|\{n\}^T [T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}\right|} \right) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \tau_{\max}^{(i)} \tag{1.82}$$

Where $\tau_{xy/\max}$ is the maximum value of stress τ_{xy} in torsion shear test. In eq. (1.82), the term in the bracket can be computed for all the springs giving:

$$\frac{1}{192} \sum_{i=1}^{192} [T_\sigma^{(i)}]^{-1} \frac{\{n\}^T [T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}}{\left|\{n\}^T [T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}\right|} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & -0.2708 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.2507 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0201 & 0.0000 & 0.0000 \\ 0.0104 & -0.0104 & 0.0000 & 0.0000 & 0.3749 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0305 & 0.0201 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0072 & -0.0201 \end{bmatrix} \tag{1.83}$$

Thus, eq. (1.82) can also be written as:

$$\begin{Bmatrix} \sigma_x - p \\ \sigma_y - p \\ \sigma_z - p \\ \tau_{xy} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & -0.2708 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.2507 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0201 & 0.0000 & 0.0000 \\ 0.0104 & -0.0104 & 0.0000 & 0.0000 & 0.3749 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0305 & 0.0201 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0072 & -0.0201 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \tau_{\max}^{(i)} \quad (1.84)$$

Simplification of this equation leads to:

$$\begin{Bmatrix} \sigma_x - p \\ \sigma_y - p \\ \sigma_z - p \\ \tau_{xy/\max} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.3749 \\ 0.0201 \\ -0.0072 \end{Bmatrix} \tau_{\max}^{(i)} \quad (1.85)$$

Relating the corresponding elements on either side of eq. (1.85), the shear strength of the soil when the shear applied in xy direction is given by:

$$\tau_{\max} \approx 0.375 \tau_{\max}^{(i)} \quad (1.86)$$

Torsional shear applied in yz direction

For shear strain applied in xy direction, eq. (1.81) can be more elaborated as:

$$\begin{Bmatrix} \sigma_x - p \\ \sigma_x - p \\ \sigma_x - p \\ 0 \\ \tau_{yz/\max} \\ 0 \end{Bmatrix} = \left(\frac{1}{n} \sum_{i=1}^{n} [T_\sigma^{(i)}]^{-1} \frac{\{n\}^T [T_\varepsilon^{(i)}] \{\varepsilon_{xyz}\}}{\left| \{n\}^T [T_\varepsilon^{(i)}] \{\varepsilon_{xyz}\} \right|} \right) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \tau_{\max}^{(i)} \quad (1.87)$$

Where $\tau_{yz/\max}$ is the maximum value of stress τ_{yz} in torsion shear test. The computation of term in the bracket around all the springs gives:

$$\frac{1}{192}\sum_{i=1}^{192}[T_\sigma^{(i)}]^{-1}\frac{\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}}{\left|\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}\right|}=\begin{bmatrix}0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0258 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0236 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0022 & 0.0000\\ 0.0208 & -0.0208 & 0.0000 & -0.0338 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3836 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0590 & 0.0000 & 0.0000\end{bmatrix} \tag{1.88}$$

Substitution of the result obtained in eq. (1.89) into eq. (1.88) leads to:

$$\begin{Bmatrix}\sigma_x-p\\ \sigma_y-p\\ \sigma_z-p\\ 0\\ \tau_{xy}\\ 0\end{Bmatrix}=\begin{bmatrix}0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0258 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0236 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0022 & 0.0000\\ 0.0208 & -0.0208 & 0.0000 & -0.0338 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3836 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0590 & 0.0000 & 0.0000\end{bmatrix}\begin{Bmatrix}0\\0\\0\\0\\1\\0\end{Bmatrix}\tau_{\max}^{(i)} \tag{1.89}$$

Simplification of this equation gives:

$$\begin{Bmatrix}\sigma_x-p\\ \sigma_y-p\\ \sigma_z-p\\ 0\\ \tau_{yz/\max}\\ 0\end{Bmatrix}=\begin{Bmatrix}0.0258\\ -0.0236\\ -0.0022\\ 0\\ 0.3836\\ 0\end{Bmatrix}\tau_{\max}^{(i)} \tag{1.90}$$

Relating the corresponding elements on either side of eq. (1.90), the shear strength of the soil when the shear applied in yz direction is given by:

$$\tau_{\max}\approx 0.384\tau_{\max}^{(i)} \tag{1.91}$$

Torsional shear applied in zx direction

For shear strain applied in zx direction, eq. (1.82) can be more elaborated as:

$$\begin{Bmatrix}\sigma_x-p\\ \sigma_x-p\\ \sigma_x-p\\ 0\\ 0\\ \tau_{zx/\max}\end{Bmatrix}=\left(\frac{1}{n}\sum_{i=1}^{n}[T_\sigma^{(i)}]^{-1}\frac{\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}}{\left|\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}\right|}\right)\begin{Bmatrix}0\\0\\0\\0\\1\\0\end{Bmatrix}\tau_{\max}^{(i)} \tag{1.92}$$

Where $\tau_{zx/\max}$ is the maximum value of stress τ_{zx} in torsion shear test. The term in the bracket can be computed for all the springs.

$$\frac{1}{192}\sum_{i=1}^{192}[T_\sigma^{(i)}]^{-1}\frac{\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}}{\left|\{n\}^T[T_\varepsilon^{(i)}]\{\varepsilon_{xyz}\}\right|}=\begin{bmatrix}-0.0312 & -0.0312 & 0.0000 & 0.0265 & 0.0000 & 0.0000\\ -0.0312 & -0.0312 & 0.0000 & 0.0025 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & -0.0625 & -0.0291 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0270 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & -0.0844 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3776 & 0.0000\end{bmatrix} \tag{1.93}$$

Substitution of this into eq. (1.93) results in:

$$\begin{Bmatrix}\sigma_x-p\\ \sigma_y-p\\ \sigma_z-p\\ 0\\ 0\\ \tau_{zx}\end{Bmatrix}=\begin{bmatrix}-0.0312 & -0.0312 & 0.0000 & 0.0265 & 0.0000 & 0.0000\\ -0.0312 & -0.0312 & 0.0000 & 0.0025 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & -0.0625 & -0.0291 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0270 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & -0.0844 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.3776 & 0.0000\end{bmatrix}\begin{Bmatrix}0\\0\\0\\0\\1\\0\end{Bmatrix}\tau_{\max}^{(i)} \tag{1.94}$$

Simplification of this equation leads to:

$$\begin{Bmatrix}\sigma_x-p\\ \sigma_y-p\\ \sigma_z-p\\ 0\\ 0\\ \tau_{zx/\max}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\-0.02703\\0\\0.3775\end{Bmatrix}\tau_{\max}^{(i)} \tag{1.95}$$

Relating the corresponding elements on either side of eq. (1.86), the shear strength of the soil when the shear applied in zx direction in terms of the shear strength of the springs is given by:

$$\tau_{\max}\approx 0.378\tau_{\max}^{(i)} \tag{1.96}$$

The relationship $p = (\sigma_x+\sigma_y+\sigma_z)/3$ remains valid for all directions. This can be checked by adding the first three rows of eqs. (1.85), (1.90) and (1.95) which should result in zero.

Eqs. (1.85) , (1.90) and (1.95) concern deviatoric stresses. Instead of producing a global shear stress corresponding to the applied global shear strain, also other terms are non-zero, namely eq. (1.90) for γ_{yz} gives normal stress errors and eqs. (1.85) and (1.95) for γ_{xy} and γ_{zx} give shear stress errors. The relative errors in the deviatoric stress vector in each direction relative to the deviatoric stress in the direction of the applied global strain can be summarized in the following table.

Table 1.2. Relative values of deviatoric stresses in torsion shear test relative to those applied global in the direction of global shear strain.

Deviatoric strain in torsion shear test	Global shear strain direction		
	XY	YZ	ZX
σ_{xx}-p'	0	0.0673	0
σ_{yy}-p'	0	-0.0615	0
σ_{zz}-p'	0	-0.0057	0
τ_{xy}	1	0	-0.0716
τ_{yz}	0.0536	1	0
τ_{zx}	-0.0192	0	1

As can be seen from the table, there is an error associated to every direction. This error would affect the performance of the model to simulate real case or laboratory observations.

1.9.3. Mean stress dependency of parameters

With the help of eqs. (1.66) and (1.91), the reference strain γ_r is given as a function of the overall shear strength, $\tau_{\max}$, as:

$$\gamma_r \approx \frac{2.607\tau_{\max}}{k_{\max} p'} \tag{1.97}$$

The parameters k_{max} given in eq. (1.74) and γ_r given in eq. (1.97) are functions of the mean effective stress. The common assumption for cohesionless materials states that shear modulus G_{max} at small strain is proportional to square root of mean effective stress p'. This dependence can be expressed by:

$$\frac{G_{\max}}{G_{\max,o}} = \sqrt{\frac{p'}{p_o'}} \tag{1.98}$$

Where p_o' is a reference mean effective stress and $G_{\max,o}$ is the corresponding small-strain shear modulus at p_o'. Thus, if k_{max} given by eq. (1.74) is defined in terms of $k_{max,o}$ at the reference mean effective stress, p_o',then k_{max} at a given mean effective stress, p', satisfies:

$$\frac{k_{\max}}{k_{\max o}} = \frac{G_{\max} / p'}{G_{\max,o} / p_o'} = \frac{p_o' G_{\max}}{p' G_{\max,o}} = \frac{p_o'}{p'}\sqrt{\frac{p'}{p_o'}} = \sqrt{\frac{p_o'}{p'}} \tag{1.99}$$

A similar equation can also be written for γ_r using eqs. (1.97) and (1.99).

$$\frac{\gamma_r}{\gamma_{r,o}} = \frac{2.65\tau_{\max} / k_{\max} p'}{2.65\tau_{\max,o} / k_{\max,o} p_o'} = \frac{\tau_{\max}}{\tau_{\max,o}} \frac{k_{\max,o}}{k_{\max}} \frac{p_o'}{p'} = \frac{\tau_{\max}}{\tau_{\max,o}} \sqrt{\frac{p'}{p_o'}} \frac{p_o'}{p'} = \frac{\tau_{\max}}{\tau_{\max,o}} \sqrt{\frac{p_o'}{p'}} \qquad (1.100)$$

where $\gamma_{r,o}$ and $\tau_{\max,o}$ respectively are γ_r and $\tau_{\max}$ at $p' = p_o'$. For cohesionless materials the maximum global shear stress τ_{max} is proportional to the mean effective stress *p'* through the expression involving a function of ϕ_{max} and θ, by:

$$\tau_{\max} = p' f(\phi_{\max}, \theta) \qquad (1.101)$$

In which, ϕ_{max} is the maximum friction angle and θ is Lode angle in π-plane. The function is constant for every mean effective stress. Hence eq. (1.100) can be written as:

$$\frac{\gamma_r}{\gamma_{r,o}} = \frac{\tau_{\max}}{\tau_{\max,o}} \sqrt{\frac{p_o'}{p'}} = \frac{p'}{p_o'} \sqrt{\frac{p_o'}{p'}} = \sqrt{\frac{p'}{p_o'}} \qquad (1.102)$$

1.9.4. Hysteresis loop

The equation of the skeleton curve for regular loading is given by eq. (1.68). When irregular or asymmetric loading is applied, the origin of the skeleton curve might need to be shifted so as to reproduce cumulative strain on one side. For this reason, and for the sake of generality and flexibility of the model, the origin of the skeleton curve is made movable by introducing a parameter $\gamma_o^{(i)}$.

$$R^{(i)} = \frac{k_{\max} (\gamma^{(i)} - \gamma_o^{(i)})}{1 + \dfrac{\left|\gamma^{(i)} - \gamma_o^{(i)}\right|}{\gamma_r}} \qquad (1.103)$$

By introducing the parameter $\gamma_o^{(i)}$, the skeleton curve starts from a different point than the origin as shown in fig 1.13.

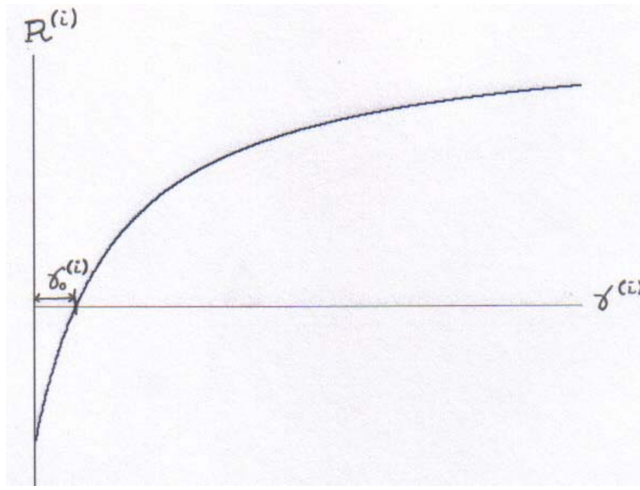


Figure 1.13. The skeleton curve on the normalized space.

The original Masing's rules states that the hysteresis loop is given as the hyperbolic curve which has doubled size of the skeleton curve and passes through the most recent reversal point $(\gamma_{rev}^{(i)}, R_{rev}^{(i)})$. In that case, the curve which is shown in fig 1.14 is given by the equation

$$R^{(i)} = R_{rev}^{(i)} + \frac{k_{\max}(\gamma^{(i)} - \gamma_{rev}^{(i)})}{1 + \dfrac{\left|\gamma^{(i)} - \gamma_{rev}^{(i)}\right|}{2\gamma_r}} \qquad (1.104)$$

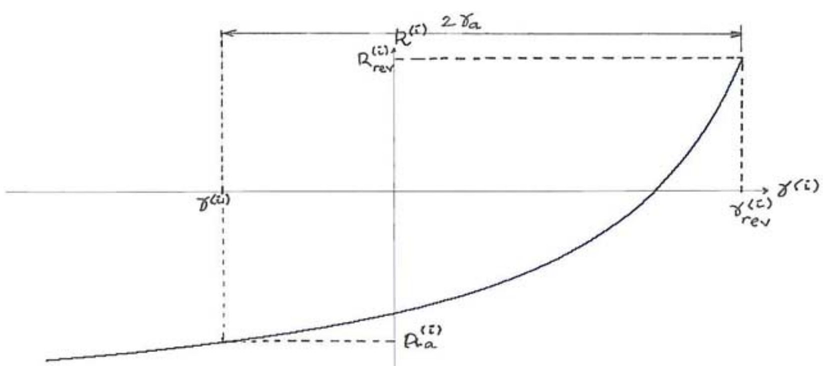


Figure 1.14. The skeleton curve on the normalized space.

When $R^{(i)} = R_a^{(i)}$, the difference $\gamma^{(i)} - \gamma_{rev}^{(i)}$ in this model is equal to twice the value of the strain amplitude $\gamma_a^{(i)}$ as illustrated in fig. 1.14. Hence eq. (1.104) becomes

$$R_a^{(i)} = R_{rev}^{(i)} + \frac{k_{\max} 2\gamma_a^{(i)}}{1 + \dfrac{\left|2\gamma_a^{(i)}\right|}{2\gamma_r}} \quad \Rightarrow \quad \frac{R_a^{(i)} - R_{rev}^{(i)}}{2} = \frac{k_{\max}\gamma_a^{(i)}}{1 + \dfrac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} \qquad (1.105)$$

where $\dfrac{R_a^{(i)} - R_{rev}^{(i)}}{2}$ is the stress ratio amplitude.

From this , the expression for strain amplitude can be derived as:

For $\gamma_a^{(i)}>0$, in which $\left|\gamma_a^{(i)}\right| = \gamma_a^{(i)}$ $$\frac{\gamma_a^{(i)}}{\gamma_r} = \frac{R_a^{(i)} - R_{rev}^{(i)}}{2k_{\max}\gamma_r - (R_a^{(i)} - R_{rev}^{(i)})} \tag{1.106a}$$

For $\gamma_a^{(i)}<0$, in which $\left|\gamma_a^{(i)}\right| = -\gamma_a^{(i)}$ $$\frac{\gamma_a^{(i)}}{\gamma_r} = \frac{R_a^{(i)} - R_{rev}^{(i)}}{2k_{\max}\gamma_r + (R_a^{(i)} - R_{rev}^{(i)})} \tag{1.106b}$$

Eq. (1.104) can also be written in another form by adding two terms which add up to zero, namely:

$$R^{(i)} = R_{rev}^{(i)} + \frac{k_{\max}(\gamma^{(i)} - \gamma_{rev}^{(i)})}{1 + \dfrac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} + \left\{ \frac{k_{\max}(\gamma^{(i)} - \gamma_{rev}^{(i)})}{1 + \dfrac{\left|\gamma^{(i)} - \gamma_{rev}^{(i)}\right|}{2\gamma_r}} - \frac{k_{\max}(\gamma^{(i)} - \gamma_{rev}^{(i)})}{1 + \dfrac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} \right\} \tag{1.107}$$

Now we consider eq. (1.107) in more detail. The second term in eq. (1.107) expresses the line connecting both ends of a loop and the last terms in the parentheses represent deviation of a loop from that line as can be proved below by referring to Fig. 1.15.

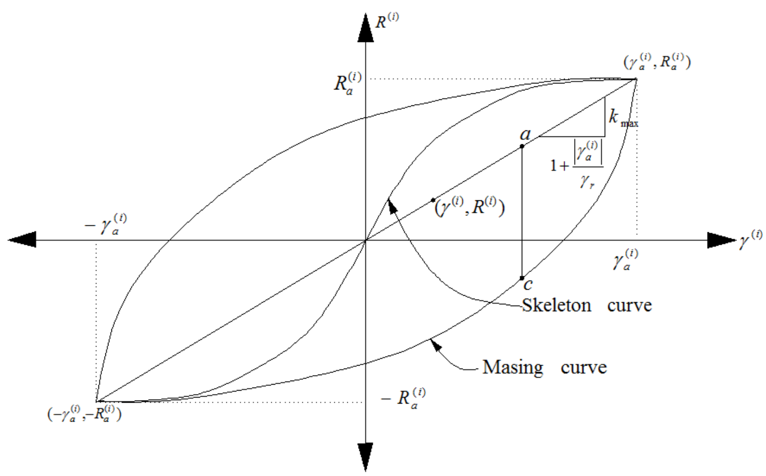


Figure 1.15. Illustration of skeleton curve and Masing curve for $R_{rev}^{(i)} = -R_a^{(i)}$ *and* $\gamma_{rev}^{(i)} = -\gamma_a^{(i)}$

The corner points are on the skeleton curve. Hence, the coordinates satisfy the skeleton curve equation given by eq. (1.68). The respective equations for the right and left corner points are given by:

$$R_a^{(i)} = \frac{k_{\max}\gamma_a^{(i)}}{1+\frac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} \quad and \quad -R_a^{(i)} = -\frac{k_{\max}\gamma_a^{(i)}}{1+\frac{\left|-\gamma_a^{(i)}\right|}{\gamma_r}} \tag{1.108}$$

The slope of the line connecting these corner points is given as ratio between vertical increment and horizontal increment of any two points. Taking the two corner points, the slope is given by :

$$slope = \frac{R_a^{(i)} - (-R_a^{(i)})}{\gamma_a^{(i)} - (-\gamma_a^{(i)})} = \frac{R_a^{(i)}}{\gamma_a^{(i)}} = \frac{k_{\max}}{1+\frac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} \tag{1.109}$$

If $(\gamma^{(i)}, R^{(i)})$ represents a coordinate of any point on this line and if the reversal point is $(\gamma_{rev}^{(i)}, R_{rev}^{(i)})$, then the following equation can be written:

$$\frac{R^{(i)} - (R_{rev}^{(i)})}{\gamma^{(i)} - (\gamma_{rev}^{(i)})} = \frac{k_{\max}}{1+\frac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} \tag{1.110}$$

Then eq. (1.110) can be written as:

$$R^{(i)} = R_{rev}^{(i)} + \frac{k_{\max}\left(\gamma^{(i)} - \gamma_{rev}^{(i)}\right)}{1+\frac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} \tag{1.111}$$

This equation is identical to eq. (1.107) without the terms in the parenthesis. Hence, it can be concluded that the first two terms of eq. (1.107) represent the equation of the diagonal line connecting the left hand side and the right hand side corners of figure 1.15 if .

$$R_{rev}^{(i)} = R_a^{(i)} and \quad \gamma_{rev}^{(i)} = \gamma_a^{(i)} \qquad \text{[right corner point]} \tag{1.112a}$$

$$R_{rev}^{(i)} = -R_a^{(i)} and \quad \gamma_{rev}^{(i)} = -\gamma_a^{(i)} \qquad \text{[left corner point]} \tag{1.112b}$$

By substituting these values in eq. (1.111), it can be checked that the resulting expression equals the expression of the skeleton curve given by eq. (1.65).

The resulting lines for eq. (1.111) can also be illustrated in figure below for two Masing curves with the same reversal point but different strain amplitude.

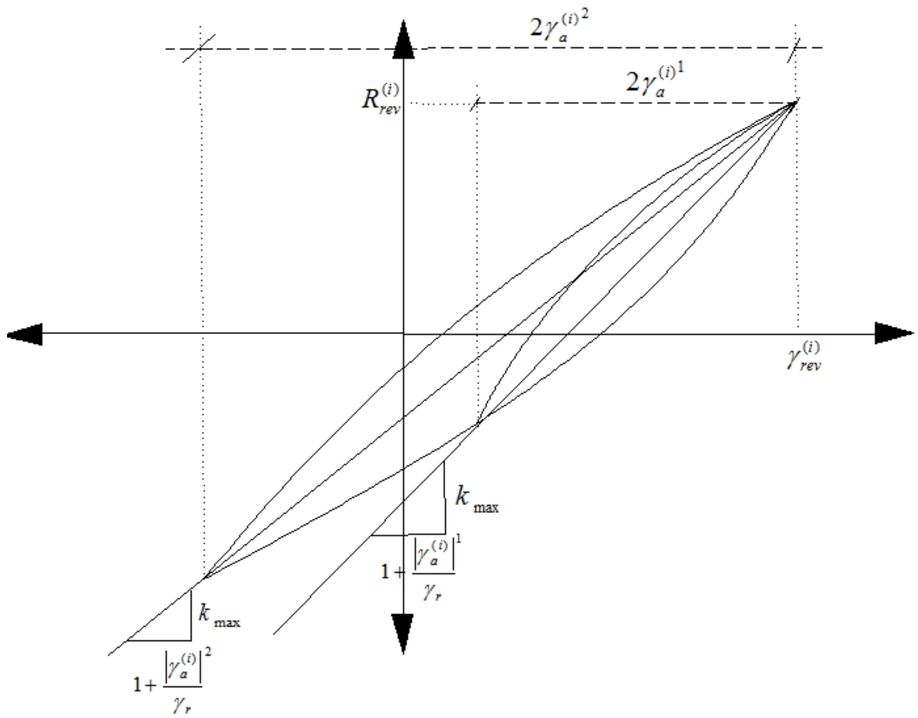


Fig. 1.15b. inclined line with slope : $\dfrac{k_{\max}}{1+\dfrac{\left|\gamma_a^{(i)}\right|^x}{\gamma_r}}$ through $(\gamma_{rev}^{(i)}, R_{rev}^{(i)})$

For an arbitrary reversal point $(\gamma_{rev}^{(i)}, R_{rev}^{(i)})$, part of eq. (1.107) excluding the terms in the parenthesis represent a line through both this reversal point and point on the corresponding Masing's curve representing the other end point of a closed loop with double amplitude $\gamma_a^{(i)}$. From this , it can be clearly observed that the terms in the parenthesis represent the difference of shear stress ratio obtained by eqs. (1.107) and (1.111). This difference is denoted by line $\overline{ac}$ in fig. (1.15). In a physical sense, the diagonal line stands for the deformation associated with secant modulus, while the deviation is related to energy dissipation.

The combination of the original Masing's rule and a hyperbolic skeleton curve yield exaggerated damping ratio. In this model, the damping ratio is reduced by multiplying the area of the hysteresis loop by an arbitrary factor η. Therefore, the damping ratio can be reduced to an arbitrary level by scaling the terms in the parenthesis of eq. (1.83) with a factor η.

$$R^{(i)} = R_{rev}^{(i)} + \frac{k_{\max}(\gamma^{(i)} - \gamma_{rev}^{(i)})}{1+\dfrac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} + \eta\left\{\frac{k_{\max}(\gamma^{(i)} - \gamma_{rev}^{(i)})}{1+\dfrac{\left|\gamma^{(i)} - \gamma_{rev}^{(i)}\right|}{2\gamma_r}} - \frac{k_{\max}(\gamma^{(i)} - \gamma_{rev}^{(i)})}{1+\dfrac{\left|\gamma_a^{(i)}\right|}{\gamma_r}}\right\} \qquad (1.113)$$

The Masing curves before and after applying the reduction factor are shown in the figure below in which point b is located on a reduced curve.

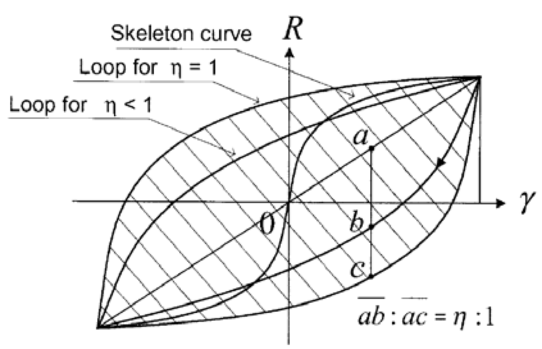


Figure 1.16. Modification of damping ratio

When irregular loading with $\gamma_{rev}^{(i)} \neq \gamma_a^{(i)}$ and $R_{rev}^{(i)} \neq R_a^{(i)}$ is concerned, the above equation is not sufficient to portray a closed loop. For example, if unloading occurs at point 3 in fig. (1.17), eqs. (1.106) and (1.107) would offer the path from point 3 to point 4 in fig. (1.17), while the path connecting point 3 with the point of previously maximum stress ratio on the skeleton curve is desired. This is because those equations give γ_a for this unloading in place of γ_a' in fig.1.17. Consequently, the loop is obtained by shifting the bold dotted curve in fig. 1.17 by $2\gamma_a$-$2\gamma_a$'. This fact means that the calculation of γ_a in terms of the stress ratio by eq. (1.106) is not appropriate when loading reversal at second-order loops (loops which are originating from curves other than the skeleton curve) is concerned. The same goes for a loading reversal at loops originating from the skeleton curve in case of η ≠ 1. Indicating the maximum shear strain amplitude which is going to occur by $\gamma_{\max}^{(i)}$, the amplitude of strain illustrated in fig. 1.17 as γ_a' is given in the following by:

$$\gamma_a^{(i)} = \left|\gamma_{\max}^{(i)} - \gamma_{rev}^{(i)}\right| / 2 \tag{1.114}$$

In this equation, the sign of $\gamma_{\max}^{(i)}$ will depend on the type of curve which is going to occur from the point of reversal considered. If the curve which is going to occur is unloading curve, $\gamma_{\max}^{(i)}$ will take negative value otherwise it will be positive.

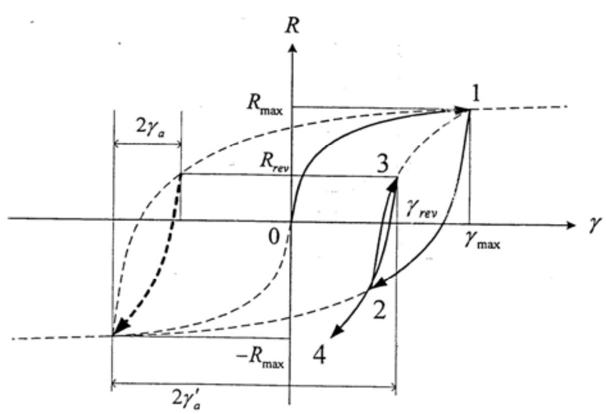


Figure 1.17. Hysteresis loops without modifying the strain amplitude.

At this stage it may be noted that the Masing's rule requires that the maximum reversal point is located on the skeleton curve, in which its coordinate is indicated by $(\gamma^{(i)}_{\max}, R^{*}_{\max})$. However, in the following formulation, instead of a scaling factor $C^{(i)}$ is introduced while the previously maximum stress ratio $R^{*}_{\max}$ is replaced by $R^{(i)}_{\max}$, the strength parameter that was defined in eq. (1.64). The scaling factor will also ensure that any reversal point $(\gamma^{(i)}_{rev}, R^{(i)}_{rev})$ is connected to the point of maximum stress ratio $(\gamma^{(i)}_{\max}, R^{(i)}_{\max})$ or $(-\gamma^{(i)}_{\max}, -R^{(i)}_{\max})$. Then the scaling factor $C^{(i)}$ is defined, while modifying eq. (1.105), by

$$\frac{k_{\max}\dfrac{2\gamma^{(i)}_a}{2}}{1+\dfrac{\left|2\gamma^{(i)}_a\right|}{2\gamma_r}} = \frac{R^{(i)}_a - R^{(i)}_{rev}}{2} = \frac{R^{(i)}_{\max} - R^{(i)}_{rev}}{2C^{(i)}} \tag{1.115}$$

In which,

$$C^{(i)} = \frac{R^{(i)}_{\max} - R^{(i)}_{rev}}{R^{(i)}_a - R^{(i)}_{rev}} = \frac{(R^{(i)}_{\max} - R^{(i)}_{rev})\left(1+\dfrac{\left|\gamma^{(i)}_a\right|}{\gamma_r}\right)}{2k_{\max}\gamma^{(i)}_a} \tag{1.116}$$

By the introduction of $C^{(i)}$, eq. (1.107) will be written as:

$$R^{(i)} = R^{(i)}_{rev} + \frac{C^{(i)}k_{\max}(\gamma^{(i)} - \gamma^{(i)}_{rev})}{1+\dfrac{\left|\gamma^{(i)}_a\right|}{\gamma_r}} + \eta\left\{\frac{C^{(i)}k_{\max}(\gamma^{(i)} - \gamma^{(i)}_{rev})}{1+\dfrac{\left|\gamma^{(i)} - \gamma^{(i)}_{rev}\right|}{2\gamma_r}} - \frac{C^{(i)}k_{\max}(\gamma^{(i)} - \gamma^{(i)}_{rev})}{1+\dfrac{\left|\gamma^{(i)}_a\right|}{\gamma_r}}\right\} \tag{1.117}$$

The whole part of the equations involved in the hysteresis loop will be summarized with the following graph:

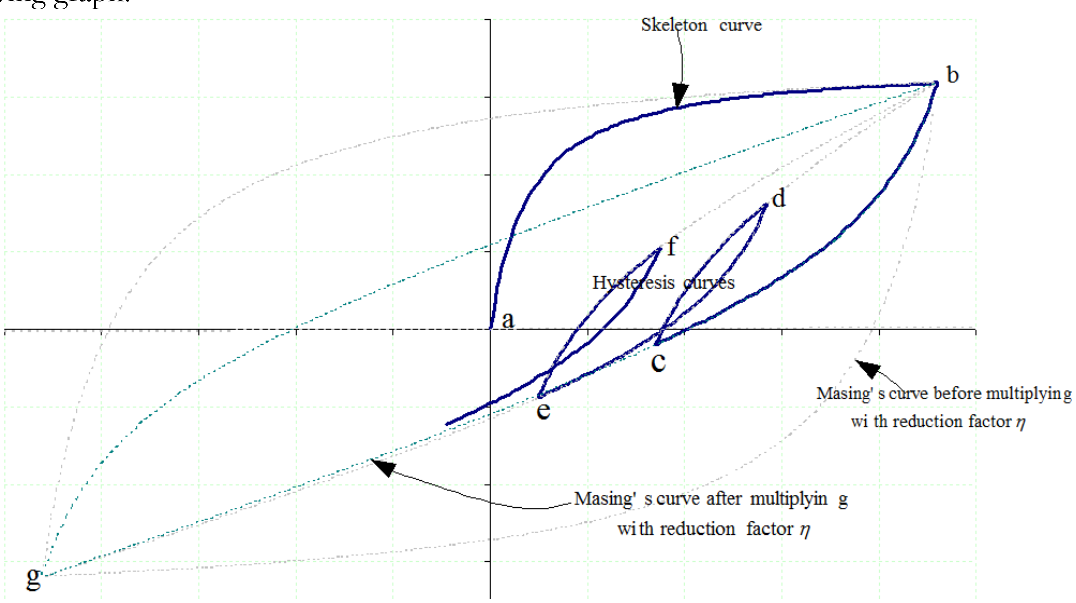


Fig. 1.18. Skeleton, Masing and hysteresis curves.

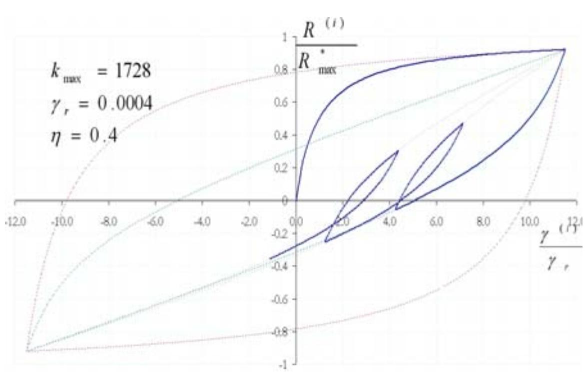
- For the initial loading shown by curve **ab** in fig. 1.18 , the curve is governed by the equation of the skeleton curve given by eq. (1.68). The maximum values of $\gamma^{(i)}$ and $R^{(i)}$ will be taken to be the values of $R^{(i)}_{\max}$ and $\gamma^{(i)}_{\max}$ respectively. The parameters $k_{\max}$ and γ_r, which will be used for calculation of the stress ratio $R^{(i)}$ from $\gamma^{(i)}$ should be given at the beginning.

- When unloading starts, the consequent part of the curve will be a Masing curve governed by eq. (1.117). This part is described by the curve **bc** in the figure. This curve heads to point g which is the reflection of the point where the skeleton curve ended. Here the reversal point will be the point where the skeleton curve ended and the values of $R^{(i)}_{rev}$ and $\gamma^{(i)}_{rev}$ will be the values $R^{(i)}_{\max}$ and $\gamma^{(i)}_{\max}$ obtained from the skeleton curve.

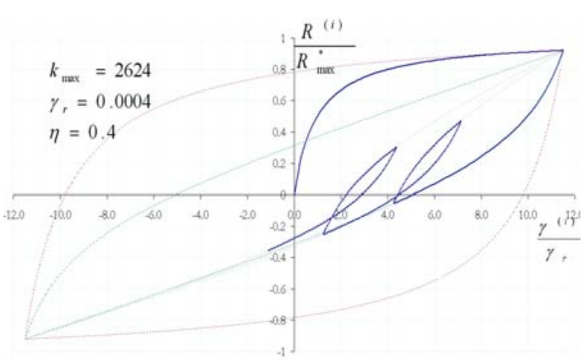
  Other variables which will be used in this equation are the amplitude of shear strain $\gamma^{(i)}_a$ calculated by eq. (1.114) the scaling factor $C^{(i)}$ calculated by eq. (1.116). The reduction factor η should be given at the beginning. For the unloading part of the Masing curve, $\gamma^{(i)}_a$ is always equal to $-\gamma^{(i)}_{\max}$ and for the re-loading part it will be equal to $\gamma^{(i)}_{\max}$. The value of $C^{(i)}$ is always 1 for the Masing curve.

- If re-loading occurs before the curve reaches point g, it will be a hysteresis curve governed by eq. (1.117). The only change with the Masing curve is that the values of $R^{(i)}_{rev}$ and $\gamma^{(i)}_{rev}$ will no more be the values $R^{(i)}_{\max}$ and $\gamma^{(i)}_{\max}$. The hysteresis curves are also directed towards the point of maximum stress ratio or its reflection.

The effect of the three parameters $k_{\max}$,γ_r and η on the value of stress ratio $R^{(i)}$ will be shown in the graphs below. The ordinate of the graphs represent the value of stress ratio.



(a)



(b)

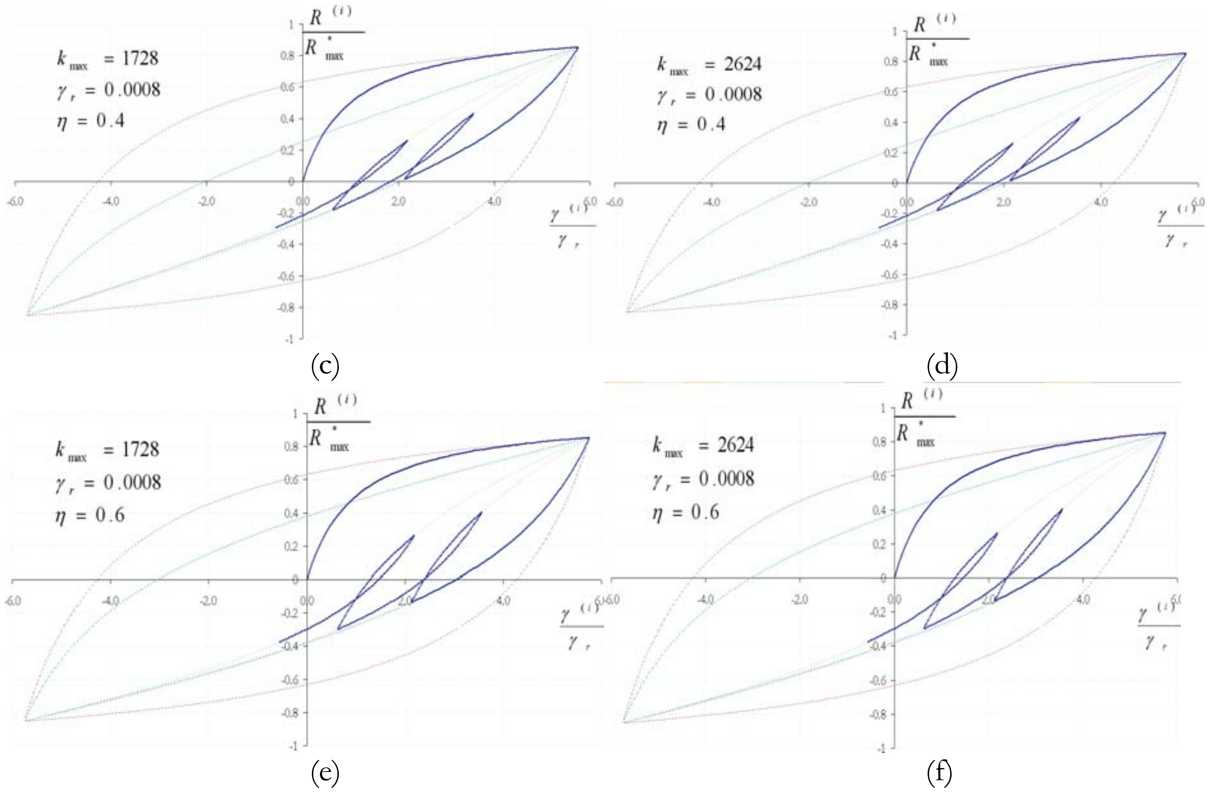


Fig. 1.19.(a)-(f) effect of the parameters $k_{\max}$ $, \gamma_r$ and η on normalized stress ratio $R^{(i)}$.

Figures 1.19. (a) and (b) represent normalized stress ratio- normalized strain graphs for different values of $k_{\max}$;keeping the other two parameters constant. The graphs show that the spring stiffness $k_{\max}$ is directly proportional to the stress ratio because both graphs with normalized qantities are identical in accordance with $R^*_{\max} = k_{\max}\gamma_r$. Comparing (a) and (c) or (b) and (d), the effect of reference shear strain γ_r on normalized stress ratio $R^{(i)} / R^*_{\max}$ can be noticed. From the graphs, it can be again observed that with increasing γ_r the normalized stress ratio increases in accordance with eq. (1.103) due to the effect of normalized strain ratio $\gamma^{(i)} / \gamma_r$. The effect of the reduction factor η can be observed by comparing the graphs (c) and (e) or (d) and (f). Larger value of reduction factor means wider Masing or hysteresis curves.

In eq. (1.117), the term in the parentheses along with the reduction factor for the damping ratio η gives the deviation of the hysteresis loop from the diagonal line in fig. 1.16. This deviation will be denoted by $\widetilde{R}$ and will be given by the equation:

$$\tilde{R} = \eta \left\{ \frac{C^{(i)} k_{\max} (\gamma^{(i)} - \gamma_{rev}^{(i)})}{1 + \dfrac{\left|\gamma^{(i)} - \gamma_{rev}^{(i)}\right|}{2\gamma_r}} - \frac{C^{(i)} k_{\max} (\gamma^{(i)} - \gamma_{rev}^{(i)})}{1 + \dfrac{\left|\gamma_a^{(i)}\right|}{\gamma_r}} \right\} \tag{1.118}$$

Then the area inside the hysteresis loop can be given by:

$$\Delta W = 2 \int_{-\gamma_{\max}^{(i)}}^{\gamma_{\max}^{(i)}} R' d\gamma^{(i)} \tag{1.119}$$

The factor two is introduced because the area should be calculated for the hysteresis loop on both sides of the diagonal. The hysteresis curve is bounded by curves governed by eq. (1.80). Thus, the following relations can be written for the curve on the top of the diagonal:

$$C^{(i)} = 1 \quad , \gamma_{rev}^{(i)} = -\gamma_{\max}^{(i)} \quad , \; R_{rev}^{(i)} = -R_{\max}^{(i)} \quad , \; \gamma_a^{(i)} = \frac{\gamma_{\max}^{(i)} - (-\gamma_{\max}^{(i)})}{2} = \gamma_{\max}^{(i)} \tag{1.120}$$

By substituting these values, eq. (1.118) can be simplified into:

$$\tilde{R} = \eta k_{\max} \left\{ \frac{(\gamma^{(i)} + \gamma_a^{(i)})}{1 + \dfrac{\left|\gamma^{(i)} + \gamma_a^{(i)}\right|}{2\gamma_r}} - \frac{(\gamma^{(i)} + \gamma_a^{(i)})}{1 + \dfrac{\gamma_a^{(i)}}{\gamma_r}} \right\} \tag{1.121}$$

In this case, for any value of $\gamma^{(i)}$, the expression $\gamma^{(i)} + \gamma_a^{(i)}$ is always positive. Thus, the absolute value in the expression for *R'* can be removed without bringing any change in the final outcome. Now, the area inside the hysteresis loop can be explicitly expressed as:

$$\Delta W = 2\eta k_{\max} \int_{-\gamma_a^{(i)}}^{\gamma_a^{(i)}} \left\{ \frac{(\gamma^{(i)} + \gamma_a^{(i)})}{1 + \dfrac{\gamma^{(i)} + \gamma_a^{(i)}}{2\gamma_r}} - \frac{(\gamma^{(i)} + \gamma_a^{(i)})}{1 + \dfrac{\gamma_a^{(i)}}{\gamma_r}} \right\} d\gamma^{(i)} \tag{1.122}$$

The integration will finally result in the area inside the hysteresis loop to be given by:

$$\Delta W = 2\eta k_{\max} \left\{ 4\gamma_a \gamma_r - 4\gamma_r^2 \ln\left(1 + \frac{\gamma_a}{\gamma_r}\right) - \frac{2\gamma_a^2}{1 + \dfrac{\gamma_a}{\gamma_r}} \right\} \tag{1.123}$$

The elastic work done *W* is given by:

$$W = \frac{1}{2}\left(\frac{k_{\max}\gamma_a^2}{1+\dfrac{\gamma_a}{\gamma_r}}\right) \tag{1.124}$$

The damping ratio h is given by the formula:

$$h = \frac{\Delta W}{4\pi W} = \frac{2\eta}{\pi}\left\{1+\frac{2\gamma_r}{\gamma_a}-2\left(\frac{\gamma_r}{\gamma_a}\right)^2\left(1+\frac{\gamma_a}{\gamma_r}\right)\ln\left(1+\frac{\gamma_a}{\gamma_r}\right)\right\} \tag{1.125}$$

Thus, damping ratio h can be plotted against the amplitude of strain γ_a/γ_r for various values of η as:

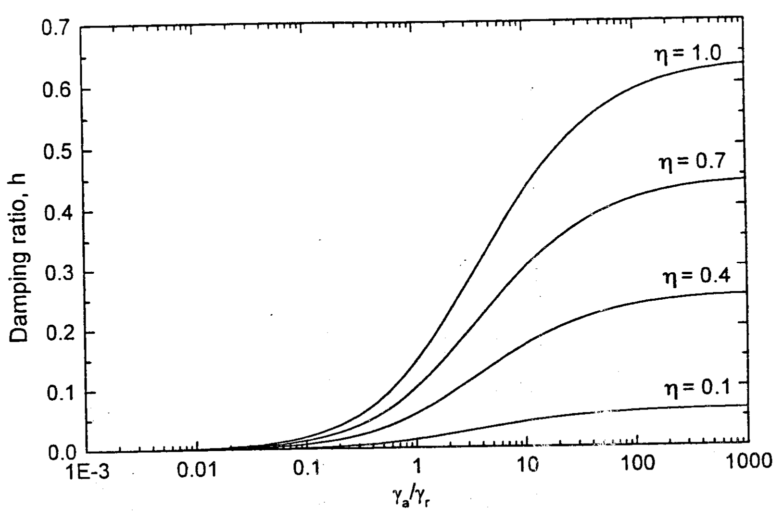


Figure 1.20. Damping ratio for various values of η.

1.9.5. Modeling of hardening due to shear loading

Sand exhibits gradual hardening when subjected to drained cyclic loading. The hardening can be attributed to a densification and influence of shear history. The effects of shear history on hardening of sand means that sand which has experienced some extent of shearing exhibits harder response than virgin sand at the same density does. In order to model this hardening phenomenon, a correlation between stress amplitude for a constant strain amplitude and accumulated volumetric strain was experimentally investigated by Shahnazari and Towhata (2000) leading to the linear relationship in the fig. 1.21 below. In the figure, the maximum shear stress in each cycle normalized by the maximum shear stress in the initial loading for a given constant strain amplitudes versus the volumetric strain.

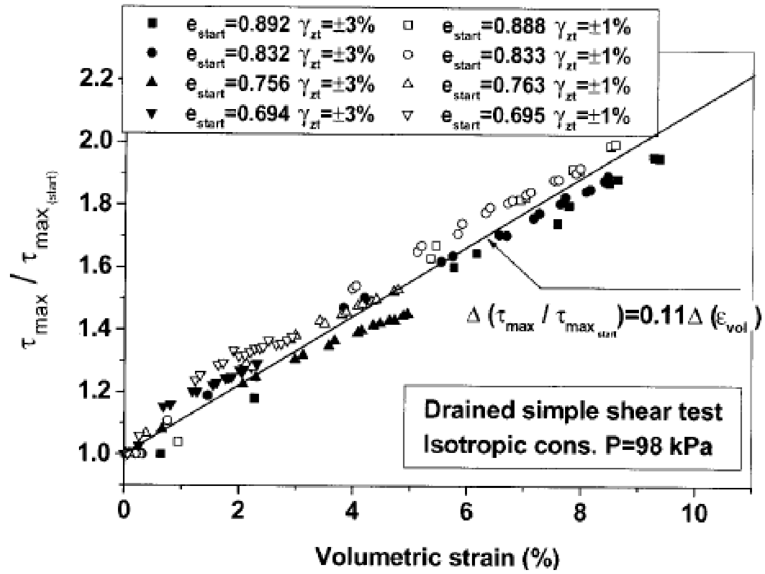


Figure 1.21.Correlation between shear stress amplitude and volumetric strain.

The linear relationship suggested the following equation:

$$R' = (1 + H_p \varepsilon_v)R \tag{1.126}$$

in which *R* is the stress ratio amplitude before hardening is considered and *R'* is the value after modification for hardening and H_p is a coefficient which corresponds to the gradient of the line in the fig. 1.21 above. Hence, in this model, the hardening effect is reproduced by multiplying the shear stress of each shear mechanism by a factor of $(1 + H_p \varepsilon_v)$.

1.9.6. Tangent shear stiffness

The tangential stiffness of each one-dimensional stress-strain relationship $G_{\tan}^{(i)}$ is required in order to form the overall tangent stiffness matrix. Its expression can be obtained from eq. (1.49). If the effect of hardening is included,

$$G_{\tan}^{(i)} = \frac{d\tau^{(i)}}{d\gamma^{(i)}} = \frac{d\left(R^{(i)} p'\left(1 + H_p \varepsilon_v\right)\right)}{d\gamma^{(i)}} \tag{1.127}$$

When *p'* is constant, the tangent stiffness of the one-dimensional relationship is given as:

$$G_{\tan}^{(i)} = \frac{p'\left(1 + H_p \varepsilon_v\right) dR^{(i)}}{d\gamma^{(i)}} \tag{1.128}$$

The expression of $R^{(i)}$ for the skeleton curve is given by eq. (1.103). Substitution of this equation in eq. (1.128) and after some algebraic manipulation we will get:

$$G_{\tan}^{(i)} = \frac{k_{\max} p'\left(1 + H_p \varepsilon_v\right)}{\left(1 + \frac{\left|\gamma - \gamma_o\right|}{\gamma_r}\right)^2} \tag{1.129}$$

For the hysteresis loops, the expression of $R^{(i)}$ is given by eq. (1.117). Similarly, the expression of the tangent shear stiffness when *p'* is constant can be derived as:

$$G_{\tan}^{(i)} = p'\left(1 + H_p \varepsilon_v\right) C k_{\max} \left\{ \frac{1-\eta}{1 + \frac{\gamma_a}{\gamma_r}} + \frac{\eta}{\left(1 + \frac{\left|\gamma - \gamma_{rev}\right|}{2\gamma_r}\right)^2} \right\} \tag{1.130}$$

When the effective principal stress *p'* is not constant, the tangent stiffness matrix is obtained by the following equation:

$$G_{\tan}^{(i)} = \left(\left(1 + H_p \varepsilon_v\right) R^{(i)}\right) \frac{dp'}{d\gamma^{(i)}} + p'\left(1 + H_p \varepsilon_v\right) \frac{dR^{(i)}}{d\gamma^{(i)}} + p' R^{(i)} H_p \frac{d\varepsilon_v}{d\gamma^{(i)}} \tag{1.131}$$

Sand exhibits an elastic response when subjected to small unloading or reloading. Thus, an equivalent elastic shear stiffness $G_{eq}^{(i)}$ can be defined to be the tangent stiffness which appears immediately after loading reversals. Its expression can be derived by inserting the value $\gamma = \gamma_{rev}$ in eq. (1.130).

$$G_{eq}^{(i)} = p'\left(1 + H_p \varepsilon_v\right) C k_{\max} \left\{ \frac{1-\eta}{1 + \frac{\gamma_a}{\gamma_r}} + \eta \right\} \tag{1.132}$$

1.10. Dilatancy and isotropic compression/swelling

1.10.1. Modeling of stress-dilatancy relationship

The two-dimensional Towhata-Iai model uses a correlation between excess pore water pressure and shear work done to sand to calculate the development of excess pore water pressure. This assumption renders the model to be used only for undrained conditions. However, the present three dimensional model enables modeling of volumetric change with a stress-dilatancy relation which will be applicable for general drainage conditions.

The total volumetric strain increment, $d\varepsilon_v$, is assumed to consist of two components; dilatancy component , $d\varepsilon_v^d$, induced by plastic shear strain increment and consolidation component $d\varepsilon_v^c$ induced by the change in *p'*.

$$d\varepsilon_v = d\varepsilon_v^d + d\varepsilon_v^c \tag{1.133}$$

The volume change under drained condition is related to variation of excess pore water pressure under undrained condition via a consolidation curve, based on a postulation that undrained condition is equivalent to constant volume condition.

Since dilatancy is closely related to shear deformation, this model applies a stress-dilatancy relation to all the constituent shear mechanisms. The plastic shear strain increment $d\gamma^{p,(i)}$ of each one-dimensional shear mechanism is given by:

$$d\gamma^{p,(i)} = d\gamma^{(i)} - d\gamma^{e,(i)} \tag{1.134}$$

in which $d\gamma^{(i)}$ is the total shear strain increment and $d\gamma^{e,(i)}$ is the elastic shear increment calculated by the following formula:

$$d\gamma^{e,(i)} = \frac{d\tau}{G_{eq}} \tag{1.135}$$

The ratio between $d\varepsilon_v^d$ and $d\gamma^{p,(i)}$ is termed as dilatancy ratio. A linear relationship between the stress ratio and dilatancy ratio as shown in the figure below will be employed in this model. The ratio between the stress ratio and dilatancy ratio is called stress-dilatancy relationship.

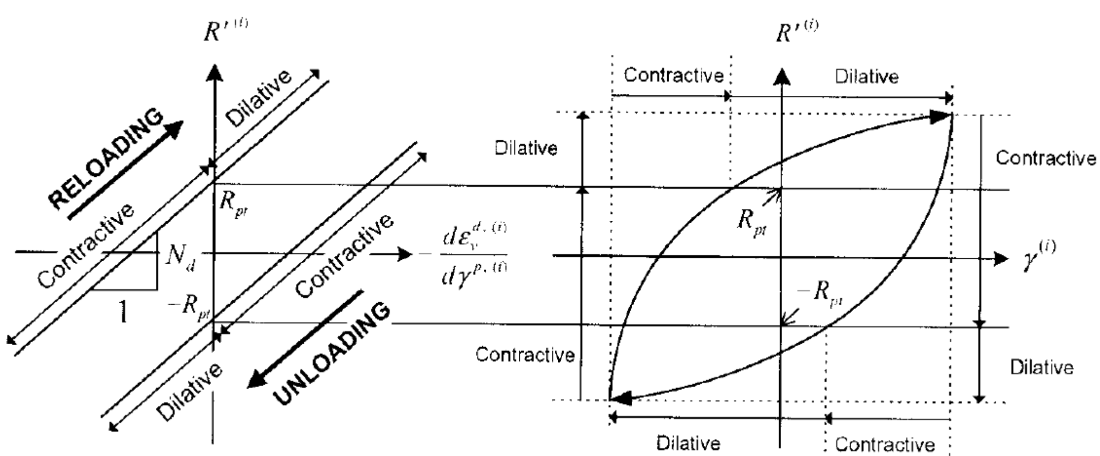


Figure 1.22. Stress dilatancy diagram.

The following equations can be written from the linear stress-dilatancy relationship.

For loading in positive direction or $d\gamma^{p,(i)} > 0$

$$R'^{(i)} = \frac{\tau^{(i)}}{\sigma'^{(i)}} = N_d^{(i)}\left(-\frac{d\varepsilon_v^{d,(i)}}{d\gamma^{p,(i)}}\right) + R_{pt}^{(i)} \tag{1.136}$$

For loading in negative direction or $d\gamma^{p,(i)}<0$

$$R'^{(i)} = \frac{\tau^{(i)}}{\sigma'^{(i)}} = N_d^{(i)}\left(-\frac{d\varepsilon_v^{d,(i)}}{d\gamma^{p,(i)}}\right) - R_{pt}^{(i)} \tag{1.137}$$

Here, $N_d^{(i)}$ and $R_{pt}^{(i)}$ are constants as shown in the fig. 1.22 above. Their values can be determined directly by performing a drained torsion shear from isotropic consolidation test if those parameters are assumed to be identical for all shear mechanisms.

A value of $R_{pt}^{(i)}$ varies for initial and subsequent loading cycles in a cyclic problem. Thus, the present model employs $R_{pt,i}^{(i)}$ and $R_{pt,s}^{(i)}$ for the initial and subsequent loadings respectively. In this model, a threshold strain parameter γ_{th} is introduced for all of the shear mechanisms. If loading in one direction generates a plastic shear strain which exceeds γ_{th} measured from a point of the last loading reversal, $R_{pt,i}^{(i)}$ is switched to $R_{pt,s}^{(i)}$ after the next loading reversal. Otherwise, the loading in this direction is considered to be still minor and the initial stress-dilatancy relation is kept unchanged even after the next loading reversal.

The dilatancy-induced volumetric strain of a soil element is calculated from the average of contributions from all the shear mechanisms.

$$d\varepsilon_v^d = \frac{1}{n}\sum_{i=1}^{n} d\varepsilon_v^{d,(i)} \tag{1.138}$$

1.10.2. Stiffness matrix with dilatancy

In section 1.9.6, the stiffness matrix without the contribution from dilatancy was derived. The stiffness matrix under existence of dilatancy can also be derived using eqs. (1.134), (1.135) and (1.43).

$$\begin{aligned} d\gamma^{p,(i)} &= d\gamma^{(i)} - d\gamma^{e,(i)} \\ &= d\gamma^{(i)} - \frac{d\tau}{G_{eq}^{(i)}} \\ &= d\gamma^{(i)} - \frac{G_{\tan}^{(i)}}{G_{eq}^{(i)}} d\gamma^{(i)} \\ &= \left(1 - \frac{G_{\tan}^{(i)}}{G_{eq}^{(i)}}\right) d\gamma^{(i)} \\ &= \left(1 - \frac{G_{\tan}^{(i)}}{G_{eq}^{(i)}}\right) \{n\}^T [T_\varepsilon^{(i)}] \{d\varepsilon\} \end{aligned} \tag{1.139}$$

If the plastic shear strain is approximated to be equal to the total shear strain, then the plastic shear strain will be given by:

$$d\gamma^{p,(i)} \approx \{n\}^T [T_\varepsilon^{(i)}]\{d\varepsilon\} \tag{1.140}$$

From eqs. (1.136) and (1.137), the increment of volumetric strain due to dilatancy can be given as:

$$d\varepsilon_v^{d,(i)} = -d\gamma^{p,(i)}\left(\frac{R'^{(i)} \pm R_{pt}^{(i)}}{N_d^{(i)}}\right) \tag{1.141}$$

Using eqs. (1.138), (1.139) and (1.140) and considering the hardening effect, the volumetric strain vector due to dilatancy for the overall soil element, $d\varepsilon_v^d$, can be obtained as:

$$\left\{d\varepsilon_v^d\right\} = \{m\}\frac{1}{nN_d}\sum_{i=1}^{n}\left[-(1+H_p\varepsilon_v)R^{(i)}\left(1-\frac{G_{\tan}^{(i)}}{G_{eq}^{(i)}}\right)\{n\}^T\left[T_\varepsilon^{(i)}\right] \pm R_{pt}\left(1-\frac{G_{\tan}^{(i)}}{G_{eq}^{(i)}}\right)\{n\}^T\left[T_\varepsilon^{(i)}\right]\right]\{d\varepsilon\} \tag{1.142}$$

In which ,

$$\{m\} = \{1/3 \quad 1/3 \quad 1/3 \quad 0 \quad 0 \quad 0\}^T \tag{1.143}$$

Rearranging the above equation,

$$\{d\varepsilon_v^d\} = [D]\{d\varepsilon\} \tag{1.144}$$

where ,

$$[D] = \{m\}\frac{1}{n}\sum_{i=1}^{n}\frac{-(1+H_p\varepsilon_v)R^{(i)} \pm R_{pt}}{N_d}\left(1-\frac{G_{\tan}^{(i)}}{G_{eq}^{(i)}}\right)\{n\}^T\left[T_\varepsilon^{(i)}\right] \tag{1.145}$$

The total strain vector is given by the following equation:

$$\{d\varepsilon\} = \{d\gamma\} + \left\{d\varepsilon_v^c\right\} + \left\{d\varepsilon_v^d\right\} \tag{1.146}$$

If [C] is the inverse of the tangential stiffness matrix, then the total strain vector can also be given as:

$$\{d\varepsilon\} = [C]\{d\sigma\} + [D]\{d\varepsilon\} \tag{1.147}$$

by inverting and re-arranging the terms this equation,

$$\{d\sigma\} = [K]([I]-[D])\{d\varepsilon\} \tag{1.148}$$

where [*I*] is the unit matrix and thus the stiffness matrix with dilatancy, [*K'*], is

$$[K'] = [K]([I]-[D]) \tag{1.149}$$

The previous assumption of approximating the plastic shear strain increment $d\gamma^{p,(i)}$ by the total shear strain increment $d\gamma^{(i)}$ leads to the expression for $\{d\varepsilon_v^d\}$ to be given by:

$$\{d\varepsilon_v^d\} = \{m\}\frac{1}{nN_d}\sum_{i=1}^{n}\left(-(1+H_p\varepsilon_v)R^{(i)}\{n\}^T\left[T_\varepsilon^{(i)}\right] \pm R_{pt}\{n\}^T\left[T_\varepsilon^{(i)}\right]\right)\{d\varepsilon\} \tag{1.150}$$

The first term of eq. (1.150) is associated with shear stress vector normalized by mean effective stress *p'* in the original xyz coordinate system. Thus, this term becomes

$$\{m\}\frac{1}{nN_d}\{d\varepsilon\}^T\sum_{i=1}^{n}\left(-(1+H_p\varepsilon_v)R^{(i)}\left[T_\varepsilon^{(i)}\right]^{-1}\{n\}\right) = \frac{1}{N_d}\{d\varepsilon\}^T\frac{\{\tau\}}{p'} \tag{1.151}$$

When loading in positive direction in isotropic torsion shear mode is considered,

$$\{d\varepsilon\}^T = \{d\varepsilon_v/3 \quad d\varepsilon_v/3 \quad d\varepsilon_v/3 \quad 0 \quad 0 \quad \gamma_{zx}\}^T \tag{1.152}$$

Hence the product $\{d\varepsilon\}^T\{\tau\}$ gives,

$$\{d\varepsilon\}^T\{\tau\} = \tau_{zx}d\gamma_{zx} \tag{1.153}$$

The second term in eq. (1.150) is numerically calculated for extended icosahedral distribution with n=192.

$$\{d\varepsilon_v^d\} = \frac{R_{pt}}{nN_d}\sum_{i=1}^{n}\left(\pm\{n\}^T\left[T_\varepsilon^{(i)}\right]\right)\{d\varepsilon\} \approx 0.384\frac{R_{pt}}{N_d}d\gamma_{zx} \tag{1.154}$$

Then eq. (1.150) will become,

$$\{d\varepsilon_v^d\} = \frac{\tau_{zx}d\gamma_{zx}}{N_d p'} + 0.384\frac{R_{pt}}{N_d}d\gamma_{zx} \tag{1.155}$$

If the aforementioned assumption of approximating the plastic shear strain increment by the total shear strain increment is again applied in eq. (1.136) and (1.137), the stress-dilatancy equation of an overall element undergoing isotropic torsion shear is derived as:

$$\frac{\tau_{zx}}{p'} = N_d \left(-\frac{d\varepsilon_v^d}{d\gamma_{zx}^p} \right) + 0.384 R_{pt} \tag{1.156}$$

For loading in negative direction,

$$\frac{\tau_{zx}}{p'} = N_d \left(-\frac{d\varepsilon_v^d}{d\gamma_{zx}^p} \right) - 0.384 R_{pt} \tag{1.157}$$

1.10.3. Modeling of isotropic compression and swelling

Isotropic compression and swelling are modeled using the conventional linear ε-log*p'* curve as shown in fig. (1.23) . The expression of the volumetric strain due to isotropic compression and swelling can be given as:

$$\varepsilon_v^c = \frac{0.434 C_c}{1 + e_o} \ln p' + \beta \tag{1.158}$$

Where C_c and e_o are compression index of sand and initial void ratio respectively, and β is constant. The bulk modulus of sand can be derived from C_c as:

$$B = \frac{dp'}{d\varepsilon_v^c} = \frac{1 + e_o}{0.434 Cc} p' \tag{1.159}$$

The bulk modulus is dependent on the level of mean effective stress. The bulk modulus of the sand skeleton is given as B_o at reference mean effective principal stress, p'_o , namely

$$B_o = \frac{1 + e_o}{0.434 C_c} p'_o \tag{1.160}$$

By relating eqs. (1.159) and (1.160), the bulk modulus *B* at mean effective principal stress, *p'*, is given by:

$$B = B_o \frac{p'}{p'_o} \tag{1.161}$$

Then eq. (1.158) becomes:

$$\varepsilon_v^c = \frac{p_o'}{B_o} \ln p' + \beta \tag{1.162}$$

If $(p'_y, \varepsilon_{v,y}^c)$ represents the point of isotropic compression yield stress as shown in fig. 1.23, the value of the constant β can be obtained as:

$$\beta = \varepsilon_{v,y}^{c} - \frac{p_o'}{B_o} \ln p_y' \tag{1.163}$$

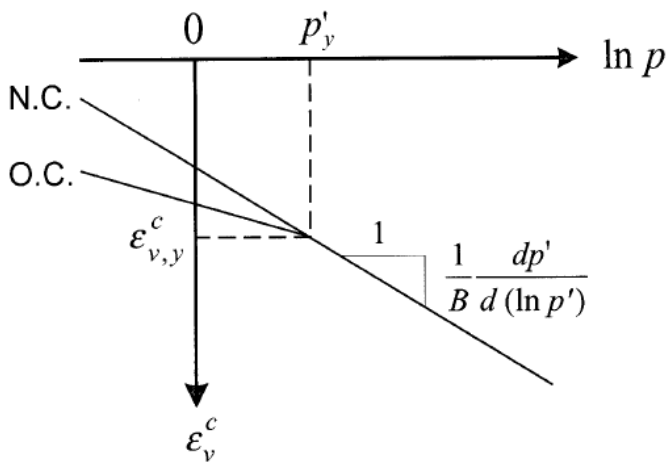


Figure 1.23. Bilinear elasto-plastic isotropic compression curve.

Two different values for Bo should be prepared in order to describe normal isotropic compression (N.C.) and over-consolidated isotropic consolidation (O.C.). Those are referred to in the present model as $B_{c,o}$ and $B_{s,o}$ respectively. These parameters respectively correspond to compression and swelling indices in the ε-log*p'* plot.
Taking the derivative of both sides of eq. (1.126) with respect to *p'* gives:

$$\frac{d\varepsilon_v^c}{dp'} = \frac{p_o'}{p' B_o} \qquad \Rightarrow \qquad dp' = \frac{p' B_o}{p_o'} d\varepsilon_v^c \tag{1.164}$$

Substitution of the value of *p'* from eq. (1.162) in eq. (1.164) leads to the relation:

$$dp' = \frac{p'_y B_o}{p'_o} \exp\left\{ \frac{B_o}{p'_o} (\varepsilon_v^c - \varepsilon_{v,y}^c) \right\} . d\varepsilon_v^c \tag{1.165}$$

Under undrained condition where $d\varepsilon_v = 0$, eq. (1.133) leads to:

$$d\varepsilon_v^c = -d\varepsilon_v^d \tag{1.166}$$

The mean effective stress can be derived from eqs. (1.162) and (1.163).

$$p' = p_y' \exp\left(\frac{B_o}{p_o'} \left(\varepsilon_v^c - \varepsilon_{v,y}^c \right) \right) \tag{1.167}$$

The variation of effective stress under undrained condition is reproduced via the isotropic compression model by first calculating virtual dilatancy, $d\varepsilon_v^d$, and subsequently calculating increment of mean effective stress from stress-dilatancy relation as follows.

$$dp' = -\frac{p'_y\, B_o}{p'_o}\exp\left\{\frac{B_o}{p'_o}(\varepsilon_v^c - \varepsilon_{v,y}^c)\right\}.d\varepsilon_v^d \tag{1.168}$$

1.11. Overview of material and state parameters of the model

The material parameters describe the properties of the material for which the analysis is to be executed. Thus, their values are constant throughout the computation process. The state parameters are variable quantities in the model which should be updated each time as the computation progresses. The material and state parameters which are necessary for the model are summarized in the following table.

A. Material parameters

There are thirteen parameters which will have constant value throughout the analysis process. They are listed in the following table along with the equation number in which the parameter is involved.

Table 1.3. Material parameters of the model.

Parameters symbol	Description	Equation number
$k_{max,o}$	Spring stiffness at initial mean effective stress	1.99
$\gamma_{r,o}$	Reference shear strain at initial mean effective stress	1.102
η	Reduction factor for damping ratio	1.117
$B_{c,o}$	Bulk modulus of compression at mean effective stress	1.165
$B_{s,o}$	Bulk modulus of swelling at mean effective stress	1.165
p'_o	Reference mean effective stress	1.98 ,1.99 1.102 ,1.161
$R_{pt,i}$	Stress ratio at phase transformation point for initial loading	1.136, 1.137
$R_{pt,s}$	Stress ratio at phase transformation point for subsequent loading	1.136, 1.137
N_d	Slope of the curve of stress ratio versus dilatancy ratio diagram	1.136 ,1.137
H_p	Factor to consider hardening effect	1.126
γ_{th}	Threshold value for transfer of stress-dilatancy relation	1.136,1.137
NN	Coefficient of stress dependency of bulk modulus (=1)	1.161
MM	Coefficient of stress dependency of k_{max} and γ_r (=0.5)	1.99 , 1.100

B. State parameters of the model

The parameters which should be updated during the computation process are listed in the following table.

Table 1.4. State parameters of the model.

Parameters symbol	Description	The quantity belongs to:	Equation number
ε_v^d	Total volumetric strain due to dilatancy	global	1.133,1.141
$\varepsilon_{v,y}^c$	Volumetric strain due to consolidation at yield stress	global	1.165,1.167
$p_y^{'}$	Yield stress	global	1.165,1.167
p'	Mean stress at the previous step	Plane	1.167
wp	Total plastic shear work	Spring	-
p'	Normal stress to each plane	plane	1.167
G_{max}	Shear stiffness	Spring	1.72
γ_{max}	Maximum shear strain in the past	Spring	1.114
γ_{rev}	Shear strain at reversal point	Spring	1.117
γ_1	Shear strain at the end of previous step	Spring	-
R_{max}	Maximum stress ratio attained in the past	Spring	1.116
R_{rev}	Stress ratio at reversal point	Spring	1.117
R	Stress ratio	Spring	1.64,1.68,1.117
$C^{(i)}$	Scaling factor	Spring	1.116
γ_a	Amplitude of shear strain.	spring	1.68,1.114,1.117

CHAPTER TWO

ANALYSIS PROCESS

2.1. Flow of analysis

Based on the constitutive model described in chapter one, the flow of the analysis for liquefaction analyis can be given. Since the finite element procedure adopted is a displacement based finite element procedure, only the strain controlled flow of analysis is relevant for understanding the source code. The flow diagram for the strain controlled analysis is shown in the flow chart below.

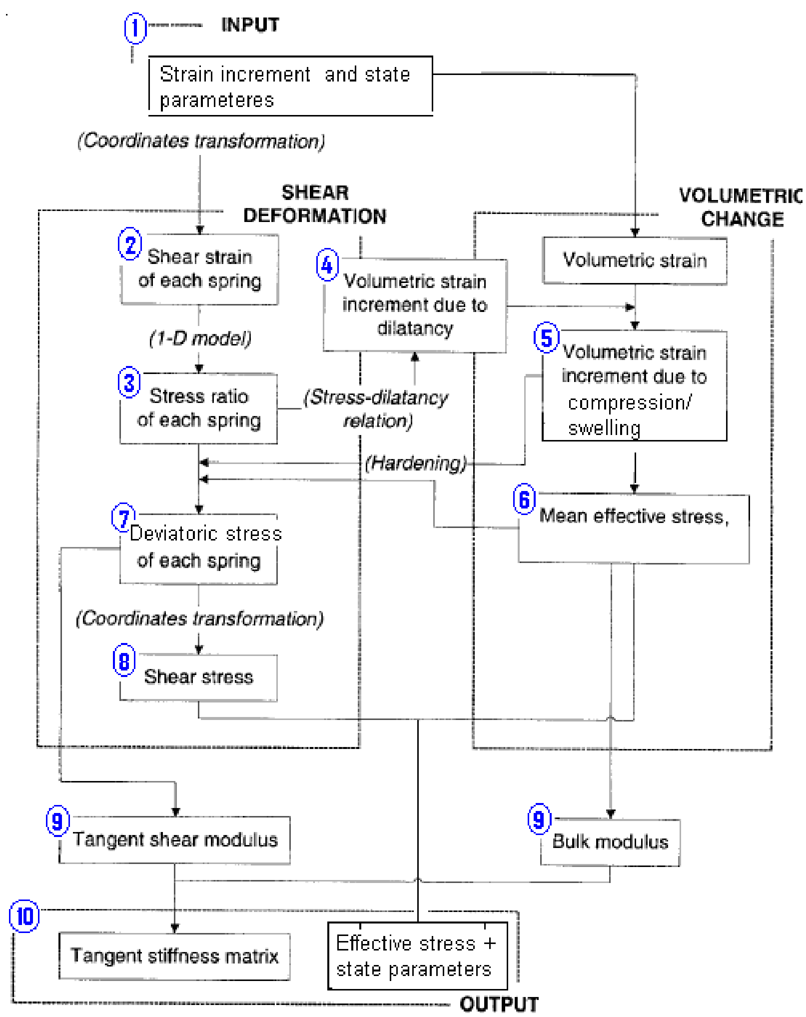


Fig. 2.1. Flow of calculation for strain controlled case

Depending on this flow of analysis, the source code for the 3-D liquefaction analysis will follow the following task orders.

1. The prescribed value of strain increment, the material parameters, the status parameters and the initial mean effective stress will be read.
2. The shear component of strain for each spring will be computed from the given strain using coordinate transformation.
3. From Masing's rule, the stress ratio of each spring will be determined
4. Using stress-dilatancy relationships, the volumetric strain increment due to dilatancy will be computed.
5. The volumetric strain increment due to isotropic compression and swelling will be reproduced from the "virtual" dilatancy.
6. The mean effective stress will be computed from isotropic compression and swelling curve.
7. The volumetric strain along with the hardening rule and the mean effective stress will be used to compute the shear stress in each spring.
8. The total deviatoric shear stress in the soil element will be calculated from shear stresses in each spring by coordinate transformation.
9. The tangent shear modulus and the bulk modulus will be calculated from the shear stress of each spring and from the mean effective stress respectively.
10. The tangent shear modulus and the bulk modulus will finally be used to compute the tangent stiffness matrix and the effective stresses in the soil.

2.2. Review of the source code

The source code to be reviewed in this report is used at integration point level. The matrix manipulation facilities to assemble the computed quantities for the whole soil mass are obtained by linking the user supplied subroutine with DIANA environment.

2.2.1. List of quantities used in the program.

- *BETA:* the angle ζ which determines the orientation of each spring on the plane.
- *BULK:* compression (or swelling) stiffness matrix.[*B*]
- *BULK0:* bulk modulus of compression*,* $B_{c,o}$
- *BULKS:* current bulk modulus
- *BULKS0:* bulk modulus of swelling *,*$B_{s,o}$
- *COOR (NPLANE*2):* the angles θ and ϕ of the normal vectors for planes in icosahedral distribution.
- *COORD(3) :* coordinates of integration point
- *COR(SPRING):* correction factor C
- *DEPS(NSTR)[Intent: in] :* total strain increment (xx,yy,zz,xy,yz,zx)
- *DEV:* incremental volumetric strain.
- *DEVC:* incremental volumetric strain due to consolidation, $d\varepsilon_v^c$.
- *DEVD:* incremental volumetric strain due to dilatancy, $d\varepsilon_v^d$.
- *DMODE(SPRING): User indicator*
- *DPEPS:* incremental plastic shear strain of each spring, $d\gamma^{p,(i)}$.

- *DPSIG(NPLANE):* incremental mean effective stress, *dp'*.
- *DPWORK:* incremental plastic shear work.
- *DTIME :* time increment
- *ELEMEN :* current element number
- *EPS0(NSTR)[Intent:in] :* strain vector at the start of the increment
- *EVCSUM:* volumetric strain due to consolidation.
- *EVDSUM:* volumetric strain due to dilatancy.
- *EVSUM:* total volumetric strain.
- *EVY:* value of consolidation strain at the yield point.
- *G(SPRING,1):* the shear strain of each spring at previous step.
- *G(SPRING,2):* the shear strain of each spring at the current step.
- *GAMP(SPRING):* amplitude of shear strain.
- *GLAST(3):* carries the last three diagonal elements of the stiffness matrix.
- *GMAX(SPRING):* the maximum tangential shear stiffness.
- *GORI(SPRING):* factor for the shift of skeleton curve when the maximum stress ratio in the past is exceeded.
- *GR:* reference shear strain, γ_r.
- *GR0:* reference shear strain $\gamma_{r,o}$
- *GREV(SPRING):* shear strain at the recent reversal point, γ_{rev}.
- *GTH:* threshold shear strain = 0.0001
- *GTYPE=1: parameter.*
- *HP:* a parameter used to compute the factor for hardening effects ,H_p.
- *HPEV:* factor to account for hardening effect, $1+\varepsilon_v H_p$.
- *I:* simple counter.
- *IDEVD:* incremental volumetric strain of each spring, $d\varepsilon_v^{d,(i)}$
- *IEPS(NSTR):* strain vector at the current step.
- *INTPT :* current integration point number.
- *ISIG(NSTR):* shear stress vector of each spring, $\tau^{(i)}$.
- ISTIFF(NSTR,NSTR): tangential stiffness matrix of each spring.
- *ITER :* current iteration number
- *J:* simple counter
- *K:* simple counter
- *KEQU:* equivalent elastic shear stiffness.
- *KMAX:* the stiffness of each spring at small strain.
- *KMAX0:* initial stiffness of each spring, $k_{max,o}$
- *KTAN(SPRING):* tangent stiffness matrix each spring.
- *L:* simple counter.
- *LTDEPS=E-4:* minimum possible total initial strain.
- *LTDGAM=E-14:* minimum allowable value for the difference between shear strains at consecutive steps.
- *LTDPEP=E-14:* minimum possible value for plastic shear strain.

- *LTEVCS:* minimum possible value of volumetric strain due to consolidation.
- *LTGTAN=E4:* minimum possible value for GLAST.
- *LTMEAN=E-4:* the minimum possible mean effective stress.
- *LTPSIG=E-4:* the minimum possible mean effective stress.
- *LTRATI=E-3:* threshold value for the rate of stress dependency.
- *LTSTP=100:* maximum possible number of steps.
- *M:* simple counter.
- *MAXSTP:* maximum number of steps
- *MEPS:* mean effective isotropic stress.
- *MEPSO:* initial effective isotropic stress σ_o'
- *MM:* the rate of stress dependency.
- *MMODE(SPRING): User indicator*
- *N:* simple counter.
- *NDI(3):* vector containing the values of the gradient of stress-dilatancy relationship.
- *NINDIC[Intent :in] :* number of status indicators
- *NDIV=6:* number of springs on each plane in icosahedral distribution.
- *NDN:* a gradient of stress-dilatancy relationships for each mechanism, N_d
- *NDS(3):* vector containing the values of the gradient of stress-dilatancy relationship.
- *NN:* the rate of stress dependency.
- *NOWSTP:* current step number.
- *NPLANE=32:* total number of planes for 3-D modeling.
- *NSTATE [Intent : in]:* number of user indicators.
- *NSTR [Intent :in]:* number of stress components.
- *NUSRVL :* number of user parameters to be defined.
- *OFFSET=40:* parameter.
- *PAI:* set to the value of π.
- *PHI:* the angle ϕ of the normals to each plane.
- *PLN0:* current plane number
- *PSIG(NPLANE):* mean effective stress vector.
- *PTAN(SPRING):* overall tangent compression (or swelling) stiffness matrix.
- *PWORK:* cumulative plastic shear work.
- *R(SPRING,1):* stress ratio at the previous step.
- *R(SPRING,2):* stress ratio at the current step.
- *RMAX(SPRING):* maximum stress ratio.
- *RPT(1):* stress ratio, $R_{pt,i}$
- *RPT(2):* stress ratio, $R_{pt,s}$
- *RREV(SPRING):* stress ratio at recent reversal point.
- *SE (NSTR,NSTR) :* elastic stiffness matrix.
- *SIGMA(NSTR):* stress vector.
- *SIG(NSTR):* vector containing total shear stress vector imposed to the soil.
- *SIGMA:* stress vector.

- *SIGMAB(NSTR)[Intent : in/out] :* vector containing total normal stress.
- *SIGY:* the value of mean effective stress at the yield point.
- *SPN0:* the spring number.
- *SPRING (=192):* the total number of springs in 3-D icosahedral distribution.
- *STIFF(NSTR,NSTR) [Intent: in/out] :* current tangent stiffness
- *THETA:* the angle θ of the normal lines to each plane.
- *TIME0 :* time
- *USRIND(NINDIC)[Intent : in/out]:* user supplied status indicators
- *USRSTA(NSTATE) :* user state variables at start of the increment.
- *USRVAL(NUSRVL)[Intent : in] :* user-supplied material parameters.
- *WIDTH=9:* parameter
- *XDEPS(NSTR):* incremental strain vector per each step
- *XEPS0(NSTR):* accumulated strain vector from previous step
- *YETA:* factor to control damping ratio η

2.2.2. Subroutines used in the program.

- SPLOCA (COOR) : gives the angles θ and ϕ of the normal vectors for planes in icosahedral distribution.
- TRANSMAT (THETA, PHI, BETA, TA, TB, TC): establishes the transformation matrix.
- TRANSFER (CASE, THETA, PHI,BETA,ISIG): transfers strains and stresses from one coordinate system to another.
- RENEW (KMAX,GR,MM,PSIG(PLNO),MEPSO,KMAXO,GRO): determines the current values of the spring stiffness and reference shear strain.
- SPMAT (THETA,PHI,BETA,ISTIFF): computes the matrix product $[T_\varepsilon]^T[N][T_\varepsilon]$.
- MASING (PSIG(PLNO),KMAX,GR,YETA,GORI(SPNO)) : establishes the 1-D shear stress-strain relationships using Masing's rule.
- DILATANCY(IDEVD, DPEPS, DMODE(SPNO), G(SPNO,1), G(SPNO,2), R(SPNO,1), R(SPNO,2), NDI, NDS, KEQU , GTH, RPT, LTDPEP, GREV(SPNO) , RREV(SPNO) , HPEV, PSIG(PLNO))) : calculates the volumetric strain due to dilatancy.
- SUBSIG(MEPS, MEPSO, BULKSO, BULKO, EVCSUM, EVY, SIGY, SIGI, DEVC, BULK, NN, LTRATI): computes the mean effective principal stress from the component of volumetric strain due to consolidation.
- ZEROM (STIFF): forms the pattern of the elastic stiffness matrix.
- EXTHD (R, KTAN, KMAX, GAM(2), GR, MEAN, JUDGE, GREV, RREV, COR, GAMP, YETA): calculates the stress ratio and tangent stiffness matrix for regular and general loading patterns.
- CORRECT (GAMP, COR, GREV, GTMAX, RTMAX, RREV, GR, KMAX, 1): determines the strain amplitude and the correction factor for regular as well as for general loading patterns.

- EQUIV (KEQU,KMAX,COR,YETA,GAMP,GR,MEAN,MMODE): computes the equivalent elastic shear stiffness.

The inter-relation of the main source code USRLIQ, the subroutines and the DIANA environment can be summarized as in fig below. Some of the subroutines also require other subroutines to accomplish their tasks.

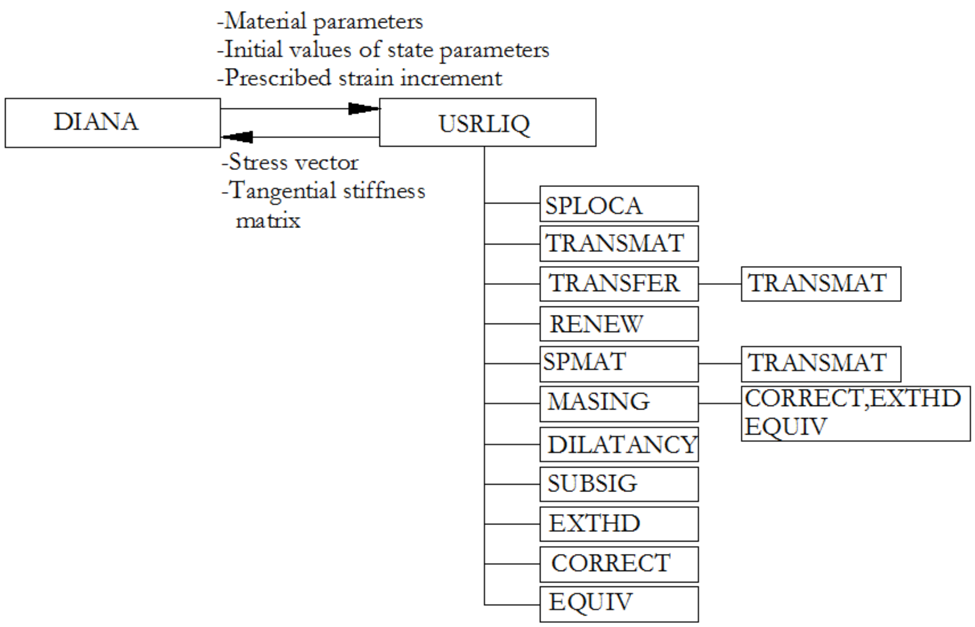


Fig. 2.2. Structure of the main source code

2.2.3. Detailed overview of the subroutines.

The subroutines outlined above will be explained in detail in the next section by referring back the theoretical background given in chapter one.

1. SPLOCA (COOR):

For even distribution of virtual planes so as to have an isotropic model, an icosahedron is proven to be the best option for the multi-spring model (Nishimura 2002). A plane can be uniquely described by a (normal) line and a point on the plane. In this model, 32 planes will be used. These planes are described by perpendicular lines originating from the center of the icosahedron towards the center of gravity of the facets and apices will be used. Hence, there will also be 32 normal lines. These normal lines can be uniquely described by two angles θ and ϕ in space as shown in fig 1.2a. The aim of this subroutine is to give the values of θ and ϕ for each normal line.

A double precision array `COOR(64)` is used to store these values. This array contains 64 elements: 32 of which belong to the value of θ and the rest 32 for the values of ϕ.

`COOR (1)` up to `COOR (32)` stores the values of θ for line 1 up to line 32.

`COOR (33)` up to `COOR (64)` stores the values of ϕ for line 1 up to line 32.

The values of θ and ϕ are given in appendix A. There is agreement between the values given in the source code and in the original model.

2. TRANSMAT (THETA, PHI, BETA, TA, TB, TC):

In this multiple mechanism model, the strain imposed to the soil should be distributed to each spring. For this, the transformation matrices given in eqs. (1.15) ,(1.17) and (1.19) are necessary. This subroutine establishes the transformation matrix for each rotation operation of strain. The statements used in the program are:

```
     DOUBLE PRECISION THETA, PHI, BETA, TA(6,6), TB(6,6), TC(6,6)
       DOUBLE PRECISION CTHETA, STHETA, CPHI, SPHI, CBETA, SBETA
CC
       CTHETA=DCOS(THETA) ; STHETA=DSIN(THETA)
       CPHI  =DCOS(PHI)   ; SPHI  =DSIN(PHI)
       CBETA =DCOS(BETA)  ; SBETA =DSIN(BETA)
       TA(1,1)=CTHETA*CTHETA;TA(1,2)=STHETA*STHETA;TA(1,3)=0.D0
       TA(1,4)=CTHETA*STHETA;TA(1,5)=0.D0;TA(1,6)=0.D0
CC
       TA(2,1)=STHETA*STHETA;TA(2,2)=CTHETA*CTHETA;TA(2,3)=0.D0
       TA(2,4)=(-1.D0)*CTHETA*STHETA;TA(2,5)=0.D0;TA(2,6)=0.D0
CC
       TA(3,1)=0.D0;TA(3,2)=0.D0;TA(3,3)=1.D0
       TA(3,4)=0.D0;TA(3,5)=0.D0;TA(3,6)=0.D0
CC
       TA(4,1)=(-2.D0)*CTHETA*STHETA;TA(4,2)=2.D0*CTHETA*STHETA
       TA(4,3)=0.D0;TA(4,4)=CTHETA*CTHETA-STHETA*STHETA
       TA(4,5)=0.D0;TA(4,6)=0.D0
CC
       TA(5,1)=0.D0;TA(5,2)=0.D0;TA(5,3)=0.D0
       TA(5,4)=0.D0;TA(5,5)=CTHETA;TA(5,6)=-1.D0*STHETA
CC
       TA(6,1)=0.D0;TA(6,2)=0.D0;TA(6,3)=0.D0
       TA(6,4)=0.D0;TA(6,5)=STHETA;TA(6,6)=CTHETA
       TB(1,1)=CPHI*CPHI;TB(1,2)=0.D0;TB(1,3)=SPHI*SPHI
       TB(1,4)=0.D0;TB(1,5)=0.D0;TB(1,6)=(-1.D0)*CPHI*SPHI
CC
       TB(2,1)=0.D0;TB(2,2)=1.D0;TB(2,3)=0.D0
       TB(2,4)=0.D0;TB(2,5)=0.D0;TB(2,6)=0.D0
CC
       TB(3,1)=SPHI*SPHI;TB(3,2)=0.D0;TB(3,3)=CPHI*CPHI
       TB(3,4)=0.D0;TB(3,5)=0.D0;TB(3,6)=CPHI*SPHI
CC
       TB(4,1)=0.D0;TB(4,2)=0.D0;TB(4,3)=0.D0
       TB(4,4)=CPHI;TB(4,5)=(-1.D0)*SPHI;TB(4,6)=0.D0
CC
       TB(5,1)=0.D0;TB(5,2)=0.D0;TB(5,3)=0.D0
       TB(5,4)=SPHI;TB(5,5)=CPHI;TB(5,6)=0.D0
CC
       TB(6,1)=2.D0*CPHI*SPHI;TB(6,2)=0.D0
       TB(6,3)=(-2.D0)*CPHI*SPHI;TB(6,4)=0.D0
       TB(6,5)=0.D0;TB(6,6)=CPHI*CPHI-SPHI*SPHI
       TC(1,1)=CBETA*CBETA;TC(1,2)=SBETA*SBETA;TC(1,3)=0.D0
       TC(1,4)=CBETA*SBETA;TC(1,5)=0.D0;TC(1,6)=0.D0
CC
       TC(2,1)=SBETA*SBETA;TC(2,2)=CBETA*CBETA;TC(2,3)=0.D0
```

```
        TC(2,4)=(-1.D0)*CBETA*SBETA;TC(2,5)=0.D0;TC(2,6)=0.D0
CC
        TC(3,1)=0.D0;TC(3,2)=0.D0;TC(3,3)=1.D0
        TC(3,4)=0.D0;TC(3,5)=0.D0;TC(3,6)=0.D0
CC
        TC(4,1)=(-2.D0)*CBETA*SBETA;TC(4,2)=2.D0*CBETA*SBETA;
      TC(4,3)=0.D0;TC(4,4)=CBETA*CBETA-SBETA*SBETA;
      TC(4,5)=0.D0;TC(4,6)=0.D0
CC
        TC(5,1)=0.D0;TC(5,2)=0.D0;TC(5,3)=0.D0
        TC(5,4)=0.D0;TC(5,5)=CBETA;TC(5,6)=-1.D0*SBETA
CC
        TC(6,1)=0.D0;TC(6,2)=0.D0;TC(6,3)=0.D0
        TC(6,4)=0.D0;TC(6,5)=SBETA;TC(6,6)=CBETA
CC
```

Here, TA, TB and TC represent $[T_{\varepsilon,\theta}]$, $[T_{\varepsilon,\phi}]$ and $[T_{\varepsilon,\zeta}]$ respectively with THETA ,PHI and BETA representing θ, ϕ and ζ respectively in eqs. (1.15), (1.17) and (1.19). The expressions given here are consistent with the theory.

3. SPMAT (THETA,PHI,BETA,ISTIFF):

The product $[T_\varepsilon]^T[N][T_\varepsilon]$ appears in many parts for the computation of the stiffness matrices. This subroutine computes the value of this expression. Before looking into the Fortran format to accomplish this task, the mathematical simplification of the matrix product will be reviewed.

Writing the matrices in index format, will result in $[T_\varepsilon]_{ij}$ and N_{kl}. From the definition of matrix N, $N_{kl} = 0$, if k≠5 or l≠5. Hence, the expansion of the product $[N_{ki}][T_\varepsilon]_{ij}$ will yield:

$$[N_{ki}][T_\varepsilon]_{ij} = [N_{k1}][T_\varepsilon]_{1j} + [N_{k2}][T_\varepsilon]_{2j} + [N_{k3}][T_\varepsilon]_{3j} + [N_{k4}][T_\varepsilon]_{4j} + [N_{k5}][T_\varepsilon]_{5j}$$
$$[N_{k6}][T_\varepsilon]_{6j}$$

for $k \neq 5$ *or* $j \neq 5$, *all the terms except the fifth term will be zero. Hence*

$$[N_{ki}][T_\varepsilon]_{ij} = [N_{55}][T_\varepsilon]_{5j}$$

and the value of $[N_{55}]$ *is equal to* 1 *which finally simplifies the expression to*:

$$[N_{ki}][T_\varepsilon]_{ij} = [T_\varepsilon]_{5j}$$

therefore, the whole product $[T\varepsilon]^T[N][T_\varepsilon]$ *will become*

$$[T_\varepsilon]^T[N][T_\varepsilon] = [T_\varepsilon]_{ij}[T_\varepsilon]_{5k}$$

In this subroutine, the expression to compute $[T_\varepsilon]$ is :

```
      DO 100 I=1,6
              DO 200 J=1,6
                TEMP1(I,J)=0.D0
                TEMP2(I,J)=0.D0
                STIFF(I,J)=0.D0
        200   CONTINUE
        100 CONTINUE
```

```
CC
      CALL TRANSMAT(THETA, PHI, BETA, TA, TB, TC)
CC
      DO 1100 I=1,6
        DO 1200 J=1,6
          DO 1300 K=1,6
            TEMP1(I,J)=TEMP1(I,J)+TB(I,K)*TA(K,J)
 1300     CONTINUE
 1200   CONTINUE
 1100 CONTINUE
CC
      DO 1400 I=1,6
        DO 1500 J=1,6
          DO 1600 K=1,6
            TEMP2(I,J)=TEMP2(I,J)+TC(I,K)*TEMP1(K,J)
 1600     CONTINUE
 1500   CONTINUE
 1400 CONTINUE
CC
```

To have a complete product, the other term $[N_{ki}][T\varepsilon]_{ij} = [T\varepsilon]_{5j}$ should also be calculated as follows:

```
DO 1700 I=1,6
        DO 1800 J=1,6
          TEMP1(I,J)=0.D0
 1800   CONTINUE
 1700 CONTINUE
CC
      DO 1900 I=1,6
        TEMP1(5,I)=TEMP2(5,I)
 1900 CONTINUE
```

Here `TEMP2(I,J)` represents $[T_\varepsilon]$ and the final product is written in the program as:

```
      DO 2000 I=1,6
        DO 2100 J=1,6
          DO 2200 K=1,6
            STIFF(I,J)=STIFF(I,J)+TEMP2(K,I)*TEMP1(K,J)
 2200     CONTINUE
 2100   CONTINUE
 2000 CONTINUE
```

4. TRANSFER (CASE, THETA, PHI,BETA,ISIG):

This subroutine transfers stress and strain quantities from the original coordinate system to the new coordinate system or the other way round by multiplying the quantities with the right transformation matrix. The total transformation matrices for strain and stress are obtained by the products of the transformation matrices for each rotation angles θ, ϕ and ζ according to eqs. (1.20) and (1.35).

In this subroutine, three cases will be selected for the product.

TCASE = 1 : to multiply global strain vector with the overall transformation matrix for strain to get the strain vector in the rotated coordinate system[eq.(1.9).].
TCASE = 2 : to calculate the stress vector in the global coordinate system from strain vector in the springs according to eq.(1.23).
TCASE = 3 : to transfer stress vector from xyz to $x'''y'''z'''$ coordinate system.

When stress quantities are transferred from the xyz coordinate system to the new system, the angles should be reversed and therefore have opposite sign in which the overall transformation matrix will be calculated when TCASE = 3. The angles are defined for the three cases as follows in the subroutine.

```
IF(TCASE.EQ.3) THEN
        THETA=(-1.D0)*ITHETA
        PHI  =(-1.D0)*IPHI
        BETA =(-1.D0)*IBETA
      ELSE
        THETA=ITHETA
        PHI  =IPHI
        BETA =IBETA
      END IF
```

The first case which transforms the strain quantity TMAT(J) is:

```
CASE(1)
          DO 1100 I=1,6
            DO 1200 J=1,6
              TEMP1(I)=TEMP1(I)+TA(I,J)*TMAT(J)
 1200       CONTINUE
 1100     CONTINUE
          DO 1300 I=1,6
            DO 1400 J=1,6
              TEMP2(I)=TEMP2(I)+TB(I,J)*TEMP1(J)
 1400       CONTINUE
 1300     CONTINUE
          DO 1500 I=1,6
            TMAT(I)=0.D0
 1500     CONTINUE
          DO 1600 I=1,6
            DO 1700 J=1,6
              TMAT(I)=TMAT(I)+TC(I,J)*TEMP2(J)
 1700       CONTINUE
 1600     CONTINUE
```

In compact form, the above statements compute the product [TC]*[TB]*[TA]*[TMAT] which gives [T_ε]*[TMAT]. Under Case 2, the following statements can be found.

```
CASE(2)
          DO 2100 I=1,6
            DO 2200 J=1,6
              TEMP1(I)=TEMP1(I)+TC(J,I)*TMAT(J)
 2200       CONTINUE
 2100     CONTINUE
          DO 2300 I=1,6
            DO 2400 J=1,6
              TEMP2(I)=TEMP2(I)+TB(J,I)*TEMP1(J)
 2400       CONTINUE
 2300     CONTINUE
```

```
            DO 2500 I=1,6
               TMAT(I)=0.D0
2500        CONTINUE
            DO 2600 I=1,6
              DO 2700 J=1,6
                TMAT(I)=TMAT(I)+TA(J,I)*TEMP2(J)
2700          CONTINUE
2600        CONTINUE
```

These statements aim to get the result of the product `[TA]ᵀ*[TB]ᵀ*[TC]ᵀ*[TMAT]` which is $[T_{\sigma}]^{-1}$`*[TMAT]`. The third case which is aimed to decompose the stress components from the soil into each spring in the original coordinate system is written as follows.

```
CASE(3)
            DO 3100 I=1,6
              DO 3200 J=1,6
                TEMP1(I)=TEMP1(I)+TA(J,I)*TMAT(J)
 3200         CONTINUE
 3100       CONTINUE
            DO 3300 I=1,6
              DO 3400 J=1,6
                TEMP2(I)=TEMP2(I)+TB(J,I)*TEMP1(J)
 3400         CONTINUE
 3300       CONTINUE
            DO 3500 I=1,6
               TMAT(I)=0.D0
 3500       CONTINUE
            DO 3600 I=1,6
              DO 3700 J=1,6
                TMAT(I)=TMAT(I)+TC(J,I)*TEMP2(J)
 3700         CONTINUE
 3600       CONTINUE
```

The outcome of these statements being `[TC]ᵀ*[TB]ᵀ*[TA]ᵀ*[TMAT]` which is the overall transformation matrix from $x'''y'''z'''$ to the original xyz coordinate system. Here it is worth noticing that the angles of rotations will be in reverse order.

5. RENEW (KMAX, GR, MM, PSIG (PLNO),MEPSO,KMAXO,GRO):

The magnitude of effective stress has a significant role in establishing stress-strain relationships of sand. Thus, this subroutine determines the current values of spring stiffness and reference shear strain. The Fortran expression used in the source code to calculate these values is:

```
KMAX=KMAXO*(MEPS/MEPSO)**(DABS(MM)-1.D0)
        IF(MM.GT.0.D0) THEN
          GR=GRO*(MEPS/MEPSO)**(1.D0-MM)
        ELSE IF(MM.LE.0.D0) THEN
          GR=GRO*(MEPS/MEPSO)**(-1.D0*DABS(MM))
        END IF
```

If the value of `MM` is defined to be 0.5 as an input, this expression will be in good agreement with the equations given in the model [eqs.(1.99) and (1.100)]. However, the expression in the source code will be applicable for any values of `MM` as the rate of stress dependency may vary for various soil types.

6. ZEROM (STIFF):

In the liquefaction analysis, the shear stiffness matrix is often involved at different stages of computation. This matrix will always have the same pattern for isotropic elastic materials as shown eq.(1.55). Due to numerical instabilities, some of the elements in the matrix which should be zero may attain non-zero values. Pre-establishment of the matrix pattern in a subroutine will ensure that the off-diagonal deviatoric elements will only have a zero value. The objective of this subroutine will be defining the pattern of the stiffness matrix.

In the subroutine, the statements to form this are:

```
      DO 100 I=1,3
              DO 200 J=1,3
                MAT(I,J+3)=0.D0
                MAT(I+3,J)=0.D0
  200   CONTINUE
  100 CONTINUE
          MAT(4,5)=0.D0 ; MAT(4,6)=0.D0
          MAT(5,4)=0.D0 ; MAT(5,6)=0.D0
          MAT(6,4)=0.D0 ; MAT(6,5)=0.D0
```

7. EQUIV (KEQU,KMAX,COR,YETA,GAMP,GR,MEAN,MMODE):

Sand exhibits elastic response when subjected to small unloading or reloading and an equivalent elastic stiffness $G_{eq}^{(i)}$ was defined by stiffness in this elastic region as given in eq. (1.132). In this model, the elastic shear strain is calculated based on this stiffness.

Two cases should be defined to calculate the elastic shear stiffness i.e. when $\eta = 1$ and when $\eta \neq 1$. When $\eta = 1$, $G_{eq}^{(i)}$ is simply equal to $p'k_{max}$ otherwise the expression given in eq. (1.132) will be stipulated. In the subroutine, this case happens when `ECASE = 1`. The Fortran statements for this task are:

```
IF(ECASE.EQ.1) THEN
  KEQU=MEAN*KMAX
ELSE
  KEQU=MEAN*KMAX*COR*((1.D0-YETA)/(1.D0+GAMP/GR)+YETA)
END IF
```

8. CORRECT (GAMP, COR, GREV, GTMAX, RTMAX, RREV, GR, KMAX, 1):

The correction factor C and the shear strain amplitude γ_a are important quantities in the formation of the hysteretic loops. This subroutine aims in computing these values. Three cases (`CCASE`) will be defined in the subroutine to write the equations.

`CCASE = 1`, formulation for regular loading in which C = 1 and $\gamma_a = |\gamma_{max}|$

`CCASE = 2`, formulation for general loading and when p' is constant in which C and γ_a

will be given by eqs. (1.116) and (1.114) respectively.

`CCASE = 3`, formulation for general loading and when *p'* is constant in which C and γ_a will be given by eqs. (1.116) and (1.114) respectively.

The Fortran statements which are written in the subroutine for this task are:

```
IF(CCASE.EQ.1) THEN
        GAMP=DABS(GMAX)
        COR=1.D0
      ELSE IF(CCASE.EQ.2.OR.CCASE.EQ.3) THEN
        GAMP=0.5D0*DABS(GMAX-GREV)
        COR=DABS(RMAX-RREV)*(1.D0+GAMP/GR)/(2.D0*KMAX*GAMP)
      END IF
```

9. EXTHD (R, KTAN, KMAX, GAM(2), GR, MEAN, JUDGE, GREV, RREV, COR, GAMP, YETA):

Calculation of the stress ratio R and the tangent shear stiffness of each spring $G_{\tan}^{(i)}$ is mandatory for each hysteretic loop . This subroutine calculates these values. Three cases (`MCASE`) will be defined.

`MCASE = 1-` calculates R and $G_{\tan}^{(i)}$ for skeleton curve as per eqs. (1.68) and (1.129).

`MCASE = 2-` calculates R and $G_{\tan}^{(i)}$ for unloading hysteretic loops as per eqs. (1.117) and (1.130).

`MCASE = 3-` calculates R and $G_{\tan}^{(i)}$ for re-loading hysteretic loops as per eqs. (1.117) and (1.130).

The fortran statements for these computations are:

```
IF(MCASE.EQ.1) THEN
        R=(KMAX*GAM/(1.D0+DABS(GAM)/GR))
        KTAN=KMAX*MEAN/(1.D0+DABS(GAM)/GR)**2.D0
      ELSE IF(MCASE.EQ.2.OR.MCASE.EQ.3) THEN
        R=RREV+COR*KMAX*(GAM-GREV)/(1.D0+GAMP/GR)+
     $    YETA*(COR*KMAX*(GAM-GREV)/
     $    (1.D0+DABS(GAM-GREV)/(2.D0*GR))-
     $    COR*KMAX*(GAM-GREV)/(1.D0+GAMP/GR))
        KTAN=KMAX*MEAN*COR*((1.D0-YETA)/(1.D0+GAMP/GR)+
     $       YETA/(1.D0+DABS(GAM-GREV)/(2.D0*GR))**2.D0)
END IF
```

These Fortran statements are in good agreement with the expression given in chapter one.

10. MASING (PSIG (PLNO), KMAX, GR, YETA, GORI (SPNO)) :

Establishing the shear-stress strain relationship is one of the core components of the whole model. This relationship in the model is formulated using Masing's rule along with a few details.

The following four stages of the curve of stress ratio strain relationship are important to notice. The shear strains in the previous iteration (γ_1) and current iteration (γ_2) are given by `GAM(1)` and `GAM(2)` respectively. Their difference `GAM(2) - GAM(1)` is represented by `DGAM`.

1. Loading from the origin along the skeleton (back-bone curve) depicted by the curve a-b in the Fig. (1.18). If γ_1 and γ_2 are shear strains at the beginning and at the end of a curve, the condition for this case to occur is $(\gamma_2 - \gamma_1) * \gamma_1 \geq 0$. In the subroutine, this case is identified by `JUDGE = 1`.

```
IF(DGAM*(GAM(1)).GE.0.D0) THEN
JUDGE = 1
```

2. Unloading curves are depicted by the curves b-c and d-e in Fig. (1.18). It is important to notice that the unloading curve always connects the point of recent stress reversal to the point of maximum stress ratio. The condition for this case to occur is $(\gamma_2 - \gamma_1) < 0$. In the subroutine, this case is assigned as: `JUDGE = 2.`

```
ELSE IF(DGAM.LT.0.D0) THEN
          JUDGE = 2
```

3. Re-loading curve delineated by the curve c-d in Fig. (1.18). This curve also connects the recent reversal point with the point of maximum stress ratio. The condition for this case to occur is $(\gamma_2 - \gamma_1) \geq 0$. This case is identified in the subroutine when `JUDGE = 3`

```
ELSE IF(DGAM.GE.0.D0) THEN
          JUDGE = 3
```

4. The remaining case which should be enumerated here is the case when the calculated stress ratio exceeds the maximum stress ratio ever experienced in the soil. In such cases, the hysteresis loop follows again the skeleton curve. This condition occurs whenever $R > R_{max}$.

The steps followed in order to construct the hysteresis loops are as follows:

1. The value of `MMODE` will be read from the previous step. `MMODE` is an integer which attains values from 1 to 4. Its value indicates where the previous iteration step ends on the hysteresis loop. This is done by assigning different values for `MMODE`. `MMODE = 1`, `MMODE = 2` and `MMODE = 3` will respectively indicate that the previous iteration step ended on the skeleton, unloading and reloading part of the curve .The process starts from `MMODE = 1.`
2. For each value of `MMODE` , the value of `JUDGE` will be decided depending on the values of γ_1 and γ_2,. This helps to decide the type of curve as explained above. For all type of curves, γ_1 and R values from previous calculation will be set to γ_{rev} and R_{rev} values. For initial loading, these values will have a value of zero.
3. The stress ratio R and the tangent shear stiffness matrix will be calculated using the subroutine `EXTHD`; from which the new value shear strain γ_2 will be calculated.

4. The values of R_{max} and γ_{max} will be set from the previous calculations. For the skeleton curve, R_{max} and γ_{max} will respectively be the value of R and γ_2 calculated at the current step. For the hysteresis loops R_{max} and γ_{max} are taken to be the corresponding maximum values ever experienced.
5. The value of MMODE for next iteration will be assigned so as to indicate the location of the current point where the computation ended up for the current iteration.

When the maximum stress experienced in the soil is exceeded, the transfer from a hysteresis loop to a skeleton curve considered to occur so that the calculated stress ratio never exceeds the limit determined by the skeleton curve. This adjustment is achieved by shifting the origin of the skeleton curve. When the stress ratio exceeds the maximum value in the past,.

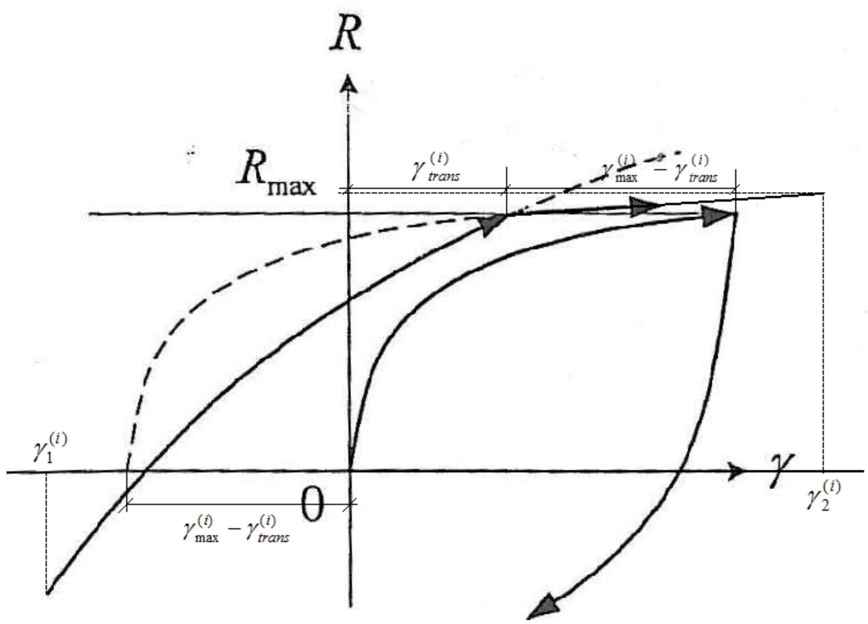


Fig. 2.2. Adjustment of Masing's curve when the stress ratio is exceeded

The point at which the transfer occurs is the point at which the current loop crosses the value of R_{max}. As shown in fig. , the curve will lead to new value of R_{max}. However the maximum stress ratio limit should be determined by the skeleton curve equation given by eq. (1.68). Hence, the position of the skeleton curve should be adjusted by shifting its origin so that it passes through the transfer point.

Since the current curve is on the hysteresis loop, the distance from the origin to the transfer point γ_{trans} will be determined from eq. (1.117) by equating R to R_{max} and determining the value of γ. The procedure of determination of the value of $\gamma_{trans.}$

$$R_{\max}^{(i)} = R_{rev}^{(i)} + \frac{C^{(i)}k_{\max}(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{|\gamma_a^{(i)}|}{\gamma_r}} + \eta\left\{\frac{C^{(i)}k_{\max}(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{|\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)}|}{2\gamma_r}} - \frac{C^{(i)}k_{\max}(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{|\gamma_a^{(i)}|}{\gamma_r}}\right\} \quad (2.1)$$

$$R_{\max}^{(i)} - R_{rev}^{(i)} = \frac{(1-\eta)C^{(i)}k_{\max}(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{|\gamma_a^{(i)}|}{\gamma_r}} + \eta\frac{C^{(i)}k_{\max}(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{|\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)}|}{2\gamma_r}} \quad (2.2)$$

A. For reloading, $\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)} > 0$ and $|\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)}| = \gamma_{trans}^{(i)} - \gamma_{rev}^{(i)}$. In addition, $|\gamma_a^{(i)}| = \gamma_a^{(i)}$

$$\frac{R_{\max}^{(i)} - R_{rev}^{(i)}}{C^{(i)}k_{\max}} = \frac{(1-\eta)(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{\gamma_a^{(i)}}{\gamma_r}} + \frac{\eta(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)}}{2\gamma_r}} \quad (2.3)$$

Solving this equation, the value of $\gamma_{trans}^{(i)}$ will be given as:

$$\gamma_{trans}^{(i)} = \gamma_{rev}^{(i)} + \frac{\gamma_r + \gamma_a^{(i)}}{2(1-\eta)}\left(\frac{R_{\max}^{(i)} - R_{rev}^{(i)}}{C^{(i)}k_{\max}\gamma_r} - \frac{2(\gamma_r + \eta\gamma_a^{(i)})}{\gamma_r + \gamma_a^{(i)}} + \sqrt{\left(\frac{2(\gamma_r + \eta\gamma_a^{(i)})}{\gamma_r + \gamma_a^{(i)}} - \frac{R_{\max}^{(i)} - R_{rev}^{(i)}}{C^{(i)}k_{\max}\gamma_r}\right)^2 + 8\frac{(R_{\max}^{(i)} - R_{rev}^{(i)})(1-\eta)}{C^{(i)}k_{\max}(\gamma_r + \gamma_a^{(i)})}}\right) \quad (2.4)$$

B. For unloading, $\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)} < 0$ and $|\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)}| = -(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})$ .In addition, $|\gamma_a^{(i)}| = \gamma_a^{(i)}$

$$\frac{R_{\max}^{(i)} - R_{rev}^{(i)}}{C^{(i)}k_{\max}} = \frac{(1-\eta)(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 + \frac{\gamma_a^{(i)}}{\gamma_r}} + \frac{\eta(\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)})}{1 - \frac{\gamma_{trans}^{(i)} - \gamma_{rev}^{(i)}}{2\gamma_r}} \quad (2.5)$$

Solving this equation, the value of $\gamma_{trans}^{(i)}$ will be given as:

$$\gamma_{trans}^{(i)} = \gamma_{rev}^{(i)} + \frac{\gamma_r + \gamma_a^{(i)}}{2(1-\eta)}\left(\frac{R_{\max}^{(i)} - R_{rev}^{(i)}}{C^{(i)}k_{\max}\gamma_r} + \frac{2(\gamma_r + \eta\gamma_a^{(i)})}{\gamma_r + \gamma_a^{(i)}} + \sqrt{\left(\frac{2(\gamma_r + \eta\gamma_a^{(i)})}{\gamma_r + \gamma_a^{(i)}} + \frac{R_{\max}^{(i)} - R_{rev}^{(i)}}{C^{(i)}k_{\max}\gamma_r}\right)^2 - 8\frac{(R_{\max}^{(i)} - R_{rev}^{(i)})(1-\eta)}{C^{(i)}k_{\max}(\gamma_r + \gamma_a^{(i)})}}\right) \quad (2.6)$$

The values given by eqs. (2.4) and (2.6) are the distance between the origin and the transfer point. Thus, the origin of the skeleton curve should be shifted back by a distance of $\gamma_{\max}^{(i)} - \gamma_{trans}^{(i)}$. Then the skeleton curve will be continued. Here it has to be noted that the new calculated shear strain and stress ratio will respectively be the γ_{max} and R_{max}.

These values of the scaling factor $C^{(i)}$ and the shear strain amplitude $\gamma_a^{(i)}$ in eqs. (2.4) and (2.6) are determined by eqs. (1.114) and (1.116) by replacing R_{max} by the new value of R and γ_{max} by the new value of shear strain γ. However, the new value of R is not yet determined. Hence it is better to devide the case into two parts: part of the curve till the stress ratio reaches the previous Rmax value and the remaining part.

11.DILATANCY(IDEVD,DPEPS,DMODE(SPNO),G(SPNO,1),G(SPNO,2), R(SPNO,1), R(SPNO,2), NDI, NDS, KEQU , GTH, RPT, LTDPEP, GREV(SPNO) , RREV(SPNO) , HPEV, PSIG(PLNO))) :

In contrast to the 2D Towhata-Iai model which suits only for undrained conditions, this model utilizes stress-dilatancy relations to deal with drained analysis as well. This subroutine calculates the contribution of volumetric strain due to dilatancy.

The following sequence of operations will be performed in this subroutine to calculate the volumetric strain due to dilatancy.

1. Calculate the incremental plastic shear strain `DPEPS` and the plastic shear strain increment measured from the last reversal point `PEPS` according to eq. (1.134) and the current value of stress ratio `RHPEV` as:

```
DPEPS=G2-G1-(R2-R1)*MEAN/KEQU
PEPS =G2-GREV-(R2-RREV)*MEAN/KEQU
RHPEV=HPEV*R2
```

2. Choose between "initial" and "subsequent" stress dilatancy relationship by comparing the values of `PEPS` and `GTH`.

If the value of `PEPS` is between `GTH` and –`GTH`, then the "initial" stress dilatancy relationship will be considered and the initial value of $R_{pt}^{(i)}$, `RPT(1)`, will be used in the stress dilatancy relationship given in eq. (136) and (137). Otherwise, `RPT(2)` will be used to calculate the volumetric strain increment due to dilatancy `IDEVD`.

3. Depending on the value of `DMODE` from the previous step, set the value of `DMODE` for the current step. This value of `DMODE` helps to make a selection whether the strain level is:

- on the initial loading curve or on the subsequent loading curve
- the loading is in the positive direction or on the negative direction.

Thus there are four combinations which will be assigned with four `DMODE` values.

`DMODE = 0` for initial loading in both negative and positive direction.
`DMODE = 1` for initial loading in the positive direction or subsequent loading in the negative direction.
`DMODE = 2` for initial loading in the negative direction or subsequent loading in the positive direction.
`DMODE = 3` for subsequent loading in both negative and positive direction.

4. Choose which value of $R_{pt}^{(i)}$ should be used to calculate `IDEVD` from the value of `DMODE` of the current step.

```
DMODE = 0 - RPTI(1)=RPT(1);RPTI(2)=RPT(1)
DMODE = 1- RPTI(1)=RPT(1);RPTI(2)=RPT(2)
DMODE = 2 - RPTI(1)=RPT(2);RPTI(2)=RPT(1)
DMODE = 3 - RPTI(1)=RPT(2);RPTI(2)=RPT(2)
```

5. Set new values of `DMODE` depending on the direction of loading.

For loading in the positive direction:

```
IF(G2.GE.G1) THEN
                DMODE=DMODE+10
```

For loading in the negative direction:

```
ELSE
                DMODE=DMODE+20
```

This new value of `DMODE` will be assigned to (`USRIND((SPNO-1)*2+2)` value to be used as an input `DMODE` value for the next step.

6. Depending on the new value of `DMODE`, calculate the incremental strain due to dilatancy `IDEVD`.

For initial loading:

```
IF(DMODE.LT.20) THEN
            IDEVD=(RPTI(1)-RHPEV)*DPEPS/NDN
```

For subsequent loading:

```
ELSE IF(DMODE.GE.20) THEN
            IDEVD=(-RPTI(2)-RHPEV)*DPEPS/NDN
```

In any case, if the absolute value of `IDEVD` value should not be less than the minimum allowable value `LTDPEP`.

This sequence of operations will be performed for all the springs at each increment of strains.

12. SUBSIG(MEPS, MEPSO, BULKSO, BULKO, EVCSUM, EVY, SIGY, SIGI, DEVC, BULK, NN, LTRATI):

This subroutine computes the mean effective principal stress by using eq. (1.167) and the updated bulk modulus using eq. (1.161). Two conditions must be considered for the normally consolidated part and for the over consolidated part as shown in Fig. (1.17). however in this program an over-consolidation ratio value of 1 is assumed. Hence the initial stress will be equal to the preconsolidation stress. In this case, the bulk modulus of compression can be used for the loading part and the bulk modulus of swelling can be used for unloading part.

When $p' < p_y'$, the bulk modulus for compression $B_{c,o}$ will be used and the coordinate of the yield point $(p_y', \varepsilon_{v,c}^y)$ will be taken from `USRSTA (3)` and `USRSTA(4)` values. Hence, the Fortran statements for this case are:

```
              SIG=SIGY*DEXP(KKS0/SIG0*(EV-EVY))
              KK=KKS0*(SIG/SIG0)**(NN)
                IF((SIG.GT.SIGY) .AND.
         $         (SIGI*DEXP(KK0/SIG0*(EV-0.D0)).GT.SIGY)) THEN
                  SIG=SIGI*DEXP(KK0/SIG0*(EV-0.D0))
                  KK=KK0*(SIG/SIG0)**(NN)
                  SIGY=SIG
                  EVY=EV
                END IF
```

When $p' > p_y'$ and the origin of the curve given in Fig. 1.17 will be considered as a yield point. Hence, the Fortran statements for this case are:

```
              SIG=SIGI*DEXP(KK0/SIG0*(EV-0.D0))
              KK=KK0*(SIG/SIG0)**(NN)
              SIGY=SIG
              EVY=EV
```

In both conditions, the stress and strains after the computation are set to represent the yield coordinates $(p_y', \varepsilon_{v,c}^y)$ for the next step.

2.2.4. Review of the main source code

The main source code is programmed for strain controlled analysis. For convenience, each line of this main source code is numbered to help the explanations of the source code in the subsequent parts with the same sequential order as it is written in the program.

2.2.4.1. Definition of variables and parameters

The first major task in the main subroutine is definition of the quantities used in the program. The quantities are defined according to their type. The parameters, variables or arrays defined can be integers or double precision reals. The definition runs from line 6 to line 49 of the main subroutine.

Line 51 of the source code sets the value of π according to the formula:

```
PAI=DATAN(1.D0)*4.D0
```

$$\pi = 4\arctan(1.0)$$

2.2.4.2. User defined parameters

There are thirteen values describing the properties of the soil for which 3D liquefaction analysis is to be executed to be defined by the user. These values are collected in the `USERVAL (NUSRVL)` array. `NUSRVL` representing the number of user defined parameters which is thirteen. The following table outlines these parameters. Table 1.3 or section 2.2.1 can be referred for the description of the parameters.

Table 2.2. User defined parameters

Symbol in the source code	Position in the `USRVAL` array	Conventional symbol
KMAXO	USRVAL (1)	$k_{max,o}$
GRO	USRVAL (2)	$\gamma_{r,o}$
YETA	USRVAL (3)	η
BULKO	USRVAL (4)	$B_{c,o}$
BULKSO	USRVAL (5)	$B_{s,o}$
MEPSO	USRVAL (6)	p'_o
NN	USRVAL (7)	n
MM	USRVAL (8)	m
RPT(1)	USRVAL (9)	$R_{pt,i}$
RPT(2)	USRVAL (10)	$R_{pt,s}$
NDN	USRVAL (11)	N_d
HP	USRVAL (12)	H_p
GTH	USRVAL (13)	γ_{th}

2.2.4.3. Computation of incremental strains

For this displacement controlled analysis, there will be a predefined strain vector at each integration point at the beginning of the calculation process. A method of incremental strains is adopted in the source code. The number of increments should be calculated first to calculate the value of the incremental strains at each step. The statements to calculate the number of increments `MAXSTP` in the source code are:

```
      NOWSTP=0
       MAXSTP=1
       DO   1000  "I=1,NSTR"
       IF(INT(DABS(DEPS(I))/LTDEPS).GE.MAXSTP)  THEN
            MAXSTP=INT(DABS(DEPS(I))/LTDEPS)
```

```
          END   IF
          1000  CONTINUE
     CC
          IF(MAXSTP.GE.LTSTP)     THEN
          MAXSTP=LTSTP
          END   IF
```

The above statements dictate that the number of strain increments for each component of strain vector is calculated by dividing that strain component by `LTSTP` = 10^{-4}. However, whenever these values are smaller than the minimum number of strain increments (= 1) or larger than the maximum number (=100), these values will be stipulated for `MAXSTP`.

The strain increment for each strain vector component `XDEPS(I)` will then be computed by dividing the total strain increment applied `DEPS(I)` by the number of strain increments.

```
     DO   1100  "I=1,NSTR"
     XDEPS(I)=DEPS(I)/MAXSTP
     1100 CONTINUE
```

Then the whole analysis process shown in Fig 2.1 will be carried out for each strain increment. The accumulated strain at the beginning of each step is calculated in the program as:

```
     DO   1200  "I=1,NSTR"
     XEPS0(I)=EPS0(I)+DEPS(I)*(NOWSTP-1)/MAXSTP
     1200 CONTINUE
```

2.2.4.4. Assigning initial values

To begin the analysis for each strain increment, some of the quantities are given initial values while the expression for others is given.

```
     DO   1300  "I=1,NSTR"
     DO   1400  "J=1,NSTR"
     "STIFF(I,J)=0.D0"
     1400 CONTINUE
     SIG(I)=0.D0
     SIGMAB(I)=SIGMA(I)
     1300 CONTINUE
     CC
     DEVD =     0.D0
     EVCSUM     =     0.D0
     DPWORK     =     0.D0
     MEPS =     -1.D0*(     SIGMA(1)    +
     SIGMA(2)   +     SIGMA(3)    )/3.D0
     EVSUM      =     -1.D0*(     XEPS0(1)    +
     XEPS0(2)   +     XEPS0(3)    )
     DEV  =     -1.D0*(     XDEPS(1)    +
     XDEPS(2)   +     XDEPS(3)    )
     HPEV =(EVSUM+DEV)*HP+1.D0
```

It can be observed that, the stiffness matrix, stress vector, volumetric strain due to dilatancy and volumetric strain due to consolidation are nullified at the beginning. The mean effective stress, the cumulative volumetric strain and the incremental volumetric strain are given by the following formulae.

`Line 102` : `MEPS` – mean effective stress as given by eq. (1.40).

`Line 103` : `EVSUM` – cumulative volumetric strain as given by eq. (1.37).

`Line 104` : `DEV` – incremental volumetric strain as given by eq. (1.37).

`Line 105` : `HPEV` – factor for considering hardening as given by eq. (1.126).

2.2.4.5. Initialization of the analysis process

Before proceeding into the time dependent analysis, there is initialization analysis which is executed for each spring. This stage of analysis runs as long as the following criterion is met.

```
108          IF(DTIME.EQ.0.D0  .AND. USRIND(385) .LE.  USRIND(386))
```

This loop runs from line 108 to line 170 of the source code. Hence whenever `DTIME` ≠ 0 or `USRIND(385)` value is greater than `USRIND(386)`, this loop of analysis is terminated to proceed to the next step. At the beginning of the program, there is a statement to add a unit value to `USRIND(385)` . After one step of initialization, the value of `USRIND(385)` will be higher than that of `USRIND(386)` and the requirement `USRIND(385).LE.USRIND(386)` will be violated after one step which allows to proceed to the main analysis process.

The computations which are made in this loop will be briefed line by line.

```
110                         BULKS=BULKSO*(MEPS/MEPSO)**NN
```

This expression calculates the current value of the bulk modulus according to eq. (1.161) with the value of `NN` to be equal to 1.

The next sequence of statements aim at computing the shear stresses, mean effective stresses and the overall tangent shear stiffness matrix for each spring on each plane. The subroutine `SPLOCA` gives the orientation of the normal lines to each plane. The spring number `SPNO` will be calculated according to eq. (1.61) in line 114. The angles `THETA` (θ) and `PHI` (ϕ) will be extracted from the subroutine `SPLOCA` in lines 115 and 116.

The angle `BETA` in line 117 gives the distribution of springs on each plane. Since, the springs are distributed evenly, only the half plane distribution of the six springs is required. The expression `BETA=PAI*(J-1)/NDIV+0.5D0*PAI/NDIV` states that the six springs are oriented with 30^0 interval between them and the first spring making an angle of 15^0 with the center line of the plane. The orientation of the springs is shown in the fig below.

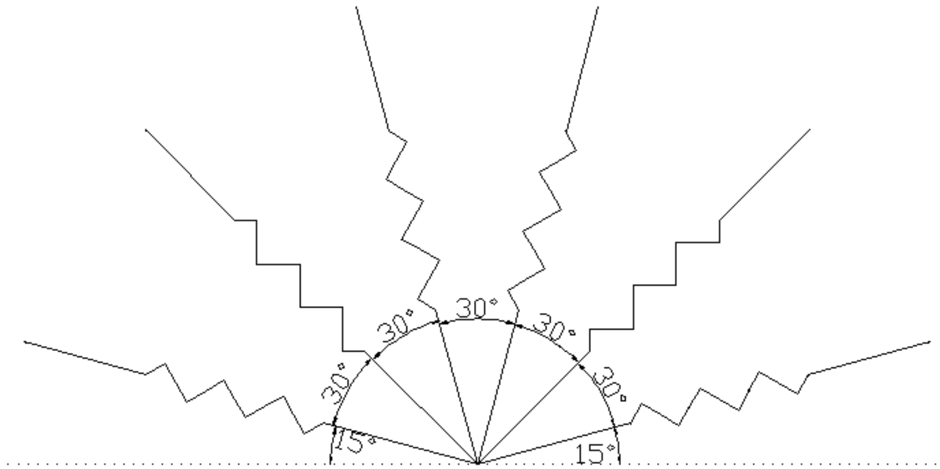


Fig 2.1. Orientation of springs on half plane

The next task will be determining the initial stresses components of the stress vector for each spring.

```
DO    2200  "K=1,NSTR"
ISIG(K)=-1.D0*SIGMA(K)
2200  CONTINUE
```

Since the mean effective stress for each plane is the same, the mean effective stress will be computed for each plane not for individual spring. The criteria to switch from springs on one plane to the next is `IF(MOD(SPNO-1,NDIV).EQ.0)`. Then the statement `CALL TRANSFER (3,THETA,PHI,BETA,ISIG)` computes the total initial stress on the soil. Then the mean effective stress of each plane is taken to be the third component of the total stress because the other normal stress components will be zero due to the coordinate transformation.

```
IF(MOD(SPNO-1,NDIV).EQ.0)    THEN
CALL TRANSFER(3, THETA,      PHI,BETA,   ISIG)
PSIG(PLNO)=ISIG(3)
END  IF
```

The minimum value of the initial effective stress should not be less than the minimum allowable value `LTPSIG.`

```
IF(PSIG(PLNO).LE.LTPSIG)      THEN
PSIG(PLNO)=LTPSIG
END   IF
```

The subsequent statements are aimed at constructing the overall tangent shear stiffness matrix for each spring. The subroutines `RENEW` and `SPMAT` will calculate the current k_{max} value and its product with $[T_\varepsilon]^T[N][T_\varepsilon]$ respectively. Then, the overall tangent shear stiffness matrix will be computed according to eq. (1.54).

```
DO  2300 L=1,NSTR
```

```
              DO  2400 M=1,NSTR
      STIFF(L,M)=STIFF(L,M)+KMAX*PSIG(PLNO)*ISTIFF(L,M)/SPRING
              2400    CONTINUE
              2300    CONTINUE
```

In which `ISTIFF` represents the matrix product $[T_\varepsilon]^T[N][T_\varepsilon]$. This shear stiffness matrix will have the pattern shown in eq. (1.55).

Then the last three diagonal elements of the overall tangent stiffness matrix is assigned to the array `GLAST (3).` For isotropy of the model, these values should be equal.

Since the matrices [G] and [B] have the formats as shown in eqs. (1.55) and (1.57) respectively and since the last three diagonal elements of [G] are stored in the array `GLAST(3)`, the remaining non-zero elements of both matrices will be the first three rows and columns of both. Hence, both matrices can be compressed into `3X3` matrix consisting of these non-zero elements. In the source code, these statements are written as:

```
                DO     2500  L=1,3
                  DO   2600  M=1,3
                             STIFF(L,M)=STIFF(L,M)+BULKS
                  2600 CONTINUE
                2500   CONTINUE
```

The next statements aim at determining the initial values of the status parameters and user indicators. The description of these parameters along with their initial values is given in the following table.

Table 2.3. initial values of the user status parameters and the user indicators.

`USRSTA/ USRIND` value	Represented quantity	Symbol of the quantity in the source code	Initial value
`USRSTA(1)`	ε_v^d	`EVDSUM`	0
`USRSTA(2)`	$\varepsilon_{v,y}^c$	`EVY`	0
`USRSTA(3)`	$p_y^{'}$	`SIGY`	p'_o
`USRSTA(4)`	p'_i	`SIGI`	p'_o
`USRSTA(5)`	`wp`	`PWORK`	0
`USRSTA(6)` ➔ `USRSTA(37)`	p'_1 ➔ p'_{32}	`PSIG(PLNO)`	p'_o
`USRSTA(38)`	`GLAST (1)`	`GLAST (1)`	`GLAST (1)`
`USRSTA(39)`	`GLAST (2)`	`GLAST (2)`	`GLAST (2)`
`USRSTA(40)`	`GLAST (3)`	`GLAST (3)`	`GLAST (3)`
`USRIND((SPNO-1)*2 + 1)`	`MMODE (SPNO)`	`MMODE (SPNO)`	1
`USRIND((SPNO-1)*2 + 2)`	`DMODE (SPNO)`	`DMODE (SPNO)`	0

USRSTA((SPNO-1)*9 + 41)	GORI (SPNO)	GORI (SPNO)	*0*
USRSTA((SPNO-1)*9 + 42)	γ_{max}	GMAX(SPNO)	*0*
USRSTA((SPNO-1)*9 + 43)	γ_{rev}	GREV (SPNO)	*0*
USRSTA((SPNO-1)*9 + 44)	γ_1	G (SPNO,1)	*0*
USRSTA((SPNO-1)*9 + 45)	R_{max}	RMAX (SPNO)	*0*
USRSTA((SPNO-1)*9 + 46)	R_{rev}	RREV (SPNO)	*0*
USRSTA((SPNO-1)*9 + 47)	R	R (SPNO,1)	*0*
USRSTA((SPNO-1)*9 + 48)	c	COR(SPNO)	*1*
USRSTA((SPNO-1)*9 + 49)	γ_a	GAMP(SPNO)	*0*

2.2.4.6. The main analysis process

When the condition for the initialization is violated, the main liquefaction analysis process follows. The analysis starts by setting the initial values of the status parameters and the user indicators.

```
EVDSUM     =     USRSTA(1)
EVY  =     USRSTA(2)
SIGY =     USRSTA(3)
SIGI =     USRSTA(4)
PWORK      =     USRSTA(5)
DO   5000  "PLNO=1,NPLANE"
PSIG(PLNO)=      USRSTA(5+PLNO)
5000 CONTINUE
GLAST(1)=USRSTA(38)
GLAST(2)=USRSTA(39)
GLAST(3)=USRSTA(40)
DO   5100  "SPNO=1,SPRING"
MMODE(SPNO)      =     USRIND((SPNO-1)*2+1)
DMODE(SPNO)      =     USRIND((SPNO-1)*2+2)
GORI(SPNO) =     USRSTA((SPNO-1)*WIDTH+OFFSET+1     )
GMAX(SPNO) =     USRSTA((SPNO-1)*WIDTH+OFFSET+2     )
GREV(SPNO) =     USRSTA((SPNO-1)*WIDTH+OFFSET+3     )
G(SPNO,1)  =     USRSTA((SPNO-1)*WIDTH+OFFSET+4     )
RMAX(SPNO) =     USRSTA((SPNO-1)*WIDTH+OFFSET+5     )
RREV(SPNO) =     USRSTA((SPNO-1)*WIDTH+OFFSET+6     )
R(SPNO,1)  =     USRSTA((SPNO-1)*WIDTH+OFFSET+7     )
 COR(SPNO) =     USRSTA((SPNO-1)*WIDTH+OFFSET+8     )
GAMP(SPNO) =     USRSTA((SPNO-1)*WIDTH+OFFSET+9     )
5100  CONTINUE
```

The main analysis process will be explained with the same numerical order as given in Fig 2.1.

1. Input value of strains

As explained in section 2.2.4.4, the strain increment vector `XDEPS(I)` will be given at the beginning of each step. From this the cumulative strain is computed as:

```
IEPS(K)=-1.D0*(XEPS0(K)+XDEPS(K))
```

2. shear strain of each spring

The cumulative strain increment at the beginning of the calculation step is decomposed into shear strains in each spring. Hence the transformation matrix will be called by the subroutine `TRANSFER.`

```
        CALL  TRANSFER(1,      THETA,PHI,  BETA, IEPS)
```

Then the fifth component of the decomposed strain represents the current one-dimensional shear strain of each spring. The one-dimensional shear strain in the previous step `G(SPNO,1)` was stored in `USRSTA ((SPNO-1)*9+44)` and the corresponding value at the current step will be extracted from the strain vector as:

```
     G(SPNO,2)=IEPS(5)
```

Here, `G(SPNO,2)` represents the value of shear strain at the current step γ_2.

3. Stress ratio of each spring

Calculation of the shear stress ratio of each spring will be the next step. For this, the effective stress on each plane should be calculated first. The stress vector of the current step is taken as.

```
              ISIG(K)=-1.D0*SIGMA(K)
```

This stress is decomposed for each sprig using the subroutine `TRANSFER`.

```
              CALL TRANSFER (3,THETA,PHI,  BETA, ISIG)
```

Note that each spring is aligned with the z'''-axis of the new coordinate system. Thus, the effective stress of each one-dimensional spring will then be the stress component in that direction.

```
     PSIG(PLNO)=ISIG(3)
```

The subroutine `RENEW` calculates the current values of spring stiffness and reference shear strain.

```
        CALL  RENEW(KMAX,GR,MM,PSIG(PLNO),MEPSO,KMAXO,GRO)
```

The previous value of the stress ratio was stored in `USRSTA ((SPNO-1)*9+47)` as `R(SPNO,1)`. This value will be used as `RREV` in the `MASING` subroutine to calculate the current value of the stress ratio.

```
            R(SPNO,2)=R(SPNO,1)
            CALL   MASING(PSIG(PLNO),KMAX,GR,YETA,GORI(SPNO),
            $ GMAX(SPNO),GREV(SPNO), G(SPNO,1),  G(SPNO,2),
            $ GAMP(SPNO),COR(SPNO),  RMAX(SPNO), RREV(SPNO),
            $ R(SPNO,2), KTAN(SPNO), KEQU, MMODE(SPNO))
```

4. Volumetric strain due to dilatancy

The stress-dilatancy relation is used to determine the volumetric strain due to dialatancy. The subroutine DILATANCY calculates the incremental value of this quantity for each spring.

```
            CALL DILATANCY(IDEVD,DPEPS,DMODE(SPNO), G(SPNO,1),
            $    G(SPNO,2), R(SPNO,1),  R(SPNO,2),  NDI,  NDS,
            $    KEQU,GTH,RPT,LTDPEP,GREV(SPNO),
            $    RREV(SPNO),HPEV,PSIG(PLNO)   )
```

The total volumetric strain increment due to dilatancy DEVD is calculated according to eq. (1.138) as:

```
                 DEVD=DEVD+IDEVD/SPRING
```

5. Volumetric strain due to consolidation

The volumetric strain due to consolidation is derived from the total volumetric strain and the volumetric strain due to dilatancy as per eq. (1.133).

```
                 DEVC =EVSUM      -     EVDSUM
                 EVDSUM = EVDSUM +      DEVD
                 EVCSUM =(EVSUM+DEV)- EVDSUM
                 DEVC = EVCSUM - DEVC
```

If the value of NN is different from one, then the volumetric strain due to consolidation can be given by:

```
           IF(DABS(NN-1.D0).GT.LTRATI)  THEN
       LTEVCS=(MEPSO**NN)/BULKSO*1.D0/(1.D0-NN)*(LTMEAN**(1.D0-
NN)-
         $       SIGI**(1.D0-NN))
         EVCSUM=MAX(EVCSUM,LTEVCS)
         END     IF
```

6. Mean effective stress

After calculating the strain due to consolidation, the conventional ε- log p' curve can be used to compute the effective stress according to eq. (1.131) for the next strain increment. The subroutine SUBSIG will be called to accomplish this task. The mean effective stress calculated at this strain increment, MEPS, will be added to the mean effective stress in the previous step to give the value of mean effective stress for the next step SIG.

```
                 CALL SUBSIG(MEPS,MEPSO,BULKSO,BULKO,
                 EVCSUM,EVY,SIGY, SIGI,
                 $    DEVC, BULK, NN,    LTRATI)
                 MEPS=MAX(MEPS,LTMEAN)

                 DO   6400  L=1,3
                 SIG(L)=SIG(L)+MEPS
                 6400 CONTINUE
```

7. Shear stress of each spring

The shear stress of each spring considering the hardening effect is given by the product of the new stress ratio calculated by `MASING` subroutine and the mean effective stress.

The one-dimensional shear stress of each spring which is expressed by eq. (1.47) is determined as:

```
ISIG(5)=R(SPNO,2)*PSIG(PLNO)*HPEV
ISIG(1)=0.D0
ISIG(2)=0.D0
ISIG(3)=0.D0
ISIG(4)=0.D0
ISIG(6)=0.D0
```

8. Total shear stress

The total shear stress in the soil is calculated according to eq. (1.48).

```
 CALL TRANSFER(2, THETA,PHI,BETA,ISIG)
DO   6300  L=1,NSTR
SIG(L)=SIG(L)+ISIG(L)/SPRING
6300 CONTINUE
```

9. Tangential shear stiffness matrix

The tangential shear stiffness matrix for each spring is established by the subroutine `EXTHD`. To consider the hardening effect, this stiffness will be multiplied by the factor to consider this hardening effect.

```
KTAN(SPNO)=KTAN(SPNO)*HPEV
```

When the value of *p'* is not constant the tangential shear stiffness matrix is given by eq. (1.131). In that case, the value of the parameter `GTYPE` will be 3. In eq. (1.131), the last two terms represent the tangential shear stiffness matrix computed so far, `KTAN`. The first term is calculated separately as `PTAN` and will be added to the value of `KTAN` afterwards.

`PTAN` can be obtained by dividing the change in effective stress multiplied by stress ratio by the change in shear strain between two consecutive steps. The effective stress at the current strain increment and at the previous strain increment are `ISIG(3)` and `PSIG(PLNO)` respectively. The corresponding change in shear strain is `G(SPNO,2)- G(SPNO,1)`. This change in shear strain should not be less than the minimum allowable value `LTDGAM`. In the program the statements for this computation are:

```
   IF(DABS(G(SPNO,2)-G(SPNO,1)).GE.LTDGAM)THEN
   PTAN(SPNO)=DPSIG(PLNO)*R(SPNO,2)*HPEV/
   $    (G(SPNO,2)-G(SPNO,1))"
   ELSE
   IF(G(SPNO,2)-G(SPNO,1).GE.0.D0)   THEN
   PTAN(SPNO)=DPSIG(PLNO)*R(SPNO,2)*HPEV/LTDGAM
   ELSE
PTAN(SPNO)=DPSIG(PLNO)*R(SPNO,2)*HPEV/(-1.D0*LTDGAM)
   END  IF
```

```
                END  IF
```

Then the tangential stiffness of each spring will then be the summation of `KTAN` and `PTAN`.

```
                          IF(PTAN(SPNO).GE.0.D0)  THEN
                          KTAN(SPNO)=KTAN(SPNO)+PTAN(SPNO)
                          END   IF
```

The total tangential shear stiffness of the soil is given by the formula in eq. (1.54) and in the program, this is given as:

```
                          DO    7300  "L=1,NSTR"
                          DO    7400  "M=1,NSTR"
             STIFF(L,M)=STIFF(L,M)+KTAN(SPNO)*ISTIFF(L,M)/SPRING
                          7400  CONTINUE
                          7300  CONTINUE
```

The last three diagonal elements of the tangential shear stiffness matrix represent the current shear modulus of the soil. These values which are stored in `GLAST` array are computed as the ratio between change in shear stress and change in volumetric strain. These values should be greater than or equal to the minimum allowable value `LTGTAN`.

```
                     DO   8200  I=4,6
                     IF(DABS(DEPS(I)).GE.LTDGAM)  THEN
                     GLAST(I-3)=(-1.D0*SIGMAB(I)-SIG(I))/DEPS(I)
                     ELSE
                     IF(DEPS(I).GE.LTDGAM)  THEN
                     GLAST(I-3)=(-1.D0*SIGMAB(I)-SIG(I))/LTDGAM
                     ELSE
                     GLAST(I-3)=(-1.D0*SIGMAB(I)-SIG(I))/(-1.D0*LTDGAM)
                     END  IF
                     END  IF
                     8200 CONTINUE
```

The change in volumetric strain should not be less than the minimum allowable value `LTDGAM`.

Any of the diagonal elements of the tangential shear stiffness matrix cannot be less than the value attained by the current shear modulus values which are store in the `GLAST` array.

10. Tangential stiffness matrix

The tangential stiffness matrix with dilatancy [K'] is given by eq. (1.149). The tangential stiffness matrix without dilatancy effect [K] is given by eqs. (1.60). In this calculation process the stiffness matrices which do not include the effects of dilatancy are used. The dilatancy is calculated independently along the shear deformation and then added to the results obtained by using [K].

The tangential stiffness matrix will be given as the summation between the tangential shear stiffness matrix and the compression (or swelling) stiffness matrix.

```
      DO   7500  L=1,3
      DO   7600  M=1,3
      STIFF(L,M)=STIFF(L,M)+BULK
7600  CONTINUE
7500  CONTINUE
```

The final pattern of the tangential stiffness matrix should have similar form as given by eq. (1.55). The subroutine ZEROM ensures this.

```
      CALL  ZEROM(STIFF)
```

The whole loop for a single strain increment will be tied by setting the new values of the status parameters , user indicators and stress vectors to be ready for computations for the next strain increment. The same string of procedures will then follow for the next strain increment till the strain increments add up to the strain applied.

CHAPTER THREE
INPUT DATA FOR ANALYSIS

3.1. Input data file

In the previous chapters, the multiple mechanism model and the source code are discussed. This chapter aims at briefing another component to execute a liquefaction analysis which is the input data file. The source code uses the input data file to read in the element and material properties and the initial values of some of the state parameters. This data file contains five main components and they will be briefly discussed in the next sections. A procedure to determine some of the material parameters will also be discussed.

3.1.1. Element properties

For the validation of the source code, an eight nodded single brick element shown in the fig. 3.1 below will be used. In DIANA, this element is named as `HX24L`. For this element, the strain ε_{xx} and stress σ_{xx} are constant in x direction and vary linearly in y and z direction. The strain ε_{yy} and stress σ_{yy} are constant in y direction and vary linearly in x and z direction. The strain ε_{zz} and stress σ_{zz} are constant in z direction and vary linearly in x and y direction. The coordinates of the nodes of the element will be given in the data file as depicted in the figure 3.1. In the data file this is given as:

```
'COORDINATES'
   1   0.000000E+00   0.000000E+00   0.000000E+00
   2   1.000000E+00   0.000000E+00   0.000000E+00
   3   0.000000E+00   1.000000E+00   0.000000E+00
   4   1.000000E+00   1.000000E+00   0.000000E+00
   5   0.000000E+00   0.000000E+00   1.000000E+00
   6   1.000000E+00   0.000000E+00   1.000000E+00
   7   0.000000E+00   1.000000E+00   1.000000E+00
   8   1.000000E+00   1.000000E+00   1.000000E+00
```

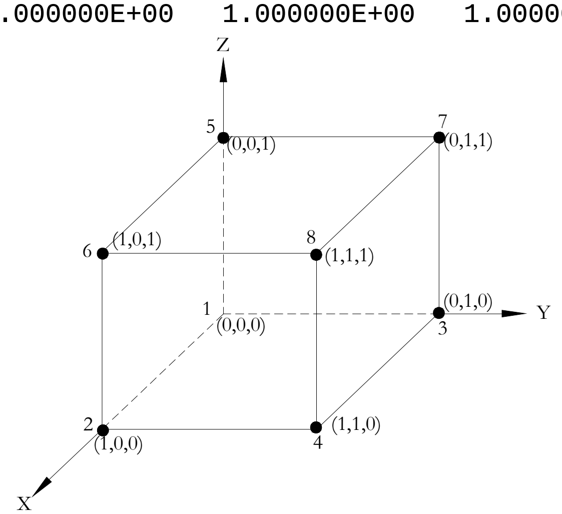


Figure 3.1. An eight nodded brick element

The connectivity of the element should also be given. For the solid element considered, the node numbering sequence in DIANA will be governed by the right hand rule in a counter clockwise direction. Hence the type and connectivity of element 1 is specified as:

```
CONNECT
   1 HX24L  1 2 4 3 5 6 8 7
```

The material properties of the element should also be assigned. Since there is one element property in this analysis, that property should be assigned to the element. This is done as:

```
 MATERI
 / 1 / 1
 DATA
 / 1 / 1
'DATA'
   1  NINTEG  1 1 1
      NOCSHE
      NUMINT GAUSS  GAUSS  GAUSS
```

3.1.2. Material properties

The properties of the soil for which the liquefaction analysis to be undergone will also be introduced. These data include the general properties of the soil such as the Young's modulus, Poisson's ratio and bulk modulus of water. The other data are the thirteen `USERVAL` values which are directly related to the liquefaction analysis. A user has to define these soil and spring parameters to carry-out the 3D liquefaction analysis using this model. The elaboration and determination of these parameters will follow in the next section.

i. $k_{max,o}$ [USRVAL (1)]

k_{max} is the non-dimensional stiffness of each spring. $k_{max,o}$ corresponds to the initial stiffness of each spring at the mean effective stress $p_o^{'}$. Its value can be determined using the shear modulus of the sand $G_{max,o}$ which is measured at $p = p_o^{'}$ by using eq. (1.99). The shear modulus of the sand can be determined from any standard test.

ii. $\gamma r,o$ [USRVAL (2)]

$\gamma_{r,o}$ is the reference shear strain at initial mean effective stress $p_o^{'}$. It is also a parameter associated with the springs. If the shear strength of the sand is determined as τ_{max}, the value of $\gamma_{r,o}$ is determined by using eqs. (1.100). The value of $k_{max,o}$ obtained above will be used in this equation.

iii. η [USRVAL (3)]

This is the factor which controls the damping ratio of sand. Its value can be determined either from eq. (1.125) or from fig. 1.20 for a given values of shear strain amplitude, reference shear strain and damping ratio. The values of these three soil parameters can be determined from simple cyclic shear or torsion tests.

iv. Bc,o [USRVAL (4)]

$B_{c,o}$ represent the bulk modulus of compression at reference mean effective stress p'_o. Its value is obtained by performing consolidation tests. From these tests, a graph of volumetric strain versus *p'* can be drawn. The slope of the resulting curve for the normally consolidated part at *p'* = p'_o will give the value of $B_{c,o}$.

v. Bs,o [USRVAL (5)]

$B_{c,o}$ represent the bulk modulus of swelling at reference mean effective stress p'_o. From consolidation tests, a graph of volumetric strain versus *p'* can be drawn. The slope of the resulting curve for the over consolidated part at *p'* = p'_o will give the value of $B_{s,o}$.

vi. p'_o [USRVAL (6)]

p'_o is the initial mean effective stress to be decided by the user. In most cases, a value is 100kPa used.

vii. NN [USRVAL (7)]

This parameter is the coefficient of stress dependency. In this model its value is set to 0.5.

viii. MM [USRVAL (8)]

This parameter is the coefficient of stress dependency. In this model, its value is equal to 1.0.

ix. Rpt,i [USRVAL (9)]

$R_{pt,i}$ is the stress ratio at phase transformation point in terms of each dilatancy mechanism for initial loading. Its value can be determined by drained cyclic simple shear tests.

x. Rpt,s [USRVAL (10)]

$R_{pt,s}$ is the stress ratio at phase transformation point in terms of each dilatancy mechanism for the subsequent loadings. Its value can be determined by drained cyclic simple shear tests.

xi. Nd [USRVAL (11)]

N_d is the gradient of stress-dilatancy relationship as shown in fig. 1.22. Its value can be determined by drained simple shear tests.

xii. Hp [USRVAL (12)]

H_p is a factor of hardening effects as shown in fig.1.21. Its value can be determined by drained simple shear tests.

xiii. γ_{th} [USRVAL (13)]

It is the threshold plastic strain introduced for the sake of numerical stability. In this model its value is given to be 0.0001.

Along with the soil and spring parameters discussed above, in this part, the initial values of the `USERSTA` and `USERIND` are given. These values will be used for the initialization stage of the liquefaction analysis. The initial value of `USERSTA` is for all the springs is zero. There will be 1768 (=(192-1)*9+40+9) `USERSTA` values. The initial `USERIND` values for all the springs are zero. But for the sake of computational suitability as discussed in section 2.2.4.5 two more values of `USERIND` are given. These extra values ,`USERIND(385)` and `USERIND(386)` are zero and one respectively.

3.1.3. Loading condition

The loading consists of two stages given by `CASE 1` and `CASE 2` in the data file. The first stage of loading given under `CASE 1` is the constant normal stress in the z-direction and the initial stresses at each node. The value of the constant normal stress is $98kN/m^2$ in the negative z-direction applied on the `ZETA2` face of the cube. `ZETA2` face is the face containing the nodes 5-6-8-7. In DIANA, z-direction is denoted by direction 3.

The second stage of loading given under `CASE 2` is the translational deformation of the upper face of the element. The nodes 5, 6, 7 and 8 will be deformed by 10^{-5} in x-direction.

3.1.4. Support condition

Nodes 1 through 8 are supported in x- and y-directions. Additionally, nodes 1 through 4 are supported in z-direction. A tying is also applied for nodes 5 -8 so that they will have the same deformation in the z-direction.

```
'SUPPORTS'
/ 1-8 / TR 1 2
/ 1-4 / TR 3
'TYINGS'
EQUAL TR 3
/ 6-8 / 5
```

3.1.5. Global directions

The global direction to define the directions of loadings and supports. It will be given in matrix of three columns. The first ,second and third columns representing the x-, y- and z-directions respectively. In this analysis, direction 1 described by a value of 1 in the first column i.e x-direction and zero in the others. Which indicates that direction 1 is oriented along the positive x global direction. Similarly, directions 2 and 3 are oriented in positive y and z global directions respectively.

```
'DIRECTIONS'
    1   1.00000E+00   0.00000E+00   0.00000E+00
    2   0.00000E+00   1.00000E+00   0.00000E+00
   3   0.00000E+00   0.00000E+00   1.00000E+00
```

3.2. Determination of the user defined parameters from laboratory tests

The parameters which the user has to define are discussed in section 3.1.2. These users defined parameters can be completely determined from three set of tests:

1. Cyclic simple shear (or torsion shear) test
2. Standard traixial test and
3. Consolidation(or Oedometer) test

Some of the data necessary for the calculations of the parameters are given in the appendix D and E. The determination of these parameters will be discussed briefly in the next sections.

i. $k_{max,o}$ [USRVAL (1)]

$k_{max,o}$ is calculated by eq. (1.99) for given values of shear modulus G and initial mean effective stress. In this case, the initial mean effective stress is assumed to be 98kPa. The shear modulus at this mean effective stress level is determined by the formula:

$$G_{\max,o} = \frac{\Delta\tau}{\Delta\gamma} = \frac{0.953kPa - 0kPa}{0.00002 - 0.00001} = 9.53E4kPa$$

Then substituting these values in eq. (1.99) the value of $k_{max,o}$ can be determined.

$$k_{\max,o} = \frac{5G_{\max,o}}{p'_o} = \frac{5*9.53E4kPa}{98kPa} = 4862$$

ii. $\gamma_{r,o}$ [USRVAL (2)]

The value of this parameter will be determined by eq. (1.100). In this equation, the parameters τ_{max}, $k_{max,o}$ and p'_o will be used as an input. τ_{max} is the shear strength of the sand obtained by standard triaxial test. From the result in appendix E, the values of cohesion C and friction angle ϕ are 0kPa and 30_0 respectively. Then the shear strength of the soil is calculated using Mohr-Coulomb equation for the reference normal stress as:

$$\tau_{\max} = c + \sigma \tan\phi = 0 + 98\tan 30 = 56.58kPa$$

Then the reference shear strain is calculated as:

$$\gamma_{r,o} \approx \frac{2.65 * 56.85kPa}{4862 * 98kPa} = 0.000315$$

iii. η [USRVAL (3)]

The reduction factor η is determined by using eq. (1.91) or fig. 1.11 for given values of damping ratio ,reference shear strain and shear strain amplitude. For a particular soil, these values can be obtained from simple shear test results.

The reference shear strain can be taken from calculation above. For the data given in appendix D, the damping ratio h is calculated by the equation:

$$h = \frac{1}{4\pi}\frac{\Delta W}{W}$$

In which, ΔW is the area inside the hysteresis loop and W is the elastic energy. As shown in fig. 3.2, the hysteresis loop for the data is not a closed loop for each cycle. Hence, it should be adjusted to be a closed loop. For the ease of adjustment each hysteresis loop will be subdivided into four regions.

Region 1 : R ≥ 0 , and dγ > 0
Region 2 : R > 0 , and dγ < 0
Region 3 : R < 0 , and dγ < 0
Region 4 : R < 0 , and dγ > 0

To form a closed loop, the last point of region 4 for each hysteresis loop should be connected to the first point of region 1. But this is not usually the case. Hence an adjustment will be made on the fourth region of the hysteresis loop so that these two points will connect to each other. After the adjustment is made for each loop, the damping ratio will be calculated from the total and elastic energies calculated for all the loops.

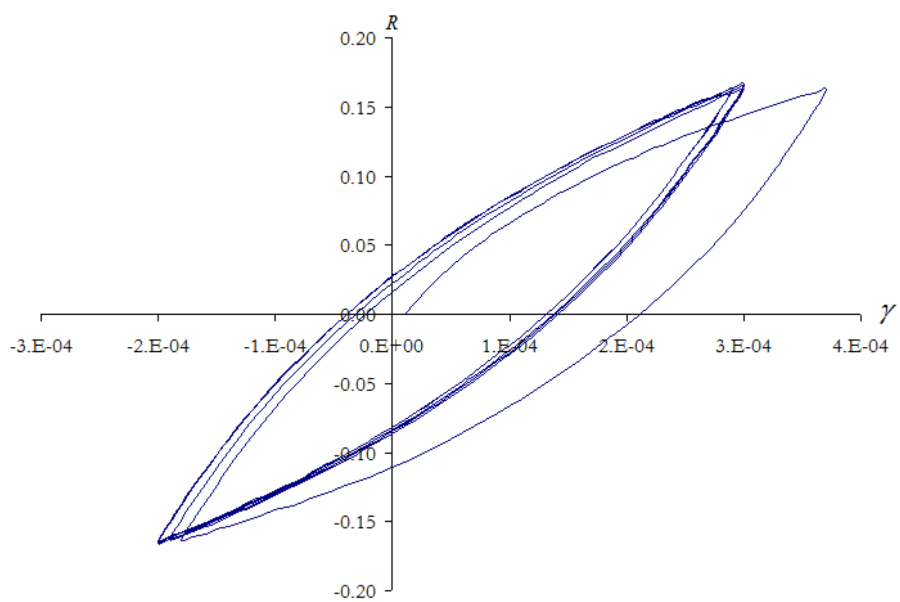


Fig. 3.2. Stress ratio versus shear strain graphs.

Here the adjustment will be done only for the first loop. First, the coordinates of the first point and the last point of the loop should be determined. The first point of the loop has coordinates of (γ, R) = (10^{-5}, 0). This loop ends somewhere between after 108th and 109th cycle. The exact value where R = 0 should be determined by interpolation. The coordinates of the hysteresis loop for the 108th cycle is (γ, R) = (-3X10^{-5}, -0.00609) and for the 109th cycle is (γ, R) = (-2X10^{-5}, 0.001545).

After interpolation, the coordinates of the last of point of the hysteresis loop is (γ, R) = (-2.202X10^{-5}, 0). Hence the value of γ deviates by 10^{-5} - -2.202X10^{-5} = 3.202X10^{-5} from the starting point. A proportion of this deviation will be applied to all points of the curve in the region 4. After application of the adjustment the curve looks like:

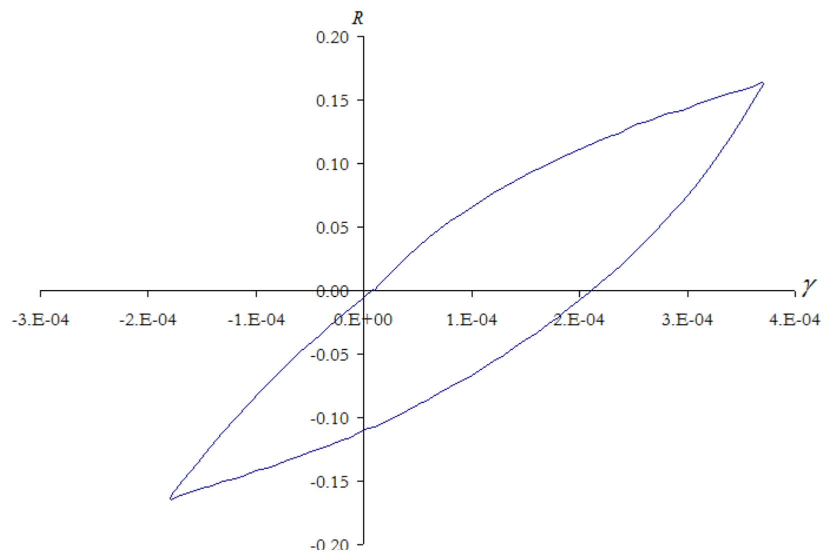


Fig. 3.3. A single hysteresis loop after adjustment

Now the area inside the hysteresis loop shown in fig3.3 which is the total strain energy for a single loop can be computed by any appropriate method. Using trapezoidal rule, this area inside the hysteresis loop it ΔW = 4.6023E-05. The value of the elastic energy W for this loop is part of the shaded area shown in fig 3.4. Its area is computed to be 7.77E-06.

.

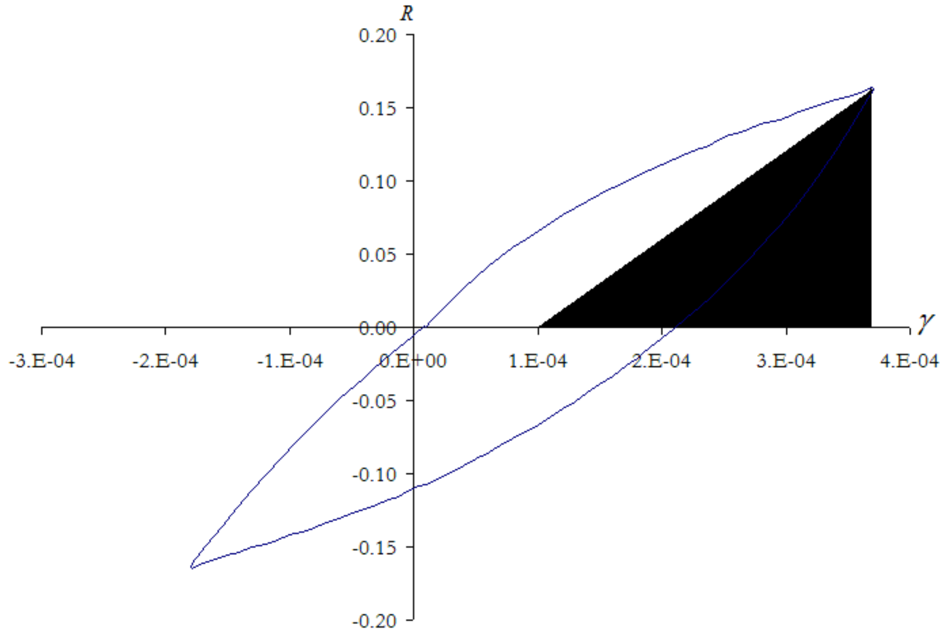


Fig 3.4. Elastic strain energy

Then the damping ratio is calculated to as:

$$h = \frac{1}{4\pi}\frac{\Delta W}{W} = \frac{1}{4\pi}\frac{4.6023E-5}{7.77E-6} = 0.47$$

The value of $\gamma_{r,o}$ is calculated to be 0.000315 and the value of $\gamma_a = \gamma_{max}$ is 0.00037. Substituting these values in eq. (1.125).

$$h = \frac{\Delta W}{4\pi W} = \frac{2\eta}{\pi}\left\{1 + \frac{2\gamma_r}{\gamma_a} - 2\left(\frac{\gamma_r}{\gamma_a}\right)^2\left(1 + \frac{\gamma_a}{\gamma_r}\right)\ln\left(1 + \frac{\gamma_a}{\gamma_r}\right)\right\}$$

$$0.47 = \frac{2\eta}{\pi}\left\{1 + \frac{2 * 0.000315}{0.00037} - 2\left(\frac{0.000315}{0.00037}\right)^2\left(1 + \frac{0.00037}{0.000315}\right)\ln\left(1 + \frac{0.00037}{0.000315}\right)\right\}$$

$$\Rightarrow \eta = 0.8$$

Hence the value of the reduction factor η is 0.8.

iv. $B_{c,o}$ [USRVAL (4)]

The bulk modulus of swelling $B_{c,o}$ is determined from consolidation curve. A data for consolidation curve is given in appendix D. The consolidation curve for this data can be drawn and shown below:

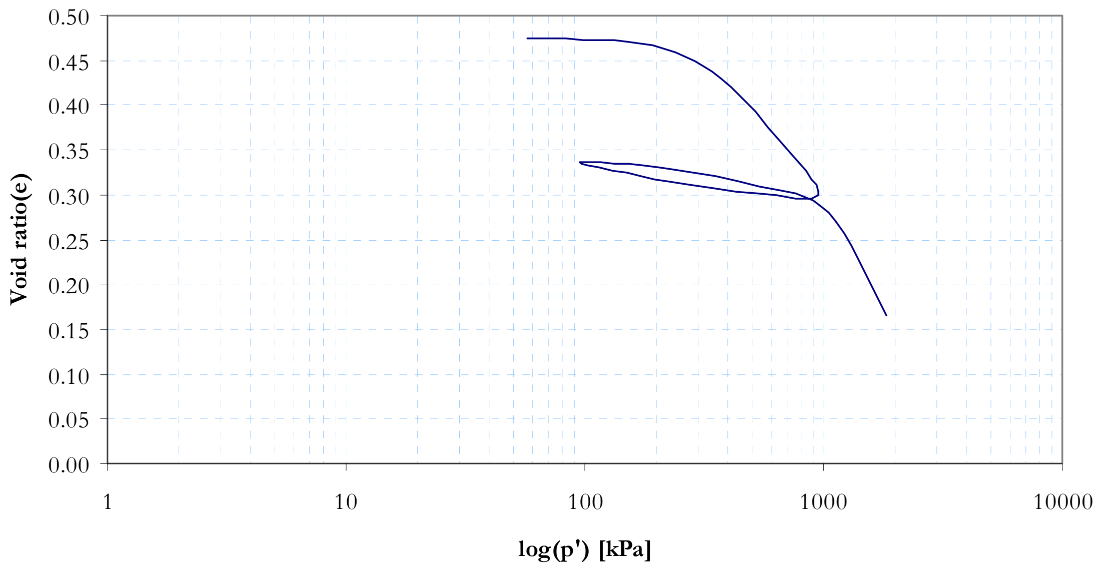


Fig. 3.5. Void ratio versus logarithm of effective stress curve.

An overconsolidation stress is the stress point at which the slop of the consolidation curve changes. From the graph, this value is determined to be around 250kPa. The parameter is $B_{c,o}$ the slope of this curve for the normally consolidated part i.e. $p' > p'_o$. This slope is determined to be 6.88E3. Hence the value of $B_{c,o}$ is 6.88E3.

v. $B_{s,o}$ [USRVAL (4)]

The bulk modulus of compression $B_{s,o}$ is also determined from consolidation curve. Its value is equal to the slope of this curve for the over consolidated part i.e. $p' < p'_o$. The consolidation curve above suggests that this slope is 1.79E4 which is the value of $B_{s,o}$.

vi. p'_o [USRVAL (6)]

The initial mean effective stress p'_o in this case is fixed to the value of to 98kPa.

vii. NN [USRVAL (7)]

This parameter is the coefficient of stress dependency will have a value of 0.5.

viii. MM [USRVAL (8)]

In the model the value of this parameter is equal to 1.0.

x. $R_{pt,i}$ [USRVAL (9)]

$R_{pt,i}$ is the stress ratio at phase transformation point in terms of each dilatancy mechanism for initial loading. Its value can be determined from stress-dilatancy diagram. $R_{pt,i}$ is the stress ratio when the value of dilatancy ratio is zero for initial loading.

xi. $R_{pt,s}$ [USRVAL (10)]

$R_{pt,s}$ is the stress ratio at phase transformation point in terms of each dilatancy mechanism for the subsequent loadings. Its value can be determined stress-dilatancy diagrams. $R_{pt,s}$ is the stress ratio when the value of dilatancy ratio is zero for subsequent loading.

xii. N_d [USRVAL (11)]

N_d is the gradient of stress-dilatancy relationship as shown in fig. 1.22. Its value can be determined by drained simple shear tests.

xxv. H_p [USRVAL (12)]

H_p is a factor of hardening effects as shown in fig.1.21 . Its value can be determined by drained simple shear tests.

xxvi. γ_{th} [USRVAL (13)]

It is the threshold plastic strain introduced for the sake of numerical stability. In this model its value is given to be 0.0001.

CHAPTER FOUR

ANALYTICAL VALIDATION OF THE MODEL

In this chapter, the consistency of the 3D liquefaction analysis using DIANA with analytical results will be checked. For this, a small prescribed elastic strain will be applied on a soil element and the resulting stresses will be calculated using the multiple spring model analytically. Later the results will be compared with results obtained DIANA under similar conditions. To have a similar model in both the analytical and DIANA analysis, there will be minor modifications in the main source code.

At the end of the chapter, the clarification of the relationship between stresses and strains in torsion shear test and DIANA results will be discussed. Since the DIANA analysis uses an icosahedral distribution, the values given in Table 1.2 will be recalled for comparison.

4.1. Shear applied in the xy direction

To apply the multiple shear mechanism method for analysis of a simple shear model shown in fig. 4.1 below, the flow of analysis given in Fig 2.1 will be used. The analysis procedure will be explained in detail below.

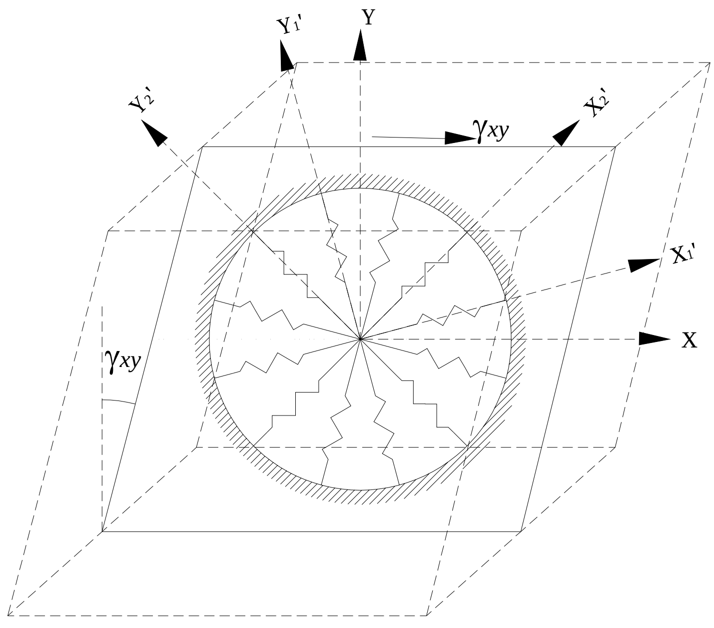


Fig 4.1. The simple shear multiple-spring model [shear strain applied in xy direction].

The material and spring parameters which will be used for the analysis in both the analytical and DIANA computations are indicated in table 5.1.

Table 5.1. Material and spring parameters

Symbol	Value

Dr [%]	22
$k_{\max,0}$	1728
$\gamma_{r,0}$	0.0008
η	0.4
$B_{\varepsilon,0}$ [kPa]	43200
$B_{s,0}$ [kPa]	54000
H_p	22
$R_{pt,i}$	1.3
$R_{pt,s}$	0.85
N_d	1.3
γ_{th}	0.0001
p'_0 [kPa]	100
n	1
m	0.5

For the verification of the results, shear strain applied in two directions will be considered; in the xy and zx directions.

4.1.1. Analytical calculation

1. A prescribed value of strain will be given.

The strain vector of the soil can be given by:

$$\{\varepsilon\} = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{yy} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}\}^T \tag{4.1}$$

Particularly, for simple shear case at constant volume shown in fig 4.1 the strain vector can be written as:

$$\{\varepsilon\} = \{0 \quad 0 \quad 0 \quad \gamma_{xy} \quad 0 \quad 0\}^T \tag{4.2}$$

Since the analysis will be displacement controlled one, the values of γ_{xy} should be given from the beginning. In this example, its value will be assumed to be 10^{-4}. Hence the strain vector for this particular analysis will be:

$$\{\varepsilon\} = \{0 \quad 0 \quad 0 \quad 10^{-3} \quad 0 \quad 0\}^T \tag{4.3}$$

2. The shear component of strain for each spring will be computed from the given strain using coordinate transformation.

The one-dimensional shear strain of each spring will be determined from the strain of the overall soil element. For this the transformation matrices for each spring should be determined. Referring to fig. 4.1, the transformation matrices between the global coordinate system xyz and the rotated coordinate system *x'y'z'* for each spring can be computed. The rotation is made only around z-axes. Thus, the transformation matrix for strain [T_ε] can be given by eq. (1.19).

For this model, the coordinates will be rotated around the origin so that the x'-axis of the new coordinate system will be aligned with the springs. There will be six springs in total. For the spring orientation given in fig. 4.1, the transformation matrices for each spring can be obtained by substituting the corresponding values of ζ in eq. (1.19). The values of ζ are as given in fig. 2.1. The springs will be numbered from 1 – 6 and their transformation matrices will be:

For spring no. 1, $\zeta = 15^o$

$$[T_\varepsilon^1] = \begin{bmatrix} 0.933013 & 0.066987 & 0 & 0.25 & 0 & 0 \\ 0.066987 & 0.933013 & 0 & -0.25 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0.866025 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.965926 & -0.258819 \\ 0 & 0 & 0 & 0 & 0.258819 & 0.965926 \end{bmatrix}$$

For spring no. 2, $\zeta = 45^o$

$$[T_\varepsilon^2] = \begin{bmatrix} 0.500000 & 0.500000 & 0 & 0.50 & 0 & 0 \\ 0.500000 & 0.500000 & 0 & -0.50 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1.0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.707107 & -0.707107 \\ 0 & 0 & 0 & 0 & 0.707107 & 0.707107 \end{bmatrix}$$

For spring no. 3, $\zeta = 75^o$

$$[T_\varepsilon^3] = \begin{bmatrix} 0.066987 & 0.933013 & 0 & 0.25 & 0 & 0 \\ 0.933013 & 0.066987 & 0 & -0.25 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & -0.866025 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.258819 & -0.965926 \\ 0 & 0 & 0 & 0 & 0.965926 & 0.258819 \end{bmatrix}$$

For spring 4, ζ= 105°

$$[T_\varepsilon^4] = \begin{bmatrix} 0.066987 & 0.933013 & 0 & -0.25 & 0 & 0 \\ 0.933013 & 0.066987 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & -0.866025 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.258819 & -0.965926 \\ 0 & 0 & 0 & 0 & 0.965926 & -0.258819 \end{bmatrix}$$

For spring no. 5, ζ= 135°

$$[T_\varepsilon^5] = \begin{bmatrix} 0.500000 & 0.500000 & 0 & -0.50 & 0 & 0 \\ 0.500000 & 0.500000 & 0 & 0.50 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1.0 & -1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.707107 & -0.707107 \\ 0 & 0 & 0 & 0 & 0.707107 & -0.707107 \end{bmatrix}$$

For spring no.6, ζ= 165°

$$[T_\varepsilon^6] = \begin{bmatrix} 0.933013 & 0.066987 & 0 & -0.25 & 0 & 0 \\ 0.066987 & 0.933013 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0.866025 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.965926 & -0.258819 \\ 0 & 0 & 0 & 0 & 0.258819 & -0.965926 \end{bmatrix}$$

Multiplication of the strain vector by the transformation matrix of each spring gives the strain vector of each spring. The strain vector of each spring will be obtained by multiplying the strain vector of the soil by the corresponding strain transformation matrix.

$$\{\varepsilon^{(i)}\} = [T_\varepsilon^{(i)}]\{\varepsilon\} = \{\varepsilon_{xx}^{(i)} \quad \varepsilon_{yy}^{(i)} \quad \varepsilon_{zz}^{(i)} \quad \gamma_{xy}^{(i)} \quad \gamma_{yz}^{(i)} \quad \gamma_{zx}^{(i)}\}^T \tag{4.6}$$

This will result in the following:

$$\{\varepsilon^{(1)}\} = [T_\varepsilon^1]\{\varepsilon\} = 10^{-3} * \{0.25 \quad -0.25 \quad 0 \quad 0.866 \quad 0 \quad 0\}^T$$
$$\{\varepsilon^{(2)}\} = [T_\varepsilon^2]\{\varepsilon\} = 10^{-3} * \{0.5 \quad -0.5 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\varepsilon^{(3)}\} = [T_\varepsilon^3]\{\varepsilon\} = 10^{-3} * \{0.25 \quad -0.25 \quad 0 \quad -0.866 \quad 0 \quad 0\}^T$$
$$\{\varepsilon^{(4)}\} = [T_\varepsilon^4]\{\varepsilon\} = 10^{-3} * \{-0.25 \quad -0.25 \quad 0 \quad -0.866 \quad 0 \quad 0\}^T$$
$$\{\varepsilon^{(5)}\} = [T_\varepsilon^5]\{\varepsilon\} = 10^{-3} * \{-0.5 \quad 0.5 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\varepsilon^{(6)}\} = [T_\varepsilon^6]\{\varepsilon\} = 10^{-3} * \{-0.25 \quad 0.25 \quad 0 \quad 0.866 \quad 0 \quad 0\}^T$$

The springs are assumed to carry only one dimensional shear strain. Hence, the shear components of the springs should be extracted from the total strain vector of the springs. The springs are oriented in a xy plane rotating along the z' axis. Thus, the strain component in the springs will be $\gamma_{x'y'}$. Hence, one-dimensional strains in each spring will be:

$$\{\gamma^{(1)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0.866 \quad 0 \quad 0\}^T$$
$$\{\gamma^{(2)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\gamma^{(3)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad -0.866 \quad 0 \quad 0\}^T$$
$$\{\gamma^{(4)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad -0.866 \quad 0 \quad 0\}^T$$
$$\{\gamma^{(5)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\gamma^{(6)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0.866 \quad 0 \quad 0\}^T$$

3. From Masing's rule, the stress ratio of each spring will be determined

The shear stiffness matrix of the soil element and that of each spring are related by eq. (1.56). Here it is assumed that the shear modulus of all the springs is the same.

$$[G] = \frac{G_{\tan}^{(i)}}{n} \sum_{i=1}^{n} \left([T_{\varepsilon}^{(i)}]^T [N][T_{\varepsilon}^{(i)}] \right) \tag{4.7}$$

For the skeleton curve, the value of the shear modulus of each spring can be determined from eq. (1.129).

$$G_{\tan}^{(i)} = \frac{k_{\max} p' \left(1 + H_p \varepsilon_v\right)}{\left(1 + \frac{|\gamma - \gamma_o|}{\gamma_r}\right)^2}$$

For this simple shear case, the volumetric strain is zero and the value of the initial strain γ_o is zero. Hence the tangent shear modulus of each spring can be computed as:

$$G_{\tan}^{(1)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(1)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4 kN/m^2$$

$$G_{\tan}^{(2)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(2)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0|}{0.0008}\right)^2} = k_{\max} p' = 1.7825E5 kN/m^2$$

$$G_{\tan}^{(3)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(3)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|-0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4kN/m^2$$

$$G_{\tan}^{(4)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(4)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|-0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4kN/m^2$$

$$G_{\tan}^{(5)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(5)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0|}{0.0008}\right)^2} = k_{\max} p' = 1.7825E5kN/m^2$$

$$G_{\tan}^{(6)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(6)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4kN/m^2$$

Substituting these values in eq. (4.7), the expression for shear stiffness matrix of the soil element can be obtained as:

$$[G] = \frac{G_{\tan}^{(i)}}{n} \sum_{i=1}^{n} \left([T_\varepsilon^{(i)}]^T [N][T_\varepsilon^{(i)}] \right) = \begin{bmatrix} 6.4241 & -6.4241 & 0.000 & -0.5751 & 0.000 & 0.000 \\ -6.4241 & 6.4241 & 0.000 & 0.5751 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.5751 & 0.5751 & 0.000 & 1.9922 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix} E4kN/m^2$$

Assuming there will be insignificant change in the mean effective stress, the bulk modulus of the soil element is equal to $B_{c,o}$ =4.32E4kPa. Thus the bulk stiffness matrix is given by:

$$[B] = \begin{bmatrix} 4.320 & 4.320 & 4.320 & 0.000 & 0.000 & 0.000 \\ 4.320 & 4.320 & 4.320 & 0.000 & 0.000 & 0.000 \\ 4.320 & 4.320 & 4.320 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix} E4kN/m^2 \qquad (4.8)$$

The tangential stiffness matrix which is the summation of the shear stiffness matrix and the bulk stiffness matrix is given by:

$$[K]=[G]+[B]=\begin{bmatrix} 1.0744 & -0.2104 & 0.4320 & -0.0575 & 0.000 & 0.000 \\ -0.2104 & 1.0744 & 0.4320 & 0.0575 & 0.000 & 0.000 \\ 0.4320 & 0.4320 & 0.4320 & 0.000 & 0.000 & 0.000 \\ -0.0575 & 0.0575 & 0.000 & 0.1992 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix} E5kN/m^2$$

Now the stress ratio of the each spring can be calculated. Here it is assumed that the relationship between the stress ratio and shear strain can be given by the back bone curve hence the following relationship can be used eq. (1.68) to calculate the stress ratio:

$$R^{(i)}=\frac{k_{\max}\gamma^{(i)}}{1+\frac{|\gamma^{(i)}|}{\gamma_r}} \tag{4.9}$$

The spring stiffness k_{max}, shear strain and the reference shear strain γ_r of each spring is given Table 4.1. If the change in mean effective stress is also assumed to be insignificant, then the reference shear strain can be taken to be equal to the initial reference shear strain. Hence, the stress ratio of the springs can be calculated as:

$$R^{(1)}=\frac{k_{\max}\gamma^{(1)}}{1+\frac{|\gamma^{(1)}|}{\gamma_r}}=\frac{1728*0.866x10^{-3}}{1+\frac{0.866x10^{-3}}{0.0008}}=0.7186$$

$$R^{(2)}=\frac{k_{\max}\gamma^{(2)}}{1+\frac{|\gamma^{(2)}|}{\gamma_r}}=\frac{1728*0}{1+\frac{0}{0.0008}}=0$$

$$R^{(3)}=\frac{k_{\max}\gamma^{(3)}}{1+\frac{|\gamma^{(3)}|}{\gamma_r}}=\frac{1728*-0.866x10^{-3}}{1+\frac{|-0.866x10^{-3}|}{0.0008}}=-0.7186$$

$$R^{(4)}=\frac{k_{\max}\gamma^{(4)}}{1+\frac{|\gamma^{(4)}|}{\gamma_r}}=\frac{1728*-0.866x10^{-3}}{1+\frac{|-0.866x10^{-3}|}{0.0008}}=-0.7186$$

$$R^{(5)}=\frac{k_{\max}\gamma^{(5)}}{1+\frac{|\gamma^{(5)}|}{\gamma_r}}=\frac{1728*0}{1+\frac{0}{0.0008}}=0$$

$$R^{(6)}=\frac{k_{\max}\gamma^{(6)}}{1+\frac{|\gamma^{(6)}|}{\gamma_r}}=\frac{1728*0.866x10^{-3}}{1+\frac{0.866x10^{-3}}{0.0008}}=0.7186$$

4. ***Using stress-dilatancy relationships, the volumetric strain increment due to dilatancy will be computed.***

The volumetric strain increment due to dilatancy for loading is calculated from the following equation given by eq. (1.136)

$$R'^{(i)} = \frac{\tau^{(i)}}{\sigma'^{(i)}} = N_d^{(i)}\left(-\frac{d\varepsilon_v^{d,(i)}}{d\gamma^{p,(i)}}\right) + R_{pt}^{(i)} \tag{4.10}$$

The values of $N_d^{(i)}$ and $R_{pt}^{(i)}$ are given to be to be 2.0 and 1.65. The plastic shear strain is calculated by eq. (1.139)

$$d\gamma^{p,(i)} = \left(1 - \frac{G_{\tan}^{(i)}}{G_{eq}^{(i)}}\right) d\gamma^{(i)} \tag{4.11}$$

$G_{eq}^{(i)}$ is calculated using eq. (1.101)

$$G_{eq}^{(i)} = p'\left(1 + H_p \varepsilon_v\right) C k_{\max}\left\{\frac{1-\eta}{1+\dfrac{\gamma_a}{\gamma_r}} + \eta\right\} \tag{4.12}$$

For each spring the values of η and Hp is given in table 4.1 to be 0.4 and 22 respectively. For the backbone curve, the value of C is 1.0. Hence, the expression for the elastic shear stiffness $G_{eq}^{(i)}$ can be simplified into:

$$G_{eq}^{(i)} = p' k_{\max}\left\{\frac{0.6}{1+\dfrac{\gamma_a}{\gamma_r}} + 0.4\right\} \tag{4.13}$$

Since the strain level is assumed to be on the skeleton curve, the amplitude of shear strain will be equal to the maximum shear strain.

$$\gamma_a^{(i)} = \left|\gamma_{\max}^{(i)}\right| \tag{4.14}$$

$$\gamma_a^{(1)} = \left|\gamma_{\max}^{(1)}\right| = 0.000866$$
$$\gamma_a^{(2)} = \left|\gamma_{\max}^{(1)}\right| = 0$$

$$\gamma_a^{(3)} = \left|\gamma_{\max}^{(3)}\right| = 0.000866$$
$$\gamma_a^{(4)} = \left|\gamma_{\max}^{(4)}\right| = 0.000866$$
$$\gamma_a^{(5)} = \left|\gamma_{\max}^{(5)}\right| = 0$$
$$\gamma_a^{(6)} = \left|\gamma_{\max}^{(6)}\right| = 0.000866$$

Then the value of the elastic shear stiffness and the corresponding ratio term in eq. (4.11) can be computed as:

$$G_{eq}^{(1)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(1)}}{G_{eq}^{(1)}}\right) = 0.6649$$

$$G_{eq}^{(2)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0}{0.0008}}+0.4\right\} = p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(2)}}{G_{eq}^{(2)}}\right) = 0$$

$$G_{eq}^{(3)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(3)}}{G_{eq}^{(3)}}\right) = 0.6649$$

$$G_{eq}^{(4)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(4)}}{G_{eq}^{(4)}}\right) = 0.6649$$

$$G_{eq}^{(5)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0}{0.0008}}+0.4\right\} = p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(5)}}{G_{eq}^{(5)}}\right) = 0$$

$$G_{eq}^{(6)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(6)}}{G_{eq}^{(6)}}\right) = 0.6649$$

Then from eq. (4.11) ,the plastic shear strain increments are calculated as:

$$d\gamma_p^{(1)} = \left(1-\frac{G_{\tan}^{(1)}}{G_{eq}^{(1)}}\right)\{n\}^T\left[T_\varepsilon^{(1)}\right]\{d\varepsilon\} = 0.0005758$$

$$d\gamma_p^{(2)} = \left(1 - \frac{G_{\tan}^{(2)}}{G_{eq}^{(2)}}\right)\{n\}^T \left[T_\varepsilon^{(2)}\right]\{d\varepsilon\} = 0$$

$$d\gamma_p^{(3)} = \left(1 - \frac{G_{\tan}^{(3)}}{G_{eq}^{(3)}}\right)\{n\}^T \left[T_\varepsilon^{(3)}\right]\{d\varepsilon\} = -0.0005758$$

$$d\gamma_p^{(4)} = \left(1 - \frac{G_{\tan}^{(4)}}{G_{eq}^{(4)}}\right)\{n\}^T \left[T_\varepsilon^{(4)}\right]\{d\varepsilon\} = -0.0005758$$

$$d\gamma_p^{(5)} = \left(1 - \frac{G_{\tan}^{(5)}}{G_{eq}^{(5)}}\right)\{n\}^T \left[T_\varepsilon^{(5)}\right]\{d\varepsilon\} = 0$$

$$d\gamma_p^{(6)} = \left(1 - \frac{G_{\tan}^{(6)}}{G_{eq}^{(6)}}\right)\{n\}^T \left[T_\varepsilon^{(6)}\right]\{d\varepsilon\} = 0.0005758$$

Now the volumetric strain due to dilatancy can be calculated using eq. (1.141).

$$d\varepsilon_v^{d,(i)} = -d\gamma^{p,(i)}\left(\frac{R'^{(i)} \pm R_{pt}^{(i)}}{N_d^{(i)}}\right)$$

$R_{pt}^{(i)}$ will be negative for $d\gamma^{p,(i)} > 0$ and positive for $d\gamma^{p,(i)} < 0$. Since the loading is initial loading the value of $R_{pt,i}^{(i)}$ =1.3 , in accordance with Table 4.1 will be used.

$$d\varepsilon_v^{d,(1)} = -d\gamma^{p,(1)}\left(\frac{R'^{(1)} - R_{pt}^{(1)}}{N_d^{(1)}}\right) = 0.0002575$$

$$d\varepsilon_v^{d,(2)} = -d\gamma^{p,(2)}\left(\frac{R'^{(2)} + R_{pt}^{(2)}}{N_d^{(2)}}\right) = 0$$

$$d\varepsilon_v^{d,(3)} = -d\gamma^{p,(3)}\left(\frac{R'^{(3)} + R_{pt}^{(3)}}{N_d^{(3)}}\right) = -0.0002575$$

$$d\varepsilon_v^{d,(4)} = -d\gamma^{p,(4)}\left(\frac{R'^{(4)} + R_{pt}^{(4)}}{N_d^{(4)}}\right) = -0.0002575$$

$$d\varepsilon_v^{d,(5)} = -d\gamma^{p,(5)}\left(\frac{R'^{(5)} + R_{pt}^{(5)}}{N_d^{(5)}}\right) = 0$$

$$d\varepsilon_v^{d,(6)} = -d\gamma^{p,(6)}\left(\frac{R'^{(6)} + R_{pt}^{(6)}}{N_d^{(6)}}\right) = 0.000275$$

The shear stresses in each spring will then be calculated from the product of the stress ratio of the particular spring as amended by the hardening effect according to eq. (1.126) and mean effective stress of p' = 100kPa. Here, for the applied small strain the mean effective stress will be assumed to be constant.

$$\tau^{(1)} = R^{(1)} p'(1 + H_p \varepsilon_v) = 71.86kPa$$
$$\tau^{(2)} = R^{(2)} p'(1 + H_p \varepsilon_v) = 0kPa$$
$$\tau^{(3)} = R^{(3)} p'(1 + H_p \varepsilon_v) = -71.86kPa$$
$$\tau^{(3)} = R^{(3)} p'(1 + H_p \varepsilon_v) = -71.86kPa$$
$$\tau^{(5)} = R^{(5)} p'(1 + H_p \varepsilon_v) = 0kPa$$
$$\tau^{(6)} = R^{(6)} p'(1 + H_p \varepsilon_v) = 71.86kPa$$

Note that this is the only the fourth component of the stress vector of each spring. All the remaining components are zero. Hence the stress vector of each spring will then be given as:

$$\{\tau^{(1)}\} = \{0 \quad 0 \quad 0 \quad 71.86 \quad 0 \quad 0\} kPa$$
$$\{\tau^{(2)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\} kPa$$
$$\{\tau^{(3)}\} = \{0 \quad 0 \quad 0 \quad -71.86 \quad 0 \quad 0\} kPa$$
$$\{\tau^{(4)}\} = \{0 \quad 0 \quad 0 \quad -71.86 \quad 0 \quad 0\} kPa$$
$$\{\tau^{(5)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\} kPa$$
$$\{\tau^{(6)}\} = \{0 \quad 0 \quad 0 \quad 71.86 \quad 0 \quad 0\} kPa$$

The global deviatoric stress will then be given by eq. (1.48).

$$\{\tau\}_{xyz}^{(i)} = \frac{1}{6}\sum_{i=1}^{6}[T_\sigma^{(i)}]^{-1}\{\tau^{(i)}\} = \frac{1}{6}\sum_{i=1}^{6}[T_\varepsilon^{(i)}]^{T}\{\tau^{(i)}\}$$

After calculation, the deviatoric stress is obtained to be:

$$\{\tau\}_{xyz}^{(i)} = \{0 \quad 0 \quad 0 \quad 41.49 \quad 0 \quad 0\} \text{kPa}$$

Here, it has to be noted that only the magnitudes of the shear stress will be considered to determine the aggregate shear stress of the soil element.

4.1.2. Analysis by DIANA:

For the analysis using DIANA, the 3D soil element shown in fig. 4.1 will be used. The components of the data file which will be used for DIANA liquefaction analysis were explained in chapter 3. Some modifications will be made in the data file to create the model.

To align the origin of the coordinate system with the center of the virtual plane, the coordinates of the nodes should be changed to:

```
'COORDINATES'
            1  -5.000000E-01   -5.000000E-01   -5.000000E-01
            2   5.000000E-01   -5.000000E-01   -5.000000E-01
            3  -5.000000E-01    5.000000E-01   -5.000000E-01
            4   5.000000E-01    5.000000E-01   -5.000000E-01
            5  -5.000000E-01   -5.000000E-01    5.000000E-01
            6   5.000000E-01   -5.000000E-01    5.000000E-01
            7  -5.000000E-01    5.000000E-01    5.000000E-01
            8   5.000000E-01    5.000000E-01    5.000000E-01
```

A prescribed shear strain value of $\gamma_{xy} = 10^{-4}$ is used and the analysis is made. This will be entered in the `LOAD` case 2.

```
CASE     2
DEFORM

/ 3-4, 7-8 / TR 1 -1.0E-4
```

The stress vector which is obtained by DIANA is:

$$\{\sigma\} = \begin{Bmatrix} 90.97 \\ 90.97 \\ 90.97 \\ 41.49 \\ 0 \\ 0 \end{Bmatrix} kN/m^2$$

There is good agreement between the resulting shear stress between the analytical and DIANA deviatoric stresses are in good agreement.

4.2. Shear applied in the zx direction

To analyze a similar multiple shear mechanism as shown in fig. 4.1 for a simple shear in zx direction, the same values of material and spring parameters as given in table 4.1 will be used. The calculation of the stress vector from a strain vector applied in zx direction will be discussed briefly step by step in the next sections.

A prescribed value of strain will be given.

For simple shear case shown in fig 4.2, the prescribed strain vector can be written as:

$$\{\varepsilon\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 10^{-3}\}^T \tag{4.15}$$

The shear component of strain for each spring will be computed from the given strain using coordinate transformation.

The one-dimensional shear strain of each spring will be determined by multiplying the strain of the overall soil element by the transformation matrix. In this case, the rotation is made only about y-axis. Thus, the transformation matrix for strain [T_ε] can be given by eq. (1.18).

For this model, the coordinates will be rotated around the y axis so that the x'-axis of the new coordinate system will be aligned with the springs. There will be six springs in total. For the spring orientation given in fig. 4.1, the transformation matrices for each spring can be obtained by substituting the corresponding values of ϕ in eq. (1.18). The values of ϕ are as given in fig. 2.1. The springs will be numbered from 1 – 6 and their transformation matrices will be:

For spring no. 1, ϕ= 15°

$$[T_\varepsilon^1]=\begin{bmatrix} 0.933013 & 0 & 0.066987 & 0 & 0 & -0.25 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.066987 & 0 & 0.933013 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0.965926 & -0.258819 & 0 \\ 0 & 0 & 0 & 0.258819 & 0.965926 & 0 \\ 0.5 & 0 & -0.5 & 0 & 0 & 0.866025 \end{bmatrix}$$

For spring no. 2, ϕ= 45°

$$[T_\varepsilon^2]=\begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.707107 & -0.707107 & 0 \\ 0 & 0 & 0 & 0.707107 & 0.707107 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

For spring no. 3, ϕ= 75°

$$[T_\varepsilon^3]=\begin{bmatrix} 0.066987 & 0 & 0.933013 & 0 & 0 & -0.25 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.933013 & 0 & 0.066987 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0.258819 & -0.965926 & 0 \\ 0 & 0 & 0 & 0.965926 & 0.258819 & 0 \\ 0.5 & 0 & -0.5 & 0 & 0.965926 & -0.866025 \end{bmatrix}$$

For spring 4, ϕ= 105°

$$[T_\varepsilon^4]=\begin{bmatrix} 0.066987 & 0 & 0.933013 & 0 & 0 & -0.25 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.933013 & 0 & 0.066987 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0.258819 & -0.965926 & 0 \\ 0 & 0 & 0 & 0.965926 & 0.258819 & 0 \\ 0.5 & 0 & -0.5 & 0 & 0 & -0.866025 \end{bmatrix}$$

For spring no. 5, ϕ= 135°

$$[T_\varepsilon^5] = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -0.707107 & -0.707107 & 0 \\ 0 & 0 & 0 & 0.707107 & -0.707107 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For spring no.6, ϕ= 165°

$$[T_\varepsilon^6] = \begin{bmatrix} 0.933013 & 0 & 0.066987 & 0 & 0 & 0.25 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.066987 & 0 & 0.933013 & 0 & 0 & -0.25 \\ 0 & 0 & 0 & -0.965926 & -0.258819 & 0 \\ 0 & 0 & 0 & 0.258819 & -0.965926 & 0 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0.866025 \end{bmatrix}$$

Multiplication of the strain vector by the transformation matrix of each spring gives the strain vector of each spring. The strain vector of each spring will be obtained by multiplying the strain vector of the soil by the corresponding strain transformation matrix.

This will result in the following:

$$\{\varepsilon^{(1)}\} = [T_\varepsilon^1]\{\varepsilon\} = 10^{-3} * \{-0.25 \quad 0.25 \quad 0 \quad 0 \quad 0 \quad 0.866\}^T$$
$$\{\varepsilon^{(2)}\} = [T_\varepsilon^2]\{\varepsilon\} = 10^{-3} * \{-0.5 \quad 0 \quad 0.5 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\varepsilon^{(3)}\} = [T_\varepsilon^3]\{\varepsilon\} = 10^{-3} * \{-0.25 \quad 0 \quad 0.25 \quad 0 \quad 0 \quad -0.866\}^T$$
$$\{\varepsilon^{(4)}\} = [T_\varepsilon^4]\{\varepsilon\} = 10^{-3} * \{-0.25 \quad 0 \quad 0.25 \quad 0 \quad 0 \quad -0.866\}^T$$
$$\{\varepsilon^{(5)}\} = [T_\varepsilon^5]\{\varepsilon\} = 10^{-3} * \{0.5 \quad 0 \quad -0.5 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\varepsilon^{(6)}\} = [T_\varepsilon^6]\{\varepsilon\} = 10^{-3} * \{0.25 \quad 0 \quad -0.25 \quad 0 \quad 0 \quad 0.866\}^T$$

The springs are assumed to carry only one dimensional shear strain. Hence, the shear components of the springs should be extracted from the total strain vector of the springs. The springs are oriented in a xy plane rotating along the z' axis. Thus, the strain component in the springs will be $\gamma_{x'z'}$. Hence, one-dimensional strains in each spring will be:

$$\{\gamma^{(1)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.866\}^T$$
$$\{\gamma^{(2)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\gamma^{(3)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.866\}^T$$

$$\{\gamma^{(4)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.866\}^T$$
$$\{\gamma^{(5)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T$$
$$\{\gamma^{(6)}\} = 10^{-3} * \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.866\}^T$$

From Masing's rule, the stress ratio of each spring will be determined

For the skeleton curve, the value of the shear modulus of each spring can be determined from eq. (1.129). For this simple shear case, the volumetric strain is zero and the value of the initial strain γ_o is zero. Hence the tangent shear modulus of each spring can be computed as:

$$G_{\tan}^{(1)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(1)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4 kN / m^2$$

$$G_{\tan}^{(2)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(2)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0|}{0.0008}\right)^2} = k_{\max} p' = 1.7825E5 kN / m^2$$

$$G_{\tan}^{(3)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(3)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|-0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4 kN / m^2$$

$$G_{\tan}^{(4)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(4)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|-0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4 kN / m^2$$

$$G_{\tan}^{(5)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(5)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0|}{0.0008}\right)^2} = k_{\max} p' = 1.7825E5 kN / m^2$$

$$G_{\tan}^{(6)} = \frac{k_{\max} p'}{\left(1 + \frac{|\gamma^{(6)}|}{\gamma_r}\right)^2} = \frac{k_{\max} p'}{\left(1 + \frac{|0.000866|}{0.0008}\right)^2} = 0.230584 k_{\max} p' = 3.9845E4 kN / m^2$$

Now the stress ratio of the each spring can be calculated. Here it is assumed that the relationship between the stress ratio and shear strain can be given by the back bone curve and the relationship which is already used in eq. (4.9) will be adopted here as well.

The spring stiffness, shear strain and the reference shear strain of each spring is given in Table 4.1 and previous computations. If the change in mean effective stress is also assumed to be insignificant, then the reference shear strain can be taken to be equal to the initial reference shear strain. Hence, the stress ratio of the springs can be calculated as:

$$R^{(1)} = \frac{k_{\max}\gamma^{(1)}}{1+\frac{\left|\gamma^{(1)}\right|}{\gamma_r}} = \frac{1728 * 0.866x10^{-3}}{1+\frac{0.866x10^{-3}}{0.0008}} = 0.7186$$

$$R^{(2)} = \frac{k_{\max}\gamma^{(2)}}{1+\frac{\left|\gamma^{(2)}\right|}{\gamma_r}} = \frac{1728 * 0}{1+\frac{0}{0.0008}} = 0$$

$$R^{(3)} = \frac{k_{\max}\gamma^{(3)}}{1+\frac{\left|\gamma^{(3)}\right|}{\gamma_r}} = \frac{1728 * -0.866x10^{-3}}{1+\frac{\left|-0.866x10^{-3}\right|}{0.0008}} = -0.7186$$

$$R^{(4)} = \frac{k_{\max}\gamma^{(4)}}{1+\frac{\left|\gamma^{(4)}\right|}{\gamma_r}} = \frac{1728 * -0.866x10^{-3}}{1+\frac{\left|-0.866x10^{-3}\right|}{0.0008}} = -0.7186$$

$$R^{(5)} = \frac{k_{\max}\gamma^{(5)}}{1+\frac{\left|\gamma^{(5)}\right|}{\gamma_r}} = \frac{1728 * 0}{1+\frac{0}{0.0008}} = 0$$

$$R^{(6)} = \frac{k_{\max}\gamma^{(6)}}{1+\frac{\left|\gamma^{(6)}\right|}{\gamma_r}} = \frac{1728 * 0.866x10^{-3}}{1+\frac{0.866x10^{-3}}{0.0008}} = 0.7186$$

Using stress-dilatancy relationships, the volumetric strain increment due to dilatancy will be computed.

The volumetric strain increment due to dilatancy for loading is calculated by using eq. (4.10). Before the calculation of the volumetric strain due to dilatancy, the plastic shear strain should be calculated first. The values of the input parameters in the calculations, $N_d^{(i)}$ and $R_{pt}^{(i)}$,will be taken from table 4.1. The plastic shear strain is calculated by eq. (4.11) which requires the value of equivalent shear stiffness. The equivalent shear stiffness $G_{eq}^{(i)}$ for each spring is calculated using eq. (4.12). Substituting the right values of the reduction factor η and H_p from table 4.1 , a simplified version of this equation can be obtained as given in eq. (4.13).

The amplitude of shear strain for the back bone curve is half of the maximum shear strain which is given by eq. (4.14). Substituting the maximum shear strain of each spring in this equation, the amplitude of shear strain can be determined as:

$$\gamma_a^{(1)} = \left|\gamma_{\max}^{(1)}\right| = 0.000866$$
$$\gamma_a^{(2)} = \left|\gamma_{\max}^{(1)}\right| = 0$$
$$\gamma_a^{(3)} = \left|\gamma_{\max}^{(3)}\right| = 0.000866$$
$$\gamma_a^{(4)} = \left|\gamma_{\max}^{(4)}\right| = 0.000866$$
$$\gamma_a^{(5)} = \left|\gamma_{\max}^{(5)}\right| = 0$$
$$\gamma_a^{(6)} = \left|\gamma_{\max}^{(6)}\right| = 0.000866$$

Then the value of the elastic shear stiffness can be computed as:

$$G_{eq}^{(1)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(1)}}{G_{eq}^{(1)}}\right) = 0.6649$$

$$G_{eq}^{(2)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0}{0.0008}}+0.4\right\} = p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(2)}}{G_{eq}^{(2)}}\right) = 0$$

$$G_{eq}^{(3)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(3)}}{G_{eq}^{(3)}}\right) = 0.6649$$

$$G_{eq}^{(4)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(4)}}{G_{eq}^{(4)}}\right) = 0.6649$$

$$G_{eq}^{(5)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0}{0.0008}}+0.4\right\} = p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(5)}}{G_{eq}^{(5)}}\right) = 0$$

$$G_{eq}^{(6)} = p'k_{\max}\left\{\frac{0.6}{1+\dfrac{0.000866}{0.0008}}+0.4\right\} = 0.6881p'k_{\max} \Rightarrow \left(1-\frac{G_{\tan}^{(6)}}{G_{eq}^{(6)}}\right) = 0.6649$$

Then from eq. (4.11) ,the plastic shear strain increments are calculated as:

$$d\gamma_p^{(1)} = \left(1 - \frac{G_{\tan}^{(1)}}{G_{eq}^{(1)}}\right)\{n\}^T \left[T_\varepsilon^{(1)}\right]\{d\varepsilon\} = 0.0005758$$

$$d\gamma_p^{(2)} = \left(1 - \frac{G_{\tan}^{(2)}}{G_{eq}^{(2)}}\right)\{n\}^T \left[T_\varepsilon^{(2)}\right]\{d\varepsilon\} = 0$$

$$d\gamma_p^{(3)} = \left(1 - \frac{G_{\tan}^{(3)}}{G_{eq}^{(3)}}\right)\{n\}^T \left[T_\varepsilon^{(3)}\right]\{d\varepsilon\} = -0.0005758$$

$$d\gamma_p^{(4)} = \left(1 - \frac{G_{\tan}^{(4)}}{G_{eq}^{(4)}}\right)\{n\}^T \left[T_\varepsilon^{(4)}\right]\{d\varepsilon\} = -0.0005758$$

$$d\gamma_p^{(5)} = \left(1 - \frac{G_{\tan}^{(5)}}{G_{eq}^{(5)}}\right)\{n\}^T \left[T_\varepsilon^{(5)}\right]\{d\varepsilon\} = 0$$

$$d\gamma_p^{(6)} = \left(1 - \frac{G_{\tan}^{(6)}}{G_{eq}^{(6)}}\right)\{n\}^T \left[T_\varepsilon^{(6)}\right]\{d\varepsilon\} = 0.0005758$$

Now the volumetric strain due to dilatancy can be calculated using eq. (1.141).

$$d\varepsilon_v^{d,(i)} = -d\gamma^{p,(i)}\left(\frac{R'^{(i)} \pm R_{pt}^{(i)}}{N_d^{(i)}}\right)$$

$R_{pt}^{(i)}$ will be negative for $d\gamma^{p,(i)} > 0$ and positive for $d\gamma^{p,(i)} < 0$. Since the loading is initial loading the value of $R_{pt,i}^{(i)}$ =1.3 , in accordance with Table 4.1 will be used.

$$d\varepsilon_v^{d,(1)} = -d\gamma^{p,(1)}\left(\frac{R'^{(1)} - R_{pt}^{(1)}}{N_d^{(1)}}\right) = 0.0002575$$

$$d\varepsilon_v^{d,(2)} = -d\gamma^{p,(2)}\left(\frac{R'^{(2)} + R_{pt}^{(2)}}{N_d^{(2)}}\right) = 0$$

$$d\varepsilon_v^{d,(3)} = -d\gamma^{p,(3)}\left(\frac{R'^{(3)} + R_{pt}^{(3)}}{N_d^{(3)}}\right) = -0.0002575$$

$$d\varepsilon_v^{d,(4)} = -d\gamma^{p,(4)}\left(\frac{R'^{(4)} + R_{pt}^{(4)}}{N_d^{(4)}}\right) = -0.0002575$$

$$d\varepsilon_v^{d,(5)} = -d\gamma^{p,(5)}\left(\frac{R'^{(5)} + R_{pt}^{(5)}}{N_d^{(5)}}\right) = 0$$

$$d\varepsilon_v^{d,(6)} = -d\gamma^{p,(6)}\left(\frac{R'^{(6)} + R_{pt}^{(6)}}{N_d^{(6)}}\right) = 0.000275$$

The shear stresses in each spring will then be calculated from the product of the stress ratio of the particular spring and mean effective stress. Here, for the applied small strain the mean effective stress will be assumed to be constant and equal to the initial mean effective stress.

$$\tau^{(1)} = R^{(1)} p'(1 + H_p \varepsilon_v) = 71.86 kPa$$
$$\tau^{(2)} = R^{(2)} p'(1 + H_p \varepsilon_v) = 0 kPa$$
$$\tau^{(3)} = R^{(3)} p'(1 + H_p \varepsilon_v) = -71.86 kPa$$
$$\tau^{(3)} = R^{(3)} p'(1 + H_p \varepsilon_v) = -71.86 kPa$$
$$\tau^{(5)} = R^{(5)} p'(1 + H_p \varepsilon_v) = 0 kPa$$
$$\tau^{(6)} = R^{(6)} p'(1 + H_p \varepsilon_v) = 71.86 kPa$$

Note that this is the only the fourth component of the stress vector of each spring. All the remaining components are zero. Hence the stress vector of each spring will then be given as:

$$\{\tau^{(1)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 71.86\} kPa$$
$$\{\tau^{(2)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\} kPa$$
$$\{\tau^{(3)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -71.86\} kPa$$
$$\{\tau^{(4)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -71.86\} kPa$$
$$\{\tau^{(5)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\} kPa$$
$$\{\tau^{(6)}\} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 71.86\} kPa$$

The global deviatoric stress will then be given by eq. (1.47).

$$\{\tau\}_{xyz}^{(i)} = \frac{1}{6}\sum_{i=1}^{6}[T_\sigma^{(i)}]^{-1}\{\tau^{(i)}\} = \frac{1}{6}\sum_{i=1}^{6}[T_\varepsilon^{(i)}]^{T}\{\tau^{(i)}\}$$

After calculation, the deviatoric stress is obtained to be:

$$\{\tau\}_{xyz}^{(i)} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 41.49\}$$

Here, it has to be noted that only the magnitudes of the shear stress will be considered to determine the aggregate shear stress of the soil element.

This is the resulting stress vector for the analysis. Similar analysis under the same conditions will be carried out by DIANA and the results will be compared with this one.

4.2.2. *Analysis by DIANA:*

For the analysis using DIANA, the 3D soil element shown in fig. 4.1 will be used. The components of the data file which will be used for DIANA liquefaction analysis were explained in chapter 3. Some modifications will be made in the data file to create the model.

To align the origin of the coordinate system with the center of the virtual plane, the coordinates of the nodes should be changed to:

```
'COORDINATES'
            1  -5.000000E-01   -5.000000E-01   -5.000000E-01
            2   5.000000E-01   -5.000000E-01   -5.000000E-01
            3  -5.000000E-01    5.000000E-01   -5.000000E-01
            4   5.000000E-01    5.000000E-01   -5.000000E-01
            5  -5.000000E-01   -5.000000E-01    5.000000E-01
            6   5.000000E-01   -5.000000E-01    5.000000E-01
            7  -5.000000E-01    5.000000E-01    5.000000E-01
            8   5.000000E-01    5.000000E-01    5.000000E-01
```

A prescribed shear strain value of $\gamma_{xy} = 10^{-4}$ is used and the analysis is made. This will be entered in the `LOAD` case 2.

```
CASE     2
DEFORM
/ 5-8 / TR 1 1.0E-4
```

And all the nodes will be fixed in the three directions which is done through the following statement:

```
'SUPPORTS'
/ 1-8 / TR 1 2 3
```

In this case, the rotation of the springs is done only around y axis. Hence, the values of θ and ζ will be zero for all the springs while the value of ϕ changes according to the orientation of the spring.

The results are shown for 20 load steps to the same strain level as the analytical solution is given below.

$$\{\sigma\} = \begin{Bmatrix} 90.97 \\ 90.97 \\ 90.97 \\ 41.49 \\ 0 \\ 0 \end{Bmatrix} kN/m^2$$

The shear stress values resulting from the analytical and DIANA are again in good agreement.

4.3. Clarification of the torsion shear-icosahedral distribution deviatoric stress relationship with DIANA analysis.

To compare the relative values given in table 1.2 of chapter one with Diana results, a drained analysis is performed for different values of shear strain level. The resulting relative values for the deviatoric shear part are summarized in the table below:

Table 4.2. Normal and shear stresses at different strain levels [Global shear strain direction zx]

normalized strain in zx direction(%)	σ_{xx}	σ_{yy}	σ_{zz}	τ_{xy}	τ_{yz}	τ_{xz}	p'	τ_{max}	τ_{xy}/τ_{max}	τ_{yz}/τ_{max}	τ_{zx}/τ_{max}
12.5	-91.57	-85.97	-98.00	-0.472	-0.500	41.610	-91.847	-126.969	0.004	0.004	-0.328
25	-91.42	-84.80	-98.00	-0.551	-0.789	45.420	-91.407	-126.361	0.004	0.006	-0.359
37.5	-91.31	-84.30	-98.00	-0.520	-0.983	47.180	-91.203	-126.079	0.004	0.008	-0.374
50	-91.20	-83.93	-98.00	-0.451	-1.122	48.300	-91.043	-125.858	0.004	0.009	-0.384
62.5	-91.11	-83.62	-98.00	-0.366	-1.222	49.120	-90.910	-125.674	0.003	0.010	-0.391
75	-91.04	-83.34	-98.00	-0.279	-1.298	49.770	-90.793	-125.513	0.002	0.010	-0.397
87.5	-90.98	-83.09	-98.00	-0.194	-1.358	50.310	-90.690	-125.370	0.002	0.011	-0.401
100	-90.94	-82.86	-98.00	-0.113	-1.405	50.780	-90.600	-125.245	0.001	0.011	-0.405
112.5	-90.92	-82.64	-98.00	-0.037	-1.443	51.190	-90.520	-125.135	0.000	0.012	-0.409
125	-90.90	-82.44	-98.00	0.034	-1.475	51.560	-90.447	-125.033	0.000	0.012	-0.412
1000	-101.50	-91.53	-98.00	-2.907	-0.708	47.840	-97.010	-134.107	0.022	0.005	-0.357
3000	-101.80	-91.78	-98.00	-2.749	-0.597	47.980	-97.193	-134.360	0.020	0.004	-0.357
Relative value of shear stress with respect to shear stress in the direction of global shear strain for the last two strain levels				**-0.061**	**-0.015**	**1.000**			**-0.061**	**-0.015**	**1.000**
				-0.057	**-0.012**	**1.000**			**-0.057**	**-0.012**	**1.000**

Table 4.3. Normal and shear stresses at different strain levels [Global shear strain direction yz]

Normalized strain in yz direction(%)	σ_{xx}	σ_{yy}	σ_{zz}	τ_{xy}	τ_{yz}	τ_{xz}	p'	τ_{max}	τ_{xy}/τ_{max}	τ_{yz}/τ_{max}	τ_{zx}/τ_{max}
12.5	-84.19	-98.00	-90.60	0.000	41.480	0.000	-90.930	-125.702	0.000	-0.330	0.000
25	-82.32	-98.00	-90.66	0.000	45.290	0.000	-90.327	-124.868	0.000	-0.363	0.000
37.5	-97.74	-98.00	-97.79	0.000	5.926	0.000	-97.843	-135.259	0.000	-0.044	0.000
50	-80.92	-98.00	-90.88	0.000	48.240	0.000	-89.933	-124.324	0.000	-0.388	0.000
62.5	-80.48	-98.00	-90.89	0.000	49.070	0.000	-89.790	-124.126	0.000	-0.395	0.000
75	-80.12	-98.00	-90.87	0.000	49.730	0.000	-89.663	-123.951	0.000	-0.401	0.000
87.5	-79.80	-98.00	-90.84	0.000	50.270	0.000	-89.547	-123.789	0.000	-0.406	0.000
100	-79.52	-98.00	-90.80	0.000	50.740	0.000	-89.440	-123.642	0.000	-0.410	0.000
112.5	-79.27	-98.00	-90.75	0.000	51.140	0.000	-89.340	-123.504	0.000	-0.414	0.000
125	-79.04	-98.00	-90.70	0.000	51.500	0.000	-89.247	-123.375	0.000	-0.417	0.000
1000	-82.67	-98.00	-92.22	0.000	45.430	0.000	-90.963	-125.748	0.000	-0.361	0.000
3000	-82.68	-98.00	-92.24	0.000	45.480	0.000	-90.973	-125.762	0.000	-0.362	0.000

Relative value of shear stress with respect to shear stress in the direction of global shear strain for the last two strain levels	**0.000**	**1.000**	**0.000**			**0.000**	**1.000**	**0.000**
	0.000	**1.000**	**0.000**			**0.000**	**1.000**	**0.000**

Table 4.4. Normal and shear stresses at different strain levels [Global shear strain direction xy]

Normalized strain in xy direction(%)	σ_{xx}	σ_{yy}	σ_{zz}	τ_{xy}	τ_{yz}	τ_{xz}	p'	τ_{max}	τ_{xy}/τ_{max}	τ_{yz}/τ_{max}	τ_{zx}/τ_{max}
12.5	-98.00	-90.07	-85.45	41.070	0.382	-0.243	-91.173	-126.038	-0.326	-0.003	0.002
25	-98.00	-89.90	-83.77	44.730	0.720	-0.333	-90.557	-125.186	-0.357	-0.006	0.003
37.5	-98.00	-89.90	-83.77	44.730	0.720	-0.333	-90.557	-125.186	-0.357	-0.006	0.003
50	-98.00	-90.04	-82.44	47.560	1.010	-0.391	-90.160	-124.637	-0.382	-0.008	0.003
62.5	-98.00	-90.08	-82.08	48.380	1.078	-0.426	-90.053	-124.490	-0.389	-0.009	0.003
75	-98.00	-90.13	-81.83	49.050	1.118	-0.475	-89.987	-124.398	-0.394	-0.009	0.004
87.5	-98.00	-90.19	-81.66	49.630	1.140	-0.534	-89.950	-124.347	-0.399	-0.009	0.004
100	-98.00	-90.25	-81.56	50.130	1.150	-0.601	-89.937	-124.328	-0.403	-0.009	0.005
112.5	-98.00	-90.31	-81.50	50.590	1.151	-0.673	-89.937	-124.328	-0.407	-0.009	0.005
125	-98.00	-90.38	-81.48	51.010	1.145	-0.749	-89.953	-124.351	-0.410	-0.009	0.006
1000	-98.00	-93.10	-86.50	45.400	0.764	-1.040	-92.523	-127.904	-0.355	-0.006	0.008
3000	-98.00	-93.10	-86.50	45.450	0.760	-1.034	-92.527	-127.909	-0.355	-0.006	0.008
Relative value of shear stress with respect to shear stress in the direction of global shear strain for the last two strain levels				**1.000**	**0.017**	**-0.023**			**1.000**	**0.017**	**-0.023**
				1.000	**0.017**	**-0.023**			**1.000**	**0.017**	**-0.023**

The values written in bold in each table, show more or less similar trend as given in table 1.2 in chapter one. But there is difference which can be attributed to the differences in the loading and boundary conditions between the DIANA analysis and when originally establishing table 1.2. in the DIANA analysis, there is a vertical loading and volumetric strain. Where as both these conditions were not assumed in chapter one.

CHAPTER FIVE
VERIFICATION OF THE MODEL FOR DIFFERENT SOIL PARAMETERS

In chapter two, the agreement between the source code for the 3-D liquefaction analysis and the theoretical background of the model was proved to be acceptable. In this chapter, the consistency of the analysis results with laboratory results and with another numerical analysis result will be checked. For this, 3-D liquefaction analysis will be carried out by DIANA on a soil element for different soil parameters.

The analysis will be carried out with equal material and state parameters as the tests. The list of user defined parameters to be used for the analysis will be given in tables.

Table 5.1. Data for the drained analysis

Symbol	Value			
Dr [%]	75	57	38	22
$k_{max,0}$	3072	2624	2176	1728
$\gamma_{r,0}$	0.0008			
η	0.4			
$B_{e,0}$ [kPa]	76800	65600	54400	43200
$B_{s,0}$ [kPa]	96000	82000	68000	54000
H_p	22			
$R_{pt,i}$	1.3			
$R_{pt,s}$	0.85			
N_d	1.3			
γ_{th}	0.0001			
p'_0 [kPa]	100			
n	1			
m	0.5			

5.1 DRAINED MONOTONIC ONE-WAY SIMPLE SHEAR

Stress and strain components in simple shear for the monotonic one-way simple shear are illustrated in Fig. 5.1. The normal stress in z direction is kept constant and normal strains in x and y directions are always zero. In this case, strain-controlled analysis will be carried out and γ_{xy} and γ_{yz} are zero while γ_{zx} is controlled. The top nodes of the cube element will be deformed by a strain of 10^{-5} per step.

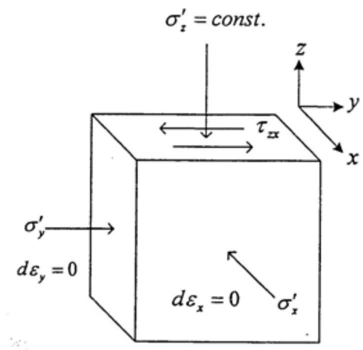


Fig 5.1. Stress-strain components in simple shear mode

Shear stress versus shear strain graphs

For monotonic loading the shear stress converges to a certain value. This state corresponds to critical state and the simulation by the analysis shows this phenomenon. The figures below show similar plot by Nishimura and also the comparison between his simulation and laboratory results. It can be noted that the computed results overestimate the observed ones. This is because the computed results retained larger mean effective principal stress *p'* than sand test did. The larger *p'* led to larger G_{max} and consequently resulted in the overestimation of stress-strain relation.

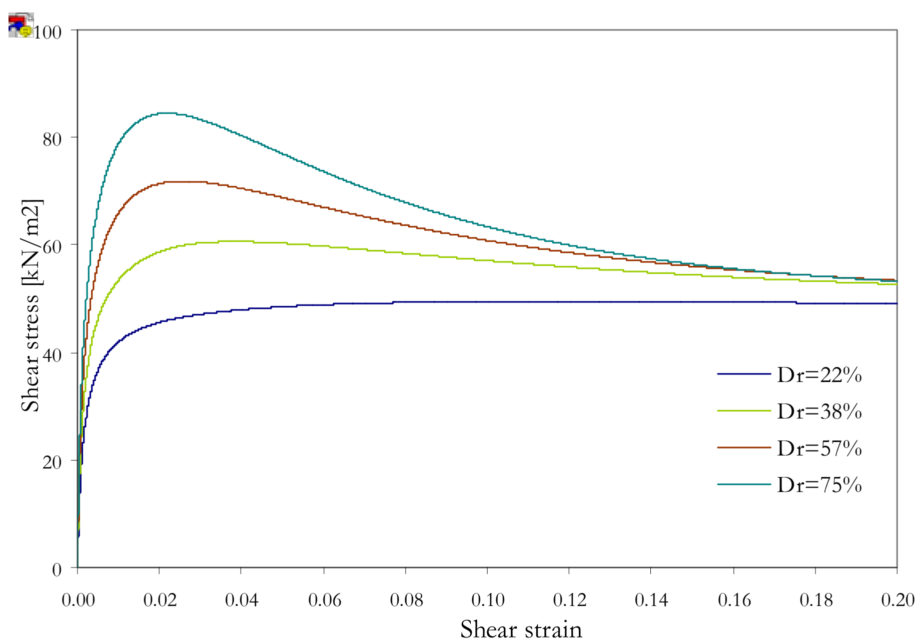


Fig 5.2. Stress strain relationship result by DIANA

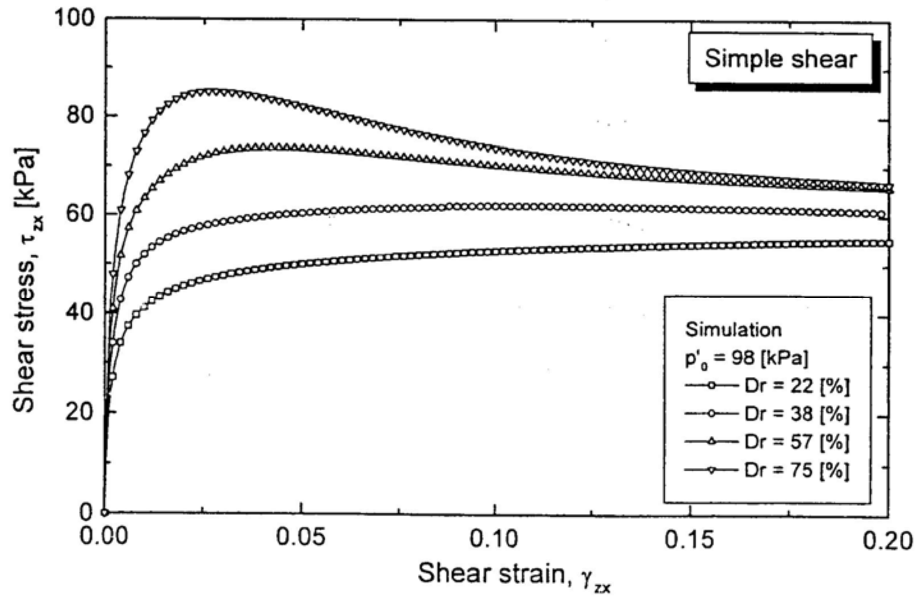


Fig 5.3. Computed stress strain relationship result by Nishimura (2002)

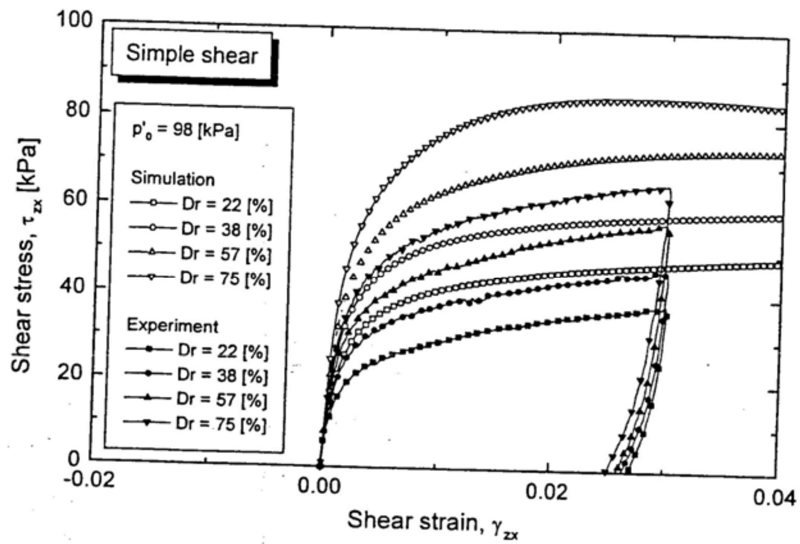


Fig 5.4. Comparison of computed stress strain relationship result by Nishimura(2002) and laboratory observation by Shahnazari(2001).

Referring to eq. (1.167), it can be deduced that this difference of the mean effective stress values between the simulation and the test arose from the difference between the values of the volumetric strain due to dilatancy. This effect can be reduced by two ways to have agreeable plots between the simulated and observed ones. They are:

i. normalizing the shear stress by mean effective stress

With the results of the analysis, the stress ratio (τ/p')-strain graph can also be drawn. These graphs are shown below. Comparing the resulting graphs from DIANA with the observation results for Dr = 22% , a better agreement exists between the two.

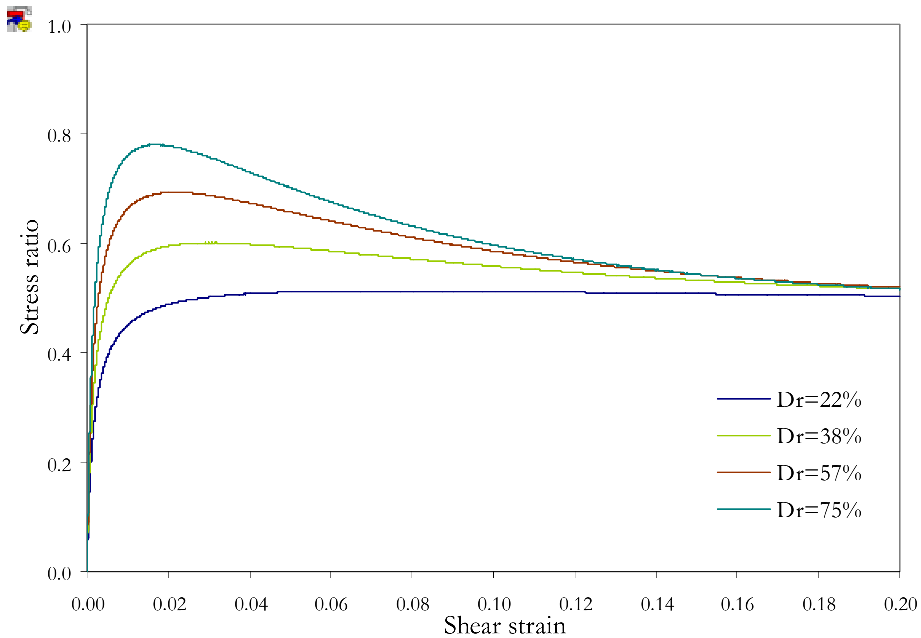


Fig 5.5. Stress ratio strain relationship result by DIANA

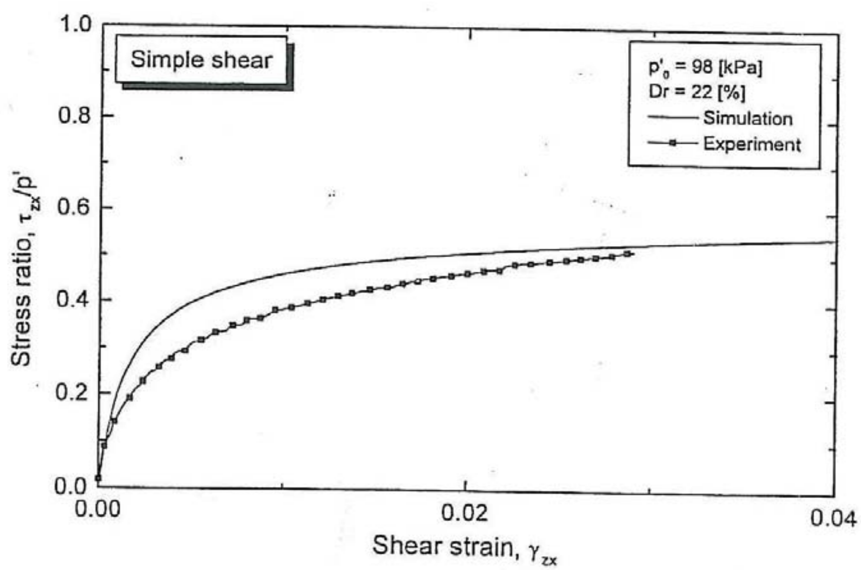


Fig 5.6. Comparison of computed stress ratio strain relationship result by Nishimura and laboratory observation.

Volumetric strain versus shear strain graphs

The graphs below show that the volumetric strain versus the shear strain curve. Here, positive volumetric strain is positive dilatancy and negative volumetric strain represents contraction or negative dilatancy for the DIANA results. The reverse is true for the results by Nishimura and observed ones.

The result from DIANA underestimates the contraction of sand as compared to the computed result by Nishimura or by laboratory tests. Due to this, there will be less swelling due to change in *p'*. The relationship between these two quantities for the observation is only given till shear strain reaches 0.03.

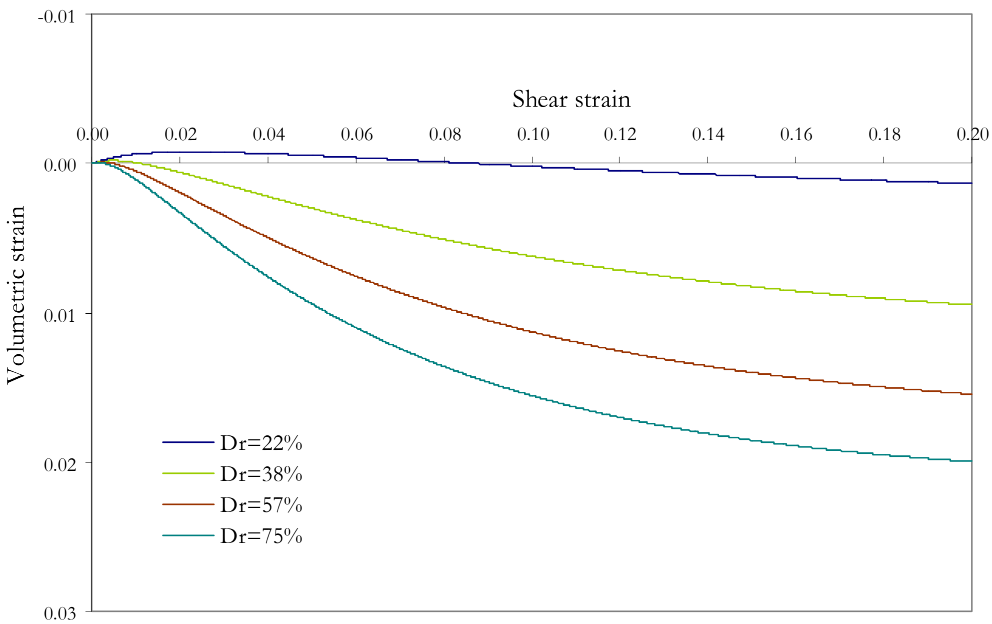


Fig 5.7. Volumetric change along with shearing result by DIANA

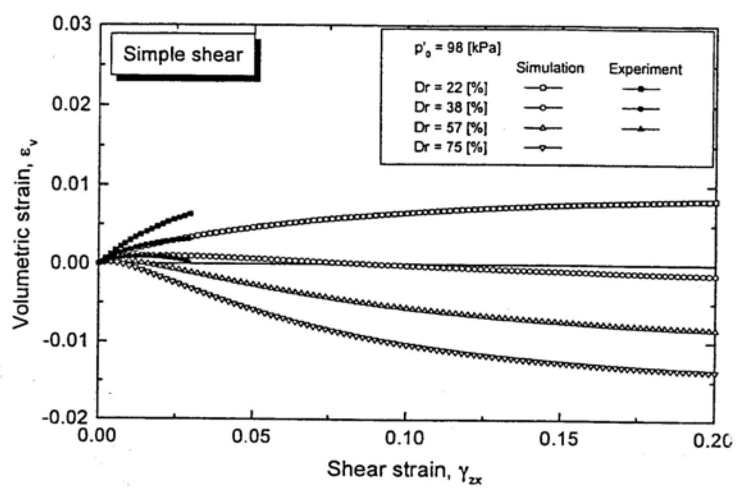


Fig 5.8. Volumetric strain change along with shearing result by Nishimura and laboratory results.

5.2. DRAINED CYCLIC SIMPLE SHEAR

This curve shows that volumetric change with shearing for different soil densities. The shear strain amplitude is kept constant at the value of 0.03. The volumetric strain progresses and converges at a certain value as expected. The shapes of the curves are also in good agreement with that of the computed results by Nishimura and laboratory observations.

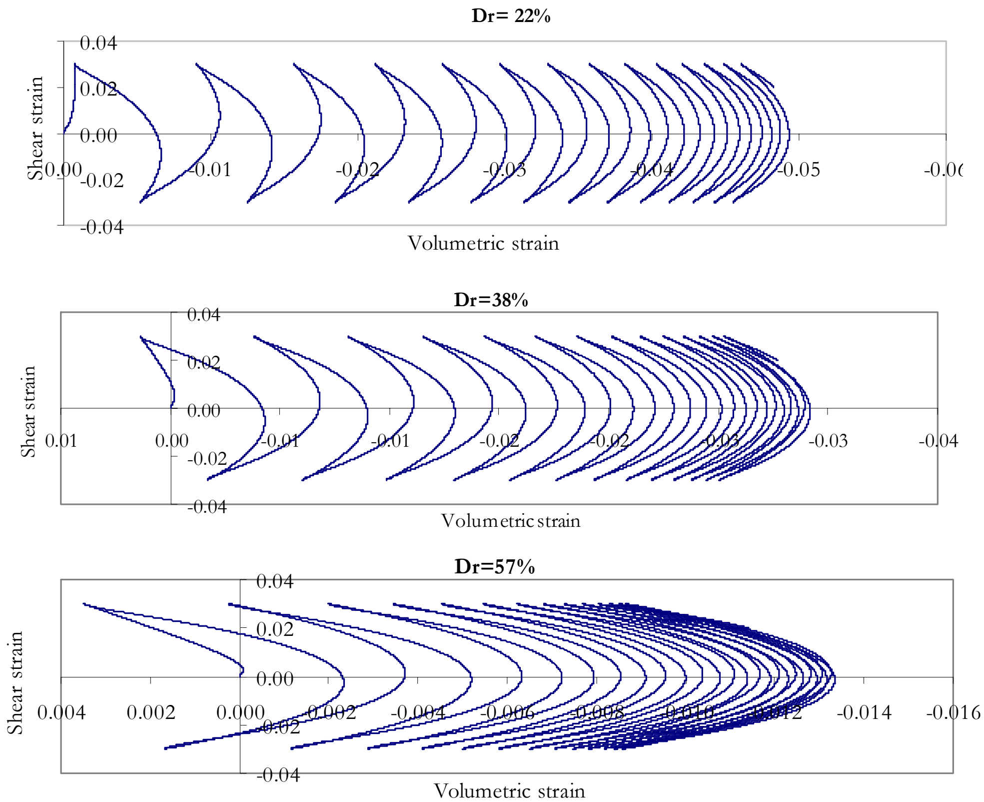


Fig 5.9 Shear strain versus volumetric strain result by DIANA

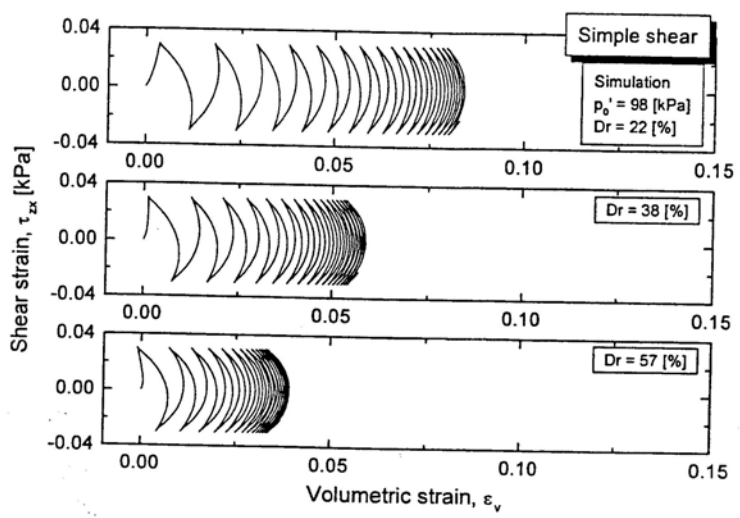


Fig 5.10 Shear strain versus volumetric strain result by Nishimura(2002)

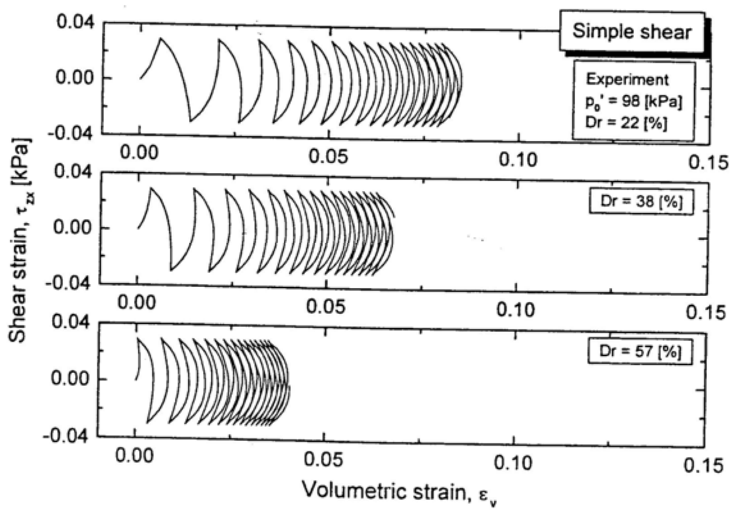
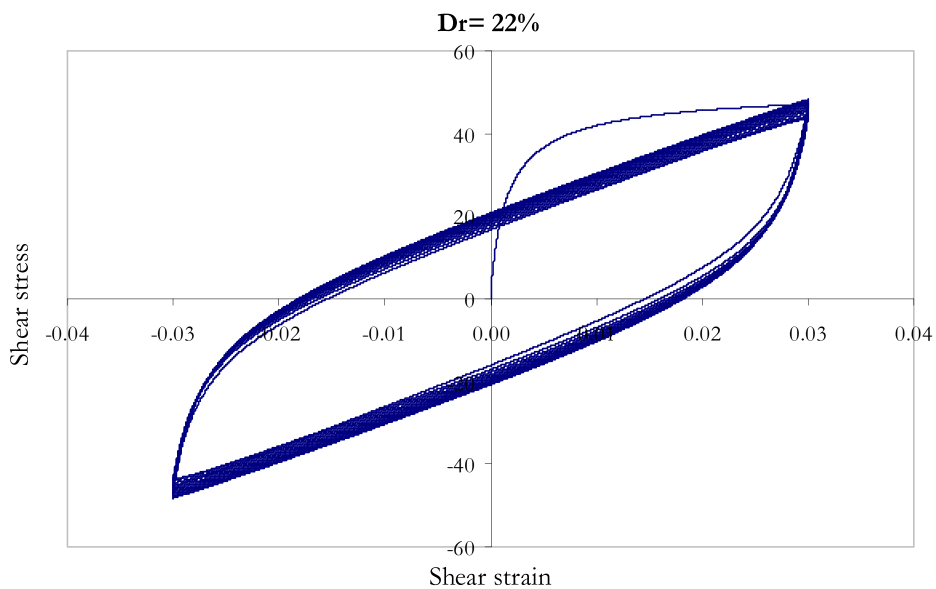


Fig 5.11 Shear strain versus volumetric strain result from laboratory observation by Shahnazari (2001).

Shear stress versus shear strain graphs

These graphs show the calculated stress-strain relationships for three different densities. The skeleton curve and the hysteresis loops are well delineated in the graphs. As the number of cycle increases, the stress amplitude will also increase. This is the effect of hardening. In addition in both results by DIANA and Nishimura the effect of hardening is more pronounced for the soils with lower densities which is logical.



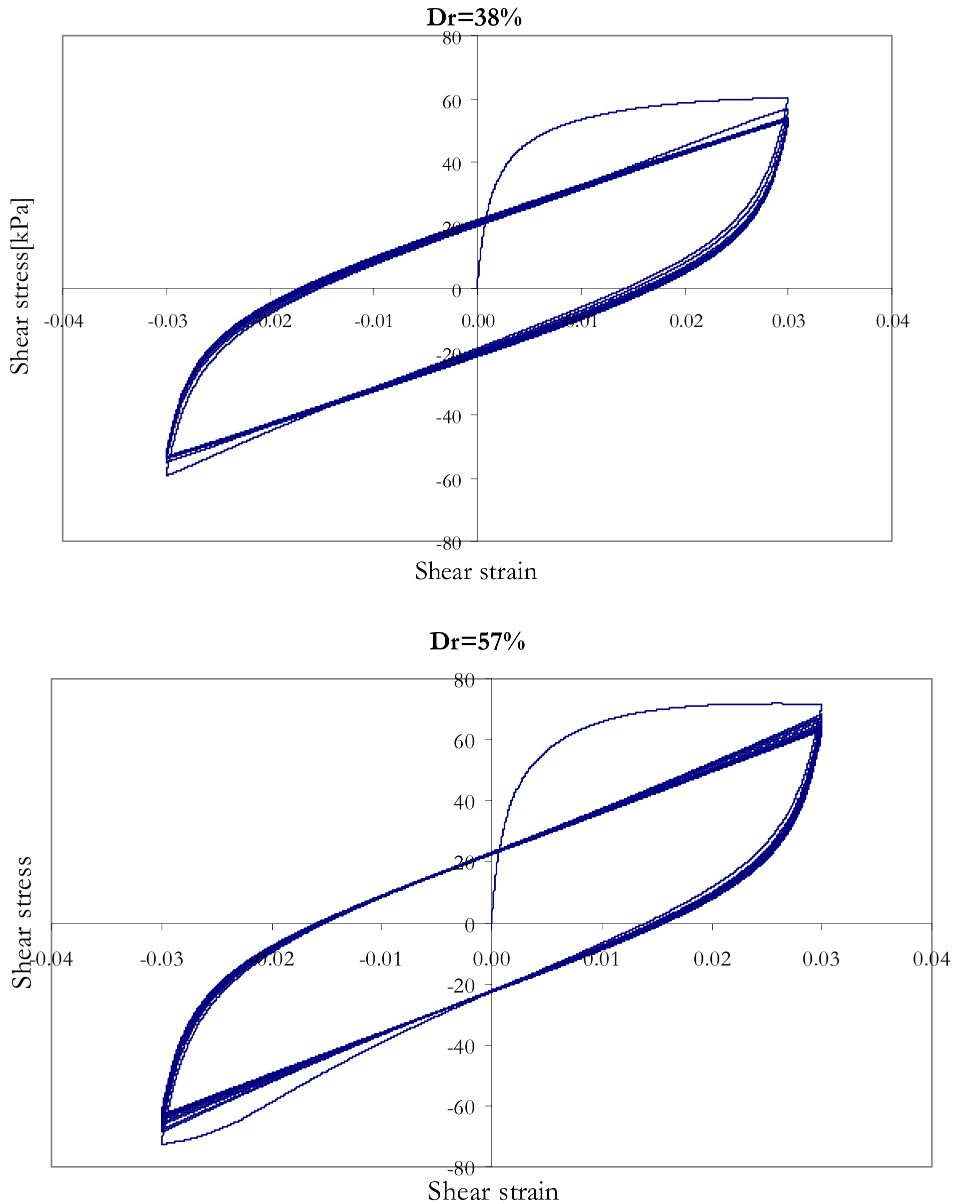


Fig 5.12. Shear strain versus volumetric strain result by DIANA.

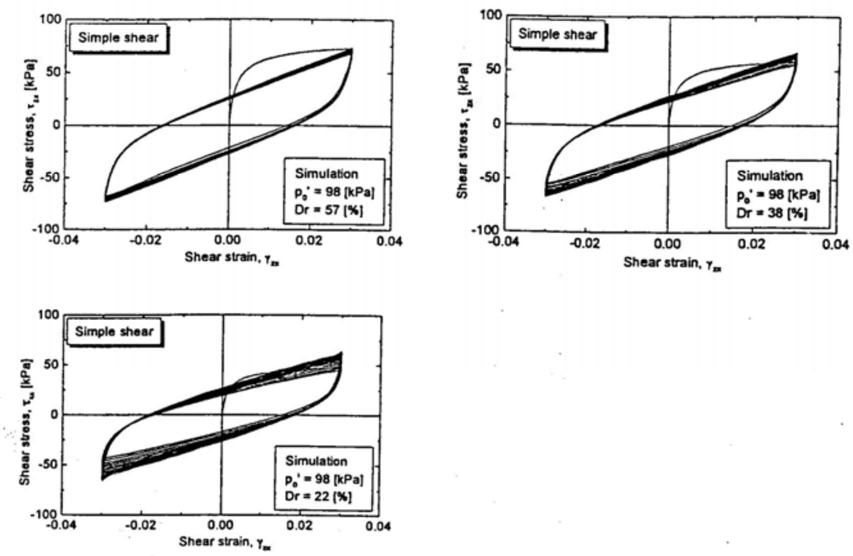


Fig 5.13 Shear stress versus shear strain result by Nishimura (2002).

5.3. UNDRAINED MONOTONIC SIMPLE SHEAR

The stress-strain relationships for different soil densities are shown below. As expected, the soil with low density, complete flow occurs while the soil with the higher density has some shear resistance. This is also supported by laboratory observation as shown in the subsequent graph.

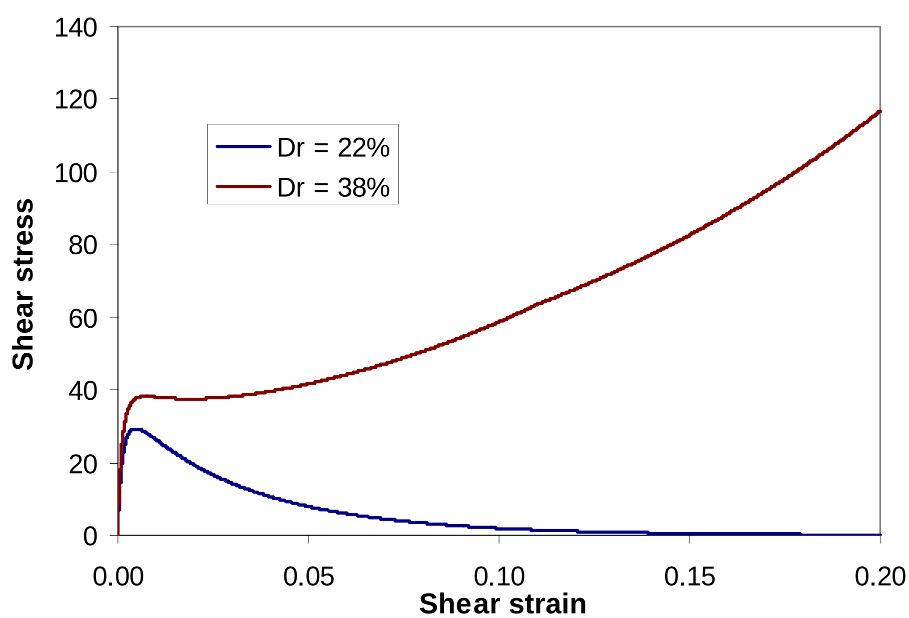


Fig.5.14. Shear stress strain relationship result by DIANA for undrained monotonic loading

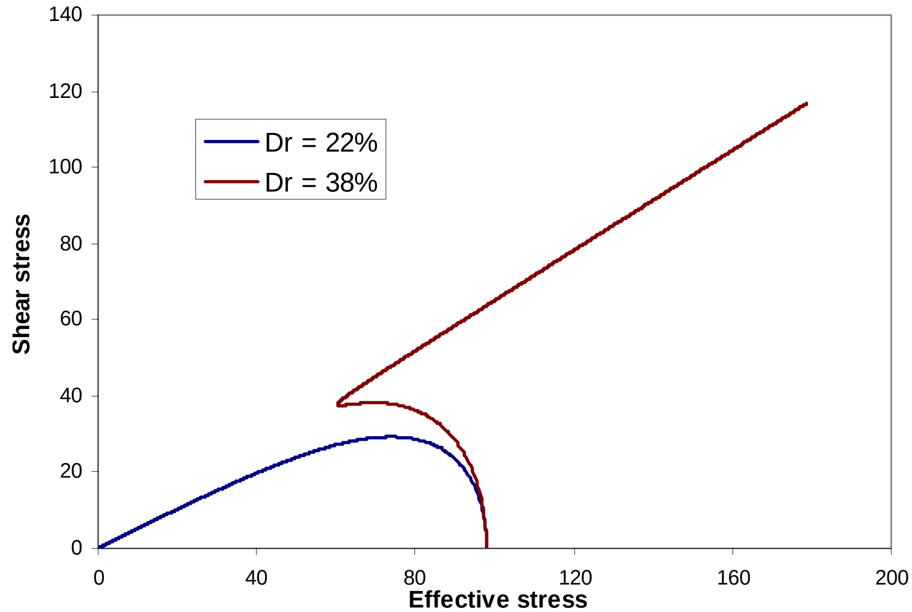


Fig.5.15 Stress path by DIANA for undrained monotonic loading

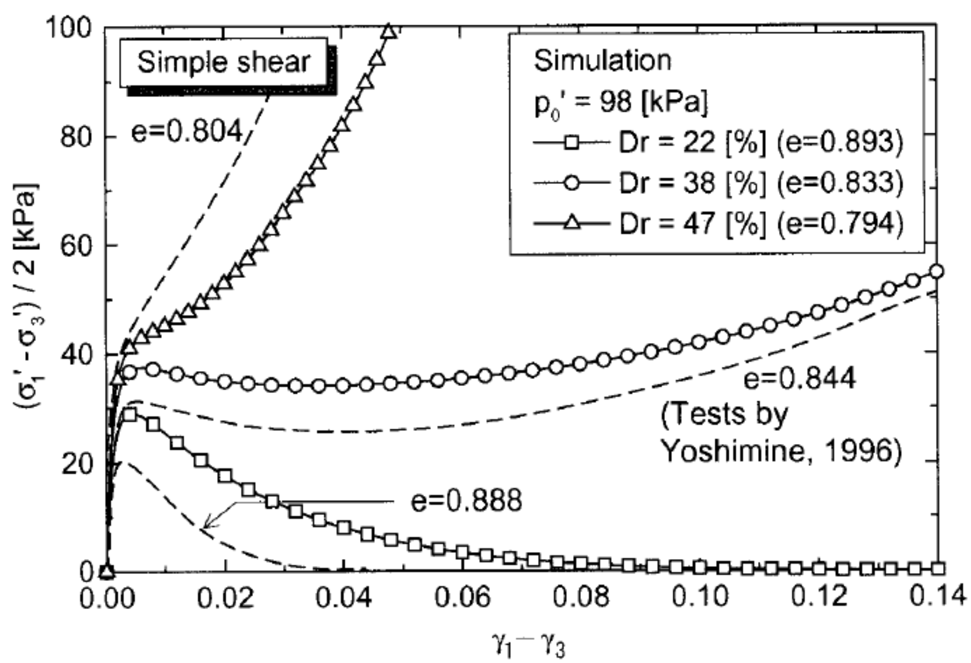
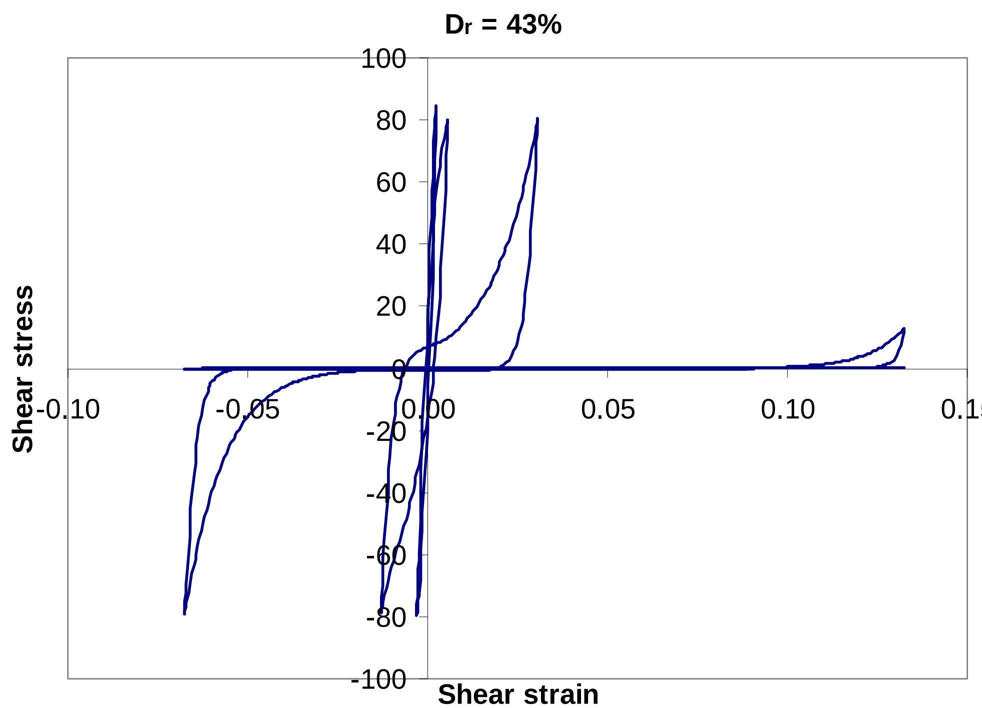


Fig.5.16. Deviatoric stress versus shear strain curves by Nishimura(2002) and Yoshimine (1996)

5.4. UNDRAINED CYCLIC SIMPLE SHEAR

Good agreement between the calculated results by DIANA and laboratory observation exists. In addition, the diagrams below suggest that the model is capable of predicting softening of loose sands and the cyclic mobility of dense sands. Abrupt softening is observed on the soils with less density. The number of cycles required to zero effective stress is smaller for looser sands. All these features of the result are consistent with the laboratory observation.



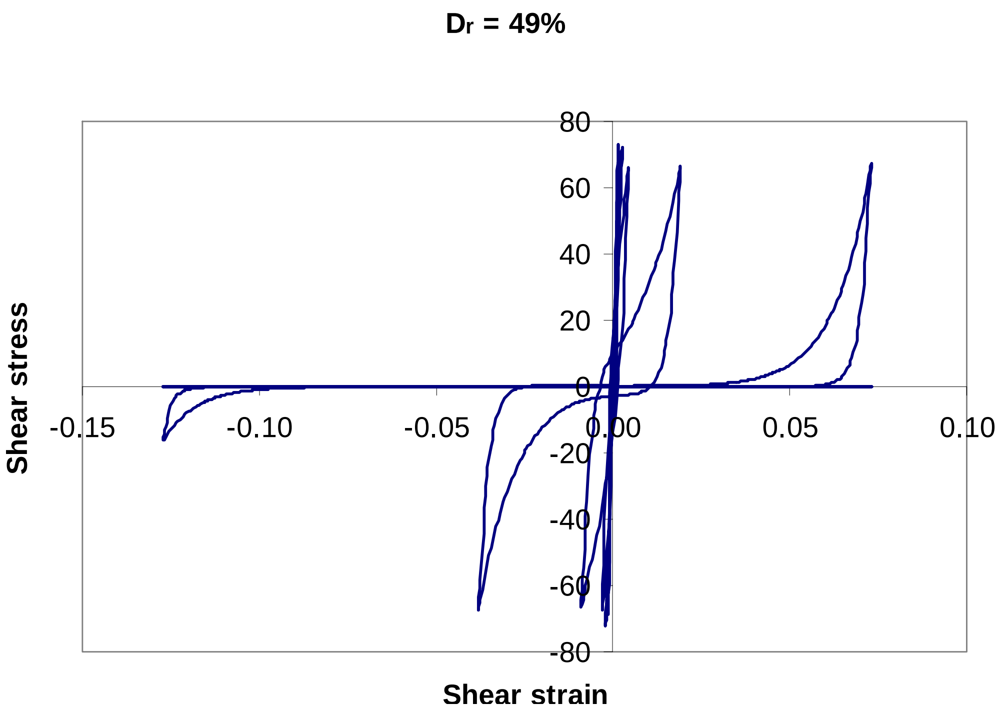


Fig.5.17. shear stress versus shear strain curves by DIANA

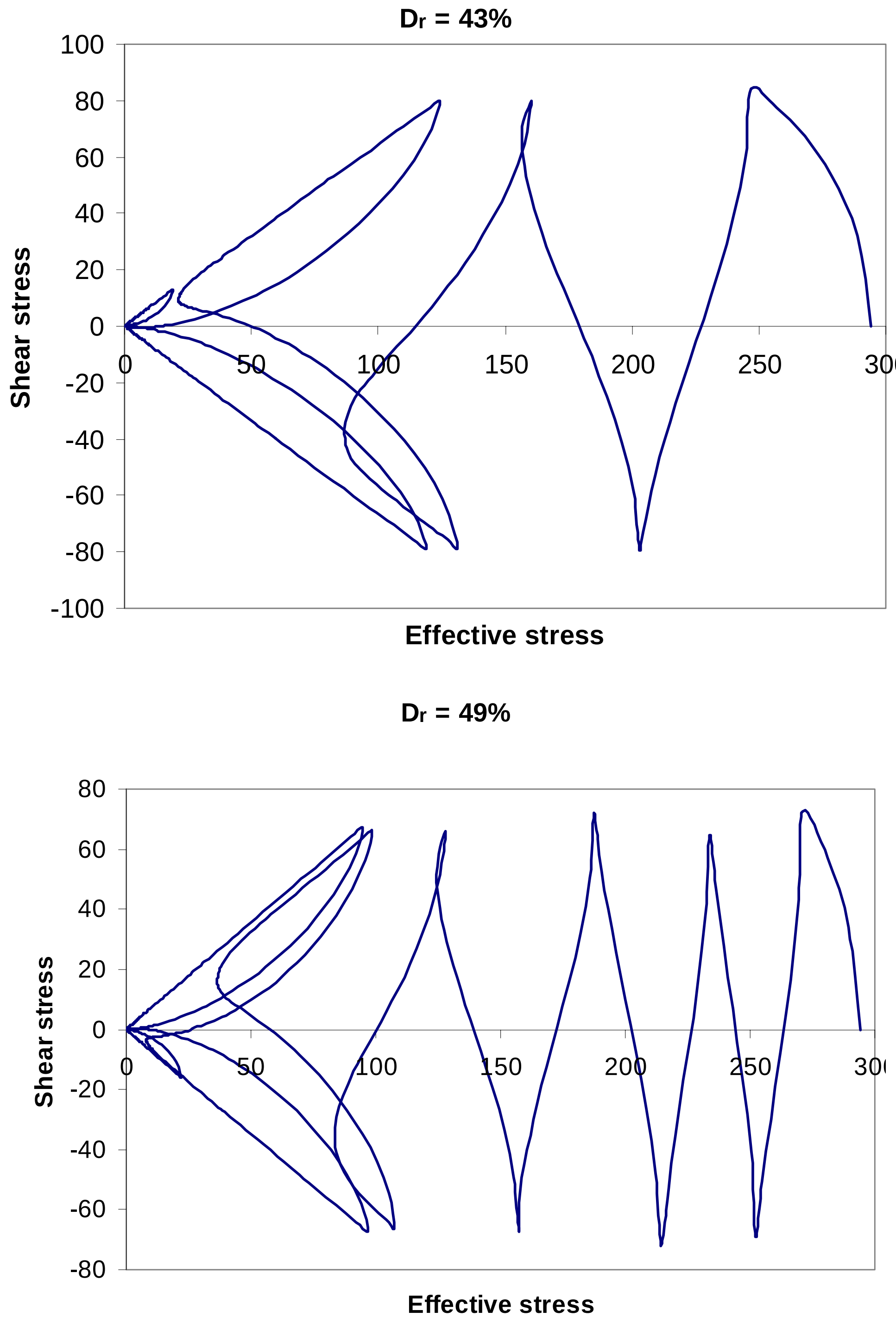


Fig.5.17. shear stress versus effective stress curves by DIANA

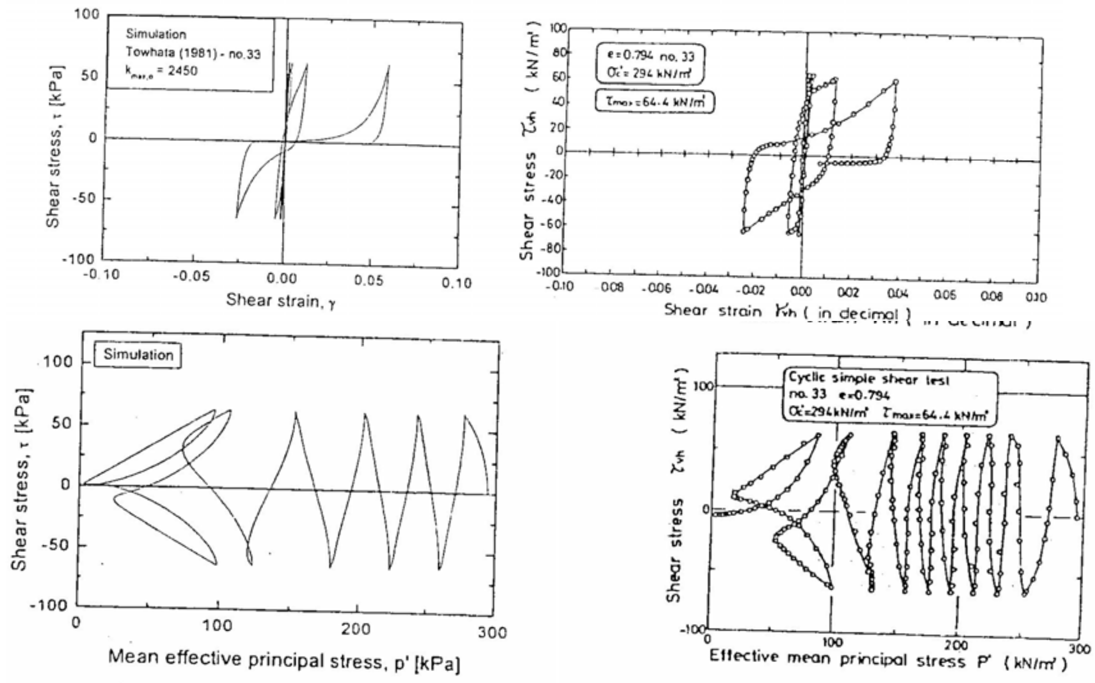


Fig.5.18. shear stress versus shear strain and shear stress versus effective stress curves by Nishimura (2002) and Shahnazari (2001)

CHAPTER SIX

CONCLUSION AND RECOMMENDATION

6.1. CONCLUSION

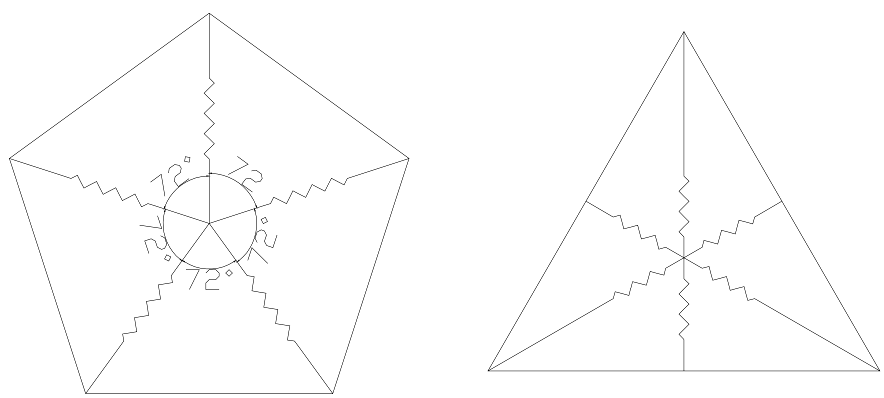
- Stress and strain vectors can be transformed between one coordinate system and another coordinate system. The inverse of transformation matrix for stress is equal to the transpose of strain transformation matrix.
- An icosahedron is the best geometrical element to distribute the planes. Nishimuara (2002) suggests that there will be distribution of six springs on each plane. However, comparison of maximum deviatoric stresses and (1.95)) in isotropic torsion shear test and from icosahedral distribution shows some disagreements between the two (as described in eq. (1.85), (1.90)). The likely reason for the discrepancy is the distribution of the springs on each plane with respect to an arbitrary coordinate system on the planes.
- The one-dimensional stress-strain relationships of the model give good prediction of the stress ratio. However, the stress dilatancy relationship results in an overestimation of contraction which there by affects the mean effective stress. The effect is also propagated into shear stress which is calculated from the product of mean effective stress and stress ratio.
- There is good agreement between the analytical calculation of stresses for a simple shear case and Diana result. This indicates that the fortran program is written correctly in accordance with the theoretical background given in chapter one.
- The comparison of shear stress-shear strain and volumetric strain-shear strain graphs from DIANA result and laboratory investigations show that there is a considerable differences. The main sources of these differences are:
    - anisotropic distribution of the springs on the virtual planes and
    - overestimation of contraction by the stress-dilatancy relationship
    - drawbacks in the MASING subroutine of the report for the case when the stress ratio in the past is exceeded.

6.2. RECOMMENDATION

- The stress-dilatancy relationship should be modified. The effect of the problem with the stress-dilatancy relationship is discussed in the stress-strain graphs in section 5.1.
- The springs on each plane should be distributed isotropically so that orientation is unique. One way to achieve a unique distribution of springs will be discussed below.

In chapter one, it is stated that the virtual planes are oriented perpendicular to the normal lines from the center of an icosahedron towards the corners and centroids of each face. Hence looking into an icosahedral element given in fig. 1.7, it can be noticed that each corner point is surrounded by five other corners. The lines connecting the corner point under consideration and the surrounding corners are unique for that particular corner point. Hence, the springs on the planes at the corners can be oriented along those lines. Hence there will be five springs on the planes which have their normal line pointing towards the corner points of an icosahedron.

The faces of an icosahedral element are equilateral triangles. Hence, the line directing from the centroids of each face towards each corner point and to the centers of each side are unique for each face. Thus six springs can be aligned along these directions: three towards the corners and three towards the bisectors of the sides. In this manner complete isotropy of the model can be achieved.



(a) (b)

Fig. 6.1 Distribution of springs (a) on the planes around the corner (b) on the planes at the centroid of each face.

- The MASING subroutine in the source-code should be modified for the case when the maximum stress ratio in the past is exceeded. Some suggestions are made in section 2.2.3 of the report.

- **References:**

1. Nishimura Satoshi, 2002:“Development of three-dimensional stress-strain model of sand undergoing cyclic undrained loading and stress-axes rotation” , Master Thesis

2. Nishimura Satoshi, Ikuo Towhata (2004): “ A three-dimensional stress-strain model of sand undergoing cyclic rotation of principal stress axes” ,Japanese Geotechnical Society, Vol. 44, No. 2, 103-116.

3. Susumu Iai (1993): “three-dimensional formulation and objectivity of a strain space multiple mechanism model for sand”, Japanese society of soil mechanics and foundation engineering, Vol. 33, No. 1, 192-199.

4. Habib Shahnazari, Ikuo Towhata (2002): ”Torsion shear tests on cyclic stress-dilatancy Relationship of sand” , Japanese Geotechnical society, Vol. 42, No. 1, 105-119.

Appendix A. Values of θ and ϕ for normal lines to each virtual plane.

Line number	θ	ϕ
1	0.00000000000	0.00000000000
2	1.57079632680	1.10714871780
3	0.31415926536	1.10714871780
4	5.34070751110	1.10714871780
5	4.08407044970	1.10714871780
6	2.82743338820	1.10714871780
7	0.94247779608	2.03444393580
8	5.96902604180	2.03444393580
9	4.71238898040	2.03444393580
10	3.45575191890	2.03444393580
11	2.19911485750	2.03444393580
12	0.00000000000	3.14159265360
13	0.94247779608	0.65235813978
14	5.96902604180	0.65235813978
15	4.71238898040	0.65235813978
16	3.45575191890	0.65235813978
17	2.19911485750	0.65235813978
18	1.57079632680	2.48923451380
19	0.31415926536	2.48923451380
20	5.34070751110	2.48923451380
21	4.08407044970	2.48923451380
22	2.82743338820	2.48923451380
23	0.94247779608	1.38208579600
24	5.96902604180	1.38208579600
25	4.71238898040	1.38208579600
26	3.45575191890	1.38208579600
27	2.19911485750	1.38208579600
28	0.31415926536	1.75950685760
29	5.34070751110	1.75950685760
30	4.08407044970	1.75950685760
31	2.82743338820	1.75950685760
32	1.57079632680	1.75950685760

Appendix B. Proof of transformation matrix properties using MAPLE program

The transformation matrix M between the unit vectors of two coordinate systems contains the direction cosines between those coordinate systems.It is defined as :

```
> with(LinearAlgebra) :
> M := Matrix([[l1, m1, n1], [ l2, m2, n2], [l3, m3, n3]])
```

$$M := \begin{bmatrix} l1 & m1 & n1 \\ l2 & m2 & n2 \\ l3 & m3 & n3 \end{bmatrix} \quad (1)$$

The strain tensor(considering its symmetry), denoted by E here is given as :

```
> E := Matrix([[Exx, Exy, Exz], [ Exy, Eyy, Eyz], [Exz, Eyz, Ezz]]);
```

$$E := \begin{bmatrix} Exx & Exy & Exz \\ Exy & Eyy & Eyz \\ Exz & Eyz & Ezz \end{bmatrix} \quad (2)$$

The stress tensor(considering its symmetry), denoted by S is given as :

```
> S := Matrix([[Sxx, Sxy, Szx], [ Sxy, Syy, Syz], [Szx, Syz, Szz]]);
```

$$S := \begin{bmatrix} Sxx & Sxy & Szx \\ Sxy & Syy & Syz \\ Szx & Syz & Szz \end{bmatrix} \quad (3)$$

The strain in the new coordinate system is obtained by the product $M \cdot E \cdot M^T$. This is calculated as :

```
> Enew := simplify(MatrixMatrixMultiply(M, MatrixMatrixMultiply(E, Transpose(M))));
```

$$\begin{aligned} Enew := [& [Exx\, l1^2 + 2\, l1\, Exy\, m1 + 2\, l1\, Exz\, n1 + Eyy\, m1^2 + 2\, m1\, Eyz\, n1 + Ezz\, n1^2, l1\, Exx\, l2 + l1\, Exy\, m2 + l1\, Exz\, n2 + m1\, Exy\, l2 + m1\, Eyy\, m2 + m1\, Eyz\, n2 \\ & + n1\, Exz\, l2 + n1\, Eyz\, m2 + n1\, Ezz\, n2, l1\, Exx\, l3 + l1\, Exy\, m3 + l1\, Exz\, n3 + m1\, Exy\, l3 + m1\, Eyy\, m3 + m1\, Eyz\, n3 + n1\, Exz\, l3 + n1\, Eyz\, m3 + n1\, Ezz\, n3], \\ & [l1\, Exx\, l2 + l1\, Exy\, m2 + l1\, Exz\, n2 + m1\, Exy\, l2 + m1\, Eyy\, m2 + m1\, Eyz\, n2 + n1\, Exz\, l2 + n1\, Eyz\, m2 + n1\, Ezz\, n2, Exx\, l2^2 + 2\, l2\, Exy\, m2 + 2\, l2\, Exz\, n2 \\ & + Eyy\, m2^2 + 2\, m2\, Eyz\, n2 + Ezz\, n2^2, l2\, Exx\, l3 + l2\, Exy\, m3 + l2\, Exz\, n3 + m2\, Exy\, l3 + m2\, Eyy\, m3 + m2\, Eyz\, n3 + n2\, Exz\, l3 + n2\, Eyz\, m3 + n2\, Ezz\, n3], \\ & [l1\, Exx\, l3 + l1\, Exy\, m3 + l1\, Exz\, n3 + m1\, Exy\, l3 + m1\, Eyy\, m3 + m1\, Eyz\, n3 + n1\, Exz\, l3 + n1\, Eyz\, m3 + n1\, Ezz\, n3, l2\, Exx\, l3 + l2\, Exy\, m3 + l2\, Exz\, n3 \\ & + m2\, Exy\, l3 + m2\, Eyy\, m3 + m2\, Eyz\, n3 + n2\, Exz\, l3 + n2\, Eyz\, m3 + n2\, Ezz\, n3, Exx\, l3^2 + 2\, l3\, Exy\, m3 + 2\, l3\, Exz\, n3 + Eyy\, m3^2 + 2\, m3\, Eyz\, n3 + Ezz\, n3^2]] \end{aligned} \quad (4)$$

Note that , this is a 3 X3 matrix and the strian components given are not the engineering strains.

Each strain components of the transformed strain tensor can be determined as follows :

```
> Enewxx := Row(Matrix([Column(Enew, 1)]), 1);
```

$$Enewxx := \left[\, Exx\, l1^2 + 2\, l1\, Exy\, m1 + 2\, l1\, Exz\, n1 + Eyy\, m1^2 + 2\, m1\, Eyz\, n1 + Ezz\, n1^2 \,\right] \quad (5)$$

```
> Enewyy := Row(Matrix([Column(Enew, 2)]), 2);
```

$$Enewyy := \left[\, Exx\, l2^2 + 2\, l2\, Exy\, m2 + 2\, l2\, Exz\, n2 + Eyy\, m2^2 + 2\, m2\, Eyz\, n2 + Ezz\, n2^2 \,\right] \quad (6)$$

```
> Enewzz := Row(Matrix([Column(Enew, 3)]), 3);
```

$$Enewzz := \left[\, Exx\, l3^2 + 2\, l3\, Exy\, m3 + 2\, l3\, Exz\, n3 + Eyy\, m3^2 + 2\, m3\, Eyz\, n3 + Ezz\, n3^2 \,\right] \quad (7)$$

```
> Enewxy := Row(Matrix([Column(Enew, 2)]), 1);
```

$$Enewxy := \left[\, l1\, Exx\, l2 + l1\, Exy\, m2 + l1\, Exz\, n2 + m1\, Exy\, l2 + m1\, Eyy\, m2 + m1\, Eyz\, n2 + n1\, Exz\, l2 + n1\, Eyz\, m2 + n1\, Ezz\, n2 \,\right] \quad (8)$$

```
> Enewyz := Row(Matrix([Column(Enew, 3)]), 2);
```

$$Enewyz := \left[\, l2\, Exx\, l3 + l2\, Exy\, m3 + l2\, Exz\, n3 + m2\, Exy\, l3 + m2\, Eyy\, m3 + m2\, Eyz\, n3 + n2\, Exz\, l3 + n2\, Eyz\, m3 + n2\, Ezz\, n3 \,\right] \quad (9)$$

```
> Enewzx := Row(Matrix([Column(Enew, 3)]), 1);
```

$$Enewzx := \left[\, l1\, Exx\, l3 + l1\, Exy\, m3 + l1\, Exz\, n3 + m1\, Exy\, l3 + m1\, Eyy\, m3 + m1\, Eyz\, n3 + n1\, Exz\, l3 + n1\, Eyz\, m3 + n1\, Ezz\, n3 \,\right] \quad (10)$$

From symmetry, the full strain tensor can be written.

Let us define the strain vector in the original coordinate system :

```
> Ev := Matrix([[Exx], [Eyy], [Ezz], [2· Exy], [2· Eyz], [2· Ezx]]);
```

$$Ev := \begin{bmatrix} Exx \\ Eyy \\ Ezz \\ 2\, Exy \\ 2\, Eyz \\ 2\, Ezx \end{bmatrix} \quad (11)$$

> Let us also define the engineering strain vector in the rotated coordinate system containing the elements of the strain tensor as :

>

>

> Evnew := Matrix([[Enewxx], [Enewyy], [Enewzz], [2· Enewxy], [2· Enewyz], [2· Enewzx]]);

$$Evnew := \begin{bmatrix} Exx\, l1^2 + 2\, l1\, Exy\, m1 + 2\, l1\, Exz\, n1 + Eyy\, m1^2 + 2\, m1\, Eyz\, n1 + Ezz\, n1^2 \\ Exx\, l2^2 + 2\, l2\, Exy\, m2 + 2\, l2\, Exz\, n2 + Eyy\, m2^2 + 2\, m2\, Eyz\, n2 + Ezz\, n2^2 \\ Exx\, l3^2 + 2\, l3\, Exy\, m3 + 2\, l3\, Exz\, n3 + Eyy\, m3^2 + 2\, m3\, Eyz\, n3 + Ezz\, n3^2 \\ 2\, l1\, Exx\, l2 + 2\, l1\, Exy\, m2 + 2\, l1\, Exz\, n2 + 2\, m1\, Exy\, l2 + 2\, m1\, Eyy\, m2 + 2\, m1\, Eyz\, n2 + 2\, n1\, Exz\, l2 + 2\, n1\, Eyz\, m2 + 2\, n1\, Ezz\, n2 \\ 2\, l2\, Exx\, l3 + 2\, l2\, Exy\, m3 + 2\, l2\, Exz\, n3 + 2\, m2\, Exy\, l3 + 2\, m2\, Eyy\, m3 + 2\, m2\, Eyz\, n3 + 2\, n2\, Exz\, l3 + 2\, n2\, Eyz\, m3 + 2\, n2\, Ezz\, n3 \\ 2\, l1\, Exx\, l3 + 2\, l1\, Exy\, m3 + 2\, l1\, Exz\, n3 + 2\, m1\, Exy\, l3 + 2\, m1\, Eyy\, m3 + 2\, m1\, Eyz\, n3 + 2\, n1\, Exz\, l3 + 2\, n1\, Eyz\, m3 + 2\, n1\, Ezz\, n3 \end{bmatrix} \quad (12)$$

> A transformation matrix Te can be set up which stransfers the strain vector in the oringinal coordinate system (Ev) into strain vector in the rotated coordinate system (Evnew) . The elements of this transformation matrix can be obtained as;

> Te11 := (coeff(Enewxx, Exx));

$$Te11 := \left[\, l1^2 \,\right] \quad (13)$$

> Te12 := (coeff(Enewxx, Eyy));

$$Te12 := \left[\, m1^2 \,\right] \quad (14)$$

> Te13 := (coeff(Enewxx, Ezz));

$$Te13 := \left[\, n1^2 \,\right] \quad (15)$$

> Te14 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewxx}, \mathrm{Exy}\right)\right)$;

$$Te14 := \left[\, l1\, m1 \,\right] \quad (16)$$

> Te15 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewxx}, \mathrm{Eyz}\right)\right)$;

$$Te15 := \left[\, m1\, n1 \,\right] \quad (17)$$

> Te16 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewxx}, \mathrm{Exz}\right)\right)$;

$$Te16 := \left[\, l1\, n1 \,\right] \quad (18)$$

> Te21 := (coeff(Enewyy, Exx));

$$Te21 := \left[\, l2^2 \,\right] \quad (19)$$

> Te22 := (coeff(Enewyy, Eyy));

$$Te22 := \left[\, m2^2 \,\right] \quad (20)$$

> Te23 := (coeff(Enewyy, Ezz));

$$Te23 := \left[\, n2^2 \,\right] \quad (21)$$

> Te24 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewyy}, \mathrm{Exy}\right)\right)$;

$$Te24 := \left[\, l2\, m2 \,\right] \quad (22)$$

> Te25 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewyy}, \mathrm{Eyz}\right)\right)$;

$$Te25 := \left[\, m2\, n2 \,\right] \quad (23)$$

> Te26 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewyy}, \mathrm{Exz}\right)\right)$;

$$Te26 := \left[\, l2\, n2 \,\right] \quad (24)$$

> Te31 := (coeff(Enewzz, Exx));

$$Te31 := \left[\, l3^2 \,\right] \quad (25)$$

> Te32 := (coeff(Enewzz, Eyy));

$$Te32 := \left[\, m3^2 \,\right] \quad (26)$$

> Te33 := (coeff(Enewzz, Ezz));

$$Te33 := \left[\, n3^2 \,\right] \quad (27)$$

> Te34 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewzz}, \mathrm{Exy}\right)\right)$;

$$Te34 := \left[\, l3\, m3 \,\right] \quad (28)$$

> Te35 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewzz}, \mathrm{Eyz}\right)\right)$;

$$Te35 := \left[\, m3\, n3 \,\right] \quad (29)$$

> Te36 := $\left(\mathrm{coeff}\left(\frac{1}{2}\cdot \mathrm{Enewzz}, \mathrm{Exz}\right)\right)$;

$$Te36 := \left[\, l3\, n3 \,\right] \quad (30)$$

> Te41 := (coeff(2· Enewxy, Exx));

> Te36 := (coeff(1/2 · Enewzz, Exz));

$$Te36 := \left[\, l3\, n3 \,\right] \quad (30)$$

> Te41 := (coeff(2· Enewxy, Exx));

$$Te41 := \left[\, 2\, l1\, l2 \,\right] \quad (31)$$

> Te42 := (coeff(2·Enewxy, Eyy));

$$Te42 := \left[\, 2\, m1\, m2 \,\right] \quad (32)$$

> Te43 := (coeff(2·Enewxy, Ezz));

$$Te43 := \left[\, 2\, n1\, n2 \,\right] \quad (33)$$

> Te44 := (coeff(Enewxy, Exy));

$$Te44 := \left[\, l1\, m2 + m1\, l2 \,\right] \quad (34)$$

> Te45 := (coeff(Enewxy, Eyz));

$$Te45 := \left[\, m1\, n2 + n1\, m2 \,\right] \quad (35)$$

> Te46 := (coeff(Enewxy, Exz));

$$Te46 := \left[\, l1\, n2 + n1\, l2 \,\right] \quad (36)$$

> Te51 := (coeff(2·Enewyz, Exx));

$$Te51 := \left[\, 2\, l2\, l3 \,\right] \quad (37)$$

> Te52 := (coeff(2·Enewyz, Eyy));

$$Te52 := \left[\, 2\, m2\, m3 \,\right] \quad (38)$$

> Te53 := (coeff(2·Enewyz, Ezz));

$$Te53 := \left[\, 2\, n2\, n3 \,\right] \quad (39)$$

> Te54 := (coeff(Enewyz, Exy));

$$Te54 := \left[\, l2\, m3 + m2\, l3 \,\right] \quad (40)$$

> Te55 := (coeff(Enewyz, Eyz));

$$Te55 := \left[\, m2\, n3 + n2\, m3 \,\right] \quad (41)$$

> Te56 := (coeff(Enewyz, Exz));

$$Te56 := \left[\, l2\, n3 + n2\, l3 \,\right] \quad (42)$$

> Te61 := (coeff(2·Enewzx, Exx));

$$Te61 := \left[\, 2\, l1\, l3 \,\right] \quad (43)$$

> Te62 := (coeff(2·Enewzx, Eyy));

$$Te62 := \left[\, 2\, m1\, m3 \,\right] \quad (44)$$

> Te63 := (coeff(2·Enewzx, Ezz));

$$Te63 := \left[\, 2\, n1\, n3 \,\right] \quad (45)$$

> Te64 := (coeff(Enewzx, Exy));

$$Te64 := \left[\, l1\, m3 + m1\, l3 \,\right] \quad (46)$$

> Te65 := (coeff(Enewzx, Eyz));

$$Te65 := \left[\, m1\, n3 + n1\, m3 \,\right] \quad (47)$$

> Te66 := (coeff(Enewzx, Exz));

$$Te66 := \left[\, l1\, n3 + n1\, l3 \,\right] \quad (48)$$

>
>
> **Then the 6 X6 strain transformation matrix Te can be written as :**
>
> *Te* := *Matrix*([[Te11, Te12, Te13, Te14, Te15, Te16], [Te21, Te22, Te23, Te24, Te25, Te26], [Te31, Te32, Te33, Te34, Te35, Te36], [Te41, Te42, Te43, Te44, Te45, Te46], [Te51, Te52, Te53, Te54, Te55, Te56], [Te61, Te62, Te63, Te64, Te65, Te66]])

$$Te := \begin{bmatrix} l1^2 & m1^2 & n1^2 & l1\, m1 & m1\, n1 & l1\, n1 \\ l2^2 & m2^2 & n2^2 & l2\, m2 & m2\, n2 & l2\, n2 \\ l3^2 & m3^2 & n3^2 & l3\, m3 & m3\, n3 & l3\, n3 \\ 2\, l1\, l2 & 2\, m1\, m2 & 2\, n1\, n2 & l1\, m2 + m1\, l2 & m1\, n2 + n1\, m2 & l1\, n2 + n1\, l2 \\ 2\, l2\, l3 & 2\, m2\, m3 & 2\, n2\, n3 & l2\, m3 + m2\, l3 & m2\, n3 + n2\, m3 & l2\, n3 + n2\, l3 \\ 2\, l1\, l3 & 2\, m1\, m3 & 2\, n1\, n3 & l1\, m3 + m1\, l3 & m1\, n3 + n1\, m3 & l1\, n3 + n1\, l3 \end{bmatrix} \quad (49)$$

> **This is the transformation matrix engineering strains between the original and the new coordinate system . The inverse transformation matrix for engineering strains can also be developed in a similar manner** $(M \cdot S.M^T)$ **:**
>
>
> **First let us define the stress in the rotated coordinate system as :**
>
> En := *Matrix*([[Enxx, Enxy, Enzx], [Enxy, Enyy, Enyz], [Enzx, Enyz, Enzz]]);

$$En := \begin{bmatrix} Enxx & Enxy & Enzx \\ Enxy & Enyy & Enyz \\ Enzx & Enyz & Enzz \end{bmatrix} \quad (50)$$

> **Then the stress tensor in the original coordinate system can be calculated by $(M^T \cdot Sn \cdot M)$:**

> EE := simplify(MatrixMatrixMultiply(Transpose(M), MatrixMatrixMultiply(En, M)));

$$EE := [[Enxx\, l1^2 + 2\, l1\, Enxy\, l2 + 2\, l1\, Enzx\, l3 + Enyy\, l2^2 + 2\, l2\, Enyz\, l3 + Enzz\, l3^2, l1\, Enxx\, m1 + l1\, Enxy\, m2 + l1\, Enzx\, m3 + l2\, Enxy\, m1 + l2\, Enyy\, m2 + l2\, Enyz\, m3 + l3\, Enzx\, m1 + l3\, Enyz\, m2 + l3\, Enzz\, m3, l1\, Enxx\, n1 + l1\, Enxy\, n2 + l1\, Enzx\, n3 + l2\, Enxy\, n1 + l2\, Enyy\, n2 + l2\, Enyz\, n3 + l3\, Enzx\, n1 + l3\, Enyz\, n2 + l3\, Enzz\, n3], [l1\, Enxx\, m1 + l1\, Enxy\, m2 + l1\, Enzx\, m3 + l2\, Enxy\, m1 + l2\, Enyy\, m2 + l2\, Enyz\, m3 + l3\, Enzx\, m1 + l3\, Enyz\, m2 + l3\, Enzz\, m3, Enxx\, m1^2 + 2\, m1\, Enxy\, m2 + 2\, m1\, Enzx\, m3 + Enyy\, m2^2 + 2\, m2\, Enyz\, m3 + Enzz\, m3^2, m1\, Enxx\, n1 + m1\, Enxy\, n2 + m1\, Enzx\, n3 + m2\, Enxy\, n1 + m2\, Enyy\, n2 + m2\, Enyz\, n3 + m3\, Enzx\, n1 + m3\, Enyz\, n2 + m3\, Enzz\, n3], [l1\, Enxx\, n1 + l1\, Enxy\, n2 + l1\, Enzx\, n3 + l2\, Enxy\, n1 + l2\, Enyy\, n2 + l2\, Enyz\, n3 + l3\, Enzx\, n1 + l3\, Enyz\, n2 + l3\, Enzz\, n3, m1\, Enxx\, n1 + m1\, Enxy\, n2 + m1\, Enzx\, n3 + m2\, Enxy\, n1 + m2\, Enyy\, n2 + m2\, Enyz\, n3 + m3\, Enzx\, n1 + m3\, Enyz\, n2 + m3\, Enzz\, n3, Enxx\, n1^2 + 2\, n1\, Enxy\, n2 + 2\, n1\, Enzx\, n3 + Enyy\, n2^2 + 2\, n2\, Enyz\, n3 + Enzz\, n3^2]] \quad (51)$$

> **This is a 3 X3 matrix containing elements of the stress tensor in the original coordinate system. Each element can be extracted from this matrix as :**

> EExx := Row(Matrix([Column(EE, 1)]), 1);

$$EExx := \left[\, Enxx\, l1^2 + 2\, l1\, Enxy\, l2 + 2\, l1\, Enzx\, l3 + Enyy\, l2^2 + 2\, l2\, Enyz\, l3 + Enzz\, l3^2 \,\right] \quad (52)$$

> EEyy := Row(Matrix([Column(EE, 2)]), 2);

$$EEyy := \left[\, Enxx\, m1^2 + 2\, m1\, Enxy\, m2 + 2\, m1\, Enzx\, m3 + Enyy\, m2^2 + 2\, m2\, Enyz\, m3 + Enzz\, m3^2 \,\right] \quad (53)$$

> EEzz := Row(Matrix([Column(EE, 3)]), 3);

$$EEzz := \left[\, Enxx\, n1^2 + 2\, n1\, Enxy\, n2 + 2\, n1\, Enzx\, n3 + Enyy\, n2^2 + 2\, n2\, Enyz\, n3 + Enzz\, n3^2 \,\right] \quad (54)$$

> EExy := Row(Matrix([Column(EE, 2)]), 1);

$$EExy := \left[\, l1\, Enxx\, m1 + l1\, Enxy\, m2 + l1\, Enzx\, m3 + l2\, Enxy\, m1 + l2\, Enyy\, m2 + l2\, Enyz\, m3 + l3\, Enzx\, m1 + l3\, Enyz\, m2 + l3\, Enzz\, m3 \,\right] \quad (55)$$

> EEyz := Row(Matrix([Column(EE, 3)]), 2);

$$EEyz := \left[\, m1\, Enxx\, n1 + m1\, Enxy\, n2 + m1\, Enzx\, n3 + m2\, Enxy\, n1 + m2\, Enyy\, n2 + m2\, Enyz\, n3 + m3\, Enzx\, n1 + m3\, Enyz\, n2 + m3\, Enzz\, n3 \,\right] \quad (56)$$

> EEzx := Row(Matrix([Column(EE, 3)]), 1);

$$EEzx := \left[\, l1\, Enxx\, n1 + l1\, Enxy\, n2 + l1\, Enzx\, n3 + l2\, Enxy\, n1 + l2\, Enyy\, n2 + l2\, Enyz\, n3 + l3\, Enzx\, n1 + l3\, Enyz\, n2 + l3\, Enzz\, n3 \,\right] \quad (57)$$

> **If we define a stress vector in the rotated (Svrotated) and original (SSv)coordinate system as :**

> Evrotated := Matrix([[Erxx], [Eryy], [Erzz], [2· Erxy], [2· Eryz], [2· Erzx]]);

$$Evrotated := \begin{bmatrix} Erxx \\ Eryy \\ Erzz \\ 2\, Erxy \\ 2\, Eryz \\ 2\, Erzx \end{bmatrix} \quad (58)$$

> EEv := Matrix([[EExx], [EEyy], [EEzz], [2· EExy], [2· EEyz], [2· EEzx]]);

$$EEv := \begin{bmatrix} Enxx\, l1^2 + 2\, l1\, Enxy\, l2 + 2\, l1\, Enzx\, l3 + Enyy\, l2^2 + 2\, l2\, Enyz\, l3 + Enzz\, l3^2 \\ Enxx\, m1^2 + 2\, m1\, Enxy\, m2 + 2\, m1\, Enzx\, m3 + Enyy\, m2^2 + 2\, m2\, Enyz\, m3 + Enzz\, m3^2 \\ Enxx\, n1^2 + 2\, n1\, Enxy\, n2 + 2\, n1\, Enzx\, n3 + Enyy\, n2^2 + 2\, n2\, Enyz\, n3 + Enzz\, n3^2 \\ 2\, l1\, Enxx\, m1 + 2\, l1\, Enxy\, m2 + 2\, l1\, Enzx\, m3 + 2\, l2\, Enxy\, m1 + 2\, l2\, Enyy\, m2 + 2\, l2\, Enyz\, m3 + 2\, l3\, Enzx\, m1 + 2\, l3\, Enyz\, m2 + 2\, l3\, Enzz\, m3 \\ 2\, m1\, Enxx\, n1 + 2\, m1\, Enxy\, n2 + 2\, m1\, Enzx\, n3 + 2\, m2\, Enxy\, n1 + 2\, m2\, Enyy\, n2 + 2\, m2\, Enyz\, n3 + 2\, m3\, Enzx\, n1 + 2\, m3\, Enyz\, n2 + 2\, m3\, Enzz\, n3 \\ 2\, l1\, Enxx\, n1 + 2\, l1\, Enxy\, n2 + 2\, l1\, Enzx\, n3 + 2\, l2\, Enxy\, n1 + 2\, l2\, Enyy\, n2 + 2\, l2\, Enyz\, n3 + 2\, l3\, Enzx\, n1 + 2\, l3\, Enyz\, n2 + 2\, l3\, Enzz\, n3 \end{bmatrix} \quad (59)$$

> **A transformation matrix which transforms the stress vector in the rotated coordinate sytem to stress vector in the original coordinate system can be set up . This transformation matrix is nothing but the inverse of the previously established stress transformation matrix (Ts) . Hence it will be called Tsinverse. The elements of Tsinverse can be determined as :**

> TEinverse11 := (coeff(EExx, Enxx));

$$TEinverse11 := \left[\, l1^2 \,\right] \quad (60)$$

> TEinverse12 := (coeff(EExx, Enyy));

$$TEinverse12 := \left[\, l2^2 \,\right] \quad (61)$$

> TEinverse13 := (coeff(EExx, Enzz));

> TEinverse13 := (coeff(EExx, Enzz));

$$TEinverse13 := [\, l3^2 \,] \tag{62}$$

> TEinverse14 := (coeff(1/2 · EExx, Enxy));

$$TEinverse14 := [\, l1\, l2 \,] \tag{63}$$

> TEinverse15 := (coeff(1/2 · EExx, Enyz));

$$TEinverse15 := [\, l2\, l3 \,] \tag{64}$$

> TEinverse16 := (coeff(1/2 · EExx, Enzx));

$$TEinverse16 := [\, l1\, l3 \,] \tag{65}$$

> TEinverse21 := (coeff(EEyy, Enxx));

$$TEinverse21 := [\, m1^2 \,] \tag{66}$$

> TEinverse22 := (coeff(EEyy, Enyy));

$$TEinverse22 := [\, m2^2 \,] \tag{67}$$

> TEinverse23 := (coeff(EEyy, Enzz));

$$TEinverse23 := [\, m3^2 \,] \tag{68}$$

> TEinverse24 := (coeff(1/2 · EEyy, Enxy));

$$TEinverse24 := [\, m1\, m2 \,] \tag{69}$$

> TEinverse25 := (coeff(1/2 · EEyy, Enyz));

$$TEinverse25 := [\, m2\, m3 \,] \tag{70}$$

> TEinverse26 := (coeff(1/2 · EEyy, Enzx));

$$TEinverse26 := [\, m1\, m3 \,] \tag{71}$$

> TEinverse31 := (coeff(EEzz, Enxx));

$$TEinverse31 := [\, n1^2 \,] \tag{72}$$

> TEinverse32 := (coeff(EEzz, Enyy));

$$TEinverse32 := [\, n2^2 \,] \tag{73}$$

> TEinverse33 := (coeff(EEzz, Enzz));

$$TEinverse33 := [\, n3^2 \,] \tag{74}$$

> TEinverse34 := (coeff(1/2 · EEzz, Enxy));

$$TEinverse34 := [\, n1\, n2 \,] \tag{75}$$

> TEinverse35 := (coeff(1/2 · EEzz, Enyz));

$$TEinverse35 := [\, n2\, n3 \,] \tag{76}$$

> TEinverse36 := (coeff(1/2 · EEzz, Enzx));

$$TEinverse36 := [\, n1\, n3 \,] \tag{77}$$

> TEinverse41 := (coeff(2 · EExy, Enxx));

$$TEinverse41 := [\, 2\, l1\, m1 \,] \tag{78}$$

> TEinverse42 := (coeff(2 · EExy, Enyy));

$$TEinverse42 := [\, 2\, l2\, m2 \,] \tag{79}$$

> TEinverse43 := (coeff(2 · EExy, Enzz));

$$TEinverse43 := [\, 2\, l3\, m3 \,] \tag{80}$$

> TEinverse44 := (coeff(EExy, Enxy));

$$TEinverse44 := [\, l1\, m2 + m1\, l2 \,] \tag{81}$$

> TEinverse45 := (coeff(EExy, Enyz));

$$TEinverse45 := [\, l2\, m3 + m2\, l3 \,] \tag{82}$$

> TEinverse46 := (coeff(EExy, Enzx));

$$TEinverse46 := [\, l1\, m3 + m1\, l3 \,] \tag{83}$$

> TEinverse51 := (coeff(2 · EEyz, Enxx));

$$TEinverse51 := [\, 2\, m1\, n1 \,] \tag{84}$$

> TEinverse52 := (coeff(2 · EEyz, Enyy));

$$TEinverse52 := [\, 2\, m2\, n2 \,] \tag{85}$$

> TEinverse53 := (coeff(2 · EEyz, Enzz));

$$TEinverse53 := [\, 2\, m3\, n3 \,] \tag{86}$$

> TEinverse54 := (coeff(EEyz, Enxy));

$$TEinverse54 := \left[\, m1\, n2 + n1\, m2 \,\right] \tag{87}$$

> TEinverse55 := (coeff(EEyz, Enyz));

$$TEinverse55 := \left[\, m2\, n3 + n2\, m3 \,\right] \tag{88}$$

> TEinverse56 := (coeff(EEyz, Enzx));

$$TEinverse56 := \left[\, m1\, n3 + n1\, m3 \,\right] \tag{89}$$

> TEinverse61 := (coeff(2·EEzx, Enxx));

$$TEinverse61 := \left[\, 2\, l1\, n1 \,\right] \tag{90}$$

> TEinverse62 := (coeff(2·EEzx, Enyy));

$$TEinverse62 := \left[\, 2\, l2\, n2 \,\right] \tag{91}$$

> TEinverse63 := (coeff(2·EEzx, Enzz));

$$TEinverse63 := \left[\, 2\, l3\, n3 \,\right] \tag{92}$$

> TEinverse64 := (coeff(EEzx, Enxy));

$$TEinverse64 := \left[\, l1\, n2 + n1\, l2 \,\right] \tag{93}$$

> TEinverse65 := (coeff(EEzx, Enyz));

$$TEinverse65 := \left[\, l2\, n3 + n2\, l3 \,\right] \tag{94}$$

> TEinverse66 := (coeff(EEzx, Enzx));

$$TEinverse66 := \left[\, l1\, n3 + n1\, l3 \,\right] \tag{95}$$

>

> **Then the inverse of the stress transformation matrix can be given as :**

> *TEinverse* := *Matrix*([[TEinverse11, TEinverse12, TEinverse13, TEinverse14, TEinverse15, TEinverse16], [TEinverse21, TEinverse22, TEinverse23, TEinverse24, TEinverse25, TEinverse26], [TEinverse31, TEinverse32, TEinverse33, TEinverse34, TEinverse35, TEinverse36], [TEinverse41, TEinverse42, TEinverse43, TEinverse44, TEinverse45, TEinverse46], [TEinverse51, TEinverse52, TEinverse53, TEinverse54, TEinverse55, TEinverse56], [TEinverse61, TEinverse62, TEinverse63, TEinverse64, TEinverse65, TEinverse66]])

$$TEinverse := \begin{bmatrix} l1^2 & l2^2 & l3^2 & l1\, l2 & l2\, l3 & l1\, l3 \\ m1^2 & m2^2 & m3^2 & m1\, m2 & m2\, m3 & m1\, m3 \\ n1^2 & n2^2 & n3^2 & n1\, n2 & n2\, n3 & n1\, n3 \\ 2\, l1\, m1 & 2\, l2\, m2 & 2\, l3\, m3 & l1\, m2 + m1\, l2 & l2\, m3 + m2\, l3 & l1\, m3 + m1\, l3 \\ 2\, m1\, n1 & 2\, m2\, n2 & 2\, m3\, n3 & m1\, n2 + n1\, m2 & m2\, n3 + n2\, m3 & m1\, n3 + n1\, m3 \\ 2\, l1\, n1 & 2\, l2\, n2 & 2\, l3\, n3 & l1\, n2 + n1\, l2 & l2\, n3 + n2\, l3 & l1\, n3 + n1\, l3 \end{bmatrix} \tag{96}$$

>

>

> **This is the inverse transformation matrix for engineering strains between the original and the new coordinate system . The transformation matrix for stresses can also be developed in a similar manner** $(M \cdot S.M^T)$.

>

> *Snew* := *simplify*(*MatrixMatrixMultiply*(*M*, *MatrixMatrixMultiply*(*S*, *Transpose*(*M*))));

$$\begin{aligned} Snew := [&[Sxx\, l1^2 + 2\, l1\, Sxy\, m1 + 2\, l1\, Szx\, n1 + Syy\, m1^2 + 2\, m1\, Syz\, n1 + Szz\, n1^2, l1\, Sxx\, l2 + l1\, Sxy\, m2 + l1\, Szx\, n2 + m1\, Sxy\, l2 + m1\, Syy\, m2 + m1\, Syz\, n2 \\ &+ n1\, Szx\, l2 + n1\, Syz\, m2 + n1\, Szz\, n2, l1\, Sxx\, l3 + l1\, Sxy\, m3 + l1\, Szx\, n3 + m1\, Sxy\, l3 + m1\, Syy\, m3 + m1\, Syz\, n3 + n1\, Szx\, l3 + n1\, Syz\, m3 + n1\, Szz\, n3], \\ &[l1\, Sxx\, l2 + l1\, Sxy\, m2 + l1\, Szx\, n2 + m1\, Sxy\, l2 + m1\, Syy\, m2 + m1\, Syz\, n2 + n1\, Szx\, l2 + n1\, Syz\, m2 + n1\, Szz\, n2, Sxx\, l2^2 + 2\, l2\, Sxy\, m2 + 2\, l2\, Szx\, n2 \\ &+ Syy\, m2^2 + 2\, m2\, Syz\, n2 + Szz\, n2^2, l2\, Sxx\, l3 + l2\, Sxy\, m3 + l2\, Szx\, n3 + m2\, Sxy\, l3 + m2\, Syy\, m3 + m2\, Syz\, n3 + n2\, Szx\, l3 + n2\, Syz\, m3 + n2\, Szz\, n3], \\ &[l1\, Sxx\, l3 + l1\, Sxy\, m3 + l1\, Szx\, n3 + m1\, Sxy\, l3 + m1\, Syy\, m3 + m1\, Syz\, n3 + n1\, Szx\, l3 + n1\, Syz\, m3 + n1\, Szz\, n3, l2\, Sxx\, l3 + l2\, Sxy\, m3 + l2\, Szx\, n3 \\ &+ m2\, Sxy\, l3 + m2\, Syy\, m3 + m2\, Syz\, n3 + n2\, Szx\, l3 + n2\, Syz\, m3 + n2\, Szz\, n3, Sxx\, l3^2 + 2\, l3\, Sxy\, m3 + 2\, l3\, Szx\, n3 + Syy\, m3^2 + 2\, m3\, Syz\, n3 + Szz\, n3^2]] \end{aligned} \tag{97}$$

> **This is a 3 X3 matrix containing the components of stress tensor in the new coordinate system . Each component can be exracted as follows :**

>

> *Snewxx* := *Row*(*Matrix*([*Column*(Snew, 1)]), 1);

$$Snewxx := \left[\, Sxx\, l1^2 + 2\, l1\, Sxy\, m1 + 2\, l1\, Szx\, n1 + Syy\, m1^2 + 2\, m1\, Syz\, n1 + Szz\, n1^2 \,\right] \tag{98}$$

> *Snewyy* := *Row*(*Matrix*([*Column*(Snew, 2)]), 2);

$$Snewyy := \left[\, Sxx\, l2^2 + 2\, l2\, Sxy\, m2 + 2\, l2\, Szx\, n2 + Syy\, m2^2 + 2\, m2\, Syz\, n2 + Szz\, n2^2 \,\right] \tag{99}$$

> Snewzz := *Row*(*Matrix*([*Column*(Snew, 3)]), 3);

$$Snewzz := \left[\, Sxx\, l3^2 + 2\, l3\, Sxy\, m3 + 2\, l3\, Szx\, n3 + Syy\, m3^2 + 2\, m3\, Syz\, n3 + Szz\, n3^2 \,\right] \tag{100}$$

> *Snewxy* := *Row*(*Matrix*([*Column*(Snew, 2)]), 1);

$$Snewxy := \left[\, l1\, Sxx\, l2 + l1\, Sxy\, m2 + l1\, Szx\, n2 + m1\, Sxy\, l2 + m1\, Syy\, m2 + m1\, Syz\, n2 + n1\, Szx\, l2 + n1\, Syz\, m2 + n1\, Szz\, n2 \,\right] \tag{101}$$

> *Snewyz* := *Row*(*Matrix*([*Column*(Snew, 3)]), 2);

$$Snewyz := \left[\, l2\, Sxx\, l3 + l2\, Sxy\, m3 + l2\, Szx\, n3 + m2\, Sxy\, l3 + m2\, Syy\, m3 + m2\, Syz\, n3 + n2\, Szx\, l3 + n2\, Syz\, m3 + n2\, Szz\, n3 \,\right] \tag{102}$$

> Snewzx := *Row*(*Matrix*([*Column*(Snew, 3)]), 1);

$$Snewzx := \left[\, l1\, Sxx\, l3 + l1\, Sxy\, m3 + l1\, Szx\, n3 + m1\, Sxy\, l3 + m1\, Syy\, m3 + m1\, Syz\, n3 + n1\, Szx\, l3 + n1\, Syz\, m3 + n1\, Szz\, n3 \,\right] \tag{103}$$

> The stress vector in the original coordinate system can be written as :

> Sv := Matrix([[Sxx], [Syy], [Szz], [Sxy], [Syz], [Szx]]);

$$Sv := \begin{bmatrix} Sxx \\ Syy \\ Szz \\ Sxy \\ Syz \\ Szx \end{bmatrix} \tag{104}$$

> Svnew := Matrix([[Snewxx], [Snewyy], [Snewzz], [Snewxy], [Snewyz], [Snewzx]]);

$$Svnew := \begin{bmatrix} Sxx\, l1^2 + 2\, l1\, Sxy\, m1 + 2\, l1\, Szx\, n1 + Syy\, m1^2 + 2\, m1\, Syz\, n1 + Szz\, n1^2 \\ Sxx\, l2^2 + 2\, l2\, Sxy\, m2 + 2\, l2\, Szx\, n2 + Syy\, m2^2 + 2\, m2\, Syz\, n2 + Szz\, n2^2 \\ Sxx\, l3^2 + 2\, l3\, Sxy\, m3 + 2\, l3\, Szx\, n3 + Syy\, m3^2 + 2\, m3\, Syz\, n3 + Szz\, n3^2 \\ l1\, Sxx\, l2 + l1\, Sxy\, m2 + l1\, Szx\, n2 + m1\, Sxy\, l2 + m1\, Syy\, m2 + m1\, Syz\, n2 + n1\, Szx\, l2 + n1\, Syz\, m2 + n1\, Szz\, n2 \\ l2\, Sxx\, l3 + l2\, Sxy\, m3 + l2\, Szx\, n3 + m2\, Sxy\, l3 + m2\, Syy\, m3 + m2\, Syz\, n3 + n2\, Szx\, l3 + n2\, Syz\, m3 + n2\, Szz\, n3 \\ l1\, Sxx\, l3 + l1\, Sxy\, m3 + l1\, Szx\, n3 + m1\, Sxy\, l3 + m1\, Syy\, m3 + m1\, Syz\, n3 + n1\, Szx\, l3 + n1\, Syz\, m3 + n1\, Szz\, n3 \end{bmatrix} \tag{105}$$

> A transformation matrix Ts can be set up which stransfers the stress vector in the oringinal coordinate system (Sv) into strain vector in the rotated coordinate system(Svnew) . The elements of this transformation matrix can be obtained as :

> Ts11 := (coeff(Snewxx, Sxx));

$$Ts11 := \left[\, l1^2 \,\right] \tag{106}$$

> Ts12 := (coeff(Snewxx, Syy));

$$Ts12 := \left[\, m1^2 \,\right] \tag{107}$$

> Ts13 := (coeff(Snewxx, Szz));

$$Ts13 := \left[\, n1^2 \,\right] \tag{108}$$

> Ts14 := (coeff(Snewxx, Sxy));

$$Ts14 := \left[\, 2\, l1\, m1 \,\right] \tag{109}$$

> Ts15 := (coeff(Snewxx, Syz));

$$Ts15 := \left[\, 2\, m1\, n1 \,\right] \tag{110}$$

> Ts16 := (coeff(Snewxx, Szx));

$$Ts16 := \left[\, 2\, l1\, n1 \,\right] \tag{111}$$

> Ts21 := (coeff(Snewyy, Sxx));

$$Ts21 := \left[\, l2^2 \,\right] \tag{112}$$

> Ts22 := (coeff(Snewyy, Syy));

$$Ts22 := \left[\, m2^2 \,\right] \tag{113}$$

> Ts23 := (coeff(Snewyy, Szz));

$$Ts23 := \left[\, n2^2 \,\right] \tag{114}$$

> Ts24 := (coeff(Snewyy, Sxy));

$$Ts24 := \left[\, 2\, l2\, m2 \,\right] \tag{115}$$

> Ts25 := (coeff(Snewyy, Syz));

$$Ts25 := \left[\, 2\, m2\, n2 \,\right] \tag{116}$$

> Ts26 := (coeff(Snewyy, Szx));

$$Ts26 := \left[\, 2\, l2\, n2 \,\right] \tag{117}$$

> Ts31 := (coeff(Snewzz, Sxx));

$$Ts31 := \left[\, l3^2 \,\right] \tag{118}$$

> Ts32 := (coeff(Snewzz, Syy));

$$Ts32 := \left[\, m3^2 \,\right] \tag{119}$$

> Ts33 := (coeff(Snewzz, Szz));

$$Ts33 := \left[\, n3^2 \,\right] \tag{120}$$

> Ts34 := (coeff(Snewzz, Sxy));

$$Ts34 := \left[\, 2\, l3\, m3 \,\right] \tag{121}$$

> Ts35 := (coeff(Snewzz, Syz));

$$Ts35 := \left[\, 2\, m3\, n3 \,\right] \tag{122}$$

$$Ts35 := \left[\ 2\, m3\, n3\ \right] \tag{122}$$

```
> Ts36 := (coeff( Snewzz, Szx));
```

$$Ts36 := \left[\ 2\, l3\, n3\ \right] \tag{123}$$

```
> Ts41 := (coeff(Snewxy, Sxx));
```

$$Ts41 := \left[\ l1\, l2\ \right] \tag{124}$$

```
> Ts42 := (coeff(Snewxy, Syy));
```

$$Ts42 := \left[\ m1\, m2\ \right] \tag{125}$$

```
> Ts43 := (coeff(Snewxy, Szz));
```

$$Ts43 := \left[\ n1\, n2\ \right] \tag{126}$$

```
> Ts44 := (coeff( Snewxy, Sxy));
```

$$Ts44 := \left[\ l1\, m2 + m1\, l2\ \right] \tag{127}$$

```
> Ts45 := (coeff(  Snewxy, Syz));
```

$$Ts45 := \left[\ m1\, n2 + n1\, m2\ \right] \tag{128}$$

```
> Ts46 := (coeff(  Snewxy, Szx));
```

$$Ts46 := \left[\ l1\, n2 + n1\, l2\ \right] \tag{129}$$

```
> Ts51 := (coeff(Snewyz, Sxx));
```

$$Ts51 := \left[\ l2\, l3\ \right] \tag{130}$$

```
> Ts52 := (coeff(Snewyz, Syy));
```

$$Ts52 := \left[\ m2\, m3\ \right] \tag{131}$$

```
> Ts53 := (coeff(Snewyz, Szz));
```

$$Ts53 := \left[\ n2\, n3\ \right] \tag{132}$$

```
> Ts54 := (coeff(  Snewyz, Sxy));
```

$$Ts54 := \left[\ l2\, m3 + m2\, l3\ \right] \tag{133}$$

```
> Ts55 := (coeff(  Snewyz, Syz));
```

$$Ts55 := \left[\ m2\, n3 + n2\, m3\ \right] \tag{134}$$

```
> Ts56 := (coeff(  Snewyz, Szx));
```

$$Ts56 := \left[\ l2\, n3 + n2\, l3\ \right] \tag{135}$$

```
> Ts61 := (coeff( Snewzx, Sxx));
```

$$Ts61 := \left[\ l1\, l3\ \right] \tag{136}$$

```
> Ts62 := (coeff( Snewzx, Syy));
```

$$Ts62 := \left[\ m1\, m3\ \right] \tag{137}$$

```
> Ts63 := (coeff(Snewzx, Szz));
```

$$Ts63 := \left[\ n1\, n3\ \right] \tag{138}$$

```
> Ts64 := (coeff(  Snewzx, Sxy));
```

$$Ts64 := \left[\ l1\, m3 + m1\, l3\ \right] \tag{139}$$

```
> Ts65 := (coeff(  Snewzx, Syz));
```

$$Ts65 := \left[\ m1\, n3 + n1\, m3\ \right] \tag{140}$$

```
> Ts66 := (coeff(  Snewzx, Szx));
```

$$Ts66 := \left[\ l1\, n3 + n1\, l3\ \right] \tag{141}$$

Then the 6 X6 stress transformation matrix Ts can be written as :

```
> Ts := Matrix([[Ts11, Ts12, Ts13, Ts14, Ts15, Ts16], [ Ts21, Ts22, Ts23, Ts24, Ts25, Ts26], [Ts31, Ts32, Ts33, Ts34, Ts35, Ts36], [Ts41, Ts42, Ts43, Ts44, Ts45,
    Ts46], [Ts51, Ts52, Ts53, Ts54, Ts55, Ts56], [Ts61, Ts62, Ts63, Ts64, Ts65, Ts66]])
```

$$Ts := \begin{bmatrix} l1^2 & m1^2 & n1^2 & 2\, l1\, m1 & 2\, m1\, n1 & 2\, l1\, n1 \\ l2^2 & m2^2 & n2^2 & 2\, l2\, m2 & 2\, m2\, n2 & 2\, l2\, n2 \\ l3^2 & m3^2 & n3^2 & 2\, l3\, m3 & 2\, m3\, n3 & 2\, l3\, n3 \\ l1\, l2 & m1\, m2 & n1\, n2 & l1\, m2 + m1\, l2 & m1\, n2 + n1\, m2 & l1\, n2 + n1\, l2 \\ l2\, l3 & m2\, m3 & n2\, n3 & l2\, m3 + m2\, l3 & m2\, n3 + n2\, m3 & l2\, n3 + n2\, l3 \\ l1\, l3 & m1\, m3 & n1\, n3 & l1\, m3 + m1\, l3 & m1\, n3 + n1\, m3 & l1\, n3 + n1\, l3 \end{bmatrix} \tag{142}$$

This is the transformation matrix to transform the stress vector in the original coordinate system to stress vector in the rotated coordinate system.The inverse of this transformation matrix can also be computed as follows :

First let us define the stress in the rotated coordinate system as :

```
> Sn := Matrix([[Snxx, Snxy, Snzx], [ Snxy, Snyy, Snyz], [Snzx, Snyz, Snzz]]);
```

$$Sn := \begin{bmatrix} Snxx & Snxy & Snzx \\ Snxy & Snyy & Snyz \\ Snzx & Snyz & Snzz \end{bmatrix} \tag{143}$$

> Then the stress tensor in the original coordinate system can be calculated by $(M^T \cdot Sn \cdot M)$:

> SS := simplify(MatrixMatrixMultiply(Transpose(M), MatrixMatrixMultiply(Sn, M)));

$$SS := [[Snxx\, l1^2 + 2\, l1\, Snxy\, l2 + 2\, l1\, Snzx\, l3 + Snyy\, l2^2 + 2\, l2\, Snyz\, l3 + Snzz\, l3^2, l1\, Snxx\, m1 + l1\, Snxy\, m2 + l1\, Snzx\, m3 + l2\, Snxy\, m1 + l2\, Snyy\, m2 + l2\, Snyz\, m3 + l3\, Snzx\, m1 + l3\, Snyz\, m2 + l3\, Snzz\, m3, l1\, Snxx\, n1 + l1\, Snxy\, n2 + l1\, Snzx\, n3 + l2\, Snxy\, n1 + l2\, Snyy\, n2 + l2\, Snyz\, n3 + l3\, Snzx\, n1 + l3\, Snyz\, n2 + l3\, Snzz\, n3], [l1\, Snxx\, m1 + l1\, Snxy\, m2 + l1\, Snzx\, m3 + l2\, Snxy\, m1 + l2\, Snyy\, m2 + l2\, Snyz\, m3 + l3\, Snzx\, m1 + l3\, Snyz\, m2 + l3\, Snzz\, m3, Snxx\, m1^2 + 2\, m1\, Snxy\, m2 + 2\, m1\, Snzx\, m3 + Snyy\, m2^2 + 2\, m2\, Snyz\, m3 + Snzz\, m3^2, m1\, Snxx\, n1 + m1\, Snxy\, n2 + m1\, Snzx\, n3 + m2\, Snxy\, n1 + m2\, Snyy\, n2 + m2\, Snyz\, n3 + m3\, Snzx\, n1 + m3\, Snyz\, n2 + m3\, Snzz\, n3], [l1\, Snxx\, n1 + l1\, Snxy\, n2 + l1\, Snzx\, n3 + l2\, Snxy\, n1 + l2\, Snyy\, n2 + l2\, Snyz\, n3 + l3\, Snzx\, n1 + l3\, Snyz\, n2 + l3\, Snzz\, n3, m1\, Snxx\, n1 + m1\, Snxy\, n2 + m1\, Snzx\, n3 + m2\, Snxy\, n1 + m2\, Snyy\, n2 + m2\, Snyz\, n3 + m3\, Snzx\, n1 + m3\, Snyz\, n2 + m3\, Snzz\, n3, Snxx\, n1^2 + 2\, n1\, Snxy\, n2 + 2\, n1\, Snzx\, n3 + Snyy\, n2^2 + 2\, n2\, Snyz\, n3 + Snzz\, n3^2]] \quad (144)$$

> This is a 3 X3 matrix containing elements of the stress tensor in the original coordinate system. Each element can be extracted from this matrix as :

> SSxx := Row(Matrix([Column(SS, 1)]), 1);

$$SSxx := [\, Snxx\, l1^2 + 2\, l1\, Snxy\, l2 + 2\, l1\, Snzx\, l3 + Snyy\, l2^2 + 2\, l2\, Snyz\, l3 + Snzz\, l3^2 \,] \quad (145)$$

> SSyy := Row(Matrix([Column(SS, 2)]), 2);

$$SSyy := [\, Snxx\, m1^2 + 2\, m1\, Snxy\, m2 + 2\, m1\, Snzx\, m3 + Snyy\, m2^2 + 2\, m2\, Snyz\, m3 + Snzz\, m3^2 \,] \quad (146)$$

> SSzz := Row(Matrix([Column(SS, 3)]), 3);

$$SSzz := [\, Snxx\, n1^2 + 2\, n1\, Snxy\, n2 + 2\, n1\, Snzx\, n3 + Snyy\, n2^2 + 2\, n2\, Snyz\, n3 + Snzz\, n3^2 \,] \quad (147)$$

> SSxy := Row(Matrix([Column(SS, 2)]), 1);

$$SSxy := [\, l1\, Snxx\, m1 + l1\, Snxy\, m2 + l1\, Snzx\, m3 + l2\, Snxy\, m1 + l2\, Snyy\, m2 + l2\, Snyz\, m3 + l3\, Snzx\, m1 + l3\, Snyz\, m2 + l3\, Snzz\, m3 \,] \quad (148)$$

> SSyz := Row(Matrix([Column(SS, 3)]), 2);

$$SSyz := [\, m1\, Snxx\, n1 + m1\, Snxy\, n2 + m1\, Snzx\, n3 + m2\, Snxy\, n1 + m2\, Snyy\, n2 + m2\, Snyz\, n3 + m3\, Snzx\, n1 + m3\, Snyz\, n2 + m3\, Snzz\, n3 \,] \quad (149)$$

> SSzx := Row(Matrix([Column(SS, 3)]), 1);

$$SSzx := [\, l1\, Snxx\, n1 + l1\, Snxy\, n2 + l1\, Snzx\, n3 + l2\, Snxy\, n1 + l2\, Snyy\, n2 + l2\, Snyz\, n3 + l3\, Snzx\, n1 + l3\, Snyz\, n2 + l3\, Snzz\, n3 \,] \quad (150)$$

> If we define a stress vector in the rotated (Svrotated) and original (SSv)coordinate system as :

> Svrotated := Matrix([[Srxx], [Sryy], [Srzz], [Srxy], [Sryz], [Srzx]]);

$$Svrotated := \begin{bmatrix} Srxx \\ Sryy \\ Srzz \\ Srxy \\ Sryz \\ Srzx \end{bmatrix} \quad (151)$$

> SSv := Matrix([[SSxx], [SSyy], [SSzz], [SSxy], [SSyz], [SSzx]]);

$$SSv := \begin{bmatrix} Snxx\, l1^2 + 2\, l1\, Snxy\, l2 + 2\, l1\, Snzx\, l3 + Snyy\, l2^2 + 2\, l2\, Snyz\, l3 + Snzz\, l3^2 \\ Snxx\, m1^2 + 2\, m1\, Snxy\, m2 + 2\, m1\, Snzx\, m3 + Snyy\, m2^2 + 2\, m2\, Snyz\, m3 + Snzz\, m3^2 \\ Snxx\, n1^2 + 2\, n1\, Snxy\, n2 + 2\, n1\, Snzx\, n3 + Snyy\, n2^2 + 2\, n2\, Snyz\, n3 + Snzz\, n3^2 \\ l1\, Snxx\, m1 + l1\, Snxy\, m2 + l1\, Snzx\, m3 + l2\, Snxy\, m1 + l2\, Snyy\, m2 + l2\, Snyz\, m3 + l3\, Snzx\, m1 + l3\, Snyz\, m2 + l3\, Snzz\, m3 \\ m1\, Snxx\, n1 + m1\, Snxy\, n2 + m1\, Snzx\, n3 + m2\, Snxy\, n1 + m2\, Snyy\, n2 + m2\, Snyz\, n3 + m3\, Snzx\, n1 + m3\, Snyz\, n2 + m3\, Snzz\, n3 \\ l1\, Snxx\, n1 + l1\, Snxy\, n2 + l1\, Snzx\, n3 + l2\, Snxy\, n1 + l2\, Snyy\, n2 + l2\, Snyz\, n3 + l3\, Snzx\, n1 + l3\, Snyz\, n2 + l3\, Snzz\, n3 \end{bmatrix} \quad (152)$$

> A transformation matrix which transforms the stress vector in the rotated coordinate sytem to stress vector in the original coordinate system can be set up . This transformation matrix is nothing but the inverse of the previously established stress transformation matrix(Ts) . Hence it will be called Tsinverse. The elements of Tsinverse can be determined as :

> Tsinverse11 := (coeff(SSxx, Snxx));

$$Tsinverse11 := [\, l1^2 \,] \quad (153)$$

> Tsinverse12 := (coeff(SSxx, Snyy));

$$Tsinverse12 := [\, l2^2 \,] \quad (154)$$

$$Tsinverse12 := \left[\, l2^2 \,\right] \quad (154)$$

```
> Tsinverse13 := (coeff(SSxx, Snzz));
```

$$Tsinverse13 := \left[\, l3^2 \,\right] \quad (155)$$

```
> Tsinverse14 := (coeff( SSxx, Snxy));
```

$$Tsinverse14 := \left[\, 2\, l1\, l2 \,\right] \quad (156)$$

```
> Tsinverse15 := (coeff( SSxx, Snyz));
```

$$Tsinverse15 := \left[\, 2\, l2\, l3 \,\right] \quad (157)$$

```
> Tsinverse16 := (coeff( SSxx, Snzx));
```

$$Tsinverse16 := \left[\, 2\, l1\, l3 \,\right] \quad (158)$$

```
> Tsinverse21 := (coeff(SSyy, Snxx));
```

$$Tsinverse21 := \left[\, m1^2 \,\right] \quad (159)$$

```
> Tsinverse22 := (coeff(SSyy, Snyy));
```

$$Tsinverse22 := \left[\, m2^2 \,\right] \quad (160)$$

```
> Tsinverse23 := (coeff(SSyy, Snzz));
```

$$Tsinverse23 := \left[\, m3^2 \,\right] \quad (161)$$

```
> Tsinverse24 := (coeff( SSyy, Snxy));
```

$$Tsinverse24 := \left[\, 2\, m1\, m2 \,\right] \quad (162)$$

```
> Tsinverse25 := (coeff( SSyy, Snyz));
```

$$Tsinverse25 := \left[\, 2\, m2\, m3 \,\right] \quad (163)$$

```
> Tsinverse26 := (coeff( SSyy, Snzx));
```

$$Tsinverse26 := \left[\, 2\, m1\, m3 \,\right] \quad (164)$$

```
> Tsinverse31 := (coeff(SSzz, Snxx));
```

$$Tsinverse31 := \left[\, n1^2 \,\right] \quad (165)$$

```
> Tsinverse32 := (coeff(SSzz, Snyy));
```

$$Tsinverse32 := \left[\, n2^2 \,\right] \quad (166)$$

```
> Tsinverse33 := (coeff(SSzz, Snzz));
```

$$Tsinverse33 := \left[\, n3^2 \,\right] \quad (167)$$

```
> Tsinverse34 := (coeff( SSzz, Snxy));
```

$$Tsinverse34 := \left[\, 2\, n1\, n2 \,\right] \quad (168)$$

```
> Tsinverse35 := (coeff( SSzz, Snyz));
```

$$Tsinverse35 := \left[\, 2\, n2\, n3 \,\right] \quad (169)$$

```
> Tsinverse36 := (coeff( SSzz, Snzx));
```

$$Tsinverse36 := \left[\, 2\, n1\, n3 \,\right] \quad (170)$$

```
> Tsinverse41 := (coeff(SSxy, Snxx));
```

$$Tsinverse41 := \left[\, l1\, m1 \,\right] \quad (171)$$

```
> Tsinverse42 := (coeff(SSxy, Snyy));
```

$$Tsinverse42 := \left[\, l2\, m2 \,\right] \quad (172)$$

```
> Tsinverse43 := (coeff(SSxy, Snzz));
```

$$Tsinverse43 := \left[\, l3\, m3 \,\right] \quad (173)$$

```
> Tsinverse44 := (coeff( SSxy, Snxy));
```

$$Tsinverse44 := \left[\, l1\, m2 + m1\, l2 \,\right] \quad (174)$$

```
> Tsinverse45 := (coeff(  SSxy, Snyz));
```

$$Tsinverse45 := \left[\, l2\, m3 + m2\, l3 \,\right] \quad (175)$$

```
> Tsinverse46 := (coeff(  SSxy, Snzx));
```

$$Tsinverse46 := \left[\, l1\, m3 + m1\, l3 \,\right] \quad (176)$$

```
> Tsinverse51 := (coeff(SSyz, Snxx));
```

$$Tsinverse51 := \left[\, m1\, n1 \,\right] \quad (177)$$

```
> Tsinverse52 := (coeff(SSyz, Snyy));
```

$$Tsinverse52 := \left[\, m2\, n2 \,\right] \quad (178)$$

```
> Tsinverse53 := (coeff(SSyz, Snzz));
```

$$Tsinverse53 := \left[\, m3\, n3 \,\right] \quad (179)$$

```
> Tsinverse54 := (coeff(  SSyz, Snxy));
```

$$Tsinverse54 := \left[\, m1\, n2 + n1\, m2 \,\right] \quad (180)$$

```
> Tsinverse55 := (coeff(  SSyz, Snyz));
```

$$Tsinverse55 := \left[\, m2\, n3 + n2\, m3 \,\right] \quad (181)$$

```
> Tsinverse56 := (coeff(  SSyz, Snzx));
```

$$Tsinverse56 := \left[\, m1\, n3 + n1\, m3 \,\right] \quad (182)$$

> Tsinverse61 := (coeff(SSzx, Snxx));

$$Tsinverse61 := [\ l1\ n1\ ] \quad (183)$$

> Tsinverse62 := (coeff(SSzx, Snyy));

$$Tsinverse62 := [\ l2\ n2\ ] \quad (184)$$

> Tsinverse63 := (coeff(SSzx, Snzz));

$$Tsinverse63 := [\ l3\ n3\ ] \quad (185)$$

> Tsinverse64 := (coeff(SSzx, Snxy));

$$Tsinverse64 := [\ l1\ n2 + n1\ l2\ ] \quad (186)$$

> Tsinverse65 := (coeff(SSzx, Snyz));

$$Tsinverse65 := [\ l2\ n3 + n2\ l3\ ] \quad (187)$$

> Tsinverse66 := (coeff(SSzx, Snzx));

$$Tsinverse66 := [\ l1\ n3 + n1\ l3\ ] \quad (188)$$

>

> Then the inverse of the stress transformation matrix can be given as :

> *Tsinverse* := *Matrix*([[Tsinverse11, Tsinverse12, Tsinverse13, Tsinverse14, Tsinverse15, Tsinverse16], [Tsinverse21, Tsinverse22, Tsinverse23, Tsinverse24, Tsinverse25, Tsinverse26], [Tsinverse31, Tsinverse32, Tsinverse33, Tsinverse34, Tsinverse35, Tsinverse36], [Tsinverse41, Tsinverse42, Tsinverse43, Tsinverse44, Tsinverse45, Tsinverse46], [Tsinverse51, Tsinverse52, Tsinverse53, Tsinverse54, Tsinverse55, Tsinverse56], [Tsinverse61, Tsinverse62, Tsinverse63, Tsinverse64, Tsinverse65, Tsinverse66]])

$$Tsinverse := \begin{bmatrix} l1^2 & l2^2 & l3^2 & 2\,l1\,l2 & 2\,l2\,l3 & 2\,l1\,l3 \\ m1^2 & m2^2 & m3^2 & 2\,m1\,m2 & 2\,m2\,m3 & 2\,m1\,m3 \\ n1^2 & n2^2 & n3^2 & 2\,n1\,n2 & 2\,n2\,n3 & 2\,n1\,n3 \\ l1\,m1 & l2\,m2 & l3\,m3 & l1\,m2 + m1\,l2 & l2\,m3 + m2\,l3 & l1\,m3 + m1\,l3 \\ m1\,n1 & m2\,n2 & m3\,n3 & m1\,n2 + n1\,m2 & m2\,n3 + n2\,m3 & m1\,n3 + n1\,m3 \\ l1\,n1 & l2\,n2 & l3\,n3 & l1\,n2 + n1\,l2 & l2\,n3 + n2\,l3 & l1\,n3 + n1\,l3 \end{bmatrix} \quad (189)$$

>

> The transpose of the engineering strain transformation matrix can be determined as :

> *Tetranspose* := *Transpose*(*Te*);

$$Tetranspose := \begin{bmatrix} l1^2 & l2^2 & l3^2 & 2\,l1\,l2 & 2\,l2\,l3 & 2\,l1\,l3 \\ m1^2 & m2^2 & m3^2 & 2\,m1\,m2 & 2\,m2\,m3 & 2\,m1\,m3 \\ n1^2 & n2^2 & n3^2 & 2\,n1\,n2 & 2\,n2\,n3 & 2\,n1\,n3 \\ l1\,m1 & l2\,m2 & l3\,m3 & l1\,m2 + m1\,l2 & l2\,m3 + m2\,l3 & l1\,m3 + m1\,l3 \\ m1\,n1 & m2\,n2 & m3\,n3 & m1\,n2 + n1\,m2 & m2\,n3 + n2\,m3 & m1\,n3 + n1\,m3 \\ l1\,n1 & l2\,n2 & l3\,n3 & l1\,n2 + n1\,l2 & l2\,n3 + n2\,l3 & l1\,n3 + n1\,l3 \end{bmatrix} \quad (190)$$

>

> It is worth noting that the Tsinvers is equal to Tetranspose. This can also be checked by performing the following computaion.

>

> *evalm*(*Tsinverse* − *Tetranspose*);

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (191)$$

>

> which leads to a conclusion that the transpose of engineering strain transformation matrix is the inverse of stress transformation matrix :

>

> This is the transformation matrix for engineering strains between the original and the new coordinate system . The transformation matrix for stresses can also be developed in a similar manner$(M \cdot S.M^T)$.

>

> *Tstranspose* := *Transpose*(*Ts*);

$$Tstranspose := \begin{bmatrix} l1^2 & l2^2 & l3^2 & l1\,l2 & l2\,l3 & l1\,l3 \\ m1^2 & m2^2 & m3^2 & m1\,m2 & m2\,m3 & m1\,m3 \\ n1^2 & n2^2 & n3^2 & n1\,n2 & n2\,n3 & n1\,n3 \\ 2\,l1\,m1 & 2\,l2\,m2 & 2\,l3\,m3 & l1\,m2 + m1\,l2 & l2\,m3 + m2\,l3 & l1\,m3 + m1\,l3 \\ 2\,m1\,n1 & 2\,m2\,n2 & 2\,m3\,n3 & m1\,n2 + n1\,m2 & m2\,n3 + n2\,m3 & m1\,n3 + n1\,m3 \\ 2\,l1\,n1 & 2\,l2\,n2 & 2\,l3\,n3 & l1\,n2 + n1\,l2 & l2\,n3 + n2\,l3 & l1\,n3 + n1\,l3 \end{bmatrix} \quad (192)$$

> *evalm*(*TEinverse* − *Tstranspose*);

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{193}$$

\>

\>

\>

\> This result implies that the traspose of stress transformation matrix is equal
to the inverse of engineering strain transformation matrix.

\> This orthogonality of stress and engineering strain transformation matrices can be checked
by performing the following computations

\>

\> *evalm*(*Tetranspose* − *TEinverse*);

$$\begin{bmatrix} 0 & 0 & 0 & l1\, l2 & l2\, l3 & l1\, l3 \\ 0 & 0 & 0 & m1\, m2 & m2\, m3 & m1\, m3 \\ 0 & 0 & 0 & n1\, n2 & n2\, n3 & n1\, n3 \\ -l1\, m1 & -l2\, m2 & -l3\, m3 & 0 & 0 & 0 \\ -m1\, n1 & -m2\, n2 & -m3\, n3 & 0 & 0 & 0 \\ -l1\, n1 & -l2\, n2 & -l3\, n3 & 0 & 0 & 0 \end{bmatrix} \tag{194}$$

\>

\> The difference of the transpose and the inverse of engineering strain transformation matrix doesn't result
in zero matrix which suggests that these two matrices are not equal. It can be deduced that the transformation matrix
for engineering strain is not orthogonal.

\>

\> Similarly for stress transformation matrix

\> Similarly for stress transformation matrix

\>

\> *evalm*(*Tstranspose* − *Tsinverse*);

$$\begin{bmatrix} 0 & 0 & 0 & -l1\, l2 & -l2\, l3 & -l1\, l3 \\ 0 & 0 & 0 & -m1\, m2 & -m2\, m3 & -m1\, m3 \\ 0 & 0 & 0 & -n1\, n2 & -n2\, n3 & -n1\, n3 \\ l1\, m1 & l2\, m2 & l3\, m3 & 0 & 0 & 0 \\ m1\, n1 & m2\, n2 & m3\, n3 & 0 & 0 & 0 \\ l1\, n1 & l2\, n2 & l3\, n3 & 0 & 0 & 0 \end{bmatrix} \tag{195}$$

\>

\>

\> This also implies that the transformation matrix for stress is not orthogonal.

Appendix C. Fortran 77 program for matrix computations of icosahedra distribution of springs

```
    PROGRAM TRANS2
      INTEGER           NPLANE,    NDIV
      PARAMETER        ( NPLANE=32, NDIV=6 )
C
      INTEGER          I, J, K, IPLANE, JDIV, SPNO
      DOUBLE PRECISION PAI, G(6,6)
      DOUBLE PRECISION COOR(NPLANE*2), ETHETA, EPHI, EBETA,
     $                 N(6,6), DIR(6), MAG
      DOUBLE PRECISION INTMAT(6,6), TRANS(6,6), EE(6)
      DOUBLE PRECISION TA(6,6), TB(6,6), TC(6,6),
     $                 TEM(6,6), TE(6,6), TSM(6,6), TS(6,6)
CC.............................................................
.
      PAI=ATAN(1.D0)*4.D0
CC.............................................................
.
CC    *****READING THE ORINETATION OF EACH SPRING FOR AN ICOSAHEDRAL
DISTRIBUTION*****
CC
      CALL SPLOCA(COOR)
CC
            DO 600 I=1,6
              DO 700 J=1,6
                TRANS(I,J)=0.D0
                G(I,J)=0.D0
                TE(I,J)   =0.D0
                TS(I,J)   =0.D0
  700         CONTINUE
  600       CONTINUE

      DO 2000 IPLANE=1,NPLANE
        DO 2100 JDIV=1,NDIV
            SPNO=(IPLANE-1)*NDIV+JDIV
            ETHETA=COOR(IPLANE)
            EPHI  =COOR(IPLANE+32)
            EBETA =PAI*(JDIV-1)/NDIV+0.5D0*PAI/NDIV
CC
CC      **** INITIALIZATION OF
MATRICES***********************************************
            DO 100 I=1,6
              DO 200 J=1,6
                TEM(I,J)     =0.D0
                TE(I,J)      =0.D0
                INTMAT(I,J)  =0.D0
                TSM(I,J)     =0.D0
                TS(I,J)      =0.D0
                DIR(I)       =0.D0
CC                G(I,J)       =0.D0
CC                TRANS(I,J)   =0.D0
  200         CONTINUE
  100       CONTINUE
CC
CC       ***** READING THE TRANSFORMATION MATRICES AFTER EACH ROTATION
********
CC
            CALL TRANSM(ETHETA, EPHI, EBETA, TA, TB, TC)
CC
CC       **** CALCULATION OF THE OVERALL ENGINEERING STRAIN
TRANSFORMAION MATRIX ********
CC
            DO 1100 I=1,6
              DO 1200 J=1,6
```

```
                DO 1300 K=1,6
                  TEM(I,J)=TEM(I,J)+TB(I,K)*TA(K,J)
 1300           CONTINUE
 1200         CONTINUE
 1100       CONTINUE
CC
            DO 1400 I=1,6
              DO 1500 J=1,6
                DO 1600 K=1,6
                  TE(I,J)=TE(I,J)+TC(I,K)*TEM(K,J)
 1600           CONTINUE
 1500         CONTINUE
 1400       CONTINUE
CC
CC    ***** DEFINING THE SELECTOR MATRIX N
***************************************
CC
            DO 1700 I=1,6
              DO 1800 J=1,6
                N(I,J)=0.D0
                N(5,5)=1.D0
 1800         CONTINUE
 1700       CONTINUE
CC
CC   *** DEFINING THE COEFFICIENT TO CALCULATE ONLY THE MAGNITUDES OF
CC          STRESSES AND STRAINS AS PER EQ. 1.73 IN THE
REPORT**********************
           DO 6000, I=1, 6
              EE(I)=0.D0
              EE(6)=1.D0
 6000      CONTINUE
CC
           DO 6100, I=1, 6
             DO 6200 J=1, 6
                DIR(I)=DIR(I)+TE(I,J)*EE(J)
 6200        CONTINUE
 6100      CONTINUE

                IF (ABS(DIR(5)) .GT. 1E-10) THEN
                MAG=DIR(5)/ABS(DIR(5))
                ELSE
                MAG=0.D0
                END IF

CC
CC    **** CALCULATION OF THE OVERALL STRESS TRANSFORMATION MATRIX
***************
CC
            CALL TRANSM(ETHETA, EPHI, EBETA, TA, TB, TC)
CC
            DO 3100 I=1,6
              DO 3200 J=1,6
                DO 3300 K=1,6
                  TSM(I,J)=TSM(I,J)+TB(K,I)*TC(J,K)
 3300           CONTINUE
 3200         CONTINUE
 3100       CONTINUE
CC
            DO 3400 I=1,6
              DO 3500 J=1,6
                DO 3600 K=1,6
                  TS(I,J)=TS(I,J)+TA(K,I)*TSM(K,J)
 3600           CONTINUE
 3500         CONTINUE
```

```
 3400       CONTINUE
CC
CC      *** CALCULATION OF THE MATRIX PRODUCT
transpose(TE)*N*TE*****************
CC
            DO 3700 I=1,6
              DO 3800 J=1,6
                DO 3900 K=1,6
                  INTMAT(I,J)=INTMAT(I,J)+N(I,K)*TE(K,J)
 3900           CONTINUE
 3800         CONTINUE
 3700       CONTINUE
CC
            DO 4000 I=1,6
              DO 4100 J=1,6
                DO 4200 K=1,6
                  TRANS(I,J)=TRANS(I,J)+TS(I,K)*INTMAT(K,J)
 4200           CONTINUE
 4100         CONTINUE
 4000       CONTINUE
CC     **** PRINTING THE RESULTS
************************************************
CC
             WRITE(*,*) 'spring number= ', SPNO
             DO 5200, I = 1, 6
                 DO 5300, J = 1, 6
                 G(I,J)=G(I,J)+TS(I,J)*MAG
                 WRITE(*,*)I,J, G(I,J)
 5300            CONTINUE
 5200        CONTINUE
 2100     CONTINUE
 2000   CONTINUE
             DO 6020, I = 1, 6
                 DO 6010, J = 1, 6
                   G(I,J)= G(I,J)/192.D0
CC                   call primat(G(I,J), 6, 6, 'G =')
 6010            CONTINUE
 6020        CONTINUE
CC        call primat(G(I,J), 6, 6, 'G =')
      END
```

Appendix D. Part of Simple shear data for determination of liquefaction parameters

Number of steps	Effective stress(kPa)	Shear strain	Shear stress(kPa)	Stress ratio
1.00E+00	9.80E+01	1.00E-05	0.00E+00	0.00E+00
2.00E+00	9.80E+01	2.00E-05	9.53E-01	9.72E-03
3.00E+00	9.79E+01	3.00E-05	1.82E+00	1.86E-02
4.00E+00	9.79E+01	4.00E-05	2.62E+00	2.68E-02
5.00E+00	9.78E+01	5.00E-05	3.36E+00	3.44E-02
6.00E+00	9.78E+01	6.00E-05	4.05E+00	4.14E-02
7.00E+00	9.78E+01	7.00E-05	4.70E+00	4.81E-02
8.00E+00	9.78E+01	8.00E-05	5.31E+00	5.43E-02
9.00E+00	9.78E+01	9.00E-05	5.90E+00	6.03E-02
1.00E+01	9.78E+01	1.00E-04	6.45E+00	6.60E-02
1.10E+01	9.78E+01	1.10E-04	6.98E+00	7.14E-02
1.20E+01	9.78E+01	1.20E-04	7.48E+00	7.65E-02
1.30E+01	9.78E+01	1.30E-04	7.96E+00	8.14E-02
1.40E+01	9.78E+01	1.40E-04	8.43E+00	8.62E-02
1.50E+01	9.78E+01	1.50E-04	8.87E+00	9.07E-02
1.60E+01	9.77E+01	1.60E-04	9.30E+00	9.52E-02
1.70E+01	9.77E+01	1.70E-04	9.71E+00	9.94E-02
1.80E+01	9.77E+01	1.80E-04	1.01E+01	1.03E-01
1.90E+01	9.77E+01	1.90E-04	1.05E+01	1.07E-01
2.00E+01	9.77E+01	2.00E-04	1.09E+01	1.12E-01
2.10E+01	9.76E+01	2.10E-04	1.12E+01	1.15E-01
2.20E+01	9.76E+01	2.20E-04	1.16E+01	1.19E-01
2.30E+01	9.76E+01	2.30E-04	1.19E+01	1.22E-01
2.40E+01	9.75E+01	2.40E-04	1.22E+01	1.25E-01
2.50E+01	9.75E+01	2.50E-04	1.26E+01	1.29E-01
2.60E+01	9.75E+01	2.60E-04	1.29E+01	1.32E-01
2.70E+01	9.74E+01	2.70E-04	1.32E+01	1.36E-01
2.80E+01	9.74E+01	2.80E-04	1.35E+01	1.39E-01
2.90E+01	9.73E+01	2.90E-04	1.37E+01	1.41E-01
3.00E+01	9.73E+01	3.00E-04	1.40E+01	1.44E-01
3.10E+01	9.72E+01	3.10E-04	1.43E+01	1.47E-01
3.20E+01	9.72E+01	3.20E-04	1.46E+01	1.50E-01
3.30E+01	9.71E+01	3.30E-04	1.48E+01	1.52E-01
3.40E+01	9.71E+01	3.40E-04	1.51E+01	1.56E-01

Appendix E. consolidation test data for determination of liquefaction parameters

Vertical stress(kN)	void ratio
57.4564	0.4748
117.3067	0.4725
191.5212	0.4673
287.2818	0.4500
406.9826	0.4200
658.3541	0.3600
957.6060	0.3000
526.6833	0.3023
220.2494	0.3150
95.7606	0.3375
227.4314	0.3285
538.6534	0.3098
1053.3666	0.2798
1843.3916	0.1650

Appendix F. The source code for liquefaction analysis.

```
      SUBROUTINE USRLIQ( EPS0   , DEPS   , EPSVEL , NSTR          ,   TIME0    ,
     $                   DTIME  , ELEMEN , INTPT  , COORD         ,   SE       ,
     $                   ITER   , USRVAL , NUSRVL , USRSTA        ,   NSTATE   ,
     $                   USRIND , NINDIC , SIGMA  , STIFF                      )
      CC.......................................................  ..  .......  .....
      IMPLICIT NONE
      INTEGER, PARAMETER          :: NPLANE=32,
     $                                NDIV=6,
     $                                SPRING=192,
     $                                OFFSET=40,
     $                                WIDTH=9
      INTEGER, PARAMETER          :: LTSTP =100
      DOUBLE PRECISION, PARAMETER :: LTDEPS=1.D-4
      DOUBLE PRECISION, PARAMETER :: LTMEAN=1.D-4,
     $                                LTPSIG=1.D-4,
     $                                LTDPEP=1.D-14,
     $                                LTRATI=1.D-3,
     $                                LTDGAM=1.D-10,
     $                                LTGTAN=1.D+4
      INTEGER, PARAMETER          :: GTYPE =1
      DOUBLE PRECISION               LTEVCS
      INTEGER          NSTR, NUSRVL, NSTATE, NINDIC, ELEME        N,  INTPT,   ITER
      DOUBLE PRECISION EPS0(NSTR), DEPS(NSTR), EPSVEL(NSTR        ),  TIME0,
     $                 DTIME, COORD(3), SE(NSTR,NSTR), USR        VA  L(NUSRV  L),
     $                 USRSTA(NSTATE), SIGMA(NSTR), STIFF(        NS  TR,NSTR  )
      INTEGER          USRIND(NINDIC)
      INTEGER          I, J, K, L, M, N
      DOUBLE PRECISION PAI
      DOUBLE PRECISION KMAXO, GRO, YETA, BULKO, BULKSO, HP        ,   GTH,
     $                 RPT(2), NDN, MEPSO, NN, MM, OPT(10)
      INTEGER          NOWSTP, MAXSTP
      DOUBLE PRECISION XDEPS(NSTR), XEPS0(NSTR)
      INTEGER          SPNO, PLNO
      DOUBLE PRECISION COOR(NPLANE*2), THETA, PHI, BETA
      DOUBLE PRECISION KMAX, GR, BULK, BULKS,
     $                 NDI(3), NDS(3),
     $                 KTAN(SPRING), KEQU, PTAN(SPRING),
     $                 ISTIFF(NSTR,NSTR), GLAST(3)
      DOUBLE PRECISION GORI(SPRING), GMAX(SPRING),
     $                 GREV(SPRING), G(SPRING,2),
     $                 COR(SPRING),  GAMP(SPRING),
     $                 IEPS(NSTR),
     $                 EVDSUM, DEVD, EVSUM, DEV, EVCSUM, D        EV  C,
     $                 EVY, HPEV, IDEVD, DPEPS, PWORK, DPW        OR  K
      DOUBLE PRECISION RMAX(SPRING), RREV(SPRING), R(SPRIN        G,  2),
     $                 SIG(NSTR), ISIG(NSTR), SIGMAB(NSTR)        ,
     $                 PSIG(NPLANE), DPSIG(NPLANE),
```

```
$                 MEPS, SIGY, SIGI
INTEGER          MMODE(SPRING), DMODE(SPRING)
CC.......................................................  ..  .......  .....
PAI=DATAN(1.D0)*4.D0
USRIND(385)=USRIND(385)+1
CC.......................................................  ..  .......  .....
KMAXO  = USRVAL(1)
GRO    = USRVAL(2)
YETA   = USRVAL(3)
BULKO  = USRVAL(4)
BULKSO = USRVAL(5)
MEPSO  = USRVAL(6)
NN     = USRVAL(7)
MM     = USRVAL(8)
RPT(1) = USRVAL(9)
RPT(2) = USRVAL(10)
NDN    = USRVAL(11)
HP     = USRVAL(12)
GTH    = USRVAL(13)
CC.......................................................  ..  .......  .....
NOWSTP=0
MAXSTP=1
DO 1000 I=1,NSTR
IF(INT(DABS(DEPS(I))/LTDEPS).GE.MAXSTP) THEN
MAXSTP=INT(DABS(DEPS(I))/LTDEPS)
END IF
1000 CONTINUE
CC
IF(MAXSTP.GE.LTSTP) THEN
MAXSTP=LTSTP
END IF
CC
DO 1100 I=1,NSTR
XDEPS(I)=DEPS(I)/MAXSTP
1100 CONTINUE
CC
DO 10000 NOWSTP=1,MAXSTP
CC.......................................................  ..  .......  .....
CC
DO 1200 I=1,NSTR
XEPS0(I)=EPS0(I)+DEPS(I)*(NOWSTP-1)/MAXSTP
1200 CONTINUE
CC
DO 1300 I=1,NSTR
DO 1400 J=1,NSTR
STIFF(I,J)=0.D0
1400   CONTINUE
SIG(I)=0.D0
SIGMAB(I)=SIGMA(I)
```

```
1300 CONTINUE
CC
DEVD   =  0.D0
EVCSUM =  0.D0
DPWORK =  0.D0
MEPS   = -1.D0*( SIGMA(1) + SIGMA(2) + SIGMA(3) )/3.       D0
EVSUM  = -1.D0*( XEPS0(1) + XEPS0(2) + XEPS0(3) )
DEV    = -1.D0*( XDEPS(1) + XDEPS(2) + XDEPS(3) )
HPEV   =(EVSUM+DEV)*HP+1.D0
CALL SPLOCA(COOR)
CC.......................................................  ..  .......  .....
IF( DTIME .EQ. 0.D0 .AND. USRIND(385) .LE. USRIND(38       6)  ) THEN
CC.......................................................  ..  .......  .....
BULKS=BULKSO*(MEPS/MEPSO)**NN
CC
DO 2000 PLNO=1,NPLANE
DO 2100 J=1,NDIV
SPNO =(PLNO-1)*NDIV+J
THETA=COOR(PLNO)
PHI  =COOR(PLNO+32)
BETA =PAI*(J-1)/NDIV+0.5D0*PAI/NDIV
DO 2200 K=1,NSTR
ISIG(K)=-1.D0*SIGMA(K)
2200     CONTINUE
IF(MOD(SPNO-1,NDIV).EQ.0) THEN
CALL TRANSFER(3, THETA, PHI, BETA, ISIG)
PSIG(PLNO)=ISIG(3)
END IF
IF(PSIG(PLNO).LE.LTPSIG) THEN
PSIG(PLNO)=LTPSIG
END IF
CALL RENEW(KMAX,GR,MM,PSIG(PLNO),MEPSO,KMAXO,GRO           )
CALL SPMAT(THETA, PHI, BETA, ISTIFF)
DO 2300 L=1,NSTR
DO 2400 M=1,NSTR
STIFF(L,M)=STIFF(L,M)+KMAX*PSIG(PLNO)*ISTIFF               (L  ,M)/SPR  ING
2400       CONTINUE
2300     CONTINUE
2100   CONTINUE
2000 CONTINUE
GLAST(1)=STIFF(4,4)
GLAST(2)=STIFF(5,5)
GLAST(3)=STIFF(6,6)
DO 2500 L=1,3
DO 2600 M=1,3
STIFF(L,M)=STIFF(L,M)+BULKS
2600   CONTINUE
2500 CONTINUE
CC
```

```
MEPS=MAX(MEPS,LTMEAN)
USRSTA(1)=0.D0
USRSTA(2)=0.D0
USRSTA(3)=MEPS
USRSTA(4)=MEPS
USRSTA(5)=0.D0
DO 2700 PLNO=1,NPLANE
USRSTA(5+PLNO)=PSIG(PLNO)
2700 CONTINUE
USRSTA(38)=GLAST(1)
USRSTA(39)=GLAST(2)
USRSTA(40)=GLAST(3)
DO 2800 SPNO=1,SPRING
USRIND((SPNO-1)*2+1)=1
USRIND((SPNO-1)*2+2)=0
USRSTA((SPNO-1)*WIDTH+OFFSET+1 )=0.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+2 )=0.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+3 )=0.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+4 )=0.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+5 )=0.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+6 )=0.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+7 )=0.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+8 )=1.D0
USRSTA((SPNO-1)*WIDTH+OFFSET+9 )=0.D0
2800 CONTINUE
CC.......................................................  ..  .......  .....
ELSE
CC.......................................................  ..  .......  .....
EVDSUM  = USRSTA(1)
EVY     = USRSTA(2)
SIGY    = USRSTA(3)
SIGI    = USRSTA(4)
PWORK   = USRSTA(5)
DO 5000 PLNO=1,NPLANE
PSIG(PLNO)= USRSTA(5+PLNO)
5000 CONTINUE
GLAST(1)=USRSTA(38)
GLAST(2)=USRSTA(39)
GLAST(3)=USRSTA(40)
DO 5100 SPNO=1,SPRING
MMODE(SPNO) = USRIND((SPNO-1)*2+1)
DMODE(SPNO) = USRIND((SPNO-1)*2+2)
GORI(SPNO)  = USRSTA((SPNO-1)*WIDTH+OFFSET+1 )
GMAX(SPNO)  = USRSTA((SPNO-1)*WIDTH+OFFSET+2 )
GREV(SPNO)  = USRSTA((SPNO-1)*WIDTH+OFFSET+3 )
G(SPNO,1)   = USRSTA((SPNO-1)*WIDTH+OFFSET+4 )
RMAX(SPNO)  = USRSTA((SPNO-1)*WIDTH+OFFSET+5 )
RREV(SPNO)  = USRSTA((SPNO-1)*WIDTH+OFFSET+6 )
R(SPNO,1)   = USRSTA((SPNO-1)*WIDTH+OFFSET+7 )
```

```
COR(SPNO)   = USRSTA((SPNO-1)*WIDTH+OFFSET+8 )
GAMP(SPNO)  = USRSTA((SPNO-1)*WIDTH+OFFSET+9 )
5100 CONTINUE
NDI(1)=NDN;NDI(2)=NDN;NDI(3)=NDN
NDS(1)=NDN;NDS(2)=NDN;NDS(3)=NDN
CC
DO 6000 PLNO=1,NPLANE
DO 6100 J=1,NDIV
SPNO =(PLNO-1)*NDIV+J
THETA=COOR(PLNO)
PHI  =COOR(PLNO+32)
BETA =PAI*(J-1)/NDIV+0.5D0*PAI/NDIV
DO 6200 K=1,NSTR
ISIG(K)=-1.D0*SIGMA(K)
IEPS(K)=-1.D0*(XEPS0(K)+XDEPS(K))
6200     CONTINUE
CALL TRANSFER(1, THETA, PHI, BETA, IEPS)
G(SPNO,2)=IEPS(5)
IF(MOD(SPNO-1,NDIV).EQ.0) THEN
CALL TRANSFER(3, THETA, PHI, BETA, ISIG)
PSIG(PLNO)=ISIG(3)
END IF
IF(PSIG(PLNO).LE.LTPSIG) THEN
PSIG(PLNO)=LTPSIG
END IF
CALL RENEW(KMAX,GR,MM,PSIG(PLNO),MEPSO,KMAXO,GRO            )
R(SPNO,2)=R(SPNO,1)
CALL MASING(PSIG(PLNO), KMAX, GR, YETA, GORI(SPN            O)  ,
$                GMAX(SPNO), GREV(SPNO), G(SPNO,1), G       (S  PNO,2),
$                GAMP(SPNO), COR(SPNO), RMAX(SPNO), R       RE  V(SPNO)  ,
$                R(SPNO,2), KTAN(SPNO), KEQU, MMODE(S       PN  O))
CC
CALL DILATANCY( IDEVD, DPEPS, DMODE(SPNO), G(SPN            O,  1),
$                    G(SPNO,2), R(SPNO,1), R(SPNO,2),       N   DI, NDS  ,
$                    KEQU, GTH, RPT, LTDPEP, GREV(SPN       O)  ,
$                    RREV(SPNO), HPEV, PSIG(PLNO) )
DEVD=DEVD+IDEVD/SPRING
DPWORK=DPWORK+DABS(R(SPNO,2)*PSIG(PLNO)*HPEV*DPE            PS  )/SPRIN  G
ISIG(5)=R(SPNO,2)*PSIG(PLNO)*HPEV
ISIG(1)=0.D0
ISIG(2)=0.D0
ISIG(3)=0.D0
ISIG(4)=0.D0
ISIG(6)=0.D0
CALL TRANSFER(2, THETA, PHI, BETA, ISIG)
DO 6300 L=1,NSTR
SIG(L)=SIG(L)+ISIG(L)/SPRING
6300     CONTINUE
6100   CONTINUE
```

```
6000 CONTINUE
PWORK=PWORK+DPWORK
DEVC   =  EVSUM  - EVDSUM
EVDSUM =  EVDSUM + DEVD
EVCSUM = (EVSUM  + DEV   ) - EVDSUM
DEVC   =  EVCSUM - DEVC
IF(DABS(NN-1.D0).GT.LTRATI) THEN
LTEVCS=(MEPSO**NN)/BULKSO*1.D0/(1.D0-NN)*(LTMEAN**          (1  .D0-NN)  -
$          SIGI**(1.D0-NN))
EVCSUM=MAX(EVCSUM,LTEVCS)
END IF
CALL SUBSIG(MEPS, MEPSO, BULKSO, BULKO, EVCSUM, EVY,        S   IGY, SI  GI,
$            DEVC, BULK, NN, LTRATI)
MEPS=MAX(MEPS,LTMEAN)
DO 6400 L=1,3
SIG(L)=SIG(L)+MEPS
6400 CONTINUE
DO 7000 PLNO=1,NPLANE
DO 7100 J=1,NDIV
SPNO =(PLNO-1)*NDIV+J
THETA=COOR(PLNO)
PHI  =COOR(PLNO+32)
BETA =PAI*(J-1)/NDIV+0.5D0*PAI/NDIV
KTAN(SPNO)=KTAN(SPNO)*HPEV
IF(GTYPE.EQ.3) THEN
DO 7200 K=1,NSTR
ISIG(K)=SIG(K)
7200       CONTINUE
IF(MOD(SPNO-1,NDIV).EQ.0) THEN
CALL TRANSFER(3, THETA, PHI, BETA, ISIG)
DPSIG(PLNO)=ISIG(3)-PSIG(PLNO)
PSIG(PLNO) =ISIG(3)
END IF
IF(DABS(G(SPNO,2)-G(SPNO,1)).GE.LTDGAM) THEN
PTAN(SPNO)=DPSIG(PLNO)*R(SPNO,2)*HPEV/
$                   (G(SPNO,2)-G(SPNO,1))
ELSE
IF(G(SPNO,2)-G(SPNO,1).GE.0.D0) THEN
PTAN(SPNO)=DPSIG(PLNO)*R(SPNO,2)*HPEV/LTDG                  AM
ELSE
PTAN(SPNO)=DPSIG(PLNO)*R(SPNO,2)*HPEV/(-1.                  D0  *LTDGAM  )
END IF
END IF
IF(PTAN(SPNO).GE.0.D0) THEN
KTAN(SPNO)=KTAN(SPNO)+PTAN(SPNO)
END IF
END IF
CALL SPMAT(THETA, PHI, BETA, ISTIFF)
DO 7300 L=1,NSTR
```

```
DO 7400 M=1,NSTR
STIFF(L,M)=STIFF(L,M)+KTAN(SPNO)*ISTIFF(L,M)                /S  PRING
7400       CONTINUE
7300     CONTINUE
7100   CONTINUE
7000 CONTINUE
IF(GTYPE.EQ.1) THEN
DO 8000 I=4,6
IF(STIFF(I,I).LE.LTGTAN) THEN
IF(GLAST(I-3).GE.LTGTAN) THEN
STIFF(I,I)=GLAST(I-3)
ELSE
STIFF(I,I)=LTGTAN
END IF
END IF
8000   CONTINUE
ELSE IF(GTYPE.EQ.2) THEN
DO 8100 I=4,6
IF(STIFF(I,I).LE.LTGTAN) THEN
STIFF(I,I)=LTGTAN
END IF
8100   CONTINUE
END IF
DO 8200 I=4,6
IF(DABS(DEPS(I)).GE.LTDGAM) THEN
GLAST(I-3)=(-1.D0*SIGMAB(I)-SIG(I))/DEPS(I)
ELSE
IF(DEPS(I).GE.LTDGAM) THEN
GLAST(I-3)=(-1.D0*SIGMAB(I)-SIG(I))/LTDGAM
ELSE
GLAST(I-3)=(-1.D0*SIGMAB(I)-SIG(I))/(-1.D0*LTD              GA  M)
END IF
END IF
8200 CONTINUE
DO 7500 L=1,3
DO 7600 M=1,3
STIFF(L,M)=STIFF(L,M)+BULK
7600   CONTINUE
7500 CONTINUE
CALL ZEROM(STIFF)
USRSTA(1)  =  EVDSUM
USRSTA(2)  =  EVY
USRSTA(3)  =  SIGY
USRSTA(5)  =  PWORK
DO 7700 PLNO=1,NPLANE
USRSTA(5+PLNO)=PSIG(PLNO)
7700 CONTINUE
USRSTA(38)=GLAST(1)
USRSTA(39)=GLAST(2)
```

```
USRSTA(40)=GLAST(3)
DO 7800 SPNO=1,SPRING
USRIND((SPNO-1)*2+1)=MMODE(SPNO)
USRIND((SPNO-1)*2+2)=DMODE(SPNO)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 1) = GORI(SPNO)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 2) = GMAX(SPNO)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 3) = GREV(SPNO)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 4) = G(SPNO,2)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 5) = RMAX(SPNO)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 6) = RREV(SPNO)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 7) = R(SPNO,2)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 8) = COR(SPNO)
USRSTA((SPNO-1)*WIDTH+OFFSET+ 9) = GAMP(SPNO)
7800 CONTINUE
DO 7900 I=1,NSTR
SIGMA(I)=-1.D0*SIG(I)
7900 CONTINUE
END IF
10000 CONTINUE
RETURN
END
```