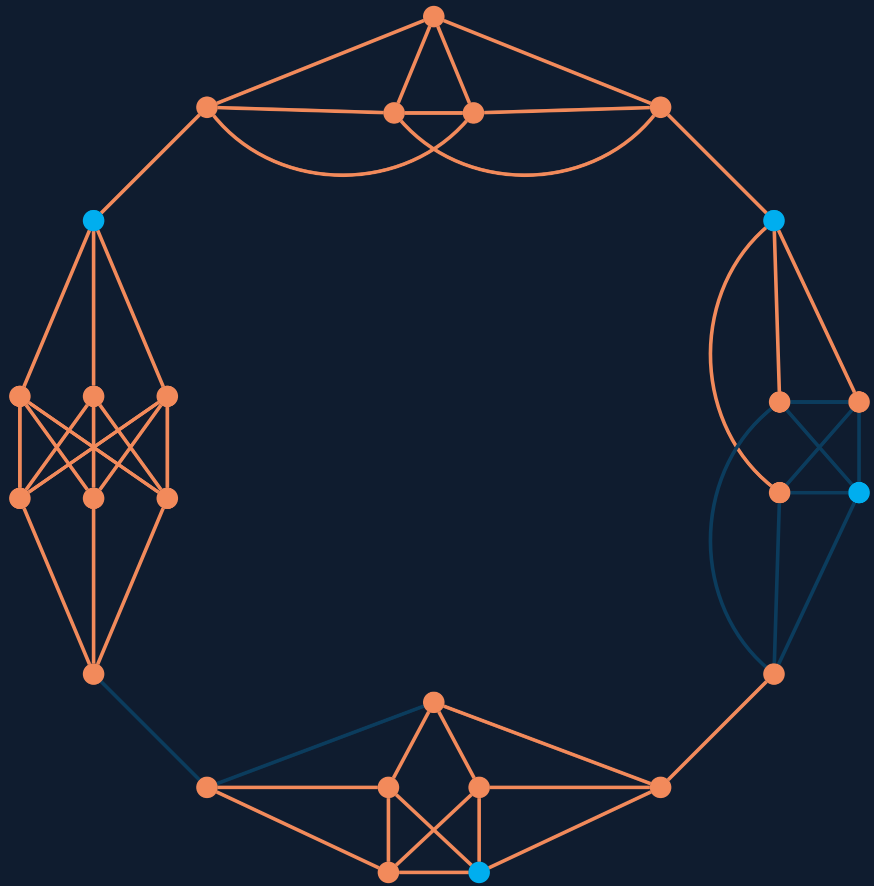
Degree-Constrained Graph Burning

Extremal Bounds, Regular Constructions, and α-Angular Trees

M.J.P. (Max) Reinders



Delft University of Technology

TU Delft

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by

M.J.P. (Max) Reinders

to obtain the degree of Master of Science
at the Delft University of Technology,
to be defended publicly on Monday May 11, 2026 at 3:45 PM.

Student number:	4899334	
Institution:	Delft University of Technology	
Faculty:	Electrical Engineering, Mathematics, and Computer Science	
Specialisation:	Discrete Mathematics and Optimization	
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Project duration:	September, 2025 – May, 2026	

An electronic version of this thesis is available at https://repository.tudelft.nl/.

TU Delft

Acknowledgements

This thesis was written within the Discrete Mathematics & Optimization research group of the Delft Institute of Applied Mathematics, part of the Faculty of Electrical Engineering, Mathematics, and Computer Science at Delft University of Technology. Between September, 2025 and May, 2026, I composed this thesis on graph burning.

I was drawn to this problem in extremal graph theory early on due to its apparent simplicity and underlying complexity. The project split into two main research directions, deriving bounds on the burning number and constructing graphs that (nearly) attain such bounds. I used computational methods to identify patterns and structural constraints, which guided the development of these constructions. This interplay between theory and computation was particularly enjoyable.

I would like to acknowledge the use of OpenAI's ChatGPT for assistance in improving the clarity and grammar of this thesis.

I would like to thank my supervisors, Yuki Murakami and Nikolaas Verhulst, for their guidance throughout this project. Their regular feedback provided structure and direction and was essential in shaping both the research and the writing of this thesis. I also greatly valued the discussions and exchange of ideas during these meetings. Finally, I would like to thank Dion Gijswijt and Rik Versendaal for being part of my graduation committee.

As this thesis concludes my time as a student at TU Delft, I look back fondly on the experiences and friendships I made during this period. I am grateful to my friends for making my student life memorable, and to my family for their continued support and encouragement.

M.J.P. (Max) Reinders
Delft, May, 2026

Abstract

Graph burning is a discrete-time process on a graph that models the spread of information or influence. At each time step, an unburned vertex may be chosen as a source of fire, after which the fire spreads from previously burned vertices to their neighbors. The minimum number of time steps required to burn all vertices is called the burning number $b(G)$ of a graph G.

In many applications, networks are subject to capacity limitations, such as a bounded number of connections per vertex. We study degree-constrained graphs and their extremal behavior.

For connected graphs of bounded maximum degree Δ and a fixed burning number b, we determine the maximum possible order. We show that this bound is of order $\mathcal{O}(\Delta^{b-1})$. We also show that this bound is tight via explicit constructions, which yields a logarithmic lower bound of the form $b(G) \geq \Omega(\log_{\Delta-1} n)$. We further show a more narrow bound in the d-regular setting and prove this is also tight via explicit constructions.

We further consider the complementary problem of constructing d-regular graphs with large burning numbers. We introduce the family of d-necklaces and show that their burning numbers match the known asymptotic upper bound of Martinsson up to an additive constant of one. These graphs also achieve the corresponding upper bound for the radius of Kim et al. up to an additive constant of one.

Finally, for restricted tree classes, we obtain improved asymptotic upper bounds on the burning number by adapting an existing framework. One consequence is an asymptotic refinement by a factor of $1/\sqrt{2}$ of the known bound for homeomorphically irreducible trees of Murakami.

Contents

1 Introduction

With the increasing importance and widespread use of the internet and social media, networks have become an integral part of modern society. As a result, the study of how information, opinions, contagions, and other processes spread through large-scale networks has emerged as a central topic of research, with applications ranging from social dynamics to the spread of infectious diseases.

In such systems, entities can be represented as vertices, while interactions between them are modeled as edges. This abstraction allows complex real-world systems to be studied using the tools of graph theory.

A central question is how quickly such processes spread through a network. One way to model such spreading phenomena is as a fire propagating through a forest, igniting at different locations and expanding locally over time [7]. Some models incorporate active intervention to slow or contain the spread. In contrast, the framework of graph burning focuses on the opposite regime, where the spread is entirely uncontrolled once initiated. This perspective is particularly relevant in real-world settings, including the viral spread of memes on social media, the dissemination of politically charged content, and large-scale influence campaigns on online platforms.

Motivated by these phenomena and earlier models of graph-based spreading, including the firefighter problem [3, 8, 10], graph cleaning [1], and graph bootstrap percolation [2], Bonato et al. [7] introduced the graph burning process. It provides a simple graph-theoretic discrete-time model for studying the speed at which influence can cover a network.

In this model, a fire starts at a selected vertex at each step and spreads to neighboring vertices over time. The process continues until all vertices are burned. The burning number of a graph is the minimum number of steps required to complete this process. The burning number captures the trade-off between the placement of sources and the speed of local spread, and thus serves as a measure of how efficiently a graph can be covered by expanding neighborhoods.

Since its introduction, the burning number has been studied extensively, with a particular focus on upper bounds. Bonato et al. [7] conjectured that every connected graph on n vertices has burning number at most $\lceil \sqrt{n} \rceil$, a bound that is tight for paths and cycles. While this conjecture remains open, a sequence of works has progressively improved the best known general upper bounds, reducing the constant factor in front of $\sqrt{n}$. More recently, Norin and Turcotte [18] showed that the burning number conjecture holds asymptotically. In addition, refined bounds have been established for graphs with further structural restrictions, such as bounded minimum degree [16].

The focus of research has largely been on general graphs or specific graph families. While general graphs impose no structural constraints and specific families are motivated by particular applications, many real-world networks are subject to capacity limitations. For example, processors in interconnection networks can only maintain a fixed number of communication links [15] and individuals in social systems have a limited number of effective contacts. This leads to the study of graphs with bounded maximum degree.

Bounding the maximum degree restricts the rate at which neighborhoods can grow, and therefore limits how quickly a spreading process can propagate through the graph. Although the burning number is not determined solely by the maximum degree, it is inherently influenced by these growth constraints. Within this setting, regular graphs represent the case in which the available degree is used uniformly across all vertices, avoiding local bottlenecks and supporting efficient spreading throughout the graph. As a result, they provide a natural framework for studying extremal behavior of the burning number. This further motivates the study of regular and near-regular graphs, which make full use of the available degree to maximize neighborhood expansion.

This thesis investigates bounds on the burning number through a combination of extremal constructions and structural arguments.

Chapters 2 and 3 introduce the necessary background and notation. Chapter 2 establishes the notation and basic concepts used throughout the thesis. Chapter 3 provides an introduction to graph burning, surveys known bounds from the literature, and develops techniques and results that will be used in later chapters.

We begin in Chapter 4 from an extremal perspective by fixing the length of a burning sequence and investigating how large a graph can be under degree constraints. Without such constraints no finite bound exists, as exemplified by star graphs, whose burning number is constant while their order is unbounded. Restricting to graphs with bounded maximum degree yields a general upper bound on the graph order, which in turn implies a lower bound on the burning number. We show that this bound is tight via explicit constructions. We further refine this bound for regular graphs, where a parity correction arises. In particular, the general bound is attained for even d, and up to an additive constant of 1 for odd d, while the refined bound is attained by explicit d-regular constructions.

We then turn in Chapter 5 to the complementary perspective of fixing the graph order and constructing regular graphs with large burning number. While paths and cycles are extremal for maximum degree two, the situation is more subtle for higher degrees. Guided by structural considerations, we construct explicit d-regular graphs that enforce slow spreading by maintaining large effective distances. These constructions match the upper bound of Martinsson [16] up to an additive constant of at most 1, and their radius matches the corresponding bound of Kim et al. [13] up to an additive constant of at most 1.

Finally, in Chapter 6, we derive an improved asymptotic upper bound on the burning number for restricted classes of trees. Building on the framework of Martinsson, we adapt the method to α-angular trees, exploiting their branching structure to enforce expansion along sufficiently long paths. This leads to an asymptotic refinement of the bound of Murakami [17] for homeomorphically irreducible trees, while extending the result to a broader class.

Chapter 7 concludes with a summary of the results and directions for further research.

2

Preliminaries

We begin by introducing the graph-theoretic concepts and notation used throughout the thesis. For standard graph-theoretic definitions and notation, we refer to [9, 19]. Throughout this thesis, we consider only unweighted, undirected, simple graphs. We omit subscripts, superscripts, and bracketed notation referring to G when the underlying graph is clear from the context.

Informally, a *graph* is a set of points, called *vertices*, connected by lines, called *edges*. For example, a train network can be modeled as a graph, where the vertices represent train stations, and two vertices are connected by an edge if there is a direct track between them.

Formally, such a network can be represented as $G = (V, E)$, where V is the set of vertices (the *vertex set*) and E is the set of edges (the *edge set*). We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively.

In an *undirected graph*, edges have no direction, and uv and vu represent the same edge, that is, $uv \in E$ if and only if $vu \in E$. Two vertices are *adjacent* if there is an edge between them. A *simple graph* is an undirected graph with no loops and no multiple edges, that is, there are no edges of the form vv for any $v \in V$, and between any two distinct vertices there is at most one edge. An *unweighted graph* is a graph where the edges have no weights assigned to them.

Let $v \in V(G)$. The *degree* of a vertex v is the number of edges incident to it, denoted by $\deg_G(v)$. The *minimum* and *maximum degree* of G are the minimum and maximum degrees over all vertices, denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph is *d-regular* if all vertices have degree d, i.e., $\deg_G(v) = d$ for all $v \in V(G)$.

A *path* is a sequence of distinct vertices $(v_0, \ldots, v_k)$ such that v_i and v_{i+1} are adjacent for all $i = 0, \ldots, k-1$. The *length* of a path is the number of edges it contains, namely k.

The *distance* between two vertices $u, v \in V(G)$, denoted by $\operatorname{dist}_G(u, v)$, is the length of a shortest path between u and v. If no such path exists, we set $\operatorname{dist}_G(u, v) = \infty$. For a vertex $v \in V(G)$ and a set $S \subseteq V(G)$, we define

$$\operatorname{dist}_G(v, S) := \min_{u \in S} \operatorname{dist}_G(v, u).$$

A graph is called *connected* if there exists a path between every pair of vertices.

The *eccentricity* of a vertex $v \in V(G)$ is defined as

$$\operatorname{ecc}_G(v) := \max_{u \in V(G)} \operatorname{dist}_G(v, u).$$

The *radius* and *diameter* of G are defined as

$$\operatorname{rad}(G) := \min_{v \in V(G)} \operatorname{ecc}_G(v) \qquad \text{and} \qquad \operatorname{diam}(G) := \max_{v \in V(G)} \operatorname{ecc}_G(v).$$

Thus, the radius measures how close a graph is to having a central vertex, while the diameter measures the largest distance between any two vertices.

A *cycle* is a sequence $(v_0, \dots, v_{k-1}, v_0)$ with $k \geq 3$ such that $(v_0, \dots, v_{k-1})$ is a path and v_{k-1} is adjacent to v_0. Its *length* is k.

A graph without cycles is called *acyclic*. A *tree* is a connected acyclic graph. If the tree has a distinguished vertex, called the *root*, the tree is called a *rooted tree*, where its *depth* or *height* is the maximum distance from the root to any other vertex. A *forest* is an acyclic graph, that is, a graph whose connected components are trees. A *rooted forest* is a forest whose components are rooted trees.

For any vertex set $S \subseteq V(G)$, the *induced subgraph*, denoted by $G[S]$, is the graph with vertex set S and all edges of G whose endpoints lie in S. A *matching* is a set of pairwise disjoint edges, meaning that no two edges share a common vertex.

The following notions are fundamental to our results and will be used throughout the thesis, so we state them explicitly.

Definition 2.1. Let G be a graph, and let $v \in V(G)$ and $r \geq 0$. The *closed r-neighbourhood* of v is

$$N_r^G[v] := \{u \in V(G) : \mathrm{dist}_G(u, v) \leq r\}.$$

For a set $S \subseteq V(G)$, we define

$$N_r^G[S] := \bigcup_{v \in S} N_r^G[v].$$

The *open neighbourhood* of v is

$$N_G(v) := N_1^G[v] \setminus \{v\}.$$

Definition 2.2. Let G be a graph and let $k \geq 0$. A set $S \subseteq V(G)$ is a *k-distance dominating set* of G if every vertex of G lies within distance k of S, i.e.

$$\mathrm{dist}_G(v, S) \leq k \quad \text{for all } v \in V(G),$$

equivalently,

$$V(G) \subseteq N_k^G[S].$$

Intuitively, the set $N_r^G[v]$ consists of all vertices within distance r of v, and a k-distance dominating set is a set of vertices whose k-neighbourhood covers the entire graph.

Since many of our results are asymptotic in nature, we recall the standard asymptotic notation used throughout this thesis. Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$. We use standard Landau notation, summarized below:

$$\begin{aligned}
f(n) &= \mathcal{O}(g(n)) && f(n) \leq Cg(n) \text{ for some } C > 0 \text{ and all sufficiently large } n, \\
f(n) &= \Omega(g(n)) && g(n) = \mathcal{O}(f(n)), \\
f(n) &= \Theta(g(n)) && f(n) = \mathcal{O}(g(n)) \text{ and } f(n) = \Omega(g(n)), \\
f(n) &= o(g(n)) && \frac{f(n)}{g(n)} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

We write $o(1)$ for a quantity tending to 0 as $n \to \infty$.

With these preliminaries established, we now introduce the graph burning process.

3

Graph Burning

Having established the necessary preliminaries, we now introduce the graph burning process and the associated burning number. We review known bounds related to the burning number conjecture and develop general techniques that will be used in subsequent chapters. Finally, we record the burning numbers of several classical graph families.

3.1. The Burning Process

We first describe the graph burning process informally. Graph burning is a discrete-time process on a graph G in which vertices are either burned or unburned. Initially, all vertices are unburned. At each time step, one unburned vertex may be selected as a *burning source* and becomes burned. At each time step, the fire spreads from all previously burned vertices to their neighbors, which become burned. This process continues until all vertices are burned.

This definition differs slightly from that of Bonato et al. [7], where at each time step an unburned vertex must be chosen. In our formulation, a burning source may be chosen at each time step, but is not required to be chosen. This modification does not affect the value of the burning number. Indeed, at any time step in which no burning source is selected, one may instead choose a vertex that is be burned by the spreading process at that time step, without changing the set of burned vertices at any subsequent time step. Allowing such steps without selecting a burning source simplifies the construction of burning sequences, as it removes the need to specify unnecessary choices of vertices at those time steps.

Definition 3.1. A sequence $(v_1, \ldots, v_k)$ of vertices of a graph G is called a *burning sequence* of length k if

$$N_{k-1}[v_1] \cup N_{k-2}[v_2] \cup \cdots \cup N_1[v_{k-1}] \cup N_0[v_k] = V(G).$$

The *burning number* $b(G)$ of G is the minimum integer k for which a burning sequence of length k exists. A burning sequence of length $b(G)$ is called *optimal*.

Remark 3.2. In a burning sequence $(v_1, \ldots, v_k)$, some entries may be left unspecified and denoted by "–". A placeholder is a notational abbreviation indicating that no new source is selected at that time step. Formally, burning sequences remain sequences of vertices. This is allowed since, at each time step, a burning source may be chosen but is not required to be chosen. This notation is used for convenience and does not affect the burning process.

Remark 3.3. In general, a burning sequence of length k is not uniquely determined.

A burning sequence records the vertices that serve as burning sources, where the index of each vertex indicates the time step at which it is chosen.

Example 3.4. Consider the path P_5 with vertices $v_1, \ldots, v_5$. The sequences (v_3, v_5, v_1) and $(v_3, -, -)$ are burning sequences of length 3, and both are optimal (by Proposition 3.9, $b(P_5) = 3$). Longer burning sequences also exist. For instance, $(v_1, -, -, -, -)$ is a burning sequence of length 5, but it is not optimal.

Figure 3.1 illustrates the burning process for these sequences over time.

Consider the sequence (v_3, v_5, v_1). At time step 1, vertex v_3 is burned. At time step 2, vertex v_5 is selected as a burning source, and the fire from v_3 spreads to its neighbors. At time step 3, vertex v_1 is selected as a burning source, and the fire continues to spread. After these steps, all vertices of the graph are burned.

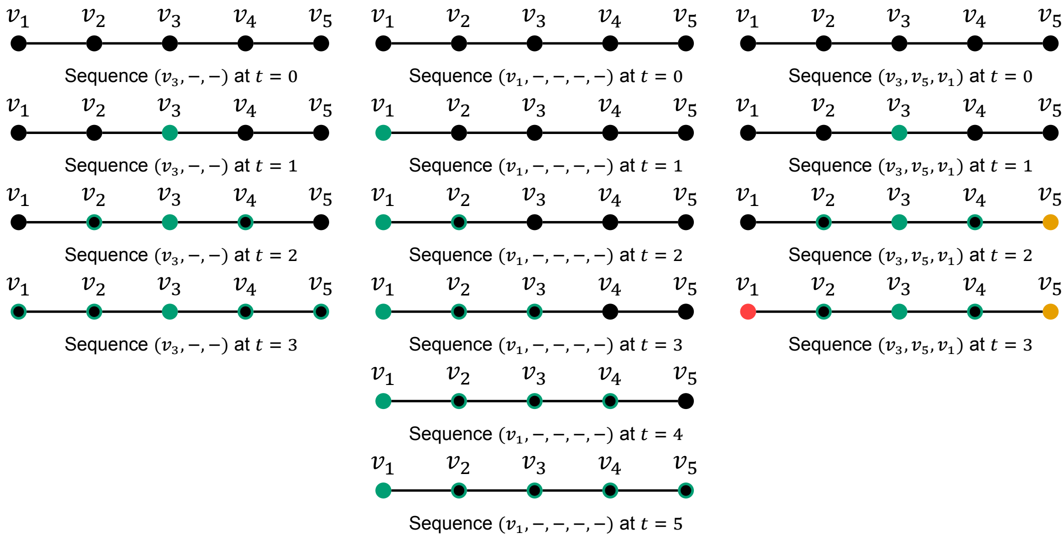


Figure 3.1: Examples of burning sequences on the path P_5. Each column corresponds to a different burning sequence, and each row shows the state of the graph at a given time step. Vertices filled in color indicate burning sources (green, orange, and red correspond to time steps 1, 2, and 3, respectively), while vertices with colored outlines indicate those reached by the spreading process from the corresponding burning source.

A high burning number indicates that the graph has large effective distances. In such graphs, no small set of vertices can rapidly cause the entire graph to become burned, and any spreading process requires many time steps. Intuitively, the graph consists of regions that are far apart in terms of graph distance.

We now turn to general bounds on the burning number and their development in the literature.

3.2. The Burning Number Conjecture

Determining the burning number of a graph is computationally difficult. It was shown by Bessy et al. [5] that deciding whether $b(G) \leq k$ is NP-complete, even when G is restricted to trees. As a result, much of the research has focused on establishing general bounds on the burning number. We summarize the main developments below.

One of the central open problems in this area is the *burning number conjecture*, introduced by Bonato et al. [7].

Conjecture 3.5 (Bonato et al., 2016, [7])**.** Let G be a connected graph on n vertices. Then

$$b(G) \leq \lceil \sqrt{n} \rceil.$$

This bound is tight for paths and cycles. While the conjecture remains open, several partial results are known.

In their original work, Bonato et al. [7] established the general upper bound

$$b(G) \leq 2\lceil \sqrt{n} \rceil - 1.$$

Subsequent work focused on improving the constant factor. In particular, Bessy et al. [6] showed that for every $0 < \varepsilon < 1$,

$$b(G) \leq \sqrt{\frac{32}{19(1-\varepsilon)}n} + \sqrt{\frac{27}{19\varepsilon}},$$

and also derived the simpler bound

$$b(G) \leq \sqrt{\frac{12}{7}n} + 2.$$

Further refinements were obtained by Land and Lu [14], who proved that

$$b(G) \leq \left\lceil \frac{\sqrt{24n+33}-3}{4} \right\rceil.$$

More recently, Bastide et al. [4] established the bound

$$b(G) \leq \left\lceil \sqrt{\frac{4}{3}n} \right\rceil + 1.$$

The following result gives the strongest known general asymptotic upper bound.

Theorem 3.6 (Norin and Turcotte, 2024, [18])**.** Let G be a connected graph on n vertices. Then, asymptotically,

$$b(G) \leq (1 + o(1))\sqrt{n}.$$

To refine such bounds under additional structural assumptions, we next introduce a general technique based on distance domination.

3.3. Distance Domination and Refined Bounds

To refine general bounds under additional structural assumptions, we use the following lemma relating burning numbers to distance dominating sets.

Lemma 3.7 (cf. Bonato et al., 2016, [7])**.** Let G be a graph and let $k \geq 0$. If $S \subseteq V(G)$ is a k-distance dominating set of G, then

$$b(G) \leq b(G[S]) + k.$$

Proof. Let $b := b(G[S])$ and let $(x_1, \dots, x_b)$ be an optimal burning sequence of $G[S]$. Let $v \in V(G)$ be arbitrary. By assumption there exists $s \in S$ with $\mathrm{dist}_G(v, s) \leq k$. Since $(x_1, \dots, x_b)$ burns $G[S]$, there exists $i \in \{1, \dots, b\}$ such that $\mathrm{dist}_{G[S]}(s, x_i) \leq b - i$. Any s–x_i path in $G[S]$ is also a path in G, so $\mathrm{dist}_G(s, x_i) \leq b - i$. Hence

$$\mathrm{dist}_G(v, x_i) \leq \mathrm{dist}_G(v, s) + \mathrm{dist}_G(s, x_i) \leq k + (b - i) = (b + k) - i,$$

and therefore $v \in N^G_{(b+k)-i}[x_i]$. Since v was arbitrary,

$$V(G) \subseteq \bigcup_{i=1}^{b} N^G_{(b+k)-i}[x_i],$$

so $(x_1, \dots, x_b)$ burns G in $b + k$ rounds and $b(G) \leq b + k$. □

Combining this idea with structural bounds on distance dominating sets yields the following refined asymptotic bound.

Theorem 3.8 (Martinsson, 2023, [16])**.** Let G be a connected graph on n vertices with minimum degree δ. Then, asymptotically,

$$b(G) \leq (1 + o(1))\sqrt{\frac{3n}{\delta + 1}}.$$

The proof of Theorem 3.8 combines a structural bound on k-distance dominating sets, Lemma 3.7, and the asymptotic estimate of Theorem 3.6. This approach will be used in later chapters to derive bounds for other graph classes.

We conclude by recording exact burning numbers for several classical graph families.

3.4. Burning Numbers of Specific Graphs

We record the burning numbers of several classical graph families, which will be used in later chapters.

Proposition 3.9 (Bonato et al., 2016, [7])**.** Let $n \geq 1$. Then,

- The path P_n satisfies $b(P_n) = \lceil\sqrt{n}\rceil$.
- The cycle C_n satisfies $b(C_n) = \lceil\sqrt{n}\rceil$.
- The complete graph K_n satisfies $b(K_n) = 2$ for $n \geq 2$.
- The star graph $K_{1,n-1}$ satisfies $b\big(K_{1,n-1}\big) = 2$ for $n \geq 2$.

Example 3.10. Figure 3.2 illustrates the burning process for these sequences over time, providing a visual example of optimal burning sequences for the graphs in Proposition 3.9, each of which has burning number 2. Notably, while a second burning source is required for P_4 and C_4, this is not the case for K_4 and $K_{1,3}$, reflecting differences in their structure.

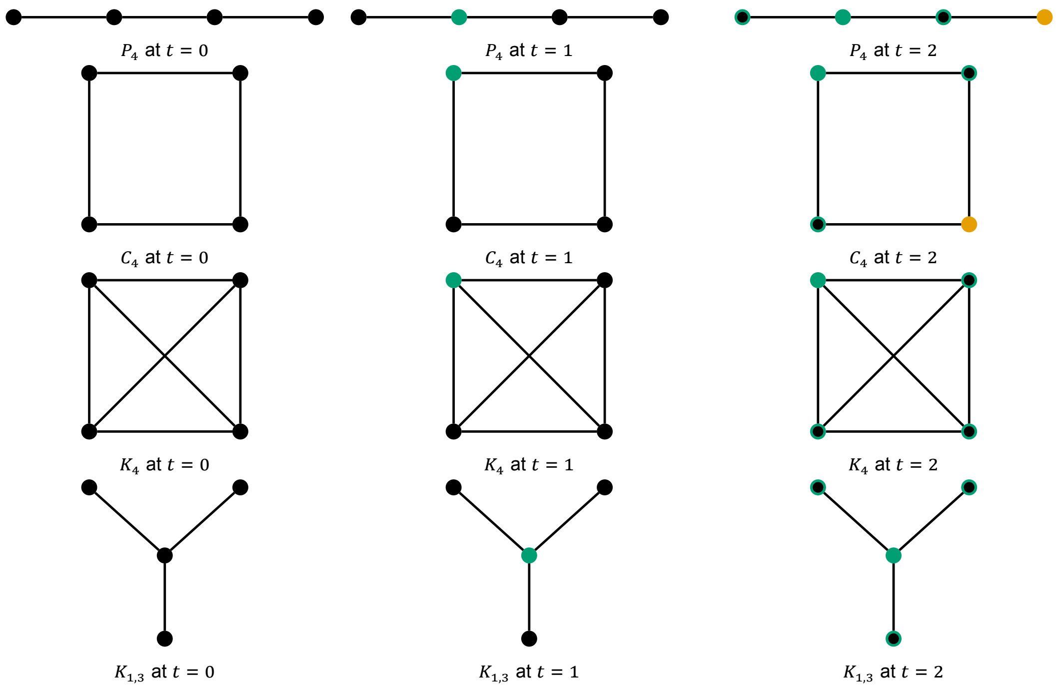


Figure 3.2: Examples of optimal burning sequences on the path P_4, the cycle C_4, the complete graph K_4, and the star graph $K_{1,3}$. Each column corresponds to a different burning sequence, and each row shows the state of the graph at a given time step. Vertices filled in color indicate burning sources (green and orange correspond to time steps 1 and 2, respectively), while vertices with colored outlines indicate those reached by the spreading process from the corresponding burning source.

These results provide the foundation for the analysis of the burning number in subsequent chapters.

4

Extremal Graphs for Fixed Burning Number

We first fix the length of a burning sequence and study how large a graph can be under degree constraints. Without such constraints, no finite bound exists, as illustrated by star graphs, whose burning number is constant while their order is unbounded. Restricting the maximum degree yields an upper bound on the graph order, which in turn implies a lower bound on the burning number.

We then specialize to regular graphs, where a parity correction arises and the bound can be sharpened accordingly. Finally, we show that this refined bound is tight by constructing d-regular graphs that admit burning sequences of the prescribed length and attain the bound.

Throughout this chapter, let $\Delta \geq 3$, $d \geq 3$, and $k \geq 2$ be integers. The cases $\Delta \leq 2$ correspond to paths and cycles, whose burning numbers are known exactly, and the case $k = 1$ is trivial, as $b(G) = 1$ if and only if $|V(G)| = 1$.

The following quantity serves as a central upper bound on the order of graphs with maximum degree Δ that admit a burning sequence of length k.

We define

$$U(\Delta, k) := \frac{\Delta(\Delta-1)^k - 2k(\Delta-2) - \Delta}{(\Delta-2)^2},$$

and

$$\widetilde{U}(d, k) := \begin{cases} U(d, k), & d \text{ even}, \\ U(d, k) - 1, & d \text{ odd}. \end{cases}$$

Definition 4.1. For $\Delta \geq 2$ and $k \geq 2$, define

$$\overline{M}(\Delta, k) := \max\{\, |V(G)| : G \text{ is connected}, \ \Delta(G) \leq \Delta, \ b(G) \leq k \,\}.$$

The quantity $\overline{M}(\Delta, k)$ measures the maximum possible order of a connected graph with maximum degree Δ and burning number at most k.

We illustrate this definition in the case $\Delta = 2$, where the extremal graphs are completely characterized by paths and cycles.

Example 4.2. For $\Delta = 2$, every connected graph G with $\Delta(G) \leq 2$ is a path or a cycle. By Proposition 3.9,

$$b(P_n) = b(C_n) = \lceil \sqrt{n} \rceil.$$

Hence $b(G) \leq k$ holds if and only if $n \leq k^2$. Therefore,

$$\overline{M}(2, k) = k^2,$$

attained for instance by P_{k^2} (or C_{k^2}).

We first establish general upper bounds on the order of graphs under degree constraints, then refine these bounds for regular graphs and show that they are tight via explicit constructions.

4.1. Upper Bounds on Graph Order

We begin by establishing a structural upper bound on the order of a graph in terms of its maximum degree and the length of a burning sequence.

Theorem 4.3. Let G be a connected graph with maximum degree $\Delta \geq 3$ and a burning sequence of length k. Then,

$$|V(G)| \leq U(\Delta, k).$$

Proof. Let $(v_1, \dots, v_k)$ be a burning sequence of G of length k. Let $i \in \{1, \dots, k\}$ be arbitrary. We partition $N_{k-i}[v_i]$ into distance layers $S_j(v_i) = \{u : \mathrm{dist}(u, v_i) = j\}$. Then $|S_0(v_i)| = 1$, $|S_1(v_i)| \leq \Delta$, and for $j \geq 2$, $|S_j(v_i)| \leq \Delta(\Delta-1)^{j-1}$, since each vertex in $S_{j-1}(v_i)$ has at most $\Delta - 1$ neighbors outside $S_{j-2}(v_i)$. Hence

$$|N_{k-i}[v_i]| = \sum_{j=0}^{k-i} |S_j(v_i)| \leq 1 + \sum_{j=1}^{k-i} \Delta(\Delta-1)^{j-1} = 1 + \Delta \sum_{j=0}^{k-i-1} (\Delta-1)^j = 1 + \Delta \frac{(\Delta-1)^{k-i} - 1}{\Delta - 2}.$$

Summing over all i yields

$$|V(G)| = \left| \bigcup_{i=1}^{k} N_{k-i}[v_i] \right| \leq \sum_{i=1}^{k} |N_{k-i}[v_i]| \leq \sum_{i=1}^{k} \left(1 + \Delta \frac{(\Delta-1)^{k-i} - 1}{\Delta - 2} \right).$$

Separating the constant terms and rewriting the remaining sum gives

$$\begin{aligned}
|V(G)| &\leq k + \frac{\Delta}{\Delta-2} \sum_{i=1}^{k} \left((\Delta-1)^{k-i} - 1 \right) \\
&= k + \frac{\Delta}{\Delta-2} \left(\sum_{j=0}^{k-1} (\Delta-1)^j - k \right) \\
&= k + \frac{\Delta}{\Delta-2} \left(\frac{(\Delta-1)^k - 1}{\Delta-2} - k \right) \\
&= U(\Delta, k).
\end{aligned}$$

□

Corollary 4.4. Let G be a connected graph with maximum degree $\Delta \geq 3$ and burning number b. Then,

$$|V(G)| \leq U(\Delta, b).$$

Proof. Let $(v_1, v_2, \dots, v_b)$ be an optimal burning sequence of G of length b. The result then follows directly from Theorem 4.3. □

Applying Theorem 4.3 to regular graphs yields the following bounds.

Corollary 4.5. Let G be a connected d-regular graph with $d \geq 3$ and a burning sequence of length k. Then,

$$|V(G)| \leq \widetilde{U}(d, k).$$

Proof. Since $\Delta(G) = d$, it follows from Theorem 4.3 that $|V(G)| \le U(d,k)$.

Suppose now that d is odd. By the Handshaking Lemma [19],

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| = d|V(G)|.$$

Since $2|E(G)|$ is even and d is odd, it follows that $|V(G)|$ is even.

Note that $U(d,k)$ is integer. We compute $U(d,k) \pmod 2$. Because d is odd,

$$d \equiv 1, \qquad d-1 \equiv 0, \qquad d-2 \equiv -1 \equiv 1 \pmod 2.$$

Hence $(d-2)^2 \equiv 1 \pmod 2$, so the denominator of $U(d,k)$ is invertible in $\mathbb{F}_2$ and we may reduce the fraction modulo 2.

Since $k \ge 1$, we have $(d-1)^k \equiv 0$, and therefore

$$d(d-1)^k \equiv 0, \qquad 2k(d-2) \equiv 0, \qquad -d \equiv -1 \equiv 1 \pmod 2.$$

Thus

$$U(d,k) = \frac{d(d-1)^k - 2k(d-2) - d}{(d-2)^2} \equiv \frac{0-0-1}{1} \equiv -1 \equiv 1 \pmod 2.$$

Hence, because $|V(G)|$ is even while $U(d,k)$ is odd, we must have $|V(G)| \le U(d,k) - 1$. □

Corollary 4.6. Let G be a connected d-regular graph with $d \ge 3$ and burning number b. Then,

$$|V(G)| \le \widetilde{U}(d,b).$$

Proof. Let $(v_1, \ldots, v_b)$ be an optimal burning sequence of G. Applying Corollary 4.5 with $k = b$ gives the stated upper bound. □

These bounds show that the order of a graph with a burning sequence of length k is tightly constrained by its maximum degree. In the next section, we show that these bounds are sharp.

4.2. Tightness of the Bounds

We now show that the bounds obtained above are tight by constructing extremal examples. To this end, we introduce a family of trees that will serve as building blocks for the construction.

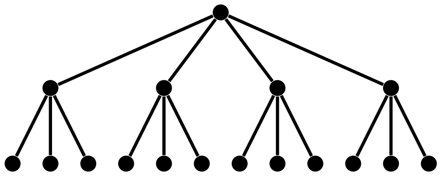
Definition 4.7 (Rooted (Δ, h)-tree)**.** Let $\Delta \ge 3$ and $h \ge 2$. A *rooted* (Δ, h)*-tree* is a rooted tree of depth h in which

- the root has exactly Δ children,
- every other non-leaf vertex has exactly $\Delta - 1$ children,
- every leaf lies at distance exactly h from the root.

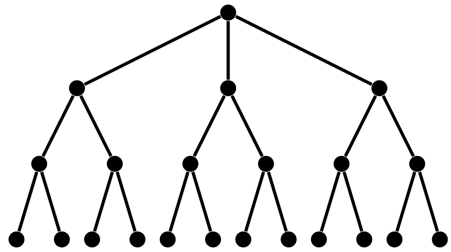
Remark 4.8. A rooted (Δ, h)-tree has exactly $\Delta(\Delta-1)^{h-1}$ leaves.

We illustrate this construction with a simple example.

Example 4.9. Figure 4.1 illustrates two examples of rooted (Δ, h)-trees for different values of Δ and h. In Figure 4.1a, we show a rooted $(4,2)$-tree, where the root has four children and each subsequent level branches by a factor of three. In Figure 4.1b, we show a rooted $(3,3)$-tree, where the root has three children and each subsequent level branches by a factor of two.



(a) A rooted $(4,2)$-tree.



(b) A rooted $(3,3)$-tree.

Figure 4.1: Examples of rooted (Δ, h)-trees for different values of Δ and h.

These trees capture the maximal branching permitted under the degree constraint and will serve as the building blocks of our extremal constructions.

Definition 4.10 (Rooted (Δ, h)-forest)**.** Let $\Delta \geq 3$ and $h \geq 2$. A *rooted* (Δ, h)*-forest* is a collection of rooted trees $F_1, \dots, F_h$, where for each $i \in \{1, \dots, h\}$, the tree F_i is a rooted $(\Delta, h+1-i)$-tree with root v_i.

Remark 4.11. Each tree F_i has $\Delta(\Delta-1)^{h-i}$ leaves, so the total number of leaves in the forest is

$$\sum_{i=1}^{h} \Delta(\Delta-1)^{h-i} = \sum_{j=0}^{h-1} \Delta(\Delta-1)^{j} = \Delta\,\frac{(\Delta-1)^h - 1}{\Delta - 2}.$$

Moreover, the total number of vertices in the forest equals $U(\Delta, h) - 1$.

Example 4.12. Figure 4.2 illustrates a rooted (Δ, h)-forest with roots v_1, v_2, v_3.

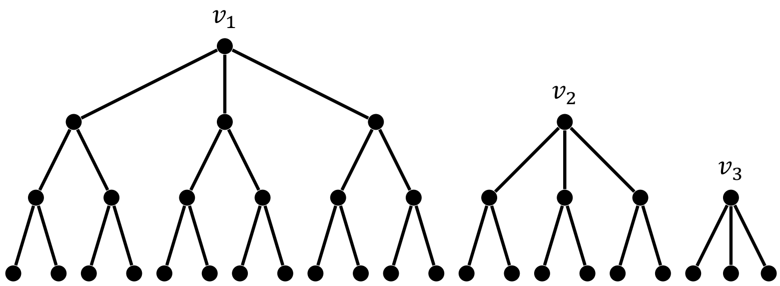


Figure 4.2: Example of a rooted $(3,3)$-forest with roots v_1, v_2, v_3.

By appropriately connecting the leaves of a rooted (Δ, h)-forest, one can obtain a connected graph whose order attains the bound $U(\Delta, h) - 1$.

We now use this construction to establish the tightness of the upper bound for regular graphs.

Theorem 4.13. Let $d \geq 3$ and $k \geq 2$ be integers. Then there exists a connected d-regular graph G with a burning sequence of length k such that

$$|V(G)| = \widetilde{U}(d, k).$$

Thus the bound in Corollary 4.5 is tight.

Proof. We construct a graph G, initially a rooted $(d, k-1)$-forest with roots $v_1, \dots, v_{k-1}$. We then complete the construction by adding edges among the leaves, distinguishing between the cases where d is even and d is odd, to obtain a connected d-regular graph with a burning sequence of length k and the desired number of vertices.

Let L denote the set of leaves of G. Then, by construction,

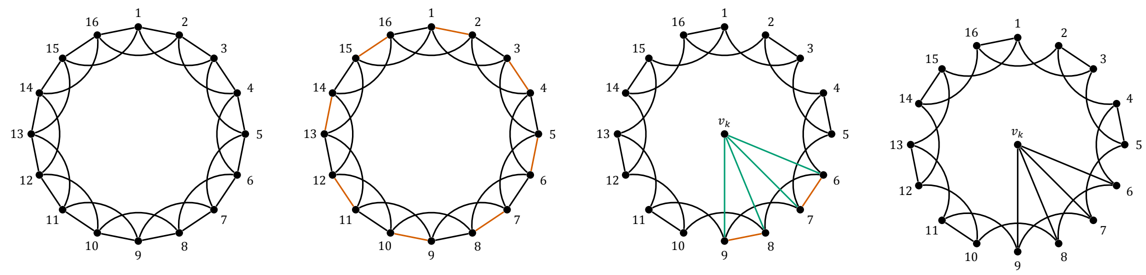
$$n_\ell := |L| = d\,\frac{(d-1)^{k-1} - 1}{d - 2}.$$

Moreover, G has $U(d,k)-1$ vertices. We add edges among the vertices of L to increase their degrees to d. Label the leaves arbitrarily by $1,2,\dots,n_\ell$.

Case d odd. Add the edges

$$(v,\, v+j \bmod n_\ell) \qquad \text{for all } v \in \{1,\dots,n_\ell\} \text{ and } j = 1,\dots,\tfrac{d-1}{2}.$$

Since $1 \le j \le \frac{d-1}{2} \le \frac{n_\ell}{2}$, no loops or parallel edges are created, and the resulting graph on L is simple. Moreover, $G[L]$ contains the cycle $(1,2,\dots,n_\ell,1)$, so $G[L]$ is connected, and therefore the whole graph G is connected. This makes $G[L]$ $(d-1)$-regular, so each leaf has total degree $1+(d-1)=d$, where the additional 1 comes from its parent in the forest. In this case all vertices already have degree d and $|V(G)| = \widetilde{U}(d,k)$, so the construction is complete.



The graph $G[L]$ after adding edges on the leaf set. The graph $G[L]$ after deleting a matching from the leaf cycle. The graph $G[L \cup \{v_k\}]$ after adding the last burning source v_k. The final graph $G[L \cup \{v_k\}]$.

Figure 4.3: Construction for the even-degree case with $d=4$ and $k=3$. The vertices in L are the leaves of the rooted $(4,2)$-forest, and v_k is the last burning source. Added edges are shown in green, while deleted edges are shown in orange.

Case d even. Figure 4.3 illustrates the following construction for the even case. Add the edges

$$(v,\, v+j \bmod n_\ell) \qquad \text{for all } v \in \{1,\dots,n_\ell\} \text{ and } j = 1,\dots,\tfrac{d}{2}.$$

Note that n_ℓ is even when d is even. Since $1 \le j \le \frac{d}{2} \le \frac{n_\ell}{2}$, no loops or parallel edges are created, and the resulting graph on L is simple. Moreover, $G[L]$ contains the cycle $(1,2,\dots,n_\ell,1)$. Remove every other edge from the cycle $(1,2,\dots,n_\ell,1)$, say the edge set $\{(1,2),(3,4),\dots,(n_\ell-1,n_\ell)\}$, which reduces the degree of all vertices by 1. This makes $G[L]$ $(d-1)$-regular, so each vertex in L has total degree $1+(d-1)=d$.

Take two consecutive edges from the set $\{(2,3),(3,4),\dots,(n_\ell,1)\}$, say $(2,3)$ and $(3,4)$. Since $d \ge 3$ and d is even, we have $\frac{d}{2} \ge 2$, and hence the edge $(1,3)$ exists, connecting these two edges. This holds for every pair of consecutive edges, and since the edge set forms a maximum matching covering all vertices of L, it follows that $G[L]$ is connected.

We now introduce a new vertex v_k, which will serve as the last burning source, and connect it to d leaves as follows. Choose $\frac{d}{2}$ edges from the set $\{(2,3),(4,5),\dots,(n_\ell,1)\}$. For each chosen edge (u,v) delete it and add the edges (v_k,u) and (v_k,v). No loops or parallel edges are created in this step, since v_k is a new vertex and the pairs (u,v) are disjoint. This operation preserves the connectivity of $G[L]$, since each removed edge is replaced by a path through v_k. As v_k is adjacent to vertices in L, it follows that the whole graph G is connected.

Moreover, this raises the degree of v_k to d. For each u,v, the deleted edge removes one incident edge and the two added edges restore the degree to d. All other vertices are unchanged. Hence the resulting graph is d-regular. One additional vertex v_k is introduced, so the total number of vertices is $\widetilde{U}(d,k)$. This completes the construction.

In both cases, adding edges among the leaves does not increase distances in the graph, so the burning argument remains valid. For each $i \in \{1,\dots,k-1\}$, every vertex of the tree F_i lies at distance at most $k-i$ from v_i, since F_i has depth $k-i$. Hence all vertices are burned by time k. Thus $(v_1,\dots,v_{k-1},-)$ is a burning sequence of length k in the odd case, and $(v_1, \ldots, v_{k-1}, v_k)$ is a burning sequence of length k in the even case.

This shows that the construction satisfies the required properties, and hence the bound in Corollary 4.5 is tight. □

To extend this result from burning sequences to the burning number, we require that $U(d,k)$ is strictly increasing in k, which we establish next.

Lemma 4.14. For every $d \geq 3$ and $k \geq 0$,

$$U(d,k+1) - U(d,k) = \frac{d(d-1)^k - 2}{d-2} > 0.$$

In particular, $U(d,k)$ is strictly increasing in k.

Proof. Compute

$$\begin{aligned} U(d,k+1) - U(d,k) &= \frac{d(d-1)^{k+1} - 2(k+1)(d-2) - d}{(d-2)^2} - \frac{d(d-1)^k - 2k(d-2) - d}{(d-2)^2} \\ &= \frac{d(d-1)^k\big((d-1)-1\big) - 2(d-2)}{(d-2)^2} \\ &= \frac{(d-2)\big(d(d-1)^k - 2\big)}{(d-2)^2} \\ &= \frac{d(d-1)^k - 2}{d-2}. \end{aligned}$$

Since $d \geq 3$ and $k \geq 0$, we have $d(d-1)^k \geq 3$, hence the final expression is positive. □

As $U(\Delta,k)$ is strictly increasing in k, the tightness result carries over to the burning number.

Corollary 4.15. Let $d \geq 3$ and $b \geq 2$ be integers. Then there exists a connected d-regular graph G with burning number b such that

$$|V(G)| = \widetilde{U}(d,b).$$

Thus the bound in Corollary 4.6 is tight.

Proof. By Theorem 4.13, there exists a connected d-regular graph G with a burning sequence of length b and $|V(G)| = \widetilde{U}(d,b)$. Hence $b(G) \leq b$.

If $b(G) \leq b-1$, then by Corollary 4.5 we have $|V(G)| \leq \widetilde{U}(d,b-1)$, contradicting the fact that $\widetilde{U}(d,b-1) < \widetilde{U}(d,b)$ (Lemma 4.14). Hence $b(G) = b$. □

This shows that the extremal bound for d-regular graphs is tight for the burning number. A similar construction yields tightness for general graphs with bounded maximum degree.

Corollary 4.16. Let $\Delta \geq 3$ and $k \geq 2$ be integers. Then there exists a connected graph G with a maximum degree Δ and a burning sequence of length k such that

$$|V(G)| = U(\Delta,k).$$

Thus the bound in Theorem 4.3 is tight.

Proof. By Theorem 4.13, there exists a connected Δ-regular graph G with a burning sequence of length k and with $|V(G)| = \widetilde{U}(\Delta,k)$. In particular, if d is even, then $\widetilde{U}(\Delta,k) = U(\Delta,k)$. Hence it remains to consider the case where Δ is odd.

Let $\Delta \geq 3$ be odd. As in the d-regular construction, begin with a rooted $(\Delta, k-1)$-forest with roots $v_1, \ldots, v_{k-1}$ and add one vertex v_k. Then, by construction, G has $U(\Delta,k)$ vertices.

Let S be the set of vertices of degree less than Δ. Then S consists of v_k together with the leaves of the (Δ, k)-forest. Label the vertices in S as $\ell_1, \ldots, \ell_{|S|}$. We now add the edges

$$\ell_1\ell_2,\ \ell_2\ell_3,\ \ldots,\ \ell_{|S|-1}\ell_{|S|},\ \ell_{|S|}\ell_1.$$

In this way, the vertices in S induce a cycle. Since S contains a vertex from each connected component of the forest and the vertex v_k, the resulting graph is connected.

Moreover, every internal vertex of the forest still has degree at most Δ, each leaf, which originally had degree 1, gains two additional edges and thus has degree $3 \le \Delta$, and v_k has degree $2 \le \Delta$. Hence the resulting graph has maximum degree Δ.

Finally, $(v_1, \ldots, v_{k-1}, v_k)$ is a burning sequence of length k. Therefore the bound is attained, and the result follows. □

Corollary 4.17. Let $\Delta \ge 3$ and $b \ge 2$ be integers. Then there exists a connected graph G with a maximum degree Δ and burning number b such that

$$|V(G)| = U(\Delta, b).$$

Thus the bound in Corollary 4.4 is tight.

Proof. By Corollary 4.16, there exists a connected graph G with maximum degree Δ, a burning sequence of length b, and $|V(G)| = U(\Delta, b)$. Hence $b(G) \le b$.

If $b(G) \le b - 1$, then by Theorem 4.3 we have $|V(G)| \le U(\Delta, b-1)$, contradicting the fact that $U(\Delta, b-1) < U(\Delta, b)$ (Lemma 4.14). Hence $b(G) = b$. □

4.3. Asymptotic Behavior

We conclude the chapter by translating these extremal bounds into asymptotic form, which reveals the growth rate of the $\overline{M}(\Delta, k)$ and yields lower bounds on the burning number.

Theorem 4.18. For all $\Delta \ge 3$ and $k \ge 2$,

$$(\Delta - 1)^{k-1} \le \overline{M}(\Delta, k) \le 6(\Delta - 1)^{k-1}.$$

In particular,

$$\overline{M}(\Delta, k) = \Theta\big((\Delta - 1)^{k-1}\big),$$

uniformly in Δ and k.

Lower bound. We first show that $\overline{M}(\Delta, k) \ge (\Delta - 1)^{k-1}$. Let T be the rooted (Δ, k)-tree. Then $\Delta(T) \le \Delta$, T is connected, and burning the root in round 1 burns all of T by round k, so $b(T) \le k$. Moreover,

$$|V(T)| = 1 + \Delta \sum_{i=0}^{k-2} (\Delta - 1)^i = 1 + \Delta \, \frac{(\Delta - 1)^{k-1} - 1}{\Delta - 2} \ge (\Delta - 1)^{k-1},$$

where the last inequality follows by multiplying by $(\Delta - 2)$ and rearranging:

$$(\Delta - 2) + \Delta\big((\Delta - 1)^{k-1} - 1\big) - (\Delta - 2)(\Delta - 1)^{k-1} = 2\big((\Delta - 1)^{k-1} - 1\big) \ge 0.$$

Hence $\overline{M}(\Delta, k) \ge |V(T)| \ge (\Delta - 1)^{k-1}$.

Upper bound. We now show that $\overline{M}(\Delta, k) \le 6(\Delta - 1)^{k-1}$. Let G be connected with $\Delta(G) \le \Delta$ and $b(G) \le k$. Then G admits a burning sequence of length k, and Theorem 4.3 gives

$$|V(G)| \le \frac{\Delta(\Delta - 1)^k - 2k(\Delta - 2) - \Delta}{(\Delta - 2)^2} \le \frac{\Delta(\Delta - 1)^k}{(\Delta - 2)^2}.$$

Rewrite the last term as

$$\frac{\Delta(\Delta-1)^k}{(\Delta-2)^2} = \frac{\Delta(\Delta-1)}{(\Delta-2)^2} \cdot (\Delta-1)^{k-1}.$$

For $\Delta \geq 3$, the factor $\frac{\Delta(\Delta-1)}{(\Delta-2)^2}$ obtains its maximum value 6 at $\Delta = 3$. Thus $|V(G)| \leq 6(\Delta-1)^{k-1}$, so $\overline{M}(\Delta, k) \leq 6(\Delta-1)^{k-1}$. □

Corollary 4.19. Let G be a connected graph on n vertices with maximum degree $\Delta \geq 3$. Then

$$b(G) \geq \left\lceil 1 + \log_{\Delta-1}\left(\frac{(\Delta-2)^2}{\Delta(\Delta-1)}\, n\right)\right\rceil.$$

In particular, $b(G) = \Omega\big(\log_{\Delta-1} n\big)$ for fixed Δ.

Proof. Proof. Let $b := b(G)$. Inspecting the proof of Theorem 4.18, we obtain

$$n = |V(G)| \leq \overline{M}(\Delta, b) \leq \frac{\Delta(\Delta-1)}{(\Delta-2)^2}(\Delta-1)^{b-1}.$$

Rearranging yields

$$b \geq 1 + \log_{\Delta-1}\left(\frac{(\Delta-2)^2}{\Delta(\Delta-1)}\, n\right).$$

Taking ceilings proves the first claim. The asymptotic statement follows immediately for fixed Δ. □

We determined the maximum possible order of a connected graph with bounded maximum degree and fixed burning number, and showed that this extremal bound is attained by explicit constructions. We further refined this bound for connected d-regular graphs, where a parity correction arises. In particular, the general bound is attained for even d, and up to an additive constant of 1 for odd d, while the refined bound $\widetilde{U}(d, b)$ is attained by explicit d-regular constructions.

We now turn to the complementary problem of constructing regular graphs with large burning number.

5

Near-Optimal Constructions for Regular Graphs

We next consider the complementary perspective of fixing the graph order and constructing regular graphs with large burning number. While paths and cycles are extremal for maximum degree 2, the situation is less straightforward for higher degrees. For $d \geq 3$, we aim to construct d-regular graphs in which the burning process spreads as slowly as possible. This suggests graphs with large effective radius, reflected in a large radius, where distant regions remain far apart in terms of graph distance, while each vertex maintains degree exactly d.

To gain insight into the structure of such graphs, we performed a computational study of all 3-regular graphs on at most 24 vertices using an integer linear programming formulation of García-Díaz et al. [11]. The extremal examples consistently exhibited a necklace-like structure, with highly connected local components arranged along a long path or cycle. Among these, cyclic variants appeared easier to generalize.

Motivated by this observation, we introduce a class of d-necklaces with slack vectors **s**, in which dense local components are arranged along a cycle so that the global structure remains elongated while preserving regularity. The slack vector allows for a controlled distribution of vertices across the construction, providing the flexibility needed to analyze extremal behavior.

We first develop the structural properties of these graphs, which we use to analyze their burning number and radius. We then determine the burning number of d-necklaces and show that the construction is asymptotically optimal. Finally, we study the radius of d-necklaces, showing that it matches the general upper bound up to an additive constant of at most 1.

As a special case, the construction with slack vector $\mathbf{s} = \mathbf{0}$ coincides with the $K_k - e$ necklace construction of Griggs and Wu [12], which we observed after finalizing our construction.

5.1. Construction of d-Necklaces

In this section, we introduce the construction used to obtain our extremal examples. The key idea is to build d-regular graphs from smaller local components, which we refer to as beads, and to connect these components in a controlled way.

We begin by defining d-beads with prescribed slack, which serve as the fundamental building blocks of our construction.

Definition 5.1 (d-beads with slack s)**.** Let $d \geq 3$ and $0 \leq s \leq d - 1$ such that $d(d + 1 + s)$ is even.

A graph $B = (V, E)$ with distinguished vertices $a, b \in V$, called the *ports*, is called a *d-bead with slack s* if the following hold:

- $|V| = d + 1 + s$ and $ab \notin E$,

- $\deg(a) = \deg(b) = d - 1$,
- $\deg(u) = d$ for all $u \in V \setminus \{a, b\}$.

Let

$$L := N(a), \qquad R := N(b), \qquad S := L \cap R.$$

Then additionally:

$$L \cup R = V \setminus \{a, b\}, \qquad |L| = |R| = d - 1, \qquad |L \cap R| = d - 1 - s,$$

and

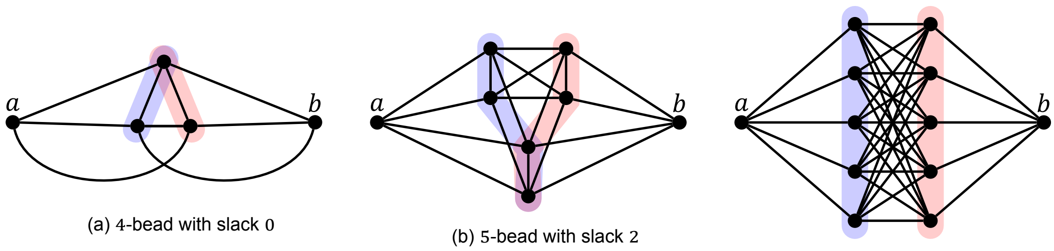
$$\operatorname{dist}(a, b) = \begin{cases} 2, & s \le d - 2, \\ 3, & s = d - 1. \end{cases}$$

Proposition 5.7 establishes the existence of such beads.

Remark 5.2. In general, a d-bead with slack s is not uniquely determined.

We illustrate the notion of d-beads with slack s with examples for different values of d and s.

Example 5.3. Figure 5.1 illustrates three examples of d-beads with slack s, namely $s = 0$, $s = 2$, and $s = d - 1$. In Figures 5.1a and 5.1b, the neighborhoods L and R intersect, so $\operatorname{dist}(a, b) = 2$. In Figure 5.1c, we have $L \cap R = \emptyset$, and thus $\operatorname{dist}(a, b) = 3$.



(a) 4-bead with slack 0

(b) 5-bead with slack 2

(c) 6-bead with slack $5 = d - 1$

Figure 5.1: Examples of d-beads with slack s. The sets $L := N_B(a)$ and $R := N_B(b)$ are shown in blue and red, respectively, with their intersection $L \cap R$ in purple. The ports are labeled a and b.

Theorem 5.4 (Erdős–Gallai [19, Ex. 3.3.29]). Let $d_1 \ge d_2 \ge \cdots \ge d_n \ge 0$ be integers. Then $(d_1, \dots, d_n)$ is *graphical* if and only if $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} \qquad \text{for all } 1 \le k \le n.$$

Corollary 5.5. Let $d_1 \ge \cdots \ge d_n \ge 0$ be integers, and let

$$m(\mathbf{d}) := \max\{\, i \in [n] : d_i \ge i - 1 \,\}.$$

Then $(d_1, \dots, d_n)$ is graphical if and only if $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} \qquad \text{for all } 1 \le k \le m(\mathbf{d}).$$

Proof. By Theorem 5.4, it remains to show that the inequalities for $k > m(\mathbf{d})$ are redundant.

For $k > m(\mathbf{d})$, we have $d_k \le k - 2$, and since the sequence is nonincreasing, $d_i \le k - 2$ for all $i \ge k$. Thus the inequality for k reduces to

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} d_i.$$

Since the inequality holds for $k-1$, we have

$$\sum_{i=1}^{k-1} d_i \le (k-1)(k-2) + \sum_{i=k}^{n} d_i,$$

Adding d_k to both sides yields

$$\sum_{i=1}^{k} d_i \le (k-1)(k-2) + \sum_{i=k+1}^{n} d_i + 2d_k \le k(k-1) + \sum_{i=k+1}^{n} d_i,$$

since $d_k \le k-2$, as required. □

Corollary 5.6. Let $1 \le d \le n-1$ and nd be even. Then there exists a d-regular graph on n vertices.

Proof. Consider the sequence $\mathbf{d} = (d, \dots, d)$. Then

$$m(\mathbf{d}) = \max\{\, i \in [n] : d \ge i-1 \,\} = d+1.$$

Hence, by Corollary 5.5, it suffices to verify the inequalities for $1 \le k \le d+1$.

If $1 \le k \le d$, then $\min(d,k) = k$, and

$$kd \le k(k-1) + (n-k)k = k(n-1),$$

which holds since $d \le n-1$.

If $k = d+1$, then $\min(d, d+1) = d$, and

$$(d+1)d \le d(d+1) + (n-d-1)d,$$

which holds since $d \le n-1$.

Since $\sum_{i=1}^{n} d_i = nd$ is even, the sequence is graphical. □

We now establish the existence of d-beads with slack s.

Proposition 5.7. For every integer $d \ge 3$ and every $0 \le s \le d-1$ with $d(d+1+s)$ even, there exists a d-bead B with slack s.

Proof. We give explicit constructions for the different values of the slack s.

Case $s = 0$. Let $H \cong K_{d-1}$ with vertex set $V(H) = L = R$. Add two new vertices a, b (the ports) and add all edges ax and bx for $x \in V(H)$. Let B be the resulting graph. Then

$$|V(B)| = |V(H)| + 2 = (d-1) + 2 = d+1 = d+1+s.$$

We have $\deg_B(a) = \deg_B(b) = d-1$. Each $x \in V(H)$ has degree $(d-2)+2 = d$ (neighbors in H plus a and b). Moreover, $ab \notin E(B)$ and $\mathrm{dist}_B(a,b) = 2$, since every vertex of $V(H)$ is a common neighbor of a and b. Finally,

$$L \cup R = V(H) = V(B) \setminus \{a,b\}, \qquad |L| = |R| = d-1, \qquad |L \cap R| = |V(H)| = d-1 = d-1-s.$$

Also, $L = N_B(a)$ and $R = N_B(b)$, as required.

Figure 5.2 illustrates this construction.

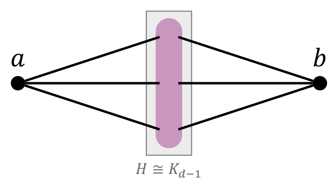


Figure 5.2: A d-bead with slack $s = 0$. The non-port vertices induce a clique $H \cong K_{d-1}$. The sets $L := N_B(a)$ and $R := N_B(b)$ coincide and are shown in blue and red, respectively, with $L \cap R$ in purple. Each vertex in $L = R$ is adjacent to both ports a and b.

Case $s = 1$**.** Since $s = 1$, we have $d + 1 + s = d + 2$, which has the same parity as d. Hence the parity condition that $d(d+1+s)$ is even implies that d is even. Let Z be a set with $|Z| = d-2$, and add two new vertices v_1, v_2. Add two new vertices a, b (the ports) and add all edges az and bz for $z \in Z$, as well as the edges av_1 and bv_2. Add all edges $v_1 z$ and $v_2 z$ for $z \in Z$, and add the edge $v_1 v_2$. Let B be the resulting graph. Then

$$|V(B)| = |Z| + 2 + 2 = (d-2) + 4 = (d+1) + 1 = d + 1 + s.$$

We have $\deg_B(a) = \deg_B(b) = d - 1$. Moreover, $\deg_B(v_1) = \deg_B(v_2) = (d-2) + 1 + 1 = d$. Each vertex $z \in Z$ has degree 4 (adjacent to a, b, v_1, v_2). We require $B[Z]$ to be $(d-4)$-regular, which is possible by Corollary 5.6, since $0 \le d - 4 \le |Z| - 1 = d - 3$ and $(d-4)|Z|$ is even. After adding these edges, every vertex of Z has degree d.

Let

$$L := Z \cup \{v_1\}, \qquad R := Z \cup \{v_2\}.$$

Moreover, $ab \notin E(B)$ and $\mathrm{dist}_B(a, b) = 2$, since every vertex of Z is a common neighbor of a and b. Finally,

$$L \cup R = Z \cup \{v_1, v_2\} = V(B) \setminus \{a, b\}, \qquad |L| = |R| = (d-2)+1 = d-1, \qquad |L \cap R| = |Z| = d-2 = d-1-s.$$

Also, $L = N_B(a)$ and $R = N_B(b)$, as required.

Figure 5.3 illustrates this construction.

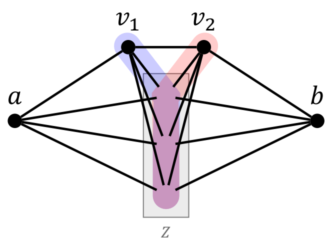


Figure 5.3: A d-bead with slack $s = 1$. The vertices Z induce a $(d-4)$-regular graph. The sets $L := N_B(a)$ and $R := N_B(b)$ are not disjoint and are shown in blue and red, respectively, with $L \cap R$ in purple. Each vertex of Z is adjacent to both ports a and b and to both v_1 and v_2. The vertices v_1 and v_2 are adjacent to each other and to a and b, respectively.

Case $s = 2 < d - 1$**.** Since $s = 2 < d - 1$, we have $d \ge 4$. Let X, Y be disjoint sets with $|X| = |Y| = 2$, and let Z be a set with $|Z| = d-3$. Add two new vertices a, b (the ports), and add all edges ax for $x \in X$ and all edges by for $y \in Y$. Add all edges between X and Y. Furthermore, add all edges az and bz for $z \in Z$, and all edges xz and yz for $x \in X$, $y \in Y$, and $z \in Z$. Let B be the resulting graph. Then

$$|V(B)| = |X| + |Y| + |Z| + 2 = 2 + 2 + (d-3) + 2 = d + 3 = d + 1 + s.$$

We have $\deg_B(a) = \deg_B(b) = |X| + |Z| = 2 + (d-3) = d - 1$. Each vertex in $X \cup Y$ has degree $2 + 1 + (d-3) = d$. Each vertex $z \in Z$ has degree 6 (adjacent to a, b and all vertices of $X \cup Y$).

It remains to adjust the degrees of the vertices in Z.

We distinguish three subcases according to the value of d. Figure 5.4 illustrates the resulting d-beads in each case.

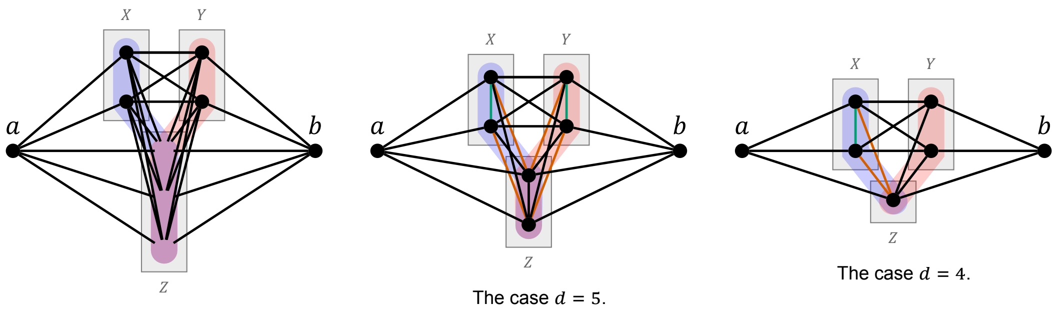


Figure 5.4: A d-bead with slack $s = 2$ in the three subcases $d \geq 6$, $d = 5$, and $d = 4$. The sets $L := N_B(a)$ and $R := N_B(b)$ are shown in blue and red, respectively, with $L \cap R$ in purple. Added edges are shown in green, while deleted edges are shown in orange.

Subcase $d \geq 6$. It suffices to require $B[Z]$ to be $(d-6)$-regular, which is possible by Corollary 5.6, since $0 \leq d-6 \leq |Z|-1 = d-4$ and $(d-6)|Z| = (d-6)(d-3)$ is even. After adding these edges, every vertex of Z has degree d.

For the remaining subcases, we modify the construction slightly.

Subcase $d = 5$. Then $Z = \{z_1, z_2\}$. Remove the edges z_1x_1 and z_2x_2, and add the edge x_1x_2. Similarly, remove the edges z_1y_1 and z_2y_2, and add the edge y_1y_2. Finally, add the edge z_1z_2. Then every vertex in Z has degree 5, and all other degrees remain unchanged.

Subcase $d = 4$. Then $Z = \{z\}$. Remove the edges zx_1 and zx_2, and add the edge x_1x_2. Then $\deg_B(z) = 4$, and all other degrees remain unchanged.

Finally, let

$$L := X \cup Z, \qquad R := Y \cup Z.$$

Then

$$L \cup R = X \cup Y \cup Z = V(B) \setminus \{a, b\}, \qquad |L| = |R| = |X| + |Z| = 2 + (d-3) = d-1,$$

and

$$|L \cap R| = |Z| = d - 3 = d - 1 - s.$$

Moreover, $L = N_B(a)$ and $R = N_B(b)$, since a is adjacent exactly to the vertices in $X \cup Z$ and b exactly to those in $Y \cup Z$. Finally, $ab \notin E(B)$ and $\text{dist}_B(a, b) = 2$, since every vertex of Z is a common neighbor of a and b.

Case $3 \leq s \leq d-2$. Set $t := d - 1 - s$ and take disjoint sets X, Y, Z with $|Z| = t$ and $|X| = |Y| = s$. Let a, b be the ports and define

$$V(B) := \{a, b\} \cup X \cup Y \cup Z, \qquad L := X \cup Z, \qquad R := Y \cup Z.$$

Add all edges au for $u \in L$ and all edges bu for $u \in R$. Then $\deg_B(a) = \deg_B(b) = d - 1$. Since $t > 0$, we have $Z \neq \emptyset$, and thus $\text{dist}_B(a, b) = 2$.

Figure 5.5 shows the following construction.

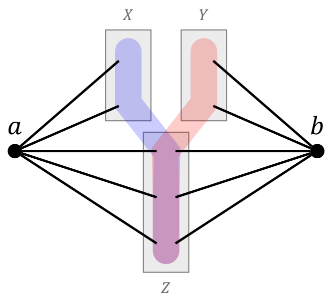


Figure 5.5: A d-bead with slack $3 \le s \le d-2$. The sets $L := N_B(a) = X \cup Z$ and $R := N_B(b) = Y \cup Z$ are shown in blue and red, respectively, with $L \cap R = Z$ in purple. Each vertex of Z is adjacent to both ports a and b, vertices in X only to a, and vertices in Y only to b.

It remains to add edges inside $W := X \cup Y \cup Z$ so that every vertex in $V(B) \setminus \{a, b\}$ has degree d. After adding the port edges, the required internal degrees in W are

$$\deg_{B[W]}(z) = d-2 \quad (z \in Z), \qquad \deg_{B[W]}(w) = d-1 \quad (w \in X \cup Y).$$

Thus it suffices to construct a simple graph H on W with degree sequence $\mathbf{d} = (d_1, \ldots, d_{d-1+s})$, where

$$d_i = \begin{cases} d-1, & 1 \le i \le 2s, \\ d-2, & 2s+1 \le i \le d-1+s. \end{cases}$$

This sequence is non-increasing. We verify that it is graphical using Corollary 5.5.

We first verify that the degree sum is even. Indeed, the degree sum equals

$$\sum_{i=1}^{d-1+s} d_i = 2s(d-1) + (d-1-s)(d-2).$$

Since

$$d(d+1+s) - (2s(d-1) + (d-1-s)(d-2)) = 4d-2$$

is even, and $d(d+1+s)$ is even by assumption, it follows that the degree sum is even.

It remains to verify the inequalities for $1 \le k \le m(\mathbf{d})$, where

$$m(\mathbf{d}) := \max\{\, i : d_i \ge i-1 \,\} = \begin{cases} d, & \text{if } 2s \ge d, \\ d-1, & \text{if } 2s \le d-1. \end{cases}$$

For $1 \le k \le m(\mathbf{d})$, we verify that

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{d+s-1} \min(d_i, k). \tag{5.1}$$

For the left-hand side (LHS), we distinguish two cases:

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{2s} (d-1) + \sum_{i=2s+1}^{k} (d-2) = k(d-2) + 2s \qquad \text{if } k > 2s \tag{5.2}$$

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} (d-1) = k(d-1) = k(d-2) + k \qquad \text{if } k \le 2s \tag{5.3}$$

Now for the right-hand side (RHS) of (5.1), we have 2 subcases

Subcase 1: $1 \le k \le d-2$.

$$\begin{aligned} RHS &= k(k-1) + \sum_{i=k+1}^{d+s-1} k \\ &= k(k-1) + (d+s-1-k)k \\ &= k(d-2) + ks. \end{aligned}$$

Now to satisfy (5.1), we distinguish two subsubcases

Subsubcase (a). $k > 2s$. Using (5.2), and since $s \ge 1$, we have $2s \le ks$. Hence

$$LHS = k(d-2) + 2s \le k(d-2) + ks = RHS.$$

Subsubcase (b). $k \le 2s$. Using (5.3), and since $s \ge 1$, we have $k \le ks$. Hence

$$LHS = k(d-2) + k \le k(d-2) + ks = RHS.$$

Therefore (5.1) holds for this subcase.

Subcase 2: $d-1 \le k \le m(\mathbf{d})$. We distinguish two subsubcases.

Subsubcase (a). $k \le 2s$. We have

$$\begin{aligned} RHS &= k(k-1) + \sum_{i=k+1}^{2s} (d-1) + \sum_{i=2s+1}^{d+s-1} (d-2) \\ &= k(k-1) + (2s-k)(d-1) + (d-s-1)(d-2) \\ &= k(k-1) + (2s-k) + (d+s-1-k)(d-2) \\ &\ge k(k-1) + (d+s-1-k)(d-2). \end{aligned}$$

Since $k \le m(\mathbf{d}) \le d$ and $s \ge 3$, we have $d+s-1-k \ge s-1 \ge 2$. Moreover, since $k \le 2s \le 2(d-2)$, it follows that

$$(d+s-1-k)(d-2) \ge 2(d-2) \ge k.$$

Hence

$$RHS \ge k(k-1) + k = k^2 \ge k(d-1) = LHS,$$

where the last inequality uses $k \ge d-1$.

Subsubcase (b). $k > 2s$. we have $2s+1 \le k \le m(\mathbf{d})$. By the definition of $m(\mathbf{d})$, this implies $d_{2s+1} \ge (2s+1)-1 = 2s$. Since $d_{2s+1} = d-2$, it follows that $2s \le d-2$. In particular, $k \le m(\mathbf{d}) = d-1$, and therefore $d+s-1-k \ge d+s-1-(d-1) = s \ge 2$. Hence

$$\begin{aligned} RHS &= k(k-1) + \sum_{i=k+1}^{d+s-1} (d-2) \\ &= k(k-1) + (d+s-1-k)(d-2) \\ &\ge k(k-1) + 2(d-2). \end{aligned}$$

Since $s \le d-2$, we have $2(d-2) \ge 2s$. Moreover, since $k \ge d-1$, it follows that

$$RHS \ge k(k-1) + 2s \ge k(d-2) + 2s = LHS.$$

Thus (5.1) holds for this subcase.

Therefore the degree sequence is graphical, and there exists a simple graph H on W with the prescribed degrees. Set $B := H \cup \{au : u \in L\} \cup \{bu : u \in R\}$. Then every vertex in $V(B) \setminus \{a, b\}$ has degree d, while $\deg_B(a) = \deg_B(b) = d - 1$, and $ab \notin E(B)$.

Finally,

$$L \cup R = X \cup Y \cup Z = V(B) \setminus \{a, b\}, \qquad |L| = |R| = |X| + |Z| = t + s = d - 1,$$

and

$$|L \cap R| = |Z| = t = d - 1 - s.$$

Moreover, $L = N_B(a)$ and $R = N_B(b)$, since a is adjacent exactly to the vertices in $X \cup Z$ and b exactly to those in $Y \cup Z$. Finally, $ab \notin E(B)$ and $\mathrm{dist}_B(a, b) = 2$, since $X \neq \emptyset$.

Subcase $s = d - 1$. Let $H \cong K_{d-1,d-1}$ with bipartition (L, R), where $|L| = |R| = d - 1$. Add two new vertices a, b (the ports) and add all edges $a\ell$ for $\ell \in L$ and all edges br for $r \in R$. Let B be the resulting graph. Then

$$|V(B)| = |L| + |R| + 2 = 2(d - 1) + 2 = d + 1 + (d - 1) = d + 1 + s.$$

We have $\deg_B(a) = \deg_B(b) = d - 1$. Each $\ell \in L$ has degree $(d - 1) + 1 = d$ (neighbors in R plus a), and similarly each $r \in R$ has degree d. Moreover, $ab \notin E(B)$ and $\mathrm{dist}_B(a, b) = 3$.

Finally,

$$L \cup R = V(B) \setminus \{a, b\}, \qquad |L| = |R| = d - 1, \qquad |L \cap R| = \emptyset,$$

so $|L \cap R| = 0 = d - 1 - s$. Moreover, $L = N_B(a)$ and $R = N_B(b)$, as required.

Figure 5.6 illustrates this construction.

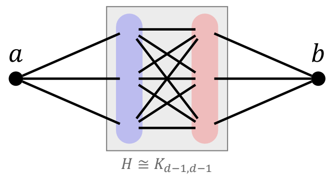


Figure 5.6: A d-bead with slack $s = d - 1$. The non-port vertices form a complete bipartite graph $H \cong K_{d-1,d-1}$ with bipartition (L, R). The sets $L := N_B(a)$ and $R := N_B(b)$ are disjoint and are shown in blue and red, respectively. Each vertex of L is adjacent to the port a, and each vertex of R to the port b.

□

Having established the existence of d-beads with slack s, we now use them as building blocks to construct d-necklaces.

Definition 5.8 (d-necklace with slack vector **s**)**.** Fix $d \geq 3$ and $k \geq 1$. Let $\mathbf{s} = (s_1, \ldots, s_k)$ be a sequence such that, for all i, $0 \leq s_i \leq d - 1$, $d(d + 1 + s_i)$ is even, and $\sum_{i=1}^{k} s_i \leq d$.

For each $i \in \{1, \ldots, k\}$, let B_i be a d-bead with slack s_i, and denote its ports by a_i, b_i.

We call s_i the *slack* of bead i. A bead with $s_i = 0$ is called *standard*, a bead with $s_i > 0$ is called *special*, and a bead with $s_i = d - 1$ has *maximal slack*.

Form a graph G from the disjoint union of $B_1, \ldots, B_k$ by adding the *connector edges*

$$b_i a_{i+1} \qquad \text{for } i = 1, \ldots, k,$$

where indices are taken modulo k.

Then each port gains exactly one additional neighbour outside its bead, and hence every vertex in G has degree d.

We call G the *d-necklace on n vertices with slack vector* **s** *and k beads*, where $n = (d + 1)k + |\mathbf{s}|$ with $|\mathbf{s}| = \sum_{i=1}^{k} s_i$

Remark 5.9. Since $\sum_{i=1}^{k} s_i \le d$, there can be at most one bead with maximal slack $s_i = d-1$, as $2(d-1) > d$ for $d \ge 3$.

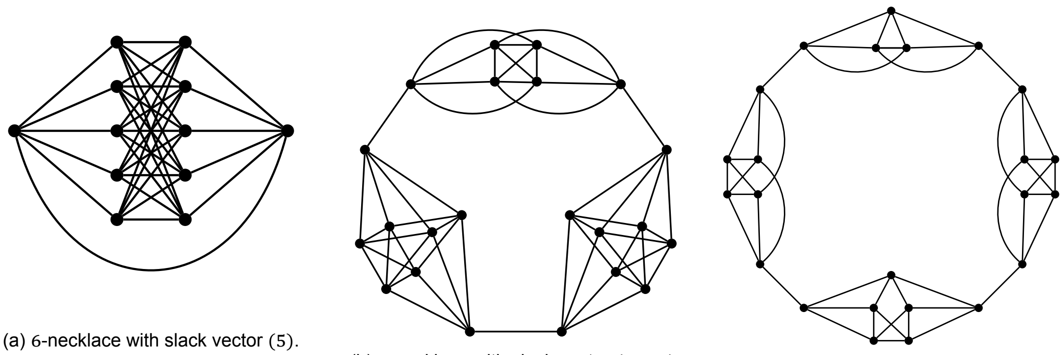
This restriction is imposed since concentrating slack in multiple maximal beads yields a shorter underlying cycle than distributing the slack to create additional beads. As the radius depends only on the underlying cycle length, and the burning number is controlled by it up to an additive constant of at most 1, such configurations are not expected to be extremal.

Remark 5.10. By cyclic symmetry, a d-necklace with slack vector $(s_1, \dots, s_k)$ can equivalently be described as a d-necklace with slack vector $(s_i, s_{i+1}, \dots, s_k, s_1, \dots, s_{i-1})$ for any $i \in \{1, \dots, k\}$.

Thus, a d-necklace is obtained by arranging d-beads cyclically and connecting their ports so that the resulting graph is d-regular.

We illustrate the notion of d-necklaces with slack vectors **s** with examples for different values of d and **s**.

Example 5.11. Figure 5.7 illustrates three examples of d-necklaces with slack vectors **s**, namely **s** $= (5)$, **s** $= (0, 2, 2)$, and **s** $= (0, 1, 2, 1)$. In Figure 5.7a, we have a single bead with maximal slack. In Figure 5.7b, there are three beads, one with slack 0 and two with slack 2. In Figure 5.7c, there are four beads, one with slack 0, two with slack 1, and one with slack 2, for a total slack of 4, which is the maximum possible total slack for a 4-necklace.



(a) 6-necklace with slack vector (5).

(b) 5-necklace with slack vector $(0, 2, 2)$.

(c) 4-necklace with slack vector $(0, 1, 2, 1)$.

Figure 5.7: Examples of d-necklaces with slack vectors **s** . Each subfigure shows a different distribution of slack values along the cycle.

To analyze distances in these graphs, we introduce a representation that captures their global structure in a simpler form.

For a d-necklace with slack vector **s**, we define an auxiliary mapping onto a cycle, that will be used to analyze the burning number and the radius. The idea is to collapse each bead to a short segment of a cycle, where beads with maximal slack occupy four consecutive vertices, while the remaining beads occupy three.

Definition 5.12 (Cycle representation (φ, ψ) of a d-necklace)**.** Let G be a d-necklace with slack vector **s**. Define

$$J := \{\, i \in \{1, \dots, k\} : s_i = d-1 \,\}, \qquad \ell := 3k + |J|.$$

Thus J indexes the beads with maximal slack $s_i = d-1$.

Let C_ℓ be the cycle on vertices $v_1, \dots, v_\ell$. Define

$$p(i) := 1 + 3(i-1) + \big|\{\, j \in J : j < i \,\}\big|.$$

The function $p(i)$ accounts for the varying bead sizes (of size 3 or 4), ensuring correct alignment of the indices.

Define maps $\varphi : V(G) \to V(C_\ell)$ and $\psi : V(C_\ell) \to V(G)$ as follows. The map φ collapses each bead to a short interval of the cycle, while ψ selects a representative vertex from each fibre.

- if $i \notin J$, then
$$\varphi : \begin{cases} a_i \mapsto v_{p(i)}, \\ V(B_i) \setminus \{a_i, b_i\} \mapsto v_{p(i)+1}, \\ b_i \mapsto v_{p(i)+2}, \end{cases} \qquad \psi : \begin{cases} v_{p(i)} \mapsto a_i, \\ v_{p(i)+1} \mapsto c_i, \\ v_{p(i)+2} \mapsto b_i, \end{cases}$$
where $c_i \in L_i \cap R_i$ is chosen arbitrarily.
- if $i \in J$, then
$$\varphi : \begin{cases} a_i \mapsto v_{p(i)}, \\ L_i \mapsto v_{p(i)+1}, \\ R_i \mapsto v_{p(i)+2}, \\ b_i \mapsto v_{p(i)+3}, \end{cases} \qquad \psi : \begin{cases} v_{p(i)} \mapsto a_i, \\ v_{p(i)+1} \mapsto c_i, \\ v_{p(i)+2} \mapsto d_i, \\ v_{p(i)+3} \mapsto b_i, \end{cases}$$
where $c_i \in L_i$, $d_i \in R_i$ arbitrarily, such that $c_i d_i \in E(B_i)$.

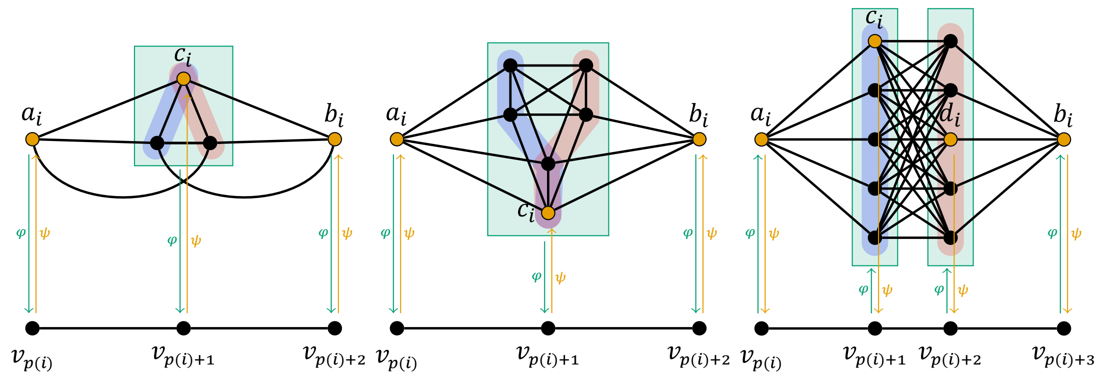
We refer to the pair (φ, ψ) as a *cycle representation* of G, with C_ℓ as its *underlying cycle*.

Remark 5.13. In general, the map ψ is not uniquely determined, while φ is.

We illustrate the construction of the maps φ and ψ on d-beads for different values of d and s.

Example 5.14. Figure 5.8 illustrates three examples of the construction of the maps φ (green) and ψ (orange) on d-beads with slack s. Each figure corresponds to a bead B_i taken from a d-necklace with slack vector **s**, where we focus on the local structure of a single bead. Accordingly, the ports are denoted by a_i and b_i, and the corresponding neighborhoods by L_i and R_i. The vertices selected for the map ψ are highlighted.

In Figure 5.8a, the intersection $L_i \cap R_i$ consists of a single vertex, so the choice of c_i is unique, and consequently the map ψ is uniquely determined on this bead. In Figures 5.8b and 5.8c, the choice of vertices is no longer unique, and multiple valid mappings can be obtained. Moreover, Figure 5.8c corresponds to a maximal slack bead, which requires four vertices on the cycle, whereas Figures 5.8a and 5.8b require only three.



(a) A 4-bead with slack 0 (b) A 5-bead with slack 2 (c) A 6-bead with slack $d-1$

Figure 5.8: Examples of the construction of the maps φ and ψ on d-beads with slack s. The sets L_i and R_i are shown in blue and red, respectively, with $L_i \cap R_i$ in purple. The vertices selected for φ are highlighted, and the maps φ (green) and ψ (orange) are indicated.

This completes the construction of d-necklaces with slack vectors **s**, together with the associated cycle representation. In the next section, we establish structural properties of these graphs that will be used in the analysis of their burning number and radius.

5.2. Structural Properties of d-Necklaces

We now establish structural properties of d-necklaces that will be used in the analysis of their burning number and radius. In particular, we relate distances in a necklace graph to those in its cycle representation, allowing us to transfer arguments to the underlying cycle.

To analyze the burning process, we use of distance domination. If a set of vertices dominates the graph within distance k, then burning the induced subgraph on that set allows us to burn the entire graph with at most an additional k steps.

To bound the burning number of a d-necklace with slack vector **s**, we first relate distances in G to those in its cycle representation, which allows us to apply this argument to the underlying cycle.

The following lemmas relate distances in G and its cycle representation and form the basis for our analysis.

Lemma 5.15. Let G be a d-necklace with cycle representation (φ, ψ) and underlying cycle C_ℓ. Then for all $u, v \in V(G)$,

$$\mathrm{dist}_{C_\ell}(\varphi(u), \varphi(v)) \le \mathrm{dist}_G(u, v).$$

Proof. We first note that φ is distance non-increasing on edges of G. Indeed, if $xy \in E(G)$, then by construction of φ, the vertices $\varphi(x)$ and $\varphi(y)$ are either equal or adjacent in C_ℓ. Hence

$$\mathrm{dist}_{C_\ell}\big(\varphi(x), \varphi(y)\big) \le 1.$$

Now let $u, v \in V(G)$, and let $P : u = x_0, x_1, \dots, x_m = v$ be a shortest u–v path in G, so that $m = \mathrm{dist}_G(u, v)$. Applying φ to the vertices of P yields a walk $\varphi(x_0), \varphi(x_1), \dots, \varphi(x_m)$ in C_ℓ, where each consecutive pair is either equal or adjacent. Therefore this is a walk in C_ℓ of length at most m, and so

$$\mathrm{dist}_{C_\ell}\big(\varphi(u), \varphi(v)\big) \le m = \mathrm{dist}_G(u, v),$$

as required. □

Lemma 5.16. Let G be a d-necklace with cycle representation (φ, ψ) and underlying cycle C_ℓ. Then for all $u', v' \in V(C_\ell)$,

$$\mathrm{dist}_G\big(\psi(u'), \psi(v')\big) \le \mathrm{dist}_{C_\ell}(u', v').$$

Proof. We first note that ψ is distance non-increasing on edges of C_ℓ. Indeed, if $x'y' \in E(C_\ell)$, then by construction of ψ, the vertices $\psi(x')$ and $\psi(y')$ are either equal or adjacent in G. Hence

$$\mathrm{dist}_G\big(\psi(x'), \psi(y')\big) \le 1.$$

Now let $u', v' \in V(C_\ell)$, and let $P : u' = x_0, x_1, \dots, x_m = v'$ be a shortest u'–v' path in C_ℓ, so that $m = \mathrm{dist}_{C_\ell}(u', v')$. Applying ψ to the vertices of P yields a walk $\psi(x_0), \psi(x_1), \dots, \psi(x_m)$ in G, where each consecutive pair is either equal or adjacent. Therefore this is a walk in G of length at most m, and so

$$\mathrm{dist}_G\big(\psi(u'), \psi(v')\big) \le m = \mathrm{dist}_{C_\ell}(u', v'),$$

as required. □

These structural properties form the basis for the analysis of the burning number and the radius of d-necklaces in the subsequent sections.

5.3. Burning Number of d-Necklaces

We now determine the burning number of d-necklaces. Using the structural properties established in the previous section, we relate the burning process in G to that in its cycle representation. This allows us to transfer the problem to the cycle C_ℓ, where the burning number is well understood.

We first establish bounds on the burning number of d-necklaces, and then show that these bounds are tight. This will allow us to compare our construction to a known asymptotic upper bound.

Theorem 5.17. Let G be a d-necklace with underlying cycle C_ℓ. Then

$$b(C_\ell) \le b(G) \le b(C_\ell) + 1.$$

Proof. Let (φ, ψ) be a cycle representation of G, with underlying cycle C_ℓ.

Upper bound. We first show that $b(G) \leq b(C_\ell) + 1$. Let $b := b(C_\ell)$, and let $(x_1, \dots, x_b)$ be an optimal burning sequence of C_ℓ. Define $z_j := \psi(x_j)$ for $j \in \{1, \dots, b\}$. We show that G can be burned in at most $b + 1$ rounds.

Let $S := \psi(V(C_\ell))$. Since every non-port vertex of G is adjacent to a port, and all ports lie in S, it follows that S is a 1-distance dominating set of G. Therefore, by Lemma 3.7,

$$b(G) \leq b(G[S]) + 1.$$

Finally, by Lemma 5.16, distances do not increase under ψ, so the lifted sequence $(z_1, \dots, z_b)$ burns $G[S]$. Hence $b(G[S]) \leq b = b(C_\ell)$. Consequently,

$$b(G) \leq b + 1 = b(C_\ell) + 1.$$

Lower bound. We now show that $b(G) \geq b(C_\ell)$. Let $b' := b(G)$, and let $(y_1, \dots, y_{b'})$ be an optimal burning sequence of G. Define $w_i := \varphi(y_i) \in V(C_\ell)$ for $i = 1, \dots, b'$. Since vertices in the same bead have the same image under φ, the sequence $(w_1, \dots, w_{b'})$ may contain repetitions. When this happens we may regard all but the first occurrence as placeholders "–": repeating a source has no effect on which vertices are burned, so it does not affect validity of the burning sequence.

We claim that $(w_1, \dots, w_{b'})$ is a burning sequence of C_ℓ.

By Lemma 5.15, for all $u, v \in V(G)$,

$$\mathrm{dist}_{C_\ell}\big(\varphi(u), \varphi(v)\big) \leq \mathrm{dist}_G(u, v).$$

Hence for each i and each $r \geq 0$,

$$\varphi\big(N_r^G[y_i]\big) \subseteq N_r^{C_\ell}[w_i].$$

Since $(y_1, \dots, y_{b'})$ is a burning sequence of G, we have

$$V(G) = \bigcup_{i=1}^{b'} N_{b'-i}^G[y_i].$$

Applying φ and using surjectivity,

$$V(C_\ell) = \varphi(V(G)) = \varphi\Big(\bigcup_{i=1}^{b'} N_{b'-i}^G[y_i]\Big) \subseteq \bigcup_{i=1}^{b'} \varphi\big(N_{b'-i}^G[y_i]\big) \subseteq \bigcup_{i=1}^{b'} N_{b'-i}^{C_\ell}[w_i].$$

Thus $(w_1, \dots, w_{b'})$ is a burning sequence of C_ℓ, and therefore $b(C_\ell) \leq b' = b(G)$.

□

Corollary 5.18. There exist a d_1-necklace G_1 and a d_2-necklace G_2, with underlying cycles C_{ℓ_1} and C_{ℓ_2}, respectively, such that

$$b(G_1) = b\big(C_{\ell_1}\big) \quad \text{and} \quad b(G_2) = b\big(C_{\ell_2}\big) + 1.$$

In particular, the bounds in Theorem 5.17 are tight.

Lower bound. Consider the d-necklace with slack vector (0) on $d + 1$ vertices. Let v be an arbitrary vertex. Then $(v, –)$ is a burning sequence of length 2, since v is adjacent to all other vertices. Thus $b(G) \leq 2$. On the other hand, since the slack vector is (0), the necklace has no maximal-slack bead, and thus $\ell = \frac{3n}{d+1}$. Therefore, by Theorem 5.17,

$$2 = \left\lceil \sqrt{\frac{3(d+1)}{d+1}} \right\rceil = \left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil = \left\lceil \sqrt{\ell} \right\rceil = b(C_\ell) \leq b(G).$$

Hence $b(G) = 2$, and the lower bound is attained.

Upper bound. Consider a 4-necklace with slack vector (2) on 7 vertices. By Theorem 5.17,

$$2 \leq b(G) \leq 3.$$

We show that $b(G) \neq 2$. Indeed, every vertex has closed neighborhood of size at most 5, so after the first burning step at least two vertices remain unburned. Therefore no burning sequence of length 2 exists. Thus $b(G) = 3$. Hence the upper bound is attained. □

Corollary 5.19. Let G be a d-necklace with slack vector **s** on $n = (d+1)k + |\mathbf{s}|$ vertices. Then

$$\left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil - 1 \leq b(G) \leq \left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil + 1.$$

Proof. Proof. Let j be the number of maximal-slack beads of G. Since $|\mathbf{s}| \leq d$, we have $j \in \{0, 1\}$. Hence the underlying cycle C_ℓ has length $\ell = 3k + j$. By Theorem 5.17,

$$\left\lceil \sqrt{3k+j} \right\rceil = \left\lceil \sqrt{\ell} \right\rceil = b(C_\ell) \leq b(G) \leq b(C_\ell) + 1 = \left\lceil \sqrt{3k+j} \right\rceil + 1.$$

Since $0 \leq |\mathbf{s}| \leq d$, we have $0 \leq \frac{3|\mathbf{s}|}{d+1} < 3$, and therefore

$$3k \leq \frac{3n}{d+1} = 3k + \frac{3|\mathbf{s}|}{d+1} < 3k + 3.$$

Lower bound. We first show that $b(G) \geq \left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil - 1$. It follows that

$$\left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil \leq \left\lceil \sqrt{3k+3} \right\rceil \leq \left\lceil \sqrt{3k} \right\rceil + 1 \leq \left\lceil \sqrt{3k+j} \right\rceil + 1,$$

and hence

$$\left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil - 1 \leq \left\lceil \sqrt{3k+j} \right\rceil \leq b(G).$$

Upper bound. We now show that $b(G) \leq \left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil + 1$. If $j = 0$, then

$$b(G) \leq \left\lceil \sqrt{3k} \right\rceil + 1 \leq \left\lceil \sqrt{\frac{3n}{(d+1)}} \right\rceil + 1.$$

If $j = 1$, then $|\mathbf{s}| \geq d - 1$, and thus

$$\frac{3|\mathbf{s}|}{d+1} \geq \frac{3(d-1)}{d+1} \geq \frac{3}{2}.$$

Hence

$$3k = \frac{3n}{d+1} - \frac{3|\mathbf{s}|}{d+1} \leq \frac{3n}{d+1} - \frac{3}{2},$$

and therefore

$$b(G) = \left\lceil \sqrt{3k+1} \right\rceil + 1 \leq \left\lceil \sqrt{\frac{3n}{d+1} - \frac{1}{2}} \right\rceil + 1 \leq \left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil + 1$$

□

We now compare our construction with the asymptotic upper bound of Martinsson (Theorem 3.8), which for fixed minimum degree d states that

$$b(G) \leq (1 + o(1))\sqrt{\frac{3n}{d+1}}.$$

For our necklace graphs, by Corollary 5.19, we have

$$\left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil - 1 \leq b(G) \leq \left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil + 1,$$

Hence the asymptotic upper bound of Martinsson is attained up to an additive constant of at most 1 within the class of d-regular graphs, and is therefore tight up to lower-order terms.

We now turn to the radius of d-necklaces, where we analyze the distance structure of these graphs more directly.

5.4. Radius of d-Necklaces

We now determine the radius of d-necklaces. Using the cycle representation, we relate distances in G to those in the underlying cycle C_ℓ, allowing us to transfer the problem to a setting where distances are easier to control.

We begin by recalling a general upper bound on the radius of graphs with given minimum degree, as established by Kim et al..

Theorem 5.20 (Kim et al., 2012, [13])**.** Let G be a connected graph on n vertices with minimum degree δ and radius at least 3. Then

$$\mathrm{rad}(G) \leq \left\lfloor \frac{3n}{2(\delta+1)} \right\rfloor.$$

Moreover, this bound is tight.

We now show that the radius of d-necklaces can be determined exactly and relates closely to this bound.

To control distances within individual beads, we first establish simple distance domination properties with respect to the ports.

Lemma 5.21. Let B be a d-bead with slack s and ports a, b. Assume that $s \leq d-2$. Then

$$V(B) \subseteq N_2[a].$$

Proof. Let $L := N_1(a)$ and $R := N_1(b)$. Every vertex in L lies at distance at most 1 from a, and the port b lies at distance 2 from a. Hence these vertices are covered.

It remains to consider vertices in $R \setminus L$. Let $v \in R \setminus L$. If v is adjacent to some vertex $u \in L$, then $\mathrm{dist}(a, v) = 2$, and we are done. So assume that v has no neighbor in L.

Since $v \in R$, it is adjacent to b, and all its remaining neighbors lie in $R \setminus L$. Hence

$$\deg(v) \leq |R \setminus L| + 1.$$

As $s \leq d-2$, we have $|R \setminus L| = s \leq d-2$, and thus

$$\deg(v) \leq (d-2) + 1 = d - 1,$$

contradicting that G is d-regular.

Therefore, v must be adjacent to some vertex in L, and hence $\mathrm{dist}(a, v) = 2$. Thus every vertex of B lies within distance at most 2 from a. ☐

Lemma 5.22. Let B be a d-bead with slack s and ports a, b. Then

$$V(B) \subseteq N_3[a].$$

Proof. If $s \leq d-2$, then the claim follows from Lemma 5.21. Thus it remains to consider the case when $s = d-1$, i.e. B has maximal slack.

Let $L := N_1(a)$ and $R := N_1(b)$. Every vertex in L lies at distance at most 1 from a, and the port b lies at distance 3 from a. Hence these vertices are covered.

It remains to consider vertices in $R \setminus L = R$, as B is maximal. Let $v \in R \setminus L$. We show that $\mathrm{dist}(a, v) \leq 3$.

Suppose for contradiction that $\mathrm{dist}_B(a, v) \geq 4$. Then v is not adjacent to any vertex of L, since otherwise $\mathrm{dist}(a, v) = 2$. Since $v \in R$, it is adjacent to b.

Because $s = d-1$, we have $L \cap R = \emptyset$, and by construction there exists at least one edge between L and R. Let $w \in R$ be a vertex adjacent to some $u \in L$. Then any neighbor of w in R lies within distance at most 3 from a. Hence v cannot be adjacent to such a vertex w.

Thus v is missing at least one possible neighbor in R, and therefore

$$\deg(v) \leq 1 + (|R| - 2).$$

Since $|R| = d-1$, it follows that

$$\deg(v) \leq 1 + (d-1) - 2 = d - 2,$$

contradicting that G is d-regular.

Therefore $\mathrm{dist}(a, v) \leq 3$, and hence every vertex of B lies within distance at most 3 from a. □

Using these local properties together with the cycle representation, we can now relate distances in G to those in the underlying cycle.

Theorem 5.23. Let G be a d-necklace with underlying cycle C_ℓ. Then

$$\mathrm{rad}(G) = \mathrm{rad}(C_\ell).$$

Proof. Let C_ℓ be the underlying cycle of G, and let (φ, ψ) be a cycle representation.

Upper bound. We first show that $\mathrm{rad}(G) \leq \mathrm{rad}(C_\ell)$. We first show that $\mathrm{rad}(G) \leq \mathrm{rad}(C_\ell)$. Let $R := \mathrm{rad}(C_\ell)$. Since C_ℓ is vertex-transitive, every vertex is a center, that is, $\mathrm{ecc}_{C_\ell}(c) = R$ for all $c \in V(C_\ell)$. Choose $c \in V(C_\ell)$ arbitrarily, and set $x := \psi(c)$. Then the set $V(C_\ell) \setminus N_{R-1}^{C_\ell}[c]$ consists of either one vertex or two adjacent vertices. We distinguish these two cases.

Case 1. The ball $N_{R-1}^{C}[c]$ misses exactly one vertex. By cyclic symmetry, we may choose c so that this omitted vertex is not mapped to a port under ψ. Hence all ports of G lie in $\psi(N_{R-1}^{C}[c])$. For any such port p, we have

$$\mathrm{dist}_G(x, p) \leq R - 1$$

by Lemma 5.16. Since every non-port vertex is adjacent to a port, it follows that every vertex of G lies within distance at most $(R-1) + 1 = R$ from x.

Case 2. The ball $N_{R-1}^{C}[c]$ misses exactly two adjacent vertices. By cyclic symmetry, we may choose c so that the two omitted vertices are mapped under ψ to two adjacent ports. Since adjacent ports lie in different beads, these ports belong to distinct beads. All other ports lie in $\psi(N_{R-1}^{C_\ell}[c])$, and hence are within distance at most $R-1$ from x. Consequently, every vertex not contained in these two beads lies within distance at most R from x.

It remains to consider the vertices in these two beads. By symmetry, it suffices to consider one of them. Let B be such a bead, and let a and b denote its ports, where $a \in \psi(N_{R-1}^{C_\ell}[c])$ and $b \notin \psi(N_{R-1}^{C_\ell}[c])$. We distinguish two subcases according to the slack of B.

Subcase (a). The bead B has non-maximal slack. Since $\mathrm{dist}_G(a,b) = 2$, by Lemma 5.15, $a \in \psi(N^C_{R-2}[c])$. Therefore

$$\mathrm{dist}_G(x,a) \le R - 2$$

by Lemma 5.16. Moreover, by Lemma 5.21, every vertex $v \in V(B)$ satisfies $\mathrm{dist}_B(a,v) \le 2$. Hence

$$\mathrm{dist}_G(x,v) \le \mathrm{dist}_G(x,a) + \mathrm{dist}_G(a,v) \le (R-2) + 2 = R$$

for all $v \in V(B)$.

Subcase (b). The bead B has maximal slack. Since $\mathrm{dist}_G(a,b) = 3$, by Lemma 5.15, $a \in \psi(N^C_{R-3}[c])$. Therefore

$$\mathrm{dist}_G(x,a) \le R - 3$$

by Lemma 5.16. Moreover, by Lemma 5.22, every vertex $v \in V(B)$ satisfies $\mathrm{dist}_B(a,v) \le 3$. Hence

$$\mathrm{dist}_G(x,v) \le \mathrm{dist}_G(x,a) + \mathrm{dist}_G(a,v) \le (R-3) + 3 = R$$

for all $v \in V(B)$.

In either case, every vertex of G lies within distance at most R from x, and therefore

$$\mathrm{rad}(G) \le R = \mathrm{rad}(C_\ell).$$

Lower bound. We now show that $\mathrm{rad}(G) \ge \mathrm{rad}(C_\ell)$. Let $x \in V(G)$ and $x' = \varphi(x) \in V(C_\ell)$. For any vertex $y' \in V(C_\ell)$, choose a vertex $y \in \varphi^{-1}(y')$, which is possible since φ is surjective. By Lemma 5.15,

$$\mathrm{dist}_{C_\ell}(x',y') = \mathrm{dist}_{C_\ell}(\varphi(x),\varphi(y)) \le \mathrm{dist}_G(x,y).$$

Taking the maximum over all $y' \in V(C_\ell)$ gives

$$\mathrm{ecc}_{C_\ell}(x') \le \mathrm{ecc}_G(x).$$

Since this holds for every $x \in V(G)$, and since φ is surjective, it follows that

$$\mathrm{rad}(C_\ell) = \min_{x' \in V(C_\ell)} \mathrm{ecc}_{C_\ell}(x') = \min_{x \in V(G)} \mathrm{ecc}_{C_\ell}(\varphi(x)) \le \min_{x \in V(G)} \mathrm{ecc}_G(x) = \mathrm{rad}(G).$$

□

As a consequence, we obtain the following bounds on the radius.

Corollary 5.24. Let G be a d-necklace with slack vector **s** on $n = (d+1)k + |\mathbf{s}|$ vertices. Then

$$\left\lfloor \frac{3n}{2(d+1)} \right\rfloor - 1 \le \mathrm{rad}(G) \le \left\lfloor \frac{3n}{2(d+1)} \right\rfloor.$$

Proof. Let j be the number of maximal-slack beads of G. Since $|\mathbf{s}| \le d$, we have $j \in \{0,1\}$. Hence the underlying cycle C_ℓ has length $\ell = 3k + j$. By Theorem 5.23,

$$\mathrm{rad}(G) = \mathrm{rad}(C_\ell) = \left\lfloor \frac{\ell}{2} \right\rfloor = \left\lfloor \frac{3k+j}{2} \right\rfloor.$$

Since $0 \le |\mathbf{s}| \le d$, we have $0 \le \frac{3|\mathbf{s}|}{2(d+1)} < \frac{3}{2}$, and therefore

$$\frac{3k}{2} \le \frac{3n}{2(d+1)} = \frac{3k}{2} + \frac{3|\mathbf{s}|}{2(d+1)} < \frac{3k}{2} + \frac{3}{2}.$$

Lower bound. We first show that $\mathrm{rad}(G) \geq \left\lfloor \frac{3n}{2(d+1)} \right\rfloor - 1$. It follows that

$$\left\lfloor \frac{3n}{2(d+1)} \right\rfloor \leq \left\lfloor \frac{3k}{2} + 1 \right\rfloor \leq \left\lfloor \frac{3k+j}{2} \right\rfloor + 1,$$

and hence

$$\left\lfloor \frac{3n}{2(d+1)} \right\rfloor - 1 \leq \left\lfloor \frac{3k+j}{2} \right\rfloor = \mathrm{rad}(G).$$

Upper bound. We now show that $\mathrm{rad}(G) \leq \left\lfloor \frac{3n}{2(d+1)} \right\rfloor$. If $j = 0$, then

$$\mathrm{rad}(G) = \left\lfloor \frac{3k}{2} \right\rfloor \leq \left\lfloor \frac{3n}{2(d+1)} \right\rfloor.$$

If $j = 1$, then $|\mathbf{s}| \geq d - 1$, and thus

$$\frac{3|\mathbf{s}|}{2(d+1)} \geq \frac{3(d-1)}{2(d+1)} \geq \frac{3}{4}.$$

Hence

$$\frac{3k}{2} = \frac{3n}{2(d+1)} - \frac{3|\mathbf{s}|}{2(d+1)} \leq \frac{3n}{2(d+1)} - \frac{3}{4},$$

and therefore

$$\mathrm{rad}(G) = \left\lfloor \frac{3k+1}{2} \right\rfloor \leq \left\lfloor \frac{3k}{2} + \frac{1}{2} \right\rfloor \leq \left\lfloor \frac{3n}{2(d+1)} - \frac{1}{4} \right\rfloor \leq \left\lfloor \frac{3n}{2(d+1)} \right\rfloor.$$

□

Corollary 5.25. There exist a d_1-necklace G_1 on n_1 vertices and a d_2-necklace G_2 on n_2 vertices such that

$$\mathrm{rad}(G_1) = \left\lfloor \frac{3n_1}{2(d_1+1)} \right\rfloor - 1 \quad \text{and} \quad \mathrm{rad}(G_2) = \left\lfloor \frac{3n_2}{2(d_2+1)} \right\rfloor.$$

In particular, the bounds in Corollary 5.24 are tight.

Proof. Let j be the number of maximal-slack beads of G. Since $|\mathbf{s}| \leq d$, we have $j \in \{0, 1\}$. Hence the underlying cycle C_ℓ has length $\ell = 3k + j$.

Lower bound. Consider the 6-necklace with slack vector (4) on 11 vertices. Since $d - 1 = 5$, no bead has maximal slack, so $j = 0$. Hence $\ell = 3$, and therefore

$$\mathrm{rad}(G) = \mathrm{rad}(C_3) = 1 = \left\lfloor \frac{3 \cdot 11}{2 \cdot (6+1)} \right\rfloor - 1 = \left\lfloor \frac{3 \cdot n}{2(d+1)} \right\rfloor - 1.$$

Thus the lower bound is attained.

Upper bound. Consider the 3-necklace with slack vector (0) on 4 vertices. Then $j = 0$, $\ell = 3$, and

$$\mathrm{rad}(G) = \mathrm{rad}(C_3) = 1 = \left\lfloor \frac{3 \cdot 4}{2 \cdot 4} \right\rfloor = \left\lfloor \frac{3 \cdot n}{2(d+1)} \right\rfloor.$$

Thus the upper bound is attained. □

We proceed by comparing our construction with the upper bound of Kim et al. (Theorem 5.20), which for fixed minimum degree d states that

$$\mathrm{rad}(G) \leq \left\lfloor \frac{3n}{2(d+1)} \right\rfloor.$$

The examples establishing the tightness of this bound are not d-regular in general. For our necklace graphs, however, Corollary 5.24 yields

$$\left\lfloor \frac{3n}{2(d+1)} \right\rfloor - 1 \leq \mathrm{rad}(G) \leq \left\lfloor \frac{3n}{2(d+1)} \right\rfloor.$$

Hence, within the class of d-regular graphs, the upper bound of Kim et al. is attained up to an additive constant of at most 1. Together with the results for the burning number, this shows that necklace constructions yield asymptotically optimal examples under degree constraints.

We now turn to trees, where the absence of cycles and additional structural restrictions lead to different extremal behaviour.

6

Asymptotic Bounds for α-Angular Trees

In contrast to the d-regular graphs considered previously, we now study restricted classes of trees, where the absence of cycles and additional structural constraints allow for stronger bounds on the burning number. We derive an improved asymptotic upper bound on the burning number for such classes of trees by adapting the asymptotic framework of Martinsson [16] to α-angular trees, which generalize homeomorphically irreducible trees.

The key idea is to exploit the branching structure to enforce expansion along sufficiently long paths. Using a connected subgraph expansion argument, we construct a small-distance dominating subtree, yielding the desired bound.

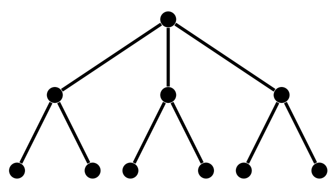
As a consequence, we obtain an asymptotic refinement of the bound of Murakami [17] for homeomorphically irreducible trees, while extending the result to the broader class of α-angular trees.

Definition 6.1. Let $\alpha \geq 2$. An *α-angular tree* is a tree in which every vertex has degree 1 or at least α.

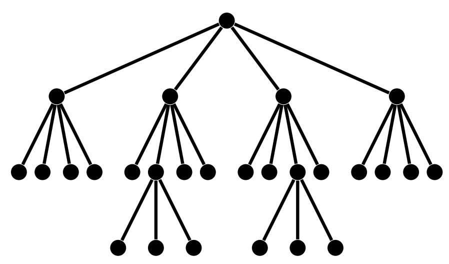
Remark 6.2. For $\alpha = 2$, every tree satisfies the definition. For $\alpha = 3$, this coincides with the standard notion of a homeomorphically irreducible tree, that is, a tree with no vertices of degree 2.

To illustrate how the parameter α influences the structure of the tree, we present representative examples for small values of α.

Example 6.3. Figure 6.1 illustrates two examples of α-angular trees, for $\alpha = 3$ and $\alpha = 4$. In Figure 6.1a, all internal vertices have degree exactly 3, and no vertex has degree 2, corresponding to the classical case of homeomorphically irreducible trees. In Figure 6.1b, every internal vertex has degree at least 4, with some vertices of higher degree, illustrating the increased branching enforced by larger values of α.



(a) A 3-angular tree.



(b) A 4-angular tree with vertices of varying degree.

Figure 6.1: Examples of α-angular trees for different values of α.

We now use this expansion argument to obtain the following upper bound for α-angular trees.

Theorem 6.4. Let T be an α-angular tree on n vertices. Then

$$b(T) \leq (1+o(1))\sqrt{\frac{n}{\alpha-1}}.$$

Proof. Fix an integer $\ell \geq 2$. We construct a sequence of connected subtrees

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_t \subseteq T$$

as follows. Let H_0 consist of a single vertex.

Given H_{i-1}, if there exists a vertex $v \in V(T)$ with $\operatorname{dist}_T(v, H_{i-1}) = \ell$, choose such a vertex and let

$$P_i : v = x_\ell, x_{\ell-1}, \ldots, x_1, x_0$$

be a shortest path from v to H_{i-1}, where $x_0 \in V(H_{i-1})$ and $x_1, \ldots, x_\ell \notin V(H_{i-1})$. Define

$$H_i := T\big[V(H_{i-1}) \cup \{x_1, \ldots, x_\ell\}\big].$$

Let t be the smallest index such that no vertex of T has distance exactly ℓ from H_t, and set $H := H_t$. By the choice of t, every vertex of T lies at distance at most $\ell - 1$ from H.

By construction, each path P_i meets H_{i-1} in exactly one vertex and contributes precisely ℓ new vertices. Hence

$$|V(H_i)| = 1 + i\ell \qquad \text{for all } i \leq t.$$

For $i \geq 0$, let $a_i := |N_1^T[V(H_i)]|$. We claim that for every $i \geq 1$,

$$a_i \geq a_{i-1} + (\alpha-1)(\ell-1).$$

Indeed, consider the newly added path P_i. For each $j \in \{1, \ldots, \ell-1\}$, the vertex x_j is not a leaf (it has two neighbors on P_i), and since T is α-angular, we have $\deg_T(x_j) \geq \alpha$. Thus each such x_j has at least $\alpha - 2$ neighbors outside $V(P_i)$.

As T is a tree, no neighbor of x_j outside $V(P_i)$ can lie in $N_1^T[V(H_{i-1})]$, since otherwise, together with the subpath of P_i from x_0 to x_j, this would create a cycle. Likewise, if a vertex $y \notin V(P_i)$ were adjacent to two distinct vertices among $x_1, \ldots, x_{\ell-1}$, a cycle would again be formed. Thus no such neighbor is adjacent to two different vertices x_j.

Finally, $x_2, \ldots, x_\ell$ do not lie in $N_1^T[V(H_{i-1})]$, since $\operatorname{dist}_T(x_j, H_{i-1}) = j \geq 2$.

Hence,

$$a_i \geq a_{i-1} + (\alpha-2)(\ell-1) + (\ell-1) = a_{i-1} + (\alpha-1)(\ell-1),$$

as claimed.

Note that $a_0 \geq 1$. Hence, by induction, for all $i \geq 0$

$$a_i \geq 1 + i(\alpha-1)(\ell-1).$$

Since $a_t \leq |V(T)| = n$, it follows that

$$t \leq \frac{n-1}{(\alpha-1)(\ell-1)}.$$

Therefore,

$$|V(H)| = 1 + t\ell \leq 1 + \frac{(n-1)\ell}{(\alpha-1)(\ell-1)}.$$

In particular,

$$|V(H)| \leq (1+o(1))\,\frac{\ell}{\ell-1} \cdot \frac{n}{\alpha-1}.$$

Therefore, by Lemma 3.7 applied with $k = \ell - 1$,

$$b(T) \leq b(H) + (\ell - 1).$$

Using Theorem 3.6 and the estimate on $|V(H)|$ above, we get

$$b(T) \leq (1 + o(1)) \sqrt{\frac{\ell}{\ell - 1} \cdot \frac{n}{\alpha - 1}} + (\ell - 1).$$

Finally, we optimize over the free parameter ℓ. The above construction and all preceding inequalities hold for every integer $\ell \geq 2$. We note that

$$\sqrt{\frac{\ell}{\ell - 1} \cdot \frac{n}{\alpha - 1}} = (1 + o(1)) \sqrt{\frac{n}{\alpha - 1}},$$

provided that $\ell \to \infty$, since in this case $\sqrt{\frac{\ell}{\ell-1}} = 1 + o(1)$. Moreover, the additive term $\ell - 1$ is negligible compared to $\sqrt{\frac{n}{\alpha-1}}$ provided that $\ell = o\left(\sqrt{\frac{n}{\alpha-1}}\right)$.

Since ℓ is a free parameter of the construction, we may choose an integer function $\ell = \ell(n)$ satisfying both conditions, for instance $\ell = \lceil \log n \rceil$. With this choice,

$$b(T) \leq (1 + o(1)) \sqrt{\frac{n}{\alpha - 1}}.$$

□

For $\alpha = 3$, corresponding to homeomorphically irreducible trees, Murakami [17] obtained the bound

$$b(T) \leq \left\lceil \sqrt{n} \right\rceil.$$

In contrast, our result yields

$$b(T) \leq (1 + o(1)) \sqrt{\frac{n}{2}},$$

which provides an asymptotic refinement by a factor of $1/\sqrt{2}$, while extending the bound to the broader class of α-angular trees.

This completes our analysis of the burning number under the considered structural constraints.

7

Conclusion and Further Research

In this thesis, we investigated the burning number under structural constraints from an extremal perspective, with a focus on bounded-degree graphs, regular constructions, and restricted classes of trees.

For connected graphs of bounded maximum degree Δ and fixed burning number b, we determined the maximum possible order, given by $U(\Delta, b)$, and showed that this bound is attained by explicit constructions (Corollary 4.4 and 4.17). This yields lower bounds on the burning number in terms of the graph order and maximum degree (Corollary 4.19). In particular, for fixed Δ, these bounds are logarithmic:

$$b(G) = \Omega(\log_{\Delta-1} n).$$

In the d-regular setting, these bounds can be refined. The maximum order equals $U(d, b)$ when d is even, and $U(d, b) - 1$ when d is odd, which are attained by explicit constructions (Corollary 4.6 and 4.15).

The complementary problem of constructing d-regular graphs of fixed order with large burning number is addressed by introducing a family of d-necklaces. These graphs achieve the asymptotic upper bound of Martinsson [16] up to an additive constant of 1 (Corollary 5.19), that is,

$$\left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil - 1 \leq b(G) \leq \left\lceil \sqrt{\frac{3n}{d+1}} \right\rceil + 1.$$

These graphs also attain the general upper bound of Kim et al. [13] for the radius up to an additive constant of 1 (Corollaries 5.24 and 5.25), yielding

$$\left\lfloor \frac{3n}{2(d+1)} \right\rfloor - 1 \leq \operatorname{rad}(G) \leq \left\lfloor \frac{3n}{2(d+1)} \right\rfloor.$$

These results suggest that d-necklaces may capture the extremal behavior of connected d-regular graphs. Determining whether d-necklaces are extremal for the burning number and the radius among connected d-regular graphs remains an interesting open problem. Therefore we pose the following conjecture.

Conjecture 7.1. Let $d \geq 3$. For every connected d-regular graph G on n vertices, there exists a d-necklace H on n vertices such that

$$b(G) \leq b(H).$$

Conjecture 7.2. Let $d \geq 3$. For every connected d-regular graph G on n vertices, there exists a d-necklace H on n vertices such that

$$\operatorname{rad}(G) \leq \operatorname{rad}(H).$$

For restricted classes of trees, adapting the framework of Martinsson [16] to α-angular trees yields improved upper bounds on the burning number (Theorem 6.4), that is,

$$b(T) \leq (1 + o(1))\sqrt{\frac{n}{\alpha - 1}}.$$

In particular, for homeomorphically irreducible trees, this gives an asymptotic refinement of the bound of Murakami [17] by a factor of $1/\sqrt{2}$, while extending the result to a broader class.

In this setting, we construct a connected subgraph that forms a distance dominating set, bound its size using structural properties of the tree, and then apply an upper bound on the burning number. This approach is not tied specifically to the asymptotic bound of Norin and Turcotte [18]. It remains of interest to determine whether alternative, possibly non-asymptotic bounds can be used in its final step to establish the burning number conjecture for specific graph classes or to obtain sharper bounds.

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