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A DICHOTOMY CONCERNING UNIFORM BOUNDEDNESS OF RIESZ TRANSFORMS ON RIEMANNIAN MANIFOLDS

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Abstract. Given a sequence of complete Riemannian manifolds (M_n) of the same dimension, we construct a complete Riemannian manifold M such that for all $p \in (1, \infty)$ the L^p-norm of the Riesz transform on M dominates the L^p-norm of the Riesz transform on M_n for all n. Thus we establish the following dichotomy: given p and d, either there is a uniform L^p bound on the Riesz transform over all complete d-dimensional Riemannian manifolds, or there exists a complete Riemannian manifold with Riesz transform unbounded on L^p.

1. Introduction

Given a Riemannian manifold M, one can consider the Riesz transform $R := \nabla(-\Delta)^{\frac{1}{2}}$, where ∇ is the Riemannian gradient and Δ is the (negative) Laplace–Beltrami operator. In the Euclidean case $M = \mathbb{R}^n$, this can be identified with the vector of classical Riesz transforms $(R_1, \ldots, R_n)$, as can be seen by writing R as a Fourier multiplier (see [12, §5.1.4]).

It is easy to show that R is bounded from $L^2(M)$ to $L^2(M; TM)$, and substantially harder to determine whether R extends to a bounded map from $L^p(M)$ to $L^p(M; TM)$ for $p \neq 2$. We let

$$R_p(M) := \sup_{\|f\|_{L^p} \leq 1} \|R(f)\|_{L^p}$$

denote the (possibly infinite) L^p-norm of the Riesz transform on M. Various conditions, often involving the heat kernel on M and its gradient, are known to imply finiteness of $R_p(M)$; see for example [2, 3, 4, 5, 6, 7, 8, 9, 13, 14]. These results usually entail finiteness of $R_p(M)$ for all $p \in (1, 2)$, or for some range of $p > 2$. On the other hand, there exist manifolds M for which $R_p(M)$ is known to be infinite for some (or all) $p > 2$: see [1, 5, 6, 7, 8, 13].

Remark 1.1. When M has finite volume we abuse notation and write $L^p(M)$ to denote the space of p-integrable functions *with mean zero.* This modification ensures that $(-\Delta)^{-1/2}$ is densely defined. When M has infinite volume, $L^p(M)$ denotes the usual Lebesgue space.

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The Euclidean case is now classical: for all $p \in (1,\infty)$ there is a constant $C_p < \infty$ such that $R_p(\mathbb{R}^n) \leq C_p < \infty$ for all $n \in \mathbb{N}$ ([16]). This behaviour is expected to persist for all complete Riemannian manifolds, at least for $p < 2$. More precisely, in [9] it is conjectured that for all $p \in (1,2)$ there exists a constant $C_p < \infty$ such that $R_p(M) \leq C_p$ for all complete Riemannian manifolds M. Such uniform bounds have been proven for all $p \in (1,\infty)$ under curvature assumptions; rather than provide an overview of the vast literature on this topic we simply point to the recent paper [10] and references therein.

One could weaken the conjecture slightly and guess that $R_p(M)$ is finite for all M, given $p \in (1,2)$. In this article we show that this can only hold if the bound is uniform among all manifolds of a fixed dimension. This observation follows from the following dichotomy.

Theorem 1.2. *Fix $d \in \mathbb{N}$ and $p \in (1,\infty)$. Then the following dichotomy holds: either*

- *there exists a constant $C_{p,d} < \infty$ such that $R_p(M) \leq C_{p,d}$ for all complete d-dimensional Riemannian manifolds M, or*
- *there exists a complete $(d+1)$-dimensional Riemannian manifold M such that $R_p(M) = \infty$.*

This follows from the following proposition, which we prove by an explicit construction.

Proposition 1.3. *Fix $d \geq 1$, and let $(M_n)_{n\in\mathbb{N}}$ be a sequence of complete d-dimensional Riemannian manifolds. Then there exists a complete Riemannian manifold M of dimension $d+1$ such that for all $p \in (1,\infty)$,*

$$R_p(M) \geq \sup_{n\in\mathbb{N}} R_p(M_n).$$

The main implication of Theorem 1.2 is as follows: to construct a manifold M for which $R_p(M) = \infty$ for some $p \in (1,2)$, it suffices to construct a sequence $(M_n)_{n\in\mathbb{N}}$ of manifolds of equal dimension such that $R_p(M_n) \to \infty$ as $n \to \infty$. Thus one is led to consider lower bounds for L^p-norms of Riesz transforms. These seem not to have been considered in the literature, excluding of course the well-known computation of the L^p-norm of the Hilbert transform (the Riesz transform on $\mathbb{R}$) [15]. We hope that our contribution will provoke further interest in such lower bounds.

2. Preliminary lemmas

We begin with some basic lemmas. The first says that the range of the Laplace-Beltrami operator is dense in L^p, and the second relates the Riesz transform on a manifold M with that on the M-cylinder $M \times \mathbb{R}$. These cylinders play a key role in the proof of our main theorem.

Lemma 2.1. *Let M be a complete Riemannian manifold. Then the set $S := \Delta(C_c^\infty(M))$ is dense in $L^p(M)$ for all $p \in (1,\infty)$ (recalling that we write $L^p(M)$ for the space of p-integrable mean zero functions when M has finite volume).*

Proof. Let $H \in L^{p'}(M)$ be such that $\langle H, F\rangle = 0$ for every $F \in S$. Then $\langle H, \Delta G\rangle = 0$ for every test function G, so H is harmonic. By [17, Theorem 3], it follows that H is constant, and the result follows. $\square$

Lemma 2.2. *Let M be a complete Riemannian manifold. Then*

$$R_p(M\times\mathbb{R})\geq R_p(M).$$

Proof. Consider the following modification of the Riesz transform on $M\times\mathbb{R}$:

$$\tilde{R}:=\nabla_M(-\Delta_{M\times\mathbb{R}})^{-\frac{1}{2}}=\nabla_M(-\Delta_M-\partial_t^2)^{-\frac{1}{2}}.$$

This is just the projection of R onto the first summand of the tangent bundle $T(M\times\mathbb{R})=TM\oplus T\mathbb{R}$, so we have that

$$\|\tilde{R}F\|_{L^p}\leq\|RF\|_{L^p}. \tag{1}$$

Let $F\in C_c^\infty(M\times\mathbb{R})$, and for all $\lambda>0$ consider the function

$$F_\lambda(x,t):=\lambda^{\frac{1}{p}}F(x,\lambda t),$$

which satisfies $\|F_\lambda\|_{L^p(M\times\mathbb{R})}=\|F\|_{L^p(M\times\mathbb{R})}$. Rescaling the operator $\tilde{R}$ in the variable t, we define

$$\tilde{R}_\lambda:=\nabla_M(-\Delta_M-\lambda^2\partial_t^2)^{-\frac{1}{2}},$$

so that

$$\|\tilde{R}F_\lambda\|_{L^p}=\|\tilde{R}_\lambda F\|_{L^p}. \tag{2}$$

Now take $f\in C_c^\infty(M)\cap D((-\Delta_M)^{-\frac{1}{2}})$ and $\rho\in C_c^\infty(\mathbb{R})$ such that $\|\rho\|_{L^p(\mathbb{R})}=1$, and consider the function $F(x,t)=f(x)\rho(t)$. Since Δ_M and ∂_t^2 commute, and the function

$$G_\lambda(x,y)=\left(\frac{x}{x+\lambda^2 y}\right)^{\frac{1}{2}}$$

is bounded by 1 for $(x,y)>0$, and $G_\lambda\to 1$ pointwise as $\lambda\to 0$, we have

$$\lim_{\lambda\to 0}(-\Delta_M-\lambda^2\partial_t^2)^{-\frac{1}{2}}F=\lim_{\lambda\to 0}G_\lambda(-\Delta_M,-\partial_t^2)(-\Delta_M)^{-\frac{1}{2}}f\otimes\rho=(-\Delta_M)^{-\frac{1}{2}}f\otimes\rho$$

in L^2, and thus also as distributions. Therefore $\tilde{R}_\lambda F\to Rf\otimes\rho$ as distributions, and so

$$\liminf_{\lambda\to 0}\|\tilde{R}_\lambda F\|_{L^p(M\times\mathbb{R})}\geq\|Rf\otimes\rho\|_{L^p(M\times\mathbb{R})}=\|Rf\|_{L^p(M)}.$$

Combining this with (2) and (1), and the fact that $C_c^\infty(M)\cap D((-\Delta_M)^{-\frac{1}{2}})$ is dense in $L^p(M)$,[1] yields $R_p(M\times\mathbb{R})\geq R_p(M)$. □

3. Proof of the main theorem

In this section we carry out the construction that proves Proposition 1.3, which implies Theorem 1.2.

Consider a sequence $(M_n)_{n\in\mathbb{N}}$ of complete d-dimensional Riemannian manifolds. We will connect the M_n-cylinders $(M_n\times\mathbb{R})_{n\in\mathbb{N}}$ along a $\mathbb{T}^d$-cylinder $\mathbb{T}^d\times\mathbb{R}$ as follows.[2] For each $n\in\mathbb{N}$ fix a coordinate chart $U_n\subset M_n\times(-1/2,1/2)$ and a small ball $B_n\subset U_n$. Similarly, for each $n\in\mathbb{N}$ choose a small coordinate chart $U'_n\subset\mathbb{T}^n\times\mathbb{R}$ such that the charts $(U'_n)_{n\in\mathbb{N}}$ are pairwise disjoint, and a small ball $B'_n\subset U'_n$. For each $n\in\mathbb{N}$, glue the manifold $(M_n\times\mathbb{R})\setminus B_n$ to $(\mathbb{T}^n\times\mathbb{R})\setminus B'_n$ along the boundaries

[1] This follows from the inclusion $D((-\Delta_M)^{-\frac{1}{2}})\supseteq D((-\Delta_M)^{-1})\supseteq\Delta_M(C_c^\infty(M))$, which is dense by Lemma 2.1. See also [11, Lemma 2.2]. Again, recall that $L^p(M)$ denotes the corresponding space of mean zero functions when M has finite volume.

[2] Of course, one could connect the M_n-cylinders to each other directly, without needing the $\mathbb{T}^d$-cylinder. This would work just as well.

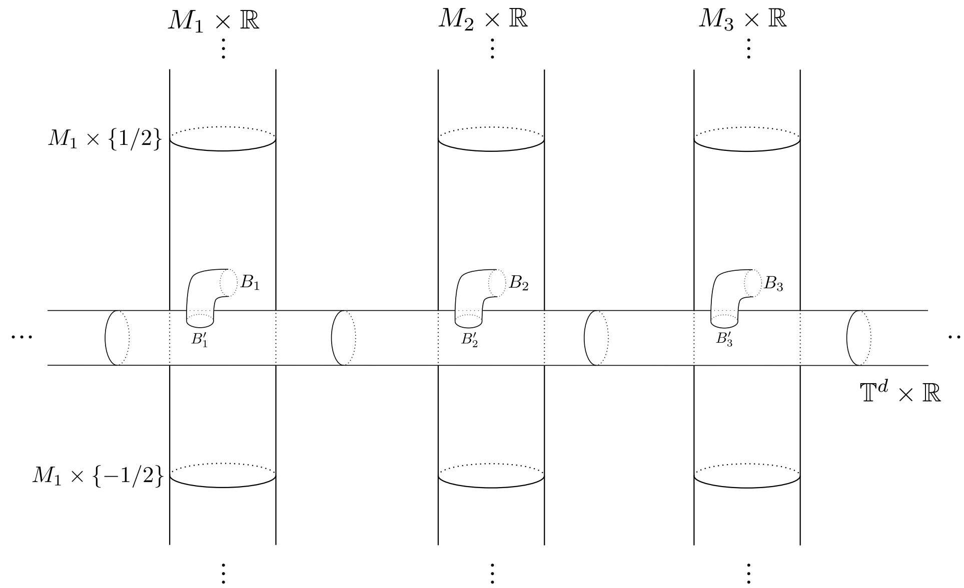

FIGURE 1. Construction of M from $(M_n)_{n\in\mathbb{N}}$.

of B_n and B'_n; this is possible since both these balls are 'Euclidean' balls sitting inside coordinate charts. This results in a C^0-Riemannian manifold (M, g'), which is C^∞ away from the set $\Sigma = \cup_n \partial B_n$ on which we glued the manifolds together. Mollify the metric to get a C^∞-Riemannian manifold (M, g) such that $g = g'$ away from the ε-neighbourhood of Σ for some very small ε. An artist's impression of this construction, with $M_n = S^1$ for each n, is shown in Figure 1.

For each $n \in \mathbb{N}$ we have an inclusion map

$$i_n \colon M_n \times (1, \infty) \to M$$

which is an isometry. From here on we fix n and just write $i = i_n$. Functions on M can be pulled back to $M_n \times (1, \infty)$; the pullback map is denoted i^*, so that for $f \colon M \to \mathbb{R}$ the function $i^* f \colon M_n \times (1, \infty) \to \mathbb{R}$ is defined by

$$i^* f(x, t) = f(i(x, t)).$$

On the other hand, for $g \colon M_n \times (1, \infty) \to \mathbb{R}$ we can define a pushforward $i_* g \colon M \to \mathbb{R}$ by setting $i_* g(i(x, t)) := g(x, t)$ on $i(M_n \times (1, \infty))$ and extending by zero to the rest of M. For a function $g \colon M_n \times \mathbb{R} \to \mathbb{R}$ and for $s \in \mathbb{R}$ we let $\tau_s g \colon M_n \times \mathbb{R} \to \mathbb{R}$ be the translated function $\tau_s g(x, t) := g(x, t - s)$. Similarly if $g \colon M_n \times (1, \infty) \to \mathbb{R}$ we can define $\tau_s g \colon M_n \times (1 + s, \infty) \to \mathbb{R}$. These concepts apply equally well to vector fields in place of functions.

We will need the following lemma, which relates the heat flow on $M_n \times \mathbb{R}$ to the one on M.

Lemma 3.1. *Let $F \colon M_n \times \mathbb{R} \to \mathbb{R}$ be smooth and compactly supported, and fix $\sigma > 0$. Then for every $(x, t) \in M_n \times \mathbb{R}$,*

$$\lim_{s \to +\infty} (e^{\sigma \Delta_M} i_* \tau_s F)(i(x, t + s)) = (e^{\sigma \Delta_{M_n \times \mathbb{R}}} F)(x, t).$$

Proof. Let $W_{x,t}(\sigma)$ be a Brownian motion on $M_n \times \mathbb{R}$ at time σ starting from the point (x,t). Since the generator $\frac{1}{2}\Delta_{M_n\times\mathbb{R}}$ satisfies $\frac{1}{2}i_* \circ \Delta_{M\times\mathbb{R}}|_{i(M_n\times(1,+\infty))} = \frac{1}{2}\Delta_M|_{i(M_n\times(1,+\infty))}$, defining the stopping time

$$T(x,t) := \inf\{s : W_{x,t}(s) \in M_n \times (-\infty, 1)\},$$

we have that $i(W_{x,t}(\sigma))$ is a Brownian motion on M for $\sigma < T(x,t)$. Therefore there exists a Brownian motion $\tilde{W}_{i(x,t)}(\sigma)$ on M such that $\tilde{W}(\sigma) = i(W(\sigma))$ for $\sigma < T$; if $\overline{W}$ is a Brownian motion on M, we can take for example

$$\tilde{W}_{i(x,t)}(\sigma) = \begin{cases} i(W_{x,t}(\sigma)) & \text{if } \sigma < T, \\ \overline{W}_{i(W_{x,t}(T))}(\sigma - T) & \text{if } \sigma \geq T. \end{cases}$$

We have that

$$\begin{aligned}
&(e^{\sigma\Delta_M} i_*\tau_s F)(i(x,t+s)) \\
&= \mathbb{E}[(i_*\tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))] \\
&= \mathbb{E}[(i_*\tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma<T}] + \mathbb{E}[(i_*\tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma\geq T}] \\
&= \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))\mathbb{1}_{2\sigma<T}] + \mathbb{E}[(i_*\tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma\geq T}] \\
&= \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))] \\
&\quad - \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))\mathbb{1}_{2\sigma\geq T}] + \mathbb{E}[(i_*\tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma\geq T}] \\
&= (e^{\sigma\Delta_{M_n\times\mathbb{R}}}\tau_s F)(x,t+s) \\
&\quad - \mathbb{E}[(\tau_s F)(W_{x,t+s}(2\sigma))\mathbb{1}_{2\sigma\geq T}] + \mathbb{E}[(i_*\tau_s F)(\tilde{W}_{i(x,t+s)}(2\sigma))\mathbb{1}_{2\sigma\geq T}].
\end{aligned}$$

Therefore

$$\left|(e^{\sigma\Delta_M} i_*\tau_s F)(i(x,t+s)) - (e^{\sigma\Delta_{M_n\times\mathbb{R}}}\tau_s F)(x,t+s)\right| \leq 2\|F\|_{L^\infty}\mathbb{P}(T(x,t+s) \leq 2\sigma).$$

Since $\Delta_{M_n\times\mathbb{R}}$ is translation invariant in the $\mathbb{R}$ coordinate, we have that

$$\begin{aligned}
\mathbb{P}(T(x,t+s) \leq 2\sigma) &\leq \mathbb{P}\big(\{W_{x,t+s}(\sigma') \in M_n \times (-\infty,1) \text{ for some } \sigma' \leq 2\sigma+1\}\big) \\
&= \mathbb{P}\big(\{W_{x,t}(\sigma') \in M_n \times (-\infty,1-s) \text{ for some } \sigma' \leq 2\sigma+1\}\big)
\end{aligned}$$

and by continuity of $W_{x,t}(\cdot)$, this tends to 0 as $s \to \infty$. Thus we find that

$$\lim_{s\to+\infty}\left((e^{\sigma\Delta_M} i_*\tau_s F)(i(x,t+s)) - (e^{\sigma\Delta_{M_n\times\mathbb{R}}}\tau_s F)(x,t+s)\right) = 0.$$

The conclusion follows from translation invariance of $\Delta_{M_n\times\mathbb{R}}$ in $\mathbb{R}$. □

We return to the proof of Proposition 1.3. Fix $\varepsilon > 0$, and choose $F = \Delta_{M_n\times\mathbb{R}}H$ for some $H \in C_c^\infty(M_n \times \mathbb{R})$ with $\|F\|_{L^p} = 1$ such that

$$\|R_{M_n\times\mathbb{R}}F\|_{L^p} \geq (R_p(M_n) - \varepsilon) \wedge \varepsilon^{-1}.$$

Such a function exists by Lemmas 2.1 and 2.2. We claim that

$$\lim_{s\to+\infty}\tau_{-s}i^* R_M(i_*\tau_s F) = R_{M_n\times\mathbb{R}}F \tag{3}$$

as distributions. Assuming (3) for the moment, we have

$$\begin{aligned}
\limsup_{s\to\infty}\|R_M(i_*\tau_s F)\|_{L^p(M)} &\geq \limsup_{s\to\infty}\|i^* R_M(i_*\tau_s F)\|_{L^p(M_n\times\mathbb{R})} \\
&= \limsup_{s\to\infty}\|\tau_{-s}i^* R_M(i_*\tau_s F)\|_{L^p(M_n\times\mathbb{R})} \\
&\geq \|R_{M_n\times\mathbb{R}}F\|_{L^p(M_n\times\mathbb{R})} \geq R_p(M_n) - \varepsilon,
\end{aligned}$$

while for all $s \in \mathbb{R}$

$$\|i_*\tau_s F\|_{L^p(M)} \leq \|\tau_s F\|_{L^p(M_n\times\mathbb{R})} = \|F\|_{L^p(M_n\times\mathbb{R})} \leq 1.$$

The result follows, so it remains to prove (3).

For s sufficiently large, we have that

$$i_*\tau_s F = i_*\tau_s(\Delta_{M_n\times\mathbb{R}} H) = i_*(\Delta_{M_n\times\mathbb{R}}\tau_s H) = \Delta_M i_*\tau_s H,$$

therefore $i_*\tau_s F \in D(\Delta_M^{-1}) \subseteq D((-\Delta_M)^{-\frac{1}{2}})$, and hence

$$R(i_*\tau_s F) = \nabla\left((-\Delta)_M^{-\frac{1}{2}} i_*\tau_s F\right)$$

as a distribution. To test the distributional convergence, let X be a smooth compactly supported vector field in $M_n \times \mathbb{R}$. For large s we have that

$$\begin{aligned}\langle \tau_{-s} i^* R_M(i_*\tau_s F), X\rangle &= \langle R_M(i_*\tau_s F), i_*\tau_s X\rangle \\ &= \left\langle (-\Delta)_M^{-\frac{1}{2}} i_*\tau_s F, \operatorname{div}(i_*\tau_s X)\right\rangle \\ &= \left\langle (-\Delta)_M^{-\frac{1}{2}} i_*\tau_s F, i_*\tau_s \operatorname{div}(X)\right\rangle.\end{aligned}$$

Therefore it is enough to show that for every $G \in C_c^\infty(M_n\times\mathbb{R})$,

$$\lim_{s\to\infty}\left\langle (-\Delta)_M^{-\frac{1}{2}} i_*\tau_s F, i_*\tau_s G\right\rangle = \left\langle (-\Delta)_{M_n\times\mathbb{R}}^{-\frac{1}{2}} F, G\right\rangle. \tag{4}$$

By the well-known formula

$$(-\Delta)^{-\frac{1}{2}} = \pi^{-\frac{1}{2}} \int_0^{+\infty} \sigma^{-\frac{1}{2}} e^{\sigma\Delta}\, d\sigma,$$

(4) is equivalent to showing that

$$\lim_{s\to\infty}\int_0^{+\infty} \sigma^{-\frac{1}{2}} \left\langle e^{\sigma\Delta_M} i_*\tau_s F, i_*\tau_s G\right\rangle d\sigma = \int_0^{+\infty} \sigma^{-\frac{1}{2}} \left\langle e^{\sigma\Delta_{M_n\times\mathbb{R}}} F, G\right\rangle d\sigma. \tag{5}$$

Note that

$$\left|\sigma^{-\frac{1}{2}} \left\langle e^{\sigma\Delta_M} i_*\tau_s F, i_*\tau_s G\right\rangle\right| \leq \sigma^{-\frac{1}{2}} \|i_*\tau_s F\|_{L^2} \|i_*\tau_s G\|_{L^2} \leq \sigma^{-\frac{1}{2}} \|F\|_{L^2}\|G\|_{L^2}$$

and

$$\left|\sigma^{-\frac{1}{2}} \left\langle e^{\sigma\Delta_M} i_*\tau_s F, i_*\tau_s G\right\rangle\right| = \left|\sigma^{-\frac{3}{2}} \left\langle e^{\sigma\Delta_M}\sigma\Delta_M i_*\tau_s H, i_*\tau_s G\right\rangle\right| \lesssim \sigma^{-\frac{3}{2}} \|H\|_{L^2}\|G\|_{L^2}.$$

Since the function $\min(\sigma^{-\frac{1}{2}}, \sigma^{-\frac{3}{2}})$ is integrable, by dominated convergence (5) will be proved if we show

$$\lim_{s\to\infty}\left\langle e^{\sigma\Delta_M} i_*\tau_s F, i_*\tau_s G\right\rangle = \left\langle e^{\sigma\Delta_{M_n\times\mathbb{R}}} F, G\right\rangle \tag{6}$$

for every $\sigma > 0$. We show (6) by writing

$$\begin{aligned}\lim_{s\to\infty}\left\langle e^{\sigma\Delta_M} i_*\tau_s F, i_*\tau_s G\right\rangle &= \lim_{s\to\infty}\left\langle \tau_{-s} i^* e^{\sigma\Delta_M} i_*\tau_s F, G\right\rangle \\ &= \lim_{s\to\infty}\int_{1-s}^{+\infty}\int_{M_n} (e^{\sigma\Delta_M} i_*\tau_s F)(i(x,t+s))G(x,t)\,dx\,dt \\ &= \int_{\mathbb{R}}\int_{M_n} (e^{\sigma\Delta_{M_n\times\mathbb{R}}} F)(x,t)G(x,t)\,dx\,dt \\ &= \left\langle e^{\sigma\Delta_{M_n\times\mathbb{R}}} F, G\right\rangle,\end{aligned}$$

using Lemma 3.1 and dominated convergence (by $\|F\|_{L^\infty} |G(x,t)|$). This completes the proof of Proposition 1.3, and hence establishes Theorem 1.2.

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